# An Operator-Algebraic Perspective on Entropy Flow in Spin Networks

Matthew Sandoz & Collaborators

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## 1 Introduction

The combinatorial "bridge-monotonicity" and "entropy-monotonicity" theorems established in [1, 2] show that inserting a spin-$j_b$ bridge across a cut $\gamma$ increases the boundary entropy by

$$\Delta S = \ln(2j_b + 1).$$

We recast those results in the language of finite von Neumann algebras. The reformulation exposes links to Jones index theory and modular flows, hinting at a uniqueness theorem for the emergent large-scale operator algebra of the universe.

**Relation to earlier subfactor constructions.** Temperley–Lieb subfactors first appeared in Jones' original index paper [3] and later in statistical-mechanics models [4], planar algebras [5], and conformal nets [6]. Our construction provides a *spin-network* realisation of the same standard invariant, motivated by loop-quantum-gravity entropy flow. This physics-driven perspective yields a concrete operator-algebraic interpretation of the entropy jump (5.1) and suggests new applications of subfactor theory to quantum information.

## 2 Subfactor background in two pages

We summarise only the notions used later; see [3, 5, 8] for full treatments.

**2.1 Jones basic construction.** Given a $\mathrm{II}_1$ subfactor $N \subset M$ with trace $\tau$, the *Jones projection* $e_N \in B(L^2(M))$ is the orthogonal projection $L^2(M) \twoheadrightarrow L^2(N)$. The von Neumann algebra $M_1 := \langle M, e_N \rangle''$ is the *basic construction* and $[M : N] = \tau(e_N)^{-1}$ is the *Jones index*. Iterating produces the *Jones tower* $N \subset M \subset M_1 \subset M_2 \subset \cdots$.

**2.2 Relative commutants and the standard invariant.** The $k$-th relative commutant $N' \cap M_k$ is finite-dimensional. The graded $*$-algebra $\mathcal{G}_\bullet(N \subset M) = \{N' \cap M_k\}_{k \geq 0}$ together with its Jones projections is called the *standard invariant*. It can be encoded diagrammatically as a *planar algebra* [5].

**2.3 Temperley–Lieb (TL) planar algebra.** For $\delta > 0$ the TL planar algebra $\mathrm{TL}_\delta$ is generated by a single idempotent $e$ obeying $e^2 = \delta^{-1} e$, $e_i e_{i\pm1} e_i = e_i$ and $e_i e_j = e_j e_i$ for $|i - j| \geq 2$. Every finite-depth subfactor with TL standard invariant is $\mathrm{TL}_\delta$ for some $\delta > 1$.

**2.4 Popa's uniqueness theorem.** If a finite-depth, amenable subfactor has the same standard invariant as $\mathcal{R} \subset \mathcal{R}$ (the hyperfinite inclusion), then it is *inner conjugate* to it [8, Thm. 4.5]. We use this in Section **??**.

# 3 Boundary von Neumann Algebras

**Definition 3.1** (Edge algebra)**.** *For a cut $\gamma$ whose intersected edges carry spins $\{j_e\}_{e\in\gamma}$, define the* edge algebra

$$\mathcal{A}_\gamma := \bigotimes_{e\in\gamma} \mathrm{End}(V_{j_e}).$$

*Here the tensor product is taken over $\mathbb{C}$. For a finite cut this is the algebraic tensor product, while for an infinite cut we take the spatial (von Neumann) completion. It is a finite (resp. properly infinite) $\mathrm{C}^*$-algebra equipped with the normalised trace* $\mathrm{tr}$*.*

**Definition 3.2** (Gauge-invariant algebra)**.** *The diagonal* $\mathrm{SU}(2)$ *action $u^{\otimes}$ on $\mathcal{A}_\gamma$ yields the* boundary algebra

$$\mathcal{N}_\gamma := \mathcal{A}_\gamma^{\mathrm{SU}(2)} = \{X \in \mathcal{A}_\gamma \mid u^{\otimes} X u^{\otimes *} = X \ \forall u \in \mathrm{SU}(2)\}.$$

# 4 Relational Entropy and Modular Hamiltonian

**Definition 4.1** (Relational state and modular generator)**.** *Let $P_\gamma \in \mathcal{N}_\gamma$ project onto the singlet subspace and set*

$$\rho_\gamma := \frac{P_\gamma}{\mathrm{tr}P_\gamma}, \qquad K_\gamma := -\ln \rho_\gamma.$$

*Then $S_\gamma = \ln \mathrm{tr} P_\gamma$ reproduces the combinatorial count, and $K_\gamma$ generates the Tomita-Takesaki flow on $(\mathcal{N}_\gamma, \rho_\gamma)$.*

**Remark 4.1** (Parity obstruction)**.** *If the cut has* odd *total spin,* $\mathrm{tr}P_\gamma = 0$ *and $\rho_\gamma$ is undefined. The operator-algebraic framework below therefore assumes $d_0 := \mathrm{tr}P_\gamma > 0$. Odd-parity cuts can be handled by first performing a Type III parity-flipping move (see Definition 7.1) and then applying the results in Proposition 7.1 and Appendix A.*

## Parity-flipping as Morita equivalence

Let $\gamma^{\mathrm{odd}}$ be a cut of odd total spin. Define the bimodule $\mathcal{H}_{\mathrm{pf}}$ by

$$\mathcal{H}_{\mathrm{pf}} := \mathrm{Inv}\Big(V_{1/2} \otimes \bigotimes_{e\in\gamma^{\mathrm{odd}}} V_{j_e}\Big),$$

on which $\mathcal{N}_{\gamma^{\mathrm{odd}}}$ acts on the right and $\mathcal{N}_{\gamma^{\mathrm{even}}}$ (obtained by attaching a spin$-\frac{1}{2}$ stub) acts on the left. This $\mathcal{H}_{\mathrm{pf}}$ is an *invertible* $\mathcal{N}_{\gamma^{\mathrm{even}}}$–$\mathcal{N}_{\gamma^{\mathrm{odd}}}$ bimodule, hence a Morita equivalence [8, Def. 2.1]. Type III/IV moves therefore transport the standard invariant unchanged, so all parity sectors share the same limit factor $\mathcal{R}$.

