Better Decremental and Fully Dynamic Sensitivity Oracles for Subgraph Connectivity

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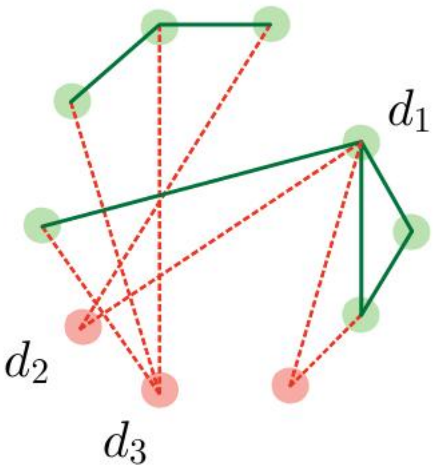
Decremental v.s. Fully Dynamic

- **Decremental Setting (Vertex-Failure)**
- There are no inactivated vertices initially, i.e., $V_{off} = \emptyset$.

- **Fully Dynamic Setting**
- No constraints

Example



Red Vertices: V_{off}
Green Vertices: V_{on}

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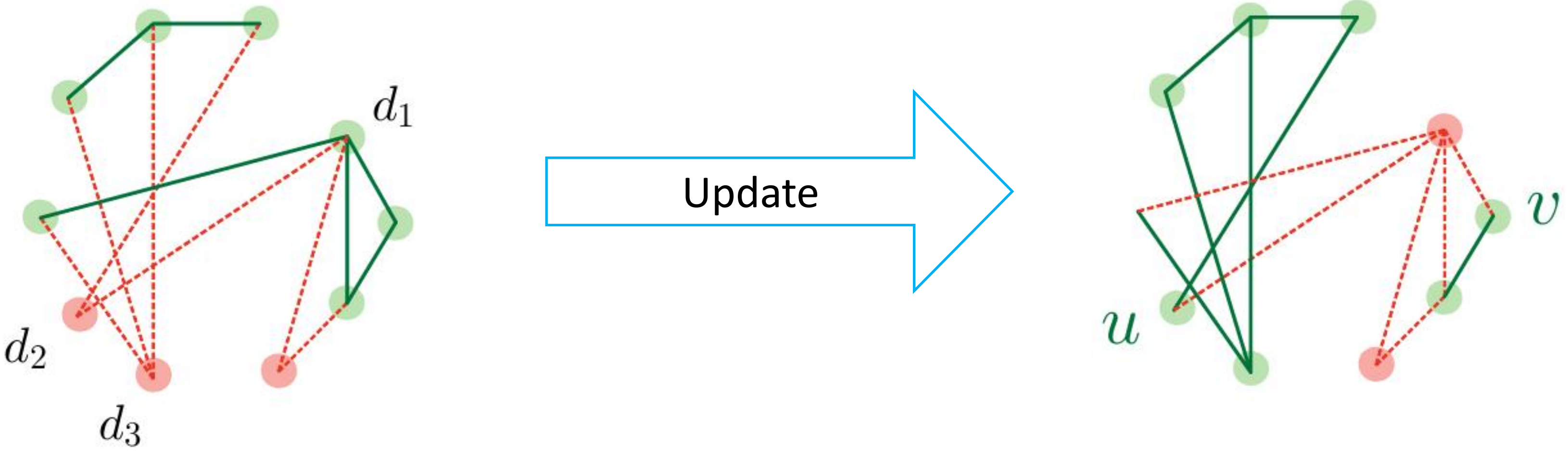

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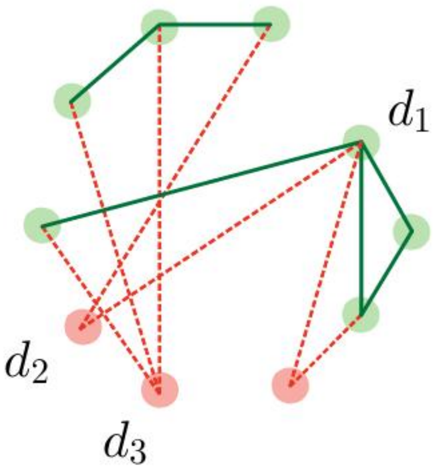

