An Operator-Algebraic Perspective on Entropy Flow in Spin Networks

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1 Introduction

The combinatorial "bridge-monotonicity" and "entropy-monotonicity" theorems established in [1, 2] show that inserting a spin-j_b bridge across a cut γ increases the boundary entropy by

$$\Delta S = \ln(2j_b + 1).$$

We recast those results in the language of finite von Neumann algebras. The reformulation exposes links to Jones index theory and modular flows, hinting at a uniqueness theorem for the emergent large-scale operator algebra of the universe.

Relation to earlier subfactor constructions. Temperley–Lieb subfactors first appeared in Jones' original index paper [3] and later in statistical-mechanics models [4], planar algebras [5], and conformal nets [6]. Our construction provides a *spin-network* realisation of the same standard invariant, motivated by loop-quantum-gravity entropy flow. This physics-driven perspective yields a concrete operator-algebraic interpretation of the entropy jump (5.1) and suggests new applications of subfactor theory to quantum information.

2 Subfactor background in two pages

We summarise only the notions used later; see [3, 5, 8] for full treatments.

2.1 Jones basic construction. Given a II_1 subfactor $N \subset M$ with trace τ, the *Jones projection* $e_N \in B(L^2(M))$ is the orthogonal projection $L^2(M) \twoheadrightarrow L^2(N)$. The von Neumann algebra $M_1 := \langle M, e_N \rangle''$ is the *basic construction* and $[M : N] = \tau(e_N)^{-1}$ is the *Jones index*. Iterating produces the *Jones tower* $N \subset M \subset M_1 \subset M_2 \subset \cdots$.

2.2 Relative commutants and the standard invariant. The k-th relative commutant $N' \cap M_k$ is finite-dimensional. The graded $*$-algebra $\mathcal{G}_\bullet(N \subset M) = \{N' \cap M_k\}_{k \geq 0}$ together with its Jones projections is called the *standard invariant*. It can be encoded diagrammatically as a *planar algebra* [5].

2.3 Temperley–Lieb (TL) planar algebra. For $\delta > 0$ the TL planar algebra TL_δ is generated by a single idempotent e obeying $e^2 = \delta^{-1} e$, $e_i e_{i\pm1} e_i = e_i$ and $e_i e_j = e_j e_i$ for $|i - j| \geq 2$. Every finite-depth subfactor with TL standard invariant is TL_δ for some $\delta > 1$.

2.4 Popa's uniqueness theorem. If a finite-depth, amenable subfactor has the same standard invariant as $\mathcal{R} \subset \mathcal{R}$ (the hyperfinite inclusion), then it is *inner conjugate* to it [8, Thm. 4.5]. We use this in Section **??**.

3 Boundary von Neumann Algebras

Definition 3.1 (Edge algebra)**.** *For a cut γ whose intersected edges carry spins $\{j_e\}_{e\in\gamma}$, define the* edge algebra

$$\mathcal{A}_\gamma := \bigotimes_{e\in\gamma} \mathrm{End}(V_{j_e}).$$

Here the tensor product is taken over $\mathbb{C}$. For a finite cut this is the algebraic tensor product, while for an infinite cut we take the spatial (von Neumann) completion. It is a finite (resp. properly infinite) C^*-algebra equipped with the normalised trace tr*.*

Definition 3.2 (Gauge-invariant algebra)**.** *The diagonal* $\mathrm{SU}(2)$ *action $u^{\otimes}$ on $\mathcal{A}_\gamma$ yields the* boundary algebra

$$\mathcal{N}_\gamma := \mathcal{A}_\gamma^{\mathrm{SU}(2)} = \{X \in \mathcal{A}_\gamma \mid u^{\otimes} X u^{\otimes *} = X \ \forall u \in \mathrm{SU}(2)\}.$$

4 Relational Entropy and Modular Hamiltonian

Definition 4.1 (Relational state and modular generator)**.** *Let $P_\gamma \in \mathcal{N}_\gamma$ project onto the singlet subspace and set*

$$\rho_\gamma := \frac{P_\gamma}{\mathrm{tr}P_\gamma}, \qquad K_\gamma := -\ln \rho_\gamma.$$

Then $S_\gamma = \ln \mathrm{tr}P_\gamma$ reproduces the combinatorial count, and K_γ generates the Tomita-Takesaki flow on $(\mathcal{N}_\gamma, \rho_\gamma)$.

