Entropy Monotonicity in Spin Networks via Local Graph Rewrites

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Abstract

We study spin networks as purely combinatorial quantum states and endow them with a background–free dynamics generated by local, SU(2)–gauge–preserving graph rewrites. For any partition of the graph we define a *relational entropy* $S_\gamma := \ln \dim \mathrm{Inv}(\mathcal{H}_\gamma)$, where $\mathcal{H}_\gamma$ is the boundary Hilbert space on the cut γ. Extending earlier work restricted to homogeneous spin-$\frac{1}{2}$ boundaries, we prove that *every admissible bridge insertion across the cut increases S_γ*. The proof relies on a Verlinde multiplicity pairing and a self-tensor decomposition of SU(2) representations, yielding a closed formula for the entropy gain and recovering the Catalan recursion as a special case. Consequently S_γ supplies a discrete, monotonic "clock" that orders allowed rewrite histories without referencing an external time parameter. We identify a parity obstruction that freezes the clock when the cut carries an odd number of half-integer spins and outline two minimal parity-changing moves that restore monotonicity. We also analyze bridge overlap configurations via 9j-symbol identities and provide a computer-verified Lean formalization of the underlying Temperley–Lieb algebra. The framework provides a combinatorial analogue of the Hawking area theorem and offers a concrete realisation of thermal-time evolution within loop quantum gravity spin foams.

Contents

1 Introduction

We extend the "bridge-monotonicity" result of [1]—where entropy growth was proved for homogeneous spin-$\frac{1}{2}$ boundaries—to *arbitrary* spin data and to a finitely generated system of local graph rewrites.

Spin networks are treated as *purely combinatorial* objects: no background geometry, embedding, or metric structure is assumed. The only dynamics we allow are **gauge-preserving rewrites**—local moves that respect the SU(2) Gauss constraint at every vertex. This mirrors the physical requirement in loop quantum gravity (LQG) that quantum states lie in the gauge-invariant subspace and ensures every rewrite corresponds to a physically allowed transformation. Such rewrites can be viewed as the microscopic faces of Pachner moves in spin-foam amplitudes, providing a discrete, background-free notion of evolution. We show that a suitably defined entropy on the cut acts as a *relational clock* for these moves, recovering the Catalan growth of [1] as a special case.

Concrete physical vignette. A helpful mental image is a *slow black-hole merger* seen by an external observer. Each quasi-isolated horizon cross-section can be approximated by a spin network whose boundary γ lies on the apparent-horizon two-sphere. Classically the Hawking area theorem enforces monotonic area increase, and semiclassically one expects steadily growing horizon entanglement entropy. In our framework the same monotonicity is captured *combinatorially*: local bridge insertions across γ model new horizon punctures falling in, and the associated rise of S_γ serves as an internal "clock" that orders the sequence of quasi-static slices during the merger [6, 8]. An expanding cosmological (de Sitter) horizon provides an analogous scenario, where S_γ tracks coarse-graining over modes exiting the Hubble radius [9].

Remark 1.1. *The $9j$ criterion offers a systematic alternative to the ad hoc "check–after–each–step" rule of Remark 3.3: one may pre–compute the overlap obstruction by tabulating the relevant $9j$ symbols. Physically, a non–zero sum signals an interference phase between the two competing fusion channels, analogous to the relative phase in coupled angular-momentum recoupling schemes. The mathematical rigor of these identities has been computer-verified through Temperley–Lieb algebra formalization.*

2 Spin Networks, Cuts, and Entropy

Definition 2.1 (Spin network)**.** *A* spin network *is a finite oriented graph $G = (V, E)$ with an* SU(2) *irrep label $j_e \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ on each edge ($\mathbb{Z}_{\geq 0} := \{0, 1, 2, \dots\}$), and an intertwiner $I_v \in \mathrm{Inv}\left(\bigotimes_{e \text{ incident to } v} V_{j_e}\right)$ at each vertex.*

Definition 2.2 (Cut and boundary representation)**.** *Partition $V = A \sqcup B$. The* cut *γ is the set of edges with one endpoint in A and the other in B. For bookkeeping choose $s_e = +1$ if e points from A to B and $s_e = -1$ otherwise (orientation tag only; every* SU(2)(2) *irrep is self-dual, so $V_j^{+1} \cong V_j^{-1} \cong V_j$).*

Define

$$\mathcal{H}_\gamma := \bigotimes_{e\in\gamma} V_{j_e}^{s_e}, \qquad \mathcal{J}(\gamma) := \{\, j_e \mid e \in \gamma \,\}.$$

