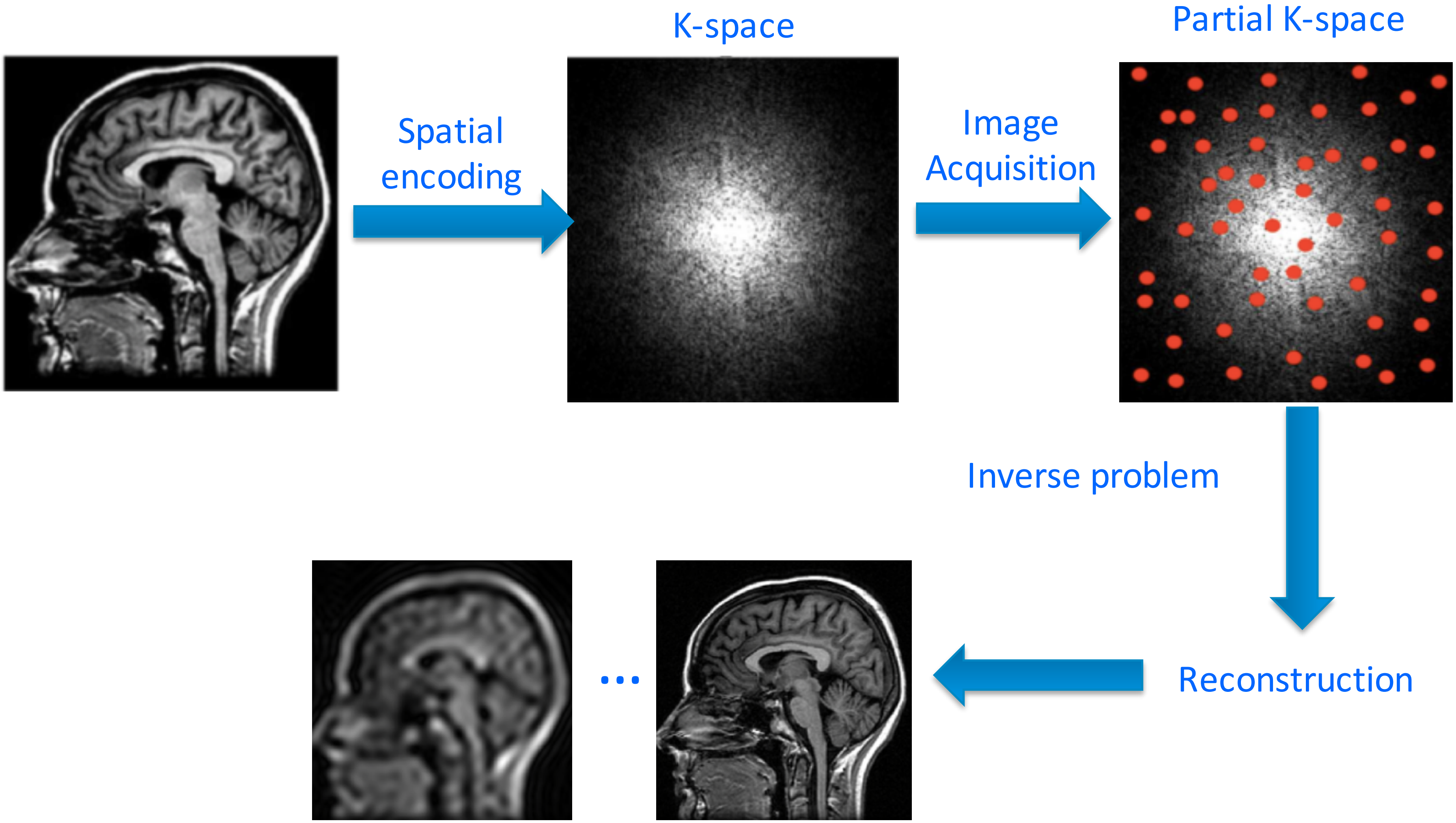
Learning Pathway:
One Month Internship Study

innovation you

OUTLINE

- MRI: Inverse problem
- Wavelet Transform
    - Continuous Wavelet Transform
    - Discrete Wavelet Transform
    - Localization properties
    - Sparity
    - Denoising
- Compressed sensing
    - A new theory of sampling
    - Solving Underdetermined Systems
    - Incoherent sampling
    - Optimization Scheme
    - Demo

MRI: Inverse problem



PHILIPS

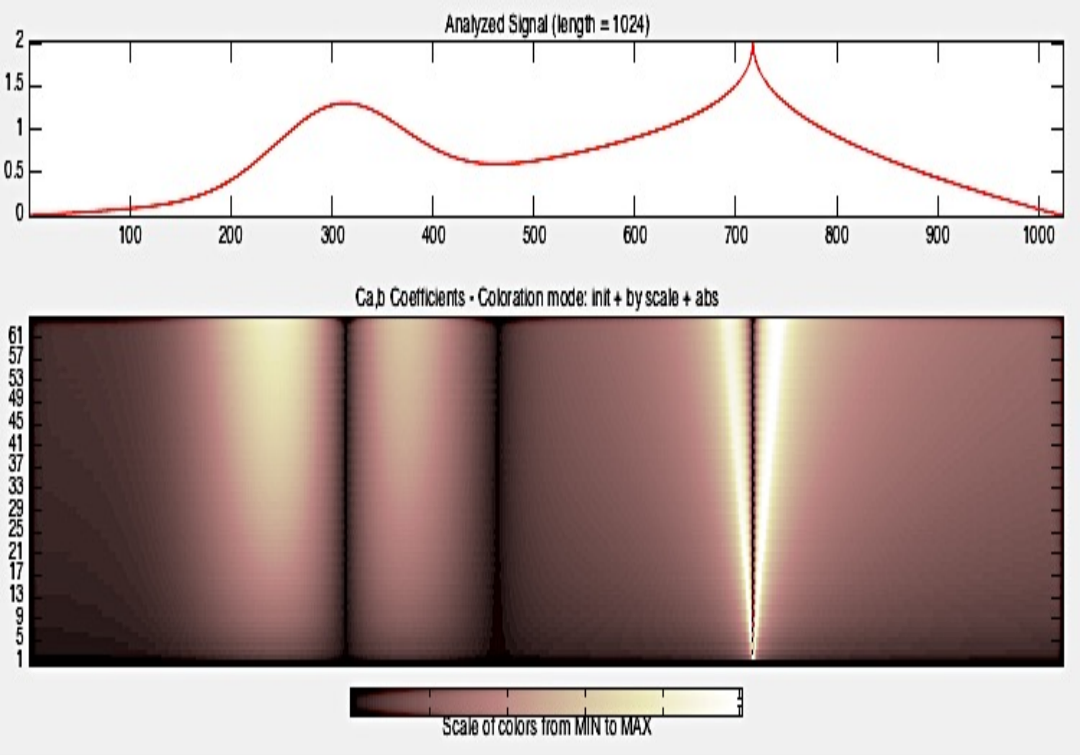
Wavelet transform: Continuous Wavelet Transform

- **CWT**: The integral sum of the signal multiplied by the Dilations and translations of the wavelet functions:

$$C(scale, position) = \int_{-\infty}^{\infty} f(t)\Psi(scale, position, t)\, dt$$

$$\psi_{\tau,s}(t) = \frac{1}{\sqrt{|s|}}\psi(\frac{t-\tau}{s})$$

- For a given wavelet function with fixed τ and s,
  $C = < f(t), \psi_{\tau,s}(t) >$
- For a set of wavelet functions with fixed s,
  $C(\tau) = f(t) * \psi_{-\tau,s}(-t)$



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Wavelet transform: Continuous Wavelet Transform

- A Wavelet is a waveform of effectively limited duration that has an average value of zero.

