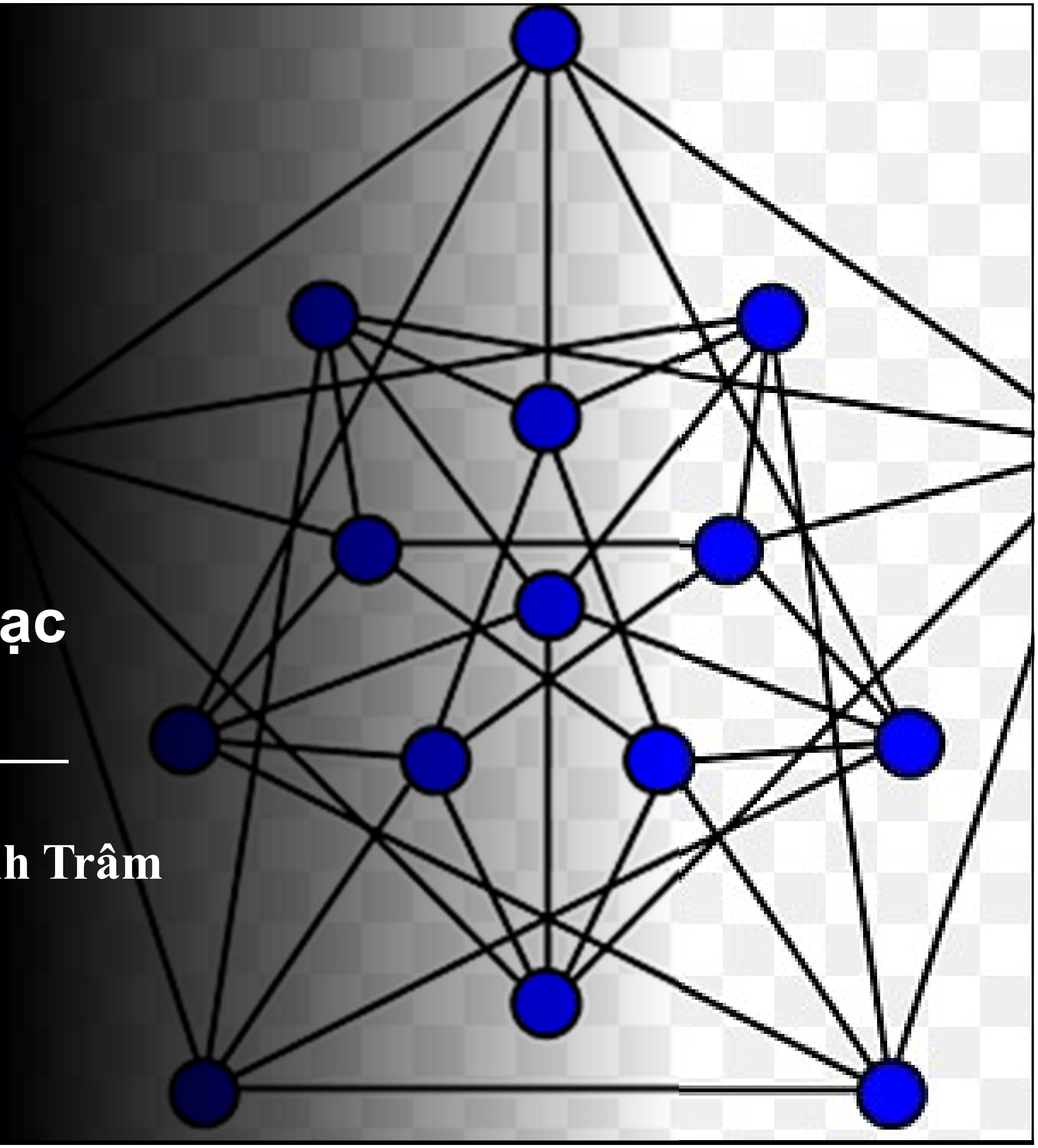
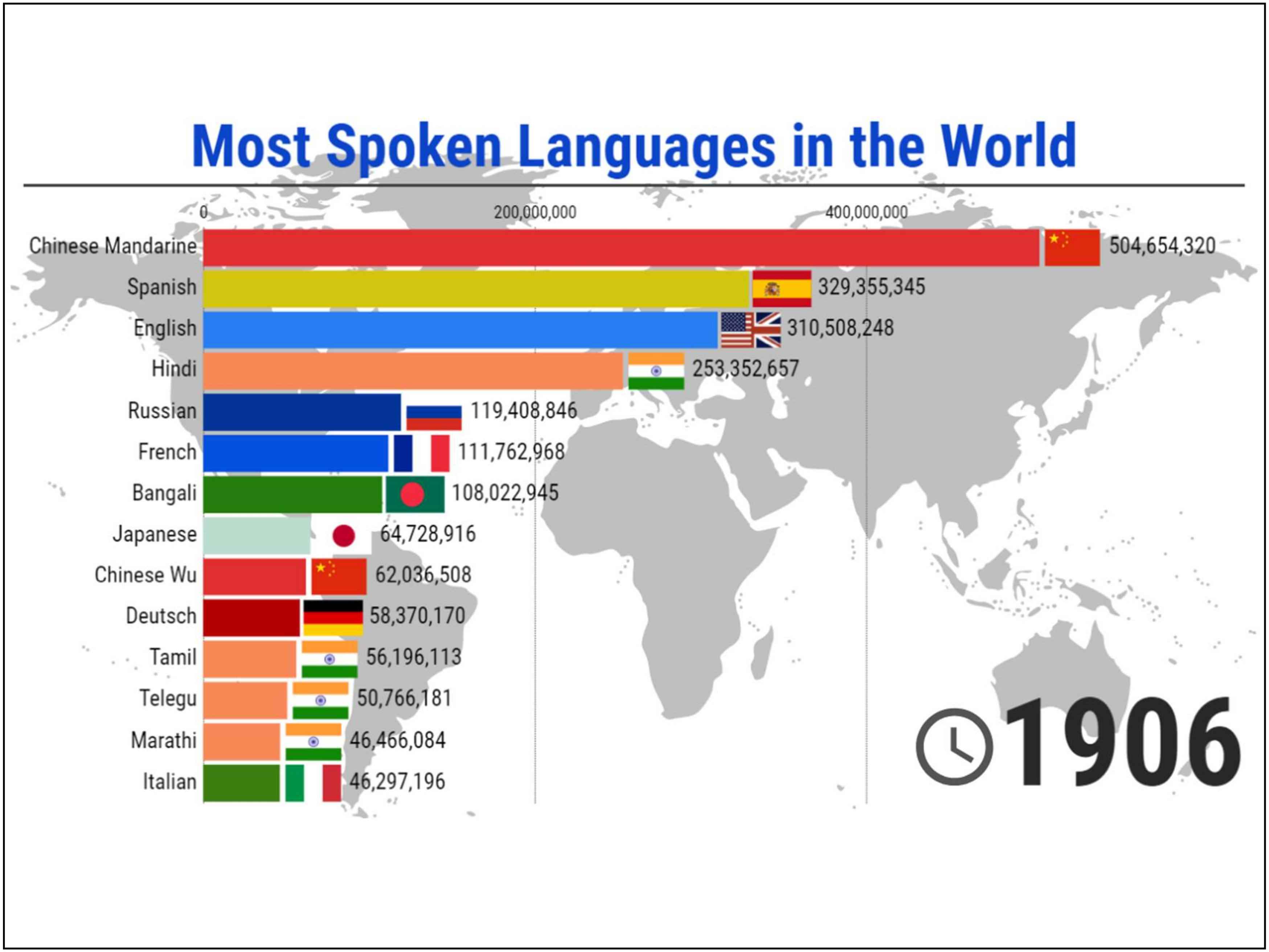
DISCRETE STRUCTURES
Cấu Trúc Rời Rạc



TS Nguyễn Thị Huỳnh Trâm

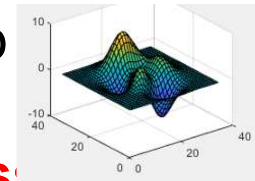

Most Spoken Languages in the World
0
200,000,000
400,000,000
Chinese Mandarine 504,654,320
Spanish 329,355,345
English 310,508,248
Hindi 253,352,657
Russian 119,408,846
French 111,762,968
Bangali 108,022,945
Japanese 64,728,916
Chinese Wu 62,036,508
Deutsch 58,370,170
Tamil 56,196,113
Telegu 50,766,181
Marathi 46,466,084
Italian 46,297,196
1906

What is discrete structures?

Web App Software AI

HTML C++ Visual Basic Java Python Matlab

PHP C# C Pascal



If, for, while, switch…case, function, class

C++	Python	Matlab

if (**dtb<5**) {
printf("**Loại yếu**");
} else {
printf("**Loại TB**");
}
	if **GPA<5**:
	printf("**Weak**")
	else:
	printf("**Average**")
		if **GPA<5**
		disp("**Weak**")
		else:
		disp("**Average**")
		end

Cấu trúc rời rạc là môn học nghiên cứu về các qui luật logic của toán học để biểu diễn các đối tượng rời rạc.

What is discrete structures?

"Discrete structures" is the study of logical principles in mathematics used to demonstrate discrete subjects

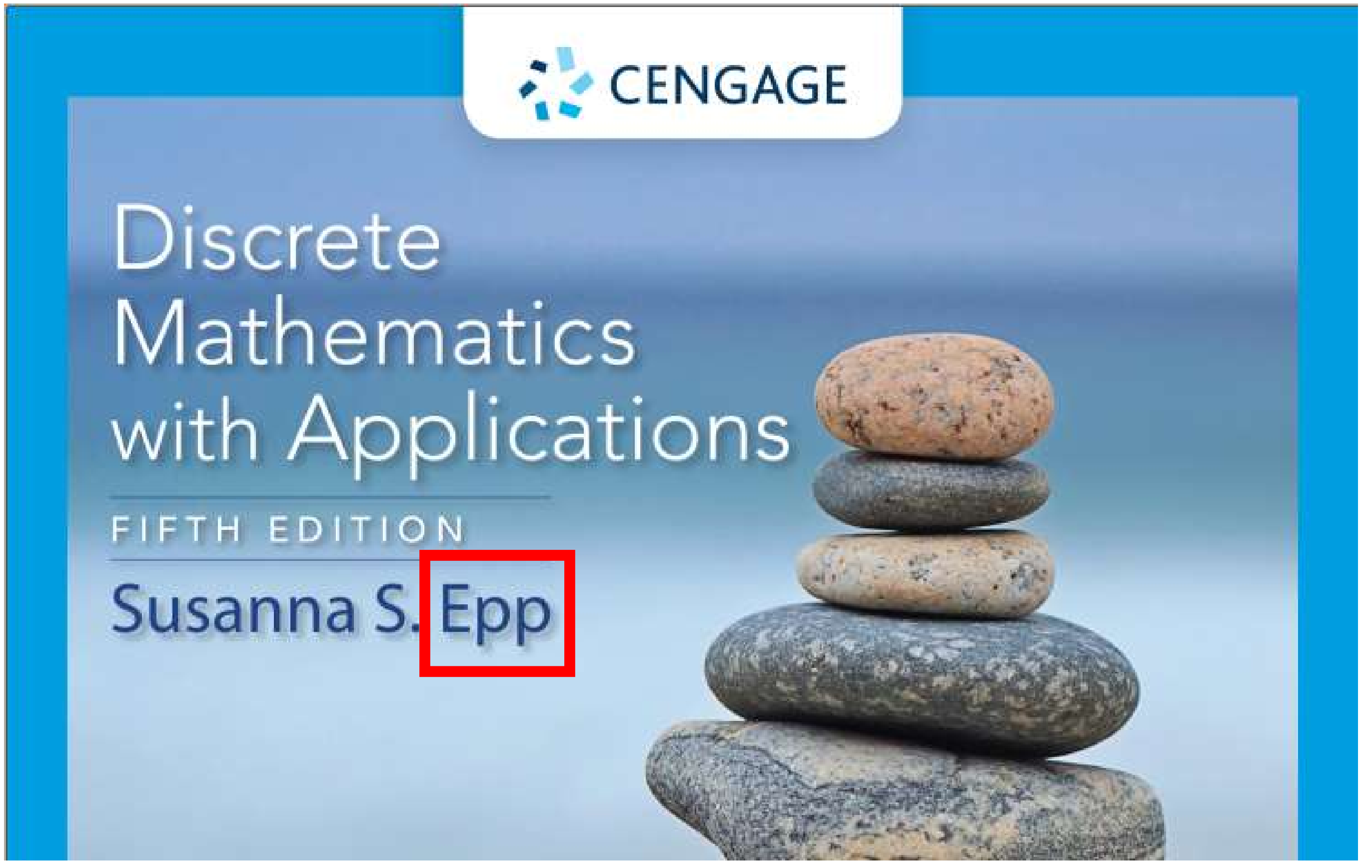
Cấu trúc rời rạc là môn học nghiên cứu về các qui luật logic của toán học để biểu diễn các đối tượng rời rạc.

