A tutorial on
Wavelet Transform & Compressed sensing

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innovation you

OUTLINE

- MRI Reconstruction: Inverse problem
- Wavelet Transform
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  - Wavelets
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  - Discrete Wavelet Transform
  - Localization properties
  - Sparity
  - Denoising
- Compressed sensing
  - A new theory of sampling
  - Solving Underdetermined Systems
  - Incoherent sampling
  - Optimization Scheme
  - Demo

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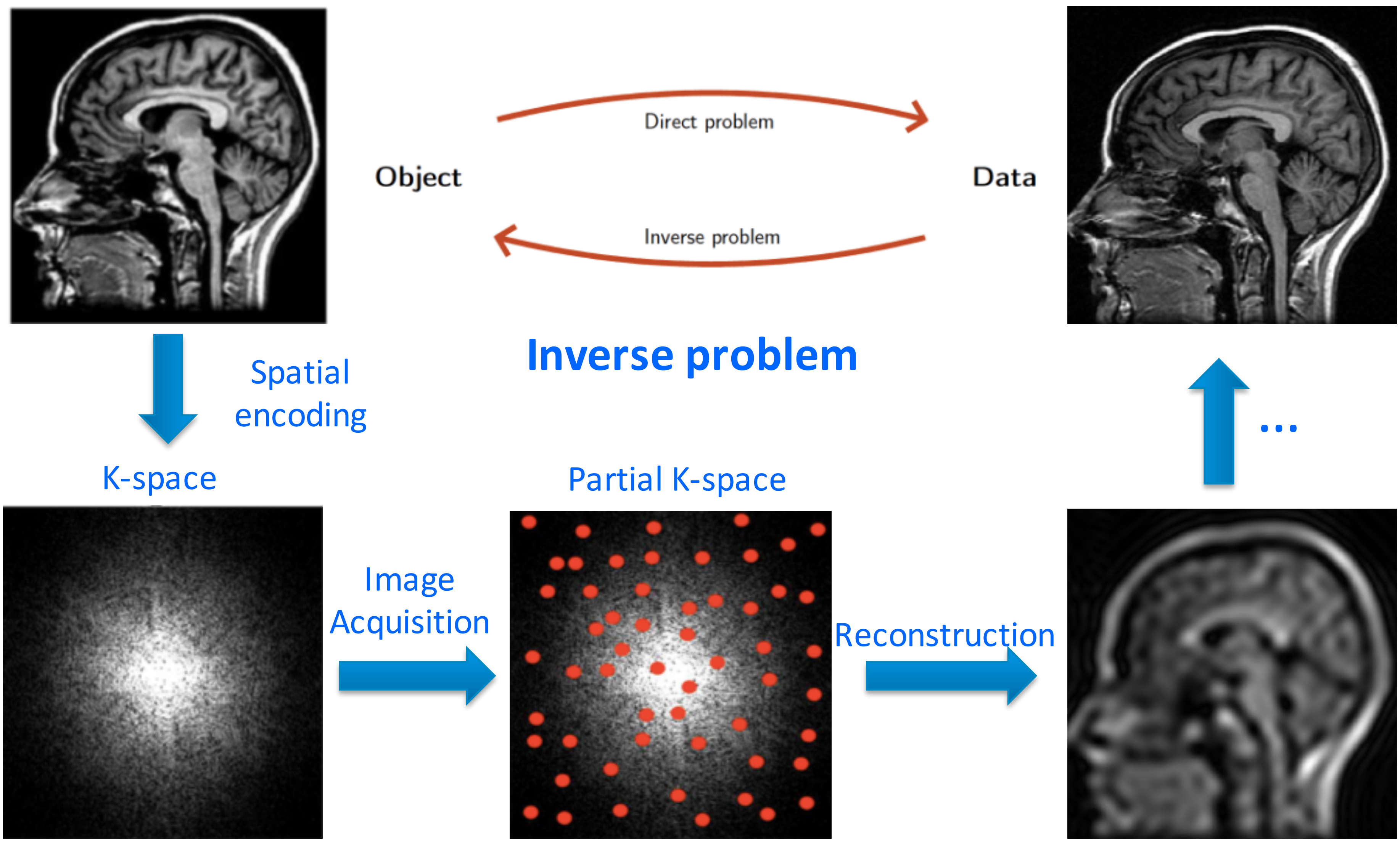
MRI Reconstruction: Inverse problem



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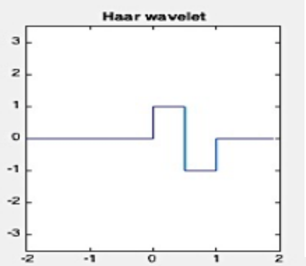
Wavelet transform: Continuous Wavelet Transform

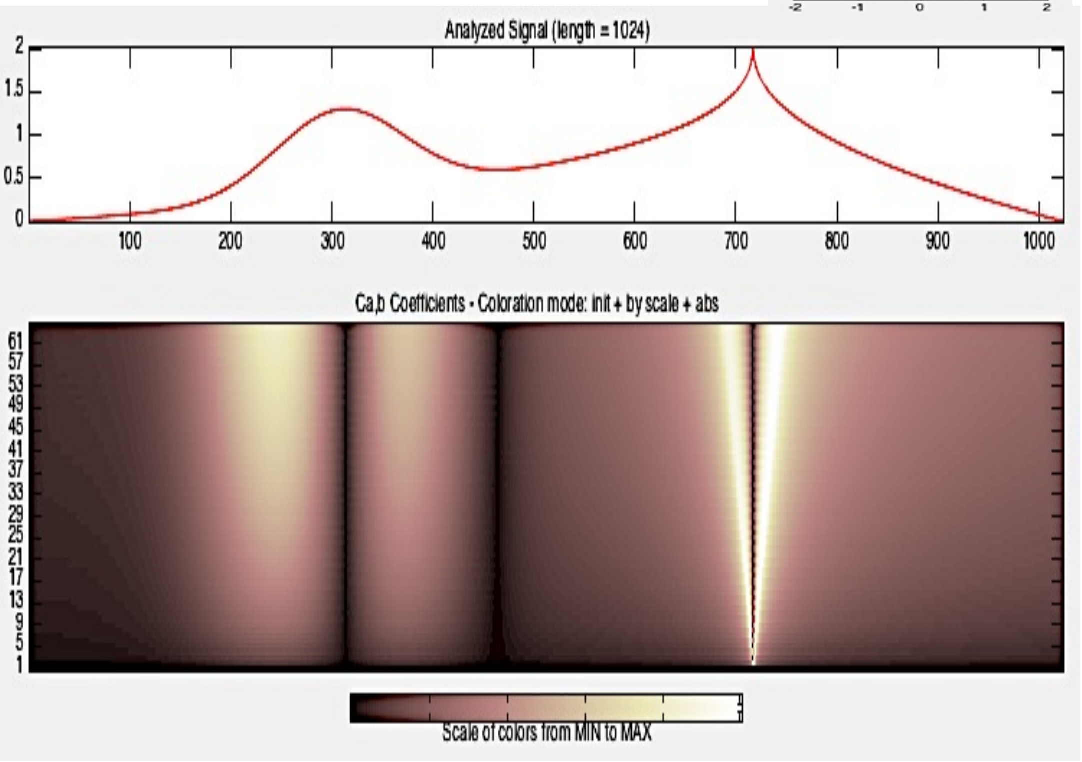
- **CWT**: The integral sum of the signal multiplied by the Dilations and translations of the wavelet functions:



$$C(scale, position) = \int_{-\infty}^{\infty} f(t)\Psi(scale, position, t)\, dt$$

$$\psi_{\tau,s}(t) = \frac{1}{\sqrt{|s|}}\psi(\frac{t-\tau}{s})$$

- For a given wavelet function with fixed τ and s,
  $C = < f(t), \psi_{\tau,s}(t) >$
- For a set of wavelet functions with fixed s,
  $C(\tau) = f(t) * \psi_{-\tau,s}(-t)$



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Wavelet transform: Wavelet function

