Beyond Persistent Homology:
A Mathematical Guide

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Abstract. This note is a survey of persistent homology that is intended for doctoral students studying pure mathematics or for anyone who knows a bit about algebraic topology and is comfortable with the language of category theory. (We refer the reader to [**Rie17**] and [**May99**] as a reference on algebraic topology and category theory respectively). This note is an attempt to engage students of pure math about the exciting work being done in applied topology. This note is also an attempt to fill a gap in the literature by providing a an exposition of the mathematically interesting contributions of applied topology over the past decade, as well as posing some soft open questions and advertising the author's own research program. This note will cover persistent homology and some of its extentions: cellular (co)sheaves and filtrations of mapping spaces. The former comes from work by Justin Curry [**Cur14**] which is connected to the author's own research program. The latter comes from work by Andrew Blumberg and Michael Mandell [**BM13**].

1. Persistent Homology

The fundamental interest of topological persistence is to calculate the fiber-wise homology of a map

$$f : X \to \mathbb{R}$$

without placing many assumptions on f. This task encompasses, by the epigraph construction (§2), a second task: calculating the homology of a filtration

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_N = X \tag{1}$$

leading to a diagram in **Ab**:

$$H_\bullet(X_1) \to H_\bullet(X_2) \to \cdots \to H_\bullet(X_N) \tag{2}$$

The second task is of great interest in topological data analysis (TDA).

Fiber-wise topology has been studied classically, and even some elements are familiar to a student in a second semester course on algebraic topology. For example, as in [**BT13**], the Serre spectral sequence calculates the total homology $H_{p+q}(X)$ for a Serre fibration

$$F \hookrightarrow X \to B$$

with fiber F by taking out the second page of the spectral sequence to convergence:

$$E^2_{p,q} = H_p(B, H_q(F)) \Rightarrow E^\infty_{p,q} = H^{p+q}(X) \tag{3}$$

In the special case of a (Hurewicz) fibrations, equivalences of spaces over B determines an equivalence of fibers:

Theorem 1.1 (Ch. 6 §5 [**May99**])**.** Let $p : X \to B$ and $q : Y \to B$ be fibrations and let $f : X \to Y$ be a map such that $p = q \circ f$. Then, f is a homotopy equivalence if and only if

$$f^{-1}(t) \simeq g^{-1}(t) \tag{4}$$

for all $t \in B$.

These classical methods fail, for example, when $p : X \to B$ is not a fibration. To be sure, we can always factor any map through a fibration (Ch. 5 §3 [**May99**]), but it is not always natural to do so, especially when working with maps that come from data since we lose information in the process.

If we make the additional relaxation, which we will take for the remainder of this note, that *homology is taken with coeficients in a field*, $\mathbb{K}$, our homology groups are vector spaces and understanding fiber-wise homology becomes a problem of representation theory. By representation theory, we mean more broadly the decomposition of functors

$$F : \mathcal{C} \to \mathbf{Vec}_{\mathbb{K}}$$

which we call **generalized persistence modules** into indecomposable parts

$$F \cong \bigoplus_i F_i$$

where $\mathcal{C}$ is a small category. Here, $\mathcal{C}$ is typically *acyclic*, that is, every diagram in $\mathcal{C}$ commutes. Equivalently, $\mathcal{C}$ is acyclic if its classifying space, $\mathbf{B}\,\mathcal{C}$, is simply-connected.

The first such functor we encounter is a **persistence module**

$$F : (\mathbb{R}, \preceq) \to \mathbf{Vec}_{\mathbb{K}} \tag{5}$$

where $(\mathbb{R}, \preceq)$ is a poset viewed as a category with objects elements of $\mathbb{R}$ and

$$\hom(s, t) = *$$

if and only if

$$s \preceq t$$

($\emptyset$ otherwise). The principal example of a persistence modules is the **persistent homology** of a filtration X_t often begotten from sub-level sets $X_t := f^{-1}(\infty, t]$ of a function $f : X \to \mathbb{R}$:

$$PH_\bullet : t \rightsquigarrow H_\bullet(X_t; \mathbb{K}) \tag{6}$$

We will encounter a second functor called a **cellular cosheaf** in §2. Finally, in §3 we will examine variations of persistent homology constructed from filtrations of mapping spaces and their discrete version, mapping complexes.

Approximating a space. Suppose we are given a finite set Z of points. Given the additional hypothesis that each $z \in Z$ was sampled from a metric space (X, d) we can ask the natural question: *what is the homology of* X? As our problem is of a *statistical* nature, we might expect our answer to be. As we will see later, our *estimate* for $H_\bullet(X)$ is given by collection of intervals in the real line called a **barcode** from which we can make an actual guess at the homology of X. Our approach is the following: suppose we knew X which is, say, compact. Then, we can construct a finite cover $\{U_z\}_{z \in Z}$ of X. From this cover, we then construct a simplicial complex from the nerve construction:

Definition 1.2. We define a functor

$$\mathcal{N} : \mathbf{Cov}_X \to \mathbf{SC}$$

where $\mathbf{Cov}_X$ is the category of covers $\{U_\alpha\}_{\alpha\in I} \supseteq X$ with morphisms given by refinement, and $\mathbf{SC}$ is the category of simplicial complexes and simplicicial maps which we should think of as monotone maps between respective face-relation posets

$$\mathbf{Fc}(K) \to \mathbf{Fc}(L)$$

of two given complexes K and L. $\mathcal{N}(\mathcal{U})$ is given dimension-wise by

$$\mathcal{N}(\{U_\alpha\}_{\alpha\in I})_k = \{J \subseteq I : \ \cap_{\alpha\in J} U_\alpha \neq \emptyset, \ |J| = k+1\} \tag{7}$$