- Admissibility condition:

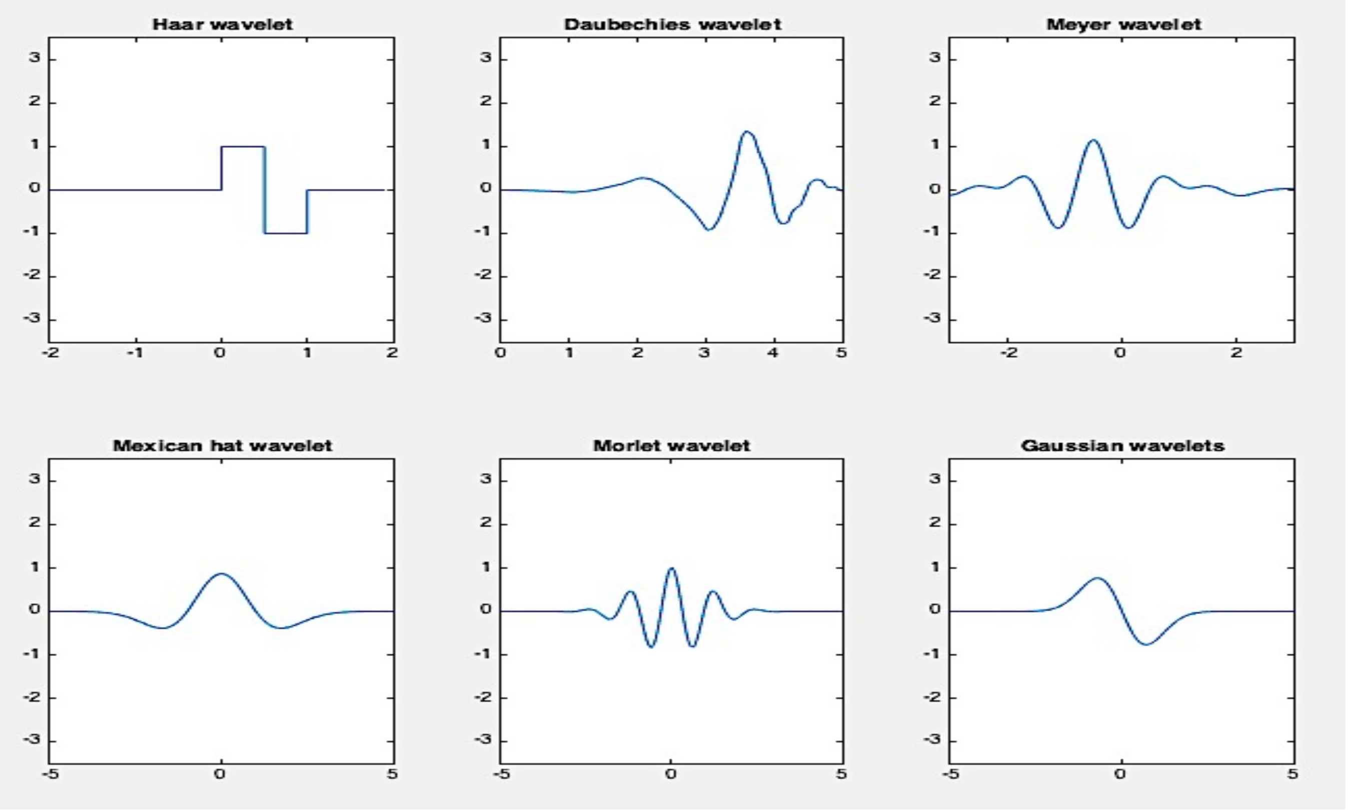
$$C_\psi = \int_{\mathbb{R}} \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < \infty$$

Zero mean:

$$\int_{-\infty}^{\infty} \Psi(t)\, \mathrm{d}\, t = 0$$

Finite energy:

$$\int_{-\infty}^{\infty} |\Psi(t)|^2 \mathrm{d}\, t \leq \infty$$



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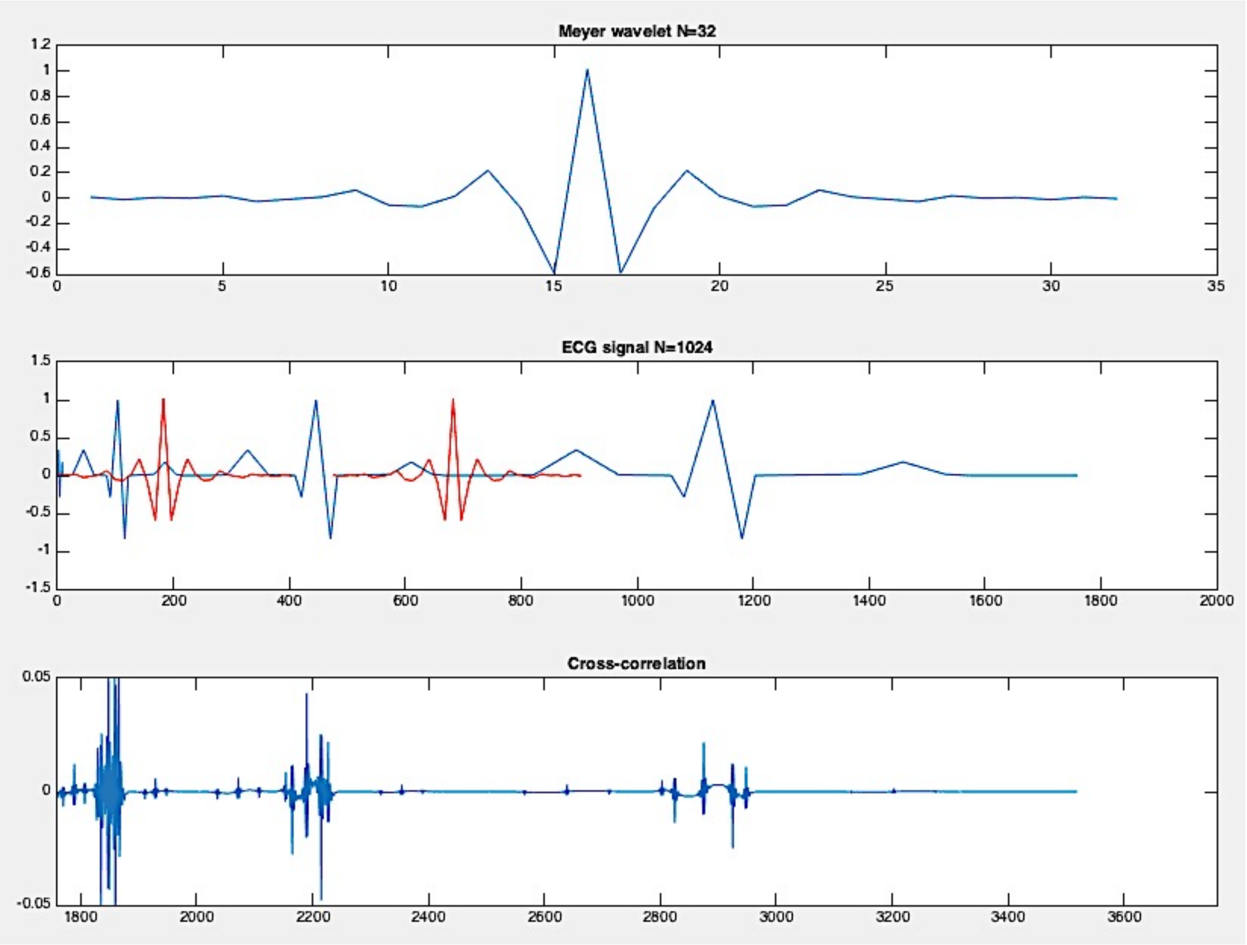
Wavelet transform: Continuous Wavelet Transform

Think wavelet in different perspectives:

- Cross correlation (inner product): A measure of the similarity between the signal and the scaled and shifted wavelet.



