Deterministic Edge Connectivity and Max Flow using Subquadratic Cut Queries

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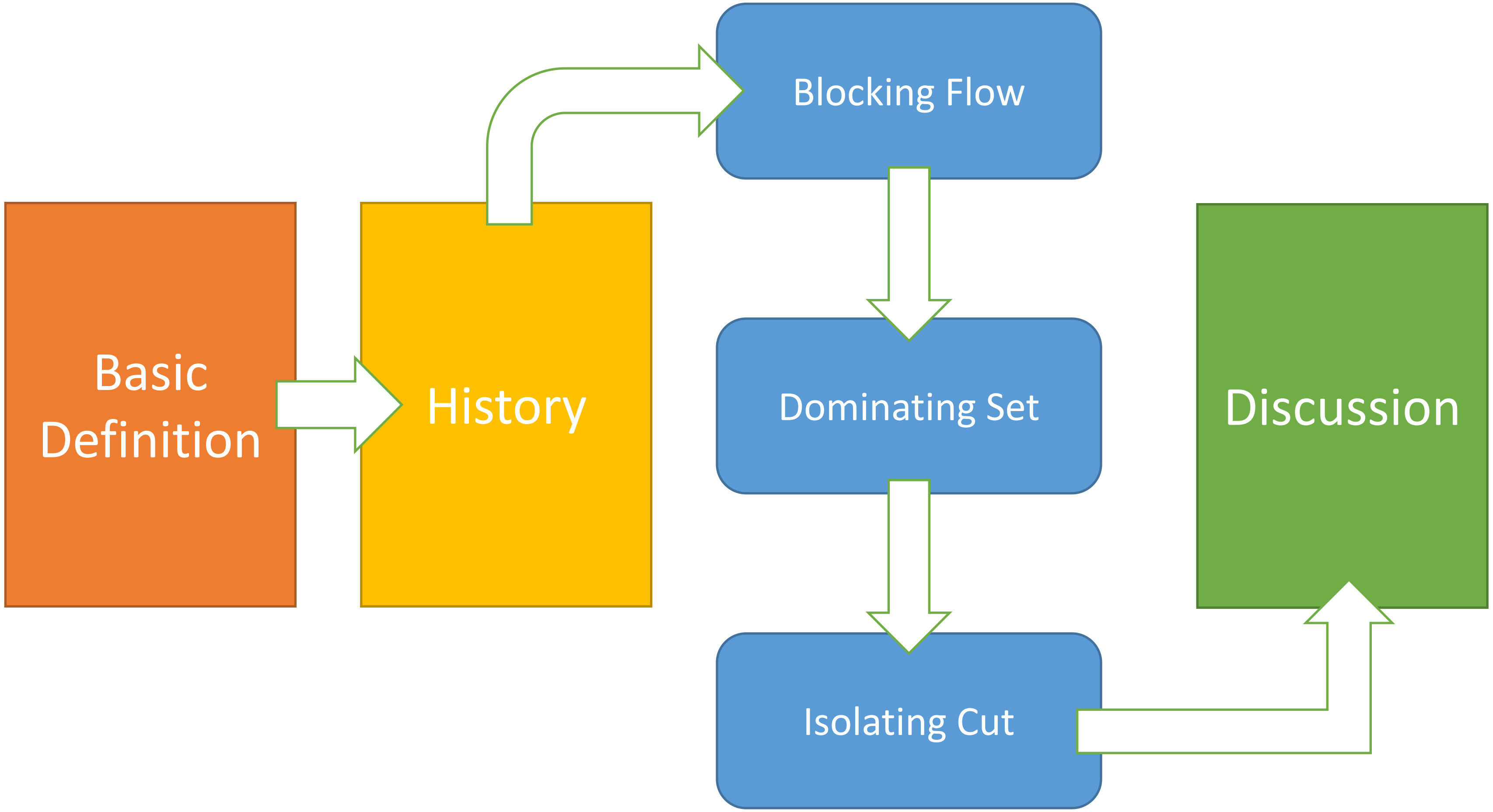
Outline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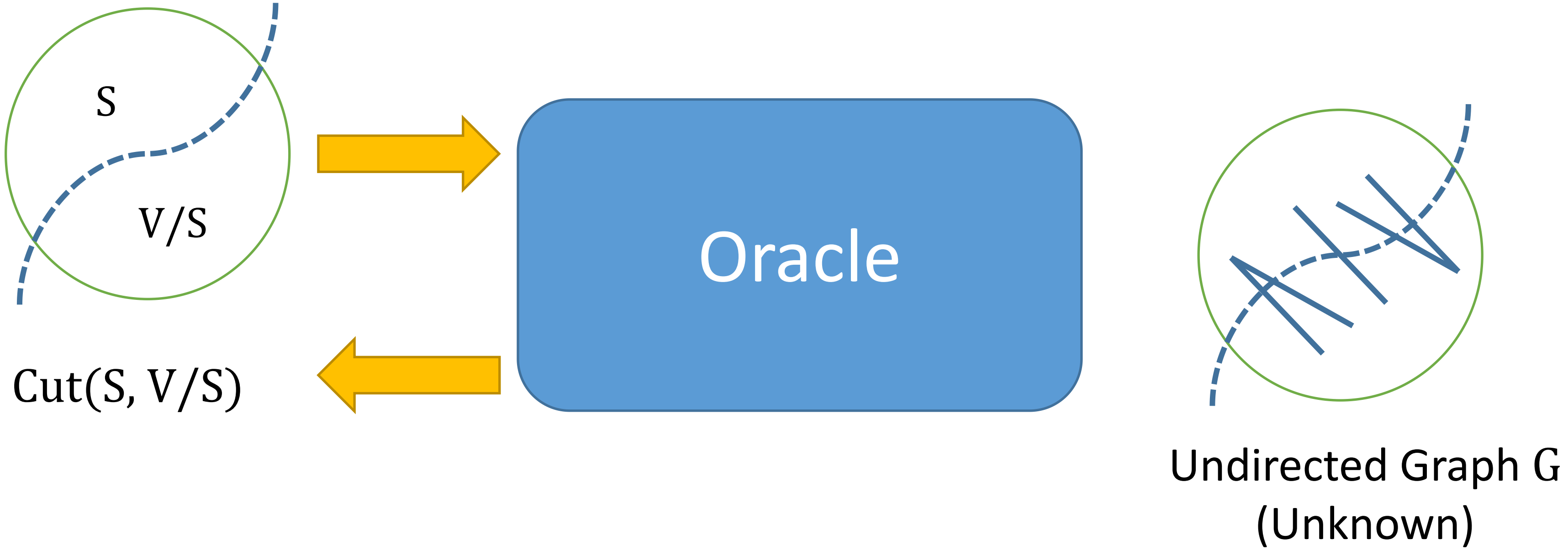
Dominating Set
Isolating Cut
Discussion

