1 Preliminary Definitions

Definition 1.1. Let (X, d) be a metric space, and let $x, y \in X$. Let $A \subseteq [0, d(x, y)]$ be such that $\{0, d(x, y)\} \subseteq A$. Then a map $\gamma : A \to X$ is called an **undergeodesic** from x to y if $\gamma(0) = x$, $\gamma(d(x, y)) = y$, and

$$d(\gamma(s), \gamma(t)) = |s - t|$$

for all $s, t \in A$. The image of γ is called an **undergeodesic segment**.

Definition 1.2. If $\gamma : A \to X$ is an undergeodesic from x to y, we say that γ is **maximal** if for every $t \in [0, d(x, y)] \setminus A$, there does not exist $z \in X$ such that the map $\gamma' : A \cup \{t\} \to X$ extending γ with $\gamma'(t) = z$ is an undergeodesic. Equivalently, γ is maximal if there does not exist $\gamma' : A' \to X$ with A a strict subset of A' such that $\gamma'|_A = \gamma$.

Proposition 1.3. *Any undergeodesic $\gamma : A \to X$ can be extended to a maximal undergeodesic $\gamma^* : A' \to X$ such that $\gamma^*|_A = \gamma$.*

Definition 1.4. Let (X, d) be a metric space, and let $\gamma : A \to X$ be an undergeodesic from x to y. Then γ is called a **dense** undergeodesic if A is dense in $[0, d(x, y)]$.

Definition 1.5. A metric space X is called **almost geodesic** if for any x, y in X there exists a dense undergeodesic $\gamma : A \to X$ from x to y.

Proposition 1.6. *Let γ be a dense undergeodesic in a metric space X. Then there exists exactly one maximal undergeodesic γ' extending γ.*

Definition 1.7. A geodesic space X is called **full** if every maximal undergeodesic in X is a geodesic. Equivalently, a geodesic space is called full if every undergeodesic lies inside the image of a geodesic. A metric space X is called **almost full** if every maximal undergeodesic in X is dense.

Proposition 1.8. *If an almost geodesic space X is complete, then it is geodesic, and every dense undergeodesic in X lies in a geodesic.*

2 Geodesic Saturation

Question 2.1. *Given a metric space X, can X be embedded into a geodesic space X' such that every undergeodesic $\gamma : A \to X$ can be written as $\gamma = \tilde{\gamma}|_A$ for some geodesic $\tilde{\gamma} : I \to X'$? Can this be a full geodesic space?*

Both questions are answered in the affirmative via the following construction.

Definition 2.2. Let (X, d_X) be a metric space. For any $x, y \in X$ and any maximal undergeodesic $\gamma : A_\gamma \to X$ from x to y, let $L_\gamma = [0, d(x, y)]$. We denote elements of L_γ by $t^{(\gamma)}$ for $t \in [0, d(x, y)]$.
Define

$$\hat{X}' = \left(X \sqcup \bigsqcup_\gamma L_\gamma \right) / \sim,$$

where the union ranges over all undergeodesics γ in X, and $\sim$ is the equivalence relation generated by the following identifications:

- For all $t \in A_\gamma$, identify $t^{(\gamma)}$ in L_γ with $\gamma(t) \in X$.
- If the image of γ lies inside the image of γ', then L_γ is glued to the corresponding portion of $L_{\gamma'}$. Formally, if $\gamma(A_\gamma)$ lies inside $\gamma'(A'_\gamma)$, then for all $t^{(\gamma)} \in A_\gamma$, $t^{(\gamma)}$ is glued to $((\gamma')^{-1}(\gamma(0)) + t^{(\gamma)})^{(\gamma')}$.

Note that every point $p \in \hat{X}'$ can be written as $t^{(\gamma)}$ for some $t \in L_\gamma$. Define a pseudometric $\hat{d}'$ on $\hat{X}'$ by defining

$$\hat{d}'(x, y) = \inf \sum_i |t_i^{(\gamma_i)} - t_{i+1}^{(\gamma_i)}|,$$

where the infimum ranges over all finite sequences $\{p_i\}_{i=0}^n$ with $p_0 = x$, $p_n = y$, and for $0 \le i < n$, $p_i = t_i^{(\gamma_i)}$ and $p_{i+1} = t_{i+1}^{(\gamma_i)}$ for some $t_i, t_{i+1} \in L_{\gamma_i}$. Let $\hat{X}$ be the metric space associated with the pseudometric space $\hat{X}'$. Then $\hat{X}$ is called the **geodesic saturation** of X.