These rules are used to distinguish between valid and invalid mathematical arguments.

Major goal of this course: understand and construct correct mathematical arguments.

Applications to computer science:

- The design of computer circuits
- The construction of computer programs
- The verification of the correctness of programs, and in many other ways.
- Software systems have been developed for constructing some, but not all, types of proofs automatically.


CENGAGE
Discrete Mathematics with Applications
FIFTH EDITION
Susanna S. Epp

Statements (Mệnh đề)

A statement (or **proposition)** is **a sentence** that is true or false but not both. Một mệnh đề là một câu mà có thể xác định là đúng hoặc sai nhưng không thể vừa đúng vừa sai

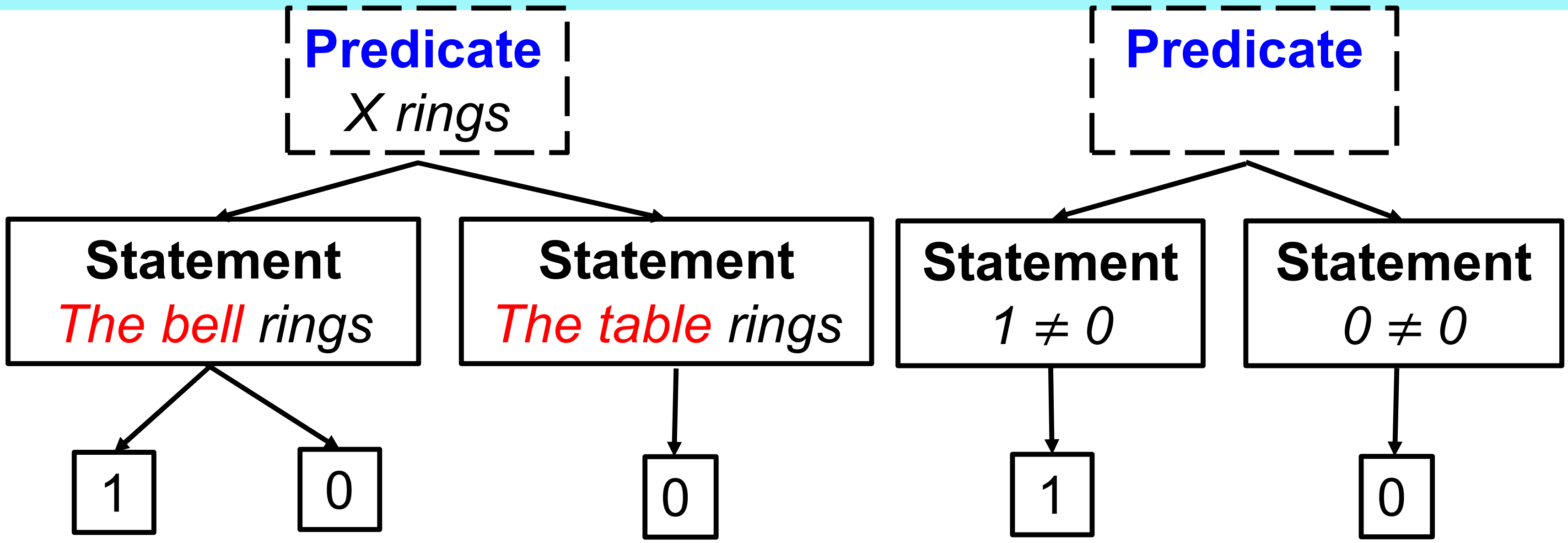
Statement *The bell rings*	**Statement** *The table rings*	**Statement** $1 \neq 0$	**Statement** $0 \neq 0$
True (T)(1) / False (F)(0)	F	T	F

The **truth value** (chân trị) of a proposition is **true**, denoted by **T** or **1**, if it is a true proposition (mệnh đề đúng).

The truth value of a proposition is **false**, denoted by **F** or **0**, if it is a false proposition.

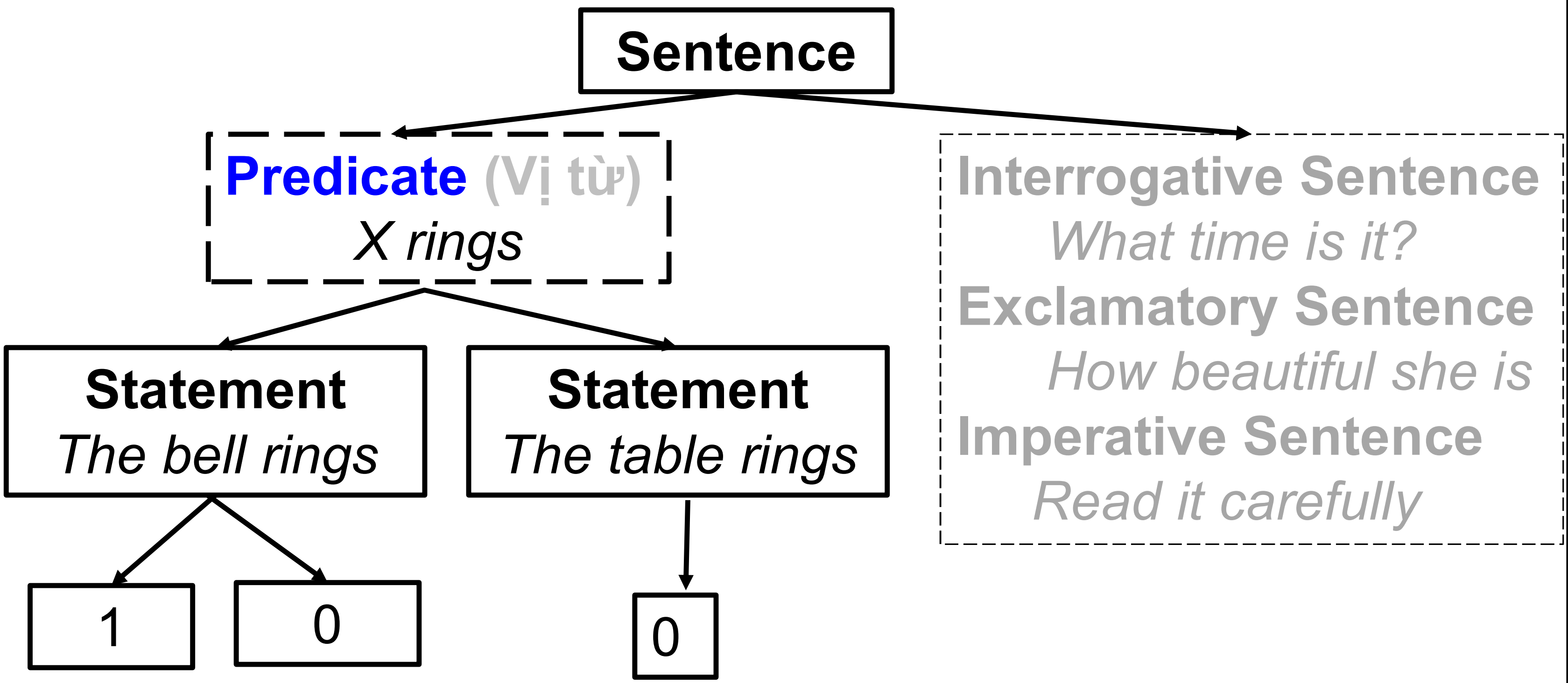
Predicate(Vị từ)

A predicate is a sentence that contains a finite number of variables and become a statement when specific values are substituted for the variables (Vị từ là một câu chứa một số hữu hạn các biến và sẽ trở thành mệnh đề khi thay giá trị cụ thể vào các biến trong câu)



Statements

A statement (or proposition) is **a sentence** that is true or false, but not both.



Argument

An argument (lý luận) is a sequence of statements aimed at demonstrating the truth of an assertion (khẳng định).

The preceding statements are called ***premises*** (tiền đề)

The assertion at the end of the sequence is called the ***conclusion***

If the bell rings or the flag drops, **then** the race is over

If $x^2 \neq 4$ **then** $x \neq 2$ and $x \neq -2$

Cho phương trình: $\mathrm{f}(\mathrm{x}, y)=\frac{1}{x}+\frac{1}{y-3}$

Điều kiện để phương trình xảy ra là $\forall x\ \mathrm{x} \neq 0$ **v**à $\forall y, y \neq 3$

Toán học	C++

Điều kiện để phương trình xảy ra là $\mathrm{x} \neq 0$ **v**à $y \neq 3$
Miền xác định $\forall x, \mathrm{x} \neq 0$, $\forall y, y \neq 3$
Nếu $\mathrm{x} \neq 0$ VÀ $y \neq 3$ thì
$\mathrm{f}(\mathrm{x}, y)=\frac{1}{x}+\frac{1}{y-3}$
Ngược lại
Nếu x=0 HAY $\mathrm{y}=3$ thì
Phương trình vô nghiệm
	`float fxy =0, x =0, y=0; int n = 100;`
	`for(x =0;  x<=n; x++)`
	`    for(y =0;  y<=n; y++)`
	`        {`
	`            if (x!=0 && y !=3)`
	`            {`
	`                fxy= (1/x)+(1/(y-3))`
	`                printf (“f(x,y) =”, fxy )`
	`            }`
	`            else // (x!=0 || y !=3)`
	`            {`
	`                printf (“Phương trình VN”)`
	`            }`
	`        }`

