

# An Operator–Algebraic Perspective on Entropy Flow in Spin Networks

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## 1 Introduction

The combinatorial “bridge-monotonicity” and “entropy-monotonicity” theorems established in [?, ?] show that inserting a spin- $j_b$  bridge across a cut  $\gamma$  increases the boundary entropy by

$$\Delta S = \ln(2j_b + 1).$$

We recast those results in the language of finite von Neumann algebras. The reformulation exposes links to Jones index theory and modular flows, hinting at a uniqueness theorem for the emergent large-scale operator algebra of the universe.

## 2 Boundary von Neumann Algebras

**Definition 2.1** (Edge algebra). *For a cut  $\gamma$  whose intersected edges carry spins  $\{j_e\}_{e \in \gamma}$ , define the edge algebra*

$$\mathcal{A}_\gamma := \bigotimes_{e \in \gamma} \text{End}(V_{j_e}).$$

*It is a finite-dimensional  $C^*$ -algebra with the normalised trace  $\text{tr}$ .*

**Definition 2.2** (Gauge-invariant algebra). *The diagonal  $\text{SU}(2)$  action  $u^\otimes$  on  $\mathcal{A}_\gamma$  yields the boundary algebra*

$$\mathcal{N}_\gamma := \mathcal{A}_\gamma^{\text{SU}(2)} = \{X \in \mathcal{A}_\gamma \mid u^\otimes X u^{\otimes*} = X \ \forall u \in \text{SU}(2)\}.$$

## 3 Relational Entropy and Modular Hamiltonian

**Definition 3.1** (Relational state and modular generator). *Let  $P_\gamma \in \mathcal{N}_\gamma$  project onto the singlet subspace and set*

$$\rho_\gamma := \frac{P_\gamma}{\text{tr} P_\gamma}, \quad K_\gamma := -\ln \rho_\gamma.$$

*Then  $S_\gamma = \ln \text{tr} P_\gamma$  reproduces the combinatorial count, and  $K_\gamma$  generates the Tomita–Takesaki flow on  $(\mathcal{N}_\gamma, \rho_\gamma)$ .*

**Remark 3.1** (Parity obstruction). *If the cut has odd total spin,  $\text{tr} P_\gamma = 0$  and  $\rho_\gamma$  is undefined. The operator–algebraic framework below therefore assumes  $d_0 := \text{tr} P_\gamma > 0$ ; extensions to odd-parity cuts require the parity-flipping moves mentioned in the Outlook.*

## 4 Bridge Insertion as an Algebra Inclusion

**Proposition 4.1** (Jones index of a bridge). *Inserting a vertex-disjoint bridge of spin  $j_b$  yields*

$$\iota_{j_b} : \mathcal{N}_\gamma \hookrightarrow \mathcal{N}_{\gamma'} \quad \text{with} \quad [\mathcal{N}_{\gamma'} : \mathcal{N}_\gamma] = 2j_b + 1.$$

*Sketch.* Write  $V_{j_b} : V_{j_b} \otimes V_{j_b} \rightarrow \bigoplus_{\ell=0}^{2j_b} V_\ell$  and define

$$P_{\gamma'} = (P_\gamma \otimes \mathbf{1}_{j_b}^{\otimes 2}) V_{j_b}^* \Pi V_{j_b},$$

where  $\Pi$  projects onto the  $\ell = 0$  singlet component of  $\bigoplus_{\ell=0}^{2j_b} V_\ell$ . Tracing gives  $\text{tr} P_{\gamma'} = (2j_b + 1) \text{tr} P_\gamma$ , so  $\Delta S = \ln(2j_b + 1)$ , which equals the Jones index of the inclusion.  $\square$

**Theorem 4.1** (Bridge-monotonicity  $\Leftrightarrow$  index additivity). *For any sequence  $\{j_b^{(i)}\}_{i=1}^n$  of disjoint bridges,*

$$S_{\gamma_n} - S_{\gamma_0} = \sum_{i=1}^n \ln(2j_b^{(i)} + 1) = \sum_{i=1}^n \ln[\mathcal{N}_i : \mathcal{N}_{i-1}] \quad (4.1)$$

**Remark 4.1** (Entropy vs. index additivity). *Equation (??) does more than tally logarithms: it unifies an information-theoretic quantity (boundary entropy) with a subfactor-theoretic invariant (Jones index), inviting cross-fertilisation between the two fields.*

## 5 Quantum-Group Extension

Replacing  $\text{Rep SU}(2)$  by  $\text{Rep SU}(2)_k$  truncates the index to

$$[\mathcal{N}_{\gamma'} : \mathcal{N}_\gamma]_k = \min(2j_b + 1, k - 2j_b + 1),$$

saturating at  $S_{\max} = \ln(k + 2)$ , as in [?].

## 6 Admissible Local Moves

**Definition 6.1** (Admissible moves). *The rewrite system consists of the four local moves of [?]:*

- I. Bridge insertion* — add a vertex-disjoint edge of spin  $j_b$  across the cut.
- II. Bridge removal* — inverse of I.
- III. Parity-flipping contraction* — contract an odd-spin boundary edge, flipping total parity.
- IV. Parity-flipping expansion* — inverse of III.

## 7 Uniqueness Conjecture

**Conjecture 7.1** (Uniqueness of the inductive-limit factor). *Let  $\{\mathcal{N}_\gamma\}$  be the directed system generated from a finite seed cut by all moves in Definition ???. Assume (i) the move graph is acyclic and (ii) indices of successive inclusions stay bounded. Then the inductive-limit algebra*

$$\mathcal{N}_\infty := \varinjlim \mathcal{N}_\gamma$$

*is  $*$ -isomorphic to the hyperfinite type  $\text{II}_1$  factor  $\mathcal{R}$ .*

## 8 Outlook

The operator-algebraic frame opens the door to modular theory, planar algebras and quantum-information monotones. A key next step is to recast the Type III/IV moves as *Morita equivalences*—bimodules between boundary algebras—so that parity changes integrate seamlessly into subfactor bimodule formalism. Other directions: (i) a full  $9j$  analysis of linked bridges; (ii) categorical extensions to arbitrary fusion categories; (iii) numerical tests of modular-flow locality.

## References

- [1] M. Sandoz, *Bridge-Monotonicity in Spin Networks*, 2025.
- [2] M. Sandoz, *Entropy Flow in Spin Networks*, 2025.