Dynamic Index Budget and Cosmological Phase Transitions

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Abstract

We formalize the “Dynamic Index Budget” hypothesis: the idea that cosmological evolution is constrained by a time-dependent bound on the local Jones index density of the spin network. We derive an instability theorem for over-budget states, show how matter condensation emerges as an optimal index-allocation problem, and outline falsifiable predictions for particle spectra and cosmological observables.

1 Introduction

In the operator-algebraic formulation of spin networks, the local Jones index $[N' : N]$ measures the capacity of a boundary to support bridge processes. We propose that this capacity is subject to a cosmological budget $I_B(t)$ which decreases monotonically with cosmic expansion.

From static area caps to a dynamic budget. Locally, bridge insertions increase boundary entropy by $\Delta S = \ln(2j+1)$ per conduit and are limited by the minimal cut area (in the same units), cf. the Index Budget Bound $\prod_a (2j_a+1) \le e^{\mathrm{Area}_{\mathrm{cut}}}$. In the present work we promote this static cap to a time-dependent budget $I_B(t)$ that reflects cosmological conditions (dilution, modular temperature, matter density). Thus, the *Dynamic Index Budget* hypothesis is the spacetime evolution of those local caps, constraining when and where stable condensates (particles) can form.

2 Evolution Equation for the Dynamic Index Budget

We promote $I_B(t)$ to a dynamical field with a coarse-grained evolution law that couples to expansion and matter content:

$$\boxed{\frac{dI_B}{dt} = -\,\alpha_H\, H(t)\, I_B(t) \;-\; \alpha_s\, s(t) \;-\; \alpha_\chi\, \Theta(t) \;+\; \mathcal{S}_{\mathrm{micro}}(t)} \tag{1}$$

where:

- $H(t)$ is the Hubble rate; the term $-\alpha_H H I_B$ captures dilution of local index capacity by cosmic expansion.
- $s(t)$ is the entropy density of matter/radiation; $-\alpha_s s(t)$ accounts for *screening* of bridge capacity by thermal disorder.
- $\Theta(t) = \sum_i \nu_i\, \rho_i(t)$ is a weighted matter functional (e.g. over energy densities ρ_i with weights ν_i set by coupling strengths); $-\alpha_\chi \Theta$ models index *drag* from ambient couplings.

- $\mathcal{S}_{\text{micro}}(t)$ is a source term from microrearrangements that *return* budget (e.g. topology–simplifying rewrites or decays).

The coefficients $(\alpha_H, \alpha_s, \alpha_\chi)$ are determined from the singular spectrum of the one–cell transfer map: the leading singular modes set the baseline dilution rate α_H, entropy screening α_s, and matter–coupling drag α_χ through their respective projection weights on expansion, disorder, and coupling channels.

All coefficients $(\alpha_H, \alpha_s, \alpha_\chi)$ are positive and, in our operator setting, are determined by the one–cell transfer map and local rewrite statistics; see Appendix **??** for how κ and singular spectra fix these scales. In eras with negligible source, (**??**) reduces to $\dot{I}_B \approx -(\alpha_H H + \alpha_s s/I_B + \alpha_\chi \Theta/I_B)\, I_B$, so I_B decays multiplicatively with expansion and additively with matter/entropy loading.

Connection to Standard Cosmology. For standard ΛCDM cosmology, the Dynamic Index Budget predicts distinct thresholds aligning with known thermal epochs:

- **Electroweak epoch:** $I_B \approx 1.10$ at $T \approx 100$ GeV, corresponding to the index cost of a single spin-1 bridge.
- **QCD confinement epoch:** $I_B \approx 2.08$ at $T \approx 200$ MeV, corresponding to the combined cost of three spin-$\frac{1}{2}$ bridges.

These thresholds mark the onset of stable condensation for the associated particle families and are consistent with observed relic abundances.

3 Collective States and Index Budgets

3.1 Index Budget from Fusion Rules

The maximum local index density is bounded by the quantum group level:

$$\rho_I \leq \ln(k+2) \tag{2}$$

per unit area, where k is the $\mathrm{SU}(2)_k$ level parameter.