Note for the future that, following the definition of the geodesic saturation, no new point in $\hat{X}$ is glued to more than one point of X.

Proposition 2.3. *The geodesic saturation of a metric space X is a metric space into which X embeds isometrically. Additionally, every undergeodesic in X extends to a geodesic in $\hat{X}$.*

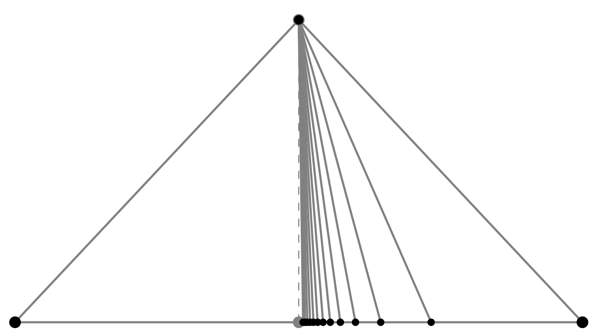
Proof. We first need to show that $\hat{X}'$ as defined above is a pseudometric space. Reflexivity and symmetry are clear from the definition of $\hat{d}$. Now we show the triangle inequality: choose $x, y, z \in \hat{X}'$, and let $\epsilon > 0$. Let $\{p_i\}$ be a path from x to y of length less than $\hat{d}'(x, y) + \epsilon/2$, and let $\{q_i\}$ be a path from y to z of length less than $\hat{d}'(y, z) + \epsilon/2$. Then concatenating the two paths gives a path from x to z of length less than $\hat{d}'(x, y) + \hat{d}'(y, z) + \epsilon$, so that

$$\hat{d}'(x, z) < \hat{d}'(x, y) + \hat{d}'(y, z) + \epsilon.$$

Taking ϵ to 0 gives the triangle inequality. Thus $\hat{d}'$ is a pseudometric, and so $(\hat{X}, \hat{d})$ is a metric space.

Note that every undergeodesic γ extends to a maximal undergeodesic γ'. It is clear from the definition of $\hat{d}$ that X embeds isometrically into $\hat{X}$, and additionally that every segment $L_{\gamma'}$ embeds isometrically into $\hat{X}$ (and is thus a geodesic in $\hat{X}$ that extends γ). $\square$

Example 2.4. *Let $X = \{(\frac{1}{n}, 0) : n \in \mathbb{N}\} \cup (-1, 0) \cup (0, 1)$, considered as a subspace of $\mathbb{R}^2$. Then $\hat{d}((0, 0), (0, 1)) = 1$, but no path from $(0, 0)$ to $(0, 1)$ achieves this length. This shows that $\hat{X}$ is not, in general, a geodesic space.*



Example 2.5. *Let $X = \mathbb{Q}$. Then $\hat{X}$ is homeomorphic to $\mathbb{R}$. The same is true if $X = \mathbb{Z}$, or in general if X is an unbounded subspace of $\mathbb{R}$.*

Example 2.6. *Let $X = \mathbb{Q}^2$. Then*

$$\hat{X} \cong \{(x, y) \in \mathbb{R}^2 : ax + by = c \text{ for some } a, b, c \in \mathbb{Q}\},$$

the set of rational lines in the plane. Note that this space is not geodesic. However, the geodesic saturation of $\hat{X}$ is isometric to $\mathbb{R}^2$.

Proposition 2.7. *$X = \hat{X}$ (meaning that the natural inclusion map from X into $\hat{X}$ is an isometry) if and only if X is a full geodesic space.*

Proof. If X is full, then any maximal undergeodesic is a geodesic. Thus, for any maximal undergeodesic γ, all of the points of L_γ will be glued to the corresponding points of γ. Thus no new points will be added.

Now suppose that X is not a full geodesic space. Then there exists a maximal undergeodesic $\gamma : A \to X$ which is not a geodesic in X (with support I). Let t be a point in $I \setminus A$. Then the point $t^{(\gamma)} \in L_\gamma$ will not be glued to any points in X (because $t \notin A$). Thus $t^{(\gamma)}$ will not be in the image of the inclusion from X to $\hat{X}$, so $X \neq \hat{X}$. $\square$

Definition 2.8. Let $\mathcal{S} : \mathbf{Met} \to \mathbf{Met}$ be the map from the objects in the category of metric spaces to itself (with morphisms being isometric embeddings) that takes a space X to $\hat{X} = \mathcal{S}X$. Note again that X embeds into $\mathcal{S}X$ via a natural inclusion map. Finally, note that the category $\mathbf{Met}$ has arbitrary direct limits.