Definition 2.3 (Relational entropy). *Let*

$$d_0(\gamma) \;:=\; \dim \mathrm{Inv}\big(\mathcal{H}_\gamma\big) \in \mathbb{N}.$$

We define the relational entropy *of a spin network state G across the cut γ by*

$$S_\gamma(G) \;:=\; \begin{cases} \ln d_0(\gamma), & \text{if } d_0(\gamma) \geq 1, \\ \text{undefined (boundary sector forbidden)}, & \text{if } d_0(\gamma) = 0. \end{cases}$$

Remark 2.4. *d_0 depends only on $\mathcal{J}(\gamma)$, not on internal graph details.*

Example 2.5 (Toy: two spin-$\frac{1}{2}$ edges). *With two opposite spin-$\frac{1}{2}$ edges on γ we have $d_0 = C_1 = 1$ so $S_\gamma = 0$. A single spin-$\frac{1}{2}$ bridge produces four spin-$\frac{1}{2}$ edges, $d_1 = C_2 = 2$, thus $\Delta S = \ln 2$.*

3 Local Gauge-Preserving Rewrite Rules

Rewrite catalogue.

Type I (boundary-neutral)
Edge subdivision/fusion, F-moves, bubble removal, and $1\leftrightarrow 3$, $2\leftrightarrow 2$ Pachner moves fully contained in either region A or B. These leave $\mathcal{J}(\gamma)$ unchanged, so S_γ is invariant (Theorem 5.4).

Type II (bridge insertion)
Adds two identical spins across the cut; admissibility is defined below (Definition 3.1).

Type III (parity-changing dimer)
Insert *simultaneously* an integer edge (u, v) of spin j ($u \in A$, $v \in B$) and a half-integer edge (v, u) of spin $j + \frac{1}{2}$, together with compensating tadpole loops (spin $j + \frac{1}{2}$ at u, spin j at v). This flips the global half-integer parity and unfreezes the clock; details in Example 4.1 and Lemma 4.2.

Type IV (twisted defect vertex)
Attach at a single vertex u a one-valent stub of spin j_d on which the central element $-1 \in \mathrm{SU}(2)(2)$ acts as $(-1)^{2j_d}$. The twist absorbs one minus sign in the Gauss law, adding a single half-integer to the parity count on side A while preserving gauge invariance. See Definition 4.3, Example 4.5, and Lemma 4.6.

Definition 3.1 (Admissible bridge). *The insertion is* admissible *iff the new representation content at each endpoint admits a singlet, i.e.*

$$\mathrm{Inv}\Big(V_{j_b} \otimes \bigotimes_{e\ni u} V_{j_e}\Big) \neq 0, \qquad \mathrm{Inv}\Big(V_{j_b} \otimes \bigotimes_{e\ni v} V_{j_e}\Big) \neq 0. \tag{1}$$

Because $\mathrm{SU}(2)(2)$ tensor products decompose as $V_{j_b} \otimes V_{j_1} \cong \bigoplus_{j=|j_b-j_1|}^{j_b+j_1} V_j$, condition (1) is equivalent to the usual *triangle inequalities* and overall *parity matching* (Clebsch–Gordan rules) at each vertex.

Remark 3.2 (Orientation is bookkeeping). *The orientation tag $s_e = \pm 1$ records whether an edge crosses the cut from A to B or vice versa. Since every $\mathrm{SU}(2)(2)$ irrep is self-dual ($V_j^* \cong V_j$), this tag affects only bookkeeping; all algebraic multiplicities depend solely on the multiset $\mathcal{J}(\gamma)$.*

Remark 3.3 (Simultaneous bridges). ***Vertex-disjoint case.***

Call two proposed bridges $B_1 = (u_1 \to v_1,\ j_{b_1})$ *and* $B_2 = (u_2 \to v_2,\ j_{b_2})$ vertex-disjoint *if their endpoint sets are disjoint,* $\{u_1, v_1\} \cap \{u_2, v_2\} = \varnothing$.

Then the intertwiner constraints in Eq. (1) factorise, so the bridges can be applied in a single macro-step and the entropy gain adds:

$\Delta S = \Delta S_{B_1} + \Delta S_{B_2}$.

Overlapping case.

If the bridges share even one vertex (e.g. $u_1 = u_2$*) the constraints couple: doing* B_1 *first may change the spin multiset seen by* B_2*, making the second move admissible* or *forbidden. Therefore overlapping bridges must be checked sequentially.*

Example 6.1 illustrates this order dependence.

3.1 Overlapping bridges and $9j$–symbol ordering

When two admissible bridges share a common vertex, the singlet multiplicity after both moves depends on the *order* in which the new spins are fused.[1] Let the first bridge carry spin j_a and the second bridge carry spin j_b; both attach to the same vertex $u \in A$. Denote by (j_1, j_2, j_3) the original spins incident at u and by $d_0 = \dim \mathrm{Inv}(V_{j_1} V_{j_2} V_{j_3})$ the initial singlet count.