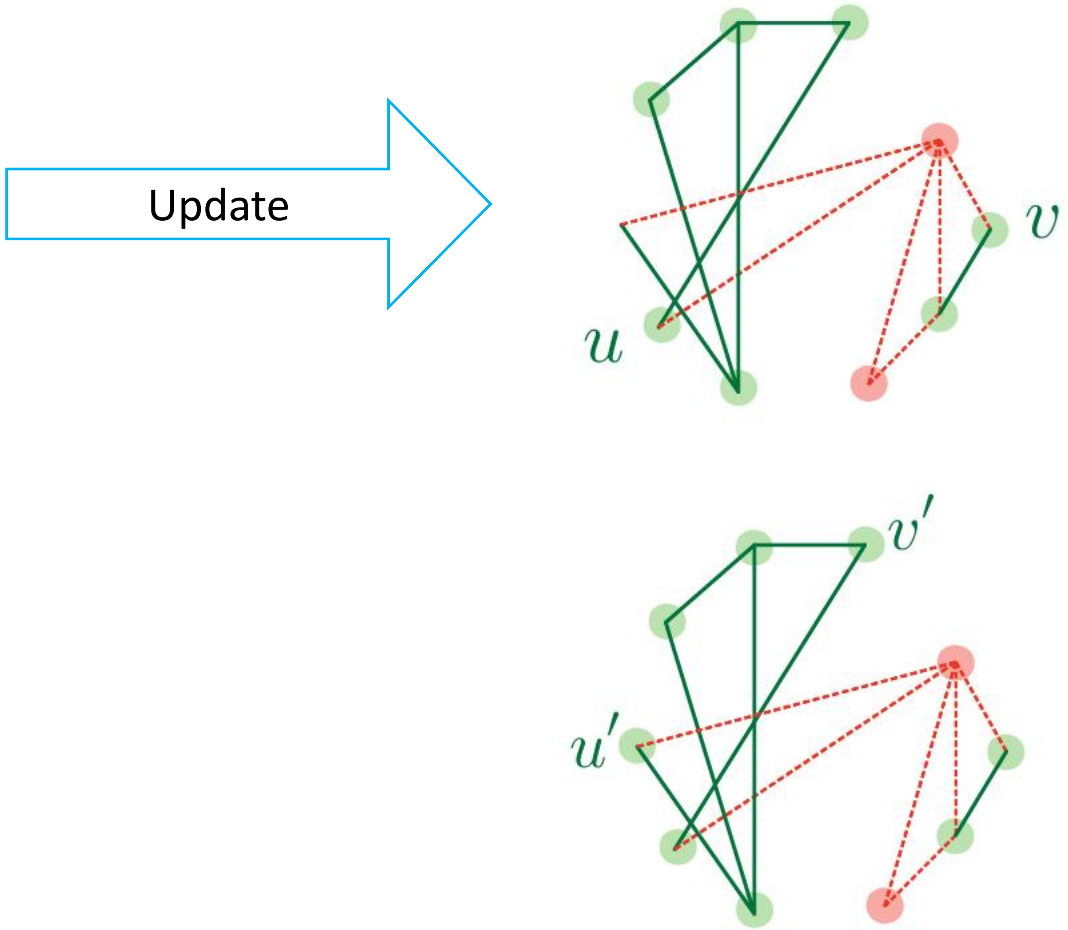

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History (Decremental)