**Proposition 4.1** (Morita equivalence of parity sectors)**.** *Let $F := \bigotimes_{e\in\gamma^{\mathrm{odd}}} V_{j_e}$ and put $H_{\mathrm{pf}} :=$ $\mathrm{Inv}_{\mathrm{SU}(2)}(V_{1/2} \otimes F)$. Then*

$$H_{\mathrm{pf}} \otimes_{\mathcal{N}_{\gamma^{\mathrm{odd}}}} \overline{H_{\mathrm{pf}}} \;\cong\; {}_{\mathcal{N}_{\gamma^{\mathrm{even}}}}\mathcal{N}_{\gamma^{\mathrm{even}}}, \qquad \overline{H_{\mathrm{pf}}} \otimes_{\mathcal{N}_{\gamma^{\mathrm{even}}}} H_{\mathrm{pf}} \;\cong\; {}_{\mathcal{N}_{\gamma^{\mathrm{odd}}}}\mathcal{N}_{\gamma^{\mathrm{odd}}},$$

*hence $\mathcal{N}_{\gamma^{\mathrm{odd}}}$ and $\mathcal{N}_{\gamma^{\mathrm{even}}}$ are Morita equivalent.*

*Proof.* Throughout, $\varepsilon : V_{1/2} \otimes V_{1/2} \to \mathbb{C}$ and $\iota : \mathbb{C} \to V_{1/2} \otimes V_{1/2}$ are the standard SU(2) cup and cap, normalised so $\varepsilon \circ \iota = \mathbf{1}$.

**1. A concrete orthonormal basis.** Fix an admissible fusion tree $(\frac{1}{2}, j_{e_1}, j_{e_2}, \dots) \rightsquigarrow (\ell_1, \ell_2, \dots)$ and denote by $\psi_{\boldsymbol{\ell}} \in H_{\mathrm{pf}}$ its Wigner basis element. The set $\{\psi_{\boldsymbol{\ell}}\}_{\boldsymbol{\ell}}$ is orthonormal and spans $H_{\mathrm{pf}}$; similarly for its complex conjugates $\overline{\psi_{\boldsymbol{\ell}}}$.

**2. First bimodule map $\Theta$.** Define

$$\Theta(\psi \otimes_{\mathcal{N}_{\gamma^{\mathrm{odd}}}} \overline{\phi}) := (\varepsilon \otimes \mathbf{1}_F)(\psi \otimes \overline{\phi}) \in \mathrm{End}(F)^{\mathrm{SU}(2)} = \mathcal{N}_{\gamma^{\mathrm{even}}}.$$

*Balanced-tensor well-definedness.* For any $a \in \mathcal{N}_{\gamma^{\mathrm{odd}}}$ we must show $\Theta(\psi a \otimes \overline{\phi}) = \Theta(\psi \otimes \overline{a^* \phi})$. Because $a$ acts only on the $F$ factor and $\varepsilon$ acts only on the two $V_{1/2}$ legs, the two expressions coincide, proving well-definedness.

*Bimodule relations.* For $b, c \in \mathcal{N}_{\gamma^{\mathrm{even}}}$, $b \cdot \Theta(\xi \otimes \overline{\eta}) \cdot c = \Theta(b \cdot \xi \otimes \overline{\eta \cdot c})$, again because $b, c$ commute with $\varepsilon$.

*Isometry.* Using the graphical inner product $\langle \psi, \phi \rangle = (\varepsilon \otimes \mathbf{1}_F)(\psi^* \phi)$, one computes

$$\langle \Theta(\psi \otimes \overline{\phi}), \Theta(\psi \otimes \overline{\phi}) \rangle = \varepsilon(\iota(1)) \, \langle \psi, \psi \rangle \langle \phi, \phi \rangle = \langle \psi \otimes \overline{\phi}, \psi \otimes \overline{\phi} \rangle,$$

so $\Theta$ preserves the bimodule inner product.

*Surjectivity.* For each fusion label $\boldsymbol{\ell}$ the image $\Theta(\psi_{\boldsymbol{\ell}} \otimes \overline{\psi_{\boldsymbol{\ell}}})$ is the minimal projection onto the $\boldsymbol{\ell}$-isotypic subspace of $F$; these projections generate $\mathcal{N}_{\gamma^{\mathrm{even}}}$, hence $\Theta$ is surjective.

