# Deterministic Edge Connectivity and Max Flow using Subquadratic Cut Queries

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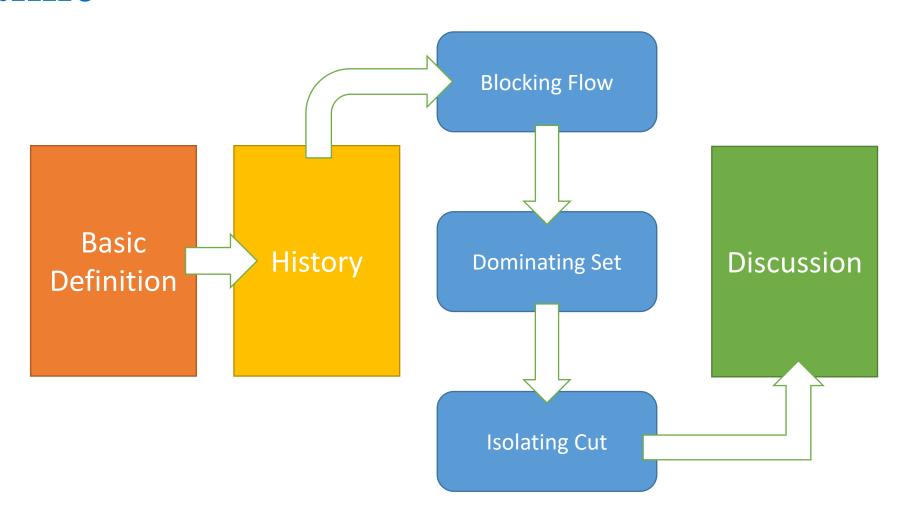
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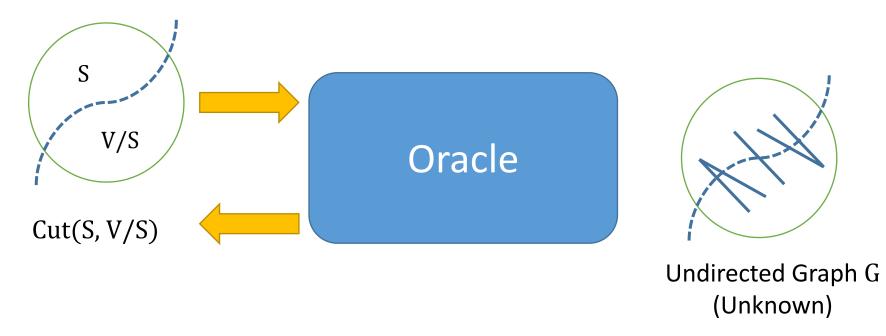
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# Outline



#### Basic Definition

Cut Query Model



Q: How many query do we need to compute the global minimum cut?

Sense of the problem



# "Learn the whole graph"

- A Naive Algorithm
- Learn each edge (u, v) by computing  $Cut(\{u\}) + Cut(\{v\}) Cut(\{u, v\})$ , which requires  $O(n^2)$  queries [Nearly Optimal]
- For dense graph, one needs at least  $\sim \Omega(n^2)$  query to learn the whole graph [Rubinstein, Schramm & Weinberg, ITCS 2018]
- One can use O(n²/log n) query to learn the whole graph[Grebinski & Kucherov, Algorithmica 2000]

#### Motivation

• Q: Can we use less information to represent the global minimum cut?



# History

#### Edge Connectivity in Cut Query Model

	Det./ Rand.	Simple/ Weighted	Query
Rubinstein, Schramm& Weinberg[RSW18]	Rand.	Simple	$O(n\log^3 n)$
Mukhopadhyay & Nanongkai[MN20]	Rand.	Weighted	$O(n \log^{O(1)} n)$
Apers et al.[AEG <sup>+</sup> 22]	Rand.	Simple	O(n)
This paper	Det.	Simple	$ ilde{O}(n^{rac{5}{3}})$