Compound Statements

If $x \neq 0$ and $y \neq 3$ then

$$f(x, y) = \frac{1}{x} + \frac{1}{y-3}$$

Vị từ

else

If x=0 or $y = 3$ then

No solution

x = 1; y =2

statement

If $\{(1 \neq 0)$ and $[\,!\,(2 = 3)]\}$ then

$$f(1,2) = \frac{1}{1} + \frac{1}{2-3} = 0$$

Statements (mệnh đề nguyên thủy, mệnh đề sơ cấp): Không được xây dựng từ các mệnh đề khác nhờ các liên từ hoặc trạng từ KHÔNG (!, ¬)

Compound Statements

If $x \neq 0$ and $y \neq 3$ then

$$f(x, y) = \frac{1}{x} + \frac{1}{y-3}$$

Vị từ

else

If x=0 or $y = 3$ then

No solution

x = 1; y =2

statement

If $\{(1 \neq 0)$ and $[\,!\,(2 = 3)]\}$ then

Compound statements

$$f(1,2) = \frac{1}{1} + \frac{1}{2-3} = 0$$

Compound Statements (mệnh đề phức hợp): Là các mệnh đề được xây dựng từ các mệnh đề khác nhờ liên từ kết chúng lại như: **Và** (AND, ∧), **Hay, Hoặc** (OR, ∨), **nếu…thì**.. , trạng từ **KHÔNG** (!, ¬, ~, NOT)

Toán học	**Các nghiệm của toán học** **Các lần thực hiện của ngôn ngữ lập trình C++**
Điều kiện để phương trình xảy ra là: $\mathrm{x} \neq 0$ **v**à $y \neq 3$ Miền xác định $\forall x, \mathrm{x} \neq 0,$ $\forall y, y \neq 3$	x = 1; y =2 $\mathrm{f}(1{,}2) = \frac{1}{1} + \frac{1}{2-3} = 0$ x = 2, y=1 $\mathrm{f}(2{,}4) = \frac{1}{2} + \frac{1}{1-3} = 0$
Nếu x ≠ 0 và $y \neq 3$ thì $\mathrm{f}(\mathrm{x}, y) = \frac{1}{x} + \frac{1}{y-3}$ Ngược lại Nếu x=0 hay $y = 3$ thì Phương trình vô nghiệm	x = 0, y =4 $\mathrm{f}(0{,}4) = \frac{1}{0} + \frac{1}{4-3}$ Phương trình vô nghiệm x = 3, y =3 $\mathrm{f}(0{,}4) = \frac{1}{3} + \frac{1}{3-3}$ Phương trình vô nghiệm

Compound Statements

If $x \neq 0$ and $y \neq 3$ then

$$f(x, y) = \frac{1}{x} + \frac{1}{y-3}$$

Vị từ

else

If x=0 or $y = 3$ then

No solution

x = 1; y =2

statement

If $\{(1 \neq 0)$ and $[\,!\,(2 = 3)]\}$ then

Compound statements

$$f(1,2) = \frac{1}{1} + \frac{1}{2-3} = 0$$

If (true statement)

action 1

else // false statement

action 2

Propositional Calculus (Phép tính mệnh đề): nghiên cứu chân trị của các mệnh đề phức hợp

Statement Variables (Biến mệnh đề)

An argument is a sequence of statements aimed at demonstrating the truth of an assertion (khẳng định).

“p”, “q”, and “r” is used to represent component sentences.

p,q,r …are statement variables (biến mệnh đề)

Ex.

$$\text{If } \overbrace{\text{the bell rings}}^{p} \text{ or } \overbrace{\text{the flag drops}}^{q}\text{, then } \overbrace{\text{the race is over}}^{r}.$$

Logical connectives

1) **Negation (<u>Phép phủ định</u>):** If p is a statement variable, the negation of p is "**not p**" or "**it is not the case that p**" and is denoted **~p** or **$\neg$p** or **!p**

(Phủ định của mệnh đề p: "không" p hay "phủ định của" p).

Ex 1: <u>**Find the negation**</u> of the proposition and express this in simple English: "Michael's PC runs Linux"

Solution:

The negation is "It is not the case that Michael's PC runs Linux."

This negation can be more simply expressed as

"Michael's PC does not run Linux.

Logical connectives

Ex 2: **Find the negation** of the proposition and express this in simple English

"Vandana's smartphone has at least 32GB of memory"

Solution: .

The negation is

This negation can also be expressed as

or even more simply as

Logical connectives

Note Java, C, and C++ use the following notations:

$\sim$	!
$\wedge$	&&
$\vee$	||

1) Negation (<u>Phép phủ định</u>):

The Truth Table for the Negation of a Propostion

(Bảng chân trị) :

p	$\neg p$
T	F
F	T

p	¬p
1	0
0	1

If (true statement)

action 1

else // false statement

action 2

If $x \neq 0$ and ! ($y = 3$) then

$$f(x,y) = \frac{1}{x} + \frac{1}{y-3}$$

Else

If x=0 or y = 3 then

No solution

x = 1; y =2

$$\text{If } \overbrace{1 \neq 0}^{1} \ \&\& \ \underbrace{!\overbrace{(2 = 3)}^{0}}_{1}$$

$$f(1,2) = \frac{1}{1} + \frac{1}{2-3} = 0$$

Logical connectives

2) Conjunction (Phép "Và"): if p and q are statement variables, the conjunction of p and q is **"p and q"**, denoted **p ∧ q**. It is true when, and only when, both p and q are true. If either p and q is false, or if both are false, p ∧ q is false.

Phép nối liền (và, hội, giao) của hai mệnh đề p, q được kí hiệu bởi p ∧ q (đọc là "p và q"), là mệnh đề được xác định bởi : p ∧ q đúng khi và chỉ khi p và q đồng thời đúng.

Truth Table for p ∧ q

p	q	p ∧ q
1	1	1
1	0	0
0	1	0
0	0	0

p but q	means	p and q
neither p nor q	means	$\sim p$ and $\sim q$.

Logical connectives

Find the conjunction of the propositions *p* and *q* where *p* is the proposition "Rebecca's PC has more than 16 GB free hard disk space" and *q* is the proposition "The processor in Rebecca's PC runs faster than 1 GHz."

Solution:

The conjunction of these propositions, $p \wedge q$, is the proposition "Rebecca's PC has more than 16 GB free hard disk space, **and** the processor in Rebecca's PC runs faster than 1 GHz."

This conjunction can be expressed more simply as "Rebecca's PC has more than 16 GB free hard disk space, **and its processor** runs faster than 1 GHz."

Logical connectives

Translating from English to Symbols: *But* and *Neither-Nor*

Write each of the following sentences symbolically, letting h = "It is hot" and s = "It is sunny."

a. It is not hot but it is sunny.

b. It is neither hot nor sunny.

p but q	means	p and q
neither p nor q	means	$\sim p$ and $\sim q$.

Solution

Logical connectives

Translating from English to Symbols: *But* and *Neither-Nor*

Write each of the following sentences symbolically, letting h = "It is hot" and s = "It is sunny."

a. It is not hot but it is sunny.

b. It is neither hot nor sunny.

p but q	means	p and q
neither p nor q	means	$\sim p$ and $\sim q$.

Solution

a. The given sentence is equivalent to "It is not hot and it is sunny,"

which can be written symbolically as $\neg h \wedge s$. ($\sim h \wedge s$)

b. To say it is neither hot nor sunny means that it is not hot and it is not sunny.

Therefore, the given sentence can be written symbolically as $\neg h \wedge \neg s$

($\sim h \wedge \sim s$)

Logical connectives

2) Conjunction

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

If $x \neq 0$ and $!\,(y = 3)$ then
$$f(x, y) = \frac{1}{x} + \frac{1}{y-3}$$
Else
If x=0 or $y = 3$ then
No solution
	x = 1; y =2
	If $\underbrace{\underbrace{1 \neq 0}_{1}\ \&\&\ \underbrace{!\,(\overbrace{2 = 3}^{0})}_{1}}_{1}$
	$$f(1,2) = \frac{1}{1} + \frac{1}{2-3} = 0$$

If (true statement)

action 1

else // false statement

action 2

Logical connectives

2) Conjunction

p	q	p ∧ q
1	1	1
1	0	0
0	1	0
0	0	0

If $x \neq 0$ and $!\,(y = 3)$ then
$$f(x, y) = \frac{1}{x} + \frac{1}{y-3}$$
Else
If x=0 or $y = 3$ then
No solution
	x = 1; y =3
	$$\text{If } \underbrace{\underbrace{1 \neq 0}_{1} \;\&\&\; \underbrace{!\,\overbrace{(3 = 3)}^{1}}_{0}}_{0}$$
	Else

If (true statement)

action 1

else // false statement

action 2

Logical connectives

3. Disjunction (phép hoặc): If p and q are statement variable, the disjunction of p and q is **"p or q",** denoted **$p \vee q$.** It is true when either p is true, or q is true, or both and q are true; it is false only when both p and q are false.

Phép hoặc (nối rời, tuyển, hợp) của hai mệnh đề p, q được kí hiệu bởi $p \vee q$ (đọc là "p hay q"), là mệnh đề được định bởi : $p \vee q$ sai khi và chỉ khi p và q đồng thời sai.

Truth Table for $p \vee q$

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

Logical connectives

p	q	p ∧ q
1	1	1
1	0	0
0	1	0
0	0	0

p	q	p ∨ q
1	1	1
1	0	1
0	1	1
0	0	0

If $x \neq 0$ and ! ($y = 3$) then

$$f(x, y) = \frac{1}{x} + \frac{1}{y-3}$$

Else

If x=0 or y = 3 then

No solution

x = 1; y =3

$$\text{If } \underbrace{\underbrace{1 \neq 0}_{1} \ \&\& \ \underbrace{!\,(\overbrace{3 = 3}^{1})}_{0}}_{0}$$

Else

$$\text{If } \underbrace{\underbrace{x=0}_{0} \ \| \ \underbrace{y = 3}_{1}}_{1}$$

No solution

Logical connectives

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

The notation for inequalities involves *and* and *or* statements. For instance, if x, a, and b are particular real numbers, then

$x \le a$	means	$x < a$	or	$x = a$
$a \le x \le b$	means	$a \le x$	and	$x \le b.$

Note that the inequality $2 \le x \le 1$ is not satisfied by any real numbers because

$$2 \le x \le 1 \quad \text{means} \quad 2 \le x \quad \text{and} \quad x \le 1,$$

and this is false no matter what number x happens to be.

Logical connectives

$x \le a$	means	$x < a$	or	$x = a$
$a \le x \le b$	means	$a \le x$	and	$x \le b.$

p	q	$p \wedge q$	p	q	$p \vee q$
1	1	1	1	1	1
1	0	0	1	0	1
0	1	0	0	1	1
0	0	0	0	0	0

And, *Or*, and Inequalities

Suppose x is a particular real number. Let p, q, and r symbolize "$0 < x$," "$x < 3$," and "$x = 3$," respectively. Write the following inequalities symbolically:

a. $x \le 3$ b. $0 < x < 3$ c. $0 < x \le 3$

Solution

a. $q \vee r$ b. $p \wedge q$ c. $p \wedge (q \vee r)$ ■

Truth Table (Bảng chân trị)

The Truth Table for a given statement form displays the truth values that correspond to all possible combinations of truth values for its component statement variables

Bảng chân trị của dạng mệnh đề E(p,q,r): là bảng ghi tất cả các trường hợp chân trị có thể xảy ra đối với dạng mệnh đề E theo chân trị của các biến mệnh đề p, q, r.

Nếu có n biến, bảng này sẽ có 2^n dòng, chưa kể dòng tiêu đề.