**3. Inverse map $\Phi$.** Define

$$\Phi(X) := \iota(1) \otimes_{\mathbb{C}} X \quad \in \; H_{\mathrm{pf}} \otimes_{\mathcal{N}_{\gamma^{\mathrm{odd}}}} \overline{H_{\mathrm{pf}}}.$$

Balanced-tensor relations are immediate and $(\varepsilon \otimes \mathbf{1}_F)(\iota(1) \otimes X) = X$, so $\Theta \circ \Phi = \mathrm{id}$. Conversely, $(\iota \otimes \mathbf{1}_F)(\varepsilon \otimes \mathbf{1}_F) = \mathbf{1}_{H_{\mathrm{pf}} \otimes \overline{H_{\mathrm{pf}}}}$, whence $\Phi \circ \Theta = \mathrm{id}$. Therefore $\Theta$ is a unitary bimodule isomorphism.

**4. Second isomorphism.** Replacing $\varepsilon$ by $\iota$ and vice-versa yields the map

$$\Xi : \overline{H_{\mathrm{pf}}} \otimes_{\mathcal{N}_{\gamma^{\mathrm{even}}}} H_{\mathrm{pf}} \longrightarrow {}_{\mathcal{N}_{\gamma^{\mathrm{odd}}}} \mathcal{N}_{\gamma^{\mathrm{odd}}}, \qquad \Xi(\overline{\phi} \otimes \psi) := (\varepsilon \otimes \mathbf{1}_F)(\overline{\phi} \otimes \psi),$$

and one verifies exactly as above that $\Xi$ is a unitary inverse to its adjoint.

Both bimodule isomorphisms being established, the two boundary algebras are Morita equivalent. □

## Relation to the combinatorial framework [1, 2]

The spin-network proofs in [1, 2] derive the entropy jump $\Delta S = \ln(2j_b + 1)$ from a counting of admissible colourings of a cut $\gamma$. Our operator-algebraic reformulation retains the same combinatorics but packages it as:

$$\Delta S = -\ln \tau(P_{\gamma'}) + \ln \tau(P_\gamma) = \ln\big[\mathcal{N}_{\gamma'} : \mathcal{N}_\gamma\big].$$

- The **advantage** is that Jones index is a stable, basis-free quantity, so the entropy formula survives parity moves and quantum-group truncation.

- Conversely, the combinatorial perspective supplies explicit TL basis vectors—fusion trees—that we exploit in the proof of Proposition 4.1.
- Thus the two viewpoints are complimentary: [1, 2] proves the raw counting formula; the present paper shows that the same formula controls the entire Jones tower and standard invariant.

Hence every odd-parity boundary algebra lies in the same Morita class as its even-parity partner; the large-scale factor $\mathcal{R}$ is therefore parity-independent.

### Verification details for the parity-flipping bimodule

**Lemma 4.1.** *Let $F = \bigotimes_{e \in \gamma^{\mathrm{odd}}} V_{j_e}$ and $H_{\mathrm{pf}} = \mathrm{Inv}(V_{1/2} \otimes F)$ as in the proposition. Then $H_{\mathrm{pf}}$ is an $\big(\mathcal{N}_{\gamma^{\mathrm{even}}}, \mathcal{N}_{\gamma^{\mathrm{odd}}}\big)$–bimodule via*

$$a_L \cdot \psi \cdot a_R := (a_L \otimes \mathbf{1}_V)\, \psi\, (\mathbf{1}_V \otimes a_R), \qquad a_L \in \mathcal{N}_{\gamma^{\mathrm{even}}},\ a_R \in \mathcal{N}_{\gamma^{\mathrm{odd}}},\ \psi \in H_{\mathrm{pf}}.$$

*Moreover the balanced tensor product relation $\psi \cdot a_R \otimes \overline{\phi} = \psi \otimes \overline{a_R^* \phi}$ holds for all $a_R, \psi, \phi$.*

*Proof.* Because $a_L$ (respectively $a_R$) acts non-trivially only on $F$, the left and right actions commute and preserve the SU(2) invariant subspace. For the balanced tensor product observe that

$$(\varepsilon \otimes \mathbf{1}_F)\big((\psi \cdot a_R) \otimes \overline{\phi}\big) = (\varepsilon \otimes \mathbf{1}_F)\big(\psi \otimes \overline{a_R^* \phi}\big),$$

because $\varepsilon$ contracts only the two $V_{1/2}$ legs; hence the two simple tensors are identified in the quotient. □

### Invertibility and balanced-tensor details

**Explicit evaluation and coevaluation.** Fix the standard weight basis $|+\rangle := |m = \frac{1}{2}\rangle$, $|-\rangle := |m = -\frac{1}{2}\rangle$ of $V_{1/2}$. Set

$$\varepsilon\big(|m_1\rangle \otimes |m_2\rangle\big) \ :=\ (-1)^{\frac{1}{2} - m_1}\, \delta_{m_1, -m_2}, \qquad \iota(1) \ :=\ |+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle.$$

Then $\varepsilon \circ \iota = \mathbf{1}_{\mathbb{C}}$ and $(\iota^\dagger \otimes \mathbf{1})(\mathbf{1} \otimes \varepsilon) = \mathbf{1}_{V_{1/2}}$, so $\varepsilon, \iota$ implement the rigid duality structure of $\mathrm{Rep}\, \mathrm{SU}(2)$.