Basic Definition

- **Cut Query Model**



Q: How many **query** do we need to compute the **global minimum cut**?

Sense of the problem

Global Minimum Cut

Submodular Function Minimization

Vector-Matrix-Vector Query

“Learn the whole graph”

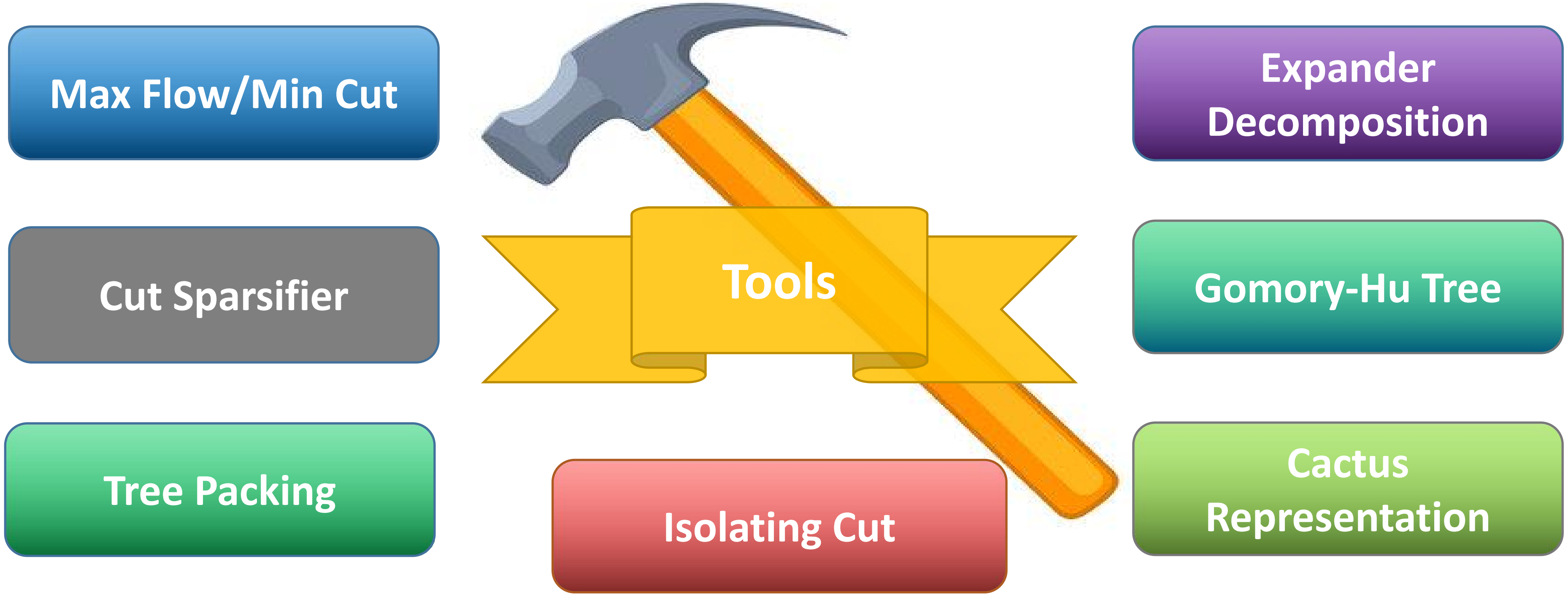
- **A Naive Algorithm**
- Learn each edge (u, v) by computing $\mathrm{Cut}(\{u\}) + \mathrm{Cut}(\{v\}) - \mathrm{Cut}(\{u, v\})$, which requires $O(n^2)$ queries [**Nearly Optimal**]

- For dense graph, one needs at least $\sim\Omega(n^2)$ **query** to learn the whole graph [Rubinstein, Schramm & Weinberg, ITCS 2018]

- One can use $O(n^2/\log n)$ **query** to learn the whole graph[Grebinski & Kucherov, Algorithmica 2000]

Motivation

- **Q**: Can we use **less information** to **represent** the **global minimum cut**?