$p \lor (q \land r)$

	p	q	r				
1							
2							
3							
4							
5							
6							
7							
8							

Truth Table

Bảng chân trị của 2 dạng mệnh đề p∨(q∧r) và (p∨q) ∧r

Thứ tự thực hiện phép logic

1) Bên trong dấu ngoặc
2) ~
3) ∧ ∨

	p	q	r				
1	0						
2	0						
3	0						
4	0						
5	1						
6	1						
7	1						
8	1						

Truth Table

Bảng chân trị của 2 dạng mệnh đề p∨(q∧r) và (p∨q) ∧r

	p	q	r	q∧r	p∨(q∧r)	p∨q	(p∨q) ∧r
1	0	0	0	0	0	0	0
2	0	0	1	0	0	0	0
3	0	1	0	0	0	1	0
4	0	1	1	1	1	1	1
5	1	0	0	0	1	1	0
6	1	0	1	0	1	1	1
7	1	1	0	0	1	1	0
8	1	1	1	1	1	1	1

Hai dạng mệnh đề **p∨(q∧r) và (p∨q) ∧r** có bảng chân trị khác nhau => Thứ tự thực hiện phép nối quan trọng và sự cần thiết của các dấu ()

Logically Equivalent (Tương đương logic)

Definition: Two statement forms are called **logically equivalent** if, and only if, they have identical truth values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms E and F is denoted by writing $E \equiv F$ **or E⇔ F** (Hai dạng mệnh đề E và F được nói là tương đương logic nếu chúng có cùng bảng chân trị)

Ex

p	q	p ∧ q	q ∧ p
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

p ∧ q and q ∧ p always have the same truth values, hence they are logically equivalent.

Logically Equivalent

p	q	p ∧ q	q ∧ p
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

(1) Dogs bark and cats meow.

(2) Cats meow and dogs bark.

If (1) is true, it follows that (2) must also be true.

On the other hand, if (1) is false, it follows that (2) must also be false.

(1) and (2) are logically equivalent statements.

Compound Statements: Order of Operations

- Order of operations:
  - ~ is performed first
  - $\wedge$ and $\vee$ are coequal in order of operation

$\sim p \wedge q = (\sim p) \wedge q$

$p \wedge q \vee r$

Ambiguous

- Use parentheses to override or disambiguate order of operations

$\sim(p \wedge q)$

Negation of $p \wedge q$

$(p \wedge q) \vee r$ $\quad$ $p \wedge (q \vee r)$

Unambiguous

Logical connectives

Inclusive-Or (HAY, OR)

(Phép nối rời , Phép tuyển)

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

Exclusive-Or (p XOR q)(HOẶC)

(Phép nối rời, Phép tuyển loại)

p	q	$p \veebar q$ $p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

"Students who have taken calculus **or** computer science can take this class."

"Students who have taken calculus **or** computer science, **but** not both, can enroll in this class."
"Soup or salad comes with an entrée,"

- Construct the truth table for this statement form:

$$(p \vee q) \wedge \sim(p \wedge q)$$

p	q	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$(p \vee q) \wedge \sim(p \wedge q)$
T	T				
T	F				
F	T				
F	F				

$(p \vee q) \wedge \sim(p \wedge q)$ is also known as exclusive-or (why?)
Denoted as $p \oplus q$ or $p \text{ XOR } q$.

Logically Equivalent

Definition: Two statement forms are called **logically equivalent** if, and only if, they have identical truth values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms E and F is denoted by writing $\boldsymbol{E \equiv F}$ **or E⇔ F** (Hai dạng mệnh đề E và F được nói là tương đương logic nếu chúng có cùng bảng chân trị)

Ex

Double negation $\sim(\sim p) \equiv p$

$\neg(\neg p) \equiv p$

p	**~p**	**~(~p)**
$\boxed{0}$	1	$\boxed{0}$
$\boxed{1}$	0	$\boxed{1}$

Logically Equivalent

Showing Non-equivalence

To show that statement forms *P* and *Q* are not logically equivalent, there are 2 ways:

a) Truth table – find at least one row where their truth values differ.

b) Find a counter example – concrete statements for each of the two forms, one of which is true and the other of which is false.

Ex: Show that the following 2 statement forms are not logically equivalent.

$\sim(p \wedge q)$ $\sim p \wedge \sim q$

a) Truth Table Method

p	q	~p	~q	p ∧ q	~(p ∧ q)	~p ∧ ~q
0	0	1	1	0	1	1
0	1	1	0	0	1	0
1	0	0	1	0	1	0
1	1	0	0	1	0	0

Logically Equivalent

Showing Non-equivalence

To show that statement forms *P* and *Q* are not logically equivalent, there are 2 ways:

a) Truth table – find at least one row where their truth values differ.

b) Find a counter example – concrete statements for each of the two forms, one of which is true and the other of which is false.

b) Counter example method:

Let p be the statement “0<1” and q the statement “1<0”

$\sim(p \wedge q)$ “Not the case that both 0<1 and 1<0” which is TRUE

$\sim p \wedge \sim q$ “Not 0<1” and “not 1<0” which is FALSE

Logically Equivalent

Ex: Write negations for each of the following:

a. John is 6 feet tall **and** he weighs at least 200 pounds.

Let p = John **is** 6 feet tall

q = he weighs **at least** 200 pounds

~p = John **is not** 6 feet tall

~q = he weighs **less than** 200 pounds

John **is** 6 feet tall **and** he weighs **at least** 200 pounds.

p ∧ q

$\sim(p \wedge q) \equiv \sim p \vee \sim q$ De Morgan's Laws

→John **is not** 6 feet tall **or** he weighs **less than** 200 pounds.

Logically Equivalent

Ex: Write negations for each of the following:

De Morgan's Laws $\sim(p \wedge q) \equiv \sim p \vee \sim q$

a. John **is** 6 feet tall **and** he weighs **at least** 200 pounds.

→John **is not** 6 feet tall **or** he weighs **less than** 200 pounds.

De Morgan's Laws $\sim(p \vee q) \equiv \sim p \wedge \sim q$

b. The bus **was** late **or** Tom's watch **was** slow.

→

→

Logically Equivalent
De Morgan's Laws

Ex: Write negations for each of the following:

b. The bus was late **or** Tom's watch was slow.

$\sim(p \vee q) \equiv \sim p \wedge \sim q$ De Morgan's Laws

Logically Equivalent
De Morgan's Laws

Ex: Write negations for each of the following:

b. The bus was late **or** Tom's watch was slow.

$\sim(p \vee q) \equiv \sim p \wedge \sim q$ De Morgan's Laws

Tautologies and Contradictions

p	q	$p \vee \neg p$	$p \wedge \neg p$
0	1	1	0
0	1	1	0
1	0	1	0
1	0	1	0

A **tautology** is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variable. A statement whose form is a tautology is **a tautological statement** (Dạng mệnh đề được gọi là hằng đúng nếu nó luôn nhận giá trị **1**)

A **contradication** is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a contradiction is **a contradictory statement.** (Dạng mệnh đề gọi là hằng sai (hay mâu thuẫn) nếu nó luôn nhận giá trị 0)

Tautologies and Contradictions

Logical equivalence involving tautologies and contradictions

Ex: If **t** is a tautology and **c** is a contradiction, show that

$p \wedge \mathbf{t} \equiv p$ and $p \wedge \mathbf{c} \equiv \mathbf{c}$

p	t	c	p∧t	p∧c
0	1	0	0	0
1	1	0	1	0

Summary of Logical Equivalences

Theorem 2.1.1 Logical Equivalence: Given any statement variables *p*, *q* and *r*, a tautology **t** and a contradiction **c**:

1	Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2	Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3	Distributive laws	$p \wedge (q \vee r)$ $\equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r)$ $\equiv (p \vee q) \wedge (p \vee r)$
4	Identity laws	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5	Negation laws	$p \vee {\sim}p \equiv \mathbf{t}$	$p \wedge {\sim}p \equiv \mathbf{c}$
6	Double negative law	${\sim}({\sim}p) \equiv p$	

Summary of Logical Equivalences

Theorem 2.1.1 Logical Equivalence: Given any statement variables *p*, *q* and *r*, a tautology **t** and a contradiction **c**:

7	Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
8	Universal bound laws	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9	De Morgan's laws	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10	Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11	Negation of **t** and **c**	$\sim \mathbf{t} \equiv \mathbf{c}$	$\sim \mathbf{c} \equiv \mathbf{t}$

Ex: Use the laws in Theorem 2.1.1 to verify the following logical equivalence:

$$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$$

if (~(~p Λq) Λ(p V q) ≡ if(p)

Ex: Use the laws in Theorem 2.1.1 to verify the following logical equivalence: $\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$

$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q)$ De Morgan's

$\equiv (p \vee \sim q) \wedge (p \vee q)$ Double Negative Law

$\equiv (p \vee (\sim q \wedge q)$ Distributive Laws

$\equiv (p \vee c)$ Negation Law

$\equiv p$ Identity Laws

1	Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2	Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3	Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4	Identity laws	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5	Negation laws	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6	Double negative law	$\sim(\sim p) \equiv p$	

7	Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
8	Universal bound laws	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9	De Morgan's laws	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10	Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11	Negation of **t** and **c**	$\sim \mathbf{t} \equiv \mathbf{c}$	$\sim \mathbf{c} \equiv \mathbf{t}$

Conditional Statements

4) Conditional (Implication, <u>Phép kéo theo)</u> If p and q are statement variables, the conditional of q by p is "**if p then q**" or **"p implies q"**, denoted **$p \to q$.**

It is false when p is true and q is false; otherwise it is true.

Mệnh đề p kéo theo q của hai mệnh đề p và q ký hiệu bởi $p \to q$ (đọc là "p kéo theo q" hay "Nếu p thì q" hay "p là điều kiện đủ của q" hay "q là điều kiện cần của p") là mệnh đề được định bởi: $p \to q$ sai khi và chỉ khi p đúng mà q sai.

p	q	p→q
0	0	1
0	1	1
1	0	0
1	1	1

Conditional Statements

4) Conditional (Implication, <u>Phép kéo theo)</u> If p and q are statement variables, the conditional of q by p is "**if p then q**" or **"p implies q",** denoted $\mathbf{p \rightarrow q}$.

It is false when p is true and q is false; otherwise it is true.

We call p the hypothesis (or antecedent) of the conditional and q the conclusion (or consequent)

p	**q**	$\mathbf{p \rightarrow q}$
0	0	1
0	1	1
1	0	0
1	1	1

If the bell rings or the flag drops, **then** the race is over

If p **then** q

$p \rightarrow \mathbf{q}$

$(r \vee s) \rightarrow \mathbf{q}$

Conditional Statements

p	q	p→q
0	0	1
0	1	1
1	0	0
1	1	1

The pledge many politicians make when running for office is

"If I am elected, then I will lower taxes."

$$p \rightarrow q$$

If the politician is elected, voters would expect this politician to lower taxes.

p	q	p→q
0	0	1
0	1	1
1	0	0
1	1	1

It is only when the politician is elected but **does not** lower taxes that voters can say that the politician has broken the campaign pledge

Conditional Statements

p	q	p→q
0	0	1
0	1	1
1	0	0
1	1	1

The pledge many politicians make when running for office is

"If I am elected, then I will lower taxes."