**Balanced-tensor identity (detail).** Let $\psi, \phi \in H_{\mathrm{pf}}$ and $a_R \in \mathcal{N}_{\gamma^{\mathrm{odd}}} = \mathrm{End}(F)^{\mathrm{SU}(2)}$. Because $a_R$ acts as $\mathbf{1}_{V_{1/2}} \otimes a_R$ on $V_{1/2} \otimes F$,

$$(\varepsilon \otimes \mathbf{1}_F)\big((\psi \cdot a_R) \otimes \overline{\phi}\big) = (\varepsilon \otimes \mathbf{1}_F)\big(\psi \otimes (\mathbf{1}_{V_{1/2}} \otimes a_R^*) \overline{\phi}\big) = (\varepsilon \otimes \mathbf{1}_F)\big(\psi \otimes \overline{a_R^* \phi}\big),$$

verifying the balanced-tensor relation required for $\Theta$.

**Invertibility — both directions.** Define $\Theta$ and $\Phi$ exactly as in the previous proof and set

$$\Xi(\overline{\phi} \otimes \psi) := (\varepsilon \otimes \mathbf{1}_F)(\overline{\phi} \otimes \psi), \qquad \Psi(X) := \overline{\iota(1)} \otimes X.$$

A direct contraction check gives $\Theta \circ \Phi = \mathrm{id}_{\mathcal{N}_{\gamma^{\mathrm{even}}}}$, $\Phi \circ \Theta = \mathrm{id}$, and similarly $\Xi \circ \Psi = \mathrm{id}_{\mathcal{N}_{\gamma^{\mathrm{odd}}}}$, $\Psi \circ \Xi = \mathrm{id}$. Thus $H_{\mathrm{pf}} \otimes_{\mathcal{N}_{\gamma^{\mathrm{odd}}}} \overline{H_{\mathrm{pf}}} \cong \mathcal{N}_{\gamma^{\mathrm{even}}}$ and $\overline{H_{\mathrm{pf}}} \otimes_{\mathcal{N}_{\gamma^{\mathrm{even}}}} H_{\mathrm{pf}} \cong \mathcal{N}_{\gamma^{\mathrm{odd}}}$, so $H_{\mathrm{pf}}$ is invertible.

**Hypotheses of Popa's conjugacy theorem.** Popa's Prop. 2.3 requires an *invertible, finite-index* $(\mathcal{N}_{\gamma^{\mathrm{even}}}, \mathcal{N}_{\gamma^{\mathrm{odd}}})$-bimodule. Invertibility is now proven. Finite index holds because $\dim_{\mathcal{N}_{\gamma^{\mathrm{even}}}} H_{\mathrm{pf}} = \mathrm{tr}_q(\iota(1)\iota(1)^\dagger) = 1$, so left and right statistical dimensions coincide and are finite. Hence all hypotheses of Popa's theorem are satisfied, justifying Corollary 1.

### Parity moves and the standard invariant

**Corollary 1** (Type III/IV moves preserve the planar algebra)**.** *Let* $\gamma^{\mathrm{odd}} \overset{\mathrm{III/IV}}{\leftrightarrow} \gamma^{\mathrm{even}}$ *be a single parity-flipping move. Tensor-conjugation by the invertible bimodule* $H_{\mathrm{pf}}$ *sends the Jones tower of* $\mathcal{N}_{\gamma^{\mathrm{odd}}}$ *to that of* $\mathcal{N}_{\gamma^{\mathrm{even}}}$*, hence their standard invariants (planar algebras) coincide.*

**Remark 4.2** (Parity-indistinguishability)**.** *The Morita equivalence means odd- and even-parity cuts differ only by an invertible defect; no low-energy observable can tell them apart. Global parity is therefore* not *a super-selection sector.*

*Proof.* By Lemma 4.1 and Proposition 4.1, $H_{\mathrm{pf}}$ is invertible. Popa's "conjugation by an invertible bimodule" theorem [9, Prop. 2.3] states that such a conjugation leaves all higher relative commutants—and therefore the planar-algebra standard invariant—unchanged. □

**Remark 4.3** (Parity-indistinguishability)**.** *Morita equivalence shows that odd- and even-parity cuts differ only by an invertible defect. No low-energy observer can distinguish the two sectors, so global parity is not a super-selection rule in the effective theory.*

Consequently the parity-flipping Type III/IV moves do not alter the Temperley–Lieb standard invariant already established for even-parity cuts; all results of Sections **??**-6 hold in both parity sectors.

## 5 Bridge Insertion as an Algebra Inclusion

**Proposition 5.1** (Jones index of a bridge)**.** *Inserting a vertex-disjoint bridge of spin* $j_b$ *yields*

$$\iota_{j_b} : \mathcal{N}_\gamma \hookrightarrow \mathcal{N}_{\gamma'} \quad \textit{with} \quad [\mathcal{N}_{\gamma'} : \mathcal{N}_\gamma] = 2j_b + 1.$$

*Proof.* Write $W := V_{j_b}$ and let $\Pi_0$ be the orthogonal projector onto the $\ell = 0$ summand of $\bigoplus_{\ell=0}^{2j_b} V_\ell$. Define $\iota_{j_b}(X) := (X \otimes \mathbf{1}_{j_b}^{\otimes 2})W^*\Pi_0 W,\ X \in \mathcal{N}_\gamma$. Because $W$ intertwines the diagonal SU(2) action, $\iota_{j_b}$ maps $\mathcal{N}_\gamma$ into $\mathcal{N}_{\gamma'}$ faithfully.

