1 Basic Definitions + Preliminaries

Definition 1.1. Let (X, d) be a metric space, and let $x, y \in X$. Let $A \subseteq [0, d(x, y)]$ be such that $\{0, d(x, y)\} \subseteq A$. Then a map $\gamma : A \to X$ is called an **undergeodesic** from x to y if $\gamma(0) = x$, $\gamma(d(x, y)) = y$, and

$$d(\gamma(s), \gamma(t)) = |s - t|$$

for all $s, t \in A$. Such a map is also called an **A-geodesic**. The image of γ is called an **undergeodesic segment** (A-geodesic segment). An A-geodesic is called **trivial** if $A = \{0, d(x, y)\}$.

Example 1.2. *Let $X = \mathbb{R}^n$, with the Euclidean metric. Then an undergeodesic segment in X is exactly a subset of a line segment that contains both of its endpoints.*

Example 1.3. *Let $X = \mathbb{Q}^2$, with the induced metric from $\mathbb{R}^2$. Then any undergeodesic segment in X is the restriction to X of an undergeodesic segment in $\mathbb{R}^2$ between points in X: that is, a subset of a line segment in $\mathbb{R}^2$, consisting solely of points with rational components, that contains both endpoints.*

Example 1.4. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed, unweighted combinatorial graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, and equip the set $\mathcal{V}$ with the shortest path metric induced by $\mathcal{E}$. Then an undergeodesic from v_1 to v_2 in $\mathcal{V}$ is exactly a subpath of a shortest path from v_1 to v_2.*

Definition 1.5. Let X be a metric space, and let $x, y \in X$. A **partial curve** from x to y is a (continuous) map $c : A \to X$, where $A \subseteq [a, b]$ for some $[a, b] \subseteq \mathbb{R}$ and $\{a, b\} \subseteq A$, such that $c(a) = x$ and $c(b) = y$. The interval $[a, b]$ is called the **support** of c. The **length** of a partial curve $c : A \to X$, with $A \subseteq [a, b]$, is

$$l(c) = \sup_{a=t_0 \leq t_1 \leq \cdots \leq t_n = b} \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1})),$$

where the supremum is taken over all $n \geq 1$ and all possible choices of $t_0, t_1, \ldots, t_n \in A$ with $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$. c is said to be **rectifiable** if its length is finite.

Proposition 1.6. *Let $c : A \to X$ be a partial curve with support $[a, b]$.*

1. *$\ell(c) \geq d(c(a), c(b))$, and $l(c) = 0$ if and only if c is a constant map.*
2. *If ϕ is a weakly monotonic map from a set $A' \subseteq [a', b']$ with $\{a', b'\} \subseteq A'$ onto A, then $l(c) = l(c \circ \phi)$.*

3. *Let $t \in A$. $l(c) = l(c|_{A\cap[a,t]}) + l(c|_{A\cap[t,b]})$.*
4. *Let $A' = \{\ell_t : \ell_t = a + b - t, t \in A\}$. Then the reverse partial curve $\bar{c} : A' \to X$ defined by $\bar{c}(\ell_t) = c(t)$ satisfies $\ell(\bar{c}) = \ell(c)$.*
5. *If c is rectifiable of length l, then the function $\lambda : A \to [0, \ell]$ defined by $\lambda(t) = l(c|_{A\cap[a,t]})$ is a continuous weakly monotonic function with $\lambda(a) = 0$ and $\lambda(b) = l(c)$.*
6. *If c and λ are as in (5), then there exists a path $\tilde{c} : \lambda(A) \to X$, such that*

$$\tilde{c} \circ \lambda = c \quad \textit{and} \quad l(\tilde{c}|_{\lambda(A)\cap[0,t]}) = t$$

   for all $t \in \lambda(A)$.

Proof. Properties (1), (2), (3), and (4) are immediate. The fact that λ is well-defined and takes values in $[0, l(c)]$ follows immediately from the definition of length. If $s, t \in A$ with $s < t$, then any partition of $[a, s]$ can be extended to a partition of $[a, t]$ by adding points in A, so $\lambda(s) \leq \lambda(t)$ and λ is weakly monotonic. It is also clear from the definition and (1) that $\lambda(a) = 0$ and $\lambda(b) = l(c)$.

To prove continuity, let $t \in A$ and let $\epsilon > 0$. By the definition of $\lambda(t)$, there exists a partition $a = t_0 < t_1 < \cdots < t_k = t$ with $t_i \in A$ such that

$$\sum_{i=0}^{k-1} d(c(t_i), c(t_{i+1})) > \lambda(t) - \epsilon/2.$$

Since c is continuous at t, there exists $\delta_1 > 0$ such that

$$|s - t| < \delta_1 \quad \Rightarrow \quad d(c(s), c(t)) < \epsilon/2.$$

Let $\delta_2 = |t - t_{k-1}|$, and set $\delta = \min(\delta_1, \delta_2)$. Now let $s \in A$ with $|s - t| < \delta$, and define a new partition

$$s_0 = t_0, \quad s_1 = t_1, \quad \ldots, \quad s_{k-1} = t_{k-1}, \quad s_k = s.$$

Then

$$\sum_{i=0}^{k-1} d(c(s_i), c(s_{i+1})) = \sum_{i=0}^{k-2} d(c(t_i), c(t_{i+1})) + d(c(t_{k-1}), c(s)).$$

If $s \leq t$, then by the reverse triangle inequality,

$$d(c(t_{k-1}), c(s)) \geq d(c(t_{k-1}), c(t)) - d(c(s), c(t)) > d(c(t_{k-1}), c(t)) - \epsilon/2.$$

Therefore,

$$\sum_{i=0}^{k-1} d(c(s_i), c(s_{i+1})) > \left(\sum_{i=0}^{k-1} d(c(t_i), c(t_{i+1}))\right) - \epsilon/2 > \lambda(t) - \epsilon,$$

so $\lambda(s) > \lambda(t) - \epsilon$. Since λ is monotonic, it follows that $|\lambda(s) - \lambda(t)| < \epsilon$.
If $s > t$, then by the triangle inequality,

$$d(c(t_{k-1}), c(s)) \leq d(c(t_{k-1}), c(t)) + d(c(s), c(t)) < d(c(t_{k-1}), c(t)) + \epsilon/2.$$

Hence,

$$\sum_{i=0}^{k-1} d(c(s_i), c(s_{i+1})) < \left(\sum_{i=0}^{k-1} d(c(t_i), c(t_{i+1}))\right) + \epsilon/2 < \lambda(t) + \epsilon,$$

so $\lambda(s) < \lambda(t) + \epsilon$. Since $\lambda(s) \geq \lambda(t)$ by monotonicity, it follows that $|\lambda(s) - \lambda(t)| < \epsilon$.
In either case, λ is continuous at t. Since $t \in A$ was arbitrary, λ is continuous on A.

(6) follows from (5) and (2). □

Definition 1.7. A partial curve is said to be **parametrized by arc length** if the map λ defined in Proposition 1.6 is the identity on A.

Proposition 1.8. *A (parametrized by arc length) partial curve from x to y is an undergeodesic if and only if its length is equal to $d(x, y)$.*

Proof. Let γ be an A-geodesic in X with support $[0, d]$. Then, for every partition $0 = x_0 \leq x_1 \leq \cdots \leq x_n = d$ with $x_i \in A$,

$$\sum_i d(\gamma(x_i), \gamma(x_{i+1})) = \sum_i |x_{i+1} - x_i| = d,$$

so γ has length $d = d(x, y)$.

Now let $\gamma : A \to X$ be a partial curve with length $d = d(x, y)$ (so that A has support $[0, d]$). Choose $s, t \in A$ with $s < t$. Then since γ is parametrized by arc length, we know that $l(\gamma|_{[s,t]}) = t - s$. Thus $d(\gamma(s), \gamma(t)) \leq t - s$. Similarly, $d(x, \gamma(s)) \leq s$ and $d(\gamma(t), y) \leq d - t$. Suppose for contradiction that $d(\gamma(s), \gamma(t)) < t - s$. Then $d(x, y) \leq d(x, \gamma(s)) + d(\gamma(s), \gamma(t)) + d(\gamma(t), y) \leq d - |t - s| + d(\gamma(s), \gamma(t)) < d - |t - s| + |t - s| = d(x, y)$, a contradiction. Thus $d(\gamma(s), \gamma(t)) = t - s$ and γ is an undergeodesic. □

Corollary 1.9. *A (parametrized by arc length) path from x to y is a geodesic if and only if its length is equal to $d(x, y)$.*

Proposition 1.10. *In a geodesic space X, every undergeodesic with finite domain lies inside the image of a geodesic.*

Proof. We will first prove the following weaker statement: if $d(x, y) = d(x, z)+d(z, y)$, and $\gamma_1 : [0, d(x, z)] \to X$ and $\gamma_2 : [d(x, z), d(x, y)] \to X$ are geodesics from x to z and from z to y respectively, then the map $\gamma : [0, d(x, y)] \to X$ defined by $\gamma = \gamma_1$ on $[0, d(x, z)]$ and $\gamma = \gamma_2$ on $[d(x, z), d(x, y)]$ is a geodesic. This follows from applying Corollary 1.9 to γ_1 and γ_2, applying Proposition 1.6 (3) to find that γ has length $d(x, y)$, then applying Corollary 1.9 again.

Now we induct on the cardinality of A. If $|A| = 2$, then the statement is trivial. Now suppose that we have proven the statement for all undergeodesics $\psi : B \to X$ from x to y with $|B| < k$, and suppose $\gamma : A \to X$ is an undergeodesic with $|A| = k$. Write $A = \{t_1, \ldots, t_k\}$ with $t_i < t_{i+1}, t_0 = 0$ and $t_k = d(x, y)$. Then $\gamma|_{A\cap[0,t_{k-1}]}$ lies in some geodesic $\rho_1 : [0, t_{k-1}] \to X$ from x to $\gamma(t_{k-1})$ by the hypothesis. Since X is geodesic, there also exists a geodesic $\rho_2 : [t_{k-1}, t_k] \to X$ from $\gamma(t_{k-1})$ to y. Since γ is an undergeodesic, we know that $d(x, \gamma(t_{k-1})) + d(\gamma(t_{k-1}), y) = d(x, y)$, so the weaker statement proven above gives that the map $\rho : [0, d(x, y)] \to X$ defined by $\rho = \rho_1$ on $[0, t_{k-1}]$ and $\rho = \rho_2$ otherwise is a geodesic. Since $\gamma(t_i) = \rho(t_i)$ for each $t_i \in A$, we conclude that γ lies in the image of ρ. This proves the statement by induction. $\square$

Remark 1.11. One would hope that Proposition 1.10 could be extended to arbitrary undergeodesics in geodesic spaces. This is not true in general, however, as will be shown later.