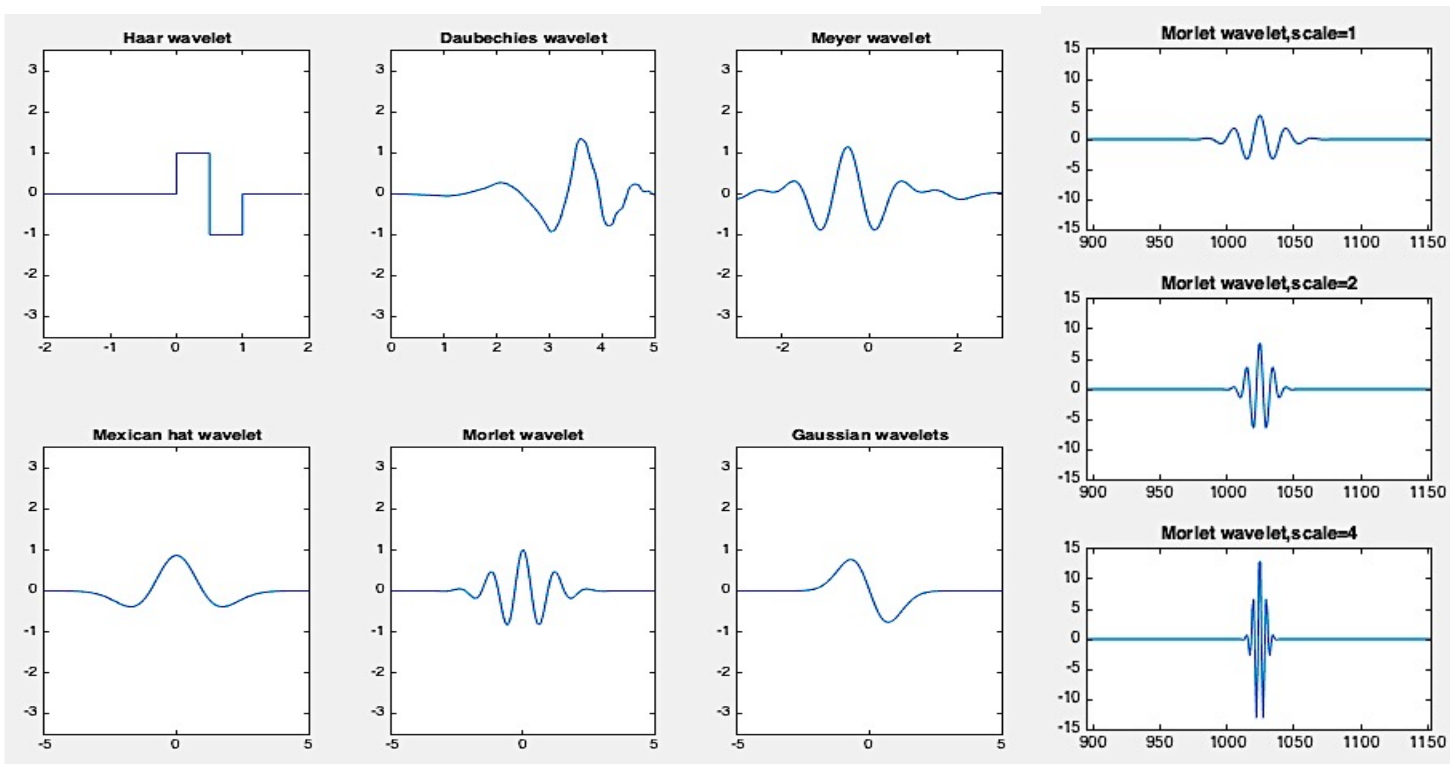
- A Wavelet is a waveform of effectively limited duration that has an average value of zero.
- Admissibility condition: $C_\psi = \int_{\mathbb{R}} \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < \infty$ $\quad \psi_{\tau,s}(t) = \frac{1}{\sqrt{|s|}} \psi(\frac{t-\tau}{s})$

Zero mean:

$$\int_{-\infty}^{\infty} \Psi(t)\,\mathrm{d}\,t = 0$$

Finite energy:

$$\int_{-\infty}^{\infty} |\Psi(t)|^2 \mathrm{d}\,t \leq \infty$$



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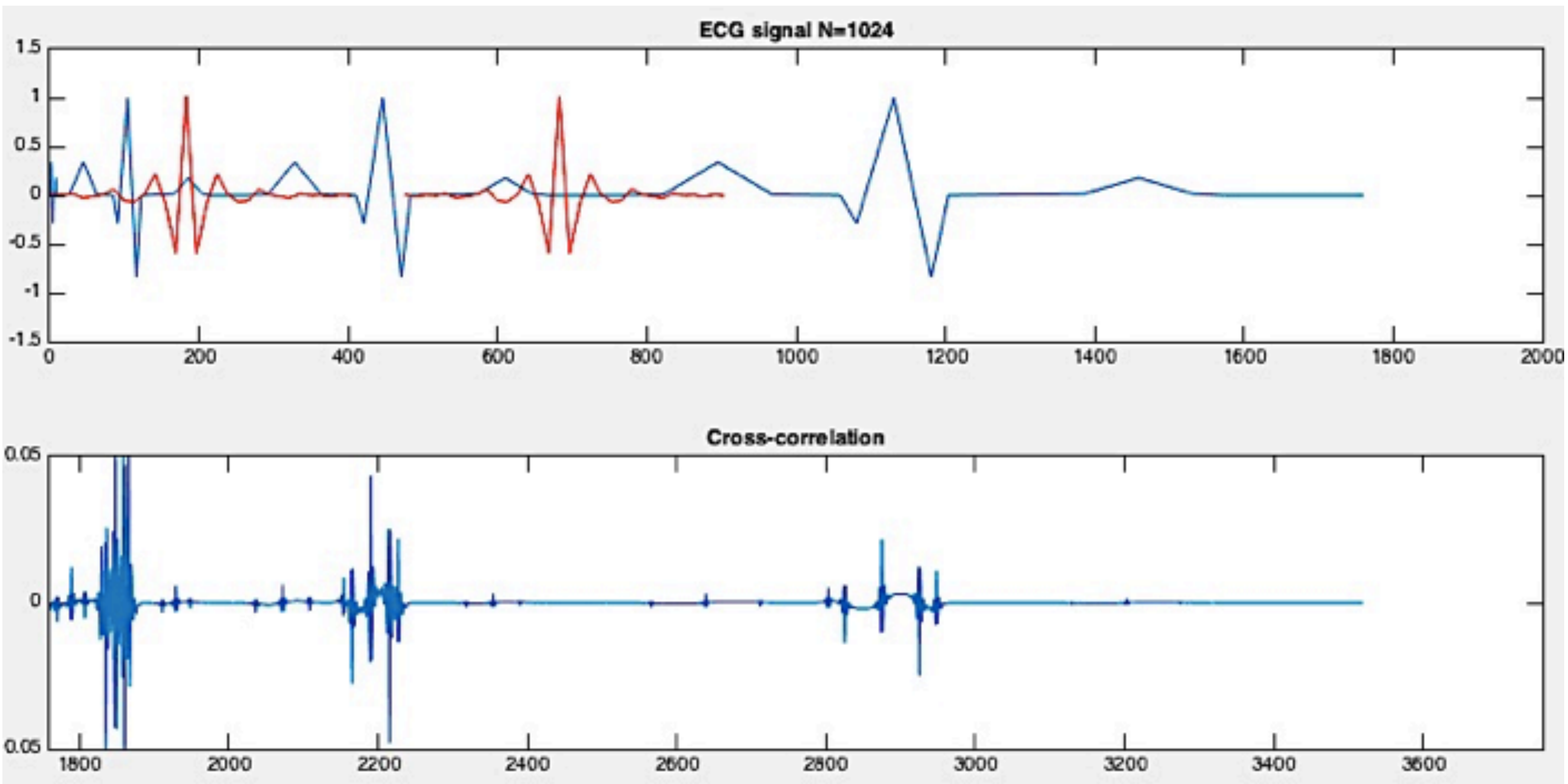
Wavelet transform: CWT — Cross Correlation

Think wavelet in different perspectives:

- Cross correlation (inner product): A measure of the similarity between the signal and the scaled and shifted wavelet.



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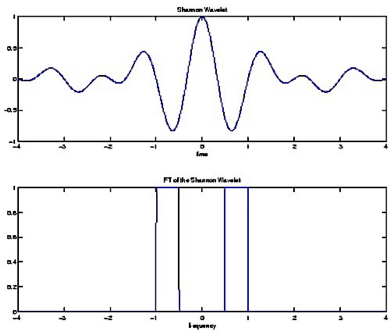
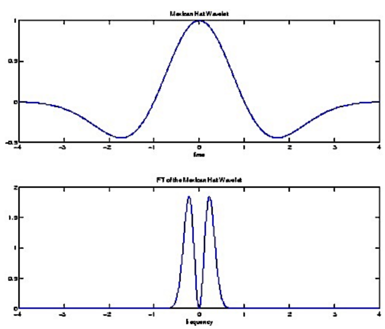
Wavelet transform: CWT — Band-pass filtering

- Band-pass filtering (convolution): For a fixed scale, the wavelet transform is the convolution of the signal and the time reversed wavelet.