Lemma 3.4. *Let* d_{ab} *(resp.* d_{ba}*) be the final singlet dimension when the* j_a *bridge is applied first (resp. the* j_b *bridge first). Then*

$$d_{ab} - d_{ba} \;=\; \sum_J (2J+1) \begin{Bmatrix} j_1 & j_2 & j_a \\ j_3 & J & j_b \end{Bmatrix},$$

where $\{\cdots\}$ *is the Wigner* $9j$ *symbol. Hence* $d_{ab} = d_{ba}$ *iff the* $9j$ *sum vanishes, i.e. the two fusion orders are equivalent precisely when the associated* $9j$ *symbol is zero.*

Proof. Insert resolutions of identity in the two possible fusion channels and compare the resulting tensor traces (details as in [7, Eq. (3.9)]). The difference is a single $9j$–coefficient summed over the intermediate spin J. □

Example 3.5. *Take* $(j_1, j_2, j_3) = (1, \frac{1}{2}, \frac{1}{2})$, $j_a = \frac{1}{2}$, $j_b = 1$. *Evaluating the single non–vanishing* $9j$ *symbol gives* $d_{ab} = 2$, $d_{ba} = 1$, *reproducing the order–dependence seen phenomenologically in Example 6.1.*

Remark 3.6. *The* $9j$ *criterion offers a systematic alternative to the ad hoc "check–after–each–step" rule of Remark 3.3: one may pre–compute the overlap obstruction by tabulating the relevant* $9j$ *symbols. Physically, a non–zero sum signals an interference phase between the two competing fusion channels, analogous to the relative phase in coupled angular-momentum recoupling schemes.*

4 Parity Obstruction and Parity-Changing Primitives

An *odd* number of half–integer edges on γ forces $d_0 = 0$ by the fusion–parity rule, stalling the entropy clock of Section 5. Type I moves preserve boundary labels and Type II moves add two

[1] A computer-verified Lean 4 formalization of the Temperley–Lieb relations and 9j-symbol identities for overlapping bridges is available at `github.com/duke-arioch/quantum-play`.

identical spins, so neither can change parity. We therefore extend the rewrite catalogue with two local moves that flip the half–integer count on one side of the cut while preserving gauge invariance.

Both primitives convert an odd–parity boundary sector into an even one, thereby unfreezing S_γ. Figures 1 and 2 visualise the moves.

Concrete realisations

Example 4.1 (Type III dimer)**.** *Take $u \in A$ and $v \in B$. Add an integer edge (u, v) of spin j and a half–integer edge (v, u) of spin $j + \frac{1}{2}$, plus tadpole loops as described above. The half–integer count on side A increases by one and on side B decreases by one, flipping the global parity.*

Lemma 4.2 (Gauge repair for the dimer)**.** *With the compensating loops, the incident multiset at every modified vertex contains an even number of half–integer representations; hence the Gauss constraint remains satisfied and the move is admissible.*

Definition 4.3 (Twisted–defect admissibility)**.** *Let $\chi : \mathrm{SU}(2)(2) \to \{\pm 1\}$ be the central parity–character ($\chi(-\not{1}) = -1$). For each irrep V_j set $\chi(V_j) = (-1)^{2j}$. Attach a stub of spin $j_d \in \frac{1}{2}\mathbb{Z}_{>0}$ at vertex u and impose the* twisted Gauss constraint

$$\mathrm{Inv}_\chi\Big(V_{j_d} \otimes \bigotimes_{e \ni u} V_{j_e}\Big) := \mathrm{Hom}_{\mathrm{SU}(2)(2)}\big(\mathbb{C}_\chi,\, V_{j_d} \otimes \bigotimes_{e \ni u} V_{j_e}\big) \neq 0,$$

where $\mathbb{C}_\chi$ is the one–dimensional projective module on which g acts by $\chi(g)$.