History

- **Edge Connectivity** in **Cut Query Model**

	Det./ Rand.	Simple/ Weighted	Query
Rubinstein, Schramm& Weinberg[RSW18]	Rand.	Simple	$O(n \log^3 n)$
Mukhopadhyay & Nanongkai[MN20]	Rand.	Weighted	$O(n \log^{O(1)} n)$
Apers et al.[AEG+22]	Rand.	Simple	$O(n)$
This paper	Det.	Simple	$\tilde{O}(n^{\frac{5}{3}})$

State-of-the-Art

- Different Settings in **Cut Query Model**

	Connectivity		Edge Connectivity	
	Lower	Upper	Lower	Upper
Deterministic	$\Omega(n)$ [HMT88]	$O(\frac{n\log n}{\log\log n})$ [LC24]	$\Omega(n)$ [HMT88]	$\tilde{O}(n^{5/3})$ **(This paper)**
Zero-error, Randomized	$\Omega(\frac{n\log\log(n)}{\log n})$ [RS95]	$O(n)$ [AEG+22]	$\Omega(n)$	$\tilde{O}(n^{5/3})$ **(This paper)**
Bounded Error, Randomized	$\Omega(\frac{n}{\log n})$ [BFS86]	$O(n)$ [AEG+22]	$\Omega(\frac{n\log\log(n)}{\log n})$ [AD21]	$O(n)$ [AEG+22]
Quantum	$\Omega(1)$	$O(\log^5(n))$ [AL21]	$\Omega(1)$	$\tilde{O}(\sqrt{n})$ [AEG+22]

Max Flow/Min Cut

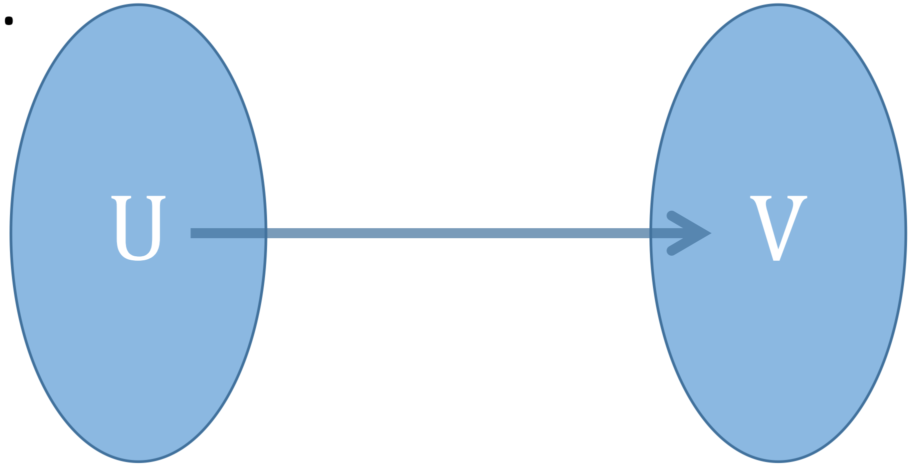
- **Max Flow Min Cut Theorem**
- No Duality Gap for $\mathrm{s}-\mathrm{t}$ max flow and $\mathrm{s}-\mathrm{t}$ min cut

  max flow value=min cut capacity

- Q: Why should we consider **Max Flow** as a start point?
- Inspiration: [RSW18] shows that, for graph G with integral weights from $[0, \mathrm{W}]$, every $\mathrm{s}-\mathrm{t}$ flow of value f can be covered by edges of at most $\mathrm{O}(\mathrm{n}\sqrt{fW})$ total weight
- **We can use $\mathbf{O}(\mathbf{n}\sqrt{n})$ edges to cover any s-t flow in simple graph!**

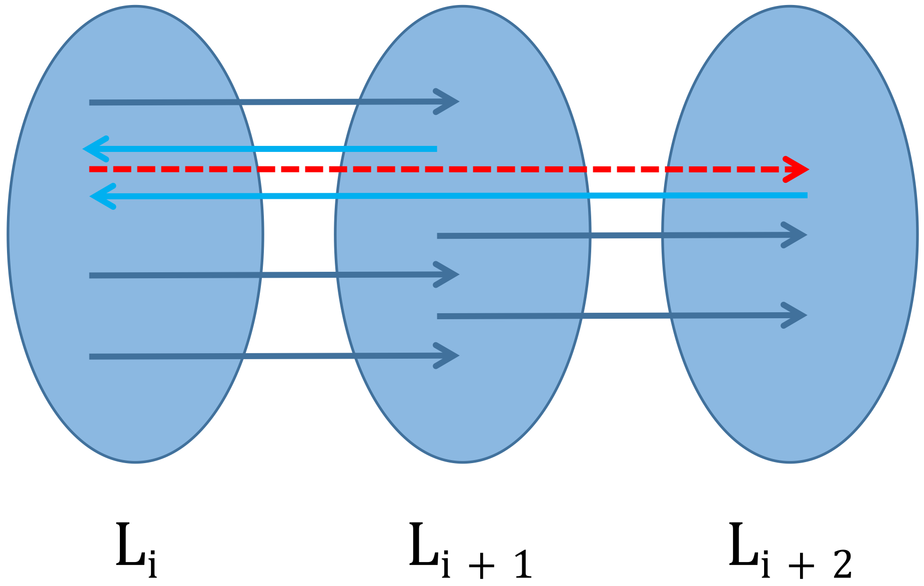
BIS/Cross Query

- A **BIS (Bipartite Independent Set)** or **Cross Query** asks whether there exists an edge between two sets U and V. In other words, it checks if there is an edge (u, v) such that $u \in U, v \in V$.
- **Fact:** A **BIS/Cross query** can be replaced by $O(1)$ Cut Query in undirected graph.