Furthermore, if the politician is not elected, then voters will not have any expectation that this person will lower taxes, although the person may have sufficient influence to cause those in power to lower taxes.

A conditional statement that is true by virtue of the fact that its hypothesis is false is often called vacuously true or true by default.

Conditional Statements

p	q	p→q
0	0	1
0	1	1
1	0	0
1	1	1

"If you show up for work Monday morning, then you will get the job" is vacuously true if you do NOT show up for work Monday morning

In general, when the "if" part of an if-then statement is false, the statement as a whole is said to be true, regardless of whether the conclusion is true or false.

Ex: A Conditional Statement with a False Hypothesis

If 0 = 1, then 1 = 2

Strange as it may seem, the statement as a whole is true!

Conditional Statements

p	q	p→q

p	q	~p	~pVq

Representation of If-Then as Or

$p \to q \equiv$

Negation of Conditional Statement

$\sim(p \to q) \equiv$

$\equiv$

$\equiv$

Conditional Statements

p	q	p→q
0	0	1
0	1	1
1	0	0
1	1	1

p	q	~p	~pVq
0	0	1	1
0	1	1	1
1	0	0	0
1	1	0	1

Representation of If-Then as Or

$$p \to q \equiv \sim p \vee q$$

Negation of Conditional Statement

$$\sim(p \to q) \equiv$$

$$\equiv$$

$$\equiv$$

Conditional Statements

p	q	p→q
0	0	1
0	1	1
1	0	0
1	1	1

p	q	~p	~pVq
0	0	1	1
0	1	1	1
1	0	0	0
1	1	0	1

Representation of If-Then as Or

$p \to q \equiv \sim p \vee q$

Negation of Conditional Statement

$\sim(p \to q) \equiv \sim(\sim p \vee q)$

$\equiv$

$\equiv$

Conditional Statements

p	q	p→q
0	0	1
0	1	1
1	0	0
1	1	1

p	q	~p	~pVq
0	0	1	1
0	1	1	1
1	0	0	0
1	1	0	1

Representation of If-Then as Or

$$p \to q \equiv \sim p \vee q$$

Negation of Conditional Statement

$$\sim(p \to q) \equiv \sim(\sim p \vee q)$$
$$\equiv \sim(\sim p) \wedge \sim(q) \quad \text{De Morgan}$$
$$\equiv p \wedge \sim q$$

Negation of Conditional Statement

$$\sim(p \to q) \equiv p \wedge \sim q$$

Conditional Statements

Representation of If-Then as Or

$$p \rightarrow q \equiv \sim p \vee q$$

Ex: Rewrite the following statement in *if-then* form:

Either you get to work on time or you are fired.

Solution: Let ~*p* be "You get to work on time" *q* be "You are fired".

Also, p is "You do not get to work on time".

Either you get to work on time or you are fired.

$\vee$

$\sim p \vee q$

$p \rightarrow q$ (Representation of If-Then as Or)

If you do not get to work on time, you are fired.

Conditional Statements

Negation of Conditional Statement

$\sim(p \rightarrow q) \equiv p \wedge \sim q$

Ex: Negation the following statement

Either you get to work on time or you are fired.

Solution: Let $\sim p$ be "You get to work on time" q be "You are fired".

Also, p is "You do not get to work on time".

~q You are not fired

Either you get to work on time or you are fired.

$\sim p \vee q$

$p \rightarrow q$ (Representation of If-Then as Or)

Conditional Statements

Negation of Conditional Statement
$\sim(p \rightarrow q) \equiv p \wedge \sim q$

Ex: Negation for the following statement

Either you get to work on time or you are fired.

Solution: Let $\sim p$ be "You get to work on time" q be "You are fired".

Also, p is "You do not get to work on time".

$\sim q$ You are not fired

Either you get to work on time or you are fired.

$\sim p \vee q$

$p \rightarrow q$ (Representation of If-Then as Or)

$\sim(p \rightarrow q) \equiv p \wedge \sim q$ Negation of Conditional Statement

You do not get to work on time and you are not fired.

Conditional Statements

Representation of If-Then as Or

$$p \to q \equiv \sim p \vee q$$

Negation of Conditional Statement

$\sim(p \to q) \equiv p \wedge \sim q$

Ex: Write negation for each of the following statements:

a. If my car is in the repair shop, then I cannot get to class.

b. If Sara lives in Athens, then she lives in Greece.

Conditional Statements

Representation of If-Then as Or

$$p \rightarrow q \equiv {\sim}p \vee q$$

Negation of Conditional Statement

$${\sim}(p \rightarrow q) \equiv p \wedge {\sim}q$$

Ex: Write negation for each of the following statements:

a. If my car is in the repair shop, then I cannot get to class.

b. If Sara lives in Athens, then she lives in Greece.

Contrapositive of a Conditional Statement

Definition (**Contrapositive**) **(Phản đảo):** The contrapositive of a conditional statement of the form “if p then q” is “if ~q then ~p”

Symbolically, the contrapostitive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$

$$p \rightarrow q \equiv \sim q \rightarrow \sim p$$

Ex: Write each of the following statements in its equivalent contrapositive form:

a. If Howard can swim across the lake, then Howard can swim to the island

b. If today is Easter, then tomorrow is Monday

Contrapositive of a Conditional Statement

Definition (**Contrapositive**) **(Phản đảo):** The contrapositive of a conditional statement of the form "if p then q" is "if ~q then ~p"

Symbolically, the contrapostitive of **$p \rightarrow q$ is $\sim q \rightarrow \sim p$**

$$p \rightarrow q \equiv \sim q \rightarrow \sim p$$

If the bell rings or the flag drops, **then** the race is over

$$(r \vee s) \rightarrow q$$

$$\sim q \rightarrow \sim(r \vee s)$$

$$\sim q \rightarrow \sim r \wedge \sim s \quad \text{De Morgan Law}$$

Contrapositive of a Conditional Statement

Definition (**Contrapositive**) **(Phản đảo):** The contrapositive of a conditional statement of the form "if p then q" is "if ~q then ~p"

Symbolically, the contrapostitive of $p \to q$ is $\sim q \to \sim p$

$$p \to q \equiv \sim q \to \sim p$$

If the bell rings or the flag drops, **then** the race is over

$$(r \lor s) \to q$$

$$\sim q \to \sim(r \lor s)$$

$$\sim q \to \sim r \land \sim s \quad \text{De Morgan Law}$$

If the race is not over **then** the bell hasn't rung or the flag hasn't dropped

Logic Form

If r **or** s, **then** q.
∴ **If** not q, **then** not r and not s.

∴ ***Therefore***

Two sentences below have very different content but their logical form is the same.

If $\overbrace{\text{the bell rings}}^{r}$ or $\overbrace{\text{the flag drops}}^{s}$, then $\overbrace{\text{the race is over}}^{q}$.

∴ If $\overbrace{\text{the race is not over}}^{\text{not } q}$, then $\overbrace{\text{the bell hasn't rung}}^{\text{not } r}$ and $\overbrace{\text{the flag hasn't dropped}}^{\text{not } s}$.

If $\overbrace{x = 2}^{r}$ or $\overbrace{x = -2}^{s}$, then $\overbrace{x^2 = 4}^{q}$.

∴ If $\overbrace{x^2 \neq 4}^{\text{not } q}$, then $\overbrace{x \neq 2}^{\text{not } s}$ and $\overbrace{x \neq -2}^{\text{not } r}$.

Logical Form

Set p = the bell rings

q = the flag drops

r = the race is over

not p = the bell hasn’t rung

not q = the flag hasn’t dropped

not r = the race is not over

If the bell rings or the flag drops, then the race is over.

If *r* **or** *s*, **then** *q*.

∴ **If** not *q*, **then** not *r* and not *s*.

∴If the race is not over, then the bell hasn’t rung and the flag hasn’t dropped

Converse and Inverse of a Conditional Statement

Definition (Converse) (Đảo): The converse of a conditional statement of the form “if p then q” is “if q then p”

Symbolically, the contrapostitive of $\mathbf{p \to q}$ **is** $\mathbf{q \to p}$

$$\mathbf{p \to q \not\equiv q \to p}$$

Definition (Inverse) (Nghịch đảo): The inverse of a conditional statement of the form “if p then q” is “if ~p then ~q”

Symbolically, the contrapostitive of $\mathbf{p \to q}$ **is** $\mathbf{\sim p \to \sim q}$

If not q then not p		if p then q		if q then p		If not p then not q
$\sim q \to \sim p$	$\equiv$	$p \to q$	$\not\equiv$	$q \to p$	$\equiv$	$\sim p \to \sim q$
contrapositive (Phản đảo)		*conditional statement*		*converse* (Đảo)		*inverse* (Nghịch Đảo)

If not q then not p		if p then q		if q then p		If not p then not q
$\sim q \to \sim p$	$\equiv$	$p \to q$	$\not\equiv$	$q \to p$	$\equiv$	$\sim p \to \sim q$
contrapositive (Phản đảo)		*conditional statement*		*converse* (Đảo)		*inverse* (Nghịch Đảo)

Ex: Write the contrapositive, converse and inverse of the following statements:

a. If Howard can swim across the lake, then Howard can swim to the island.

Contrapositive:

Converse:

Inverse:

If not q then not p	if p then q		if q then p		If not p then not q
$\sim q \to \sim p$ $\equiv$	$p \to q$	$\not\equiv$	$q \to p$	$\equiv$	$\sim p \to \sim q$
contrapositive (Phản đảo)	*conditional statement*		*converse* (Đảo)		*inverse* (Nghịch Đảo)

Ex: Write the contrapositive, converse and inverse of the following statements:

b. If today is Easter, then tomorrow is Monday.

Contrapositive:

Converse:

Inverse:

Logical Form

Fill in the blanks below so that argument Ex1 has the same form as argument Ex2. Then represent the common form of the arguments using letters to stand for component sentences.

Ex1: **If** Nam is a math major **or** Nam is a computer science major, **then** Nam will take Discrete Structure.

Nam is a computer science major.

Therefore, Nam will take Discrete Structure

Ex2: If logic is easy or ______(1)_________, then________(2)________.

I will study hard.

Therefore, I will get an A in this course

Logical Form

The common form of the arguments is	
If p or q, then r. ∴ If not r, then not p and not q.	If p or q, then r. q. Therefore, r.

Ex1: If Nam is a math major or Nam is a computer science major, then Nam will take Discrete Structure.

Nam is a computer science major.

Therefore, Nam will take Discrete Structure

Ex2: If logic is easy or I (will) study hard, then I will get an A in this course.

I will study hard.

Therefore, I will get an A in this course

Only If

To say "*p* only if *q*" means that *p* can take place only if *q* takes place also.

That is, if *q* does not take place, then *p* cannot take place.

Another way to say this is that if *p* occurs, then *q* must also occur (using contrapositive)

Definition (Only if): If p and q are statements, **"p only if q"** means **"if not q then not p"** or, equivalently **"if p then q"**

Only If

Definition (Only if): If p and q are statements, **"p only if q"** means **"if not q then not p"** or, equivalently **"if p then q"**

Ex: Rewrite the following statement in if-then form in two ways, one of which is the contrapositive of the other.

p only if q

John will break the world's record only if he runs the mile in under four minutes.