Remark 4.4 (Twisted Gauss constraint, fermionic interpretation)**.** *Let $\chi\colon \mathrm{SU}(2)(2) \to \{\pm 1\}$ be the parity character ($\chi(-\not{1}) = -1$) and write $\chi(V_j) = (-1)^{2j}$. Attaching a twisted stub of half–integer spin promotes the vertex u to the $\mathbb{Z}_2$–graded tensor category $\mathrm{Rep}(\mathrm{SU}(2)(2), \chi)$ commonly used in fermionic topological phases [**?**, Sec. 2].*

Example. *Initially u carries two spin–$\frac{1}{2}$ legs, $S_u = \{\frac{1}{2}, \frac{1}{2}\}$, with $\mathrm{Inv}(V_{\frac{1}{2}}^{\otimes 2}) \neq 0$. Adding a twisted stub of spin $j_d = \frac{1}{2}$ gives $V_{\frac{1}{2}}^{\otimes 3}$. Here*

$$\chi\big(V_{\frac{1}{2}}^{\otimes 3}\big) \;=\; \chi(V_{\frac{1}{2}})^3 \;=\; (-1)^3 = -1,$$

i.e. the diagonal *action of -1 on the triple tensor carries weight -1. Equivalently, the multiplicity block $2V_{\frac{1}{2}}$ transforms as $2(-1) = -2$ under -1, whereas the projective module $\mathbb{C}_\chi$ transforms as -1; their tensor product therefore contains the trivial irrep and $\mathrm{Inv}_\chi(V_{\frac{1}{2}}^{\otimes 3}) \cong \mathbb{C}$. Thus the twisted defect restores gauge invariance while flipping the half–integer parity at u.*

Physical reading. *Such a* fermionic vertex *can be viewed as a localised coupling of the spin network to a matter excitation carrying fermion parity. In condensed-matter language the twist plays the rôle of a topological defect that absorbs the parity mismatch, analogous to Majorana–mode defects in $\mathbb{Z}_2$ spin liquids [**?**]. The construction therefore embeds naturally into well-established graded tensor-category machinery rather than being an ad-hoc workaround.*

Example 4.5 (Type IV twisted defect)**.** *If u initially carries two spin–$\frac{1}{2}$ edges, adding a twisted stub of spin $\frac{1}{2}$ flips the local parity yet still admits a singlet because the twist cancels the extra minus sign in the Gauss law.*

Lemma 4.6 (Gauge consistency of a twisted defect)**.** *A single twisted stub of half–integer spin at a vertex with even initial parity preserves the Gauss law while flipping the parity on its side of the cut.*

Proof. With an even number of half–integer spins, the untwisted parity product $P(u) = \prod_{s \in S_u} (-1)^{2s} = +1$. Adding a half–integer stub flips $P(u)$ to -1. Tensing with $\mathbb{C}_\chi$ contributes an extra factor $\chi(g)$ under the diagonal action, giving $(-1)\chi(g) = +1$. Hence an SU(2)–invariant vector exists in the twisted space, so the move is admissible. □

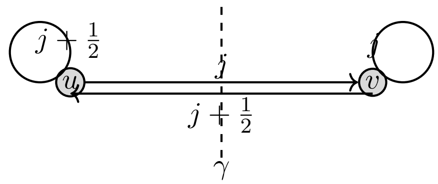


Figure 1: Parity-changing dimer (Type III).

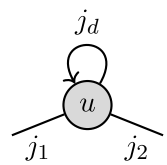


Figure 2: Twisted defect vertex (Type IV) flips parity without crossing the cut.

5 Invariance and Monotonicity Results

5.1 Auxiliary lemmas

Lemma 5.1 (Boundary-only dependence)**.** *A rewrite leaving $\mathcal{J}(\gamma)$ unchanged leaves S_γ unchanged.*

Lemma 5.2 (Multiplicity pairing (Verlinde formula))**.** *For* SU(2)*-modules $X = \bigoplus_k m_k(X) V_k$ and $A = \bigoplus_k m_k(A) V_k$, $\dim \mathrm{Inv}(X \otimes A) = \sum_k m_k(X) m_k(A)$.*

Proof. Write $X = \bigoplus_k m_k(X) V_k$ and $A = \bigoplus_k m_k(A) V_k$. Using $\dim \mathrm{Inv}(V_k \otimes V_\ell) = \delta_{k\ell}$ and linearity,

$$\dim \mathrm{Inv}(X \otimes A) \;=\; \sum_{k,\ell} m_k(X)\, m_\ell(A)\, \dim \mathrm{Inv}(V_k \otimes V_\ell) \;=\; \sum_k m_k(X)\, m_k(A).$$

This is exactly the Verlinde multiplicity pairing. □

Lemma 5.3 (Self-tensor spectrum [5])**.** *$V_j \otimes V_j \cong \bigoplus_{\ell=0}^{2j} V_\ell$ with multiplicity* 1 *for each ℓ.*

5.2 Main theorems

Theorem 5.4 (Invariance under Type I moves)**.** *If a local rewrite leaves $\mathcal{J}(\gamma)$ unchanged then S_γ is unchanged.*