We hope to recover the homology of X from the homology of this complex. The standard complex used to approximate X is the Čech complex which is defined for any given resolution ε as follows

Definition 1.3. Given a finite sub-metric space $(Z, d) \subseteq \mathbb{R}^d$, define the **Čech complex**

$$\hat{C}_\varepsilon(Z) = \mathcal{N}\left(\{B_\varepsilon(z)\}_{z\in Z}\right)$$

When can we guarantee our approximation is good? Assuming $X \subseteq \mathbb{R}^d$ for d large enough (other weaker assumptions are possible), then if for some $\varepsilon > 0$ we know that $\{B_\varepsilon(z) \cap X\}_{z\in Z}$ is a good cover in the sense described below, then $\hat{C}_\varepsilon(Z)$ is equivalent to X *up to homotopy*, a stronger guarantee than homologically necessary. This fact is given by the Nerve Lemma, which the author believes is the most important topological result in TDA, a full proof of which is missing in the literature. We say a cover $\{U_\alpha\}_{\alpha\in I}$ is *good* if for every $J \subseteq I$

$$\cap_{\alpha\in J} U_\alpha \simeq *$$

Sometimes a stronger condition which the author calls *great* is desired: each $\cap_{\alpha\in J} U_\alpha$ is *convex*. Now we are ready to state the theorem:

Theorem 1.4 (Nerve Lemma). Let $\mathcal{U}$ be a locally finite good cover of a space X. Then,

$$\mathcal{N}(\mathcal{U}) \simeq X$$

Proof. Consider a the diagram D in **Top** determined by all the inclusions

$$\cap_{\alpha\in J} U_\alpha \hookrightarrow \cap_{\alpha\in J'} U_\alpha$$

for indexing set $J \subseteq J'$. Consider also the diagram D^* in **Top** where each $\cap_{\alpha\in J} U_\alpha$ is replaced by $*$ and maps are identity. Both diagrams are indexed by the partial order category $\mathcal{N}(\mathcal{U})$ determined by face relations, that is, we have functors

$$D, D^* : (\mathcal{N}(\mathcal{U}), \preceq) \to \mathbf{Top}$$

Since $\mathcal{U}$ is good,

$$D(\sigma) \simeq D^*(\sigma)$$

and the corresponding natural transformation

$$D \Rightarrow D^*$$

induces a homotopy equivalence

$$\mathrm{hocolim} D \xrightarrow{\simeq} \mathrm{hocolim} D^* \tag{8}$$

We see that $\mathrm{hocolim} D^*$ is $|\mathrm{Bary}\,(\mathcal{N}(\mathcal{U}))\,|$, hence

$$\mathrm{hocolim} D^* \simeq |\mathcal{N}(\mathcal{U})| \tag{9}$$

We also have

$$X \cong \mathrm{colim} D \tag{10}$$

Hence, by (10), (11), (12) it suffices to define an equivalence

$$\mathrm{hocolim} D \xrightarrow{\simeq} \mathrm{colim} D$$

Such a map

$$\mathrm{hocolim} D \xrightarrow{p} \mathrm{colim} D = X$$

is defined by

$$\left(\coprod_{\tau = v_0 \to \cdots \to v_n} |\tau| \times D(v_0) \right) / \sim \xrightarrow{p} \left(\coprod_{\tau = v_0 \to \cdots \to v_n} D(v_0) \right) / \sim \tag{11}$$

By our assumption that $\mathcal{U}$ is locally finite, there exists a partition of unity $\{\phi_\alpha\}_{\alpha \in I}$. Every $\mathbf{x} \in \mathrm{hocolim} D$ can be written as (x, y) where $x \in |\sigma|^\circ$ for some $\sigma \in \mathcal{N}(\mathcal{U})$, say, $\sigma = (\alpha_0, \ldots, \alpha_k)$ and $y \in U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$. Define $q : X \to \mathrm{hocolim} D$ by

$$q(x) = \left(\sum_{\alpha \in I} \phi_\alpha(x) v_\alpha, x \right) \tag{12}$$

where v_α is the 0-simplex in $\mathcal{N}(\mathcal{U})$ assigned to U_α. Clearly, $p \circ q = \mathrm{id}_X$. We claim

$$q \circ p : \mathrm{hocolim} D \xrightarrow{\simeq} \mathrm{hocolim} D$$

This fact is trivial in the case that $\mathcal{U}$ is a *great* cover, but will require a bit of work if $\mathcal{U}$ is *good*. We refer the reader to [**Koz00**] for more details on homotopy colimts and extensions of this construction. □

Decomposition of persistence modules. Consider the category of persistence modules

$$\mathrm{Rep}(\mathbb{R}) = \mathbf{Vec}^{(\mathbb{R}, \preceq)}$$

We say $F \in \mathrm{Rep}(\mathbb{R})$ is indeomposable if $F = G \oplus H$ defined point-wise

$$F_t = G_t \oplus H_t$$

implies either $G = \mathbf{0}$ or $H = \mathbf{0}$. Which objects are indecomposable?