Remark 4.1 (Parity obstruction)**.** *If the cut has* odd *total spin,* $\mathrm{tr}P_\gamma = 0$ *and ρ_γ is undefined. The operator-algebraic framework below therefore assumes $d_0 := \mathrm{tr}P_\gamma > 0$. Odd-parity cuts can be handled by first performing a Type III parity-flipping move (see Definition 7.1) and then applying the results in Proposition 7.1 and Appendix A.*

Parity-flipping as Morita equivalence

Let γ^{odd} be a cut of odd total spin. Define the bimodule $\mathcal{H}_{\mathrm{pf}}$ by

$$\mathcal{H}_{\mathrm{pf}} := \mathrm{Inv}\Big(V_{1/2} \otimes \bigotimes_{e\in\gamma^{\mathrm{odd}}} V_{j_e}\Big),$$

on which $\mathcal{N}_{\gamma^{\mathrm{odd}}}$ acts on the right and $\mathcal{N}_{\gamma^{\mathrm{even}}}$ (obtained by attaching a spin–$\frac{1}{2}$ stub) acts on the left. This $\mathcal{H}_{\mathrm{pf}}$ is an *invertible* $\mathcal{N}_{\gamma^{\mathrm{even}}}$–$\mathcal{N}_{\gamma^{\mathrm{odd}}}$ bimodule, hence a Morita equivalence [8, Def. 2.1]. Type III/IV moves therefore transport the standard invariant unchanged, so all parity sectors share the same limit factor $\mathcal{R}$.

Proposition 4.1 (Morita equivalence of parity sectors)**.** *Let $F := \bigotimes_{e\in\gamma^{\mathrm{odd}}} V_{j_e}$ and put $H_{\mathrm{pf}} := \mathrm{Inv}_{\mathrm{SU}(2)}(V_{1/2} \otimes F)$. Then*

$$H_{\mathrm{pf}} \otimes_{\mathcal{N}_{\gamma^{\mathrm{odd}}}} \overline{H_{\mathrm{pf}}} \cong {}_{\mathcal{N}_{\gamma^{\mathrm{even}}}}\mathcal{N}_{\gamma^{\mathrm{even}}}, \qquad \overline{H_{\mathrm{pf}}} \otimes_{\mathcal{N}_{\gamma^{\mathrm{even}}}} H_{\mathrm{pf}} \cong {}_{\mathcal{N}_{\gamma^{\mathrm{odd}}}}\mathcal{N}_{\gamma^{\mathrm{odd}}},$$

hence $\mathcal{N}_{\gamma^{\mathrm{odd}}}$ and $\mathcal{N}_{\gamma^{\mathrm{even}}}$ are Morita equivalent.

Proof. Throughout, $\varepsilon : V_{1/2}\otimes V_{1/2}\to\mathbb{C}$ and $\iota : \mathbb{C}\to V_{1/2}\otimes V_{1/2}$ are the standard SU(2) cup and cap, normalised so $\varepsilon\circ\iota = \mathbf{1}$.

1. A concrete orthonormal basis. Fix an admissible fusion tree $(\frac{1}{2}, j_{e_1}, j_{e_2}, \dots) \rightsquigarrow (\ell_1, \ell_2, \dots)$ and denote by $\psi_{\boldsymbol{\ell}} \in H_{\mathrm{pf}}$ its Wigner basis element. The set $\{\psi_{\boldsymbol{\ell}}\}_{\boldsymbol{\ell}}$ is orthonormal and spans H_{pf}; similarly for its complex conjugates $\overline{\psi_{\boldsymbol{\ell}}}$.

2. First bimodule map Θ. Define

$$\Theta(\psi \otimes_{\mathcal{N}_{\gamma^{\mathrm{odd}}}} \overline{\phi}) := (\varepsilon\otimes\mathbf{1}_F)(\psi \otimes \overline{\phi}) \in \mathrm{End}(F)^{\mathrm{SU}(2)} = \mathcal{N}_{\gamma^{\mathrm{even}}}.$$

Balanced-tensor well-definedness. For any $a \in \mathcal{N}_{\gamma^{\mathrm{odd}}}$ we must show $\Theta(\psi a \otimes \overline{\phi}) = \Theta(\psi \otimes \overline{a^*\phi})$. Because a acts only on the F factor and ε acts only on the two $V_{1/2}$ legs, the two expressions coincide, proving well-definedness.

Bimodule relations. For $b, c \in \mathcal{N}_{\gamma^{\mathrm{even}}}$, $b \cdot \Theta(\xi \otimes \overline{\eta}) \cdot c = \Theta(b \cdot \xi \otimes \overline{\eta \cdot c})$, again because b, c commute with ε.

Isometry. Using the graphical inner product $\langle \psi, \phi \rangle = (\varepsilon\otimes\mathbf{1}_F)(\psi^*\phi)$, one computes

$$\langle \Theta(\psi \otimes \overline{\phi}), \Theta(\psi \otimes \overline{\phi}) \rangle = \varepsilon(\iota(1))\ \langle \psi, \psi \rangle \langle \phi, \phi \rangle = \langle \psi \otimes \overline{\phi}, \psi \otimes \overline{\phi} \rangle,$$

so Θ preserves the bimodule inner product.

Surjectivity. For each fusion label $\boldsymbol{\ell}$ the image $\Theta(\psi_{\boldsymbol{\ell}} \otimes \overline{\psi_{\boldsymbol{\ell}}})$ is the minimal projection onto the $\boldsymbol{\ell}$-isotypic subspace of F; these projections generate $\mathcal{N}_{\gamma^{\mathrm{even}}}$, hence Θ is surjective.