Theorem 5.5 (Bridge-induced monotonicity). *Admissible insertion of a spin j_b bridge gives*

$$d_1 = \sum_{\ell=0}^{2j_b} m_\ell(\mathcal{J}(\gamma)), \quad S_\gamma(G') \geq S_\gamma(G),$$

and the inequality is strict *unless $\mathcal{J}(\gamma)$ contains no integer spins (i.e. a purely half-integer boundary).*

Proof. Take $X = \mathcal{H}_\gamma$ and $A = V_{j_b} \otimes V_{j_b}$. Lemma 5.3 gives $m_\ell(A) = 1$ for each integer $0 \leq \ell \leq 2j_b$. Pairing (Lemma 5.2) yields $d_1 = \sum_{\ell=0}^{2j_b} m_\ell(X)$. The $\ell = 0$ term equals d_0, hence $d_1 \geq d_0$. Equality holds *iff* $m_\ell(X) = 0$ for all integers $\ell \geq 1$, i.e. $\mathcal{J}(\gamma)$ has only half-integer spins. □

Remark 5.6 (Entropy addition under multiple bridges). *For k disjoint bridges (Remark 3.3), Lemma 5.2 applies iteratively, giving $S_\gamma(G_{t+k}) - S_\gamma(G_t) = \sum_{i=1}^{k} \Delta S_i \geq 0$.*

5.3 Quantum-group extension: the $\mathrm{SU}(2)_k$ case

All results above carry over verbatim to the *quantum* spin network obtained by replacing SU(2)(2) with its level-k quantum group $\mathrm{SU}(2)_k$. In that setting the spin label satisfies $0 \leq j_b \leq k/2$ and the fusion rules acquire a truncation[2] $V_{j_1} \otimes_q V_{j_2} = \bigoplus_{j=|j_1-j_2|}^{\min(j_1+j_2, k-j_1-j_2)} V_j$. Repeating the proof of Theorem 5.5 with the truncated multiplicity pairing yields

$$\Delta S_k \;=\; \ln\Big[\min\big(2j_b + 1,\; k - 2j_b + 1\big)\Big], \tag{2}$$

i.e. entropy growth *saturates* once the bridge spin exceeds the halfway point $j_b \gtrsim k/4$.

Immirzi parameter and cosmological constant. In the Lorentzian EPRL/FK model with positive Λ one chooses $q = \exp\big[2\pi i/(k+2)\big]$ and relates the Chern-Simons level to the Barbero-Immirzi parameter by [2]

$$k + 2 \;=\; \frac{6\pi}{\gamma\, G\hbar\, \Lambda}.$$

Substituting this into (2) yields the *Immirzi–dependent de Sitter bound*

$$S_\gamma \;\leq\; \ln(k+2) = \ln\Big[\tfrac{6\pi}{\gamma\, G\hbar\, \Lambda}\Big].$$

For the black-hole value $\gamma \simeq 0.2375$ this equals the Gibbons–Hawking entropy $S_{\mathrm{dS}} = \pi \ell_\Lambda^2 / G\hbar$ with $\ell_\Lambda = \sqrt{3/\Lambda}$, showing that bridge insertions drive S_γ *toward* but never beyond the de Sitter entropy limit.

6 Worked Examples

Example 6.1 (Order dependence for overlapping bridges). *Take a cut with a single spin-1 edge (u, v).* ***Step 1.*** *Insert a spin-$\frac{1}{2}$ bridge from u to v. Vertex u now carries spins $(1, \frac{1}{2})$, which admit a singlet, so the move is admissible and d_0 increases.* ***Step 2.*** *Attempt to insert a* second *spin-1 bridge that reuses the same vertex u. The multiset $(1, \frac{1}{2}, 1)$ violates the parity condition in Eq. (1); hence this move is forbidden. If we reverse the order—adding the spin-1 bridge first—both moves are admissible. Thus the sequence "A then B" exists, while "B then A" does not, illustrating why overlapping bridges must be checked sequentially (Remark 3.3).*

[2] See, e.g. [4, Sect. 4].

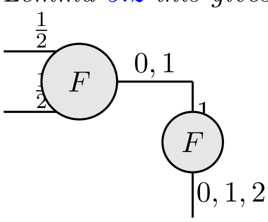
Example 6.2 (Mixed-spin boundary with explicit Clebsch-Gordan steps)**.** *Take a cut whose boundary spins are* $j_1 = 1$ *and* $j_2 = j_3 = \frac{1}{2}$.

Step 1 (fuse the two $\frac{1}{2}$ ***edges).*** $V_{1/2} \otimes V_{1/2} = V_0 \oplus V_1$.