Finally, a trivial but useful equivalence:

Definition 1.12. A metric space is called **anti-geodesic** if there exist no nontrivial undergeodesics in X. Note that a space is anti-geodesic if and only if $d(x, y) < d(x, z) + d(z, y)$ for every triple of distinct points $x, y, z \in X$.

2 Uniqueness and Maximality

Definition 2.1. Let (X, d) be a metric space, and take $x, y \in X$. X is called **uniquely undergeodesic** if for any undergeodesics $\gamma_1 : A_1 \to X$ and $\gamma_2 : A_2 \to X$ from x to y, the following hold:

1. $\gamma_1|_{A_1\cap A_2} = \gamma_2|_{A_1\cap A_2}$

2. If we define $\gamma : A_1 \cup A_2 \to X$ by

$$\gamma(t) = \begin{cases} \gamma_1(t) & t \in A_1 \\ \gamma_2(t) & t \in A_2 \end{cases}$$

then γ is an undergeodesic from x to y.

A space X is **locally uniquely undergeodesic** if every point $x \in X$ has a uniquely undergeodesic open neighborhood.

Proposition 2.2. *A space X is uniquely undergeodesic if and only if for every quadruple of points $x, y, p, q \in X$ such that $d(x,p) + d(p,y) = d(x,q) + d(q,y) = d(x,y)$, either*

$$d(x,y) = d(x,p) + d(p,q) + d(q,y), \text{ or } d(x,y) = d(x,q) + d(p,q) + d(p,y).$$

Proof. ($\Rightarrow$) Let X be a uniquely undergeodesic space, and take $x, y, p, q \in X$ satisfying the conditions. Define $\gamma_p : \{0, d(x,p), d(x,y)\}$ by $\gamma_p(d(x,p)) = p$ and define $\gamma_q : \{0, d(x,q), d(x,y)\}$ similarly. Then both are undergeodesics: all that needs to be checked is that $d(y,p) = d(x,y) - d(x,p)$ and $d(y,q) = d(x,y) - d(x,q)$, and these follow from the assumption.

If $d(x,p) = d(x,q)$, then the two undergeodesics are defined on the same set but are unequal, which contradicts the uniquely undergeodesic condition. Otherwise, the uniquely undergeodesic condition gives that the composite geodesic

$$\gamma : \{0, d(x,p), d(x,q), d(x,y)\} \to X$$

is an undergeodesic. Now suppose without loss of generality that $d(x,p) < d(x,q)$. Then we have from the fact that γ is an undergeodesic that $d(p,q) = d(x,q) - d(x,p) = d(x,y) - d(q,y) - d(x,p)$, so we get that $d(x,y) = d(x,p) + d(p,q) + d(q,y)$, as desired.

($\Leftarrow$) Suppose that X satisfies the four-point condition as above. First we show that the first condition must hold: suppose that $\gamma_1 : A_1 \to X$ and $\gamma_2 : A_2 \to X$ are undergeodesics from x to y. Take $t \in A_1 \cap A_2$, and let $p = \gamma_1(t)$ and let $q = \gamma_2(t)$. Then we have from the undergeodesic condition that $d(x,y) - t = d(x,y) - d(x,p) = d(y,p)$ and that $d(x,y) - d(x,q) = d(y,q)$, so that $d(x,p) + d(p,y) = d(x,q) + d(q,y) = d(x,y)$.

Then by assumption we either have that

$$\begin{aligned} d(x,y) &= d(x,p) + d(p,q) + d(q,y) \\ &= t + d(p,q) + d(q,y) \\ &= d(x,q) + d(p,q) + d(q,y) \\ &= d(x,y) + d(p,q), \end{aligned}$$

or we have that

$$\begin{aligned} d(x,y) &= d(x,q) + d(p,q) + d(p,y) \\ &= t + d(p,q) + d(p,y) \\ &= d(x,p) + d(p,q) + d(q,y) \\ &= d(x,y) + d(p,q), \end{aligned}$$

which proves that $d(p,q) = 0$ in either case and thus that $p = q$. This completes the proof of the first condition.

Now define $\gamma : A_1 \cup A_2 \to X$ as in the second condition. Choose $s, t \in A$ such that $t \neq s$. We need to show that $d(\gamma(s), \gamma(t)) = |s - t|$. If $\gamma(s) = \gamma_i(s)$ and $\gamma(t) = \gamma_i(t)$ for some $i \in \{1, 2\}$, then the claim is true immediately since γ_1 and γ_2 are undergeodesics. Thus suppose (WLOG) that $\gamma(s) = \gamma_1(s)$ and $\gamma(t) = \gamma_2(t)$. Let $p = \gamma(s)$ and $q = \gamma(t)$. Since γ_1 and γ_2 are undergeodesics, we have that $d(x,y) = d(x,p) + d(p,y) = d(x,q) + d(q,y)$. Note also that $d(\gamma(s), \gamma(t)) = d(p,q)$, $s = d(x,p)$, and $t = d(x,q)$. By the assumption, we have that either

$$d(x,y) = d(x,p) + d(p,q) + d(q,y), \text{ or } d(x,y) = d(x,q) + d(p,q) + d(p,y).$$

In the first case, we get that

$$\begin{aligned} d(\gamma(s), \gamma(t)) &= |d(p,q)| \\ &= |d(x,y) - d(x,p) - d(q,y)| \\ &= |(d(x,y) - d(q,y)) - d(x,p)| \\ &= |d(x,q) - d(x,p)| = |t - s|. \end{aligned}$$

In the second case, we similarly get that

$$\begin{aligned} d(\gamma(s), \gamma(t)) &= |d(p,q)| \\ &= |d(x,y) - d(x,q) - d(p,y)| \\ &= |(d(x,y) - d(p,y)) - d(x,q)| \\ &= |d(x,p) - d(x,q)| = |s - t|. \end{aligned}$$

This completes the proof. $\square$

Proposition 2.3. *A metric space X is uniquely geodesic if and only if it is geodesic and uniquely undergeodesic.*

Proof. If X is geodesic and uniquely undergeodesic, then it is certainly geodesic, as for any $x, y \in X$, any two geodesics $\gamma : [0, d] \to X$ and $\gamma' : [0, d] \to X$ from x to y must agree on the intersection of their domains, which is the whole interval.

Now suppose X is uniquely geodesic, and take $x, y, p, q \in X$ such that $d(x, p) + d(p, y) = d(x, q) + d(q, y) = d(x, y)$. X is certainly geodesic, so following Proposition 2.2, we need only to show that either

$$d(x, y) = d(x, p) + d(p, q) + d(q, y), \text{ or } d(x, y) = d(x, q) + d(p, q) + d(p, y).$$

if either $d(x, p)$ or $d(x, q)$ is equal to 0 or $d(x, y)$, then this follows immediately from the assumption on p and q. Now suppose without loss of generality that $0 < d(x, p) < d(x, q) < d(x, y)$. Then, by the assumption on p, we have that $\gamma_1 : \{0, d(x, p), d(x, y)\}$ is an undergeodesic, so Proposition 1.10 guarantees the existence of a geodesic $\tilde{\gamma}_1$ extending γ_1. We similarly get a geodesic $\tilde{\gamma}_2$ extending the undergeodesic $\gamma_2 : \{0, d(x, q), d(x, y)\}$. By geodesic uniqueness, $\tilde{\gamma}_1 = \tilde{\gamma}_2$. Thus $d(p, q) = d(\tilde{\gamma}_1(d(x, p)), \tilde{\gamma}_2(d(x, q))) = d(\tilde{\gamma}_1(d(x, p)), \tilde{\gamma}_1(d(x, q))) = d(x, q) - d(x, p)$, so $d(x, q) = d(x, p) + d(p, q)$. This means that $d(x, y) = d(x, q) + d(q, y) = d(x, p) + d(p, q) + d(q, y)$, which completes the proof. □

Definition 2.4. If X is a uniquely undergeodesic space, and $x, y \in X$, we can define the **maximal undergeodesic** from x to y as follows: let $\mathcal{U}$ denote the set of undergeodesics from x to y. Define the map $\phi : \mathcal{U} \to \mathcal{P}([0, d(x, y)])$ that takes γ to its domain of definition A_γ. Set $A = \bigcup_{\gamma \in \mathcal{U}} A_\gamma$. Then define $\tilde{\gamma} : A \to X$ by

$$\tilde{\gamma}(t) = \gamma(t) \quad \text{for } t \in A_\gamma.$$

By the uniquely undergeodesic condition, $\tilde{\gamma}$ is well-defined and an undergeodesic from x to y (a full proof of this is given in the following proposition). This is the maximal undergeodesic from x to y. In a uniquely geodesic space X, we write $[x, y]$ to denote the image of the maximal geodesic from x to y.

More generally, if $\gamma : A \to X$ is an undergeodesic from x to y, we say that γ is **maximal** if for every $t \in [0, d(x, y)] \setminus A$, there does not exist $z \in X$ such that the map $\gamma' : A \cup \{t\} \to X$ extending γ with $\gamma'(t) = z$ is an undergeodesic. Equivalently, γ is maximal if there does not exist $\gamma' : A' \to X$ with A a strict subset of A' such that $\gamma'|_A = \gamma$.

Proposition 2.5. *In any uniquely undergeodesic space X, and for any $x, y \in X$, the maximal undergeodesic $\tilde{\gamma}$ is well-defined and is an undergeodesic.*

Proof. Define A as above. We need to show that $\tilde{\gamma}$ does not depend on the choice of A_γ associated with each t. Thus we need to show that if $t \in A$ is such that $t \in A_{\gamma_1}$ and $t \in A_{\gamma_2}$, then $\gamma_1(t) = \gamma_2(t)$. This follows immediately from the undergeodesic uniqueness property (since $t \in A_{\gamma_1} \cap A_{\gamma_2}$), so $\tilde{\gamma}$ is well-defined.