Now, to affirmatively answer Question 2.1, we can do the following:

Proposition 2.9. *Every metric space X can be isometrically embedded into a geodesic space X' such that every undergeodesic $\gamma : A \to X$ can be written as $\gamma = \tilde{\gamma}|_A$ for some geodesic $\tilde{\gamma} : I \to X'$.*

Proof. Define a sequence of metric spaces and inclusions by

$$X_0 \;=\; X, \qquad X_{n+1} \;=\; \mathcal{S}(X_n), \quad i_n \colon X_n \;\hookrightarrow\; X_{n+1}.$$

Now form the direct limit in $\mathbf{Met}$ of this diagram:

$$X' \;=\; \varinjlim_{n<\omega} X_n \quad \text{with the canonical inclusion maps} \quad j_n \colon X_n \;\longrightarrow\; X'.$$

In particular $j_0 \colon X \to X'$ is an isometric embedding. Given any two points $p, q \in X'$, there exists a finite N such that $p, q \in j_N(X_N)$. But, by construction, every two points in X_N are connected by a geodesic segment in $X_{N+1} = \mathcal{S}(X_N)$. Hence there is a geodesic

$$\widetilde{\gamma} \colon [0, d(p,q)] \;\longrightarrow\; X_{N+1}$$

with $\widetilde{\gamma}(0) = j_N(p)$ and $\widetilde{\gamma}(d(p,q)) = j_N(q)$. Composing with the isometric inclusion $j_{N+1} \colon X_{N+1} \to X'$ shows that X' is geodesic.

The fact that every undergeodesic $\gamma : A \to X$ can be written as $\gamma = \tilde{\gamma}|_A$ for some geodesic $\tilde{\gamma} : I \to X'$ follows from the fact that such an extension can be found in X_1, and X_1 isometrically embeds into X'. Therefore X' is a geodesic space, $X \hookrightarrow X'$ is isometric, and every undergeodesic in X extends to a geodesic in X', as required. $\square$

We can in fact take this further:

Theorem 2.10. *Every metric space can be isometrically embedded into a full geodesic metric space.*

Proof. Let $c = 2^{\aleph_0}$ be the cardinality of the continuum, and fix an ordinal α with cofinality strictly greater than c (say, $\alpha = (c)^+$, the successor cardinal of c). We build by transfinite recursion a chain

$$X_0 = X, \quad X_{\beta+1} = \mathcal{S}(X_\beta), \quad X_\lambda = \varinjlim_{\beta<\lambda} X_\beta \quad (\lambda \text{ a limit ordinal} \leq \alpha),$$

where each inclusion $X_\beta \hookrightarrow X_{\beta+1}$ is the canonical isometry, and the direct limits are taken in **Met**.

We claim that $\mathcal{S}(X_\alpha) = X_\alpha$. Indeed, any point of $\mathcal{S}(X_\alpha)$ lies on some geodesic segment coming from an undergeodesic $\gamma \colon A \to X_\alpha$. Note that the map $\beta \colon A \to \alpha$ where $\beta(t) = \min\{\beta : \gamma(t) \in X_\beta\}$ has image of size $\leq c$, so $\delta = \sup_{t\in A} \beta(t) \leq c$. Then since $\delta \leq c < \mathrm{cf}(\alpha)$, $\gamma(A) \subset X_\delta$ where $X_{\delta+1} \subset X_\alpha$, and hence the geodesic extension L_γ already lives in $\mathcal{S}(X_\delta) = X_{\delta+1} \subset X_\alpha$. No new points are thus added at stage α, so $\mathcal{S}(X_\alpha) = X_\alpha$. Proposition 2.7 then gives that X_α is full and geodesic.

The inclusion $X = X_0 \hookrightarrow X_\alpha$ is thus an isometric embedding into a full geodesic space. This completes the proof. □

Definition 2.11. Denote by $\tilde{X}$ the full geodesic space associated with X as constructed in Theorem 2.10.

3 What I need to prove!

I don't know how to prove the first two statements listed here.