**Trace calculation.** Write $\{e_m\}_{m=-j_b}^{j_b}$ for the weight basis of $V_{j_b}$ and set $E_{mn} := |e_m\rangle\langle e_n|$. The Clebsch–Gordan intertwiner satisfies $W^*\Pi_0 W = \delta^{-1}\sum_{m,n}(-1)^{j_b-m}\, E_{mn}\otimes E_{-m,-n}$, with $\delta = 2j_b+1$. Compute

$$\mathrm{tr}\big(W^*\Pi_0 W\big) = \delta^{-1}\sum_{m,n}(-1)^{j_b-m}\,\mathrm{tr}(E_{mn})\,\mathrm{tr}(E_{-m,-n}) = \delta^{-1}\sum_m 1 = \frac{1}{2j_b+1}.$$

Next, $P_{\gamma'} = (P_\gamma \otimes \mathbf{1})\,(W^*\Pi_0 W)$, so

$$\mathrm{tr}\,P_{\gamma'} = \mathrm{tr}\,P_\gamma\ \mathrm{tr}(W^*\Pi_0 W) = \frac{\mathrm{tr}\,P_\gamma}{2j_b+1},$$

yielding $\Delta S = \ln(2j_b + 1)$.

hence the index equals $(2j_b + 1)$ [8, Thm. 2.1].

**Remark 5.1** (Physical meaning of the index)**.** *In a spin network the index* $[\mathcal{N}_{\gamma'} : \mathcal{N}_\gamma] = 2j_b + 1$ *counts the number of orthogonal channels that can pass through the bridge. Its logarithm therefore acts as a* channel capacity *or entanglement entropy contribution.*

**Faithfulness.** If $X \neq 0$ and $\iota_{j_b}(X) = 0$, then $(X \otimes \mathbf{1})\, W^* \Pi_0 W = 0$. Because $W^* \Pi_0 W$ is a rank-one projection, this forces $X = 0$. Hence $\iota_{j_b}$ is injective and therefore a faithful $*$-homomorphism.

**Jones index.** The Pimsner–Popa basis $\{(2j_b + 1)^{1/2}\, u_i\}$ given by the matrix units satisfies the $E_{\mathcal{N}_\gamma}$–basis condition, so the index of $\iota_{j_b}$ equals $(2j_b + 1)$ [8, Thm. 2.1]. □

**Theorem 5.1** (Bridge-monotonicity ⇔ index additivity)**.** *For any sequence* $\{j_b^{(i)}\}_{i=1}^n$ *of disjoint bridges,*

$$S_{\gamma_n} - S_{\gamma_0} = \sum_{i=1}^{n} \ln(2j_b^{(i)} + 1) = \sum_{i=1}^{n} \ln[\mathcal{N}_{\gamma_i} : \mathcal{N}_{\gamma_{i-1}}] \tag{5.1}$$

**Remark 5.2** (Entropy *vs.* index additivity)**.** *Equation* (5.1) *does more than tally logarithms: it unifies an* information-theoretic *quantity (boundary entropy) with a* subfactor-theoretic *invariant (Jones index), inviting cross-fertilisation between the two fields.*

# 6 Quantum-group regularisation

Loop quantum gravity often imposes a level-$k$ cutoff by replacing $\operatorname{Rep}\mathrm{SU}(2)$ with the modular category $\operatorname{Rep}\mathrm{SU}(2)_k$ at the $q$-root of unity $q = e^{\frac{\pi i}{k+2}}$; see [10] for background. This section records how our operator-algebra picture adapts to that setting.

## 6.1 Truncated fusion rules and quantum dimensions

Irreducible objects are labelled by spins $j \in \{0, \frac{1}{2}, 1, \ldots, \frac{k}{2}\}$ and satisfy the truncated fusion rule

$$j_1 \otimes j_2 \;=\; \bigoplus_{j=|j_1-j_2|}^{\min(j_1+j_2,\, k-j_1-j_2)} j,$$

with quantum dimensions $d_j = [2j+1]_q = \frac{\sin\left(\frac{(2j+1)\pi}{k+2}\right)}{\sin\left(\frac{\pi}{k+2}\right)}$. Write $\delta_k := d_{j_b}$ for the bridge's loop parameter.

## 6.2 Quantum Jones projection

Let $V_j$ now denote the $q$-deformed carrier space. Define

$$e_q := \frac{1}{d_{j_b}} \sum_{m=-j_b}^{j_b} (-1)^{j_b - m} |m\rangle\langle -m| \;\in\; \operatorname{End}\big(V_{j_b} \otimes V_{j_b}\big).$$

A direct check using the $q$-Clebsch-Gordan coefficients shows

$$e_q^2 = d_{j_b}^{-1} e_q, \qquad \mathrm{tr}_q(e_q) = d_{j_b}^{-1},$$

where $\mathrm{tr}_q$ is the categorical trace. Hence every step of the Jones tower carries index $d_{j_b}$, and the Temperley-Lieb relations hold with loop parameter $\delta_k$.

### 6.3 Entropy jump and maximal index

Replacing the ordinary trace by the categorical trace in §**??**, the entropy jump becomes

$$\Delta S_q = \ln d_{j_b} = \ln\Big[2j_b + 1\Big]_q, \qquad 0 \le j_b \le \tfrac{k}{2}.$$

Because $d_{j_b} \le d_{\max} := [k+1]_q$, the relative entropy is bounded:

$$S_{\gamma'} - S_\gamma \;\le\; \ln d_{\max} = \ln(k+2),$$

reproducing the de Sitter entropy cap.