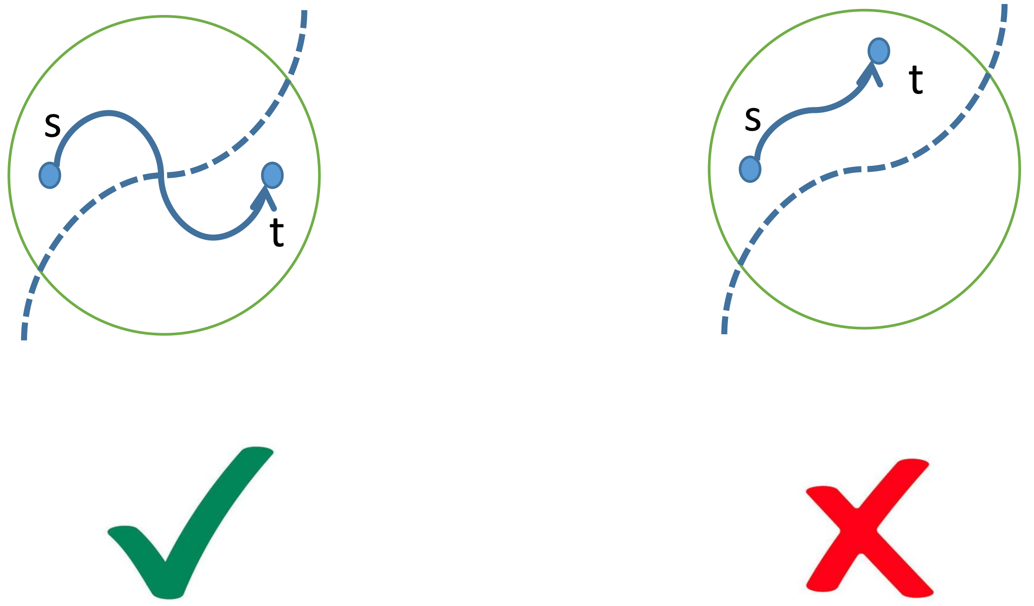


Blocking Flow

- **Theorem 1**: We can use $\sim O(n^{5/3})$ **BIS/Cross query** to obtain an explicit s-t max flow in simple graph [**Idea: Dinitz'algorithm**]

