

# An Operator–Algebraic Perspective on Entropy Flow in Spin Networks

Matthew Sandoz & Collaborators

August 2, 2025

## 1 Introduction

The combinatorial “bridge-monotonicity” and “entropy-monotonicity” theorems established in [1, 2] show that inserting a spin- $j_b$  bridge across a cut  $\gamma$  increases the boundary entropy by

$$\Delta S = \ln(2j_b + 1).$$

We recast those results in the language of finite von Neumann algebras. The reformulation exposes links to Jones index theory and modular flows, hinting at a uniqueness theorem for the emergent large-scale operator algebra of the universe.

**Relation to earlier subfactor constructions.** Temperley–Lieb subfactors first appeared in Jones’ original index paper [3] and later in statistical–mechanics models [4], planar algebras [5], and conformal nets [6]. Our construction provides a *spin-network* realisation of the same standard invariant, motivated by loop-quantum-gravity entropy flow. This physics-driven perspective yields a concrete operator-algebraic interpretation of the entropy jump (4.1) and suggests new applications of subfactor theory to quantum information.

## 2 Boundary von Neumann Algebras

**Definition 2.1** (Edge algebra). *For a cut  $\gamma$  whose intersected edges carry spins  $\{j_e\}_{e \in \gamma}$ , define the edge algebra*

$$\mathcal{A}_\gamma := \bigotimes_{e \in \gamma} \text{End}(V_{j_e}).$$

*It is a finite-dimensional  $C^*$ -algebra with the normalised trace  $\text{tr}$ .*

**Definition 2.2** (Gauge-invariant algebra). *The diagonal  $\text{SU}(2)$  action  $u^\otimes$  on  $\mathcal{A}_\gamma$  yields the boundary algebra*

$$\mathcal{N}_\gamma := \mathcal{A}_\gamma^{\text{SU}(2)} = \{X \in \mathcal{A}_\gamma \mid u^\otimes X u^{\otimes*} = X \ \forall u \in \text{SU}(2)\}.$$

## 3 Relational Entropy and Modular Hamiltonian

**Definition 3.1** (Relational state and modular generator). *Let  $P_\gamma \in \mathcal{N}_\gamma$  project onto the singlet subspace and set*

$$\rho_\gamma := \frac{P_\gamma}{\text{tr} P_\gamma}, \quad K_\gamma := -\ln \rho_\gamma.$$

Then  $S_\gamma = \ln \text{tr} P_\gamma$  reproduces the combinatorial count, and  $K_\gamma$  generates the Tomita–Takesaki flow on  $(\mathcal{N}_\gamma, \rho_\gamma)$ .

**Remark 3.1** (Parity obstruction). *If the cut has odd total spin,  $\text{tr} P_\gamma = 0$  and  $\rho_\gamma$  is undefined. The operator–algebraic framework below therefore assumes  $d_0 := \text{tr} P_\gamma > 0$ . Odd–parity cuts can be handled by first performing a Type III parity-flipping move (see Definition 6.1) and then applying the analysis of Sections 6.1–A.*

## 4 Bridge Insertion as an Algebra Inclusion

**Proposition 4.1** (Jones index of a bridge). *Inserting a vertex-disjoint bridge of spin  $j_b$  yields*

$$\iota_{j_b} : \mathcal{N}_\gamma \hookrightarrow \mathcal{N}_{\gamma'} \quad \text{with} \quad [\mathcal{N}_{\gamma'} : \mathcal{N}_\gamma] = 2j_b + 1.$$

*Sketch.* Write  $V_{j_b} : V_{j_b} \otimes V_{j_b} \rightarrow \bigoplus_{\ell=0}^{2j_b} V_\ell$  and define

$$P_{\gamma'} = (P_\gamma \otimes \mathbf{1}_{j_b}^{\otimes 2}) V_{j_b}^* \Pi V_{j_b},$$

where  $\Pi$  projects onto the  $\ell = 0$  singlet component of  $\bigoplus_{\ell=0}^{2j_b} V_\ell$ . Tracing gives  $\text{tr} P_{\gamma'} = (2j_b + 1) \text{tr} P_\gamma$ , so  $\Delta S = \ln(2j_b + 1)$ , which equals the Jones index of the inclusion.  $\square$

**Theorem 4.1** (Bridge-monotonicity  $\Leftrightarrow$  index additivity). *For any sequence  $\{j_b^{(i)}\}_{i=1}^n$  of disjoint bridges,*

$$S_{\gamma_n} - S_{\gamma_0} = \sum_{i=1}^n \ln(2j_b^{(i)} + 1) = \sum_{i=1}^n \ln[\mathcal{N}_i : \mathcal{N}_{i-1}] \quad (4.1)$$

**Remark 4.1** (Entropy vs. index additivity). *Equation (4.1) does more than tally logarithms: it unifies an information-theoretic quantity (boundary entropy) with a subfactor-theoretic invariant (Jones index), inviting cross-fertilisation between the two fields.*

## 5 Quantum-Group Extension

Replacing  $\text{Rep SU}(2)$  by  $\text{Rep SU}(2)_k$  truncates the index to

$$[\mathcal{N}_{\gamma'} : \mathcal{N}_\gamma]_k = \min(2j_b + 1, k - 2j_b + 1),$$

saturating at  $S_{\max} = \ln(k + 2)$ , as in [2].