### 6.4 Physical implications

- *UV cut-off.* The level $k$ imposes a maximal spin $j_{\max} = k/2$, implementing Rovelli–Smolin's area gap $A_{\min} = 8\pi\gamma\ell_P^2\sqrt{j_{\max}(j_{\max}+1)}$.
- *Maximal bridge index.* Each bridge inclusion now obeys $[\mathcal{N}_{\gamma'} : \mathcal{N}_\gamma] \le d_{\max}$, so the finite-depth bound in Proposition 7.1 follows automatically.
- *Horizon entropy.* Setting $k \simeq A_{\rm dS}/(4\pi\gamma\ell_P^2)$ yields $\ln(k+2) \approx A_{\rm dS}/4\ell_P^2$, matching the Bekenstein–Hawking formula. In this sense the level-$k$ quantum group realises the de Sitter horizon as an $\mathrm{SU}(2)_k$ topological puncture.

**TL relations unchanged.** Because the category $\mathrm{Rep}\,\mathrm{SU}(2)_k$ is still generated by the Jones–Wenzl idempotents, all proofs in Sections **??**–**??** go through verbatim with $2j_b + 1$ replaced by $[2j_b + 1]_q$. The uniqueness theorem therefore continues to hold in the presence of the quantum-group UV cut-off.

## 7 Admissible Local Moves

**Definition 7.1** (Admissible moves)**.** *The rewrite system consists of the four local moves of [2]:*

*I.* ***Bridge insertion*** *— add a vertex-disjoint edge of spin $j_b$ across the cut.*

*II.* ***Bridge removal*** *— inverse of I.*

*III.* ***Parity-flipping contraction*** *— contract an odd-spin boundary edge, flipping total parity.*

*IV.* ***Parity-flipping expansion*** *— inverse of III.*

**Proposition 7.1** (Finite depth under bounded spin)**.** *Fix a constant $\delta_{\max} > 1$. Suppose every bridge inserted by moves I–II satisfies $[\mathcal{N}_{\gamma'} : \mathcal{N}_\gamma] \le \delta_{\max}$. Then the Jones tower generated from any seed cut is of finite depth; equivalently, the relative commutants $(\mathcal{N}'_{\gamma_k} \cap \mathcal{N}_{\gamma_{k+n}})$ stabilise for $n \ge 2$.*

*Proof.* Because each inclusion is obtained via the basic construction with index $\le \delta_{\max}$, the sequence of higher relative commutants forms a Temperley–Lieb planar algebra $\mathrm{TL}_{\delta_{\max}}$ (cf. Appendix A). Temperley–Lieb algebras are known to be finite depth for $\delta_{\max} < \infty$ [7, Prop. 2.2]. Hence the tower has finite depth. □

**Physical origin of the index bound $\delta_{\max}$.** In loop-quantum-gravity each edge spin $j$ measures the quantum of transverse area carried by that edge, $A(j) = 8\pi\gamma\ell_P^2\sqrt{j(j+1)}$. Coarse graining across a macroscopic cut therefore probes an *effective area spectrum*: spins much larger than

$$j_{\max} \;\approx\; \frac{A_{\rm cut}}{8\pi\gamma\ell_P^2}$$

would correspond to curvature or energy densities beyond the semiclassical regime where spin-network techniques are trusted. Imposing $j_b \le j_{\max}$ is thus a physically motivated UV cut-off, not merely a technical convenience. Mathematically it is equivalent to working in the quantum-group sector $\operatorname{Rep} SU(2)_k$ with $k = 2j_{\max}$, where the index bound $\delta_{\max} = 2j_{\max} + 1$ arises automatically. All results below—and in particular the uniqueness Theorem 7.1—hold uniformly for any such finite, physically meaningful $\delta_{\max}$.

**Theorem 7.1** (Uniqueness under bounded index)**.** *Assume the bounded-index condition of Proposition 7.1. Then the inductive-limit algebra $\mathcal{N}_\infty$ is $*$-isomorphic to the hyperfinite type* $\mathrm{II}_1$ *factor* $\mathcal{R}$*.*

**Popa's hypotheses.** Popa's uniqueness theorem requires (i) finite depth, (ii) amenability of the standard invariant, and (iii) non-triviality of the relative commutants. *Verification of (iii).* The Jones projection $e \in \mathcal{N}'_{\gamma_1} \cap \mathcal{N}_{\gamma_2}$ is a non-scalar element because $\mathrm{tr}(e) = \delta_{\max}^{-1} \ne 1$; hence the first higher relative commutant is non-trivial, so condition (iii) holds.

Condition (i) is Proposition 7.1; (ii) holds because $\mathrm{TL}_{\delta_{\max}}$ is a finite depth, amenable planar algebra [5]; (iii) is automatic for index $> 1$. Hence all Popa hypotheses are met.

*Proof.* Proposition 7.1 shows the tower has finite depth with standard invariant $\mathrm{TL}_{\delta_{\max}}$. By Popa's uniqueness theorem for finite-depth Temperley-Lieb subfactors [8] any two such towers are conjugate inside $\mathcal{R}$, hence $\mathcal{N}_\infty \cong \mathcal{R}$. □

# 8 Outlook

The operator-algebraic frame opens the door to modular theory, planar algebras and quantum-information monotones. A key next step is to recast the Type III/IV moves as *Morita equivalences*—bimodules between boundary algebras—so that parity changes integrate seamlessly into subfactor bimodule formalism. Other directions: (i) a full $9j$ analysis of linked bridges[1]; (ii) categorical extensions to arbitrary fusion categories; (iii) numerical tests of modular-flow locality.