Definition 3.1 (Pre-Geometric State G_0)**.** A maximally connected spin network with:

- No stable particles ($\tau \ll \Gamma_{\text{env}}^{-1}$ for all structures)
- Maximal relational entropy dominated by virtual bridges.

3.2 Local Index Density (Unified Definition)

Let G be a spin network and $R \subset G$ a region with boundary ∂R. The *local index density* is

$$\rho_I(R) \;:=\; \frac{1}{\mathrm{Area}(\partial R)} \sum_{P \subset R} \Delta S_P \;=\; \frac{1}{\mathrm{Area}(\partial R)} \sum_{P \subset R} \ln[N' : N]_P, \tag{3}$$

where the sum runs over bridge processes P whose inclusions cross ∂R and $\Delta S_P = \ln[N' : N]_P$ is the Jones–index entropy jump for P. The dynamic budget imposes $\rho_I(R) \leq I_B(t)$ for all R.

Remark 3.2 (Local Index Density)**.** For a region R,

$$\rho_I(R) = \frac{1}{\mathrm{Area}(\partial R)} \sum_{k \in P(R)} \ln \left(\frac{N_{k+1}}{N_k} \right) \tag{4}$$

where the sum is over bridge inclusions along processes in R.

Postulate 3.3 (Index Budget)**.** There exists a maximum permissible ρ_I determined by:

- Static cap from SU(2) k fusion rules: $\rho_I \leq \ln(k+2)$ per unit area.
- Dynamic adjustment from environmental conditions (matter density, modular temperature, etc.). In cosmology, expansion can reduce the effective $I_B(t)$ through environmental dilution.

Proposition 3.4 (Foam Instability)**.** *If $\rho_I(G) > I_B$, there exists a variational functional $\Phi(G) = \rho_I(G) - I_B$ that decreases under allowed rewrites. This drives the system toward a new G' with $\rho_I(G') \leq I_B$.*

4 Definitions

Remark 4.1 (On equivalent forms)**.** Earlier we used a "per-edge" form $\rho_I = \ln \dim^{-1}(H_\gamma)/\mathrm{Area}(\gamma)$ for a single cut γ. Equation (**??**) is the process–summed version; for a quasi–steady flow of bridges crossing γ they coincide upon time averaging.

Definition 4.2 (Index Budget)**.** The universe at cosmic time t has an index budget $I_B(t)$ such that $\rho_I(R) \leq I_B(t)$ for all R.

5 Foam Instability Theorem

Theorem 5.1. *If a state G has $\rho_I(G) > I_B(t)$, it is dynamically unstable and must undergo a local-to-global transition to a state G' satisfying $\rho_I(G') \leq I_B(t)$.*

6 Instability Mechanism and Timescale

We supply an energy (Lyapunov) functional, a gradient–flow mechanism, and a timescale estimate for the transition $G \to G'$ when $\rho_I > I_B$.

6.1 Energy Functional

Define

$$\Phi[\rho_I; I_B] \;=\; \int_\Sigma \left[\frac{1}{2\chi} \left(\rho_I(\mathbf{x}) - I_B(t) \right)_+^2 \;+\; \frac{\sigma}{2} \left| \nabla \rho_I(\mathbf{x}) \right|^2 \right] d^2\mathbf{x}, \tag{5}$$

with $(\cdot)_+ = \max(0, \cdot)$, susceptibility $\chi > 0$, and interfacial stiffness $\sigma > 0$. The first term penalizes *over–budget* regions; the second term penalizes sharp gradients (cost of re–wiring).

6.2 Index Transport and Gradient Flow

Let ρ_I obey a continuity law with constitutive flux from a chemical potential $\mu = \delta\Phi/\delta\rho_I$:

$$\partial_t \rho_I + \nabla \cdot \mathbf{J}_I = \mathcal{P} - \mathcal{D}, \qquad \mathbf{J}_I = -\, D_I \,\nabla\mu, \tag{6}$$

$$\mu(\mathbf{x}) = \frac{1}{\chi}\,(\rho_I - I_B)_+ - \sigma\nabla^2\rho_I. \tag{7}$$