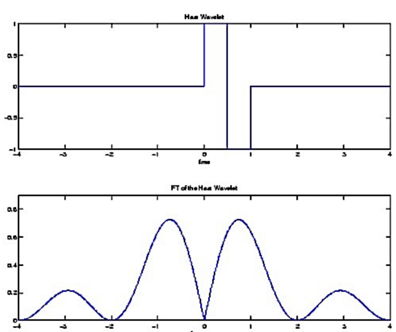
Shannon Wavelet



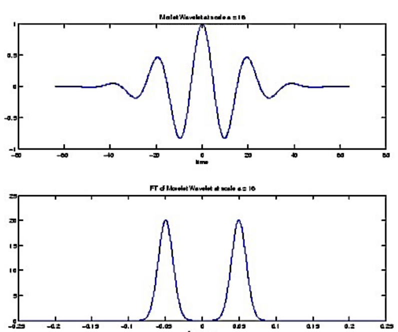
Mexican hat

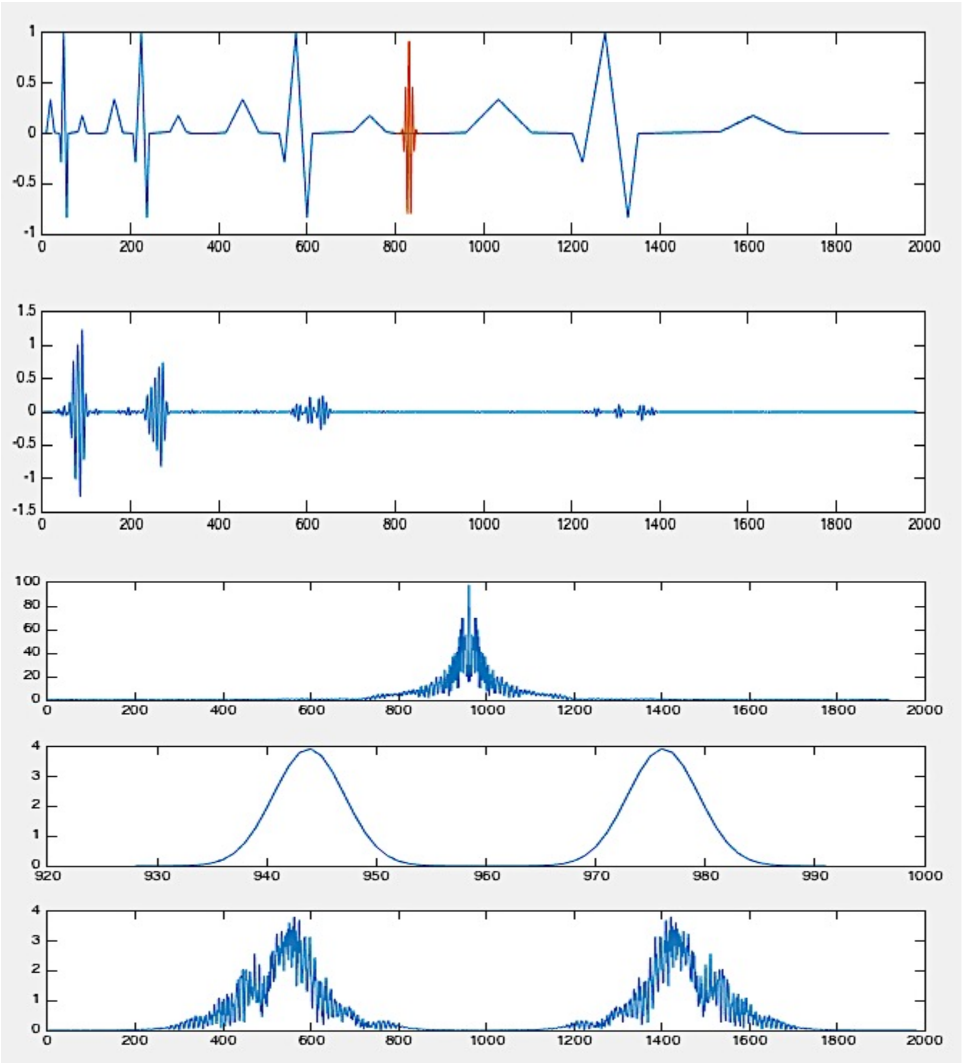


Haar



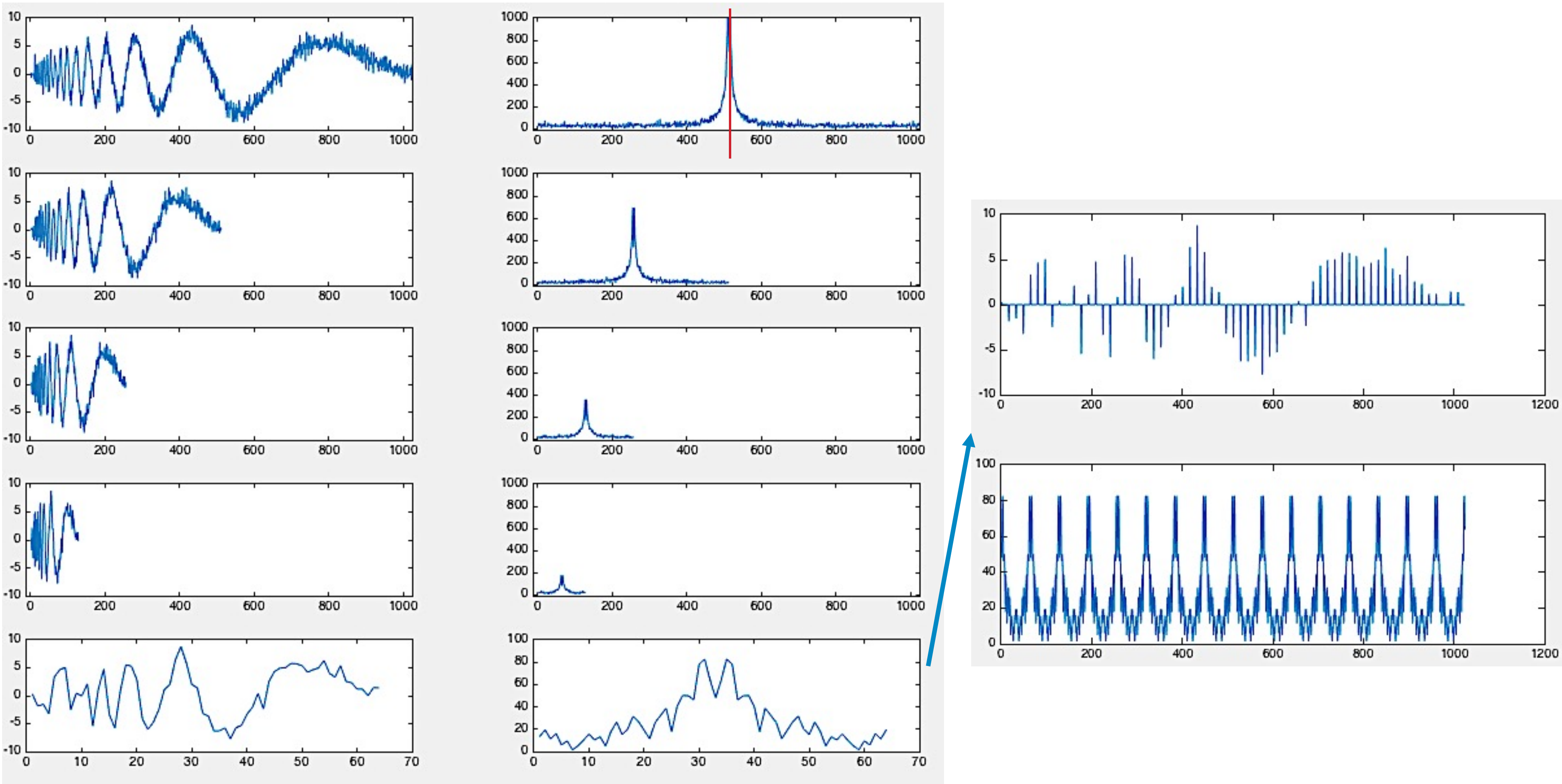
Morelet





Wavelet transform: Discrete Wavelet Transform

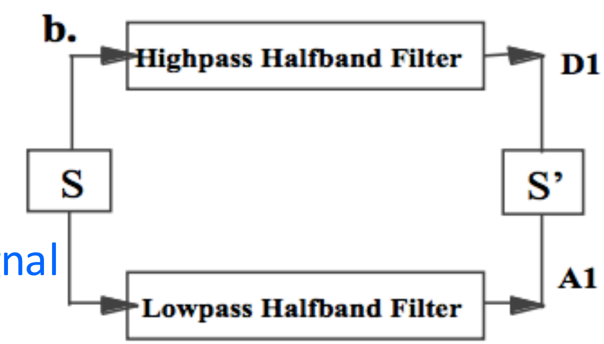
- Mutiresolution Analysis: View a signal in different numbers of points