3. Inverse map Φ. Define

$$\Phi(X) := \iota(1) \otimes_{\mathbb{C}} X \quad \in\ H_{\mathrm{pf}} \otimes_{\mathcal{N}_{\gamma^{\mathrm{odd}}}} \overline{H_{\mathrm{pf}}}.$$

Balanced-tensor relations are immediate and $(\varepsilon\otimes\mathbf{1}_F)(\iota(1) \otimes X) = X$, so $\Theta\circ\Phi = \mathrm{id}$. Conversely, $(\iota\otimes\mathbf{1}_F)(\varepsilon\otimes\mathbf{1}_F) = \mathbf{1}_{H_{\mathrm{pf}}\otimes\overline{H_{\mathrm{pf}}}}$, whence $\Phi\circ\Theta = \mathrm{id}$. Therefore Θ is a unitary bimodule isomorphism.

4. Second isomorphism. Replacing ε by ι and vice-versa yields the map

$$\Xi :\ \overline{H_{\mathrm{pf}}} \otimes_{\mathcal{N}_{\gamma^{\mathrm{even}}}} H_{\mathrm{pf}} \longrightarrow {}_{\mathcal{N}_{\gamma^{\mathrm{odd}}}}\mathcal{N}_{\gamma^{\mathrm{odd}}}, \qquad \Xi(\overline{\phi} \otimes \psi) := (\varepsilon\otimes\mathbf{1}_F)(\overline{\phi} \otimes \psi),$$

and one verifies exactly as above that Ξ is a unitary inverse to its adjoint.

Both bimodule isomorphisms being established, the two boundary algebras are Morita equivalent. □

Relation to the combinatorial framework [1, 2]

The spin-network proofs in [1, 2] derive the entropy jump $\Delta S = \ln(2j_b + 1)$ from a counting of admissible colourings of a cut γ. Our operator-algebraic reformulation retains the same combinatorics but packages it as:

$$\Delta S = -\ln \tau(P_{\gamma'}) + \ln \tau(P_\gamma) = \ln\bigl[\mathcal{N}_{\gamma'} : \mathcal{N}_\gamma\bigr].$$

- The **advantage** is that Jones index is a stable, basis-free quantity, so the entropy formula survives parity moves and quantum-group truncation.

- Conversely, the combinatorial perspective supplies explicit TL basis vectors—fusion trees—that we exploit in the proof of Proposition 4.1.
- Thus the two viewpoints are complimentary: [1, 2] proves the raw counting formula; the present paper shows that the same formula controls the entire Jones tower and standard invariant.

Hence every odd-parity boundary algebra lies in the same Morita class as its even-parity partner; the large-scale factor $\mathcal{R}$ is therefore parity-independent.

Verification details for the parity-flipping bimodule

Lemma 4.1. *Let $F = \bigotimes_{e\in\gamma^{\rm odd}} V_{j_e}$ and $H_{\rm pf} = \mathrm{Inv}(V_{1/2}\otimes F)$ as in the proposition. Then $H_{\rm pf}$ is an $\big(\mathcal{N}_{\gamma^{\rm even}}, \mathcal{N}_{\gamma^{\rm odd}}\big)$–bimodule via*

$$a_L\cdot\psi\cdot a_R := (a_L\otimes\mathbf{1}_V)\,\psi\,(\mathbf{1}_V\otimes a_R),\qquad a_L\in\mathcal{N}_{\gamma^{\rm even}},\ a_R\in\mathcal{N}_{\gamma^{\rm odd}},\ \psi\in H_{\rm pf}.$$

Moreover the balanced tensor product relation $\psi\cdot a_R\otimes\overline{\phi} = \psi\otimes\overline{a_R^*\phi}$ holds for all a_R,ψ,ϕ.

Proof. Because a_L (respectively a_R) acts non-trivially only on F, the left and right actions commute and preserve the SU(2) invariant subspace. For the balanced tensor product observe that

$$(\varepsilon\otimes\mathbf{1}_F)\big((\psi\cdot a_R)\otimes\overline{\phi}\big) = (\varepsilon\otimes\mathbf{1}_F)\big(\psi\otimes\overline{a_R^*\phi}\big),$$

because ε contracts only the two $V_{1/2}$ legs; hence the two simple tensors are identified in the quotient. □

Invertibility and balanced-tensor details

Explicit evaluation and coevaluation. Fix the standard weight basis $|+\rangle := |m=\frac{1}{2}\rangle$, $|-\rangle := |m=-\frac{1}{2}\rangle$ of $V_{1/2}$. Set

$$\varepsilon\big(|m_1\rangle\otimes|m_2\rangle\big) \ :=\ (-1)^{\frac{1}{2}-m_1}\,\delta_{m_1,-m_2},\qquad \iota(1)\ :=\ |+\rangle\otimes|-\rangle - |-\rangle\otimes|+\rangle.$$

Then $\varepsilon\circ\iota = \mathbf{1}_{\mathbb{C}}$ and $(\iota^\dagger\otimes\mathbf{1})(\mathbf{1}\otimes\varepsilon) = \mathbf{1}_{V_{1/2}}$, so ε,ι implement the rigid duality structure of $\mathrm{Rep}\,\mathrm{SU}(2)$.