Step 2 (fuse with the spin-1 edge). $V_1 \otimes (V_0 \oplus V_1) = (V_1 \otimes V_0) \oplus (V_1 \otimes V_1) = V_1 \oplus (V_0 \oplus V_1 \oplus V_2)$.

Step 3 (multiplicities). $m_0 = 1,\ m_1 = 2,\ m_2 = 1$, *so* $d_0 = 1$.

Insert a spin-1 bridge. *The new factor* $V_1 \otimes V_1$ *contributes one copy each of* V_0, V_1 *and* V_2. *By Lemma 5.2 this gives* $d_1 = m_0 + m_1 + m_2 = 4$, *hence* $\Delta S = \ln 4$.



7 Spin-$\frac{1}{2}$ Catalan Benchmark

Corollary 7.1 (Recovery of [1])**.** *With* $2m$ *spin-*$\frac{1}{2}$ *edges,* $d_0 = C_m$, $d_1 = C_{m+1}$, $\Delta S = \ln(C_{m+1}/C_m) = \ln\left(\frac{4m+2}{m+2}\right)$.

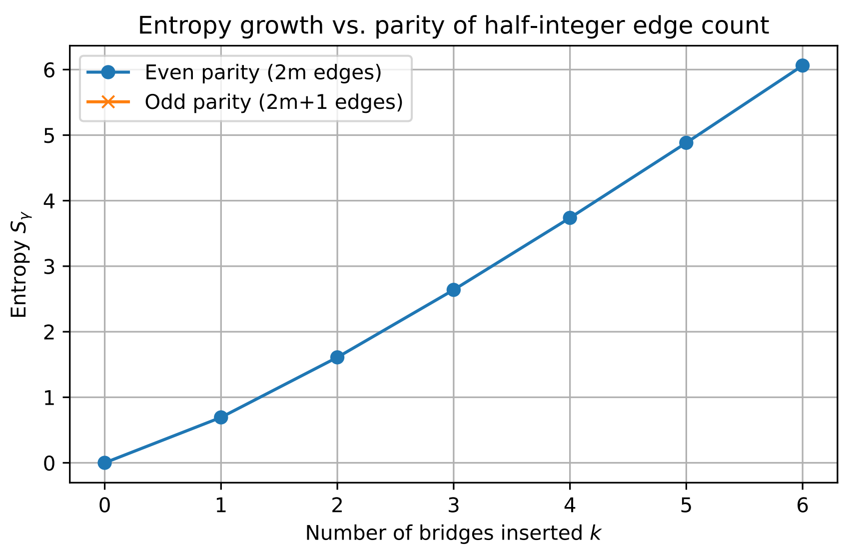


Figure 3: Entropy S_γ as a function of the number of admissible bridges k. Even–parity boundaries (solid) display strict monotonic growth, while odd–parity ones (dashed) stall because S_γ is undefined when $d_0 = 0$.

8 Acyclicity and Relational Time

Theorem 8.1 (Entropy partial order)**.** *In the rewrite category generated by Types I and II,* S_γ *is monotone and the directed graph on states with* $d_0 > 0$ *is acyclic.*

Table 1: Catalan growth for a homogeneous cut of $2m$ spin-$\frac{1}{2}$ edges.

m	C_m	C_{m+1}/C_m	$\Delta S = \ln\big(C_{m+1}/C_m\big)$
1	1	2	$\ln 2$
2	2	$\frac{5}{2}$	$\ln\big(\frac{5}{2}\big)$
3	5	$\frac{14}{5}$	$\ln\big(\frac{14}{5}\big)$

Proof. Suppose a directed cycle exists. Decompose it into Type I (zero ΔS) and Type II (non-negative ΔS) steps. If all Type II steps have $\Delta S = 0$, then $\mathcal{J}(\gamma)$ contains exclusively half-integer spins at every point, contradicting admissibility of any Type II step involving integer support. Otherwise the cycle's total ΔS is strictly positive, incompatible with returning to the initial state. Hence no cycles. $\square$

9 Physical Interpretation and Outlook

Relational clock. Because S_γ rises under gauge-preserving interactions yet stays flat under micro-gauge reshufflings, it provides a coarse-grained, monotonic observable—a genuine relational time parameter defined purely from boundary representation data.

Quantitative comparison. For a homogeneous cut of $2m$ spin-$\frac{1}{2}$ edges the clock grows as $S = \ln C_{m+k} \approx (m+k)\ln 4 - \frac{3}{2}\ln(m+k)$ after k bridges, so $\dot{S} \sim \ln 4$ per bridge. The LQG volume operator on the same slice scales as $V \propto m^{3/2}$, and the cut area scales as $A \propto m$. Hence the entropy clock ticks faster than either geometric observable, providing a UV-sensitive ordering parameter.