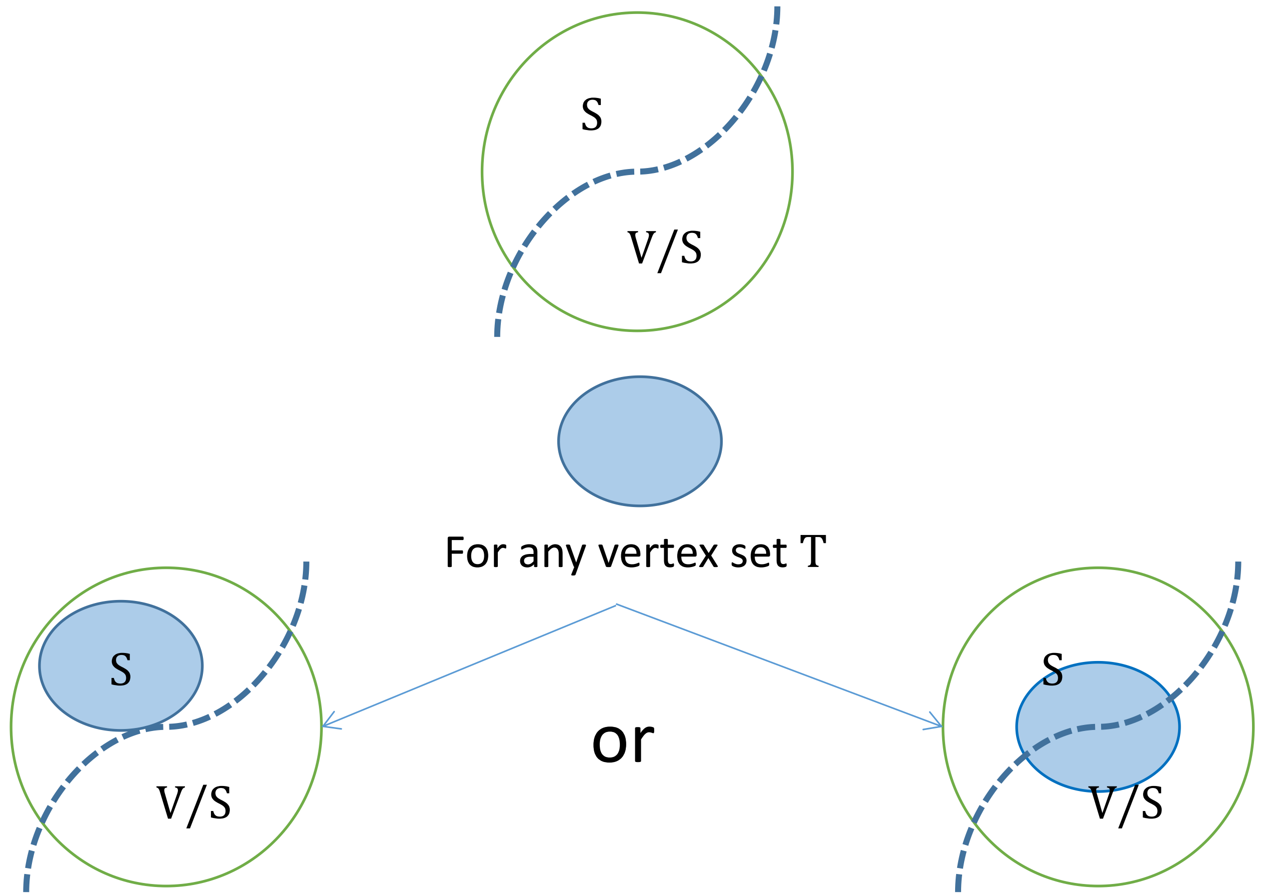


From Flow to Cut



How can we guarantee that s and t are on **different** sides of the minimum cut?

“Preserve the Minimum Cut”

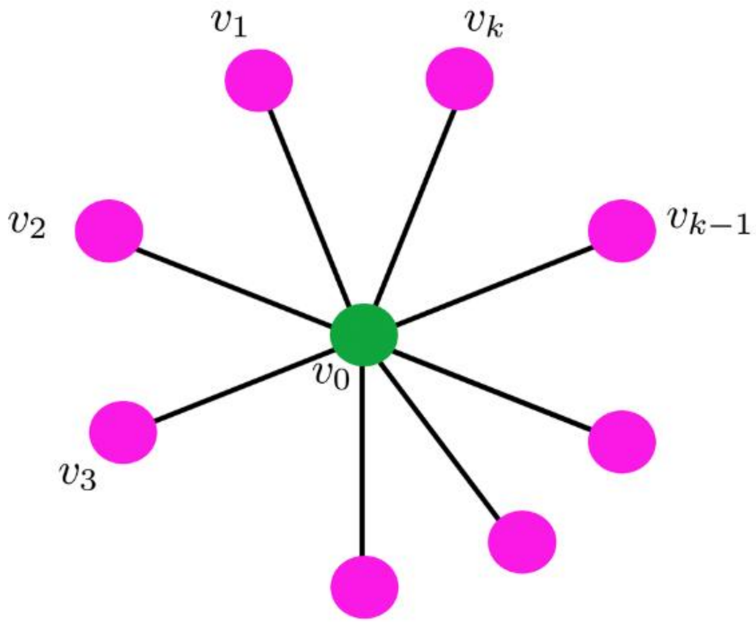


Contraction
[e.g. Karger’s Algorithm]

“Preserve the Min Cut”
[e.g. Kawarabayashi & Thorup, STOC 2015]

Dominating Set

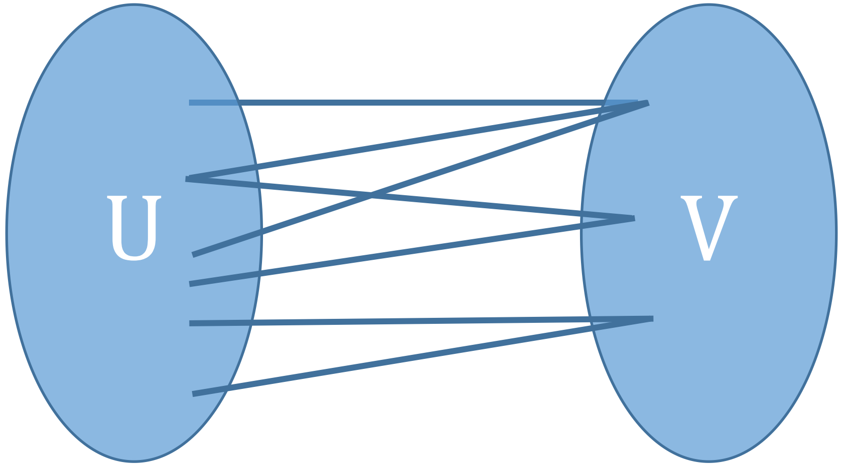
- **A Key Observation:** In simple graph, a dominating set can “preserve all non-trivial minimum cut”



Special Case: Star Graph[No non-trivial minium cut]

Dominating Set

- **Theorem 2**: If the minium degree is δ, then we can find a dominating set D with size at most $O(\frac{n}{\delta}\log\frac{n}{\delta})$ with $\sim O(n)$ cut **query**.
- Existence: Sample each vertex with probability $\sim\frac{1}{\delta}$.
- **De-randomize Idea:** Finding an element **above the average**