Balanced-tensor identity (detail). Let $\psi,\phi\in H_{\rm pf}$ and $a_R\in\mathcal{N}_{\gamma^{\rm odd}} = \mathrm{End}(F)^{\mathrm{SU}(2)}$. Because a_R acts as $\mathbf{1}_{V_{1/2}}\otimes a_R$ on $V_{1/2}\otimes F$,

$$(\varepsilon\otimes\mathbf{1}_F)\big((\psi\cdot a_R)\otimes\overline{\phi}\big) = (\varepsilon\otimes\mathbf{1}_F)\big(\psi\otimes(\mathbf{1}_{V_{1/2}}\otimes a_R^*)\overline{\phi}\big) = (\varepsilon\otimes\mathbf{1}_F)\big(\psi\otimes\overline{a_R^*\phi}\big),$$

verifying the balanced-tensor relation required for Θ.

Invertibility — both directions. Define Θ and Φ exactly as in the previous proof and set

$$\Xi(\overline{\phi}\otimes\psi) := (\varepsilon\otimes\mathbf{1}_F)(\overline{\phi}\otimes\psi),\qquad \Psi(X) := \overline{\iota(1)}\otimes X.$$

A direct contraction check gives $\Theta\circ\Phi = \mathrm{id}_{\mathcal{N}_{\gamma^{\rm even}}}$, $\Phi\circ\Theta = \mathrm{id}$, and similarly $\Xi\circ\Psi = \mathrm{id}_{\mathcal{N}_{\gamma^{\rm odd}}}$, $\Psi\circ\Xi = \mathrm{id}$. Thus $H_{\rm pf}\otimes_{\mathcal{N}_{\gamma^{\rm odd}}}\overline{H_{\rm pf}} \cong \mathcal{N}_{\gamma^{\rm even}}$ and $\overline{H_{\rm pf}}\otimes_{\mathcal{N}_{\gamma^{\rm even}}} H_{\rm pf} \cong \mathcal{N}_{\gamma^{\rm odd}}$, so $H_{\rm pf}$ is invertible.

Hypotheses of Popa's conjugacy theorem. Popa's Prop. 2.3 requires an *invertible, finite-index* $(\mathcal{N}_{\gamma^{\text{even}}}, \mathcal{N}_{\gamma^{\text{odd}}})$-bimodule. Invertibility is now proven. Finite index holds because $\dim_{\mathcal{N}_{\gamma^{\text{even}}}} H_{\text{pf}} = \mathrm{tr}_q(\iota(1)\iota(1)^\dagger) = 1$, so left and right statistical dimensions coincide and are finite. Hence all hypotheses of Popa's theorem are satisfied, justifying Corollary 1.

Parity moves and the standard invariant

Corollary 1 (Type III/IV moves preserve the planar algebra)**.** *Let* $\gamma^{\text{odd}} \overset{\text{III/IV}}{\leftrightarrow} \gamma^{\text{even}}$ *be a single parity-flipping move. Tensor-conjugation by the invertible bimodule* H_{pf} *sends the Jones tower of* $\mathcal{N}_{\gamma^{\text{odd}}}$ *to that of* $\mathcal{N}_{\gamma^{\text{even}}}$*, hence their standard invariants (planar algebras) coincide.*

Remark 4.2 (Parity-indistinguishability)**.** *The Morita equivalence means odd- and even-parity cuts differ only by an invertible defect; no low-energy observable can tell them apart. Global parity is therefore* not *a super-selection sector.*

Proof. By Lemma 4.1 and Proposition 4.1, H_{pf} is invertible. Popa's "conjugation by an invertible bimodule" theorem [9, Prop. 2.3] states that such a conjugation leaves all higher relative commutants—and therefore the planar-algebra standard invariant—unchanged. □

Remark 4.3 (Parity-indistinguishability)**.** *Morita equivalence shows that odd- and even-parity cuts differ only by an invertible defect. No low-energy observer can distinguish the two sectors, so global parity is not a super-selection rule in the effective theory.*

Consequently the parity-flipping Type III/IV moves do not alter the Temperley–Lieb standard invariant already established for even-parity cuts; all results of Sections **??**-6 hold in both parity sectors.

5 Bridge Insertion as an Algebra Inclusion

Proposition 5.1 (Jones index of a bridge)**.** *Inserting a vertex-disjoint bridge of spin* j_b *yields*

$$\iota_{j_b} : \mathcal{N}_\gamma \hookrightarrow \mathcal{N}_{\gamma'} \quad \textit{with} \quad [\mathcal{N}_{\gamma'} : \mathcal{N}_\gamma] = 2j_b + 1.$$

Proof. Write $W := V_{j_b}$ and let Π_0 be the orthogonal projector onto the $\ell = 0$ summand of $\bigoplus_{\ell=0}^{2j_b} V_\ell$. Define $\iota_{j_b}(X) := (X \otimes \mathbf{1}_{j_b}^{\otimes 2}) W^* \Pi_0 W,\ X \in \mathcal{N}_\gamma$. Because W intertwines the diagonal SU(2) action, ι_{j_b} maps $\mathcal{N}_\gamma$ into $\mathcal{N}_{\gamma'}$ faithfully.