Lemma 1.5. $F \in \mathrm{Rep}(\mathbb{R})$ is indecomposable if $F = \mathbb{K}_I$ for an interval $I \subseteq \mathbb{R}$ (read: *connected subset*) where

$$\mathbb{K}_I(t) = \begin{cases} \mathbb{K} & t \in I \\ 0 & t \notin I \end{cases} \tag{13}$$

and

$$\mathbb{K}_I(s \to t) = \begin{cases} \mathrm{id}_{\mathbb{K}} & s,\ t \in I \\ 0 & t \notin I \end{cases} \tag{14}$$

Such persistence modules are called **interval indecomposables**.

Proof. From [**Jac09**][p. 111], $\mathbb{K}_I$ is indecomposable if and only if the endomorphism ring

$$\mathrm{End}(\mathbb{K}_I) = \{\Phi : \mathbb{K}_I \Rightarrow \mathbb{K}_I\} \tag{15}$$

(check this has the structure of a ring) contains no idempotents besides 0 and 1. An easy calculation shows that in fact

$$\mathrm{End}(\mathbb{K}_I) \cong \mathbb{K} \tag{16}$$

which implies that $\mathbb{K}_I$ is indecomposable. □

Interval indecomposables have the advantage of being describable by a single $I \subseteq \mathbb{R}$, collections of which give rise to a metric space of barcodes which is crucial in answering our statistical question. Even better, if vector spaces F_t are finite dimensional for each t, we have a *decomposition into interval indecomposables* which is unique up to re-indexing. Uniqueness is by an application of a classical theorem originally attributed to Krull-Schmidt [**Jac09**][p. 115]. This slightly stronger and more recent theorem is due to William Crawley-Boevy:

Theorem 1.6 ([**CB15**])**.** Let F be a point-wise finite dimensional persistence module over $(\mathbb{R}, \preceq)$. Then,

$$F \cong \bigoplus_{I \in \mathcal{B}} \mathbb{K}_I \tag{17}$$

where $\mathcal{B}$ is a multi-set of intervals in $\mathbb{R}$ called the **barcode** of F

Persistence constructions. We now have all the necessary machinery to describe our method of approximating the homology of an unknown metric space (X, d) which we assume is embedded in some $\mathbb{R}^d$, given a sample of points Z in $\mathbb{R}^d$:

(1) Receive data $Z \subseteq \mathbb{R}^d$
(2) For $\varepsilon \in [0, \varepsilon_{\max}]$ compute
$$\hat{C}_\varepsilon(Z) \in \mathbf{SC}$$
where $\varepsilon_{\max}$ is the smallest ε for which
$$\hat{C}_\varepsilon(Z) \simeq *$$
(why does $\varepsilon_{\max}$ exist?)
(3) Previous step determines a persistence module
$$F_\bullet : \varepsilon \rightsquigarrow H_\bullet^{\text{simp.}}(\hat{C}_\varepsilon(Z); \mathbb{K}) \tag{18}$$
(4) Decompose $F_\bullet$ in each homological degree to obtain a graded barcode $\mathcal{B}_\bullet$.

Example 1.7. This pipeline is implemented for homological degree 1 where Z is a noisy sampling of S^1 with barcode as shown in Figure 1. We can interpret the longest bar that is *born* at $\varepsilon \approx 0.1$ and *dies* at $\varepsilon \approx 1.5$ as a single statistically significant class in $H^1(X)$ where X is the metric space the data is sampled from. This suggests that X is–homologically at least–a circle.

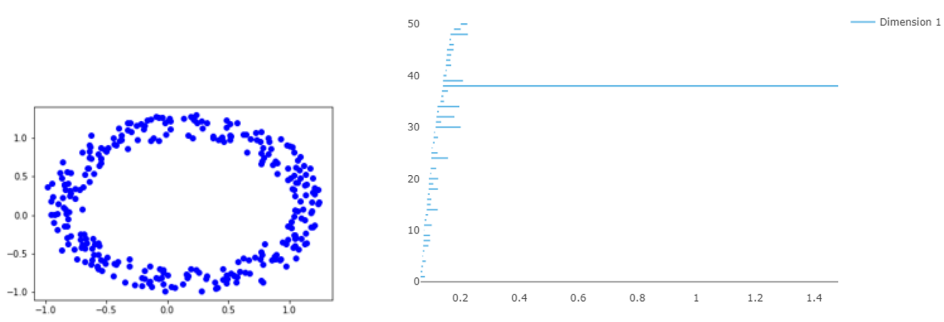

Figure 1. Barcode of noisy circle in degree 1 computed with Eirene [**HG16**]

Barcodes give a compact *topological* description of a point sample, or more generally, a filtration $\{X_t\}_{t\in\mathbb{R}}$ such as a filtration of sub-level sets

$$\left\{f^{-1}(-\infty,t]\right\}_{t\in\mathbb{R}} \tag{19}$$

of a map $f : X \to \mathbb{R}$. In fact, **sub-level set persistence** generalizes the Ĉech construction via the map

$$\begin{aligned} f &: \mathbb{R}^d \to \mathbb{R} \\ f(\mathbf{x}) &= \min_{z\in Z} ||\mathbf{x}-z|| \end{aligned} \tag{20}$$