	Det./ Rand.	Space	Preprocessing	Update	Query
Block trees, SQRT trees, and [KTDBC91] only when $d_\star \leq 3$	Det.	$O(n)$	$\widetilde{O}(m)$	$O(1)$	$O(1)$
Duan & Pettie [DP10] for $c \geq 1$	Det.	linear in preprocessing time	$\widetilde{O}(md_\star^{1-\frac{2}{c}} n^{\frac{1}{c}-\frac{1}{c\log(2d_\star)}})$	$\widetilde{O}(d^{2c+4})$	$O(d)$
Duan & Pettie [DP20]	Det.	$O(md_\star \log n)$	$O(mn \log n)$	$O(d^3 \log^3 n)$	$O(d)$
	Rand.	$O(m \log^6 n)$	$O(mn \log n)$	$\bar{O}(d^2 \log^3 n)$ w.h.p.	$O(d)$
Brand & Saranurak [vdBS19]	Rand.	$O(n^2)$	$O(n^\omega)$	$O(d^\omega)$	$O(d^2)$
Pilipczuk et al. [PSS+22]	Det.	$m2^{2^{O(d_\star)}}$	$mn^2 2^{2^{O(d_\star)}}$	$2^{2^{O(d_\star)}}$	$2^{2^{O(d_\star)}}$
	Det.	$n^2\mathrm{poly}(d_\star)$	$\mathrm{poly}(n)2^{O(d_\star \log d_\star)}$	$\mathrm{poly}(d_\star)$	$\mathrm{poly}(d_\star)$
Long & Saranurak [LS22]	Det.	$O(m \log^3 n)$	$O(mn \log n)$	$\bar{O}(d^2 \log^3 n \log^4 d)$	$O(d)$
	Det.	$O(m \log^* n)$	$\widehat{O}(m) + \widetilde{O}(d_\star m)$	$\widehat{O}(d^2)$	$O(d)$
Kosinas [Kos23]	Det.	$O(d_\star m \log n)$	$O(d_\star m \log n)$	$O(d^4 \log n)$	$O(d)$
This paper	Det.	$O(m \log^3 n)$	$\widehat{O}(m) + O(d_\star m \log^3 n)$	$O(d^2(\log^7 n + \log^5 n \log^4 d))$	$O(d)$

History (Fully Dynamic)

	Det./ Rand.	Space	Preprocessing	Update	Query
Henzinger & Neumann [HN16]	Det.	$\widetilde{O}(n_{\text{off}}^2 m)$	$\widehat{O}(n_{\text{off}}^2 m) + \widetilde{O}(d_\star n_{\text{off}}^2 m)$	$\widehat{O}(d^4)$	$O(d^2)$
Hu, Kosinas & Polak [HKP23]	Det.	$\widetilde{O}((n_{\text{off}} + d_\star)m)$	$\widetilde{O}((n_{\text{off}} + d_\star)m)$	$\widetilde{O}(d^4)$	$O(d)$
This paper	Det.	$O(\min\{(n_{\text{off}} + d_\star)m \log^2 n, n^2\})$	$\widehat{O}(m)+$ $O(\min\{(n_{\text{off}} + d_\star)m, n^\omega\} \log^2 n)$	$O(d^2 \log^7 n)$	$O(d)$

Technique Overview

- **Low Degree Hierarchy,** which can reduced to $O(\log n)$ calls to the low-degree Steiner forest decomposition.[DP20, LS22]
- We say a forest $F \subseteq E(G)$, is a spanning forest of U in G if F spans the whole U (may also span vertices not in U).