"if not q then not p"

If John does not run the mile in under four minutes, then *John will not break the world's record*.

if p then q

If *John breaks the world's record* then he will have runs the mile in under four minutes

“q whenever p”

If not q then not p	if p then q		if q then p		If not p then not q
$\sim q \rightarrow \sim p$ ≡	$p \rightarrow q$	≢	$q \rightarrow p$	≡	$\sim p \rightarrow \sim q$
contrapositive (Phản đảo)	*conditional statement*		*converse* (Đảo)		*inverse* (Nghịch Đảo)

What are the contrapositive, the converse, and the inverse of the conditional statement “The home team wins whenever it is raining?”

Solution: Because “q whenever p” is one of the ways to express the conditional statement $p \rightarrow q$, the original statement can be rewritten as

“**If** it is raining, **then** the home team wins.”

Consequently, the contrapositive of this conditional statement is

“**If** the home team **does not** win, **then** it is **not** raining.”

The converse is “If the home team wins, then it is raining.”

The inverse is “If it is not raining, then the home team does not win.”

"if *p*, then *q*"	"*p* is sufficient for *q*"	"*q* follows from *p*"
"if *p*, *q*"	"a sufficient condition for *q* is *p*"	"*q* whenever *p*"
"*q* if *p*"	"a necessary condition for *p* is *q*"	"*p* only if *q*"
"*q* when *p*"	"*q* is necessary for *p*"	
"*q* unless ~*p*"	"p is a sufficient condition for q"	
"*p* implies *q*"	"q is a necessary condition for p" mean "if not q then not p"	

If not q then not p		if p then q		if q then p		If not p then not q
$\sim q \rightarrow \sim p$	$\equiv$	$p \rightarrow q$	$\not\equiv$	$q \rightarrow p$	$\equiv$	$\sim p \rightarrow \sim q$
contrapositive (Phản đảo)		*conditional statement*		*converse* (Đảo)		*inverse* (Nghịch Đảo)

Representation of If-Then as Or

$$\mathbf{p \rightarrow q \equiv \sim p \vee q}$$

Negation of Conditional Statement

$$\mathbf{\sim(p \rightarrow q) \equiv p \wedge \sim q}$$

p	q	p→q
0	0	1
0	1	1
1	0	0
1	1	1

Let p be the statement "Maria learns discrete mathematics" and q the statement "Maria will find a good job."

Express the statement $p \rightarrow q$ as a statement in English.

Solution: From the definition of conditional statements, we see that when p is the statement "Maria learns discrete mathematics" and q is the statement "Maria will find a good job,"

$p \rightarrow q$ represents the statement

if p, then q

"**If** Maria learns discrete mathematics, **then** she will find a good job."

There are many other ways to express this conditional statement in English. Among the most natural of these are:

q when *p*

"Maria will find a good job **when** she learns discrete mathematics."

p is sufficient for *q*

"For Maria to learn discrete mathematics, **it is sufficient for** her to get a good job."

q unless ~*p*

"Maria will find a good job **unless she does not** learn discrete mathematics."

Logical connectives

Biconditionals: Given statement variables p and q, the biconditional of p and q is “p if, and only if, q” and denoted p ↔ q.

It is true if both p and q have the same truth values and is false if p and q have opposite truth values.

The words if and only if are sometimes abbreviate iff

(Phép kéo theo hai chiều): Mệnh đề p kéo theo q và ngược lại của hai mệnh đề p và q, ký hiệu bởi p ↔ q (đọc là “p nếu và chỉ nếu q” hay “p khi và chỉ khi q” hay “p là điều kiện cần và đủ của q” hay “p tương đương với q”), là mệnh đề xác định bởi:
p ↔ q đúng khi và chỉ khi p và q có cùng chân trị

p	**q**	**p ↔ q** **p → q** **q → p**
0	0	1
0	1	0
1	0	0
1	1	1

Logical connectives

p	q	$p \leftrightarrow q$ $p \to q$ $q \to p$
0	0	1
0	1	0
1	0	0
1	1	1

$p \leftrightarrow q$ $\equiv$ $(p \to q) \land (q \to p)$

"*p* is necessary and sufficient for *q*"

"if *p* then *q*, and conversely"

"*p* iff *q*"

"p if and only if"

Ex: Rewrite the following statement as a conjunction of two if-then statements.

This computer program is correct if, and only if, it produces correct answers for all possible sets of input data.

Logical connectives

$$p \leftrightarrow q \quad \equiv \quad (p \rightarrow q) \wedge (q \rightarrow p)$$

Ex: Rewrite the following statement as a conjunction of two if-then statements.

This computer program is correct if, and only if, it produces correct answers for all possible sets of input data.

If this computer program is correct, then it produces the correct answers for all possible sets of input data; and if this program produces the correct answers for all possible sets of input data, then it is correct

Necessary and Sufficient Conditions

Definition (Necessary and Sufficient Conditions)

If r and s are statements,

"p is a sufficient condition for q" mean "if p then q"

"p is a necessary condition for q" mean "if not p then not q"

- In other words, to say "p is a sufficient condition for q" means that the occurrence of **p is sufficient to guarantee the occurrence of q**

- On the other hand, to say "p is a necessary condition for q" means that **if p does not occur, then q cannot occur either**: The occurrence of p is necessary to obtain the occurrence of q

Necessary and Sufficient Conditions

Converting a Sufficient Condition to If-Then Form

Rewrite the following statement in the form "If *A* then *B*":

Pia's birth on U.S. soil is a sufficient condition for her to be a U.S. citizen.

Solution

"p is a sufficient condition for q" mean

"if p then q

If Pia was born on U.S. soil, then she is a U.S. citizen.

.

Necessary and Sufficient Conditions

Converting a Necessary Condition to If-Then Form

Use the contrapositive to rewrite the following statement in two ways:

George's attaining age 35 is a necessary condition for his being president of the United States

Solution "p is a necessary condition for q"

mean "if not p then not q"

Version 1:

If George has not attained the age of 35, then he cannot be president of the United States.

Version 2:

If George can be president of the United States, then he has attained the age of 35.

.

$p \to q$ ≡ $\sim q \to \sim p$

conditional statement *contrapositive*

Necessary and Sufficient Conditions

Note that due to the equivalence between a statement and its contrapositive: r is a necessary condition for s also means "if s then r".

Consequently: r is a necessary and sufficient condition for s means "r, if and only if, s".

Ex: "If John is eligible to vote, then he is at least 18 years old."

The truth of the condition "John is eligible to vote" is *sufficient* to ensure the truth of the condition "John is at least 18 years old.

In addition, the condition "John is at least 18 years old" is *necessary* for the condition "John is eligible to vote" to be true.

If John were younger than 18, then he would not be eligible to vote.

Ex

"p is necessary and sufficient for q"

"if p then q, and conversely"

"p iff q"

"p if and only if"

Let p be the statement "You can take the flight,"
q be the statement "You buy a ticket."

Then $p \leftrightarrow q$ is the statement
"You can take the flight if and only if you buy a ticket.

Statement Form (Dạng mệnh đề)

A statement form (or propositional form) is an expression made up of statement variable (such as p, q, and r) and logical conectives (such as ¬; ∧; ∨; →; ↔) that becomes a statement when actual statements are substituted for the component statement variable.

Biểu thức đại số	**Biểu thức logic**
Hàm số $f(x, y) = \frac{1}{x} + \frac{1}{y-3}$	Dạng mệnh đề E(p, q, r) =(p∧q) ∨((¬r→p)
Hằng số: 3; 3.5;…	Hằng mệnh đề: 1, 0
Biến số: x,y…	Biến mệnh đề p,q,r,…
Các phép toán: (); +; -;x; /.. thao tác trên hằng số và các biến số theo một thứ tự nhất định.	Các phép nối: (); ¬; ∧; ∨; →; ↔ thao tác trên các hằng mệnh đề và biến mệnh đề theo một thứ tự nhất định như trên.

Statement Form (Dạng mệnh đề)

Biểu thức đại số	Biểu thức logic
Khi thay thế các biến x,y trong hàm số bằng hằng số (VD x =1, y=2) thì kết quả thực hiện các phép toán trong biểu thức sẽ là hằng số (VD f(1,2)=0) :	Khi thay thế các biến mệnh đề trong dạng mệnh đề bằng chân trị của các biến mệnh đề (VD p: 0, q: 0, r:1)thì kết quả thực hiện các phép nối thao tác sẽ là chân trị của dạng mệnh đề (VD E(0,0,1)

Phép VÀ

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

Phép HAY

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

Phép HOẶC

p	q	$p \veebar q$ $p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

Phép PHỦ ĐỊNH

p	$\bar{p}$
1	0
0	1

Phép kéo theo
(NẾU … THÌ…, IF…THEN)

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

Phép kéo theo 2 chiều
(IF…ONLY IF, IFF, KHI VÀ CHỈ KHI

P	Q	$P \leftrightarrow Q$ $P \rightarrow Q$ $Q \rightarrow P$
0	0	1
0	1	0
1	0	0
1	1	1

Order of operations:

$\sim$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$
not	*and*	*or*	*if-then/implies*	*if and only if*

Performed first: *not*

Coequal in order: *and*, *or*

Coequal in order: *if-then/implies*, *if and only if*

Performed last

"if *p*, then *q*"	"*p* is sufficient for *q*"	"*q* follows from *p*"
"if *p*, *q*"	"a sufficient condition for *q* is *p*"	"*q* whenever *p*"
"*q* if *p*"	"a necessary condition for *p* is *q*"	"*p* only if *q*"
"*q* when *p*"	"*q* is necessary for *p*"	
"*q* unless ~*p*"	"p is a sufficient condition for q"	
"*p* implies *q*"	"q is a necessary condition for p" mean "if not q then not p"	

If not q then not p		if p then q		if q then p		If not p then not q
$\sim q \rightarrow \sim p$	$\equiv$	$p \rightarrow q$	$\not\equiv$	$q \rightarrow p$	$\equiv$	$\sim p \rightarrow \sim q$
contrapositive (Phản đảo)		*conditional statement*		*converse* (Đảo)		*inverse* (Nghịch Đảo)

Negation of Conditional Statement

$$\sim(\mathbf{p} \rightarrow \mathbf{q}) \equiv \mathbf{p} \wedge \sim\mathbf{q}$$

Representation of If-Then as Or

$$\mathbf{p} \rightarrow \mathbf{q} \equiv \sim\mathbf{p} \vee \mathbf{q}$$

p	q	p→q
0	0	1
0	1	1
1	0	0
1	1	1

Summary of Logical Equivalences

Theorem 2.1.1 Logical Equivalence: Given any statement variables *p*, *q* and *r*, a tautology **t** and a contradiction **c**:

1	Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2	Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3	Distributive laws	$p \wedge (q \vee r)$ $\equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r)$ $\equiv (p \vee q) \wedge (p \vee r)$
4	Identity laws	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5	Negation laws	$p \vee {\sim}p \equiv \mathbf{t}$	$p \wedge {\sim}p \equiv \mathbf{c}$
6	Double negative law	${\sim}({\sim}p) \equiv p$	

Summary of Logical Equivalences

Theorem 2.1.1 Logical Equivalence: Given any statement variables *p*, *q* and *r*, a tautology **t** and a contradiction **c**:

7	Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
8	Universal bound laws	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9	De Morgan's laws	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10	Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11	Negation of **t** and **c**	$\sim\mathbf{t} \equiv \mathbf{c}$	$\sim\mathbf{c} \equiv \mathbf{t}$

12. **If – then law**

$\mathbf{p} \to \mathbf{q} \Leftrightarrow \sim\mathbf{p} \vee \mathbf{q}$ Representation of If-Then as Or

$\Leftrightarrow \sim\mathbf{q} \to \sim \mathbf{p}$ Contrapositive

Logically Equivalent (Tương đương logic)

12. **If – then laws**

$$p \to q \Leftrightarrow \sim p \vee q$$
$$\Leftrightarrow \sim q \to \sim p$$

Ví dụ: **Nếu** trời mưa **thì** đường trơn

$\Leftrightarrow$ **Nếu** đường không trơn **thì** trời không mưa

Use De Morgan's laws to express the negations of "Miguel has a cellphone and he has a laptop computer" and "Heather will go to the concert or Steve will go to the concert.