Lemma 3.1. *Let X be an almost full metric space. Then $\hat{X}$ is almost full.*

Lemma 3.2. *The direct limit of a family of almost full metric spaces (where the morphisms are isometric embeddings) is almost full.*

Lemma 3.3. *X is a dense subset of $\tilde{X}$.*

Proof. Let X be a metric space. Using the notation of Theorem 2.10, we need to show that X is dense in X_λ for all ordinals $\lambda \leq \alpha$. To do this, first note that if $X_\lambda = \varinjlim_{\beta<\lambda} X_\beta$ for some limit ordinal λ and X is dense in X_β for all $\beta < \lambda$, then

X is dense in X_λ. This is because every point $x \in X_\lambda$ is in X_β for some $\beta < \lambda$, and thus every neighborhood of x intersects X.

Now suppose that $\lambda = \rho + 1$ is not a limit ordinal, and assume that X is dense in X_β for $\beta < \lambda$. Then $X_\lambda = \mathcal{S}(X_\rho)$. Take $t^{(\gamma)} \in X_\lambda$. Since X_ρ is almost full by Lemma 3.1, A_γ is dense in L_γ. Thus any neighborhood of $t^{(\gamma)}$ will contain a point of A_γ, which is in turn a point of X_ρ. Thus X_ρ is dense in X_λ, and thus X is dense in X_λ by the transitivity of denseness.

Transfinite induction then gives that X is dense in $\tilde{X}$. $\square$

4 Main Result

Lemma 4.1. *Let X be an almost full space. Then every isometric embedding $f : X \to Y$ of X into a full space Y extends to an isometric embedding $\hat{f} : \hat{X} \to Y$.*

Proof. Let $f : X \to Y$ be such an embedding. We will define $\hat{f} : \hat{X} \to Y$. Choose $t^{(\gamma)} \in \hat{X}$. Then since f is an embedding, $f(\gamma) : A \to Y$ is an undergeodesic in Y. Since Y is full, $f(\gamma)$ extends to a geodesic ϕ in Y (this geodesic is unique since $f(\gamma)$ is dense). Set $\hat{f}(t^{(\gamma)}) = \phi(t)$.

Now we need only show that $\hat{f}$ is an isometric embedding. Take $t^{(\gamma)}, s^{(\psi)} \in \hat{X}$, where $\gamma : A \to X$ and $\psi : A' \to X$ are maximal undergeodesics in X. Let $\{t_n\}$ and $\{s_n\}$ be sequences in A and A' converging to t and s respectively (which exist since γ and ψ are dense). Then

$$\begin{aligned}
d(t^{(\gamma)}, s^{(\psi)}) &= \lim_{n\to\infty} d(t_n^{(\gamma)}, s_n^{(\psi)}) \\
&= \lim_{n\to\infty} d(\gamma(t_n), \psi(s_n)) \\
&= \lim_{n\to\infty} d(f(\gamma(t_n)), f(\psi(s_n))) \\
&= \lim_{n\to\infty} d(\hat{f}(t_n^{(\gamma)}), \hat{f}(s_n^(\phi))) \\
&= d(\hat{f}(t^{(\gamma)}), \hat{f}(s^{(\phi)})),
\end{aligned}$$

which shows that $\hat{f}$ is an isometric embedding. $\square$

Theorem 4.2. *Let X be an almost full space. Then every isometric embedding $f : X \to Y$ of X into a full space Y extends to an isometric embedding $\tilde{f} : \tilde{X} \to Y$.*

Proof. As in the proof of 2.10, let

$$X_0 = X, \quad X_{\beta+1} = \mathcal{S}(X_\beta), \quad X_\lambda = \varinjlim_{\beta<\lambda} X_\beta \quad (\lambda \text{ a limit ordinal} \leq \alpha).$$

Denote by f_1 the map $\hat{f} : X_1 \to Y$ defined in 4.1. By 3.1, X_1 is almost full, so applying 4.1 again gives a map $f_2 : X_2 \to Y$. Thus we can inductively define $X_{\beta+1}$ given X_β for any β. Now, given spaces X_β and maps $\hat{f}_\beta$ for all $\beta < \lambda$ such that $\hat{f}_\beta$ extends $\hat{f}_\gamma$ for $\gamma < \beta$, we can define $f_\lambda : X_\lambda \to Y$ by setting $f_\lambda(x) = f_\beta(x)$ for $x \in X_\beta$. This map will still be an isometric embedding, as for any $x, y \in X_\lambda$ there exists a $\beta < \lambda$ such that $x, y \in X_\beta$, and so $d(f_\lambda(x), f_\lambda(y)) = d(f_\beta(x), f_\beta(y)) = d(x, y)$. Using 3.2, we can see that the limit spaces are almost full, so since every step of the process is almost full we can transfinitely induct to create a map $f_\alpha : X_\alpha \to Y$. Since $\tilde{X} = X_\alpha$, we are done. □

5 Consequences

Corollary 5.1. *For any almost full space X, $\tilde{X}$ is the smallest full space containing X (in the sense that for any full space Y and any isometric embedding $f : X \to Y$, there exists an isometric embedding $f : \tilde{X} \to Y$).*

Definition 5.2. Let **AMet** and **FullMet** denote the categories of almost full and full metric spaces, respectively, with the morphisms in both being isometric embeddings.