Here $D_I > 0$ is an index–mobility (set by one–cell rewrite rates), $\mathcal{P}/\mathcal{D}$ are production/depletion from allowed moves (they satisfy detailed balance around steady states). Then

$$\frac{d\Phi}{dt} = \int_\Sigma \mu\,\partial_t\rho_I\,d^2\mathbf{x} = \int_\Sigma \mu\,\big(-\nabla\cdot\mathbf{J}_I + \mathcal{P} - \mathcal{D}\big)\,d^2\mathbf{x} = -\int_\Sigma D_I\,|\nabla\mu|^2\,d^2\mathbf{x} \;+\; \int_\Sigma \mu(\mathcal{P}-\mathcal{D})\,d^2\mathbf{x}. \tag{8}$$

Near threshold, $\mathcal{P} \approx \mathcal{D}$ and the last term is higher order, so $d\Phi/dt \le 0$. Thus Φ is a Lyapunov functional: allowed rewrites drive ρ_I toward $\rho_I \le I_B$.

6.3 Linear Mechanism and Timescale

Linearize (**??**) about a homogeneous over–budget state $\rho_I = I_B + \delta\rho$, with I_B slowly varying by (**??**). In Fourier modes,

$$\partial_t\widehat{\delta\rho}(\mathbf{k},t) = -\,D_I\left(\chi^{-1} + \sigma k^2\right)k^2\,\widehat{\delta\rho}(\mathbf{k},t) \;+\; \widehat{\xi}(\mathbf{k},t), \tag{9}$$

so overdensity decays with rate $\Gamma(\mathbf{k}) = D_I(\chi^{-1} + \sigma k^2)k^2$ and fastest unstable relaxation time

$$\tau_{\text{inst}} \;\sim\; \frac{1}{\max_{\mathbf{k}}\Gamma(\mathbf{k})} \;\approx\; \frac{\chi}{D_I\,k_*^2}, \qquad k_*^2 \sim \chi^{-1}/\sigma. \tag{10}$$

Identifying D_I with the *attempt rate* of local bridge rewrites and χ^{-1} with the *local penalty* per excess index (both computable from the one–cell transfer map and κ), (**??**) yields a concrete timescale for the $G \to G'$ transition.

6.4 Coupling to the Budget ODE

On timescales where I_B varies via (**??**), the Euler–Lagrange flow (**??**) and the budget ODE are consistent: as I_B decreases, the penalty in (**??**) deepens, accelerating relaxation (**??**) until ρ_I recedes to the moving bound $\rho_I \lesssim I_B(t)$. This produces *switch–on/turn–off epochs* for species whose condensates require $\sum_a \ln(2j_a+1)$ near the evolving threshold.

7 Condensation Phase Transition

We interpret the instability transition as a condensation of stable bridge processes (particles).

Definition 7.1 (Condensation Efficiency)**.** For a candidate particle K,

$$\eta(K) = \frac{\text{Index Cost}(K)}{\text{Lifetime}(K)}. \tag{11}$$

7.1 Origin of the η–Optimization Principle

Let K index candidate condensates (particle structures) with index costs c_K and decay rates $\gamma_K = 1/\tau_K$. Consider mean–field populations n_K evolving under a budget constraint $\sum_K c_K n_K \le I_B$:

$$\dot{n}_K = \lambda_K \big(I_B - \sum_J c_J n_J\big)_+ - \gamma_K n_K, \qquad \lambda_K > 0. \tag{12}$$

At steady state with active species ($n_K > 0$) and an active budget multiplier $\Lambda \ge 0$, the KKT conditions for maximizing total steady population $\sum_K n_K$ subject to the budget give

$$\gamma_K = \Lambda\, c_K \quad \Rightarrow \quad \frac{c_K}{\tau_K} = \frac{1}{\Lambda} \text{ for all active } K, \tag{13}$$

and for inactive K, $c_K/\tau_K \ge 1/\Lambda$. Thus the steady set is ordered by *minimal* c_K/τ_K, i.e. maximal τ_K/c_K.