Wavelet transform: Discrete Wavelet Transform

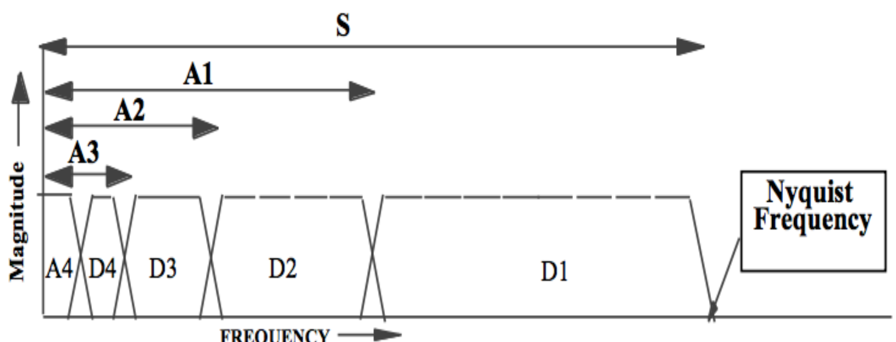
- DWT: Calculating the wavelets coefficients at only a subset of scales and shifted positions by splitting the signal to sub-bands:

Scale Discretization: s → 2^j



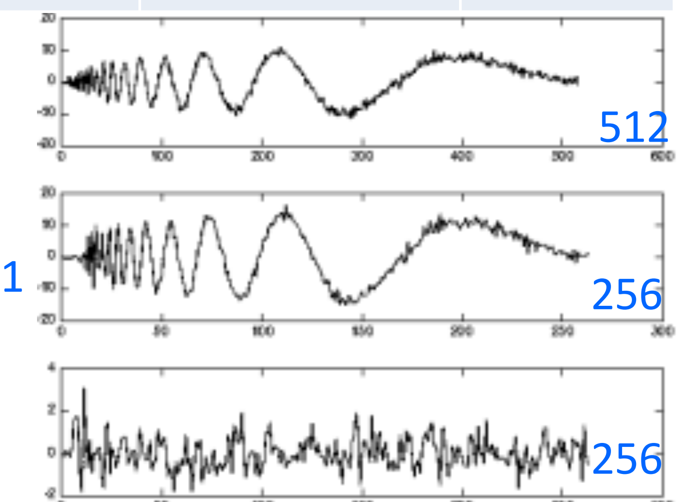
N points signal
0 ~ B Hz

Level	Frequencies	Samples
3	$0\ to\ \frac{f_n}{8}$	N/4
	$\frac{f_n}{8}\ to\ \frac{f_n}{4}$	N/4
2	$\frac{f_n}{8}\ to\ f_n/4$	N/2
1	$\frac{f_n}{2}\ to\ f_n$	N



$f_n = 2B$

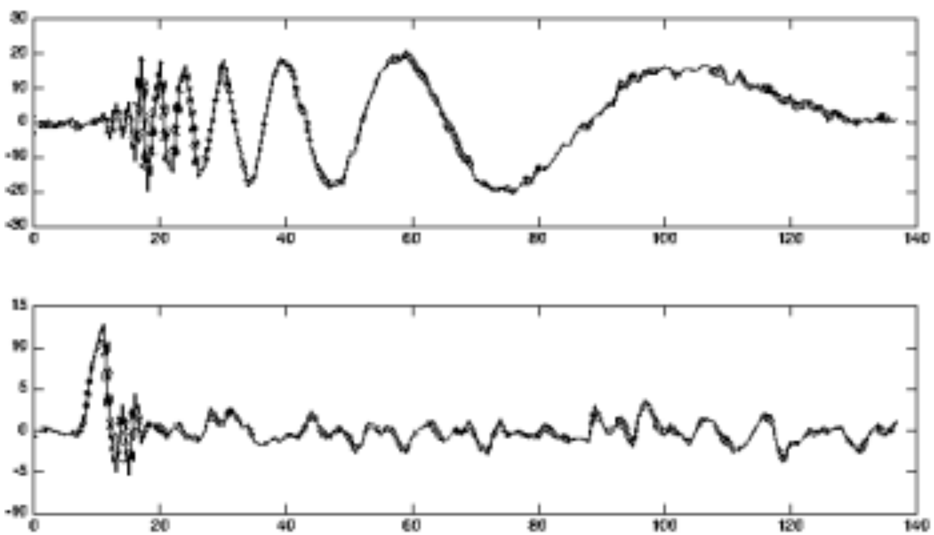
Level 1



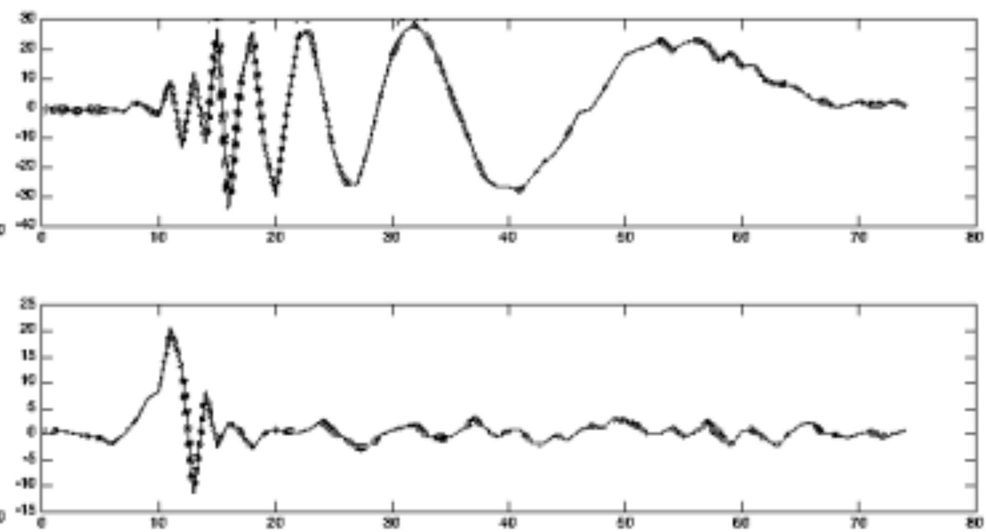
Wavelet transform: Discrete Wavelet Transform

Shifted position Discretization: $\tau \rightarrow k2^{j}$

- According to Nyquist-Shannon sampling theorem: $f_s > 2B$. Half of the samples can be discarded!



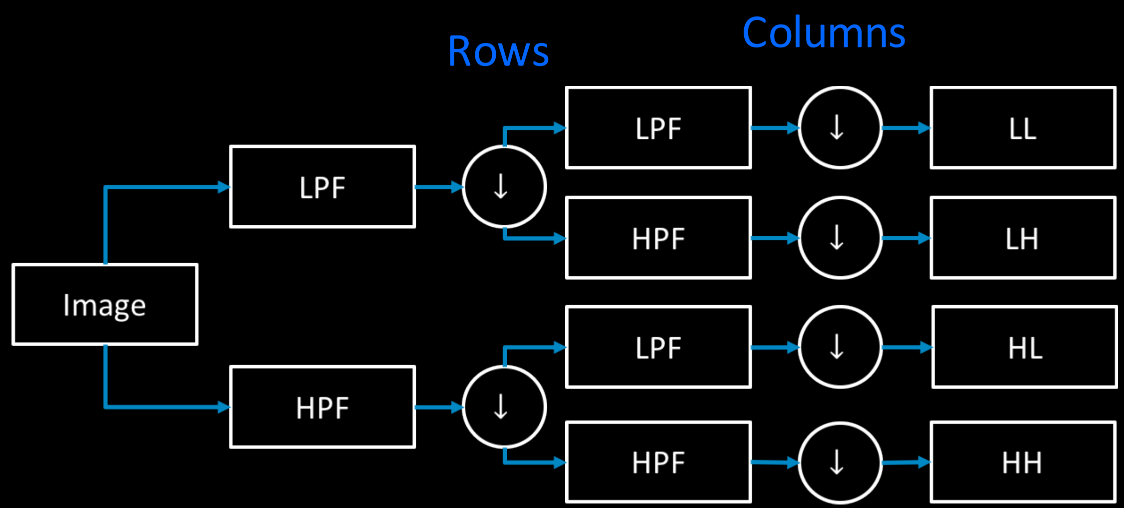
Level 2



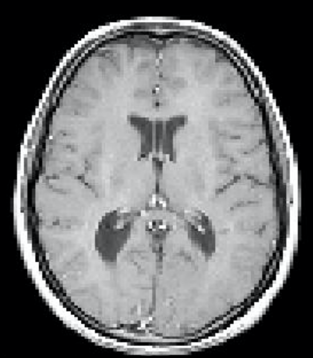
Level 3

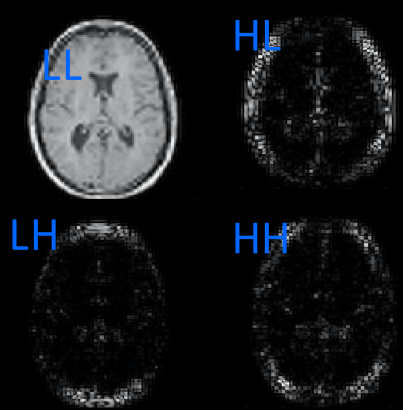
Wavelet transform: Discrete Wavelet Transform

2D-DWT: Do 1D-DWT horizontally and vertically.