Next we need to show that $\tilde{\gamma}$ is an undergeodesic. To do this, we need to show that for any $s, t \in A$, $d(\tilde{\gamma}(s), \tilde{\gamma}(t)) = |s - t|$. We have that $s \in A_{\gamma_1}$ and $t \in A_{\gamma_2}$ for some undergeodesics γ_1, γ_2. Then define $\gamma : A_{\gamma_1} \cup A_{\gamma_2} \to X$ as in the second undergeodesic uniqueness condition. Since X is uniquely undergeodesic, γ is an undergeodesic with $\gamma(s) = \tilde{\gamma(s)}$ and $\gamma(t) = \tilde{\gamma(t)}$. We then have that $d(\tilde{\gamma}(s), \tilde{\gamma}(t)) = d(\gamma(s), \gamma(t)) = |s-t|$, completing the proof. $\square$

Proposition 2.6. *Any undergeodesic $\gamma : A \to X$ can be extended to a maximal undergeodesic $\gamma^* : A' \to X$ such that $\gamma^*|_A = \gamma$.*

Proof. Let

$$\mathcal{F} = \{\phi : A_\phi \to X \mid \phi \text{ is an undergeodesic with } A \subseteq A_\phi \text{ and } \phi|_A = \gamma\}.$$

Partially order $\mathcal{F}$ by setting $\phi_1 \leq \phi_2$ if $A_{\phi_1} \subseteq A_{\phi_2}$ and $\phi_2|_{A_{\phi_1}} = \phi_1$. We show that every chain in $\mathcal{F}$ has an upper bound.

Let $\mathcal{C} \subseteq \mathcal{F}$ be a chain. Let $A^* = \bigcup_{\phi \in \mathcal{C}} A_\phi$. Then for any $t \in A^*$, there exists some $\phi \in \mathcal{C}$ with $t \in A_\phi$. Moreover, if there exist ϕ_1 and ϕ_2 in $\mathcal{C}$ such that $t \in A_{\phi_1} \cap A_{\phi_2}$, then by the total order on $\mathcal{C}$ we have (WLOG) that $A_{\phi_2} \subseteq A_{\phi_1}$, and furthermore that

$$\phi_1(t) = \phi_1|_{A_{\phi_2}}(t) = \phi_2(t).$$

Therefore the map $\phi^* : A^* \to X$ defined by $\phi^*(t) = \phi(t)$ for some choice of ϕ with $t \in A_\phi$ is well-defined. Next we must prove that ϕ^* is an undergeodesic. For any $s, t \in A^*$, we have that $s \in A_{\phi_1}$ and $t \in A_{\phi_2}$ for some $\phi_1, \phi_2 \in \mathcal{C}$. Since $\mathcal{C}$ is totally ordered, we know (WLOG) that $A_{\phi_1} \subseteq A_{\phi_2}$. Then

$$d(\phi^*(t), \phi^*(s)) = d(\phi_2(t), \phi_2(s)) = |t - s|$$

since ϕ_2 is an undergeodesic. Thus ϕ^* is an undergeodesic. It also clearly extends γ. Finally, if $\phi \in \mathcal{C}$, then $A_\phi \subseteq A^*$ and $\phi^*|_{A_\phi} = \phi$, so ϕ^* is an upper bound for $\mathcal{C}$.

Then, by Zorn's lemma, $\mathcal{F}$ has a maximal element $\gamma^* : A' \to X$. This map γ^* is an undergeodesic extending γ by construction, and is maximal since any extension $(\gamma^*)' : A \cup \{t\} \to X$ would be an element of $\mathcal{F}$ strictly greater than γ^*. $\square$

Example 2.7. *Let $X = \mathbb{R}^n$. Then X is uniquely undergeodesic and every maximal undergeodesic segment in X is a line segment.*

Example 2.8. *Let $X = \mathbb{Q}^n$. Then X is uniquely undergeodesic and every maximal undergeodesic segment in X is the restriction to $\mathbb{Q}^n$ of a line segment in $\mathbb{R}^n$.*

Example 2.9. *Let X be a combinatorial graph as defined in Example 1.4. Then the maximal undergeodesic segments from v_1 to v_2 in X are precisely the shortest paths from v_1 to v_2. In general, X is not uniquely undergeodesic, as there can be many shortest paths between any two vertices.*

3 Partial and Full Density

Definition 3.1. Let (X, d) be a metric space, and let $\gamma : A \to X$ be an undergeodesic from x to y. Then γ is called a **dense** undergeodesic if A is dense in $[0, d(x, y)]$.

Definition 3.2. For $\alpha > 0$, a metric space X is called α**-geodesic** if for any $x, y \in X$ there exists an undergeodesic $\gamma : A \to X$ from x to y such that for any $a \in [0, d(x, y)]$, there exists $b \in A$ such that $|a-b| \leq \alpha$. Such an undergeodesic is called an α-geodesic.

Lemma 3.3. *If X is an α-geodesic space, then for any $x, y \in X$ there exists an α-geodesic from x to y with finite domain.*

Proof. Choose $x, y \in X$, set $d = d(x, y)$, and let $\gamma : A \to X$ be an α-geodesic from x to y. Let $\mathcal{I}_\alpha = \{[x, x+\alpha] \subseteq [0, d] : x \in [0, d-\alpha]\}$, considered as a metric subspace of the space of compact subsets of $[0, d]$ equipped with the Hausdorff metric. For $a \in A$, define $U_a = \{I \in \mathcal{I}_\alpha : a \in I\}$. Each U_a is open in $\mathcal{I}_\alpha$, and the collection $\{U_a\}_{a \in A}$ covers A since γ is an α-geodesic. $\mathcal{I}_\alpha$ is compact as the image of $[0, 1-\alpha]$ under the map $x \mapsto [x, x+\alpha]$, so there exists a finite collection $\{U_{a_i}\}_{i=1}^n$ which covers $\mathcal{I}_\alpha$. Set $A' = \bigcup_{i=1}^n \{a_i\}$. Then $\gamma|_{A'}$ is an α-geodesic from x to y with finite domain. $\square$

Definition 3.4. A metric space X is called **almost geodesic** if for any x, y in X there exists a dense undergeodesic $\gamma : A \to X$ from x to y.

Lemma 3.5. *Let $\gamma : A \to X$ be an undergeodesic from x to y with finite domain $A = \{t_1, \ldots, t_k\}$ with $t_i < t_{i+1}$, $t_0 = 0$, and $t_k = d(x, y)$. For $1 \leq i < k$, let*

$\gamma_i : A_i \to X$ be an undergeodesic from $\gamma(t_i)$ to $\gamma(t_{i+1})$. Then the map $\gamma' : \bigcup_i A_i \to X$ defined by

$$\gamma'(t) = \gamma_i(t) \quad t \in [t_i, t_{i+1}]$$

is an undergeodesic.

Proof. We proceed similarly to the proof of Proposition 1.10. We will first prove the following weaker statement: if $d(x, y) = d(x, z) + d(z, y)$, and $\gamma_1 : [0, d(x, z)] \to X$ and $\gamma_2 : [d(x, z), d(x, y)] \to X$ are undergeodesics from x to z and from z to y respectively, then the map $\gamma : [0, d(x, y)] \to X$ defined by $\gamma = \gamma_1$ on $[0, d(x, z)]$ and $\gamma = \gamma_2$ on $[d(x, z), d(x, y)]$ is an undergeodesic. This follows from applying Proposition 1.8 to γ_1 and γ_2, applying Proposition 1.6 (3) to find that γ has length $d(x, y)$, then applying Proposition 1.8 again.

Now we induct on the cardinality of A. If $|A| = 2$, then the statement is trivial. Now suppose that we have proven the statement for all undergeodesics $\psi : B \to X$ from x to y with $|B| < k$, and suppose $\gamma : A \to X$ is an undergeodesic with $|A| = k$, and $\gamma_i : A_i \to X$ are as in the lemma. Write $A = \{t_1, \ldots, t_k\}$ with $t_i < t_{i+1}, t_0 = 0$ and $t_k = d(x, y)$. Then $\gamma'|_{\bigcup_i A_i \cap [0, t_{k-1}]}$ is an undergeodesic by the hypothesis. We also know that γ_{k-1} is an undergeodesic. Since γ is also an undergeodesic, we know that $d(x, \gamma(t_{k-1})) + d(\gamma(t_{k-1}), y) = d(x, y)$, so the weaker statement proven above gives that γ' is an undergeodesic. This proves the statement by induction. □

Proposition 3.6. *A metric space X is almost geodesic if and only if it is α-geodesic for every $\alpha > 0$.*

Proof. Suppose X is almost geodesic. For any $x, y \in X$, let $\gamma : A \to X$ be an undergeodesic with dense domain. Choose $a \in [0, d(x, y)]$. Then $A \cap B(a, \alpha) \neq \emptyset$ since A is dense, so there exists $b \in A$ such that $|a - b| \leq \alpha$. Thus X is α-geodesic for every $\alpha > 0$.

Now suppose that X is α-geodesic for every $\alpha > 0$, and choose $x, y \in X$. Set $d = d(x, y)$. By the lemma, for all $\alpha > 0$ and all $p, q \in X$ there exists an α-geodesic from p to q with finite domain.

Let γ_1 be a 1-geodesic from x to y with finite domain $\{x_{1,0}, \ldots, x_{1,n_1}\}$, where $x_{1,0} = 0$ and $x_{1,n_1} = d$. Between any $x_{1,i}$ and $x_{1,i+1}$, by the hypothesis, there exists a $\frac{1}{2}$−geodesic $\gamma_{1,i} : A_{1,i} \to X$ from $\gamma_1(x_{1,i})$ to $\gamma_1(x_{1,i+1})$ with finite domain. Define a map $\gamma_2 : A_2 \to X$, where $A_2 = \bigcup_i A_{1,i}$ by

$$\gamma_2(t) = \gamma_{1,i}(t) \quad t \in [x_{1,i}, x_{1,i+1}].$$

Note that $A_1 \subseteq A_2$ and $\gamma_2|_{A_1} = \gamma_1$. By Lemma 3.5, γ_2 is an undergeodesic. In fact, it is a $\frac{1}{2}$-geodesic: given any interval of length $\frac{1}{2}$, if it does not lie entirely within $[x_{1,i}, x_{1,i+1}]$ for some i, then it intersects A_2 as it hits one of those endpoints, and if it does lie inside one of those intervals, then it intersects A_2 as $\gamma_{1,i}$ is a $\frac{1}{2}$-geodesic for each i. Repeat this process on γ_2 by gluing a $\frac{1}{3}-$geodesic $\gamma_{2,i}$ between each of its consecutive points to define a $\frac{1}{3}$-geodesic $\gamma_3 : A_3 \to X$ with $A_2 \subseteq A_3$ and $\gamma_3|_{A_2} = \gamma_2$. Repeating this process inductively generates a sequence of nested undergeodesics γ_n. Define $\gamma : \bigcup_{i\in\mathbb{N}} A_i \to X$ by

$$\gamma(t) = \gamma_i(t) \quad t \in A_i.$$

γ is well-defined since the γ_i are nested. First, we need to check that γ is an undergeodesic. For any $s, t \in \bigcup_{i\in\mathbb{N}} A_i$, we have that $s \in A_j$ and $t \in A_k$ for some $j, k \in \mathbb{N}$. Supposing that $j > k$, we get that $s, t \in A_j$ since the A_i are nested, and so $d(\gamma(s), \gamma(t)) = d(\gamma_j(s), \gamma_j(t)) = |s - t|$ since γ_j is an undergeodesic. Finally, to see that γ is dense, take $(a, b) \subseteq [0, d(x, y)]$. Then (a, b) contains some closed interval I of length $\epsilon > 0$. Then, for N such that $\epsilon > \frac{1}{N}$, we know that I intersects A_N since γ_N is a $\frac{1}{N}$-geodesic, so I intersects $\bigcup_{i\in\mathbb{N}} A_i$. Thus (a, b) intersects $\bigcup_{i\in\mathbb{N}} A_i$ and γ is a dense undergeodesic from x to y. Therefore X is almost geodesic. □

Remark 3.7. It would be tempting to declare that every maximal undergeodesic in an almost geodesic space is dense. The first of the two following examples shows that, in full generality, this statement is false. Note that this also implies that not every undergeodesic in an arbitrary geodesic space lies inside the image of a geodesic. The second example shows that not every dense undergeodesic in a geodesic space is contained in a geodesic.