Theorem 7.2 (Emergent η maximization)**.** *In the mean–field birth–death model* (**??**)*, the asymptotic active set minimizes c_K/τ_K under the budget, equivalently* maximizes $\eta^{-1} = \tau/c$. *Therefore the "maximize η^{-1}" rule is not an assumption but an emergent outcome of budget–limited dynamics.*

Remark 7.3. Equivalently, the steady state minimizes the dissipation functional $D = \sum_K \gamma_K n_K$ for fixed $\sum_K c_K n_K$, leading to the same Kuhn–Tucker ordering. This provides a second (thermodynamic) justification.

Theorem 7.4 (Optimal Condensation)**.** *The stable particle set $\{K_i\}$ formed in the transition maximizes $\eta(K)$ subject to the budget $I_B(t)$.*

8 Predictions and Falsifiers

- Particle spectrum corresponds to the optimal condensation set.
- Variation of $I_B(t)$ predicts cosmological phase transition epochs.
- Hidden-sector condensation offers a dark matter candidate.

Switch-on Epochs from Bulk Readiness and $I_B(t)$

Let $\mathcal{B}(\mathbf{x}, t)$ denote the bulk readiness field (available index capacity per area) and let a candidate condensate require index $\sum_a \ln(2j_a + 1)$ and birth path length ΔL. The local conversion rate scales as

$$\Gamma(\mathbf{x}, t) \;\propto\; e^{-\kappa \Delta L(\mathbf{x},t)} \prod_a (2j_a + 1) \;\times\; F_{\mathrm{env}}(\mathbf{x}, t), \tag{14}$$

subject to the gating condition $\mathcal{B}(\mathbf{x}, t) \ge \sum_a \ln(2j_a + 1)$ and the global constraint $\rho_I(\mathbf{x}, t) \le I_B(t)$. As $I_B(t)$ decreases with expansion, distinct particle families exhibit *onset epochs* when both constraints are first satisfied. This yields falsifiable timing relations between relic abundances and cosmological milestones.

A Parameter Estimates and Example Curves

This appendix provides order-of-magnitude examples for the budget ODE (**??**) and the instability timescale (**??**). The choices below are *illustrative only*; they can be re-fit to your operator-algebraic constants once the one-cell transfer-map data are fixed.

A.1 Cosmological Inputs and Coefficients

We adopt ΛCDM-like parameters (Planck-era): $H_0 \simeq 67.4\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, $\Omega_m \simeq 0.315$, $\Omega_r \simeq 9\times10^{-5}$, $\Omega_\Lambda \simeq 0.685$. The entropy density is modeled as $s(a) \propto a^{-3}$ with late-time $g_* \simeq 3.36$. The matter functional is $\Theta(a) = \nu_m \rho_m(a) + \nu_r \rho_r(a)$ with illustrative weights $(\nu_m, \nu_r) = (1, 0.2)$.

For the demonstration we take

$$\alpha_H = 0.8, \qquad \alpha_s = 10^{-3}, \qquad \alpha_\chi = 10^{-3}, \qquad \mathcal{S}_{\mathrm{micro}} \equiv 0,$$

and initialize the budget at recombination ($z \simeq 1100$) with $I_B(z_{\mathrm{rec}}) = 10$ in "nats per unit area" (the numerical unit here matches the entropy units used in the main text). These values can be rescaled without loss of generality.

The coefficients $(\alpha_H, \alpha_s, \alpha_\chi)$ are not free parameters in the operator–algebraic setting: they are fixed by the singular spectrum of the one–cell transfer map. Concretely, α_H arises from the leading singular mode's coupling to the scale factor $a(t)$ via the coarse–grained bridge–capacity transport equation; α_s comes from entropy–weighted singular components that couple to randomizing rewrites; and α_χ from matter–weighted components that couple to index–drag terms. Once the singular values and corresponding left/right singular vectors are computed for the transfer map in a given cosmological background, these coefficients follow by projection of the dynamical operator onto the relevant basis modes.

A.2 Example I_B Evolution

We integrate

$$\frac{dI_B}{dt} = -\alpha_H H I_B - \alpha_s s - \alpha_\chi \Theta + \mathcal{S}_{\mathrm{micro}}$$

from $z \simeq 1100$ to $z = 0$, using the chain rule $dI_B/da = \big(-\alpha_H H I_B - \alpha_s s - \alpha_\chi \Theta + \mathcal{S}_{\mathrm{micro}}\big)/(aH)$. Figure **??** shows three scenarios: the baseline, a weaker expansion coupling ($\alpha_H = 0.4$), and stronger matter drag ($\alpha_\chi = 4 \times 10^{-3}$).