Brain MRI image



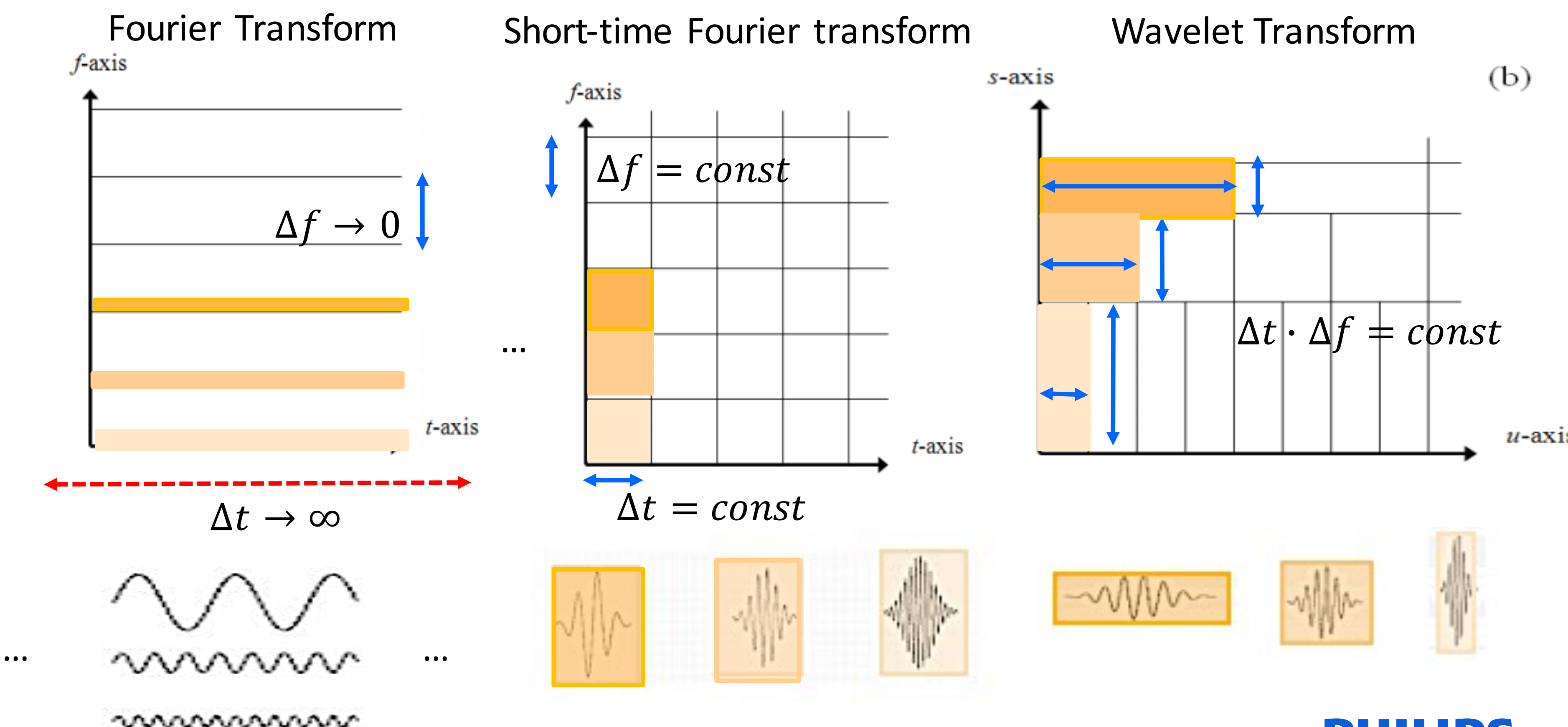


Approximation	Horizontal details
Vertical details	Diagonal details

Wavelet Transform: Localization properties

At the Time-frequency plane, the Heisenberg Uncertainty Principle told us:

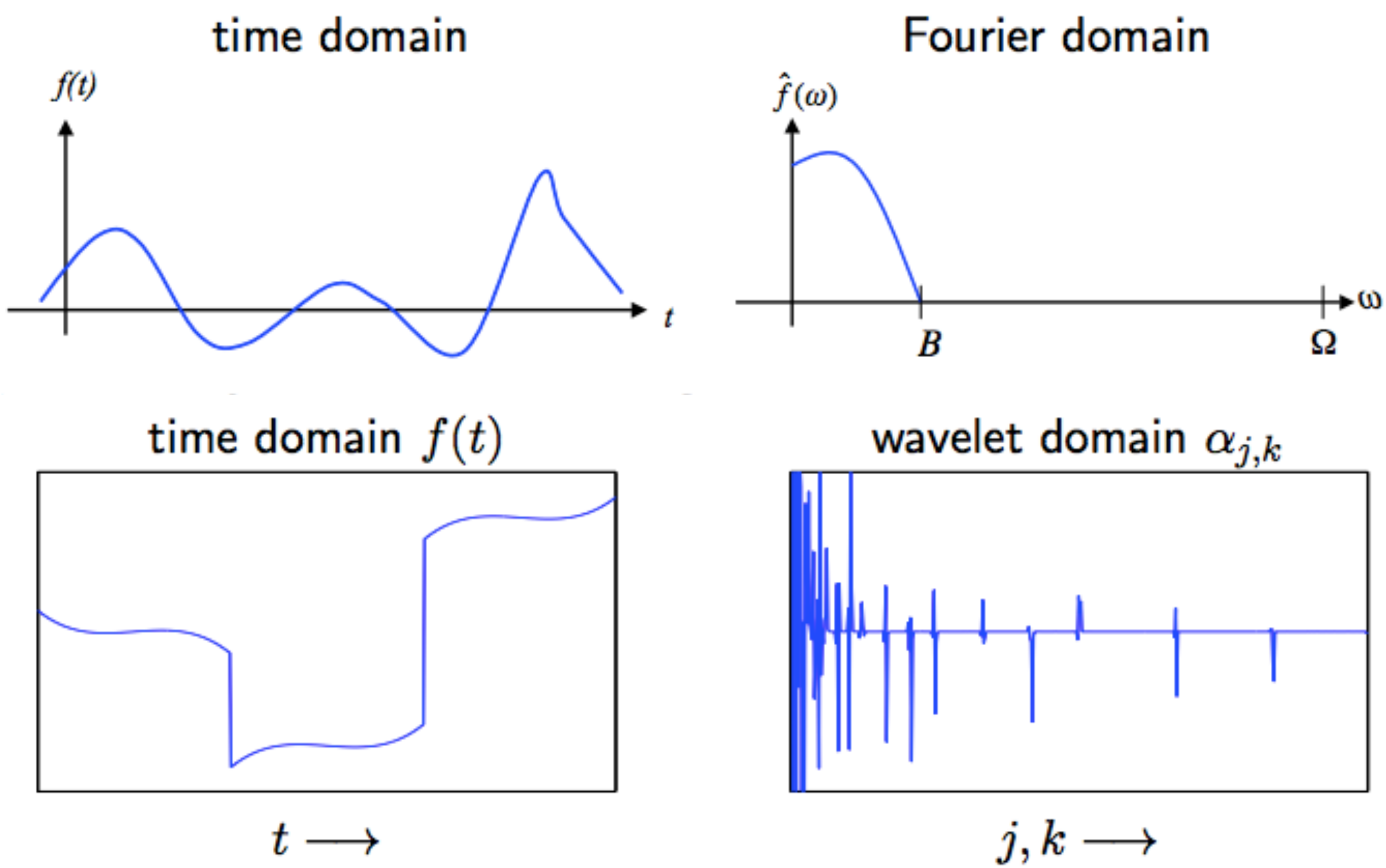
$$\Delta t \cdot \Delta f \geq \frac{1}{4\pi}$$