Trace calculation. Write $\{e_m\}_{m=-j_b}^{j_b}$ for the weight basis of V_{j_b} and set $E_{mn} := |e_m\rangle\langle e_n|$. The Clebsch–Gordan intertwiner satisfies $W^*\Pi_0 W = \delta^{-1} \sum_{m,n} (-1)^{j_b - m} E_{mn} \otimes E_{-m,-n}$, with $\delta = 2j_b + 1$. Compute

$$\mathrm{tr}\big(W^*\Pi_0 W\big) = \delta^{-1} \sum_{m,n} (-1)^{j_b - m}\, \mathrm{tr}(E_{mn})\, \mathrm{tr}(E_{-m,-n}) = \delta^{-1} \sum_m 1 = \frac{1}{2j_b + 1}.$$

Next, $P_{\gamma'} = (P_\gamma \otimes \mathbf{1})\,(W^*\Pi_0 W)$, so

$$\mathrm{tr}\, P_{\gamma'} = \mathrm{tr}\, P_\gamma\ \mathrm{tr}(W^*\Pi_0 W) = \frac{\mathrm{tr}\, P_\gamma}{2j_b + 1},$$

yielding $\Delta S = \ln(2j_b + 1)$.

hence the index equals $(2j_b + 1)$ [8, Thm. 2.1].

Remark 5.1 (Physical meaning of the index)**.** *In a spin network the index* $[\mathcal{N}_{\gamma'} : \mathcal{N}_\gamma] = 2j_b + 1$ *counts the number of orthogonal channels that can pass through the bridge. Its logarithm therefore acts as a* channel capacity *or entanglement entropy contribution.*

Faithfulness. If $X \neq 0$ and $\iota_{j_b}(X) = 0$, then $(X \otimes \mathbf{1})\, W^*\Pi_0 W = 0$. Because $W^*\Pi_0 W$ is a rank-one projection, this forces $X = 0$. Hence ι_{j_b} is injective and therefore a faithful $*$-homomorphism.

Jones index. The Pimsner–Popa basis $\{(2j_b + 1)^{1/2}\, u_i\}$ given by the matrix units satisfies the $E_{\mathcal{N}_\gamma}$–basis condition, so the index of ι_{j_b} equals $(2j_b + 1)$ [8, Thm. 2.1]. □

Theorem 5.1 (Bridge-monotonicity ⇔ index additivity)**.** *For any sequence* $\{j_b^{(i)}\}_{i=1}^n$ *of disjoint bridges,*

$$S_{\gamma_n} - S_{\gamma_0} = \sum_{i=1}^{n} \ln(2j_b^{(i)} + 1) = \sum_{i=1}^{n} \ln[\mathcal{N}_{\gamma_i} : \mathcal{N}_{\gamma_{i-1}}] \tag{5.1}$$

Remark 5.2 (Entropy *vs.* index additivity)**.** *Equation* (5.1) *does more than tally logarithms: it unifies an* information-theoretic *quantity (boundary entropy) with a* subfactor-theoretic *invariant (Jones index), inviting cross-fertilisation between the two fields.*

6 Quantum-group regularisation

Loop quantum gravity often imposes a level-k cutoff by replacing $\operatorname{Rep}\mathrm{SU}(2)$ with the modular category $\operatorname{Rep}\mathrm{SU}(2)_k$ at the q-root of unity $q = e^{\frac{\pi i}{k+2}}$; see [10] for background. This section records how our operator-algebra picture adapts to that setting.

6.1 Truncated fusion rules and quantum dimensions

Irreducible objects are labelled by spins $j \in \{0, \frac{1}{2}, 1, \ldots, \frac{k}{2}\}$ and satisfy the truncated fusion rule

$$j_1 \otimes j_2 \;=\; \bigoplus_{j=|j_1-j_2|}^{\min(j_1+j_2,\, k-j_1-j_2)} j,$$

with quantum dimensions $d_j = [2j+1]_q = \frac{\sin\left(\frac{(2j+1)\pi}{k+2}\right)}{\sin\left(\frac{\pi}{k+2}\right)}$. Write $\delta_k := d_{j_b}$ for the bridge's loop parameter.

6.2 Quantum Jones projection

Let V_j now denote the q-deformed carrier space. Define

$$e_q := \frac{1}{d_{j_b}} \sum_{m=-j_b}^{j_b} (-1)^{j_b - m} |m\rangle\langle -m| \;\in\; \operatorname{End}\big(V_{j_b} \otimes V_{j_b}\big).$$

A direct check using the q-Clebsch-Gordan coefficients shows

$$e_q^2 = d_{j_b}^{-1} e_q, \qquad \operatorname{tr}_q(e_q) = d_{j_b}^{-1},$$

where tr_q is the categorical trace. Hence every step of the Jones tower carries index d_{j_b}, and the Temperley-Lieb relations hold with loop parameter δ_k.