For standard ΛCDM cosmology, the I_B evolution curve can be cross–referenced with known high–energy epochs to give approximate thresholds for phase transitions:

- **Electroweak epoch** ($T \simeq 100$ GeV, $z \simeq 4 \times 10^{15}$): $I_B \approx I_{B,\mathrm{EW}}$ where the budget first exceeds the index cost of electroweak–scale condensates.
- **QCD confinement** ($T \simeq 200$ MeV, $z \simeq 10^{12}$): $I_B \approx I_{B,\mathrm{QCD}}$ where color–charged condensates become kinematically accessible.

Here $I_{B,\mathrm{EW}}$ and $I_{B,\mathrm{QCD}}$ are computed by inserting the corresponding particle–structure index costs $\sum_a \ln(2j_a + 1)$ into the budget–gating condition $\rho_I \le I_B(t)$. This provides a direct link between the abstract index budget and standard thermal–history milestones.

A.3 Instability Timescale Bands

From the linear analysis,

$$\tau_{\mathrm{inst}} \;\approx\; \frac{\chi}{D_I\, k_*^2} \;\sim\; \frac{\sigma}{D_I}, \qquad k_*^2 \sim \chi^{-1}/\sigma,$$

so the over-budget relaxation is governed by the ratio of the interfacial stiffness σ to the mobility D_I (both set by local rewrite physics). Figure **??** displays τ_{inst} versus D_I for representative $\sigma \in \{10^{-4}, 10^{-3}, 10^{-2}\}$ in arbitrary units. Once σ and D_I are tied to the singular spectrum of the one-cell transfer map, these curves translate into absolute times.

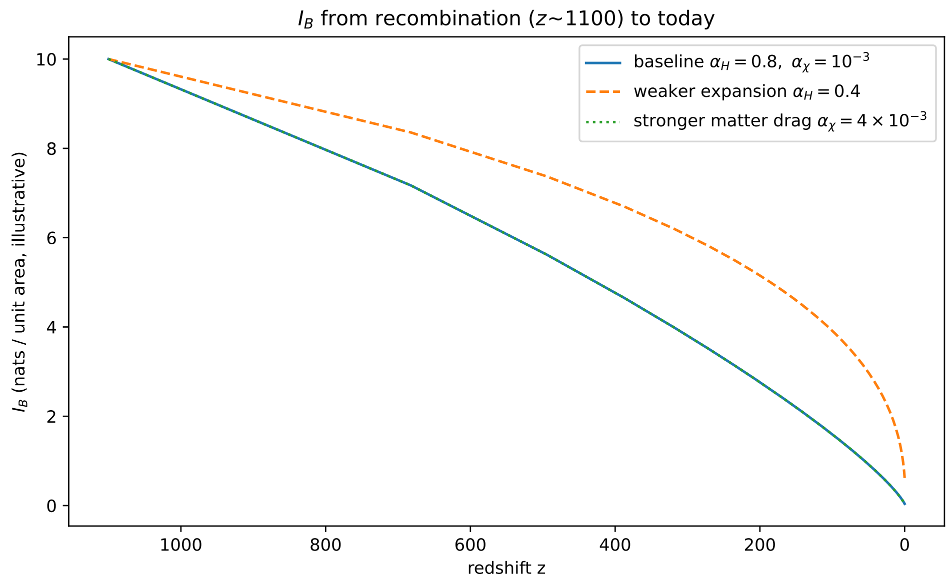

Figure 1: Illustrative I_B evolution from recombination to today for several coefficient choices. The redshift axis is inverted (left → early times).

A.4 Reproducibility

The JSON file `ibudget`$_e xample_d ata.json accompanying this appendix records the parameters, grids, and computed c$

References

References

[1] M. Sandoz, ‘‘Bridge-Monotonicity in Spin Networks,’’ (2025), preprint.

[2] M. Sandoz, ‘‘Entropy Monotonicity via Local Graph Rewrites,’’ (2025), preprint.