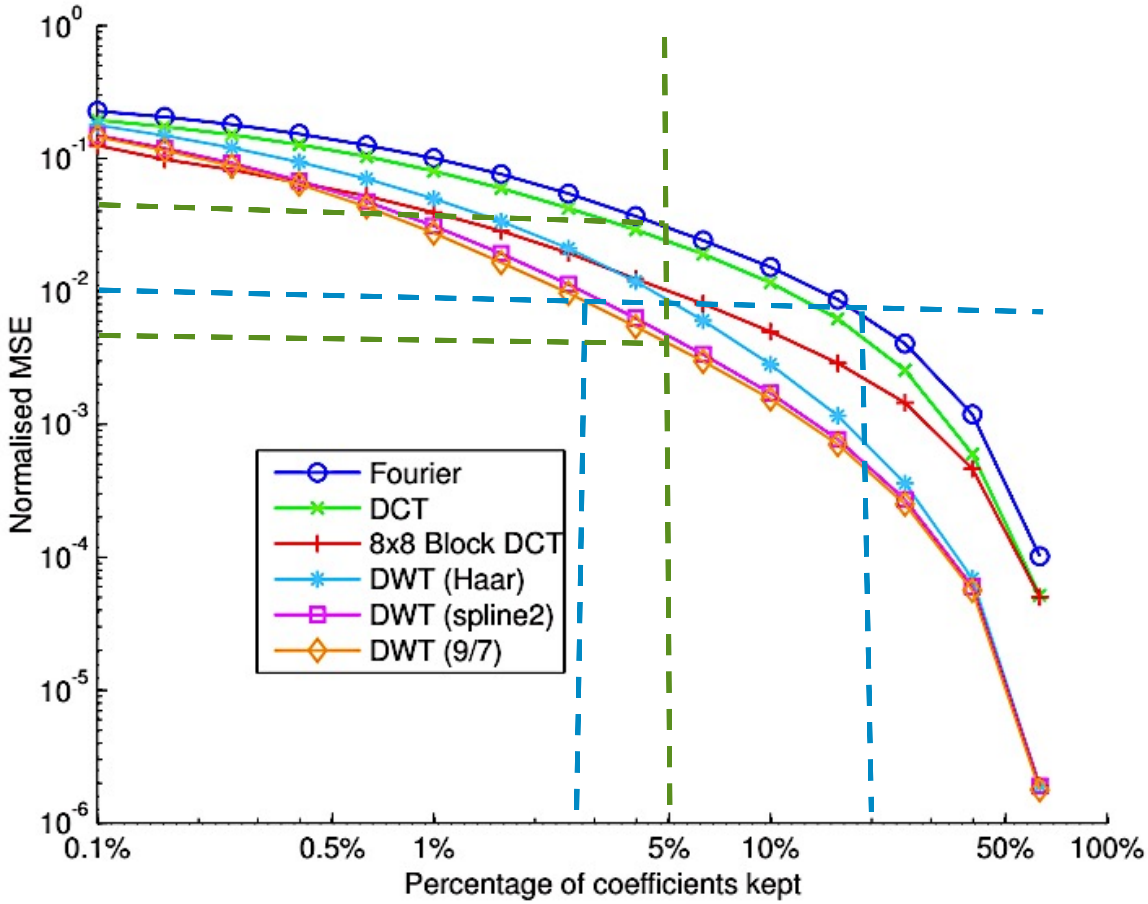
"Time" here can be replaced by space

Wavelet Transform: Sparsity

Sparse: few large coeffs, many small coeffs



Wavelet Transform: Sparsity



Guerquin–Kern
EPFL,PhD thiesis

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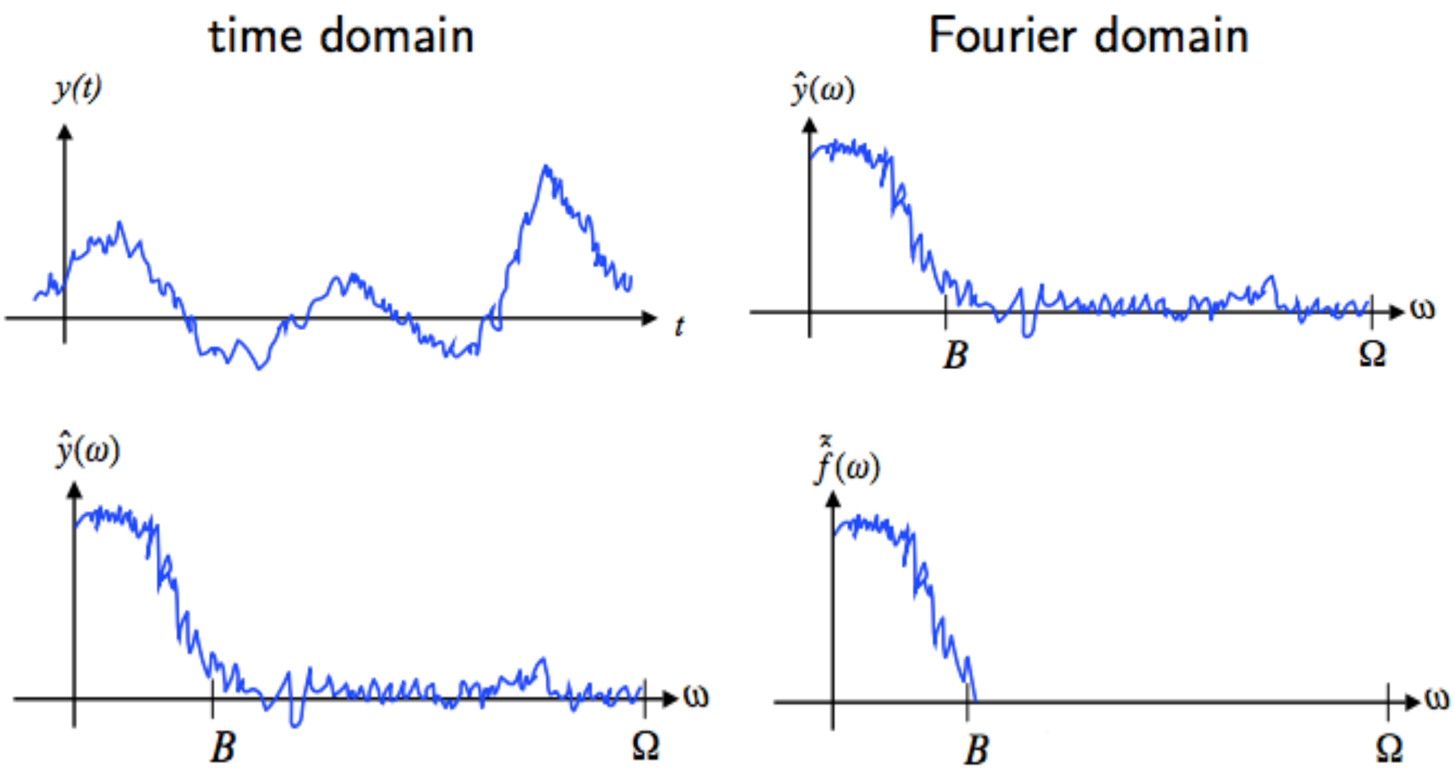
Wavelet Transform: Denoising

- Frequency domain: Denoising with low pass filtering:



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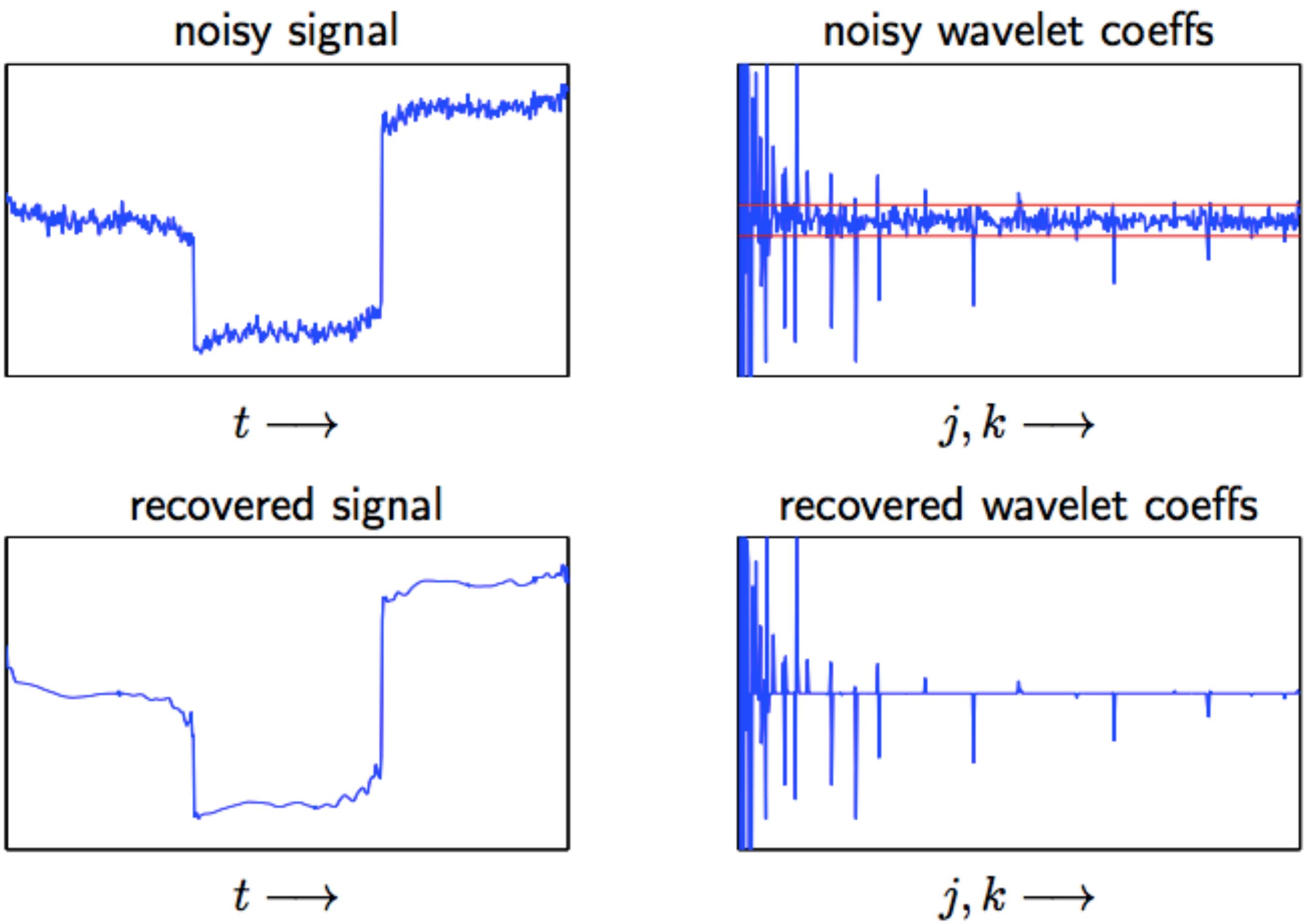
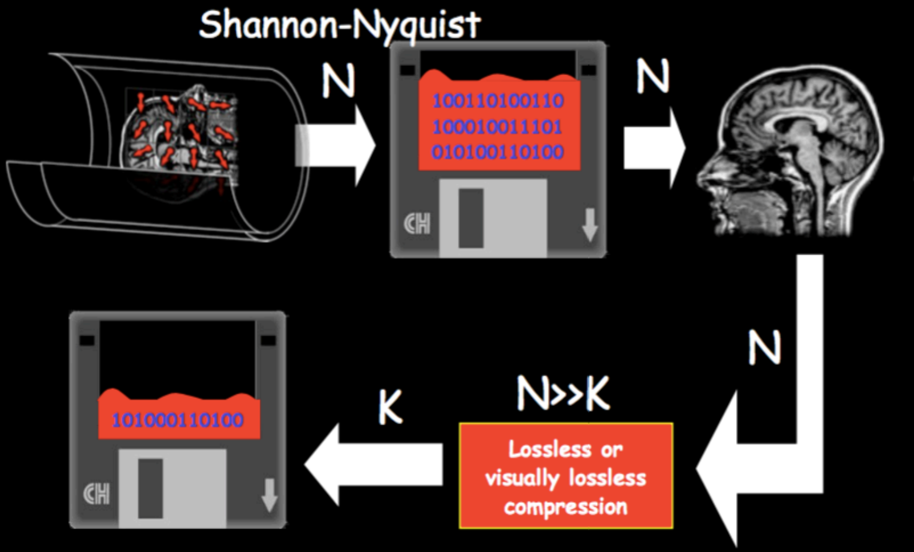
Wavelet Transform: Denoising

- Wavelet denoising: denoising with soft thresholding:

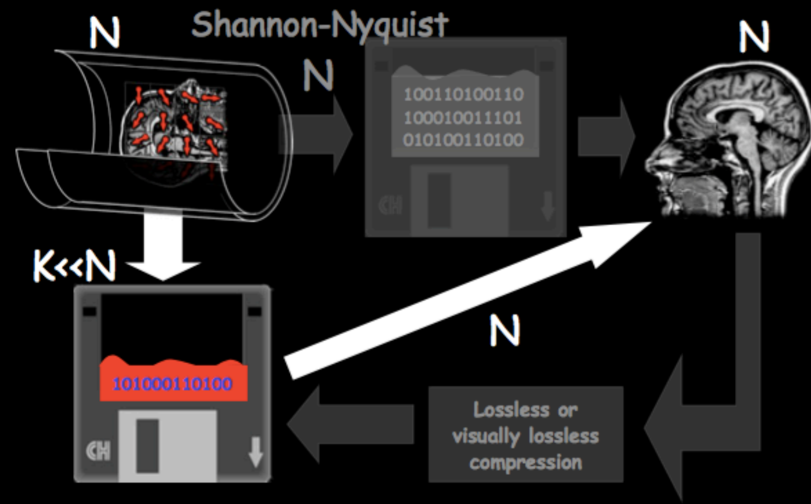


Compressed Sensing: A new theory of sampling

- First compress, then reconstruct. Instead of first collect, then compress.



Standard approach



Compressed sensing

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Compressed Sensing: Solving Underdetermined Systems

- Solve $Ax = b$ when M<<N

Linear Measurement Model

A$\in \Re^{N \times M}$

$\times$

Signal

$x \in \Re^{N}$

$=$

Observed data

b$\in \Re^{M}$

- Sparsity and Incoherency

  - X is a K-sparse signal (K<m<<n): At most K of the coefficients of x can be non-zero.
  - The sensing matrix A is incoherent: $A^* \cdot A \approx I$

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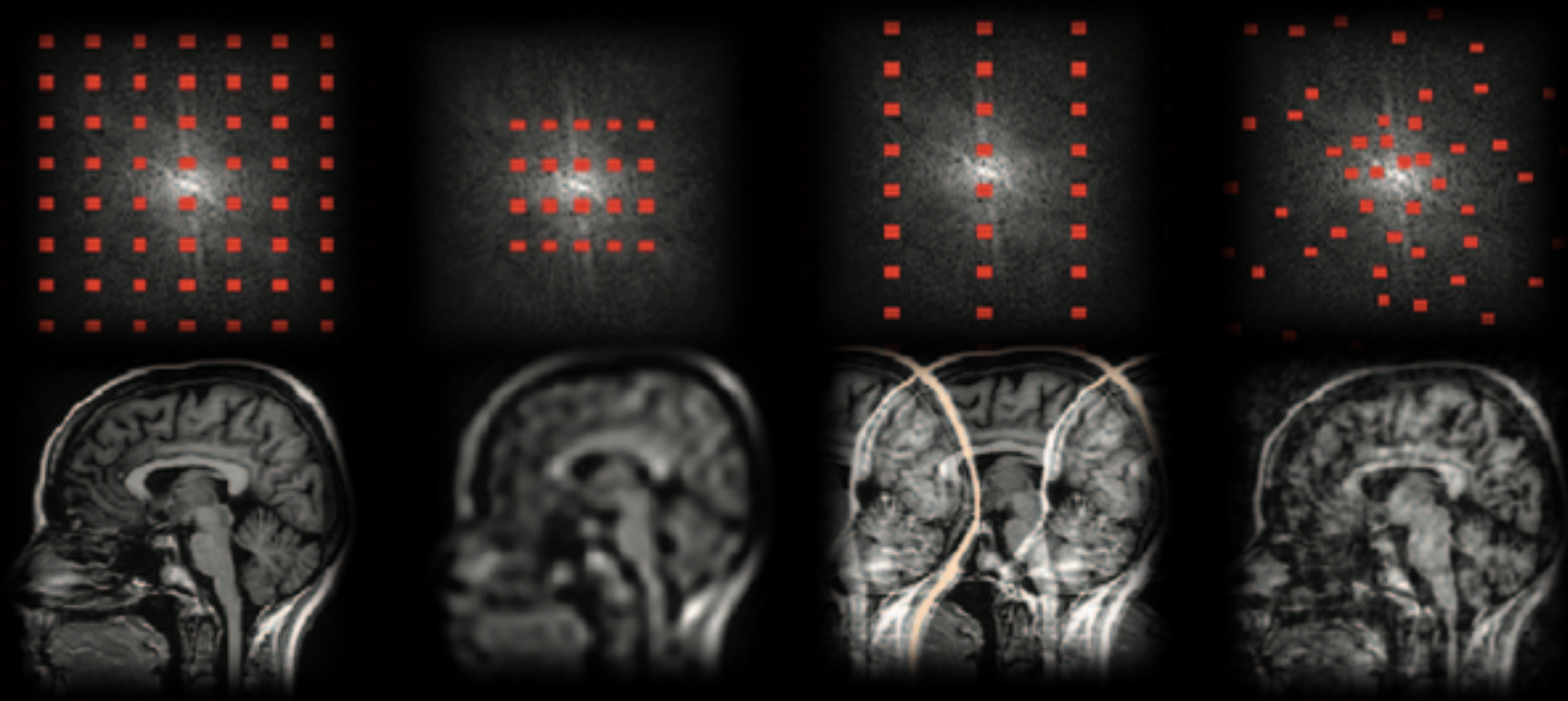
Compressed Sensing: Incoherent sampling