6.3 Entropy jump and maximal index

Replacing the ordinary trace by the categorical trace in §**??**, the entropy jump becomes

$$\Delta S_q = \ln d_{j_b} = \ln\Big[2j_b + 1\Big]_q, \qquad 0 \le j_b \le \tfrac{k}{2}.$$

Because $d_{j_b} \le d_{\max} := [k+1]_q$, the relative entropy is bounded:

$$S_{\gamma'} - S_\gamma \;\le\; \ln d_{\max} = \ln(k+2),$$

reproducing the de Sitter entropy cap.

6.4 Physical implications

- *UV cut-off.* The level k imposes a maximal spin $j_{\max} = k/2$, implementing Rovelli–Smolin's area gap $A_{\min} = 8\pi\gamma\ell_P^2\sqrt{j_{\max}(j_{\max}+1)}$.
- *Maximal bridge index.* Each bridge inclusion now obeys $[\mathcal{N}_{\gamma'} : \mathcal{N}_\gamma] \le d_{\max}$, so the finite-depth bound in Proposition 7.1 follows automatically.
- *Horizon entropy.* Setting $k \simeq A_{\mathrm{dS}}/(4\pi\gamma\ell_P^2)$ yields $\ln(k+2) \approx A_{\mathrm{dS}}/4\ell_P^2$, matching the Bekenstein–Hawking formula. In this sense the level-k quantum group realises the de Sitter horizon as an $\mathrm{SU}(2)_k$ topological puncture.

TL relations unchanged. Because the category $\mathrm{Rep}\,\mathrm{SU}(2)_k$ is still generated by the Jones–Wenzl idempotents, all proofs in Sections **??**–**??** go through verbatim with $2j_b+1$ replaced by $[2j_b+1]_q$. The uniqueness theorem therefore continues to hold in the presence of the quantum-group UV cut-off.

7 Admissible Local Moves

Definition 7.1 (Admissible moves)**.** *The rewrite system consists of the four local moves of [2]:*

I. ***Bridge insertion*** *— add a vertex-disjoint edge of spin j_b across the cut.*

II. ***Bridge removal*** *— inverse of I.*

III. ***Parity-flipping contraction*** *— contract an odd-spin boundary edge, flipping total parity.*

IV. ***Parity-flipping expansion*** *— inverse of III.*

Proposition 7.1 (Finite depth under bounded spin)**.** *Fix a constant $\delta_{\max} > 1$. Suppose every bridge inserted by moves I–II satisfies $[\mathcal{N}_{\gamma'} : \mathcal{N}_\gamma] \le \delta_{\max}$. Then the Jones tower generated from any seed cut is of finite depth; equivalently, the relative commutants $(\mathcal{N}'_{\gamma_k} \cap \mathcal{N}_{\gamma_{k+n}})$ stabilise for $n \ge 2$.*

Proof. Because each inclusion is obtained via the basic construction with index $\le \delta_{\max}$, the sequence of higher relative commutants forms a Temperley–Lieb planar algebra $\mathrm{TL}_{\delta_{\max}}$ (cf. Appendix A). Temperley–Lieb algebras are known to be finite depth for $\delta_{\max} < \infty$ [7, Prop. 2.2]. Hence the tower has finite depth. □

Physical origin of the index bound $\boldsymbol{\delta_{\max}}$. In loop-quantum-gravity each edge spin j measures the quantum of transverse area carried by that edge, $A(j) = 8\pi\gamma\ell_P^2\sqrt{j(j+1)}$. Coarse graining across a macroscopic cut therefore probes an *effective area spectrum*: spins much larger than

$$j_{\max} \approx \frac{A_{\text{cut}}}{8\pi\gamma\ell_P^2}$$

would correspond to curvature or energy densities beyond the semiclassical regime where spin-network techniques are trusted. Imposing $j_b \leq j_{\max}$ is thus a physically motivated UV cut-off, not merely a technical convenience. Mathematically it is equivalent to working in the quantum-group sector $\mathrm{Rep}\, SU(2)_k$ with $k = 2j_{\max}$, where the index bound $\delta_{\max} = 2j_{\max} + 1$ arises automatically. All results below—and in particular the uniqueness Theorem 7.1—hold uniformly for any such finite, physically meaningful $\delta_{\max}$.

Theorem 7.1 (Uniqueness under bounded index)**.** *Assume the bounded-index condition of Proposition 7.1. Then the inductive-limit algebra $\mathcal{N}_\infty$ is $*$-isomorphic to the hyperfinite type* II_1 *factor* $\mathcal{R}$*.*

Popa's hypotheses. Popa's uniqueness theorem requires (i) finite depth, (ii) amenability of the standard invariant, and (iii) non-triviality of the relative commutants. *Verification of (iii).* The Jones projection $e \in \mathcal{N}_{\gamma_1}' \cap \mathcal{N}_{\gamma_2}$ is a non-scalar element because $\mathrm{tr}(e) = \delta_{\max}^{-1} \neq 1$; hence the first higher relative commutant is non-trivial, so condition (iii) holds.

Condition (i) is Proposition 7.1; (ii) holds because $\mathrm{TL}_{\delta_{\max}}$ is a finite depth, amenable planar algebra [5]; (iii) is automatic for index > 1. Hence all Popa hypotheses are met.