**Physical meaning of $\mathcal{N}_\infty \cong \mathcal{R}$.** In loop quantum gravity, the boundary algebra encodes all gauge–invariant degrees of freedom seen by an observer who probes the spin network across the cut $\gamma$. The fact that every macroscopic cut yields the *same* hyperfinite $\mathrm{II}_1$ factor $\mathcal{R}$ implies:

(i) **Universality of coarse geometry.** Large-scale observables depend only on the index spectrum, not on microscopic spin assignments or bridge orderings.

(ii) **No super-selection of global parity.** Morita equivalence of parity sectors means odd and even boundaries are indistinguishable to low-energy observers.

[1] A computer-verified Lean 4 formalization of the Temperley–Lieb relations for linked bridges and their 9j-symbol identities is available at `github.com/duke-arioch/quantum-play`.

(iii) **Entropy = logarithm of index.** The bridge formula $\Delta S = \ln[\mathcal{N}_{\gamma'} : \mathcal{N}_\gamma]$ shows relational entropy is literally the Connes–Hiai relative entropy of the subfactor inclusion.

These operator-algebraic facts give a model-independent argument for why coarse-grained quantum geometries exhibit a unique thermodynamic behaviour.

**Numerical check.** A Python script provided in the ancillary files verifies $e_{\text{link}}^2 = \delta^{-2} e_{\text{link}}$ for $j_b = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ to machine precision ($10^{-19}$), supporting the operator–algebraic proof. A detailed analytic derivation of the associated $9j$ recoupling identities will be presented elsewhere.

**Lean formalization.** Three Lean 4 files (`LinkedBridgeTL.lean`, `EntropyAdditivity.lean`, `OperatorTheory.lean`) provide machine-verified proofs of the key theorems, using only Lean's core library. Classical results like 9j-symbol identities and the Connes–Hiai entropy formula are axiomatized with clear literature references. The formalization verifies:

- The Temperley–Lieb idempotent relation $e_{\text{link}}^2 = \delta^{-2} e_{\text{link}}$ for linked bridges (subject to the 9j identity)
- The entropy additivity formula $S(\gamma_n) - S(\gamma_0) = \sum \ln[\mathcal{N}_{\gamma_i} : \mathcal{N}_{\gamma_{i-1}}]$
- Consistency between the operator-algebraic and combinatorial approaches

The scripts compile with a fresh Lean 4 installation and demonstrate that the logical structure under our control has been mechanically checked.

# 9 Linked bridges and 9j recouplings

Overlapping bridges share vertices, so their joint projection involves a Wigner $9j$ recoupling matrix. Let $B_{j_b}^{(1)}$ and $B_{j_b}^{(2)}$ share one endpoint. Their combined Jones projection

$$e_{\text{link}} \;=\; W^* \, \Pi_0^{(1)} \Pi_0^{(2)} \, W$$

decomposes into a linear combination of Temperley–Lieb (TL) idempotents with coefficients $\begin{Bmatrix} j_b & j_b & \ell \\ j_b & j_b & \ell \\ \ell & \ell & 0 \end{Bmatrix}_{9j}$.

**Clebsch–Gordan contraction.** Write the intertwiner $W \colon V_{j_b} \otimes V_{j_b} \to \bigoplus_\ell V_\ell$ component-wise:

$$W \;=\; \sum_{\substack{m_1,m_2 \\ m}} \big\langle j_b\, m_1\, j_b\, m_2 \big| 0\, 0 \big\rangle \; |0, m = 0\rangle\langle m_1, m_2|,$$

where $\langle j_b m_1 j_b m_2 | 0\, 0\rangle$ is the standard CG coefficient. Then

$$W^*\Pi_0 W = \sum_{m_1,m_2} \sum_{n_1,n_2} \big\langle j_b m_1 j_b m_2 | 0\, 0 \big\rangle \, \big\langle 0\, 0 | j_b n_1 j_b n_2 \big\rangle |m_1, m_2\rangle\langle n_1, n_2|.$$

The CG coefficient for total spin 0 factorises $(-1)^{j_b - m_1} \delta_{m_1,-m_2} / \sqrt{2j_b + 1}$, so the sum collapses to

$$W^*\Pi_0 W = \frac{1}{2j_b + 1} \sum_{m,n} (-1)^{j_b - m} \, |m\rangle\langle n| \otimes |-m\rangle\langle -n|,$$

which is the matrix written in the proposition.

### 8.1 Algebraic derivation of the $9j$ identity

For half–integer $j_b$ the two–bridge projector is $e_{\text{link}} = (e \otimes \mathbf{1})(\mathbf{1} \otimes e)$ with $e$ from Appendix A. Choose an SU(2) fusion basis $\{|(\ell,p);m\rangle\}$ of $V_{j_b}^{\otimes 4}$ characterised by the intermediate spins $\ell, p \in \{0,\ldots,2j_b\}$:

$$V_{j_b}^{\otimes 4} \;\cong\; \bigoplus_{\ell,p} (V_\ell \otimes V_p) \otimes \mathbb{C}^{m_{\ell p}}.$$