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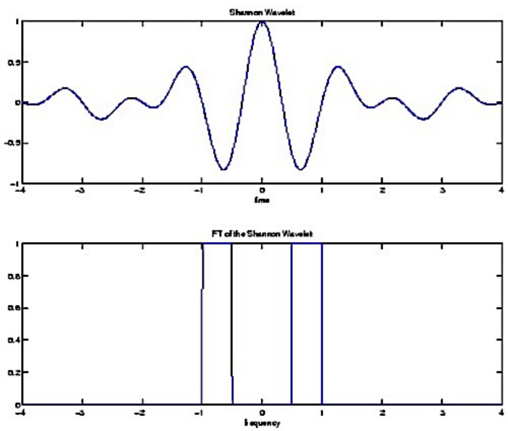
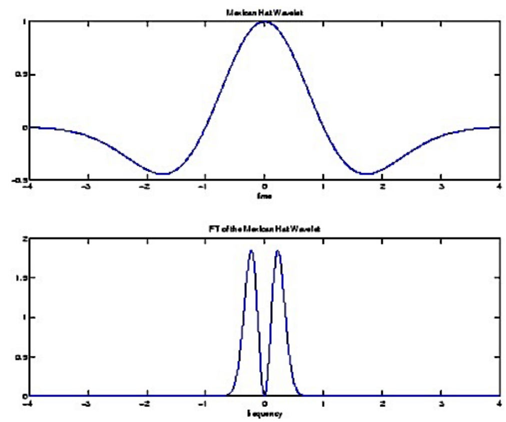
Wavelet transform: Continuous Wavelet Transform

- Band-pass filtering (convolution): For a fixed scale, the wavelet transform is the convolution of the signal and the time reversed wavelet.

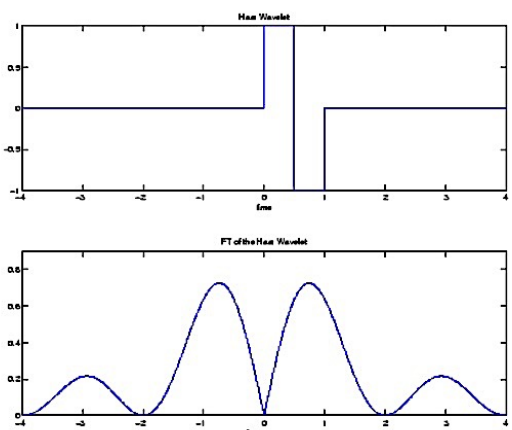
Shannon Wavelet



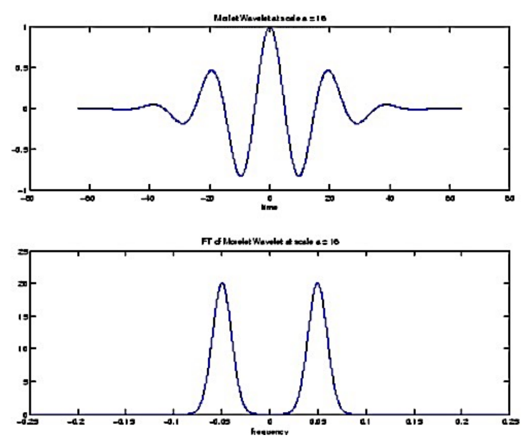
Mexican hat



Haar



Morelet



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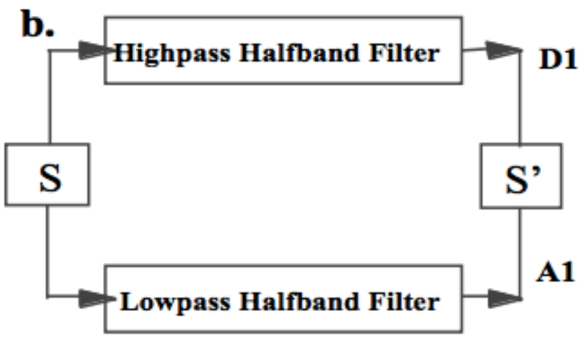
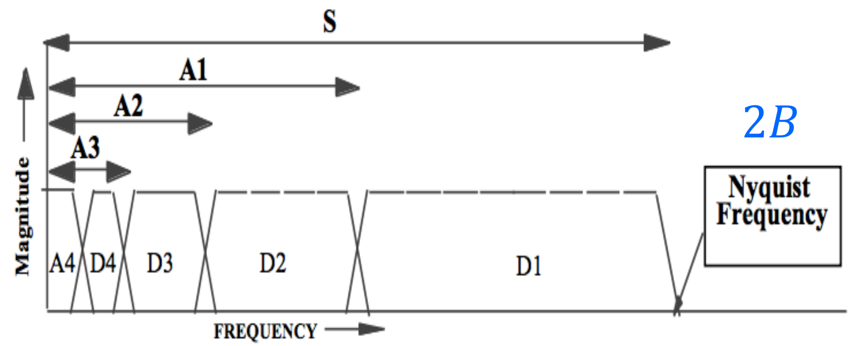
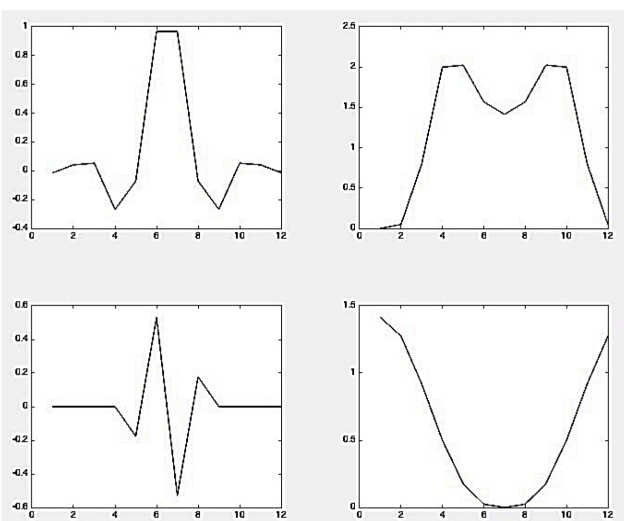
Wavelet transform: Discrete Wavelet Transform

- DWT: Calculating the wavelets coefficients at only a subset of scales and shifted positions by splitting the signal to sub-bands:

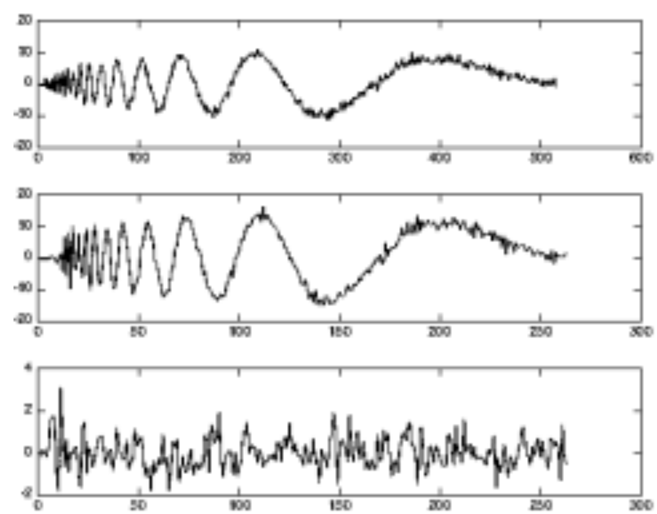
Scale Discretization:







LPF&HPF



Level 1

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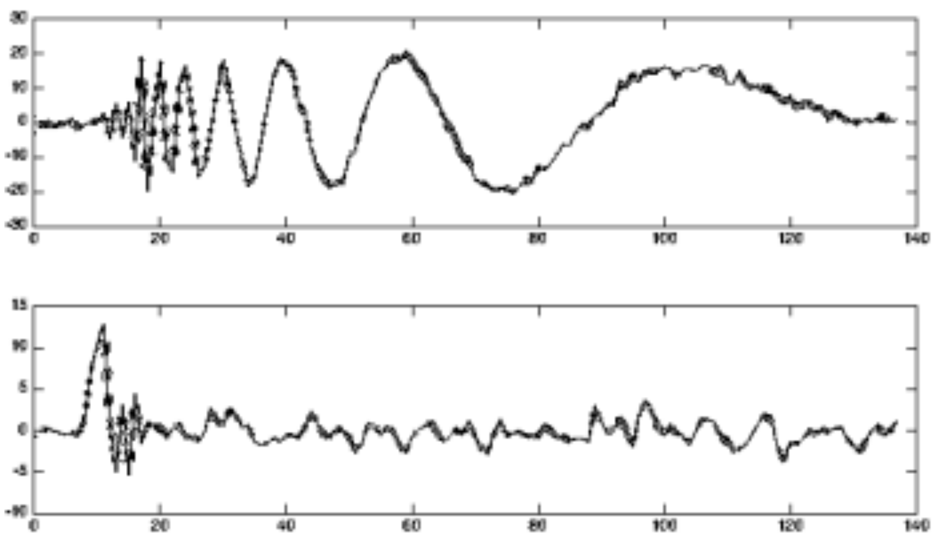
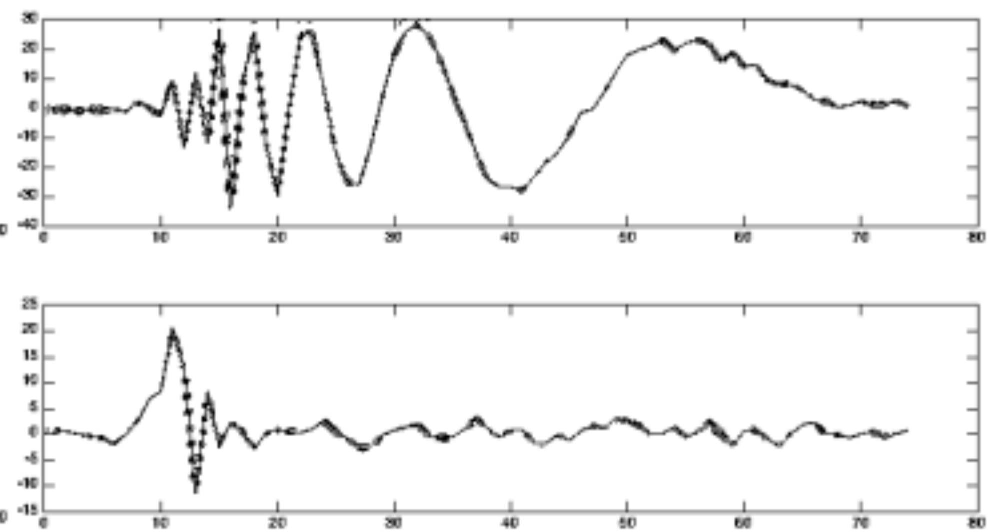
Wavelet transform: Discrete Wavelet Transform

Shifted position Discretization:

- According to Nyquist-Shannon sampling theorem: $f_s > 2B$. Half of the samples can be discarded!



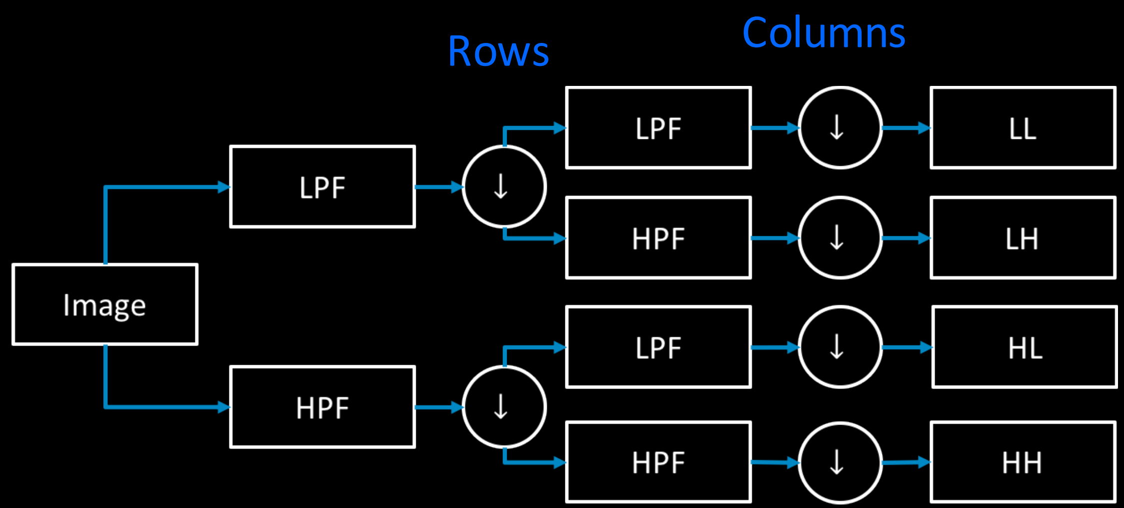
Level 2



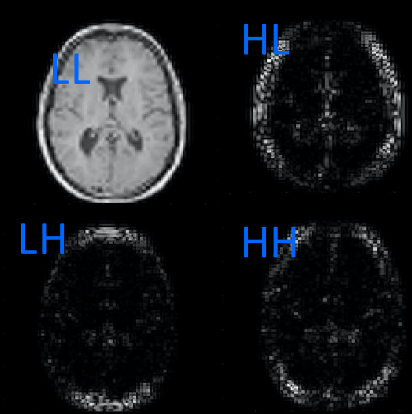
Level 3

Wavelet transform: Discrete Wavelet Transform

2D-DWT: Do 1D-DWT horizontally and vertically.