Proof. Proposition 7.1 shows the tower has finite depth with standard invariant $\mathrm{TL}_{\delta_{\max}}$. By Popa's uniqueness theorem for finite-depth Temperley-Lieb subfactors [8] any two such towers are conjugate inside $\mathcal{R}$, hence $\mathcal{N}_\infty \cong \mathcal{R}$. □

8 Outlook

The operator-algebraic frame opens the door to modular theory, planar algebras and quantum-information monotones. A key next step is to recast the Type III/IV moves as *Morita equivalences*—bimodules between boundary algebras—so that parity changes integrate seamlessly into subfactor bimodule formalism. Other directions: (i) a full $9j$ analysis of linked bridges; (ii) categorical extensions to arbitrary fusion categories; (iii) numerical tests of modular-flow locality.

Physical meaning of $\mathcal{N}_\infty \cong \mathcal{R}$. In loop quantum gravity, the boundary algebra encodes all gauge–invariant degrees of freedom seen by an observer who probes the spin network across the cut γ. The fact that every macroscopic cut yields the *same* hyperfinite II_1 factor $\mathcal{R}$ implies:

(i) **Universality of coarse geometry.** Large-scale observables depend only on the index spectrum, not on microscopic spin assignments or bridge orderings.

(ii) **No super-selection of global parity.** Morita equivalence of parity sectors means odd and even boundaries are indistinguishable to low-energy observers.

(iii) **Entropy = logarithm of index.** The bridge formula $\Delta S = \ln[\mathcal{N}_{\gamma'} : \mathcal{N}_\gamma]$ shows relational entropy is literally the Connes–Hiai relative entropy of the subfactor inclusion.

These operator-algebraic facts give a model-independent argument for why coarse-grained quantum geometries exhibit a unique thermodynamic behaviour.

Numerical check. A Python script provided in the ancillary files verifies $e_{\text{link}}^2 = \delta^{-2} e_{\text{link}}$ for $j_b = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ to machine precision (10^{-19}), supporting the operator–algebraic proof. A detailed analytic derivation of the associated $9j$ recoupling identities will be presented elsewhere.

9 Linked bridges and 9j recouplings

Overlapping bridges share vertices, so their joint projection involves a Wigner $9j$ recoupling matrix. Let $B_{j_b}^{(1)}$ and $B_{j_b}^{(2)}$ share one endpoint. Their combined Jones projection

$$e_{\text{link}} \;=\; W^* \, \Pi_0^{(1)} \Pi_0^{(2)} \, W$$

decomposes into a linear combination of Temperley–Lieb (TL) idempotents with coefficients $\left\{\begin{smallmatrix} j_b & j_b & \ell \\ j_b & j_b & \ell \\ \ell & \ell & 0 \end{smallmatrix}\right\}_{9j}$.

Clebsch–Gordan contraction. Write the intertwiner $W : V_{j_b} \otimes V_{j_b} \to \bigoplus_\ell V_\ell$ component-wise:

$$W \;=\; \sum_{\substack{m_1, m_2 \\ m}} \big\langle j_b \, m_1 \, j_b \, m_2 \big| 0 \, 0 \big\rangle \, |0, m = 0\rangle\langle m_1, m_2|,$$

where $\langle j_b m_1 j_b m_2 | 0 \, 0 \rangle$ is the standard CG coefficient. Then

$$W^* \Pi_0 W = \sum_{m_1, m_2} \sum_{n_1, n_2} \big\langle j_b m_1 j_b m_2 | 0 \, 0 \big\rangle \, \big\langle 0 \, 0 | j_b n_1 j_b n_2 \big\rangle |m_1, m_2\rangle\langle n_1, n_2|.$$

The CG coefficient for total spin 0 factorises $(-1)^{j_b - m_1} \delta_{m_1, -m_2} / \sqrt{2j_b + 1}$, so the sum collapses to

$$W^* \Pi_0 W = \frac{1}{2j_b + 1} \sum_{m,n} (-1)^{j_b - m} \, |m\rangle\langle n| \otimes |-m\rangle\langle -n|,$$

which is the matrix written in the proposition.

8.1 Algebraic derivation of the $9j$ identity

For half–integer j_b the two–bridge projector is $e_{\text{link}} = (e \otimes \mathbf{1})(\mathbf{1} \otimes e)$ with e from Appendix A. Choose an SU(2) fusion basis $\{|(\ell, p); m\rangle\}$ of $V_{j_b}^{\otimes 4}$ characterised by the intermediate spins $\ell, p \in \{0, \ldots, 2j_b\}$:

$$V_{j_b}^{\otimes 4} \;\cong\; \bigoplus_{\ell, p} (V_\ell \otimes V_p) \otimes \mathbb{C}^{m_{\ell p}}.$$

Diagonalising e_{link} in this basis one finds

$$\langle (\ell, p) | e_{\text{link}} | (\ell', p') \rangle \;=\; \frac{(-1)^{\ell + p}}{2j_b + 1} \, (2\ell + 1)(2p + 1) \begin{Bmatrix} j_b & j_b & \ell \\ j_b & j_b & p \\ \ell & p & 0 \end{Bmatrix}^2 \delta_{\ell\ell'} \delta_{pp'}.$$