### State-of-the-Art

#### • Different Settings in Cut Query Model

	Connectivity		Edge Connectivity	
	Lower	Upper	Lower	Upper
Deterministic	$\Omega(n)$ [HMT88]	$O(\frac{n\log n}{\log\log n})$ [LC24]	$\Omega(n)$ [HMT88]	$ ilde{O}(n^{5/3})$ (This paper)
Zero-error, Randomized	$\frac{\Omega(\frac{n\log\log(n)}{\log n})}{[RS95]}$	O(n) [AEG <sup>+</sup> 22]	$\Omega(n)$	$ ilde{O}(n^{5/3})$ (This paper)
Bounded Error, Randomized	$\frac{\Omega(\frac{n}{\log n})}{[BFS86]}$	O(n) [AEG <sup>+</sup> 22]	$\frac{\Omega(\frac{n\log\log(n)}{\log n})}{[\text{AD21}]}$	O(n) [AEG <sup>+</sup> 22]
Quantum	$\Omega(1)$	$O(\log^5(n))$ [AL21]	$\Omega(1)$	$\tilde{O}(\sqrt{n})$ [AEG <sup>+</sup> 22]

#### Max Flow/Min Cut

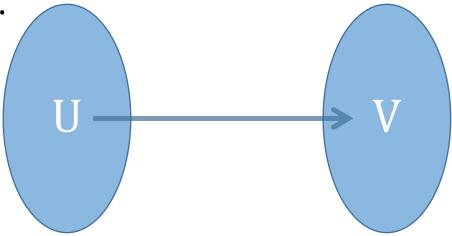
- Max Flow Min Cut Theorem
- No Duality Gap for s t max flow and s t min cut max flow value=min cut capacity
- Q: Why should we consider Max Flow as a start point?
- Inspiration: [RSW18] shows that, for graph G with integral weights from [0, W], every s t flow of value f can be covered by edges of at most  $O(n\sqrt{fW})$  total weight
- We can use  $O(n\sqrt{n})$  edges to cover any s-t flow in simple graph!

# BIS/Cross Query

• A BIS (Bipartite Independent Set) or Cross Query asks whether there exists an edge between two sets U and V. In other words, it checks if there is an edge (u, v) such that  $u \in U, v \in V$ .

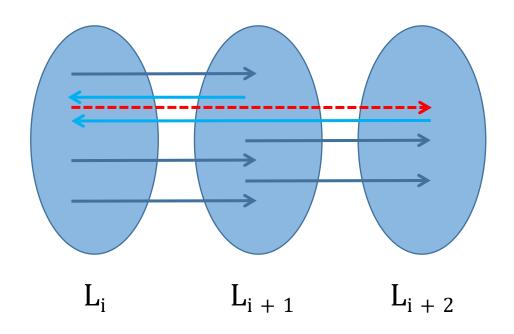
• Fact: A BIS/Cross query can be replaced by O(1) Cut Query in

undirected graph.

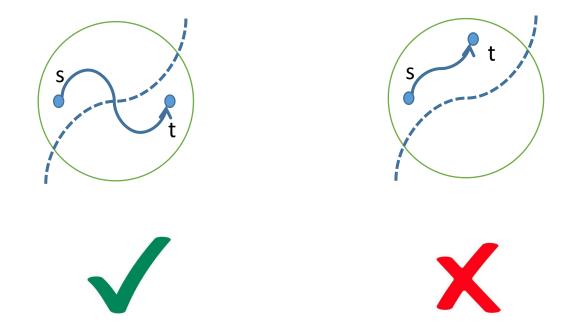


# Blocking Flow

• Theorem 1: We can use  $\sim O(n^{5/3})$  BIS/Cross query to obtain an explicit s-t max flow in simple graph [Idea: Dinitz'algorithm]

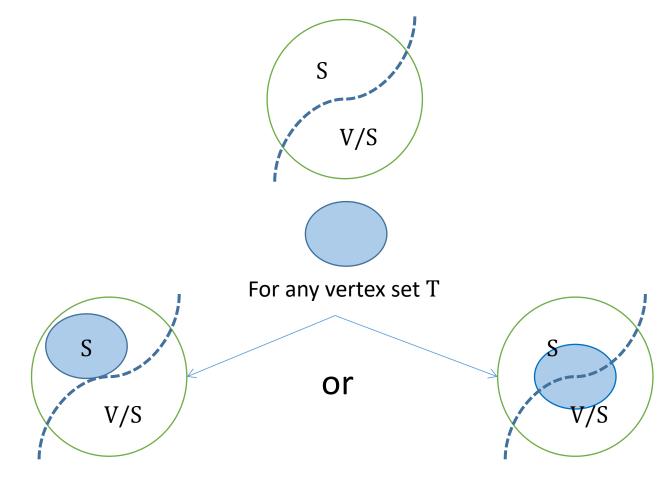


### From Flow to Cut



How can we guarantee that s and t are on **different** sides of the minimum cut?