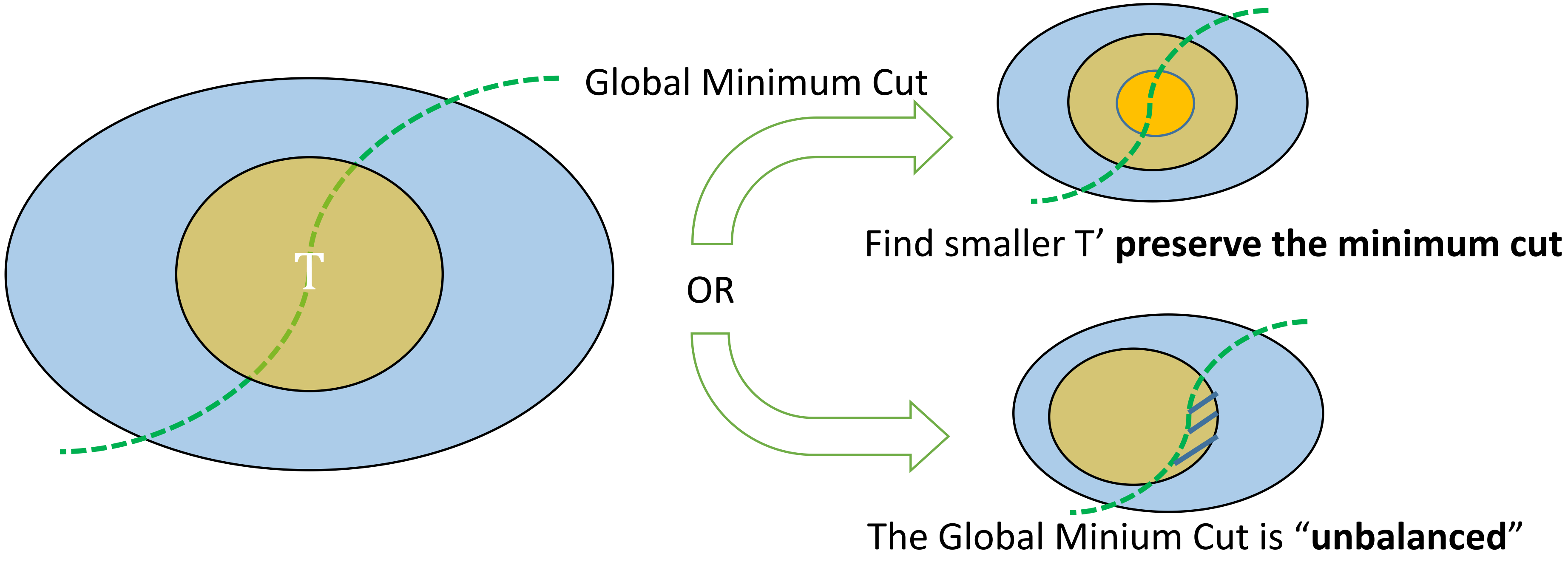


In $O(\log n)$ queries

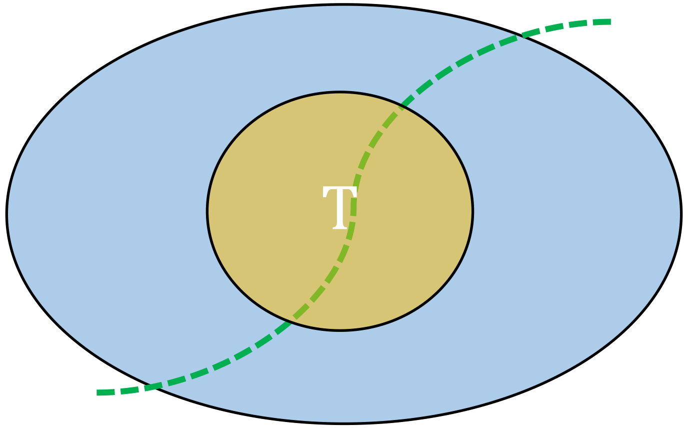
Find $u \in U$ such that $\deg_V(u) \geq |E(U,V)|/|U|$

Framework of Isolating Cut

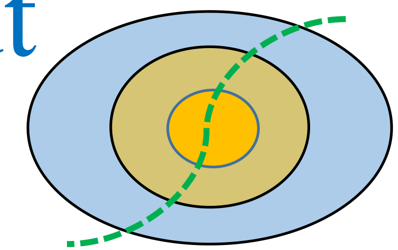
- Isolating Cut[Li & Panigrahi, FOCS 2020]



Framework of Isolating Cut

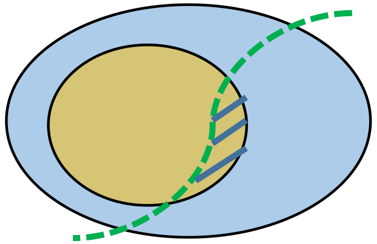


Global Minimum Cut



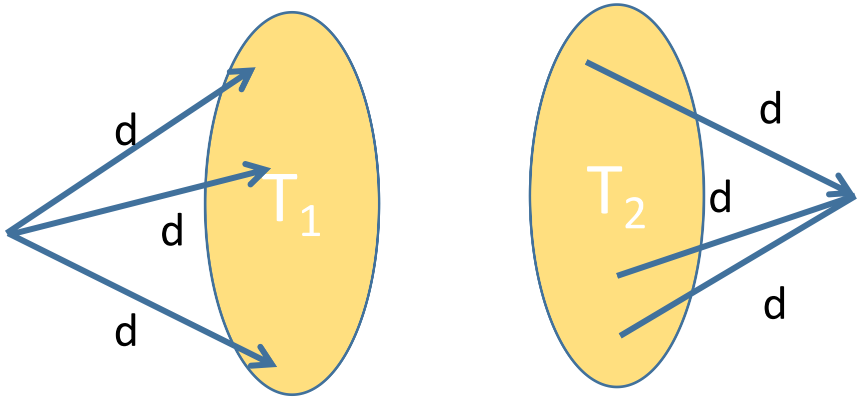
Find T’ **preserve the minimum cut**

OR



The Global Minium Cut is “**unbalanced**”