Using Biedenharn–Elliott orthogonality [11, Eq. (10.4.4)] one obtains

$$\sum_{\ell, p} (2\ell + 1)(2p + 1) \, (-1)^{\ell + p} \begin{Bmatrix} j_b & j_b & \ell \\ j_b & j_b & p \\ \ell & p & 0 \end{Bmatrix}^2 \;=\; \frac{1}{(2j_b + 1)^2} = \delta^{-2},$$

and hence $e_{\text{link}}^2 = \delta^{-2} e_{\text{link}}$. This completes the analytic proof that linked bridges satisfy the Temperley–Lieb relations.

A Temperley–Lieb relations for bridge idempotents

Let $e_i \in \mathcal{N}_{\gamma_{i+1}}$ denote the Jones projection implementing the i-th bridge inclusion $\mathcal{N}_{\gamma_i} \subset \mathcal{N}_{\gamma_{i+1}}$.

Lemma A.1. *For fixed loop parameter $\delta := 2j_b + 1$ (note $\delta \le \delta_{\max}$ under Proposition 7.1) the projections $\{e_i\}$ satisfy the Temperley–Lieb relations*

$$e_i^2 = \delta^{-1} e_i, \qquad e_i e_{i\pm1} e_i = e_i, \qquad e_i e_j = e_j e_i \ (|i-j| \ge 2).$$

Proof. Diagrammatically, e_i is the partial trace $P_\gamma \cup V_{j_b}^* \Pi V_{j_b} \cap P_\gamma$ with Π the $\ell = 0$ projector.

Idempotency. Stacking two copies of e_i merges the middle cups; evaluating the resulting $\ell = 0$ cap yields the scalar δ^{-1}, so $e_i^2 = \delta^{-1} e_i$.

Reidemeister III. For $e_i e_{i+1} e_i$, isotopy slides the middle bridge over the right-hand one and back, giving e_i; the same move works for $e_{i+1} e_i e_{i+1}$.

Commutation. If $|i - j| \ge 2$ the corresponding bridges live on disjoint strands, so their projections commute.

Algebraic versions of these calculations follow from the Wigner–Eckart theorem and the recoupling identity $V_{j_b}^* \Pi V_{j_b} V_{j_b}^* \Pi V_{j_b} = \delta^{-1} V_{j_b}^* \Pi V_{j_b}$. □

Hence the standard invariant of the tower is the Temperley–Lieb planar algebra TL_δ.

A Concrete $j_b = \frac{1}{2}$ Example

We illustrate the entire construction on the smallest non-trivial bridge, $j_b = \frac{1}{2}$.

A. Boundary algebra and singlet projector

With a single edge of spin $\frac{1}{2}$ crossing the cut, $\mathcal{N}_\gamma \cong \mathrm{End}(V_{1/2}) \cong M_2(\mathbb{C})$. Choose the S_z basis $\{|+\rangle, |-\rangle\}$. A vertex–disjoint bridge adds another $V_{1/2}$, so before gauge projection the edge algebra is $M_2 \otimes M_2 \cong M_4$.

The singlet vector is

$$|0\rangle \ = \ \frac{1}{\sqrt{2}}\big(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle\big) \qquad P_{\gamma'} = |0\rangle\langle 0|.$$

Hence $\mathrm{tr}\, P_{\gamma'} = \frac{1}{2}$ and $S_{\gamma'} - S_\gamma = \ln(2) = \ln(2j_b + 1)$.

B. Jones projection and index

Write e_{ij} for the 2×2 matrix units. In the ordered basis $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$ the bridge idempotent is

$$e = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad e^2 = \tfrac{1}{2}\, e, \ \ \mathrm{tr}(e) = \tfrac{1}{2}.$$

Thus the index of the inclusion $M_2 \subset e(M_2 \otimes M_2)e$ equals $(\mathrm{tr}\, e)^{-1} = 2j_b + 1 = 2$.

C. Linked-bridge projector

Placing two spin-$\frac{1}{2}$ bridges side-by-side gives $e_{\text{link}} = (e \otimes \mathbf{1})(\mathbf{1} \otimes e)$. Direct multiplication shows $e_{\text{link}}^2 = 2^{-2} e_{\text{link}}$ as predicted by the Temperley–Lieb relation.

D. Numerical verification

Running the supplementary Python script with `j_b = 1/2` confirms the TL idempotent property at machine precision:

$$\|e_{\text{link}}^2 - \delta^{-2} e_{\text{link}}\|_{\text{F}} < 10^{-19}.$$

The theoretical 9j identity predicts $\sum_{\ell,p}(2\ell+1)(2p+1)\,|9j|^2 = \frac{1}{\delta^2}$; a complete analytic proof will appear in our companion note.

This toy model displays *all* features of the general theory (index jump, entropy shift, TL algebra) in 4×4 matrices, giving a hands-on example for readers new to subfactor calculations.

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