Corollary 5.3. *There exists a functor $\widetilde{(-)} : \mathbf{AMet} \to \mathbf{FullMet}$ defined by taking X to $\tilde{X}$ and $f : X \to Y$ to $\tilde{f} : \tilde{X} \to \tilde{Y}$. To define $\tilde{f}$ when Y is not full, you just set $\tilde{f} = \widetilde{(i \circ f)}$, where $i : Y \to \tilde{Y}$ is the inclusion.*

Proposition 5.4. *Let $I : \mathbf{FullMet} \to \mathbf{AMet}$ denote the inclusion functor. Then the functors I and $\widetilde{(-)}$ define an adjunction between the categories* **AMet** *and* **FullMet***.*

Proof. We first need to establish, for every almost full space X and every full space Y, a bijection

$$\Phi_{X,Y} : \mathbf{FullMet}(\tilde{X}, Y) \to \mathbf{AMet}(X, Y).$$

Given a morphism $g : \tilde{X} \to Y$, we set $\Phi(g) = g|_X$. Given a morphism $f : X \to Y$, we set $\Phi^{-1}(f) = \tilde{f}$. It is clearly seen that for any map $f : X \to Y$, we have that $\tilde{f}|_X = f$. We also must have that $\widetilde{g|_X} = g$ for any $g : \tilde{X} \to Y$, as the two maps agree on a dense subset of $\tilde{X}$ (namely X).

We need next to check the naturality of Φ. First, we need to check that for any full space Y, the map $\Phi_{(-),Y}$ between $\mathbf{FullMet}(\widetilde{(-)}, Y)$ and $\mathbf{AMet}(-, Y)$ is a natural transformation. Let $u : X \to X'$ be an embedding of almost full spaces. We need to check the commutativity of the following diagram:

$$\begin{tikzcd}
\mathbf{FullMet}(\tilde{X}', Y) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "(-)\circ\tilde{u}"'] & \mathbf{AMet}(X', Y) \arrow[d, "(-)\circ u"] \\
\mathbf{FullMet}(\tilde{X}, Y) \arrow[r, "\Phi_{X,Y}"] & \mathbf{AMet}(X, Y)
\end{tikzcd}$$

Take $h : \tilde{X}' \to Y$. We need to show that $h|_X \circ u = (h \circ \tilde{u})|_X$. This is obvious since $\tilde{u}|_X = u$. Next we need to check that for any almost full space X, the map $\Phi_{X,(-)}$ is a natural transformation. Let $v : Y \to Y'$ be an embedding of full spaces. We need to check the commutativity of the following diagram:

$$\begin{tikzcd}
\mathbf{FullMet}(\tilde{X}, Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "v\circ(-)"'] & \mathbf{AMet}(X, Y) \arrow[d, "v\circ(-)"] \\
\mathbf{FullMet}(\tilde{X}, Y') \arrow[r, "\Phi_{X,Y'}"] & \mathbf{AMet}(X, Y')
\end{tikzcd}$$

Take $g : \tilde{X} \to Y$. We need to show that $v \circ g|_X = (v \circ g)|_X$. This is clearly the case. This establishes the naturality of Φ and completes the proof. $\square$

Corollary 5.5. **FullMet** *is a reflective subcategory of* **Met**.