- Applying inverse Fourier transform for reconstruction:



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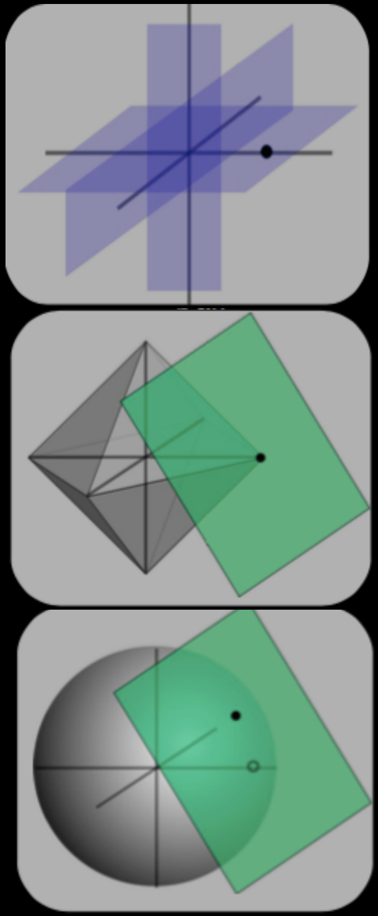
Compressed Sensing: Optimization Scheme

- Minimizes $l_p(x)$, s.t. $Ax = b$

	Definition	Description
l_0	$x^{\#} = argmin_{X:Ax=b} \lVert x \rVert_{l^0}$ $\lVert x \rVert_{l^0} = \sum_{i=1}^{N} \lvert x_i \rvert^0 = \#(1 \leq i \leq n, x_i \neq 0)$	NP-hard Problem
l_1	$x^{\#} = argmin_{X:Ax=b} \lVert x \rVert_{l^1}$ Solving non-linear convex optimization	Convex optimization Problem
l_2	$x^{\#} = argmin_{X:Ax=b} \lVert x \rVert_{l^2}$ $\lVert x \rVert_{l^2} = A^*(AA^*)^{-1} b$	Least Squares Solution

Geometry Interpretation



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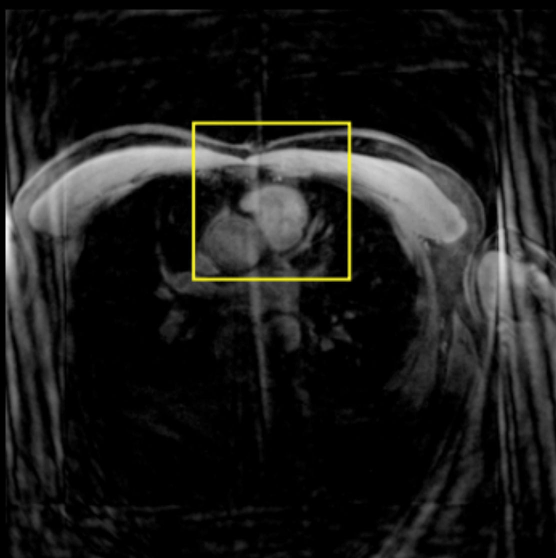
Compressed Sensing: Optimization Scheme

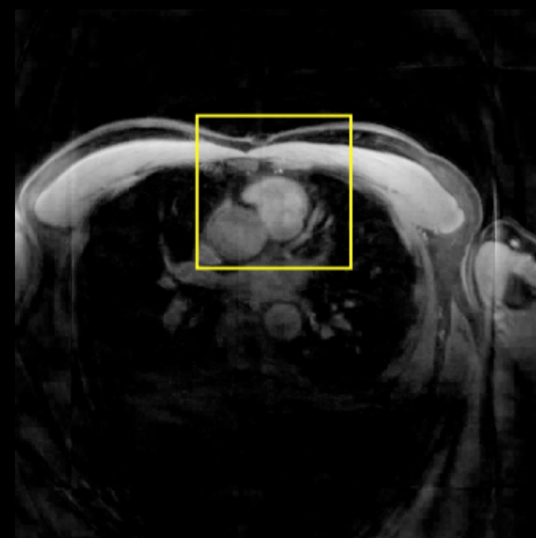
- Reconstruction as an optimization problem

$$x^{\#} = argmin\|y - Ax\|_2^2 + \lambda\|x\|_p$$

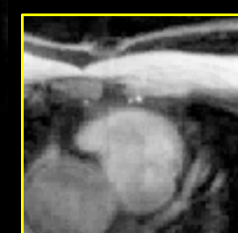
Data consistency   Regularization



l_2 *regularization*



l_1 *regularization*



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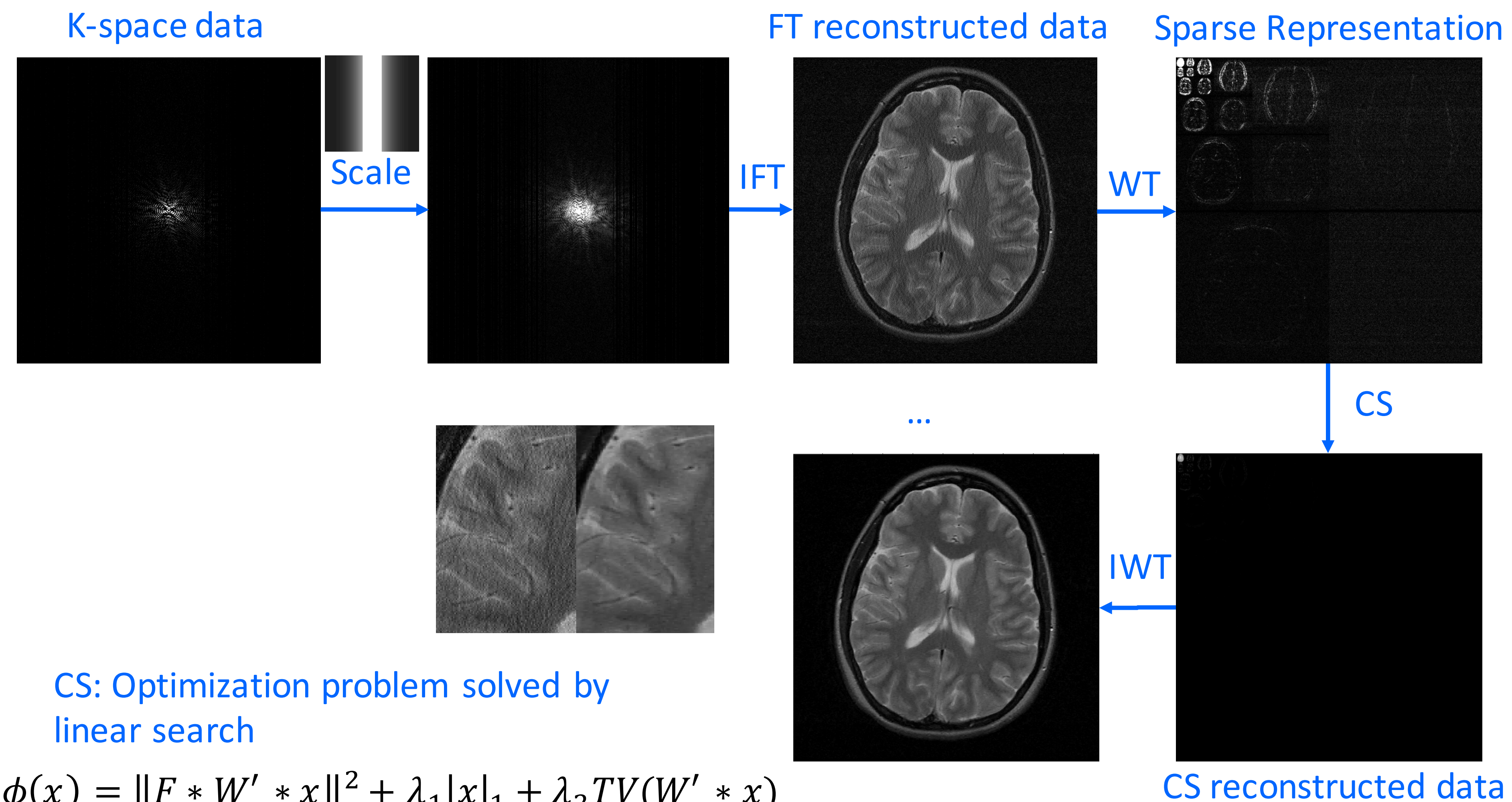
Compressed Sensing: Demo



CS: Optimization problem solved by linear search

$$\phi(x) = \|F * W' * x\|^2 + \lambda_1 |x|_1 + \lambda_2 TV(W' * x)$$

Total variation

Thank you for your attention !