Logically Equivalent

Use De Morgan's laws to express the negations of "Miguel has a cellphone and he has a laptop computer"

Solution:

Let p be "Miguel has a cellphone" and
q be "Miguel has a laptop computer."

Then "Miguel has a cellphone **and** he has a laptop computer" can be represented by $p \land q$.

By the first of De Morgan's laws, $\neg(p \land q)$ is equivalent to $\neg p \lor \neg q$. Consequently, we can express the negation of our original statement as "Miguel does not have a cellphone **or** he does not have a laptop computer

$$\neg p \lor \neg q$$

Logically Equivalent (Tương đương logic)

Use De Morgan's laws to express the negations of "Heather will go to the concert or Steve will go to the concert.

Solution:

Let r be "Heather will go to the concert" and s be "Steve will go to the concert."

Then "Heather will go to the concert **or** Steve will go to the concert" can be represented by $r \lor s$.

By the second of De Morgan's laws, $\neg(r \lor s)$ is equivalent to $\neg r \land \neg s$. Consequently, we can express the negation of our original statement as Heather will not go to the concert **and** Steve will not go to the concert.

$$\neg r \land \neg s$$

De Morgan $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$

$\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$

Distributive Law $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

If - then law $p \to q \Leftrightarrow \sim p \vee q$

$\Leftrightarrow \sim q \to \sim p$

VD Cho p, q, r là các biến mệnh đề. Chứng minh rằng:

$$\mathbf{(\sim p \to r) \wedge (q \to r) \Leftrightarrow (p \to q) \to r}$$

If (true statement)

action 1

else // false statement

action 2

Giải:

$(\sim p \to r) \wedge (q \to r)$

$\Leftrightarrow (p \vee r) \wedge (\sim q \vee r)$ (if – then law)

$\Leftrightarrow (p \wedge \sim q) \vee r$ (distributive law)

$\Leftrightarrow \sim(\sim p \vee q) \vee r$ (De Morgan)

$\Leftrightarrow \sim(p \to q) \vee r$ (if – then law)

$\Leftrightarrow (p \to q) \to r$ (if – then law)

Summary of Logical Equivalences

Theorem 2.1.1 Logical Equivalence: Given any statement variables *p*, *q* and *r*, a tautology **t** and a contradiction **c**:

1	Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2	Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3	Distributive laws	$p \wedge (q \vee r)$ $\equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r)$ $\equiv (p \vee q) \wedge (p \vee r)$
4	Identity laws	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5	Negation laws	$p \vee {\sim}p \equiv \mathbf{t}$	$p \wedge {\sim}p \equiv \mathbf{c}$
6	Double negative law	${\sim}({\sim}p) \equiv p$	

Summary of Logical Equivalences

Theorem 2.1.1 Logical Equivalence: Given any statement variables *p*, *q* and *r*, a tautology **t** and a contradiction **c**:

7	Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
8	Universal bound laws	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9	De Morgan's laws	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10	Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11	Negation of **t** and **c**	$\sim\mathbf{t} \equiv \mathbf{c}$	$\sim\mathbf{c} \equiv \mathbf{t}$

12. **If – then law**

$\mathbf{p} \rightarrow \mathbf{q} \Leftrightarrow \sim\mathbf{p} \vee \mathbf{q}$ Representation of If-Then as Or

$\Leftrightarrow \sim\mathbf{q} \rightarrow \sim \mathbf{p}$ Contrapositive

Logical Equivalences

1	Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2	Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3	Distributive laws	$p \wedge (q \vee r)$ $\equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r)$ $\equiv (p \vee q) \wedge (p \vee r)$
4	Identity laws	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5	Negation laws	$p \vee {\sim}p \equiv \mathbf{t}$	$p \wedge {\sim}p \equiv \mathbf{c}$
6	Double negative law	${\sim}({\sim}p) \equiv p$	
7	Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
8	Universal bound laws	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9	De Morgan's laws	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10	Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11	Negation of **t** and **c**	${\sim}\mathbf{t} \equiv \mathbf{c}$	${\sim}\mathbf{c} \equiv \mathbf{t}$

12. **If – then law**

$p \rightarrow q \Leftrightarrow {\sim}p \vee q$ Representation of If-Then as Or

$\Leftrightarrow {\sim}q \rightarrow {\sim}p$ Contrapositive

$$({\sim}p \rightarrow r) \wedge (q \rightarrow r) \Leftrightarrow (p \rightarrow q) \rightarrow r$$

$({\sim}p \rightarrow r) \wedge (q \rightarrow r)$

$\Leftrightarrow (p \vee r) \wedge ({\sim}q \vee r)$ (if – then law)

$\Leftrightarrow (p \wedge {\sim}q) \vee r$ (distributive law)

$\Leftrightarrow {\sim}({\sim}p \vee q) \vee r$ (De Morgan)

$\Leftrightarrow {\sim}(p \rightarrow q) \vee r$ (if – then law)

$\Leftrightarrow (p \rightarrow q) \rightarrow r$ (if – then law)

Valid and Invalid Arguments

Modus Ponens	$p \rightarrow q$ p $\therefore q$	**Elimination**	**a.** $p \vee q$, ${\sim}q$, $\therefore p$ **b.** $p \vee q$, ${\sim}p$, $\therefore q$
Modus Tollens	$p \rightarrow q$ ${\sim}q$ $\therefore {\sim}p$	**Transitivity**	$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$
Generalization	**a.** p, $\therefore p \vee q$ **b.** q, $\therefore p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \rightarrow r$ $q \rightarrow r$ $\therefore r$
Specialization	**a.** $p \wedge q$, $\therefore p$ **b.** $p \wedge q$, $\therefore q$		
Conjunction	p q $\therefore p \wedge q$	**Contradiction Rule**	${\sim}p \rightarrow \mathbf{c}$ $\therefore p$

$p \rightarrow (q \rightarrow r)$
$p \vee s$
$t \rightarrow q$
$\bar{s}$
$\therefore \bar{r} \rightarrow \bar{t}$

$p \vee s$
$\bar{s}$
$\therefore p$
Elimination

$p \rightarrow (q \rightarrow r)$
p
$\therefore q \rightarrow r$
Modus Ponens

$t \rightarrow q$
$q \rightarrow r$
$\therefore t \rightarrow r$
Transitivity

$t \rightarrow r \Leftrightarrow \bar{r} \rightarrow \bar{t}$ Contrapositive

Valid and Invalid Arguments

If p, then q	premises	If Socrates is a man, then Socrates is mortal.
p		Socrates is a man.
∴ q	conclusion	Socrates is mortal.

An Argument (argument form) is a sequence of statements (statement forms).

All statements in an argument (argument form), except for the final one, are call premises (or assumptions or hypothesis).

The final statement (statement form) is called the conclusion.

The symbol ∴, which is read "therefore", is normally placed just before the conclusion

The common form of the arguments is	
If p **or** q, **then** r. ∴ **If** not r, **then** not p and not q.	**If** p **or** q, **then** r. q. **Therefore**, r.

Valid and Invalid Arguments

If p, then q	*premises*
p	
• q	*conclusion*

If Socrates is a man, then Socrates is mortal.

Socrates is a man.

Socrates is mortal.

An argument form is called valid if, and only if, whenever statements are substituted that make all the premises true, the conclusion is also true.

Determining Validity or Invalidity: Example #1

~ ∧ ∨ → ↔

p	q	p→q
0	0	1
0	1	1
1	0	0
1	1	1

$p \to q \vee \sim r$

$q \to p \wedge r$

$\bullet\ p \to r$

Invalid argument

premises | conclusion

p	q	r	$\sim r$	$q \vee \sim r$	$p \wedge r$	$p \to q \vee \sim r$	$q \to p \wedge r$	$p \to r$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	
T	F	T	F	F	T	F	T	
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	
F	T	F	T	T	F	T	F	
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

Critical rows

Valid and Invalid Arguments

Testing an Argument Form for Validity

1. Identify the premises and conclusion of the argument form.
2. Construct a truth table showing the truth values of all the premises and the conclusion.
3. A row of the truth table in which all the premises are true is called a critical row.
   - If there is a critical row in which the conclusion is false $\Rightarrow$ the argument form is invalid.
   - If the conclusion in every critical row is true $\Rightarrow$ the argument form is valid.

Valid and Invalid Arguments

Modus Ponens Qui tắc khẳng định	$p \to q$ p $\therefore q$	**Elimination** Qui tắc tam đoạn luận rời	**a.** $p \vee q$ $\sim q$ $\therefore p$ **b.** $p \vee q$ $\sim p$ $\therefore q$
Modus Tollens Qui tắc phủ định	$p \to q$ $\sim q$ $\therefore \sim p$	**Transitivity** Qui tắc tam đoạn luận	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	**a.** p $\therefore p \vee q$ **b.** q $\therefore p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$
Specialization Qui tắc đơn giản	**a.** $p \wedge q$ $\therefore p$ **b.** $p \wedge q$ $\therefore q$		
Conjunction Qui tắc nối liền	p q $\therefore p \wedge q$	**Contradiction Rule**	$\sim p \to \mathbf{c}$ $\therefore p$

A rule of inference is a form of argument that is valid

Valid and Invalid Arguments

Modus Ponens Qui tắc khẳng định	$p \to q$ p $\therefore q$
Modus Tollens Qui tắc phủ định	$p \to q$ $\sim q$ $\therefore \sim p$

p	q	p→q
0	0	1
0	1	1
1	0	0
1	1	1

Use modus ponens or modus tollens to fill in the blanks of the following arguments so that they become valid inferences.

a. If there are more pigeons than there are pigeonholes, then at least two pigeons roost in the same hole.

There are more pigeons than there are pigeonholes.

- ____

b. If 870,232 is divisible by 6, then it is divisible by 3.

870,232 is not divisible by 3.

- ____

Valid and Invalid Arguments

Generalization	**a.** p $\therefore p \vee q$	**b.** q $\therefore p \vee q$
Specialization Qui tắc đơn giản	**a.** $p \wedge q$ $\therefore p$	**b.** $p \wedge q$ $\therefore q$

Allows you to discard extraneous information to concentrate on the particular property of interest

Nam learns discrete structures.

- (More generally) Nam learns discrete structures or Nam learns graph algorithms
- Nam learns discrete structures and Nam learns graph algorithms

(In particular) Nam learns discrete structures .

Valid and Invalid Arguments

Elimination Qui tắc tam đoan luận rời	**a.** $p \vee q$ $\sim q$ $\therefore p$	**b.** $p \vee q$ $\sim p$ $\therefore q$

When you have two possibilities and you can rule one out, the other must be the case

(x-3)(x+2)=0

x>0

Suppose you know that for a particular number x, x – 3 = 0 or x + 2 = 0

If you also know that x is not negative, then x ≠ -2, so

by elimination you can conclude that x = 3.