6 Curvature

Definition 6.1. An **undergeodesic triangle** $\Delta = \Delta(p, q, r) = \Delta(\gamma_{pq}, \gamma_{qr}, \gamma_{rp})$ in a metric space X consists of three points $p, q, r \in X$ and the union of the images of three undergeodesics

$$\gamma_{pq} : A_{pq} \to X, \quad \gamma_{qr} : A_{qr} \to X, \quad \gamma_{rp} : A_{rp} \to X.$$

An undergeodesic triangle is called **maximal** if the geodesics that compose it are all maximal. A triangle $\overline{\Delta}(p,q,r) = \Delta(\bar{p},\bar{q},\bar{r})$ in M^2_κ is called a **comparison triangle** for Δ if $d(p,q) = d(\bar{p},\bar{q})$, $d(q,r) = d(\bar{q},\bar{r})$, and $d(r,p) = d(\bar{r},\bar{p})$. Such a triangle always exists if the perimeter $d(p,q)+d(q,r)+d(r,p)$ of Δ is less than $2D_\kappa$, where $D_\kappa = \infty$ if $\kappa \leq 0$ and $D_\kappa = \pi/\sqrt{\kappa}$ otherwise. A point $\bar{x} \in [\bar{q},\bar{r}]$ is called a **comparison point** for $x \in A_{qr}$ if $d(q,x) = d(\bar{q},\bar{x})$. Comparison points on $[\bar{p},\bar{q}]$ and $[\bar{r},\bar{p}]$ are defined in the same way.

Definition 6.2. Let (X,d) be a metric space, and let $\kappa \in \mathbb{R}$. Let Δ be an undergeodesic triangle in X with perimeter less than $2D_\kappa$, and let $\overline{\Delta}$ be a comparison triangle for Δ in M^2_κ. We say that Δ satisfies the **under-CAT**(κ) **inequality** if for all $x,y \in \Delta$ and all comparison points $\bar{x},\bar{y} \in \overline{\Delta}$,

$$d(x,y) \leq d(\bar{x},\bar{y}).$$

X is called an **under-CAT**(κ) **space** (more briefly "X is under-CAT(κ)") if all of its undergeodesic triangles with perimeter less than $2D_\kappa$ satisfy the under-CAT(κ) inequality.

Remark 6.3. It is clear from the definition of under-CAT(κ) and from Proposition 1.3 that one may consider only undergeodesic triangles whose constituent undergeodesics are all maximal when deciding if a space is under-CAT(κ).

In particular, to check that a uniquely undergeodesic space X is under-CAT(κ), we need only check that for every triple of distinct points $x,y,z \in X$, the undergeodesic triangle $\Delta([x,y],[y,z],[x,z])$ satisfies the under-CAT(κ) inequality.

Proposition 6.4. *For $\kappa \in \mathbb{R}$, a geodesic metric space X is under-CAT(κ) if and only if it is CAT(κ).*

Proposition 6.5. *For $\kappa \in \mathbb{R}$, an almost full space X is under-CAT(κ) if and only if $\tilde{X}$ is CAT(κ).*

Proof. Incomplete □

Definition 6.6. An almost full space X is said to have geodesics that

7 Homotopy

The following is a crude and slightly wonky attempt to provide a homotopy theory of almost full metric spaces.

Definition 7.1. Let X be an almost full space. An **admissible path** in X is a continuous path $\gamma : [0,1] \to \tilde{X}$ such that $\gamma|_X$ is dense in γ. An **A-homotopy** in X is a homotopy $h : [0,1] \times [0,1] \to X$ such that $h(t)$ is an admissible path for every t in $[0,1]$. The **A-fundamental group** π_1^M of X based at x_0 is the group of homotopy classes of admissible loops based at x_0.

Example 7.2. *For any full space X, $\pi_1^M(X) = \pi_1(X)$.*

Example 7.3. *While $\pi_1(\mathbb{R}^2 \setminus \mathbb{Q}^2)$ is extremely large and difficult to describe, $\pi_1^M(\mathbb{R}^2 \setminus \mathbb{Q}^2) = 0$.*

Question 7.4. *What is $\pi_1(\mathbb{Q}^2)$?*

8 Boundary Behavior

Definition 8.1. Let X be an almost full space. An **admissible ray** in X is an isometric embedding $r : [0,\infty) \to \tilde{X}$ such that r_X is dense in r, and $r(0) \in X$. Two admissible rays r, r' are said to be *asymptotic* if there exists a constant K such that $d_{\tilde{X}}(r(t), r'(t)) \leq K$ for all $t \geq 0$. The set ∂X of **admissible boundary points** of X is the set of equivalence classes of admissible rays under the relation of being asymptotic.

Definition 8.2. Let X be an almost full space, and suppose that $\tilde{X}$ happens to be complete and CAT(0). Then one can define the **cone topology** on the space $\tilde{X} \cup \partial\tilde{X}$, following [BH99]. One defines the **cone topology** on X to be the induced topology on the subspace $X \cup \partial X$ of $\tilde{X} \cup \partial\tilde{X}$.