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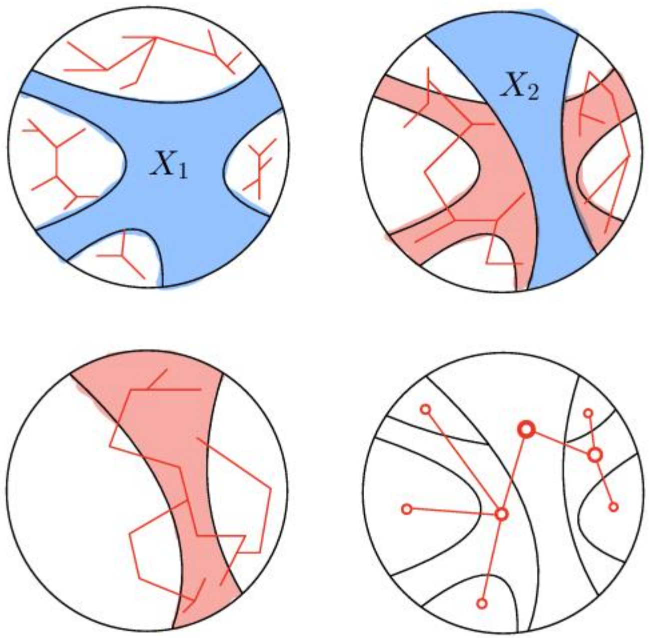
Low Degree Hierarchy

- Low Degree Hierarchy: a laminar set C of components.
- Components form a tree hierarchy.
- $X_i = \mathrm{Decomp}(G, X_{i-1})$



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Cut Matching Game

- Let G be a undirected graph with a terminal set U. Given a parameter φ and a partition (A, B) of U, there is a deterministic algorithm that computes either
- 1. A vertex cut (L, S, R) with $|R \cap U| \geq |L \cap U| \geq \min\{|A|, |B|\}/3$ and $S \leq \varphi \cdot |L \cap U|$
- 2. A matching M between A and B with size $|M| \geq \min\{|A|, |B|\}/3$ s.t. there is an embedding of M into G with congestion $O(1/\varphi)$