Brain MRI image

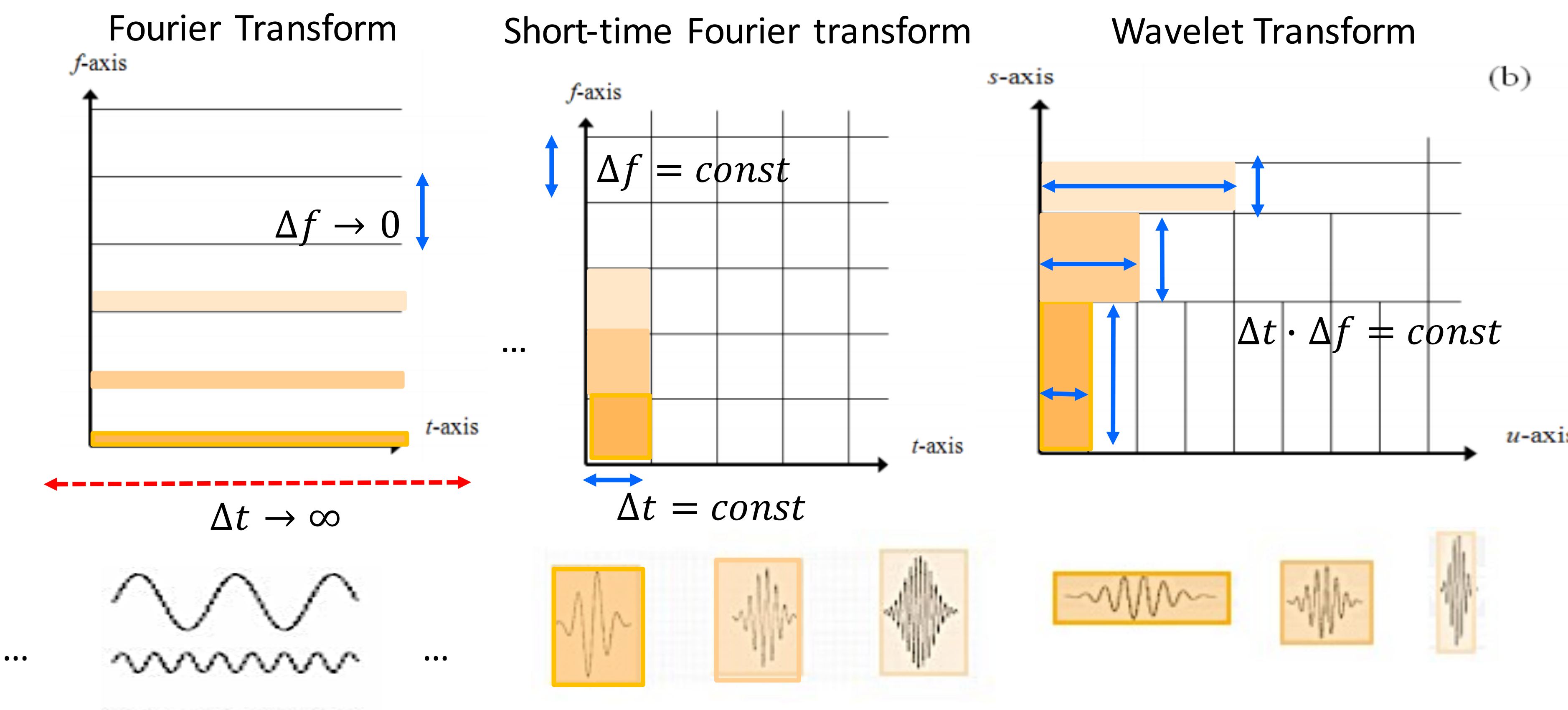


Approximation	Horizontal details
Vertical details	Diagonal details

Wavelet Transform: Localization properties

At the Time-frequency plane, the Heisenberg Uncertainty Principle told us:

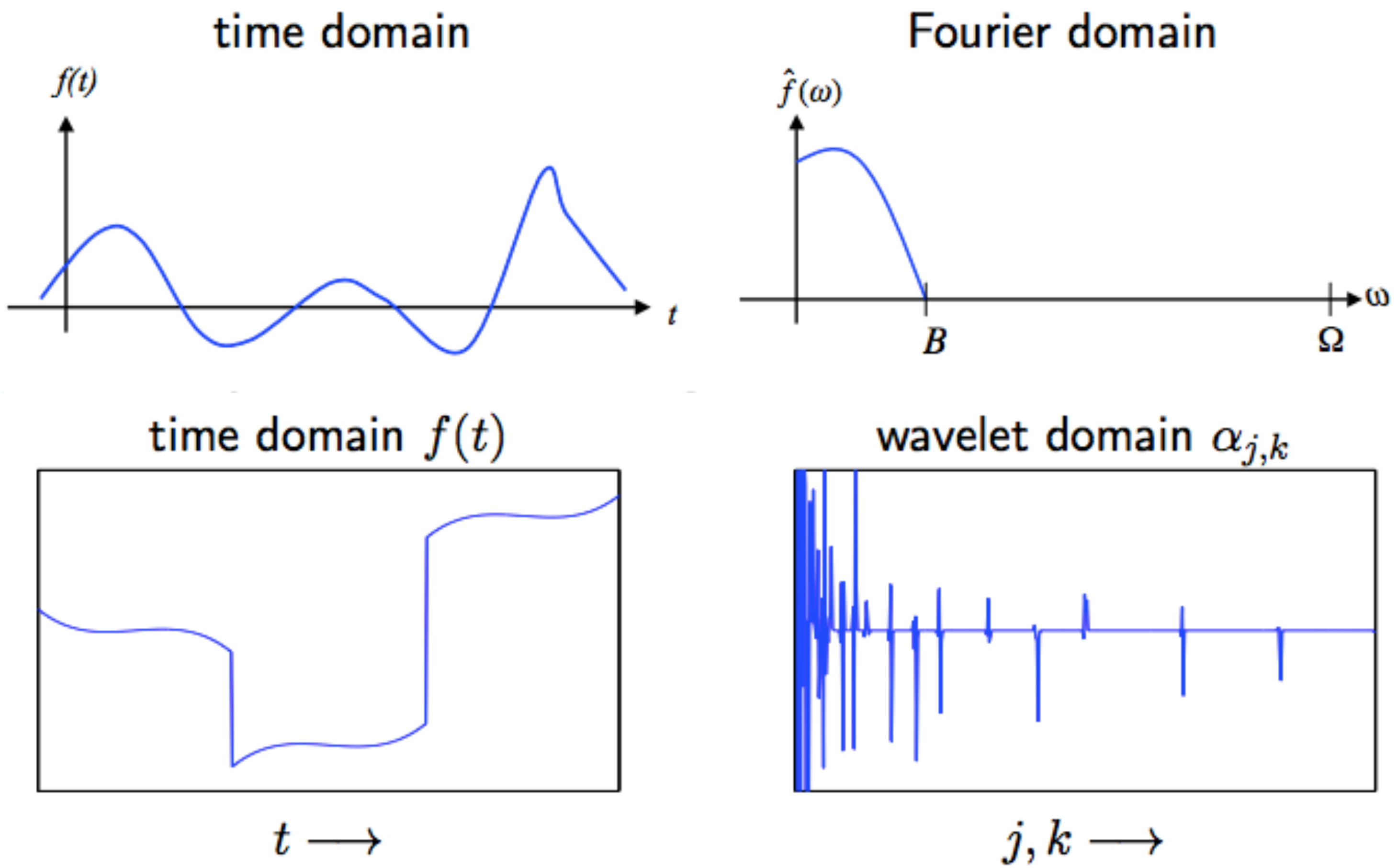
$$\Delta t \cdot \Delta f \geq \frac{1}{4\pi}$$