Filtrations can be compared by metrics on the space of barcodes. One such metric called the *bottleneck distance* is

$$d_B(\mathcal{B},\mathcal{B}') = \inf_{\Phi\in\mathrm{Bij}(\mathcal{B},\mathcal{B}')} \sup_{I\in\mathcal{B}} \max\left\{|I^- - \Phi(I)^-|, |I^+ - \Phi(I)^+|\right\} \tag{21}$$

where I^- and I^+ are the left and right endpoints of an interval $I \subseteq \mathbb{R}$, possibly $+\infty$, and Φ is a matching of multi-sets $\mathcal{B}$ and $\mathcal{B}'$. Perhaps surprisingly, the mapping of metric spaces

$$\Psi : \mathrm{Map}_{\mathbf{Top}}(X,\mathbb{R}) \to \mathbf{Bar} \tag{22}$$

$$\Psi : f \rightsquigarrow \mathcal{B}H_\bullet(f^{-1}(\infty,-]) := \mathcal{B}\left(\mathcal{PH}_f\right) \tag{23}$$

is 1-Lipschitz continuous. We conclude this section with a result by Cohen-Steiner, Edelesbrunner and Harer that may best be interpreted by the slogan: *sub-level set peristence is robust*

Theorem 1.8 ([CSEH07])**.** Given compact X and $f,g \in \mathrm{Map}_{\mathbf{Top}}(X,\mathbb{R})$ with metric

$$d(f,g) = \inf_{x\in X} |f(x)-g(x)| \tag{24}$$

and bottleneck distance on the space of barcodes d_B, then

$$d_B\left(\mathcal{B}\,\mathcal{PH}_f, \mathcal{B}\,\mathcal{PH}_g\right) \le d(f,g) \tag{25}$$

2. Sheaves and Cohseaves

Let Y be a cell complex, say, the geometric realization of a simplicial complex for sake of simplicity. Y has face relation poset $\mathbf{Fc}(Y)$ given by

$$\sigma \trianglelefteq \tau$$

if σ is a face of τ. A **cellular sheaf** is a functor

$$\overline{\mathcal{F}} : \mathbf{Fc}(Y) \to \mathbf{Vec}_{\mathbb{K}} \tag{26}$$

A **cellular cosheaf** is a functor

$$\underline{\mathcal{F}} : \mathbf{Fc}(Y)^{\mathrm{op}} \to \mathbf{Vec}_{\mathbb{K}} \tag{27}$$

A **bisheaf** is a triple $\left(\overline{\mathcal{F}}, \underline{\mathcal{F}}, F\right)$ where the diagram

$$\begin{tikzcd} \overline{\mathcal{F}}(\sigma) \arrow[r, "\overline{\mathcal{F}}_{\sigma\trianglelefteq\tau}"] \arrow[d, "F_\sigma"] & \overline{\mathcal{F}}(\tau) \arrow[d, "F_\tau"] \\ \underline{\mathcal{F}}(\sigma) & \underline{\mathcal{F}}(\tau) \arrow[l, "\underline{\mathcal{F}}_{\sigma\trianglelefteq\tau}"] \end{tikzcd} \tag{28}$$

commutes. Here, the maps $\overline{\mathcal{F}}_{\sigma \trianglelefteq \tau}$ and $\underline{\mathcal{F}}_{\sigma \trianglelefteq \tau}$ are linear maps called **(co)restriction maps** induced by functoriality. While we will not discuss bisheaves any further, the author will note that the understanding of the interplay between sheaves and cosheaves, such as with respect to a stratification [**NP18**], is still an open area.

Remark 2.1. The reader familiar with sheaves may find a point of confusion. Let $\mathbf{D}$ be a target category. Usually, we want $\mathbf{D}$ to be abelian, and in the context of topological persistence $\mathbf{D} = \mathbf{Vec}_{\mathbb{K}}$. Classically, a **presheaf** over a space X is a functor

$$\mathcal{F} : \mathbf{Open}\,(X)^{\mathrm{op}} \to \mathbf{D} \tag{29}$$

where $\mathbf{Open}\,(X)$ is the poset of opens ordered by inclusion. Said presheaf is a **sheaf** if it satisfies the following gluing condition: for every open U and every open cover $\{U_\alpha\}_{\alpha \in I_U}$ of U, the following is an equalizer

$$\mathcal{F}(U) \dashrightarrow \textstyle\prod_{\alpha \in I_U} \mathcal{F}(U_\alpha) \underset{\prod F(\iota_\beta)}{\overset{\prod F(\iota_\alpha)}{\rightrightarrows}} \textstyle\prod_{\alpha,\beta \in I_U} \mathcal{F}(U_\alpha \cap U_\beta) \tag{30}$$

Why is a cellular sheaf contravariant? The reason is that a cellular sheaf is a sheaf over a certain topology called the **Alexandrov** topology. Given a poset $(\mathcal{P}, \preceq)$ such as the *face relation poset* of a complex, $\mathbf{Fc}(Y)$, the Alexandrov topology is the topology of P with basis

$$\{U_x := \{y : \; y \succeq x\}\}_{x \in \mathcal{P}} \tag{31}$$

Under this paradigm, a *cellular sheaf is a sheaf* over Alex($\mathbf{Fc}\ Y$).

