Queueing Theory Reading Notes

Brendan Duke

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1 Definitions

2 Paper Summaries

2.1 Performance Modeling and Design of Computer Systems: Queueing Theory in Action [1]

2.1.1 Chapter 17 Networks of Queues and Jackson Product Form

A **Jackson Network** is a network with probabilistic routing where jobs are served in FCFS order, and service times are exponentially distributed. Arrivals, from inside and outside the network, are from Poisson processes.

$$\mathbf{P}\{\text{state is } (n_1, \ldots, n_k)\} = \prod_{i=1}^{k} \mathbf{P}\{n_i \text{ jobs at server } i\} = \prod_{i=1}^{k} \rho_i^{n_i}(1-\rho_i) \tag{1}$$

2.1.2 Chapter 18 Closed Networks of Queues

Solving Closed Batch Jackson Networks

1. Determine λ_i's by solving rate equations and picking a value for λ_1, e.g. $\lambda_1 = 1$.
2. Compute $\rho_i = \frac{\lambda_i}{\mu_i}$ for all i.
3. Set $\pi_{n_1,\ldots,n_k} = C' \rho_1^{n_1} \ldots \rho_k^{n_k}$
4. Use $\sum_{\substack{n_1,\ldots,n_k \\ \text{s.t.} \sum_i n_i = N}} \pi_{n_1,\ldots,n_k} = 1$ to solve for C'.

Mean Values Analysis (MVA) is a method for recursively computing the expected number of jobs at a server in a closed network with a total number of jobs M, namely $\mathbf{E}\left[N_j^{(M)}\right]$.

Theorem 1. *An arriving job to a server j in a closed Jackson network with $M > 1$ total jobs witnesses a job distribution at each server equivalent to that server's steady-state job distribution in the same network with $M - 1$ total jobs. In particular, the mean number of jobs that the arrival sees at server j is $\mathbf{E}\left[N_j^{(M-1)}\right]$.*

We would like to use the convenient variable p_j, which is independent of M, defined as,

$$p_j = \frac{\lambda_j^{(M)}}{\lambda^{(M)}} = \frac{X^{(M)} V_j}{\sum_{j=1}^{k} X^{(M)} V_j} = \frac{V_j}{\sum_{j=1}^{k} V_j}. \tag{2}$$

We use p_j in the following recursive definition of $\mathbf{E}\left[T_j^{(M)}\right]$,

$$\mathbf{E}\left[T_j^{(M)}\right] = \frac{1}{\mu_j} + \frac{p_j \cdot \lambda^{(M-1)} \mathbf{E}\left[T_j^{(M-1)}\right]}{\mu_j}. \tag{3}$$

In the above equation, we can use the fact that $\sum_{j=1}^{k} \mathbf{E}\left[N_j^{(M-1)}\right] = M - 1$ and apply Little's Law to find $\lambda^{(M-1)}$ as below,

$$\lambda^{(M-1)} = \frac{M-1}{\sum_{j=1}^{k} p_j \mathbf{E}\left[T_j^{(M-1)}\right]}. \tag{4}$$

References

[1] M. Harchol-Balter, *Performance Modeling and Design of Computer Systems: Queueing Theory in Action*, 1st ed. New York, NY, USA: Cambridge University Press, 2013.