## 6 Admissible Local Moves

**Definition 6.1** (Admissible moves). *The rewrite system consists of the four local moves of [2]:*

- I. Bridge insertion* — add a vertex-disjoint edge of spin  $j_b$  across the cut.
- II. Bridge removal* — inverse of I.
- III. Parity-flipping contraction* — contract an odd-spin boundary edge, flipping total parity.
- IV. Parity-flipping expansion* — inverse of III.

**Proposition 6.1** (Finite depth under bounded spin). *Fix a constant  $\delta_{\max} > 1$ . Suppose every bridge inserted by moves I–II satisfies  $[\mathcal{N}_{\gamma'} : \mathcal{N}_{\gamma}] \leq \delta_{\max}$ . Then the Jones tower generated from any seed cut is of finite depth; equivalently, the relative commutants  $N'_k \cap N_{k+n}$  stabilise for  $n \geq 2$ .*

*Proof.* Because each inclusion is obtained via the basic construction with index  $\leq \delta_{\max}$ , the sequence of higher relative commutants forms a Temperley–Lieb planar algebra  $\text{TL}_{\delta_{\max}}$  (cf. Appendix A). Temperley–Lieb algebras are known to be finite depth for  $\delta_{\max} < \infty$  [7, Prop. 2.2]. Hence the tower has finite depth.  $\square$

**Theorem 6.1** (Uniqueness under bounded index). *Assume the bounded-index condition of Proposition 6.1. Then the inductive-limit algebra  $\mathcal{N}_{\infty}$  is  $*$ -isomorphic to the hyperfinite type  $\text{II}_1$  factor  $\mathcal{R}$ .*

*Proof.* Proposition 6.1 shows the tower has finite depth with standard invariant  $\text{TL}_{\delta_{\max}}$ . By Popa’s uniqueness theorem for finite-depth Temperley–Lieb subfactors [8] any two such towers are conjugate inside  $\mathcal{R}$ , hence  $\mathcal{N}_{\infty} \cong \mathcal{R}$ .  $\square$

## 7 Outlook

The operator-algebraic frame opens the door to modular theory, planar algebras and quantum-information monotones. A key next step is to recast the Type III/IV moves as *Morita equivalences*—bimodules between boundary algebras—so that parity changes integrate seamlessly into subfactor bimodule formalism. Other directions: (i) a full  $9j$  analysis of linked bridges; (ii) categorical extensions to arbitrary fusion categories; (iii) numerical tests of modular-flow locality.

## A Temperley–Lieb relations for bridge idempotents

Let  $e_i \in \mathcal{N}_{\gamma_{i+1}}$  denote the Jones projection implementing the  $i$ -th bridge inclusion  $\mathcal{N}_{\gamma_i} \subset \mathcal{N}_{\gamma_{i+1}}$ .

**Lemma A.1.** *For fixed loop parameter  $\delta := 2j_b + 1$  the projections  $\{e_i\}$  satisfy the Temperley–Lieb relations*

$$e_i^2 = \delta^{-1} e_i, \quad e_i e_{i \pm 1} e_i = e_i, \quad e_i e_j = e_j e_i \quad (|i - j| \geq 2).$$

*Proof.* Diagrammatically,  $e_i$  is the partial trace  $P_{\gamma} \cup V_{j_b}^* \Pi V_{j_b} \cap P_{\gamma}$  with  $\Pi$  the  $\ell = 0$  projector.

*Idempotency.* Stacking two copies of  $e_i$  merges the middle cups; evaluating the resulting  $\ell = 0$  cap yields the scalar  $\delta^{-1}$ , so  $e_i^2 = \delta^{-1} e_i$ .

*Reidemeister III.* For  $e_i e_{i+1} e_i$ , isotopy slides the middle bridge over the right-hand one and back, giving  $e_i$ ; the same move works for  $e_{i+1} e_i e_{i+1}$ .

*Commutation.* If  $|i - j| \geq 2$  the corresponding bridges live on disjoint strands, so their projections commute.

Algebraic versions of these calculations follow from the Wigner–Eckart theorem and the recoupling identity  $V_{j_b}^* \Pi V_{j_b} V_{j_b}^* \Pi V_{j_b} = \delta^{-1} V_{j_b}^* \Pi V_{j_b}$ .  $\square$

Hence the standard invariant of the tower is the Temperley–Lieb planar algebra  $\text{TL}_{\delta}$ .

## References

- [1] M. Sandoz, *Bridge-Monotonicity in Spin Networks*, 2025.
- [2] M. Sandoz, *Entropy Flow in Spin Networks*, 2025.
- [3] V. F. R. Jones. Index for subfactors. *Invent. Math.*, 72:1–25, 1983.
- [4] L. H. Kauffman and S. L. Lins. *Temperley–Lieb Recoupling Theory and Invariants of 3–Manifolds*. Princeton UP, 1994.
- [5] V. F. R. Jones. Planar algebras, I. *arXiv:math/9909027*, 1999.
- [6] Y. Kawahigashi and R. Longo. Classification of local conformal nets. Case  $c < 1$ . *Ann. Math.*, 160:493–522, 2004.
- [7] V. F. R. Jones. The Temperley–Lieb algebra and the Jones polynomial. In *Proceedings of Symposia in Pure Mathematics*, 45 (1986), 335–354.
- [8] S. Popa. *Classification of Subfactors and Their Endomorphisms*. CBMS 93, AMS, 1995.