Lattice sheaves. Cellular sheaves valued in other categories, such as the category of lattices

$$\bar{\mathcal{F}} : \mathbf{Fc}(Y) \to \mathbf{Lat} \tag{32}$$

are of great interest to the author. A **lattice** is a poset L such that every subset $S \subseteq L$ has a greatest lower bound denoted

$$\bigwedge S$$

and a least upper bound denoted

$$\bigvee S$$

Furthermore, since S is not assumed to be non-empty, L contains a top and bottom element

$$\top = \bigwedge \emptyset$$

$$\bot = \bigvee \emptyset$$

Morphisms in the category $\mathbf{Lat}$ are maps that preserve joins:

$$f(x \vee y) = f(x) \vee f(y) \tag{33}$$

$$f(\bot) = \bot \tag{34}$$

The study of lattice sheaves is ongoing and will appear in forthcoming work. Some interesting questions about them concern their **global sections**, that is, assignments

$$\mathbf{x} \in \prod_{v \in Y_0} \bar{\mathcal{F}}_v \tag{35}$$

where Y_0 are the vertices of Y, such that

$$\bar{\mathcal{F}}_{v \trianglelefteq e}\ x_v = \bar{\mathcal{F}}_{w \trianglelefteq e}\ x_w \tag{36}$$

for all edges $e : v \leftrightarrow w$. Global sections of sheaves valued in vector spaces or other abelian categories may be calculated by simple application of homological algebra [**Wei95**]. Calculating global sections or higher "cohomology" in non-abelian settings such as lattice sheaves is still open, but may be partially answered in a book by Marco Grandis [**Gra13**]. One such example of a lattice sheaf will appear at the end of this section.

Level set persistence. We begin with an observation that favors our interest in the homology of the fibers of a real-valued map over that of sub-level sets: given a $f : X \to \mathbb{R}$, we can construct a map $\tilde{f} : X \to \mathbb{R}$ such that

$$\tilde{f}^{-1}(t) = f^{-1}\left(-\infty, t\right] \tag{37}$$

via the construction of the epigraph

$$\text{epi } f = \{(x,t) : \; f(x) \leq t\} \tag{38}$$

In words, studying the homology of the fibers of a real-valued map encompasses the study of sub-level set filtrations.

Just as persistence modules are the proper algebraic structure to describe the homology of a filtered spaces, the author believes cosheaves are the proper algebraic structure to describe the homology of the fibers of a real valued map which we vaguely refer to as **level set persistence**. Unlike sub-level sets of a real-valued map, fibers of real-valued maps do not include into larger preimages in any cannonical way. Sheaves and cosheaves describe how local data over a space extends to global data. Here, our local data comprises of the homology of fibers

$$H_\bullet(f^{-1}(t))$$

For this reason, sheaves and cosheaves are the right machinery to describe level set persistence.

For **homologically tame** maps, the homology of fibers is constant along certain open intervals of $\mathbb{R}$ and changes at only finitely many values

$$t_0 < t_1 < \cdots < t_n$$

which determine the 0-simplices and 1-simplices ($[t_0, t_1]$, $[t_1, t_2]$ etc.) of a simplicial decomposition of $\mathbb{R}$. Given a simplex σ, define its **open star**

$$\text{st } \sigma = \bigcup_{\tau \succeq \sigma} |\tau|^\circ \tag{39}$$

In other words, an open star is the interior of the geometric realization on an Alexandrov open.

Definition 2.2. For a given decomposition of $\mathbb{R}$ by homological critical values, the **Leray cosheaf** of a function $f : X \to \mathbb{R}$ is given by

$$\mathcal{L}_\bullet(\sigma) := H_\bullet\left(f^{-1}(\text{st } \sigma)\right) \tag{40}$$

$$\mathcal{L}_\bullet(\sigma \trianglelefteq \tau) := H_\bullet\left(f^{-1}(\text{st } \tau) \hookrightarrow f^{-1}(\text{st } \sigma)\right) \tag{41}$$

Leray cosheaves, like persistence modules, decompose as a direct sum of interval indecomposables which lead to a barcode. Here we consider the more general definition of an interval in a poset: $I \subseteq \mathcal{P}$ is an interval if for any $x, y \in I$,

$$z \in \mathcal{P}, \; x \preceq z \preceq y \;\Rightarrow\; z \in I \tag{42}$$

Intervals in the face relation poset of a given simplicial decomposition of the real line have four types:

- $[i, i+1] := v_i \preceq e_i \succeq v_{i+1}$

- $[i, i+1) := v_i \preceq e_i$
- $(i, i+1] := e_i \succeq v_{i+1}$
- $(i, i+1) := e_i$

Interval cosheaves are defined analogously for interval I of one of four types: $\mathbb{K}_I$ takes the value $\mathbb{K}$ on I with identity maps, $\mathbf{0}$ elsewhere. See [**Cur14**] for a additional details about this decomposition and for details on the theory of cellular cosheaves in general.

Example 2.3. Consider the height function f on the torus $\mathbb{T}^2$ embedded in $\mathbb{R}^3$ as in Figure 2. f is in fact a Morse function and is naturally stratified by its critical values which we label

$$t_0 < t_1 < t_2 < t_3$$

The open stars of vertices are $(-\infty, t_1)$, (t_0, t_2), (t_1, t_3), and (t_1, ∞) with preimages $f^{-1}(\mathrm{st}\ v)$; open stars of edges are (t_0, t_1), (t_1, t_2), (t_2, t_3) and they include into open stars of vertices drawn on the right. Applying homology yields a cellular cosheaf in degrees 0 and 1. These cosheaves decompose into interval cosheaves to yield a barcode as in Figure 3.