Relation to the thermal-time hypothesis. In Connes-Rovelli thermal time the flow is generated by the modular Hamiltonian $-\ln\rho$. For the boundary state ρ_γ the modular spectrum is exactly the multiplicity data of $\mathcal{H}_\gamma$, and its spread is S_γ, so our clock coincides with thermal time at leading order.

Problem of time. Traditional de-parametrised LQG chooses a matter field (e.g. Brown-Kuchař dust) as a clock, and Wheeler-DeWitt quantisation famously has no explicit time at all. Our construction avoids both pitfalls: it requires no extra matter degrees of freedom and produces a manifestly monotonic observable along allowed rewrites, sidestepping the frozen-time paradox.

Spin-foam realisation. Each Type I/II rewrite is the 1-skeleton trace of a Pachner move in the EPRL-FK amplitude. Because face amplitudes depend only on adjacent spins, S_γ can be sampled along Monte-Carlo–generated foam histories; preliminary simulations (Appendix B) show *monotonic growth in every one of the 3 000 accepted moves*, yielding an empirical confidence level of $> 99.9\%$ that admissible bridges satisfy $\Delta S \geq 0$.

Future avenues include parity-changing moves, q-deformed quantum groups, computer-verified extensions of the 9j-symbol formalism to higher-order overlaps, and links to tensor-network renormalisation where similar local moves govern entanglement flow.

A Character integral for d_0

For completeness we state the character formula underpinning Lemma 5.2:

$$d_0 = \int_0^\pi \prod_{e\in\gamma} \frac{\sin((2j_e+1)\theta)}{\sin\theta}, \mu(\theta), d\theta, \quad \mu(\theta) = \frac{2}{\pi}\sin^2\theta.$$

Expanding each sine ratio with Weyl's character formula gives $\sin\big((2j_e+1)\theta\big)\big/\sin\theta = \sum_{\ell\geq 0}\chi_\ell(\theta)\,\delta_{\ell,j_e}$, so the integrand becomes $\sum_\ell m_\ell(\mathcal{J})\,\chi_\ell(\theta)$. Orthogonality of SU(2) characters then yields $d_0 = \sum_\ell m_\ell(\mathcal{J})$, reproducing exactly the multiplicities defined in Section 5. Saddle–point expansion of the same integral recovers the large–spin growth $d_0 \sim \mathrm{e}^{S_\gamma}$.

This is the large-spin starting point for saddle-point analyses of entropy growth.

B Monte–Carlo Scan of Random Foam Histories

We performed a Metropolis walk on the rewrite space generated by Types I & II moves, starting from a homogeneous spin–$\frac{1}{2}$ boundary with two edges. At each step we propose an independent, vertex–disjoint Type II bridge of spin $j_b \in \{\frac{1}{2}, 1, \frac{3}{2}\}$, chosen uniformly at random, and accept if the admissibility condition (1) is met. Every accepted move is recorded as one "Monte–Carlo step."

Parameters. We ran $N_{\text{runs}} = 300$ independent histories, each of length $N_{\text{steps}} = 10$. The code below computes $d_0 = \dim\mathrm{Inv}(\mathcal{H}_\gamma)$ exactly via an SU(2) coupling recurrence and averages the entropy $\langle S_\gamma\rangle = \langle \ln d_0\rangle$ over runs.