Cut Matching Game



Almost every edge saturated

OR

Find balanced sparse cut

Polylog times

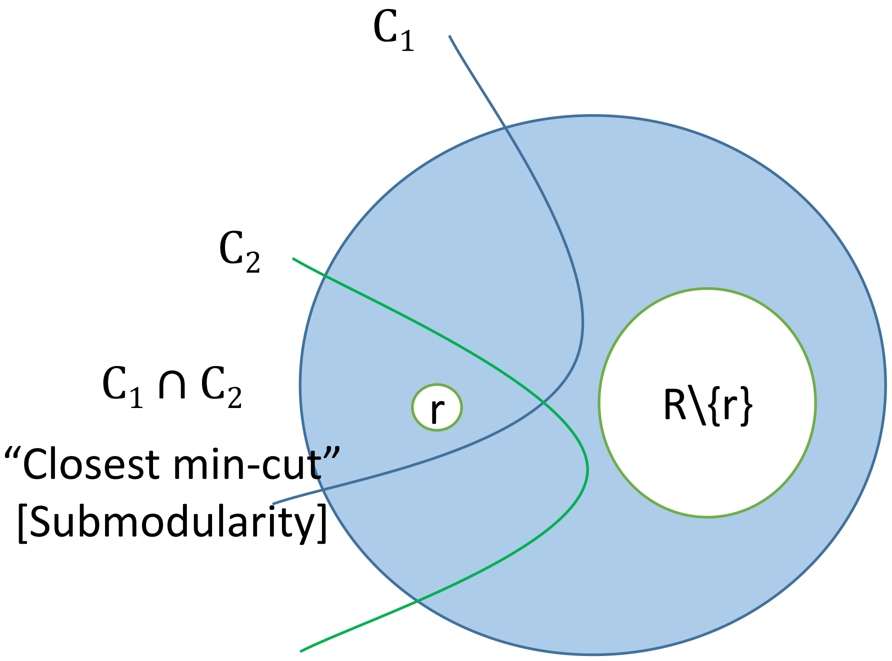
Guarantee “Almost Expander”

Demand $d \le \delta$ [Total Flow Size $\le \sim\boldsymbol{O(n)}$]

NOT exactly the same as [LP20], we don’t compute $\mathrm{T}_1 - \mathrm{T}_2$ min-cut

Minimum Isolating Cut

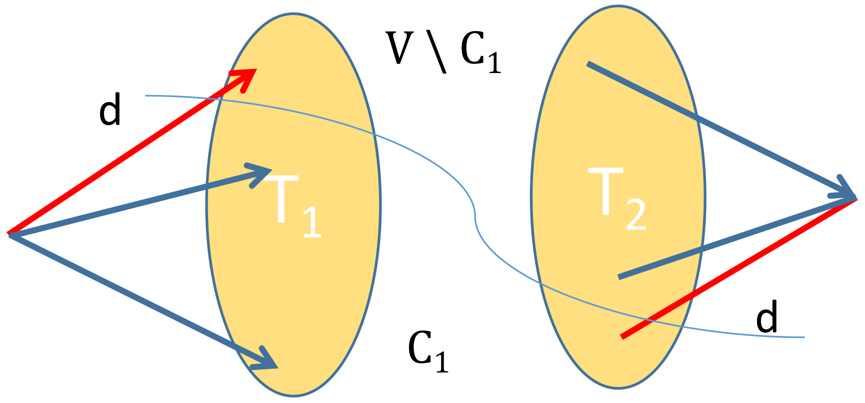
- For any set of vertices $R \subseteq V$, $r \in R$, the minimum isolating cut of r is an $\{r\} - \{R \setminus \{r\}\}$ min-cut



Subroutine

- Let $d = \tau + 1$, we will either outputs an **isolating cut** of R of size at most τ,
- or certifies that the **minimum isolating cut** of R has a size larger than τ.

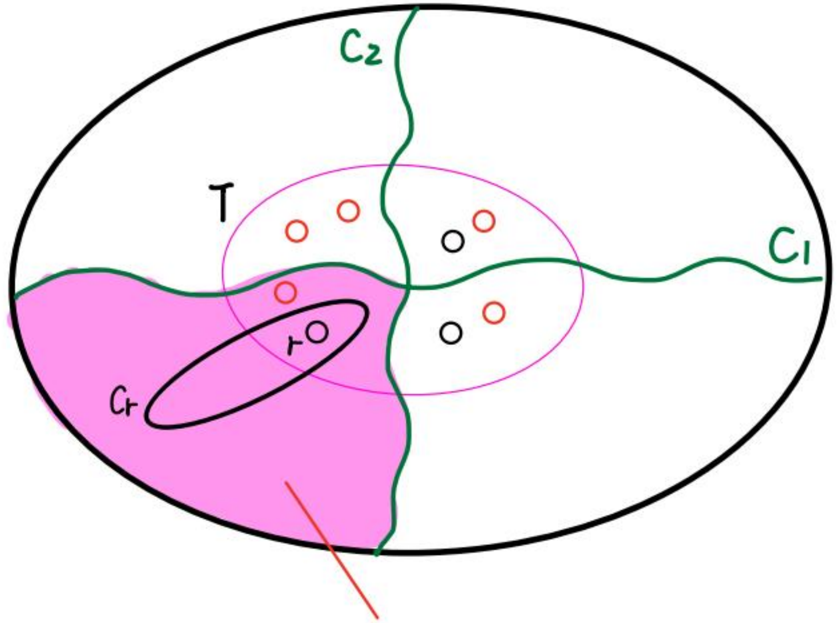
Cut Matching Game



If an edge is **saturated**, then the corresponding minimum isolating cut has size at least d.