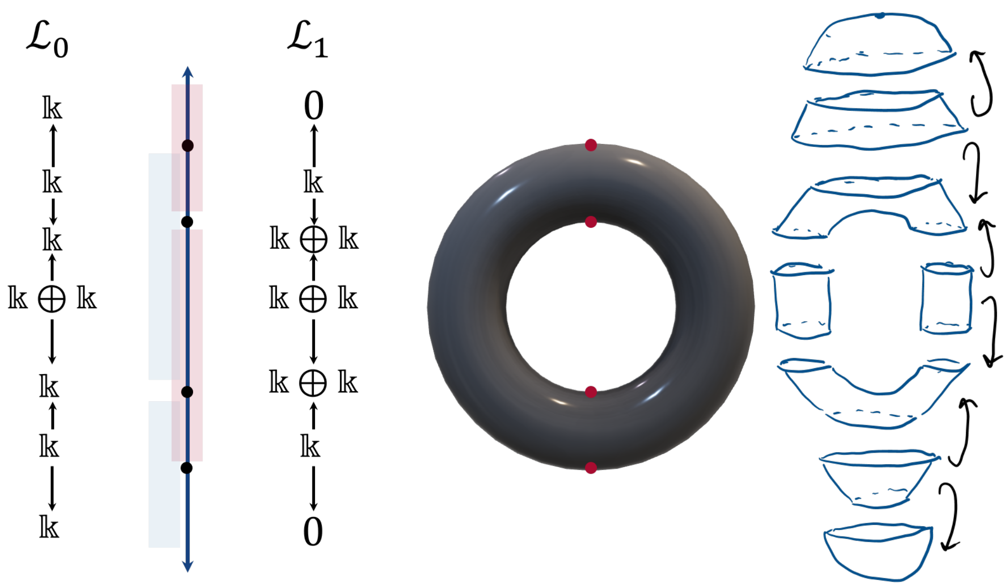

Figure 2. The Leray cosheaf of the standard height function of the torus with open stars of cells.

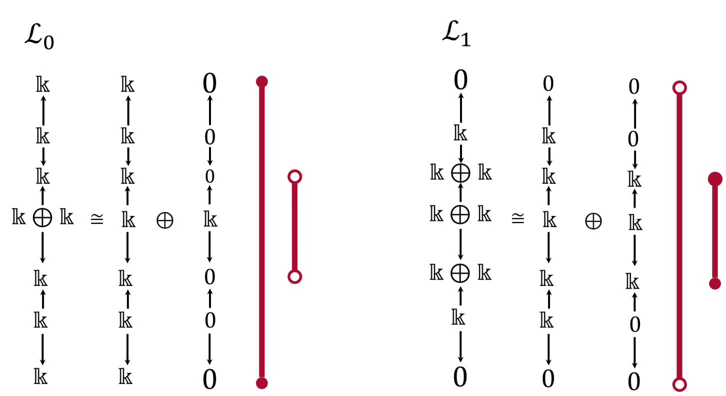

Figure 3. Barcodes $\mathcal{B}\ \mathcal{L}_0$ and $\mathcal{B}\ \mathcal{L}_1$.

Lattice embedding cosheaf. A natural question to ask at this point is the following. Fix a space X. *Given a graded cosheaf $\mathcal{F}_\bullet$ on a finite stratification of the real line, what is the class of real-valued maps out of X whose Leray cosheaf is $\mathcal{F}_\bullet$?* This question is already difficult where $X = S^2$ and our class of functions are required to be Morse [**Cur18**]. A **Morse-Smale function** on a smooth manifold is a C^∞ map

$$f : M \to \mathbb{R}$$

that has finitely many critical points, said critical points are non-degenerate and critical values are unique. Two disparate embeddings of S^2 are depicted in Figure 4. Their embeddings are different, fiber-wise, yet have the same barcode. While level-set persistence via the Leray cosheaf construction fails to distinguish these embeddings, a proposed tool called the **lattice embedding cosheaf** can distinguish them. Let

$$f = \pi_z \circ \iota : S^2 \to \mathbb{R} \tag{43}$$

be a Morse function that is a composition of an embedding

$$\iota : S^2 \hookrightarrow \mathbb{R}^3$$

and projection onto the z-axis

$$\pi_z : \mathbb{R}^3 \to \mathbb{R}$$

For a non-critical value t of f,

$$f^{-1}(t) \cong \coprod S^1 \hookrightarrow \pi_z^{-1}(t) = \mathbb{R}^2 \tag{44}$$

because non-critical fibers are compact 1-dimensional sub-manifolds of $\mathbb{R}^2$ which are completely classified as such. Over critical values, fibers are wedge products and disjoint unions of copies of S^1 and a single point. If $\mathbf{Fc}(\mathbb{R})$ is the face relation poset given by the stratification of $\mathbb{R}$ by critical values, then we define a cosheaf

$$\mathcal{E} : \mathbf{Fc}(\mathbb{R})^{\mathrm{op}} \to \mathbf{Lat}^\vee \tag{45}$$

$$\sigma \rightsquigarrow \pi_0\left(\mathbb{R}^2 - f^{-1}(\mathrm{st}\ \sigma)\right) \tag{46}$$

where $\mathbf{Lat}^\vee$ is the category of join-semilattices. We see that

$$\pi_0\left(\mathbb{R}^2 - f^{-1}(\mathrm{st}\ \sigma)\right)$$

has the structure of a join-semilattice: if e is an edge, then

$$\mathbb{R}^2 - f^{-1}(\mathrm{st}\ e)$$

can be collapsed down to

$$\mathbb{R}^2 - \iota(\coprod S^1)$$

for some embedding

$$\iota : \coprod S^1 \hookrightarrow \mathbb{R}^2$$

because by Morse theory [**Nic11**], the homotopy type of fibers between critical points do not change. Then, it is clear that