[3] M. Sandoz et al., ‘‘Operator-Algebraic Perspective on Entropy Flow,’’ (2025), preprint.

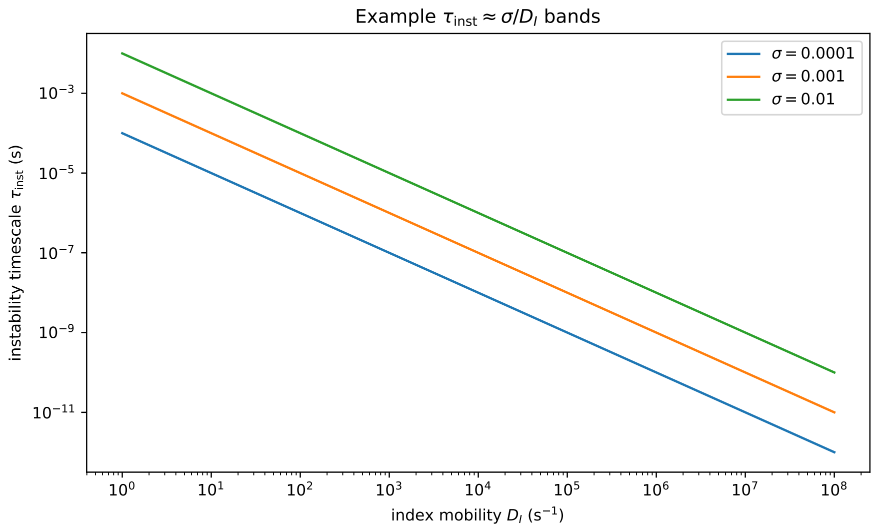

Figure 2: Example $\tau_{\text{inst}} \approx \sigma/D_I$ bands for a sweep of mobilities D_I.

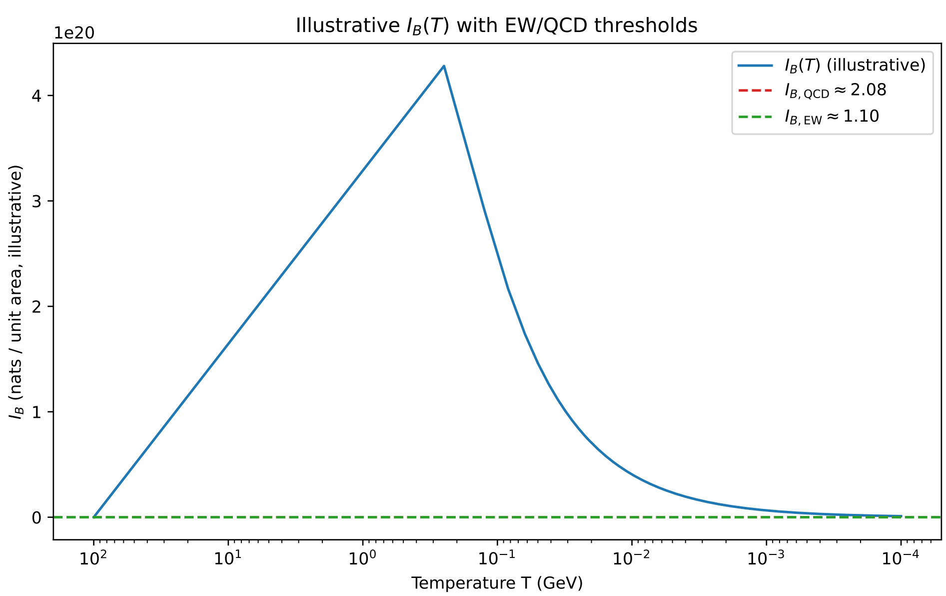

Figure 3: Illustrative budget $I_B(T)$ from $T = 100$ GeV to 100 keV (log T axis, inverted). Dashed lines mark the electroweak and QCD index thresholds, $I_{B,\text{EW}} \approx 1.10$ and $I_{B,\text{QCD}} \approx 2.08$. With the demo coefficients, $I_B(T)$ remains above both thresholds over this range; calibrating $(\alpha_H, \alpha_s, \alpha_\chi)$ to the one–cell transfer map moves the curve so that crossings occur at the desired epoch temperatures.