#### "Preserve the Minimum Cut"

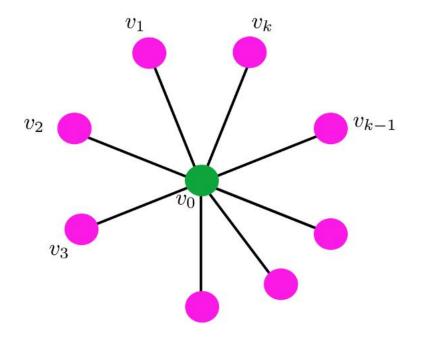


Contraction [e.g. Karger's Algorithm]

"Preserve the Min Cut"
[e.g. Kawarabayashi & Thorup, STOC 2015]

# Dominating Set

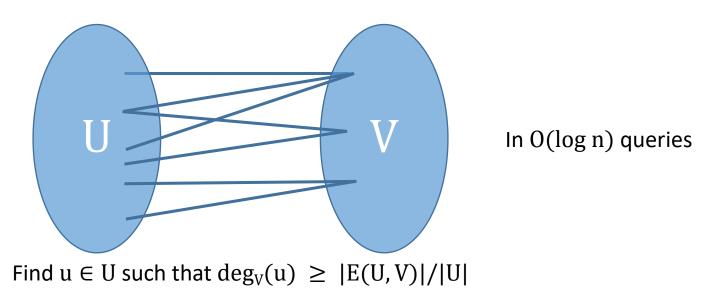
• A Key Observation: In simple graph, a dominating set can "preserve all non-trivial minimum cut"



Special Case: Star Graph[No non-trivial minium cut]

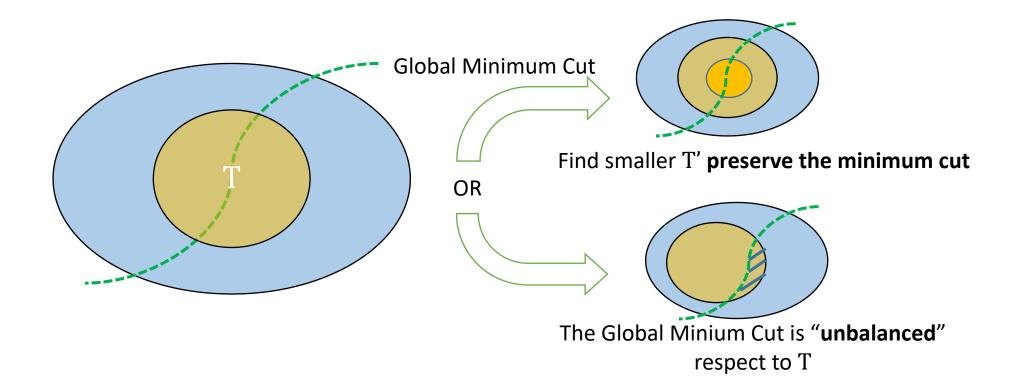
# **Dominating Set**

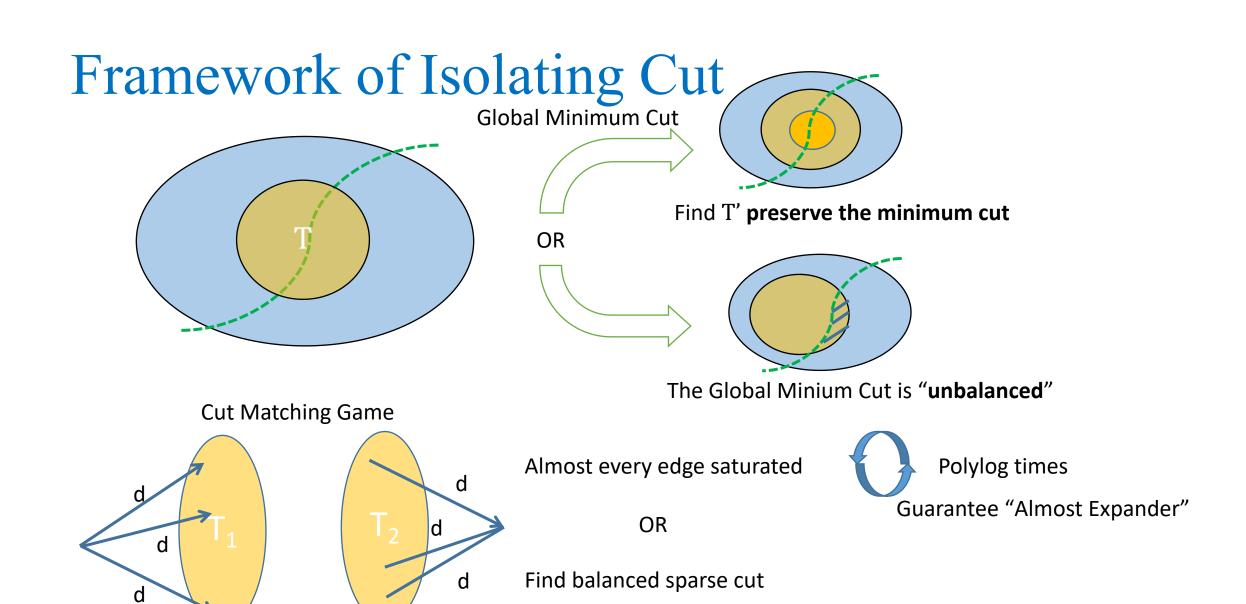
- **Theorem 2**: If the minium degree is  $\delta$ , then we can find a dominating set D with size at most  $O(\frac{n}{\delta}\log\frac{n}{\delta})$  with  $\sim O(n)$  cut query.
- Existence: Sample each vertex with probability  $\sim \frac{1}{\delta}$ .
- De-randomize Idea: Finding an element above the average