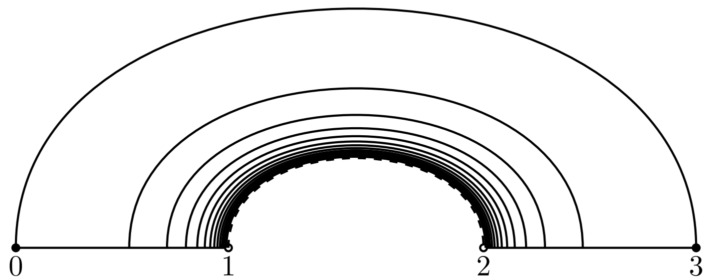
Example 3.8. *Let $X = [0, 1) \cup (2, 3]$ with the Euclidean topology. For each $n \in \mathbb{N}$, let $L_n = [0, 1 + 2/n]$. Define*

$$\hat{X} = \left(X \sqcup \bigcup_{x,y\in X} L_{x,y} \right) \setminus \sim,$$

where $\sim$ identifies $0 \in L_n$ with $1 - 1/n \in X$ and $1 + 2/n$ in L_n with $2 + 1/n \in X$. Metrize $\hat{X}$ by setting

$$d(x, y) = \begin{cases} |x - y| & x, y \in X \\ |x - 1 + 1/n| + y & x \in [0, 1) \text{ and } y \in L_n \\ |x - 2 - 1/n| + |y - 1 - 2/n| & x \in (2, 3] \text{ and } y \in L_n \\ |1/n - 1/m| + min\{x + y, 1 + 2/n - x - y\} & x \in L_n \text{ andy} \in L_n \end{cases}$$

This space looks like this:



Note that $\hat{X}$ is locally compact, that each line L_n is a geodesic segment between its endpoints, and that the space X embeds isometrically into $\hat{X}$, so that the inclusion map $i : X \to \hat{X}$ is an undergeodesic (as X is itself a subset of an interval).

$\hat{X}$ is a geodesic space: in particular, for $x, y \in X$ with $x \in [0,1)$ and $y \in (2,3]$, we can choose $N \in \mathbb{N}$ such that $1-1/N > x$ and $2+1/N < y$, and then the path that goes from x to $1-1/N$, follows L_N, and then goes from $2+1/N$ to y is a geodesic. However, i cannot be extended to a geodesic segment, and in fact i is a maximal undergeodesic. This shows that

1. *It is false that every undergeodesic in an almost geodesic space extends to a dense undergeodesic.*
2. *In particular, it is false that every undergeodesic in a geodesic space extends to a geodesic.*

This example additionally shows that neither of these statements hold even when the space is locally compact.

Example 3.9. *Let $X = [0,1) \cup (1,2]$ with the Euclidean topology, and glue line segments L_n between the points $1-1/n$ and $1+1/n$ in X as in the previous example to define a metric space $\hat{X}$. Then $\hat{X}$ is a geodesic space and the inclusion map $i : X \to \hat{X}$ is a dense undergeodesic (since X is dense in $[0,2]$), but i cannot be extended to a geodesic in $\hat{X}$. This example shows that it is false that every dense undergeodesic in a geodesic space extends to a geodesic, and that this does not hold even when the space is locally compact.*

It might be conjectured that unique undergeodesicity of a space might be controlled by its dense and almost geodesic subspaces. The following example shows that this is not true:

Example 3.10. *Let $X = S^1$, and let $Y = \{e^{2\pi i n \theta n} : n \in \mathbb{Z}\} \subset S^1$, where θ is irrational. Then Y is dense in S^1, and since Y contains no antipodal pairs, Y is uniquely undergeodesic. It is also clearly almost geodesic. However, S^1 is not uniquely undergeodesic.*

Proposition 3.11. *Let γ be a dense undergeodesic in a metric space X. Then there exists exactly one maximal undergeodesic γ' extending γ.*

Proof. Let $\gamma : A \to X$ be a dense undergeodesic, and let $\gamma_1 : A_1 \to X$ and $\gamma_2 : A_2 \to X$ be maximal undergeodesics extending γ. Choose $t \in A_1 \cap A_2$, and let $\epsilon > 0$. Since A is dense in the support of γ, we can choose $s \in A$ such that $|s - t| < \epsilon/2$. Then

$$\begin{aligned} d(\gamma_1(t), \gamma_2(t)) &\leq d(\gamma_1(t), \gamma(s)) + d(\gamma(s), \gamma_1(t)) \\ &= |t - s| + |s - t| < \epsilon. \end{aligned}$$

Taking $\epsilon \to 0$ gives that $\gamma_1(t) = \gamma_2(t)$. Now suppose for contradiction that $A_1 \triangle A_2 \neq \emptyset$, and suppose without loss of generality that $\ell \in A_1 \setminus A_2$. Take $k \in A_2$ with (WLOG) $\ell < k$, and choose some $t \in A \cap (\ell, k)$. Then

$$\begin{aligned} d(\gamma_1(\ell), \gamma_2(k)) &\leq d(\gamma_1(\ell), \gamma(t)) + d(\gamma(t), \gamma_2(k)) \\ &= t - \ell + k - t = |k - \ell|. \end{aligned}$$

For the reverse direction, let $\epsilon > 0$, and choose $s \in A$ with $\ell < s < k$ and $|s - k| < \epsilon$. Then the reverse triangle inequality gives that

$$\begin{aligned} d(\gamma_1(\ell), \gamma_2(k)) &\geq |d(\gamma_1(\ell), \gamma(s)) - d(\gamma(s), \gamma_2(k))| \\ &= |s - \ell - (k - s)| \\ &= |2s - \ell - k| \\ &= 2s - \ell - k && \text{(for } \epsilon \text{ small enough)} \\ &= 2k + 2(k - s) - \ell - k \\ &= k - \ell + 2|s - k| \\ &\geq |k - \ell|. \end{aligned}$$

Thus ℓ can be added to A_2 while keeping the extension undergeodesic, contradicting the maximality of γ_2. Thus $A_1 = A_2$ and so $\gamma_1 = \gamma_2$. □

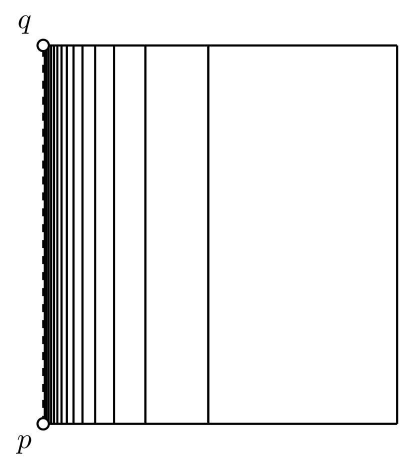
4 Embedding Theorems

Remark 4.1. It seems intuitive from the definition to think of almost geodesic spaces as in some sense dense subsets of geodesic spaces, where the geodesics are obtained by filling in the gaps of the dense undergeodesics. A natural idea, then, might be to prove that the competion of an almost geodesic space is geodesic. This cannot work, however, as the completion of even a geodesic space need not be geodesic, as is shown in the next example. The "filling in the gaps" idea does have some merit, though, as Proposition 4.4 shows.

Example 4.2 (Taken from [BH99], I.3.6(d))**.** *Consider the following subset of $\mathbb{R}^2$, equipped with the induced length metric:*

$$X = ((0,1] \times \{0,1\}) \cup \left(\{1, \frac{1}{2}, \frac{1}{3}, \dots\} \times [0,1] \right).$$

Note that X is locally compact and geodesic. The completion $\overline{X}$ is obtained from X by adding the two points $p = (0,0)$ and $q = (0,1)$.



Observe that there exists no geodesic in $\hat{X}$ from p to q, and that p admits no compact neighborhood in $\hat{X}$. Then X is a locally compact geodesic space whose completion is neither locally compact nor geodesic.

Definition 4.3. A geodesic space X is called **full** if every maximal undergeodesic in X is a geodesic. Equivalently, a geodesic space is called full if every undergeodesic lies inside the image of a geodesic. A metric space X is called **almost full** if every maximal undergeodesic in X is dense.

Proposition 4.4. *Let X be an almost geodesic metric space, and denote by $\overline{X}$ the completion of X. Then for any $x, y \in X$, there exists a geodesic in $\overline{X}$ from x to y. Furthermore, any dense undergeodesic in an arbitrary metric space X is a restriction of a geodesic in $\overline{X}$.*

Proof. Take $x, y \in X$ and set $I = [0, d(x,y)]$. By the hypothesis, there exists an undergeodesic $\gamma : A \to X$, where A is dense in I. Considering X as a dense subspace of $\overline{X}$, we can consider γ as an undergeodesic in $\overline{X}$. By a preceding proposition, there exists a maximal undergeodesic $\gamma^* : A' \to \overline{X}$ such that $\gamma^*|_A = \gamma_n$.