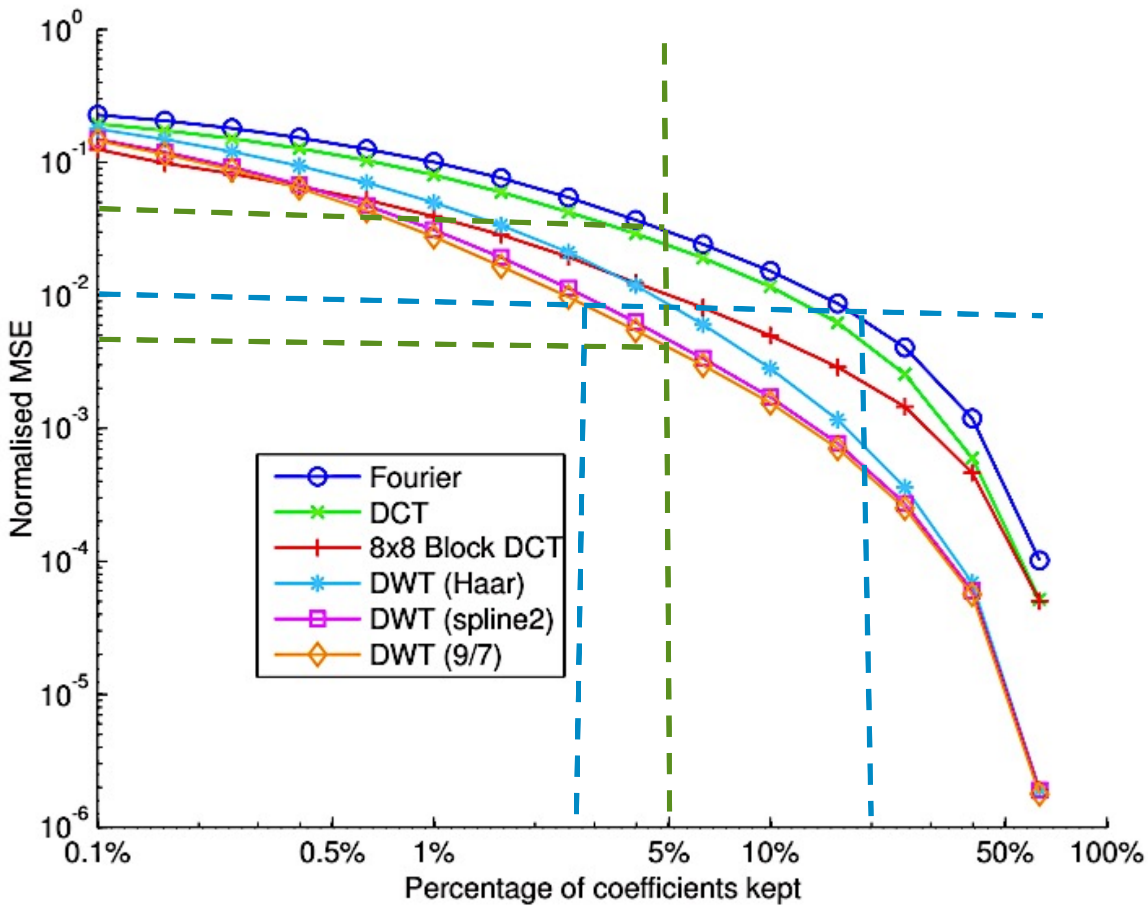
"Time" here can be replaced by space

Wavelet Transform: Sparity

Sparse: few large coeffs, many small coeffs



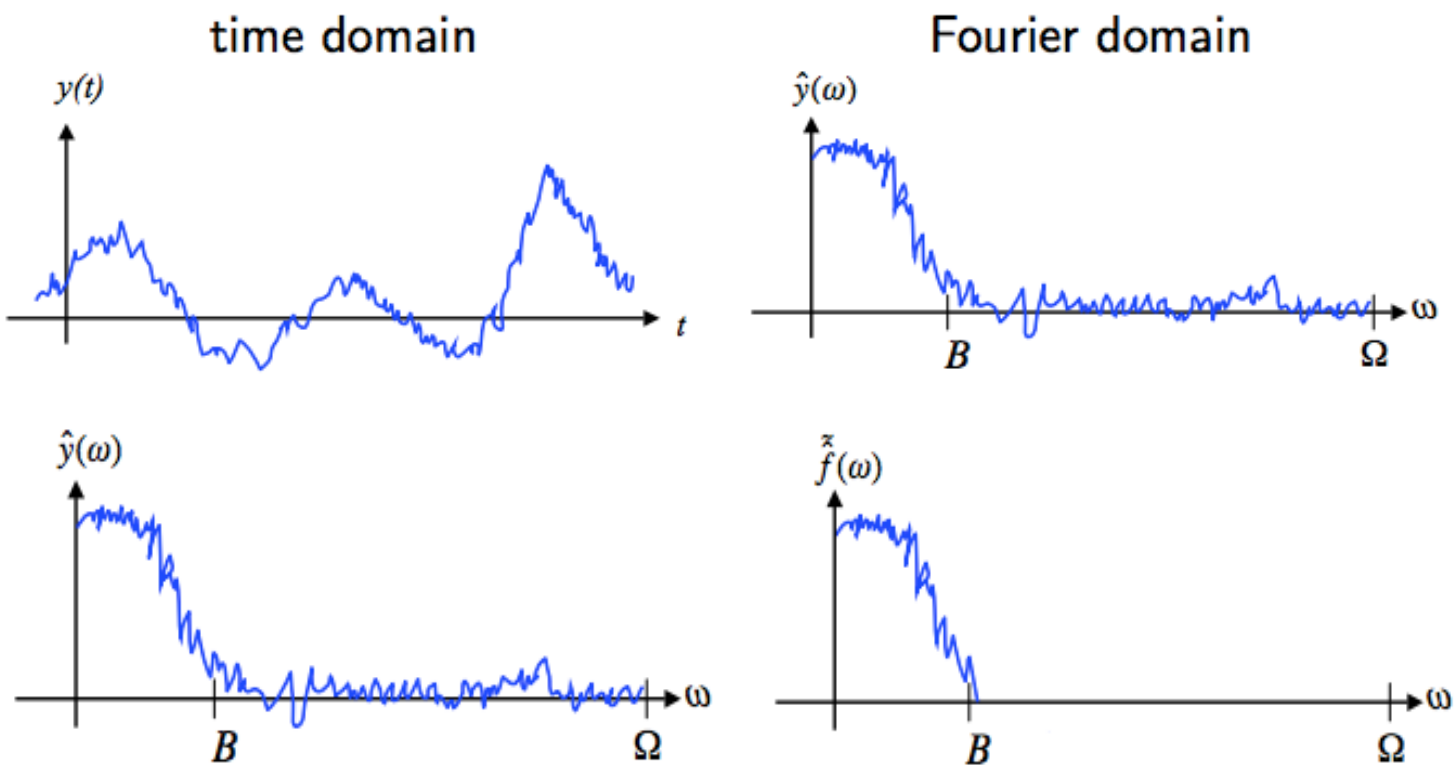
Wavelet Transform: Sparity



Guerquin−Kern
EPFL,PhD thiesis

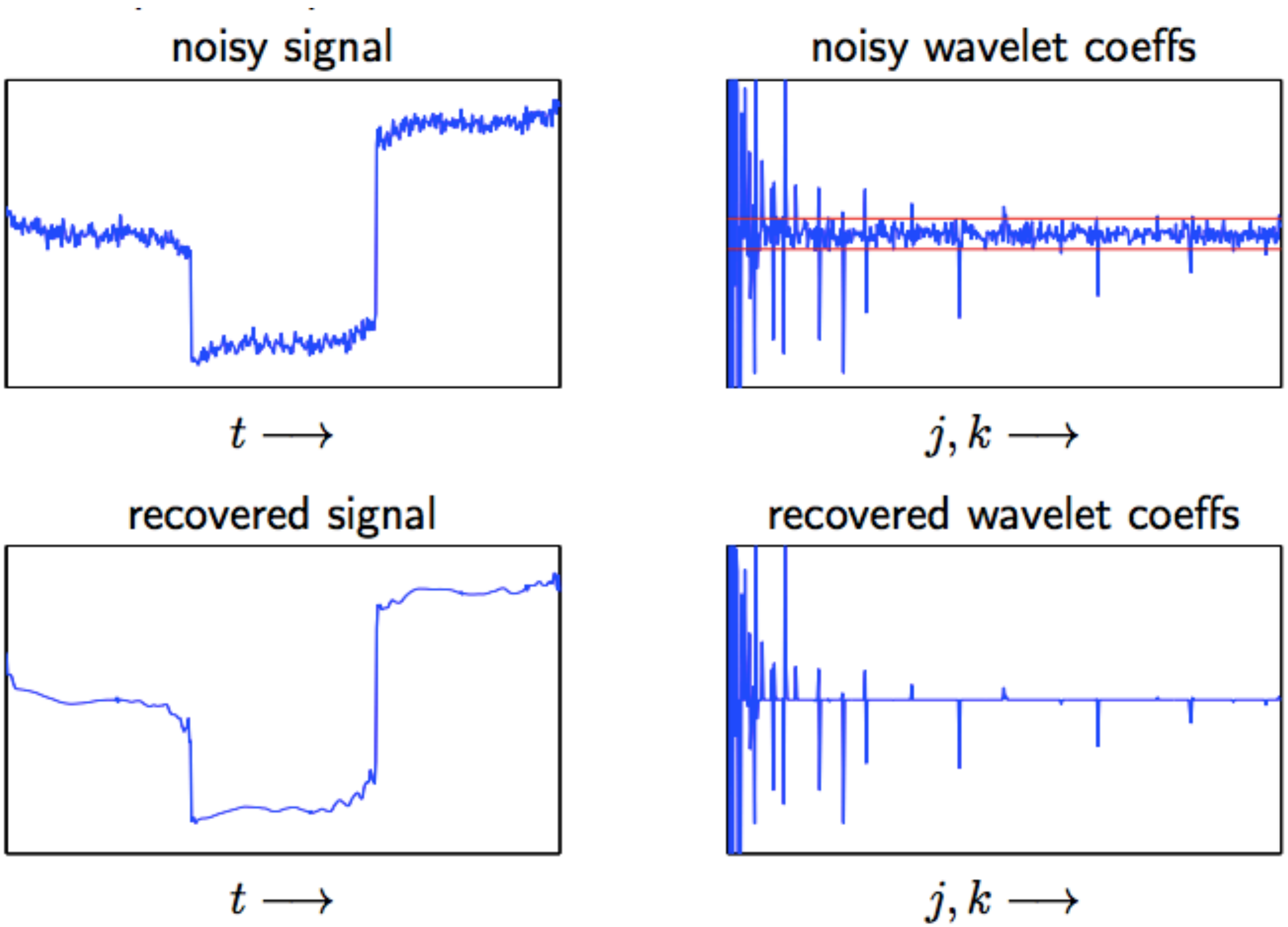
Wavelet Transform: Denoising

- Frequency domain: Denoising with low pass filtering:



Wavelet Transform: Denoising

- Wavelet denoising: denoising with soft thresholding:



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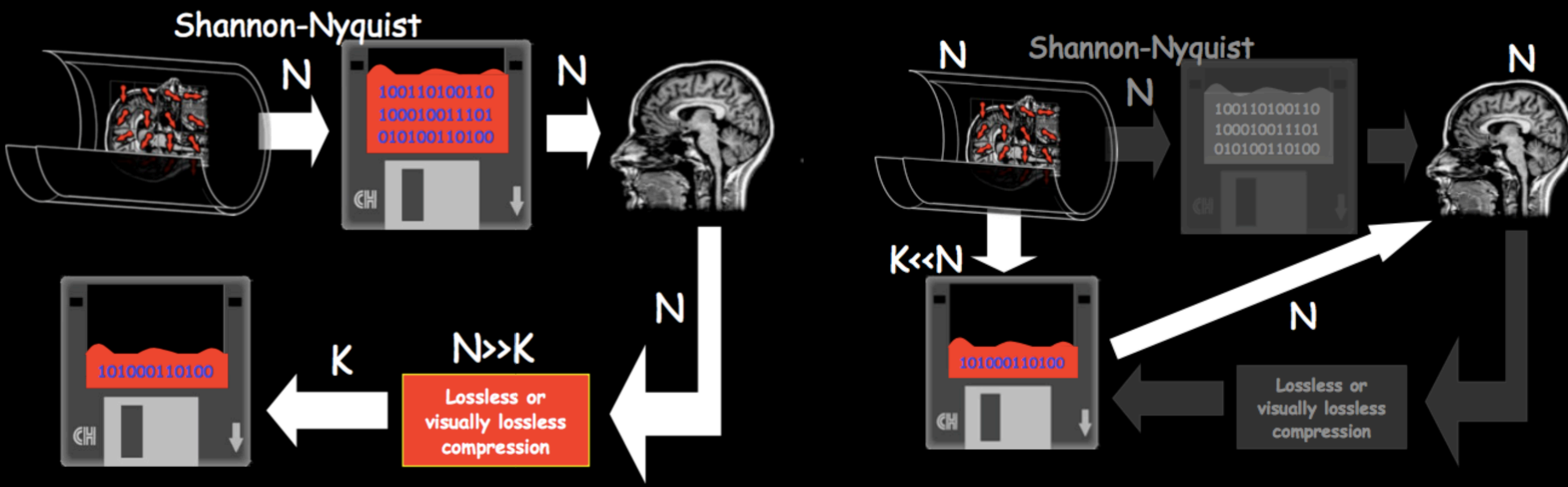
Compressed Sensing: A new theory of sampling

- First compress, then reconstruct. Instead of first collect, then compress.



Standard approach

Compressed sensing

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Compressed Sensing: Solving Underdetermined Systems

- Solve $Ax = b$ when M<<N

Linear Measurement Model

A$\in \mathfrak{R}^{N\times M}$

$\times$

Signal

$x \in \mathfrak{R}^{N}$

$=$

Observed data

b$\in \mathfrak{R}^{M}$

- Sparsity and Incoherency

  - X is a K-sparse signal (K<m<<n): At most K of the coefficients of x can be non-zero.

  - The sensing matrix A is incoherent: $A^{*} \cdot A \approx I$

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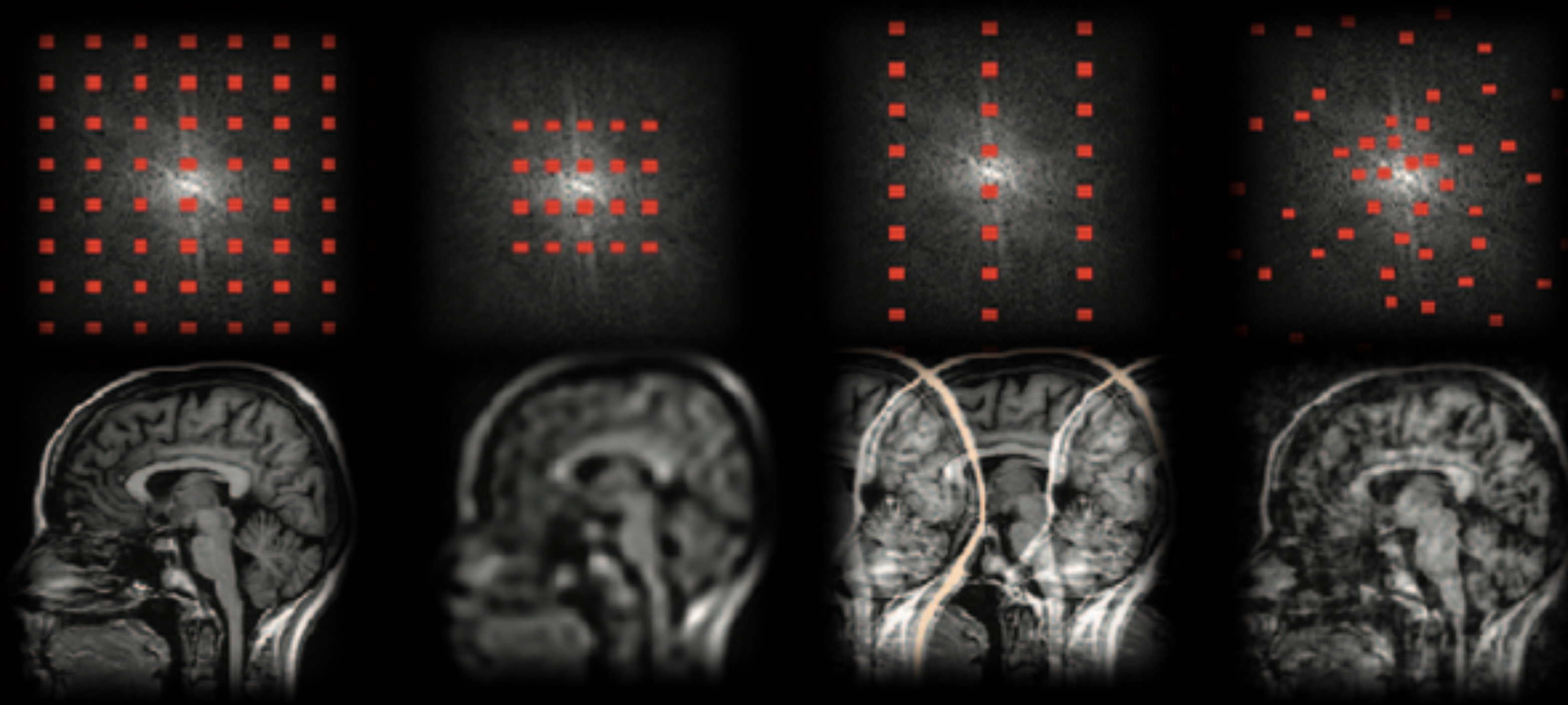
Compressed Sensing: Incoherent sampling

- Applying inverse Fourier transform for reconstruction:

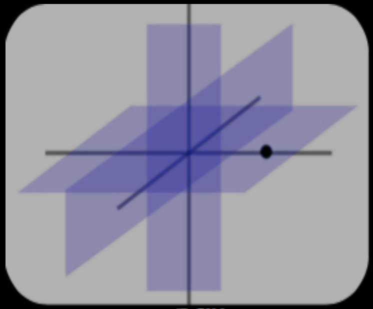


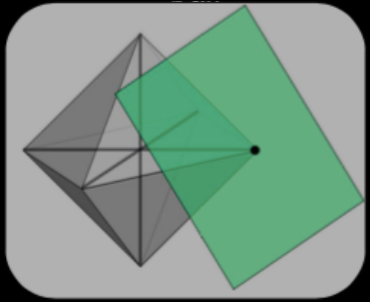
Compressed Sensing: Optimization Scheme

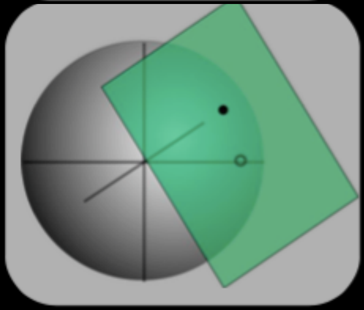
- Minimizes $l_p(x)$, s.t. $Ax = b$

	Definition	Description
l_0	$x^{\#} = argmin_{X:Ax=b} \|\|x\|\|_{l^0}$ $\|\|x\|\|_{l^0} = \sum_{i=1}^{N} \|x_i\|^0 = \#(1 \le i \le n, x_i \ne 0)$	NP-hard Problem
l_1	$x^{\#} = argmin_{X:Ax=b} \|\|x\|\|_{l^1}$ Solving non-linear convex optimization	Convex optimization Problem
l_2	$x^{\#} = argmin_{X:Ax=b} \|\|x\|\|_{l^2}$ $\|\|x\|\|_{l^2} = A^*(AA^*)^{-1}b$	Least Squares Solution

Geometry Interpretation







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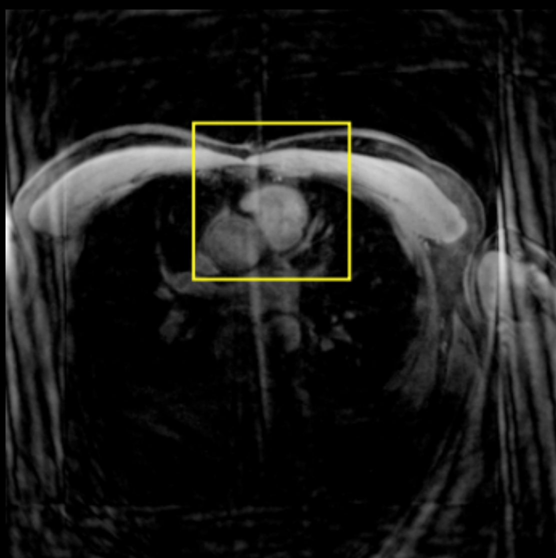
Compressed Sensing: Optimization Scheme

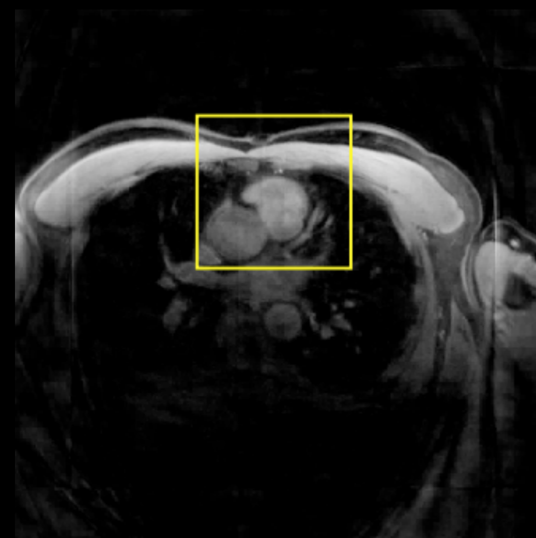
- Reconstruction as an optimization problem

$$x^{\#} = argmin\|y - Ax\|_2^2 + \lambda\|x\|_p^p$$

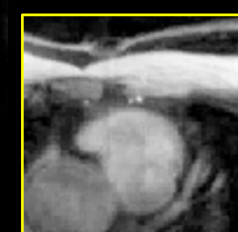
Data consistency   Regularization



l_2 *regularization*



l_1 *regularization*



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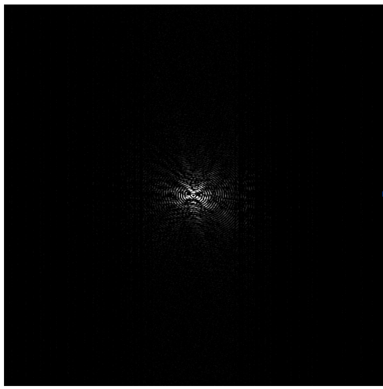
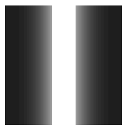
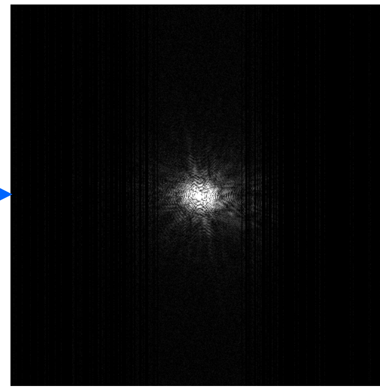
Compressed Sensing: Demo

K-space data



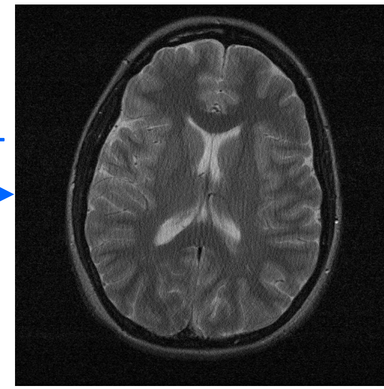


Scale



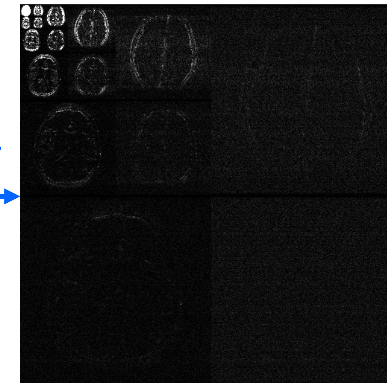
IFT

FT reconstructed data

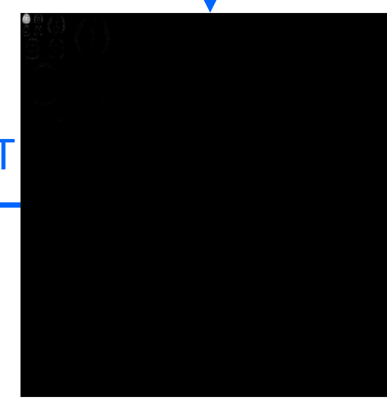


WT

Sparse Representation

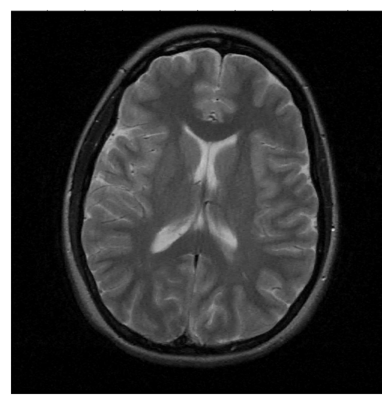


CS



CS reconstructed data

IWT



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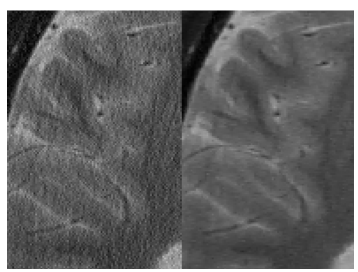

CS: Optimization problem solved by linear search

$$\phi(x) = \|F * W' * x\|^2 + \lambda_1 |x|_1 + \lambda_2 TV(W' * x)$$

Total variation

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Thank you for your attention !