# Framework of Isolating Cut

• Isolating Cut[Li & Panigrahi, FOCS 2020]



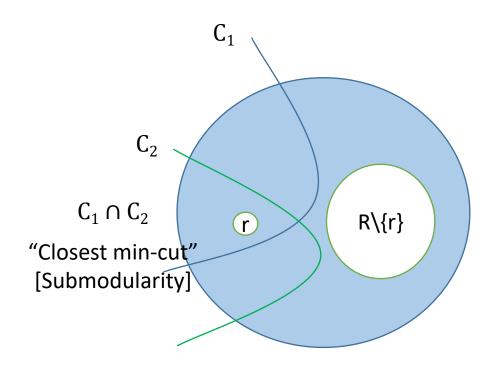


**Demand**  $d \leq \delta$  [Total Flow Size  $\leq \sim O(n)$ ]

**NOT** exactly the same as [LP20], we don't compute  $T_1 - T_2$  min-cut

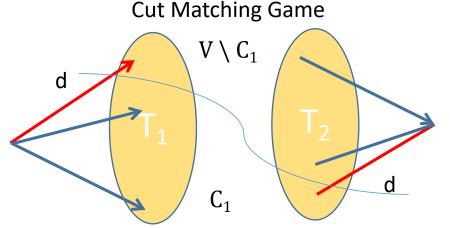
# Minimum Isolating Cut

• For any set of vertices  $R \subseteq V$ ,  $r \in R$ , the minimum isolating cut of r is an  $\{r\} - \{R \setminus \{r\}\}$  min-cut



### Subroutine

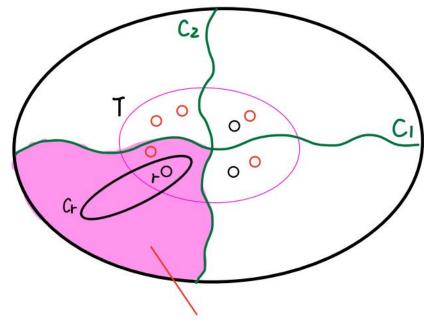
- Let  $d = \tau + 1$ , we will either outputs an **isolating cut** of R of size at most  $\tau$ ,
- or certifies that the **minimum isolating cut** of R has a size larger than  $\tau$ .



If an edge is **saturated**, then the corresponding minimum isolating cut has size at least d.

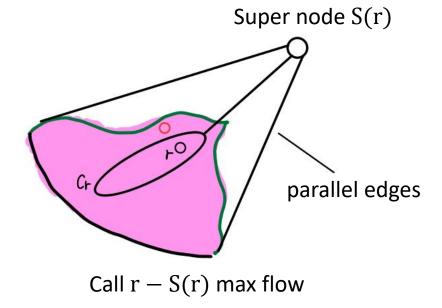
If the minimum isolating cut of R is  $C_r$  less than d, then we must have  $C_r \subseteq C_1$  or  $C_r \subseteq V/C_1$ 

#### "From Global to Local"



**Guarantee:** each part contains at most 1 black vertex

- not saturated in all max-flow call
- saturated at least once



### Discussion