We claim that $A' = I$. Suppose for contradiction that this is not the case, so that there exists a $z \in I$ such that $z \notin A'$. Since $A \subseteq A'$, A' is dense in I, and thus there exists a sequence $(z_k) \subset A'$ such that $z_k \to z$. Since γ^* is an isometric embedding and (z_k) is a Cauchy sequence, we have that $(\gamma^*(z_k))$ is a Cauchy sequence, so it corresponds to an element $\bar{z} \in \overline{X}$. Define $\gamma' : A' \cup \{z\} \to \overline{X}$ by $\gamma'|_{A'} = \gamma^*$ and $\gamma'(z) = \bar{z}$. To check that γ' is an undergeodesic, we need only to check that $d(\gamma'(p), \gamma'(z)) = |p - z|$ for all $p \in A'$. We see that

$$\begin{aligned} d(\gamma'(p), \gamma'(z)) &= d(\gamma^*(p), \bar{z}) \\ &= \lim_k d(\gamma^*(p), \gamma^*(z_k)) \\ &= \lim_k |p - z_k| \\ &= |p - z|, \end{aligned}$$

so γ' is an undergeodesic. This contradicts the maximality of γ^*, so we conclude that γ^* is a geodesic in $\overline{X}$ from x to y whose restriction to A is equal to γ. This argument does not depend on the choice of dense undergeodesic, proving that any dense undergeodesic is a restriction of a geodesic in $\overline{X}$. □

Corollary 4.5. *If an almost geodesic space X is complete, then it is geodesic, and every dense undergeodesic in X lies in a geodesic.*

Proposition 4.6. *Every complete geodesic space is full.*

Proof. By Corollary 4.5, every dense undergeodesic lies in a geodesic, so by Proposition 2.6 we need only show that every maximal undergeodesic in X is dense. Let $\gamma : A \to X$ be such an undergeodesic with support $[0, d]$. Suppose for contradiction that γ is not dense, so that $(a, b) \cap A = \emptyset$ for some $(a, b) \subset [0, d]$, where $0 < a < b < d$.

Let $\alpha = \sup\{t \in A : t < a\}$, and let $\beta = \inf\{t \in A : t > b\}$. If $\alpha \in A$ an $\beta \in A$, then the interval $[\alpha, \beta]$ only intersects A at α and β, then by Lemma 3.5 we could glue γ to some geodesic from α to β and get a larger undergeodesic, a contradiction. Now suppose that $\alpha \notin A$. Then there exists an increaing sequence (α_n) of points in A such that $\alpha_n \to \alpha$. Since γ is an isometric embedding, $(\gamma(\alpha_n))$ is a Cauchy sequence in X, so by completeness it has a limit z. Extend γ to $\gamma' : A \cup \{\alpha\} \to X$ via $\gamma'(\alpha) = z$. Then for any $p \in A$,

$$\begin{aligned} d(\gamma'(p), \gamma'(\alpha)) &= d(\gamma(p), z) \\ &= \lim_n d(\gamma(p), \gamma(\alpha_n)) \\ &= \lim_n |p - \alpha_n| \\ &= |p - \alpha|, \end{aligned}$$

so γ' is an undergeodesic, contradicting the maximality of γ. A similar argument shows that $\beta \notin A$ leads to a contradiction. Thus the assumption that A is not dense in $[0, d]$ is incorrect, so γ is dense. This completes the proof. □

Corollary 4.7. *Every proper geodesic space is full.*

Proof. This follows immediately from Proposition 4.6 and Corollary I.3.8 in [BH99], which states that every proper geodesic space is complete and locally compact. □

Example 4.8. *k Let X, p, and q be as in Example 4.2. Let X' denote the space $X \cup \{p\}$. Then X' is a geodesic space, but there does not exist a geodesic in $\overline{X'} = \overline{X}$ from p to q. This provides a counterexample to the (perhaps plausible) conjecture that for any almost geodesic space X, any point $x \in X$, and any point $y \in \overline{X}$, there exists a geodesic in $\overline{X}$ from x to y.*

Remark 4.9. It seems to me that the obstructions to the completion of an almost geodesic space being a geodesic space come with new points added "on the boundary" of the space. This idea motivates this next conjecture.

Definition 4.10. Let X be a metric space. Define the **geodesic envelope** $\tilde{X}$ of X to be the set

$$\bigcup_{\gamma, x, y} \{\gamma(t) : t \in [0, d(x, y))\},$$

where the union is over all $x \in X$, all $y \in \overline{X}$, and all geodesics γ in $\overline{X}$ from x to y, metrized with the induced metric from $\overline{X}$.

Conjecture 4.11. *Let X be an almost geodesic metric space. Then $\tilde{X}$ is a geodesic space.*

Question 4.12. *Given a metric space X, can X be embedded into a geodesic space X' such that every undergeodesic $\gamma : A \to X$ can be written as $\gamma = \tilde{\gamma}|_A$ for some geodesic $\tilde{\gamma} : I \to X'$?*

This question is answered in the affirmative via the following construction.

Definition 4.13. Let (X, d_X) be a metric space. For any $x, y \in X$ and any undergeodesic $\gamma : A_\gamma \to X$ from x to y, let $L_\gamma = [0, d(x, y)]$. We denote elements of L_γ by $t^{(\gamma)}$ for $t \in [0, d(x, y)]$.
Define

$$\hat{X}' = \left(X \sqcup \bigsqcup_{\gamma} L_\gamma \right) / \sim,$$

where the union ranges over all undergeodesics γ in X, and $\sim$ is the equivalence relation generated by the following identifications:

- For all $t \in A_\gamma$, identify $t^{(\gamma)}$ in L_γ with $\gamma(t) \in X$.
- If the image of γ lies inside the image of γ', then L_γ is glued to the corresponding portion of $L_{\gamma'}$.

Note that every point $p \in \hat{X}'$ can be written as $t^{(\gamma)}$ for some $t \in L_\gamma$. Define a pseudometric $\hat{d}'$ on $\hat{X}'$ by defining

$$\hat{d}'(x, y) = \inf \sum_i |t_i^{(\gamma_i)} - t_{i+1}^{(\gamma_i)}|,$$

where the infimum ranges over all finite sequences $\{p_i\}_{i=0}^n$ with $p_0 = x$, $p_n = y$, and for $0 \le i < n$, $p_i = t_i^{(\gamma_i)}$ and $p_{i+1} = t_{i+1}^{(\gamma_i)}$ for some $t_i, t_{i+1} \in L_{\gamma_i}$. Let $\hat{X}$ be the metric space associated with the pseudometric space $\hat{X}'$. Then $\hat{X}$ is called the **geodesic saturation** of X.

Proposition 4.14. *The geodesic saturation of a metric space X is a metric space into which X embeds isometrically. Additionally, every undergeodesic in X extends to a geodesic in $\hat{X}$.*

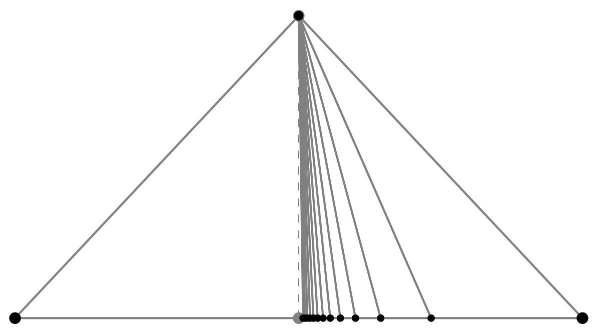
Proof. We first need to show that $\hat{X}'$ as defined above is a pseudometric space. Reflexivity and symmetry are clear from the definition of $\hat{d}$. Now we show the triangle inequality: choose $x, y, z \in \hat{X}'$, and let $\epsilon > 0$. Let $\{p_i\}$ be a path from x to y of length less than $\hat{d}'(x, y) + \epsilon/2$, and let $\{q_i\}$ be a path from y to z of length less than $\hat{d}'(y, z) + \epsilon/2$. Then concatenating the two paths gives a path from x to z of length less than $\hat{d}'(x, y) + \hat{d}'(y, z) + \epsilon$, so that

$$\hat{d}'(x, z) < \hat{d}'(x, y) + \hat{d}'(y, z) + \epsilon.$$

Taking ϵ to 0 gives the triangle inequality. Thus $\hat{d}'$ is a pseudometric, and so $(\hat{X}, \hat{d})$ is a metric space.

It is clear from the definition of $\hat{d}$ that X embeds isometrically into $\hat{X}$, and additionally that every segment L_γ embeds isometrically into $\hat{X}$ (and is thus a geodesic in $\hat{X}$ that extends γ). $\square$

Example 4.15. *Let $X = \{(\frac{1}{n}, 0) : n \in \mathbb{N}\} \cup (-1, 0) \cup (0, 1)$, considered as a subspace of $\mathbb{R}^2$. Then $\hat{d}((0,0),(0,1)) = 1$, but no path from $(0,0)$ to $(0,1)$ achieves this length. This shows that $\hat{X}$ is not, in general, a geodesic space.*



Example 4.16. *Let $X = \mathbb{Q}$. Then $\hat{X}$ is homeomorphic to $\mathbb{R}$.*

Example 4.17. *Let $X = \mathbb{Q}^2$. Then*

$$\hat{X} \cong \{(x, y) \in \mathbb{R}^2 : ax + by = c \text{ for some } a, b, c \in \mathbb{Q}\},$$

the set of rational lines in the plane. Note that this space is not geodesic. However, the geodesic saturation of $\hat{X}$ is homeomorphic to $\mathbb{R}^2$.

Proposition 4.18. *$X = \hat{X}$ (meaning that the natural inclusion map from X into $\hat{X}$ is an isometry) if and only if X is a full geodesic space.*

Proof. If X is full, then any undergeodesic lies inside a geodesic. Thus, for any undergeodesic γ, the image of γ lies inside the image of a geodesic γ', and so all of the points of L_γ will be glued to the corresponding points of L'_γ, which will in turn be glued to the points of the geodesic segment defined by γ'. Thus no new points will be added.

Now suppose that X is not full. Then there exists a maximal undergeodesic $\gamma : A \to X$ which is not a geodesic in X (with support I). Let t be a point in $I \setminus A$.

Then the point $t^{(\gamma)} \in L_\gamma$ will not be directly glued to any points in X (because $t \notin A$). It will also not be glued to any points in $L_{\gamma'}$ for any other γ' because γ is maximal. Thus $t^{(\gamma)}$ will not be in the image of the inclusion from X to $\hat{X}$, so $X \neq \hat{X}$. □

Definition 4.19. Let $\mathcal{S} : \mathbf{Met} \to \mathbf{Met}$ be the map from the objects in the category of metric spaces to itself (with morphisms being isometric embeddings) that takes a space X to $\hat{X} = \mathcal{S}X$. Note again that X embeds into $\mathcal{S}X$ via a natural inclusion map. Finally, note that the category **Met** has arbitrary direct limits.