$$\pi_0\left(\mathbb{R}^2 - \iota(\coprod S^1)\right)$$

is a poset given by nesting relations at the boundary as shown in Figure 5. In fact, the stalks, $\mathcal{E}(e)$, over edges are meet semi-lattices, the top element of which is a π_0 class "way out at infinity" in $\mathbb{R}^2$. Further work shows that the stalks, $\mathcal{E}(v)$, over vertices has a structure of a lattice, and inclusions of open stars

$$\mathrm{st}\ e \hookrightarrow \mathrm{st}\ v$$

induce a map in **Set**

$$f^{\#}: \ \pi_0\left(\mathbb{R}^2 - f^{-1}(\mathrm{st}\ e)\right) \to \pi_0\left(\mathbb{R}^2 - f^{-1}(\mathrm{st}\ v)\right) \tag{47}$$

that is in fact a join semi-lattice homomorphism.

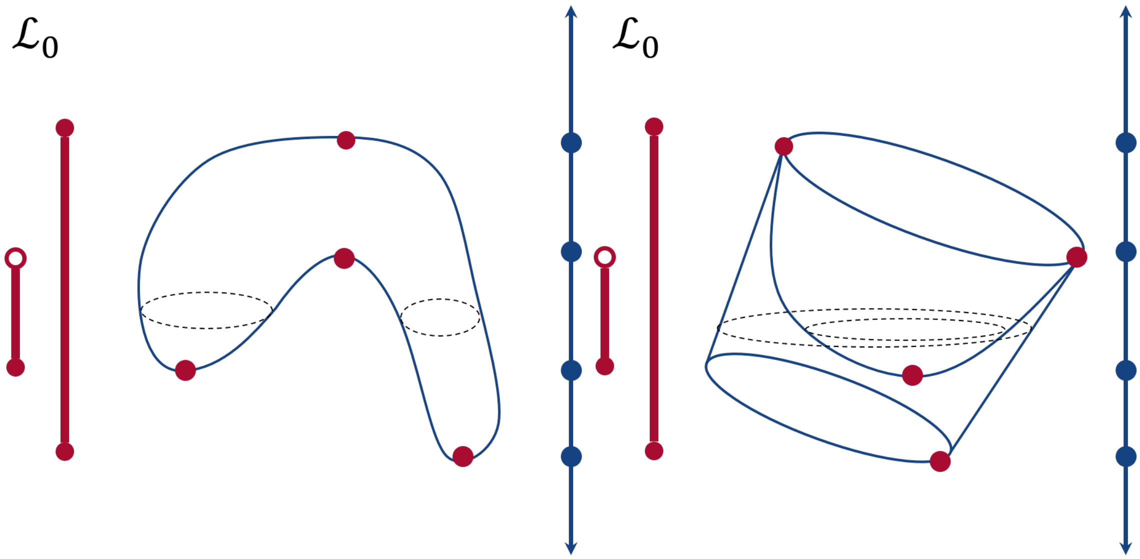

Figure 4. Two embeddings of $S^2 \subseteq \mathbb{R}^3$, the *worm* and *shotglass,* with the same Leray cosheaf, but different lattice embedding cosheaves with respect to the height function. (Note the *shotglass* is not a smooth embedding but can be easily made smooth by deforming the rim every so slightly).

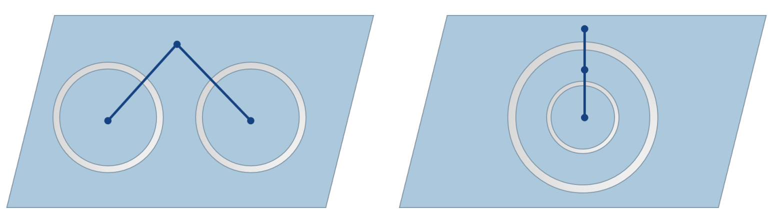

Figure 5. The meet semi-lattice structure on $\pi_0\left(\mathbb{R}^2 - \iota(\coprod S^1)\right)$ for two different embeddings.

3. Mapping spaces

As we have seen, the Leray cosheaf is a non-obvious generalization of persistent homology: $\mathcal{L}_k$ describes the fiberwise homology of a map $f : X \to \mathbb{R}$. We will now explore further generalizations. One motivation may be the following example: suppose some points Z sampled in a trivial embedding of $S^1 \coprod S^1$ are perturbed so that that $\hat{C}_\varepsilon(Z) \simeq S^1 \vee S^1$, *how can we distinguish this change in homotopy type* (See Figure 6)?

Persistent homotopy. Here is one approach to distinguish filtrations up to weak equivalence: given a pointed filtration $\{(X_t, x_t)\}_{t\in\mathbb{R}}$ which can be viewed as a functor

$$(\mathbb{R}, \preceq) \to \mathbf{Top}_*^{\mathrm{inj}}$$

we can pre-compose with

$$\pi_n : \mathbf{Top}_* \to \mathbf{Grp}$$

to obtain a persistence module of homotopy groups for $n \geq 1$

$$\Theta_n : (\mathbb{R}, \preceq) \to \mathbf{Grp} \tag{48}$$

where Θ_n maps faithfully into $\mathbf{Ab}$ for $n \geq 2$, and for $n = 0$ we have a functor

$$R : (\mathbb{R}, \preceq) \to \mathbf{Set} \tag{49}$$

which we call the **Reeb** functor. To the author's knowledge **persistent homotopy** groups have not been studied for $n > 0$. Reeb functors are studied in [**Cur19**]. The decomposition of

$$\Theta_n \otimes \mathbb{Q}_{\mathbb{R}}$$

of a filtration, for example, remains open to the author's knowledge.