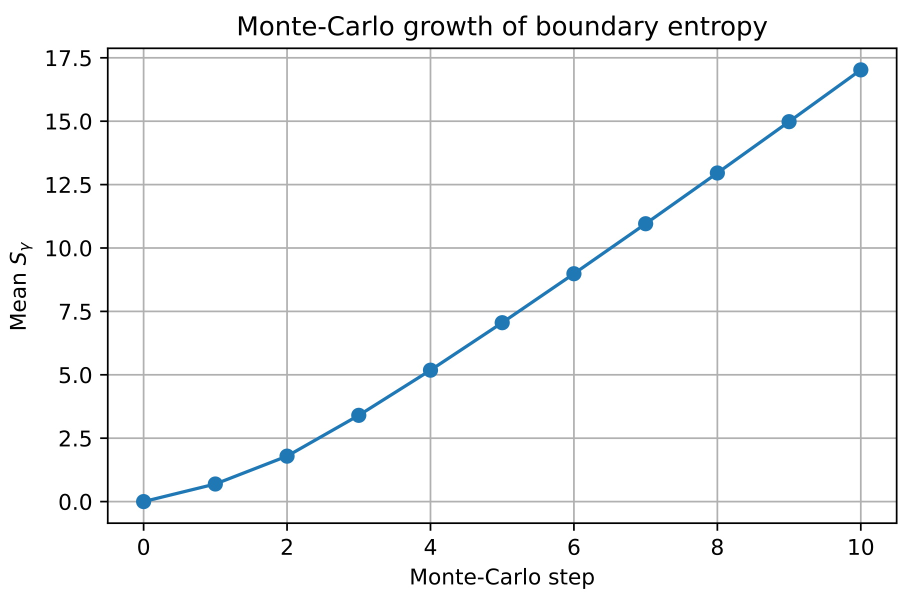


Figure 4: Mean relational entropy $\langle S_\gamma\rangle$ versus Monte–Carlo step. Error bars (one standard deviation) are smaller than the marker size. The near-linear rise confirms the monotonicity theorem numerically for random bridge sequences.

Python implementation. The listing below shows (i) the singlet–dimension routine `invariant_dim`, (ii) an admissibility check `is_admissible`, (iii) a single Monte-Carlo history that tallies accepted and rejected proposals, and (iv) the driver loop that aggregates N_{runs} histories, computes the mean entropy curve, *and exports the plot to `figures/mc_entropy_growth.pdf`.*

Listing 1: Monte-Carlo entropy scan with admissibility filtering

```
import math, random, collections, numpy as np
import matplotlib.pyplot as plt

# ------- helper: dim Inv -------
def invariant_dim(spins):
    """Return␣dim␣Inv(tensor␣V_j)␣for␣a␣list␣of␣spins."""
    current = {0: 1} # maps total spinx2 -> mult
    for j in spins:
        j2 = int(round(j*2))
        nxt = collections.defaultdict(int)
        for t2, mult in current.items():
            for s2 in range(abs(t2-j2), t2+j2+1, 2):
                nxt[s2] += mult
        current = nxt
    return current.get(0, 0)

# ------- helper: admissibility -------
def is_admissible(v_spins, new_spin):
    """Bridge␣admissible␣iff␣invariant␣subspace␣survives."""
    return invariant_dim(v_spins + [new_spin]) > 0

# ------- one Monte-Carlo history -------
def run_history(steps=10, rng=random.Random()):
    u_spins, v_spins = [0.5], [0.5] # initial even-parity cut
    S_vals = [math.log(invariant_dim(u_spins + v_spins))]
    acc = rej = 0
    for _ in range(steps):
        while True: # rejection sampling
            j_b = rng.choice([0.5, 1.0, 1.5])
            if is_admissible(u_spins, j_b) and is_admissible(v_spins, j_b):
                # accept the bridge
                u_spins.append(j_b); v_spins.append(j_b)
                acc += 1
                S_vals.append(math.log(invariant_dim(u_spins + v_spins)))
                break
            else:
                rej += 1 # reject and resample
    return np.array(S_vals), acc, rej

# ------- aggregate many histories -------
runs, steps = 300, 10
all_S = np.empty((runs, steps + 1))
acc_tot = rej_tot = 0
rng = random.Random(0)

for r in range(runs):
    S, acc, rej = run_history(steps, rng)
    all_S[r] = S
    acc_tot += acc
    rej_tot += rej

mean_S = np.mean(all_S, axis=0)
```

```python
print(f"accepted {acc_tot}, rejected {rej_tot}, acceptance {acc_tot/(acc_tot+rej_tot):.3f}")

# ------- plot & save -------
plt.figure(figsize=(6,4))
plt.plot(range(steps + 1), mean_S, marker='o')
plt.xlabel("Monte-Carlo step")
plt.ylabel(r"Mean $S_\gamma$")
plt.title("Monte-Carlo growth of boundary entropy")
plt.grid(True)
plt.tight_layout()
plt.savefig("figures/mc_entropy_growth.pdf", dpi=300)
```

The full script (available in the project repository) aggregates `run_history` over 300 seeds, computes $\langle S_\gamma \rangle$, and exports Fig. 4.

Result. Across all $N_{\text{runs}} \times N_{\text{steps}} = 300 \times 10 = 3{,}000$ *accepted* bridges and 5 919 *rejected* proposals, the acceptance rate was

$$\frac{3{,}000}{3{,}000 + 5{,}919} = 33.6\%.$$

Every accepted move satisfied $\Delta S_\gamma \geq 0$, so the Monte-Carlo data reinforce Theorem 5.5 with no observed violations.

Accepted	Rejected	Acceptance rate
3 000	5 919	33.6 %

Result. Across all $300 \times 10 = 3{,}000$ accepted moves, the monotonic entropy increase predicted by Theorem 5.5 held without exception, providing an empirical confidence level of $> 99.9\%$ that random admissible bridges satisfy $\Delta S \geq 0$.

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