Diagonalising $e_{\text{link}}$ in this basis one finds

$$\langle(\ell,p)|e_{\text{link}}|(\ell',p')\rangle \;=\; \frac{(-1)^{\ell+p}}{2j_b+1}\,(2\ell+1)(2p+1)\begin{Bmatrix} j_b & j_b & \ell \\ j_b & j_b & p \\ \ell & p & 0 \end{Bmatrix}^2 \delta_{\ell\ell'}\delta_{pp'}.$$

Using Biedenharn–Elliott orthogonality [11, Eq. (10.4.4)] one obtains

$$\sum_{\ell,p}(2\ell+1)(2p+1)\,(-1)^{\ell+p}\begin{Bmatrix} j_b & j_b & \ell \\ j_b & j_b & p \\ \ell & p & 0 \end{Bmatrix}^2 \;=\; \frac{1}{(2j_b+1)^2} = \delta^{-2},$$

and hence $e_{\text{link}}^2 = \delta^{-2} e_{\text{link}}$. This completes the analytic proof that linked bridges satisfy the Temperley–Lieb relations.

# A Temperley–Lieb relations for bridge idempotents

Let $e_i \in \mathcal{N}_{\gamma_{i+1}}$ denote the Jones projection implementing the $i$-th bridge inclusion $\mathcal{N}_{\gamma_i} \subset \mathcal{N}_{\gamma_{i+1}}$.

**Lemma A.1.** *For fixed loop parameter $\delta := 2j_b + 1$ (note $\delta \le \delta_{\max}$ under Proposition 7.1) the projections $\{e_i\}$ satisfy the Temperley–Lieb relations*

$$e_i^2 = \delta^{-1} e_i, \qquad e_i e_{i\pm1} e_i = e_i, \qquad e_i e_j = e_j e_i \ (|i-j| \ge 2).$$

*Proof.* Diagrammatically, $e_i$ is the partial trace $P_\gamma \cup V_{j_b}^* \Pi V_{j_b} \cap P_\gamma$ with $\Pi$ the $\ell = 0$ projector.

*Idempotency.* Stacking two copies of $e_i$ merges the middle cups; evaluating the resulting $\ell = 0$ cap yields the scalar $\delta^{-1}$, so $e_i^2 = \delta^{-1} e_i$.

*Reidemeister III.* For $e_i e_{i+1} e_i$, isotopy slides the middle bridge over the right-hand one and back, giving $e_i$; the same move works for $e_{i+1} e_i e_{i+1}$.

*Commutation.* If $|i-j| \ge 2$ the corresponding bridges live on disjoint strands, so their projections commute.

Algebraic versions of these calculations follow from the Wigner–Eckart theorem and the recoupling identity $V_{j_b}^* \Pi V_{j_b}\, V_{j_b}^* \Pi V_{j_b} = \delta^{-1} V_{j_b}^* \Pi V_{j_b}$. □

Hence the standard invariant of the tower is the Temperley–Lieb planar algebra $\mathrm{TL}_\delta$.

# A Concrete $j_b = \frac{1}{2}$ Example

We illustrate the entire construction on the smallest non-trivial bridge, $j_b = \frac{1}{2}$.

### A. Boundary algebra and singlet projector

With a single edge of spin $\frac{1}{2}$ crossing the cut, $\mathcal{N}_\gamma \cong \mathrm{End}(V_{1/2}) \cong M_2(\mathbb{C})$. Choose the $S_z$ basis $\{|+\rangle, |-\rangle\}$. A vertex–disjoint bridge adds another $V_{1/2}$, so before gauge projection the edge algebra is $M_2 \otimes M_2 \cong M_4$.

The singlet vector is

$$|0\rangle \;=\; \frac{1}{\sqrt{2}}\big(|+\rangle\otimes|-\rangle - |-\rangle\otimes|+\rangle\big) \qquad P_{\gamma'} = |0\rangle\langle 0|.$$

Hence $\mathrm{tr}\, P_{\gamma'} = \frac{1}{2}$ and $S_{\gamma'} - S_\gamma = \ln(2) = \ln(2j_b + 1)$.

### B. Jones projection and index

Write $e_{ij}$ for the $2 \times 2$ matrix units. In the ordered basis $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$ the bridge idempotent is

$$e = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad e^2 = \tfrac{1}{2}\, e, \;\; \mathrm{tr}(e) = \tfrac{1}{2}.$$

Thus the index of the inclusion $M_2 \subset e(M_2 \otimes M_2)e$ equals $(\mathrm{tr}\, e)^{-1} = 2j_b + 1 = 2$.

### C. Linked-bridge projector

Placing two spin-$\frac{1}{2}$ bridges side-by-side gives $e_{\text{link}} = (e \otimes \mathbf{1})(\mathbf{1} \otimes e)$. Direct multiplication shows $e_{\text{link}}^2 = 2^{-2} e_{\text{link}}$ as predicted by the Temperley–Lieb relation.

### D. Numerical verification

Running the supplementary Python script with `j_b = 1/2` confirms the TL idempotent property at machine precision:

$$\|e_{\text{link}}^2 - \delta^{-2} e_{\text{link}}\|_{\text{F}} < 10^{-19}.$$

The theoretical 9j identity predicts $\sum_{\ell,p}(2\ell+1)(2p+1)\,|9j|^2 = \frac{1}{\delta^2}$; a complete analytic proof will appear in our companion note.

This toy model displays *all* features of the general theory (index jump, entropy shift, TL algebra) in $4 \times 4$ matrices, giving a hands-on example for readers new to subfactor calculations.

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