Now, to affirmatively answer Question 4.12, we can do the following:

Proposition 4.20. *Every metric space X can be isometrically embedded into a geodesic space X' such that every undergeodesic $\gamma : A \to X$ can be written as $\gamma = \tilde{\gamma}|_A$ for some geodesic $\tilde{\gamma} : I \to X'$.*

Proof. Define a sequence of metric spaces and inclusions by

$$X_0 \ = \ X, \qquad X_{n+1} \ = \ \mathcal{S}(X_n), \quad i_n \colon X_n \ \hookrightarrow \ X_{n+1}.$$

Now form the direct limit in **Met** of this diagram:

$$X' \ = \ \varinjlim_{n<\omega} X_n \quad \text{with the canonical inclusion maps} \quad j_n \colon X_n \ \longrightarrow \ X'.$$

In particular $j_0 \colon X \to X'$ is an isometric embedding. Given any two points $p, q \in X'$, there exists a finite N such that $p, q \in j_N(X_N)$. But, by construction, every two points in X_N are connected by a geodesic segment in $X_{N+1} = \mathcal{S}(X_N)$. Hence there is a geodesic

$$\widetilde{\gamma} \colon [0, d(p,q)] \ \longrightarrow \ X_{N+1}$$

with $\widetilde{\gamma}(0) = j_N(p)$ and $\widetilde{\gamma}(d(p,q)) = j_N(q)$. Composing with the isometric inclusion $j_{N+1} \colon X_{N+1} \to X'$ shows that X' is geodesic.

The fact that every undergeodesic $\gamma : A \to X$ can be written as $\gamma = \tilde{\gamma}|_A$ for some geodesic $\tilde{\gamma} : I \to X'$ follows from the fact that such an extension can be found in X_1, and X_1 isometrically embeds into X'. Therefore X' is a geodesic space, $X \hookrightarrow X'$ is isometric, and every undergeodesic in X extends to a geodesic in X', as required. □

We can in fact take this further:

Theorem 4.21. *Every metric space can be isometrically embedded into a full geodesic metric space.*

Proof. Let $c = 2^{\aleph_0}$ be the cardinality of the continuum, and fix an ordinal α with cofinality strictly greater than c (say, $\alpha = (c)^+$, the successor cardinal of c). We build by transfinite recursion a chain

$$X_0 = X, \quad X_{\beta+1} = \mathcal{S}(X_\beta), \quad X_\lambda = \varinjlim_{\beta<\lambda} X_\beta \quad (\lambda \text{ a limit ordinal} \leq \alpha),$$

where each inclusion $X_\beta \hookrightarrow X_{\beta+1}$ is the canonical isometry, and the direct limits are taken in **Met**.

We claim that $\mathcal{S}(X_\alpha) = X_\alpha$. Indeed, any point of $\mathcal{S}(X_\alpha)$ lies on some geodesic segment coming from an undergeodesic $\gamma\colon A \to X_\alpha$. Note that the map $\beta\colon A \to \alpha$ where $\beta(t) = \min\{\beta : \gamma(t) \in X_\beta\}$ has image of size $\leq c$, so $\delta = \sup_{t\in A} \beta(t) \leq c$. Then since $\delta \leq c < \mathrm{cf}(\alpha)$, $\gamma(A) \subset X_\delta$ where $X_{\delta+1} \subset X_\alpha$, and hence the geodesic extension L_γ already lives in $\mathcal{S}(X_\delta) = X_{\delta+1} \subset X_\alpha$. No new points are thus added at stage α, so $\mathcal{S}(X_\alpha) = X_\alpha$. Proposition 4.18 then gives that X_α is full and geodesic.

The inclusion $X = X_0 \hookrightarrow X_\alpha$ is thus an isometric embedding into a full geodesic space. This completes the proof. □

Definition 4.22. Denote by $\tilde{X}$ the full geodesic space associated with X as constructed in Theorem 4.21.

Lemma 4.23. *Let X be a uniquely undergeodesic space with a dense, almost geodesic subspace. Then the space $\hat{X}$ is uniquely undergeodesic.*

Proof. Let X be such a space, and denote by $\hat{X}$ its geodesic saturation. Let $\phi_1 : A_1 \to X$ and $\phi_2 : A_2 \to X$ be undergeodesics in $\hat{X}$, choose $\epsilon > 0$, and take $a \in A_1 \cap A_2$. We then have that $\phi_1(a) = \gamma$ □

This is certainly not true without the almost geodesic assumption:

Example 4.24. *Let $X = \{x_1, x_2, x_3\}$, where $d(x_i, x_j) = 1$ for $i \neq j$. Then X is uniquely undergeodesic, but $\hat{X}$ is not uniquely undergeodesic (as there are two geodesic paths between midpoints and their opposite vertex). Thus $\tilde{X}$ is not uniquely geodesic.*

Example 4.25. *Let $X = \mathbb{Z}^2$. Then let γ_1 be the trivial undergeodesic in X from $(0,0)$ to $(1,1)$ and let γ_2 be the trivial undergeodesic in X from $(0,1)$ to $(1,0)$. Consider the points $p = (0.5)^{(\gamma_1)}$ and $q = (0.5)^{(\gamma_2)}$ in $\hat{X}$. Then the sequence that goes from p to $(0,0)$ to $(0,1)$ to q is an undergeodesic in X, as well as the sequence that goes from p to $(1,1)$ to $(1,0)$ to q. Thus $\hat{X}$ is not uniquely undergeodesic, and so $\tilde{X}$ is not uniquely geodesic.*

Lemma 4.26. *Let X be an almost full metric space. Then $\hat{X}$ is almost full.*

Proof. Let $\gamma : A \to \hat{X}$ be a maximal geodesic in $\hat{X}$ from p to q. Suppose for contradiction that $I \subseteq (0, d(p,q))$ is an interval such that $I \cap A = \emptyset$. □

Conjecture 4.27. *Let X be an almost full space. Then every isometric embedding $f : X \to Y$ of X into a full space Y extends to an isometric embedding $\hat{f} : \hat{X} \to Y$.*

Proof. Let $f : X \to Y$ be such an embedding. Define $\hat{f} : \hat{X} \to Y$ as follows: for any undergeodesic $\gamma : A \to X$. □

Remark 4.28. It is natural to consider a sort of converse to the work above: finding a condition on a subset of a geodesic space that ensures some kind of density condition on its undergeodesics. The following two examples show how this idea fails.

Example 4.29. *There exists a dense subset of $\mathbb{R}^2$ in which no three points are collinear. Such a set can be constructed inductively by taking a countable basis (B_n) of X, and for each n choosing p_n such that p_n does not lie on the line between p_i and p_j for $i, j < n$. This is an example of an anti-geodesic dense subset of a complete geodesic space.*

Example 4.30. *There also exists an uncountable dense subset of $\mathbb{R}^2$ with no three collinear points. By the Baire category theorem, since a countable union of lines in $\mathbb{R}^2$ is meager, it cannot cover any ball in $\mathbb{R}^2$, so arguing via transfinite induction as in the last example gives such a subset. This is an example of an uncountable, anti-geodesic dense subset of a complete geodesic space.*

5 Curvature

Definition 5.1. A metric space X is called **germ-saturated** if 0 is a limit point of the domain of every maximal undergeodesic in X.

Example 5.2. *$\mathbb{Q}^2$ is germ-saturated, as is $\mathbb{Q}^2 \setminus D^1$, where D^1 is the unit disk. $\mathbb{Z}^2$ is not germ-saturated.*

Example 5.3. *Every open subspace of a germ-saturated space is germ-saturated. In particular, every open subspace of a dense undergeodesic space is germ-saturated.*

Definition 5.4. Let X be a germ-saturated space, and let $c : A \to X$ and $c' : A' \to X$ be two maximal undergeodesics with $c(0) = c'(0)$. Then we can define the **upper angle** between c and c' exactly as in Definition I.1.12 of [BH99]: that is,

$$\angle(c, c') = \lim_{\epsilon \to 0} \sup_{0 < t, t' < \epsilon, t \in A, t \in A'} \overline{\angle}_{c(0)}(c(t), c(t')),$$

where $\overline{\angle}_{c(0)}(c(t), c(t'))$ is the interior angle of the comparison triangle $\overline{\Delta}(c(0), c(t), c(t'))$ in $\mathbb{R}^2$ at $c(0)$. This angle is well-defined and meaningful since X is germ-saturated, and in fact it depends only on the germs of the undergeodesics c, c' at 0.

Definition 5.5. An **undergeodesic triangle** $\Delta = \Delta(p, q, r) = \Delta(\gamma_{pq}, \gamma_{qr}, \gamma_{rp})$ in a metric space X consists of three points $p, q, r \in X$ and the union of the images of three undergeodesics

$$\gamma_{pq} : A_{pq} \to X, \quad \gamma_{qr} : A_{qr} \to X, \quad \gamma_{rp} : A_{rp} \to X.$$

An undergeodesic triangle is called **maximal** if the geodesics that compose it are all maximal. A triangle $\overline{\Delta}(p, q, r) = \Delta(\bar{p}, \bar{q}, \bar{r})$ in M^2_κ is called a **comparison triangle** for Δ if $d(p, q) = d(\bar{p}, \bar{q})$, $d(q, r) = d(\bar{q}, \bar{r})$, and $d(r, p) = d(\bar{r}, \bar{p})$. Such a triangle always exists if the perimeter $d(p, q) + d(q, r) + d(r, p)$ of Δ is less than $2D_\kappa$, where $D_\kappa = \infty$ if $\kappa \leq 0$ and $D_\kappa = \pi/\sqrt{\kappa}$ otherwise. A point $\bar{x} \in [\bar{q}, \bar{r}]$ is called a **comparison point** for $x \in A_{qr}$ if $d(q, x) = d(\bar{q}, \bar{x})$. Comparison points on $[\bar{p}, \bar{q}]$ and $[\bar{r}, \bar{p}]$ are defined in the same way.

Definition 5.6. Let (X, d) be a metric space, and let $\kappa \in \mathbb{R}$. Let Δ be an undergeodesic triangle in X with perimeter less than $2D_\kappa$, and let $\overline{\Delta}$ be a comparison triangle for Δ in M^2_κ. We say that Δ satisfies the **under-CAT**(κ) **inequality** if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \overline{\Delta}$,

$$d(x, y) \leq d(\bar{x}, \bar{y}).$$

X is called an **under-CAT**(κ) **space** (more briefly "X is under-CAT(κ)") if all of its undergeodesic triangles with perimeter less than $2D_\kappa$ satisfy the under-CAT(κ) inequality.