The following lemma suggests another method for constructing a generalized persistence module:

Lemma 3.1. Given (compactly generated weak Hausdorff) spaces X and Y

$$\pi_0 \operatorname{Map}_{\mathbf{Top}}(X, Y) \cong \operatorname{Map}_{Ho(\mathbf{Top})}(X, Y) \tag{50}$$

Proof. By definition,

$$\pi_0(-) = [*, -]$$

Hence,

$$\pi_0 \operatorname{Map}_{\mathbf{Top}}(X, Y) = \left[*, Y^X\right] \cong [X \times *, Y] \cong [X, Y] \tag{51}$$

by the Hom-Tensor adjunction, where Y^X is $\operatorname{Map}(X, Y)$ with the compact open topology. $\square$

Hence, if we fix a test space X, then a filtration $\{Y_t\}_{t\in\mathbb{R}}$ leads to a persistence module valued in $\mathbf{Set}$

$$t \rightsquigarrow \pi_0 \operatorname{Map}_{\mathbf{Top}}(X, Y_t) \tag{52}$$

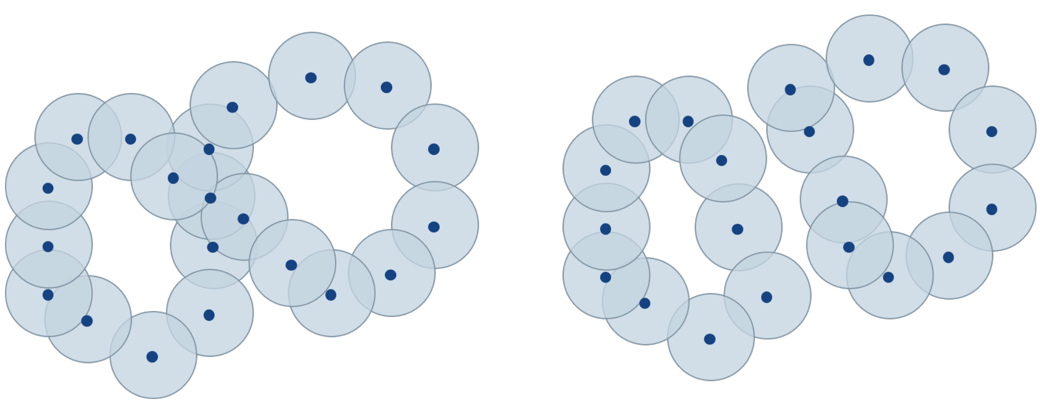

Figure 6. Čech coverings of two similar point clouds whose geometric realizations are not weakly equivalent.

since $\pi_0(-)$ and $\mathrm{Map}_{\mathbf{Top}}(X,-)$ are covariant functors. The properties of this persistence module have not yet been studied.

Mapping complexes. The Yoneda embedding [**Rie17**] provides the following lemma:

Lemma 3.2. For nice spaces Y, Y'

$$[-,Y] \cong [-,Y'] \tag{53}$$

if and only if $Y \simeq Y'$

Hence, the persistence module

$$t \rightsquigarrow \mathrm{Map}_{\mathbf{Top}}(X, Y_t)$$

cooresponding to a filtration Y_t may be better able to distinguish filtered spaces. In order to apply this intuition to Ĉech filtrations of a point sample, we need to understand mapping "spaces" of simplicial complexes.

Given a simplicial complexes X and Y, the set of simplicial maps

$$\mathrm{Map}_{\mathbf{SC}}(X, Y)$$

is a simplcial complex called the **contiguity complex**: $f_0, \ldots, f_k \in \mathrm{Map}_{\mathbf{SC}}(X, Y)$ form a k-simplex if the set

$$\{f_0(\sigma), \ldots, f_k(\sigma)\} \tag{54}$$

forms a simplex in Y for any simplex $\sigma \in X$. The following theorem of Andrew Blumberg and Michael Mandel guarantees a good enough approximation of a mapping space by a contiguity complex.

Theorem 3.3 ([**BM13**])**.** Let X be a finite simplicial complex and Y a simplicial complex. Let X_n be a sequence of subdivisions of X with compatible simplicial approximations

$$X_{n+1} \to X_n$$

If $\lim_{n\to\infty} \mathrm{Mesh} X_n = 0$, then

$$\bigcup_{\phi_n : \mathrm{Map}_{\mathbf{SC}}(X_n, Y) \to \mathrm{Map}_{\mathbf{SC}}(X_n, Y)} |\mathrm{Map}_{\mathbf{SC}}(X_n, Y)| \simeq \mathrm{Map}_{\mathbf{Top}}(|X|, |Y|) \tag{55}$$

One upshot of this theorem is that we may define a filtration of mapping complexes from a filtration of complexes $\{Y_t\}_{t\in\mathbb{R}}$:

$$\mathrm{Map}_{\mathbf{SC}}(X_n, Y_0) \hookrightarrow \mathrm{Map}_{\mathbf{SC}}(X_n, Y_1) \hookrightarrow \cdots \hookrightarrow \mathrm{Map}_{\mathbf{SC}}(X_n, Y_m) \tag{56}$$

Applying the homology functor to this filtation leads to a persistence module which leads to a barcode decomposition. The study of such barcodes has only been touched on in [**BM13**] and largely remains open.

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