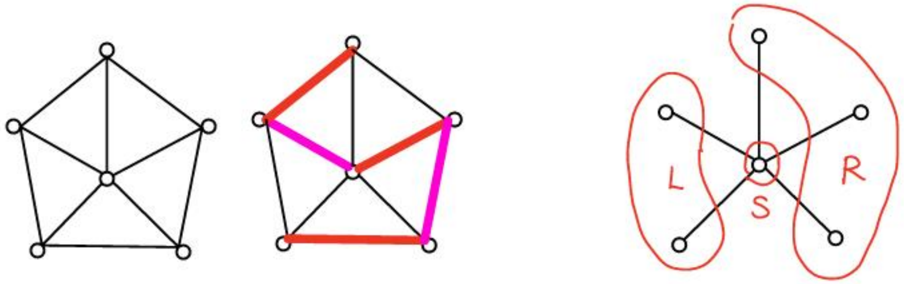

Figure Left: Low Degree Steiner Tree Right: Balanced Vertex Cut

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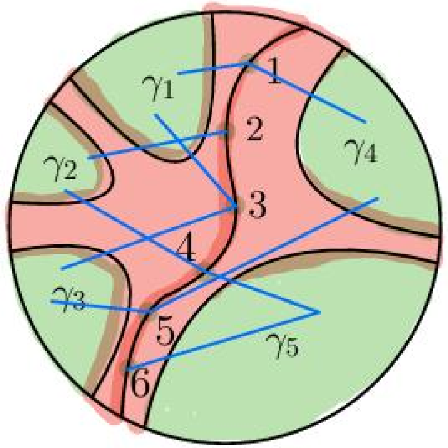
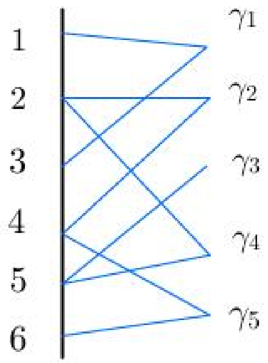
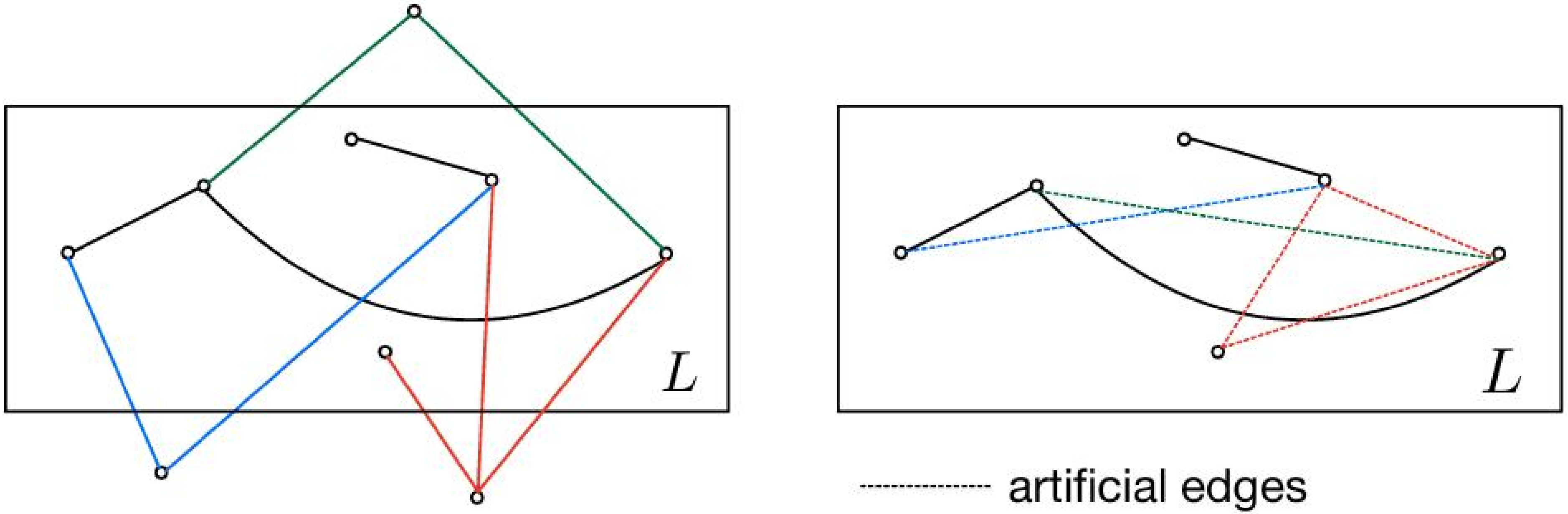




Abstract Graph

- High level idea: maintain all vertices in L and ignore the vertices in R



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Borůvka-based algorithm

- For any two vertices, we can detect whether they are connected in $\tilde{O}(d)$ time. A naive algorithm can compute all pairs connectivity in $\tilde{O}(d^3)$ time. To achieve $\tilde{O}(d^2)$, we need to consider a Borůvka-based algorithm.
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Conditional Lower Bound

1. If $t_u + t_p = f(d) \cdot n^{o(1)}$, then $S = \widehat{\Omega}(n^2)$.
2. If $t_u + t_p = f(d) \cdot n^{o(1)}$, then $t_u = \widehat{\Omega}((n_{\text{off}} + d)m)$. [HKP23]
3. If $t_u + t_p = f(d) \cdot n^{o(1)}$, then $t_u = \widehat{\Omega}(n^{\omega_{\text{bool}}})$.
4. If $t_p = \text{poly}(n)$, then $t_u + t_q = \widehat{\Omega}(d^2)$. [LS22]
5. If $t_p = \text{poly}(n)$ and $t_u = \text{poly}(dn^{o(1)})$, then $t_q = \widehat{\Omega}(d)$. [HKNS15]

- The $f(d)$ above can be an arbitrary growing function, and ω_{bool} is the exponent of Boolean matrix multiplication.

Acknowledgement

- We thank Thatchaphol Saranurak for helpful discussions.

Thank you!