Remark 5.7. It is clear from the definition of under-CAT(κ) and from Proposition 2.6 that one may consider only undergeodesic triangles whose constituent undergeodesics are all maximal when deciding if a space is under-CAT(κ).

In particular, to check that a uniquely undergeodesic space X is under-CAT(κ), we need only check that for every triple of distinct points $x, y, z \in X$, the undergeodesic triangle $\Delta([x, y], [y, z], [x, z])$ satisfies the under-CAT(κ) inequality.

Definition 5.8. We say that a metric space (X, d) has **undercurvature** bounded above by κ (or that X has undercurvature $\leq \kappa$) if every point has a neighborhood that is an under-CAT(κ) space.

Proposition 5.9. *For $\kappa \in \mathbb{R}$, a geodesic metric space X is under-CAT(κ) if and only if it is CAT(κ).*

Proof. If X is under-CAT(κ), then X is CAT(κ) by definition, since geodesic triangles are undergeodesic triangles.

Now suppose X is CAT(κ), and let $\Delta(x, y, z)$ be an undergeodesic triangle in X with perimeter less than $2D_\kappa$. Then Δ is contained in $B(x, D_\kappa)$. By Proposition II.1.4(1) in [BH99], $B(x, D_\kappa)$ is a uniquely geodesic space, so in particular every undergeodesic in $B(x, D_\kappa)$ lies in a geodesic. Therefore there exists a geodesic triangle Δ' in X such that $\Delta \subset \Delta'$. Since the CAT(κ) inequality holds for Δ', the under-CAT(κ) inequality holds for Δ. Thus X is under-CAT(κ). $\square$

Proposition 5.10. *Let X be an under-CAT(κ) space.*

1. *X is uniquely undergeodesic in any ball of radius less than D_κ.*

2. *The balls of radius smaller than $D_\kappa/2$ are convex, in the sense that any undergeodesic joining any two points in such a ball is contained in the ball.*

3. *Approximate midpoints are close to each other in the following sense: for every $\lambda < D_\kappa$ and $\epsilon > 0$ there exists $\delta = \delta(\kappa, \lambda, \epsilon)$ such that for every undergeodesic $\gamma : A \to X$ from x to y with $d(x, y) \leq \lambda$, if $t_1, t_2 \in A$ are such that*

$$max\{d(x, m_1), d(y, m_1), d(x, m_2), d(y, m_2)\} \leq \frac{1}{2}d(x, y) + \delta,$$

where $m_1 = \gamma(t_1)$ and $m_2 = \gamma(t_2)$, then $d(m_1, m_2) < \epsilon$.

Proof. First we will prove (1) using Proposition 2.2. Consider $x, y, p, q \in X$ with $d(x, y) < D_\kappa$ such that $d = d(x, p) + d(p, y) = d(x, q) + d(q, y)$. First suppose that $d(x, p) = d(x, q)$. Then define $\gamma_1 : \{0, d(x, p), d\} \to X$ and $\gamma_2 : \{0, d(x, q), d\} \to X$ as undergeodesics from x to y through p and q respectfully. Then any comparison triangle in M^2_κ for $\Delta(\gamma_2, \gamma_1|_{\{0,d(x,p)\}}, \gamma_1|_{\{d(x,p),d\}})$ is degenerate and the comparison points for $\gamma_1(d(x, p)) = \gamma_2(d(x, p))$ are the same. Therefore the under-CAT(κ) inequality implies that $d(\gamma_1(t), \gamma_2(t)) = d(p, q) = 0$. Then $d(x, y) = d(x, p) + d(p, y) = d(x, p) + d(q, y) = d(x, p) + d(p, q) + d(q, y)$ as needed. Now suppose without loss of generality that $d(x, p) < d(x, q)$. Define $c : \{0\} \to X$ by $c(0) = x$ and define γ_1

and γ_2 as earlier. Consider the undergeodesic triangle $\Delta(\gamma_1, \gamma_2, c)$. Any comparison triangle for this triangle in M_κ^2 is degenerate, so the under-CAT(κ) inequality gives that $d(p,q) \leq d(x,q) - d(x,p)$, so that $d(p,q) + d(x,p) \leq d(x,q)$. The triangle inequality implies that this is an equality, so we get that $d(x,q) = d(x,p) + d(p,q)$. Then we have that $d(x,y) = d(x,q) + d(q,y) = d(x,p) + d(p,q) + d(q,y)$, as desired. Proposition 2.2 then tells us that X is uniquely undergeodesic in the ball of radius less than D_κ.

(2) follows from the fact that balls in M_κ^2 of radius less than $D_\kappa/2$ are convex (following Proposition I.2.11 in [BH99]). In detail, given a ball B of radius $r < D_\kappa/2$ centered at $p \in X$, and given any two points $x, y \in B$ and any undergeodesic γ from x to y, we get an undergeodesic triangle $\Delta(\gamma, \gamma_{xp}, \gamma_{py})$ where γ_{xp} and γ_{py} are trivial. For any $t \in A$ with $\gamma(t) = q$, then, the under-CAT(κ) inequality gives that $d(p,q) \leq d(\bar{p}, \bar{q}) < r$ since the comparison triangle for Δ lies in a ball of radius less than $D_\kappa/2$.

Now let γ, m_1, m_2 be as in (3) where $d = d(x,y)$. Consider the undergeodesic triangles $\Delta_1 = \Delta_1(\gamma, \gamma|_{[0,m_1]\cap A}, \gamma|_{[m_1,d]\cap A})$ and $\Delta_2 = \Delta_2(\gamma, \gamma|_{[0,m_2]\cap A}, \gamma|_{[m_2,d]\cap A})$. Any comparison triangles $\overline{\Delta_1}$ and $\overline{\Delta_2}$ for Δ_1 and Δ_2 in M_κ^2 are degenerate of the same length, and thus are isometric and can be regarded as a single comparison triangle $\overline{\Delta}$. Following Lemma I.2.25 in [BH99], there exists a constant δ such that for all $x, y \in M_\kappa^2$ with $d(x,y) \leq \lambda$, if $d(x,m')$ and $d(m',y)$ are both less than $\frac{1}{2}d(x,y) + \delta$, then $d(m',m) < \epsilon/2$, where m is the midpoint of $[x,y]$. Let $\bar{m}$ be the midpoint of the segment $[x,y]$ in $\overline{\Delta}$. We then get, from the under-CAT(κ) inequality, that

$$\begin{aligned} d(m_1, m_2) &\leq d(\bar{m}_1, \bar{m}_1) \\ &\leq d(\bar{m}_1, \bar{m}) + d(\bar{m}_2, \bar{m}) \\ &< \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned}$$

completing the proof of (3). □

Proposition 5.11. *Let X be a metric space. The following conditions are equivalent (when $\kappa > 0$ we assume that the perimeter of each undergeodesic triangle considered is smaller than $2D_\kappa$):*

1. *X is an under-CAT(κ) space.*

2. *For every undergeodesic triangle $\Delta(\gamma_{pq}, \gamma_{qr}, \gamma_{rp})$ in X and every point $t \in A_{qr}$ with $x = \gamma_{qr}(t)$, the following inequality is satisfied by the comparison point*

$\bar{x} \in [\bar{q}, \bar{r}] \subset \overline{\Delta}(p,q,r) \subset M^2_\kappa$:

$$d(p,x) \leq d(\bar{p}, \bar{x}).$$

3. *For every undergeodesic triangle $\Delta(\gamma_{pq}, \gamma_{qr}, \gamma_{pr})$ in X and every pair of points $t \in A_{pq}$, $s \in A_{rp}$ with $\gamma_{pq}(t) = x$ and $\gamma_{pr}(s) = y$ such that $x \neq p$ and $y \neq p$, the angles at the vertices corresponding to p in the comparison triangles $\overline{\Delta}(\gamma_{pq}, \gamma_{qr}, \gamma_{rp}) = \overline{\Delta}(p,q,r) \subset M^2_\kappa$ and $\overline{\Delta}(\gamma_{pq}|_{[0,x]\cap A_{pq}}, \gamma_{xy}, \gamma_{pr}|_{[0,y]\cap A_{pr}}) = \overline{\Delta}(p,x,y) \subset M^2_\kappa$, where $\gamma_{xy} : \{0, d(x,y)\} \to X$ is trivial, satisfy*

$$\angle_p^{(\kappa)}(x,y) \leq \angle_p^{(\kappa)}(q,r).$$

If additionally X is germ-saturated, then the following conditions are also equivalent to the three conditions above:

4. *The upper angle between the sides of any maximal undergeodesic triangle in X with distinct vertices is no greater than the angle between the corresponding sides of its comparison triangle in M^2_κ.*

5. *For every undergeodesic triangle $\Delta(\gamma_1, \gamma_2, \gamma_3) = \Delta(p,q,r)$ in X with $p \neq q$ and $p \neq r$, if α denotes the upper angle between γ_1 and γ_3 at p and if $\Delta(\hat{p}, \hat{q}, \hat{r}) \subset M^2_\kappa$ is a geodesic triangle with $d(\hat{p}, \hat{q}) = d(p,q)$, $d(\hat{p}, \hat{r}) = d(p,r)$, and $\angle_{\hat{p}}(\hat{q}, \hat{r}) = \alpha$, then $d(q,r) \geq d(\hat{q}, \hat{r})$.*

Proof. It is clear that (1) implies (2). We will show that (3) implies (1), and then that (2) implies (3).

Let p, q, r, t, s, x, y be as in (3). We write $\bar{z}$ to denote comparison points in $\overline{\Delta} = \overline{\Delta}(p,q,r) \subset M^2_\kappa$, and $\bar{z}'$ to denote comparison points in $\overline{\Delta} = \overline{\Delta}(p,x,y) \subset M^2_\kappa$. Consider the vertex angles $\bar{\alpha} = \angle_p^{(\kappa)}(q,r)$ and $\bar{\alpha}' = \angle_p^{(\kappa)}(x,y)$ at $\bar{p}$ and $\bar{p}'$. According to the law of cosines in M^n_κ (I.2.13 in [BH99]), the inequality $d(\bar{x}, \bar{y}) \geq d(\bar{x}', \bar{y}') = d(x,y)$ holds if and only if $\bar{\alpha}' \leq \bar{\alpha}$, which is true by the assumption. Thus (3) implies (1).