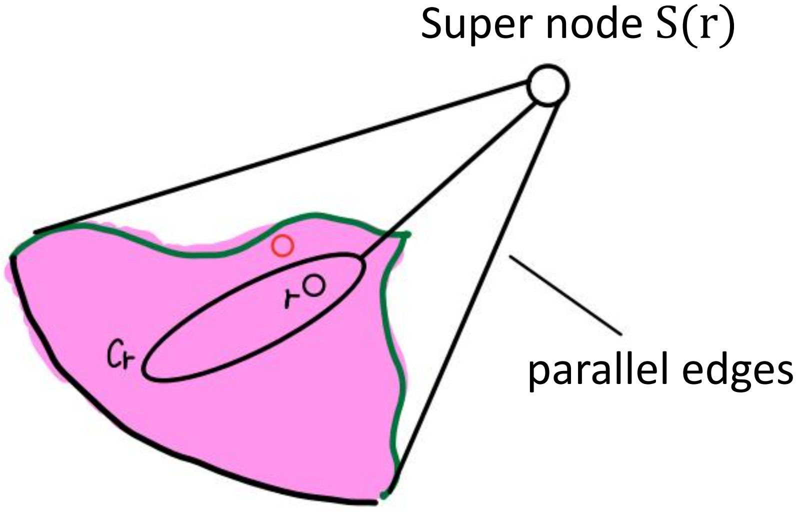
If the minimum isolating cut of R is C_r less than d, then we must have $C_r \subseteq C_1$ or $C_r \subseteq V/C_1$

“From Global to Local”



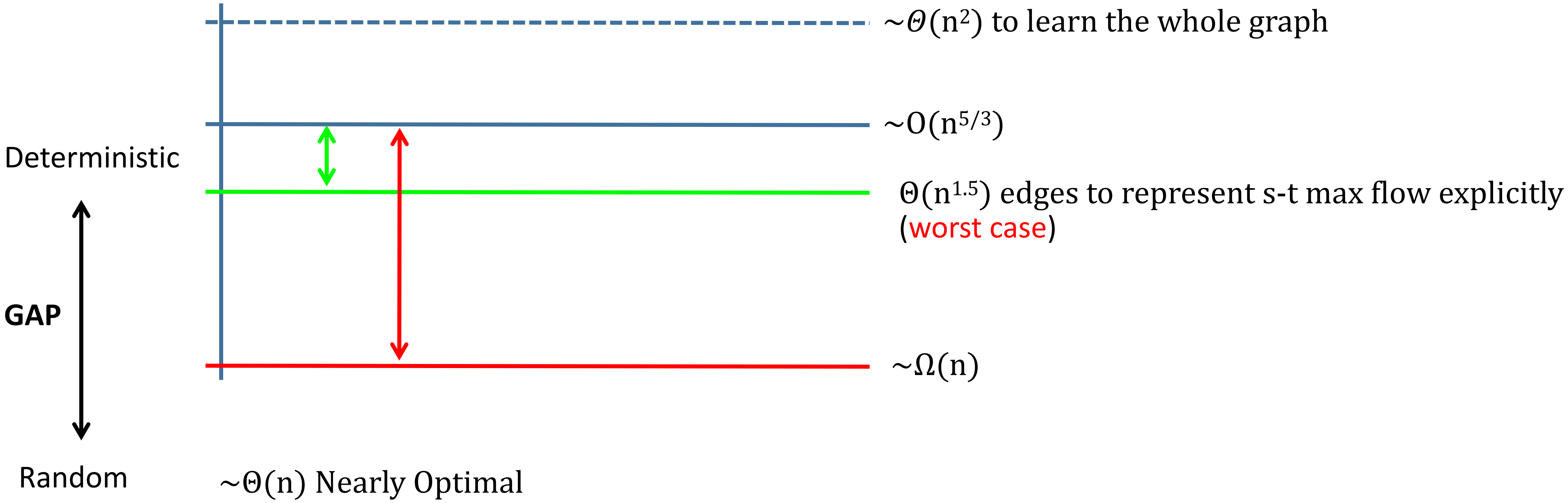
- not saturated in all max-flow call
- saturated at least once

Guarantee: each part contains at most 1 black vertex



Call $r - S(r)$ max flow

Discussion


$\sim\Theta(n^2)$ to learn the whole graph
$\sim O(n^{5/3})$
Deterministic
$\Theta(n^{1.5})$ edges to represent s-t max flow explicitly
(worst case)
GAP
$\sim\Omega(n)$
Random
$\sim\Theta(n)$ Nearly Optimal