Maintaining the notation of the previous paragraph, we will now show that (2) implies (3). Let $\Delta(\bar{p}'', \bar{x}'', \bar{r}'') \subset M^2_\kappa$ be a comparison triangle for $\Delta(p,x,r) = \Delta(\gamma_{pq}|_{[0,x]\cap A_{pq}}, \gamma_{xr}, \gamma_{pr})$, where γ_{xr} is trivial. Let $\bar{\alpha}''$ denote the vertex angle $\bar{p}''$. By (2), we have that $d(x,y) \leq d(\bar{x}'', \bar{y}'')$, where $\bar{y}'' \in [\bar{p}'', \bar{r}'']$ is a comparison point for y. Since $d(x,y) = d(\bar{x}', \bar{y}')$, we get that $\bar{\alpha}' \leq \bar{\alpha}''$. Again by (2), $d(\bar{x}'', \bar{r}'') = d(x,r) \leq d(\bar{x}, \bar{r})$, so $\bar{\alpha}'' \leq \bar{\alpha}$. Thus $\bar{\alpha}' \leq \bar{\alpha}$ as needed. Thus (2) implies (3).

It is clear that (4) is equivalent to (5). (4) also follows immediately from (3) and the observation that one can use comparison triangles in M_κ^2 rather than $\mathbb{R}^2$ when defining the upper angle. Thus we need only show that (4) implies (2). $\square$

Definition 5.12. A **subembedding in** M_κ^2 of a 4-tuple of points (x_1, x_2, y_1, y_2) from a metric space X is a 4-tuple of points $(\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2)$ in M_κ^2 such that $d(\bar{x}_i, \bar{y}_j) = d(x_i, y_j)$ for $i, j \in \{1, 2\}$, and $d(x_1, x_2) \leq d(\bar{x}_1, \bar{x}_2)$ and $d(y_1, y_2) \leq d(\bar{y}_2, \bar{y}_2)$.

X is said to satisfy the **CAT(κ) 4-point condition** if every 4-tuple of points (x_1, y_1, x_2, y_2) with $d(x_1, y_1) + d(y_2, x_1) + d(x_2, y_2) + d(y_2, x_1) < 2D_\kappa$ has a subembedding in M_κ^2.

Proposition 5.13. *If a metric space X satisfies the CAT(κ) 4-point condition, then it is under-CAT(κ).*

Proof. Let X be a space satisfying the 4-point condition. Consider an undergeodesic triangle $\Delta(x, y, z) = \Delta(\gamma_{xy}, \gamma_{yz}, \gamma_{xz}) \subset X$ of perimeter less than $2D_\kappa$, and let $t \in A_{xy}$ with $m = \gamma_{xy}(t)$. Let $(\bar{z}, \bar{x}, \bar{m}, \bar{y})$ be a subembedding of (z, x, m, y) in M_κ^2. As $d(x, y) \leq d(\bar{x}, \bar{y}) \leq d(\bar{x}, \bar{m}) + d(\bar{m}, \bar{y}) = d(x, m) + d(m, y) = d(x, y)$, we see that $d(x, y) = d(\bar{x}, \bar{y})$. Additionally, it follows immediately from the properties of a subembedding that $d(x, z) = d(\bar{x}, \bar{z})$ and that $d(y, z) = d(\bar{y}, \bar{z})$, so $\Delta(\bar{x}, \bar{y}, \bar{z})$ is a comparison triangle for $\Delta(x, y, z)$ such that $\bar{m}$ is the comparison point in $[\bar{x}, \bar{y}]$ for $m \in \gamma_{xy}(A_{xy})$. By the definition of the subembedding, $d(z, m) \leq d(\bar{z}, \bar{m})$. Thus $\Delta(x, y, z)$ satisfies the under-CAT(κ) inequality as characterized in Proposition 5.11 (2). $\square$

Corollary 5.14. *Every subspace of a CAT(κ) space is under-CAT(κ).*

Proof. This follows from the fact that every subspace of a CAT(κ) space satisfies the CAT(κ) 4-point condition (see Definition II.1.10 in [BH99]). Proposition 5.13 then gives the result. $\square$

Example 5.15. *Let $X = \{x_1, y_1, x_2, y_2\}$, and metrize X by setting $d(x_1, x_2) = 1$, $d(y_1, y_2) = \frac{7}{4}$, and $d(x_i, y_j) = 1$ for all i, j. Since this space has no nontrivial undergeodesics, it is under-CAT(κ) for every κ, so it is in particular under-CAT(0). However, this space admits no subembedding into $\mathbb{R}^2$. Thus the converse of Proposition 5.13 does not hold in general.*

Example 5.16. *Corollary 5.14 guarantees that, in particular, $\mathbb{Q}^2$ and $\mathbb{Z}^2$ are nonpositively curved as subspaces of $\mathbb{R}^2$.*

Question 5.17. *What conditions on a metric space X guarantee that the implication "CAT(κ) 4-point condition $\implies$ X is under-CAT(κ)" holds?*

Definition 5.18. Following Definition II.3.7 in [BH99], a metric space (X, d) is a **4-point limit** of a sequence of metric spaces (X_n, d_n) if, for every 4-tuple of points (x_1, x_2, x_3, x_4) from X and every $\epsilon > 0$, there exist infinitely many integers n such that there is a 4-tuple $(x_1(n), x_2(n), x_3(n), x_4(n))$ from X_n with $|d(x_i, x_j) - d_n(x_i(n), x_j(n))| < \epsilon$ for $1 \leq i, j \leq 4$.

6 Boundary Behavior

Definition 6.1. Let (X, d) be a metric space and let $x_0 \in X$. Let $A \subseteq [0, \infty)$ be an unbounded set with $0 \in A$. Then a map

$$\gamma : A \to X$$

is called an **undergeodesic ray** emanating from x_0 if

$$\gamma(0) = x_0 \quad \text{and} \quad d(\gamma(s), \gamma(t)) = |s - t|$$

for all $s, t \in A$. An undergeodesic ray is called **admissible** if it is α-geodesic for some α (in the sense that its domain intersects every interval of length α).

Definition 6.2. Let (X, d) be a metric space. We say that two admissible undergeodesic rays $\gamma_1 : A_1 \to X$ and $\gamma_2 : A_2 \to X$, both emanating from x_0, are **equivalent** if, for all $C > 0$, there exists $K > 0$ such that

$$d\big(\gamma_1(t), \gamma_2(s)\big) < K$$

whenever $|t - s| < C$. The **undervisual boundary** of X, denoted ∂X, is then defined as the set of equivalence classes of admissible undergeodesic rays in X.

Proposition 6.3. *Equivalence of admissible undergeodesic rays is an equivalence relation.*

Proof. Reflexivity and symmetry are obvious. Let $\gamma_1, \gamma_2, \gamma_3$ be such that $\gamma_1 \sim \gamma_2$ and $\gamma_2 \sim \gamma_3$. Choose $C > 0$. Then, since γ_2 is admissible, there exists an $\alpha \geq 0$ such that every interval of length α intersects A_2. Since $\gamma_1 \sim \gamma_2$ and $\gamma_3 \sim \gamma_2$, there exist numbers $K_1 > 0$ and $K_2 > 0$ such that $d(\gamma_1(x_1), \gamma_2(x_2)) < K_1$ and

$d(\gamma_2(y_1), \gamma_3(y_2)) < K_2$ whenever $|x_1 - x_2| \leq \alpha$ and $|y_1 - y_2| \leq \alpha$. Now let $t \in A_1$ and $s \in A_3$ be such that $|t - s| < C$. By the admissibility condition, we can choose $a_1, a_2 \in A_2$ such that $|t - a_1| \leq \alpha$ and $|s - a_2| \leq \alpha$. Note that

$$|a_1 - a_2| \leq |a_1 - t| + |t - s| + |s - a_2| < 2\alpha + C.$$

Then we see that

$$\begin{aligned} d(\gamma_1(t), \gamma_3(s)) &\leq d(\gamma_1(t), \gamma_2(a_1)) + d(\gamma_2(a_1), \gamma_2(a_2)) + d(\gamma_2(a_2), \gamma_3(s)) \\ &< K_1 + |a_1 - a_2| + K_2 \\ &\leq K_1 + 2\alpha + C + K_2. \end{aligned}$$

Since this quantity does not depend on the choice of t or s, we are done. □

7 Other

Definition 7.1. A **undergeodesic tree** is a metric space T such that:

1. T is uniquely undergeodesic.
2. Following the notation of Definition 2.4, if $[y, x]$ and $[x, z]$ are maximal undergeodesic segments such that $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.

Proposition 7.2. *A space X is an undergeodesic tree if and only if it is under-CAT(κ) for every κ.*

Proof. ($\Rightarrow$) First suppose that X is an undergeodesic tree. Choose $x, y, z \in X$, and consider the undergeodesic triangle $\Delta([x, y], [y, z], [x, z])$. Suppose first that $[x, y] \cap [y, z] = \{y\}$. Then $[x, y] \cup [y, z] = [x, z]$, the comparison triangle in M_κ^2 is degenerate, and the under-CAT(κ) inequality is satisfied.

Now suppose that $w \in [x, y] \cap [z, y]$ with $w \neq y$. Suppose without loss of generality that $d(x, y) \leq d(z, y)$. Then $[y, w] \cap [w, z] = \{w\}$.

($\Leftarrow$) Let X be a space that is under-CAT(κ) for every κ. X is uniquely undergeodesic by Proposition 5.10. Now let $[y, x]$ and $[x, z]$ be maximal undergeodesic segments that intersect only at x. □

Remark 7.3. The familiar Gromov hyperbolicity condition is as follows: a geodesic metric space (X, d) is called **δ-hyperbolic** if for every geodesic triangle in X with vertices x, y, z, and for every point p on any side, there exists a point q on the union of the other two sides such that

$$d(p, q) \leq \delta.$$

This condition is equivalent to the following four-point condition: for any $x, y, p, q \in X$, we have that

$$d(x, y) + d(p, q) \leq \max\{d(w, p) + d(y, q), d(x, q) + d(y, p)\} + 2\delta.$$

Importantly, these two are equivalent when X is geodesic. Of course, if (for example) no geodesics exist at all in X, then the first condition is trivially true whereas the second may well be false. Is there an undergeodesic version of Gromov hyperbolicity that is equivalent to this four-point condition in an arbitrary metric space and that reduces to the familiar version in a geodesic space?

Upon reflection, if there is to be such a condition on undergeodesics, it must be purely combinatorial in the case when undergeodesics are trivial. This is because there is no guarantee on whether the four-point condition is satisfied given only the information that no nontrivial undergeodesics exist.

8 References

References

[BH99] Martin R. Bridson and André Haefliger. *Metric Spaces of Non-Positive Curvature*. Vol. 319. Grundlehren der mathematischen Wissenschaften. Springer Berlin, Heidelberg, 1999.