Valid and Invalid Arguments

Transitivity Qui tắc tam đoan luận	$p \to q$ $q \to r$ $\therefore p \to r$

Many arguments in mathematics contain chains of if-then statements.

From the fact that one statement implies a second and the second implies the third, you can conclude that the first statement implies the third.

If 18,486 is divisible by 18, then 18,486 is divisible by 9.

If 18,486 is divisible by 9, then the sum of the digits of 18,486 is divisible by 9.

- If 18,486 is divisible by 18, then the sum of the digits of 18,486 is divisible by 9.

Valid and Invalid Arguments

Proof by Division into Cases	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$

It often happens that you know one thing or another is true. If you can show that in either case a certain conclusion follows, then this conclusion must also be true.

Ex: Suppose you know that x is a nonzero real number.

The trichotomy property of the real numbers says that any number is positive, negative, or zero. Thus (by elimination) you know that x is positive or negative.

You can deduce that $x^2 > 0$ by arguing as follows:

x is positive or x is negative.

If x is positive, then $x^2 > 0$.

If x is negative, then $x^2 > 0$.

● $x^2 > 0$.

Valid and Invalid Arguments

Modus Ponens Qui tắc khẳng định	$p \to q$ p $\therefore q$	**Elimination** Qui tắc tam đoạn luận rời	**a.** $p \vee q$ $\sim q$ $\therefore p$ **b.** $p \vee q$ $\sim p$ $\therefore q$
Modus Tollens Qui tắc phủ định	$p \to q$ $\sim q$ $\therefore \sim p$	**Transitivity** Qui tắc tam đoạn luận	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	**a.** p $\therefore p \vee q$ **b.** q $\therefore p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$
Specialization Qui tắc đơn giản	**a.** $p \wedge q$ $\therefore p$ **b.** $p \wedge q$ $\therefore q$		
Conjunction Qui tắc nối liền	p q $\therefore p \wedge q$	**Contradiction Rule**	$\sim p \to \mathbf{c}$ $\therefore p$

$$
\begin{array}{l}
p \to (q \to r) \\
p \vee s \\
t \to q \\
\overline{s} \\
\hline
\therefore \overline{r} \to \overline{t}
\end{array}
$$

$$
\begin{array}{l}
p \vee s \\
\overline{s} \\
\hline
\therefore \ p
\end{array}
$$

Elimination

$$
\begin{array}{l}
p \to (q \to r) \\
p \\
\hline
\therefore \ q \to r
\end{array}
$$

Modus Ponens

$$
\begin{array}{l}
t \to q \\
q \to r \\
\hline
\therefore \ t \to r
\end{array}
$$

Transitivity

$t \to r \Leftrightarrow \overline{r} \to \overline{t}$ Contrapositive

Fallacies

A fallacy is an error in reasoning that results in an invalid argument.

Three common fallacies:

1. Using **ambiguous premises**, and treating them as if they were unambiguous.
2. **Circular reasoning** (assuming what is to be proved without having derived it from the premises)
3. **Jumping to a conclusion** (without adequate grounds)

For an argument to be valid, every argument of the same form whose premises are all true must have a true conclusion.

It follows that for an argument to be invalid means that there is an argument of that form whose premises are all true and whose conclusion is false.

Fallacies: Converse Error

Modus Ponens Qui tắc khẳng định	$p \to q$ p $\therefore q$
Modus Tollens Qui tắc phủ định	$p \to q$ $\sim q$ $\therefore \sim p$

- Example:

If Zeke is a cheater, then Zeke sits in the back row.
Zeke sits in the back row.
• Zeke is a cheater.

$p \to q$
q
• p

$q \to p$
q
• p

Converse error is also known as the fallacy of affirming the consequence.

Fallacies: Inverse Error

- Example:

If interest rates are going up, stock market prices will go down.
Interest rates are not going up.
• Stock market prices will not go down.

$p \rightarrow q$	$\sim p \rightarrow \sim q$
$\sim p$	$\sim p$
$\bullet \sim q$	$\bullet \sim q$

Modus Ponens Qui tắc khẳng định	$p \rightarrow q$ p $\therefore q$
Modus Tollens Qui tắc phủ định	$p \rightarrow q$ $\sim q$ $\therefore \sim p$

Fallacies: A Valid Argument with a False Premise and a False Conclusion

- The argument below is valid by modus ponens. But its major premise is false, and so is its conclusion.

If John Lennon was a rock star, then John Lennon had red hair.
John Lennon was a rock star.
• John Lennon had red hair.

The argument below is invalid by the converse error, but it has a true conclusion.

If New York is a big city, then New York has tall buildings.
New York has tall buildings.
• New York is a big city.

Modus Ponens Qui tắc khẳng định	$p \to q$ p $\therefore q$
Modus Tollens Qui tắc phủ định	$p \to q$ $\sim q$ $\therefore \sim p$

Fallacies: Sound and Unsound Arguments

Definition 2.3.2 (Sound and Unsound Arguments)

An argument is called **sound** if, and only if, it is valid and all its premises are true.

An argument that is not sound is called **unsound**.

$$\left.\begin{array}{l}\text{If } p\text{, then } q \\ p\end{array}\right\}\ \textit{premises}$$

$$\bullet\ q \quad \textit{conclusion}$$

Contradictions and Valid Arguments

The concept of logical contradiction can be used to make inferences through a technique of reasoning called the contradiction rule. Suppose *p* is some statement whose truth you wish to deduce.

Contradiction Rule

If you can show that the supposition that statement *p* is false leads logically to a contradiction, then you can conclude that *p* is true.

Show that the following argument form is valid:

$\sim p \rightarrow c$ where *c* is a contradiction

$\bullet\ p$

			premise	conclusion
p	$\sim p$	c	$\sim p \rightarrow c$	p
T	F	F	T	T
F	T	F	F	

Only one critical row, and in this row the conclusion is true.
Hence this form of argument is valid.

Contradictions and Valid Arguments: Example – Contradiction Rule

- The contradiction rule is the logical heart of the method of proof by contradiction.
- A slight variation also provides the basis for solving many logical puzzles by eliminating contradictory answers:

> If an assumption leads to a contradiction, then that assumption must be false.

Valid and Invalid Arguments

Ex: You are about to leave for school in the morning and discover that you don't have your glasses. You know the following statements are true:

a. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

b. If my glasses are on the kitchen table, then I saw them at breakfast.

c. I did not see my glasses at breakfast.

d. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.

e. If I was reading the newspaper in the living room then my glasses are on the coffee table.

So, where are your glasses?

Valid and Invalid Arguments

Let RK = I was reading the newspaper in the kitchen.

GK = My glasses are on the kitchen table.

SB = I saw my glasses at breakfast.

RL = I was reading the newspaper in the living room.

GC = My glasses are on the coffee table.

a. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

$$RK \rightarrow GK$$

b. If my glasses are on the kitchen table, then I saw them at breakfast.

$$GK \rightarrow SB$$

c. I did not see my glasses at breakfast. ~SB

d. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.

$$RL \vee RK$$

e. If I was reading the newspaper in the living room then my glasses are on the coffee table.

$$RL \rightarrow GC$$

Valid and Invalid Arguments

1. $RK \rightarrow GK$ by (a)
   $GK \rightarrow SB$ by (b)
   $\therefore RK \rightarrow SB$ by transitivity

2. $RK \rightarrow SB$ by the conclusion of (1)
   $\sim SB$ by (c)
   $\therefore \sim RK$ by modus tollens

3. $RL \vee RK$ by (d)
   $\sim RK$ by the conclusion of (2)
   $\therefore RL$ by elimination

4. $RL \rightarrow GC$ by (e)
   RL by the conclusion of (3)
   $\therefore GC$ by modus ponens

Thus the glasses are on the coffee table.

BÀI TẬP

5) Dùng các qui luật logic để chứng minh các mệnh đề dưới đây

$p \vee q \equiv \neg p \rightarrow q$

$p \wedge q \equiv \neg(p \rightarrow \neg q)$

$\neg(p \rightarrow q) \equiv p \wedge \neg q$

$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

6) Chứng minh tính đúng sai của các kết luận sau

a)

a. $p \vee q$
b. $q \rightarrow r$
c. $p \wedge s \rightarrow t$
d. $\sim r$
e. $\sim q \rightarrow u \wedge s$
f. $\therefore t$

b)

a. $\sim p \vee q \rightarrow r$
b. $s \vee \sim q$
c. $\sim t$
d. $p \rightarrow t$
e. $\sim p \wedge r \rightarrow \sim s$
f. $\therefore \sim q$

c)

a. $\sim p \rightarrow r \wedge \sim s$
b. $t \rightarrow s$
c. $u \rightarrow \sim p$
d. $\sim w$
e. $u \vee w$
f. $\therefore \sim t$

d)

a. $p \rightarrow q$
b. $r \vee s$
c. $\sim s \rightarrow \sim t$
d. $\sim q \vee s$
e. $\sim s$
f. $\sim p \wedge r \rightarrow u$
g. $w \vee t$
h. $\therefore u \wedge w$

7) "If compound X is boiling, then its temperature must be at least 150° C." Assuming that this statement is true, which of the following must also be true?

a. If the temperature of compound X is at least 150°C, then compound X is boiling.

b. If the temperature of compound X is less than 150°C, then compound X is not boiling.

c. Compound X will boil only if its temperature is at least 150°C.

d. If compound X is not boiling, then its temperature is less than 150°C.

e. A necessary condition for compound X to boil is that its temperature be at least 150°C.

f. A sufficient condition for compound X to boil is that its temperature be at least 150°C

8) Rewrite each of the statements in a and b as a conjunction of two if-then statements.

a. This quadratic equation has two distinct real roots if, and only if, its discriminant is greater than zero

b. This integer is even if, and only if, it equals twice some integer.

9) Rewrite the statements in a and b in if-then form in two ways, one of which is the contrapositive of the other. Use the formal definition of "only if."

a. The Cubs will win the pennant only if they win tomorrow's game.

b. Sam will be allowed on Signe's racing boat only if he is an expert sailor.

10) In a–c, rewrite the statements in if-then form.

a Payment will be made on fifth unless a new hearing is granted.

b. Ann will go unless it rains.

c. This door will not open unless a security code is entered

d. Catching the 8:05 bus is a sufficient condition for my being on time for work.

e. Having two 45° angles is a sufficient condition for this triangle to be a right triangle

11) Use the contrapositive to rewrite the statements in a and b in if-then form in two ways.

a. Being divisible by 3 is a necessary condition for this number to be divisible by 9.

b. Doing homework regularly is a necessary condition for Jim to pass the course.

12) In 1–4 (a) use the logical equivalences $p \rightarrow q \equiv \sim p \vee q$ and $p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)$ to rewrite the given statement forms without using the symbol $\rightarrow$ or $\leftrightarrow$, and (b) use the logical equivalence $p \vee q \equiv \sim(\sim p \wedge \sim q)$ to rewrite each statement form using only $\wedge$ and $\sim$.

1. $p \wedge \sim q \rightarrow r$
2. $p \vee \sim q \rightarrow r \vee q$
3. $(p \rightarrow r) \leftrightarrow (q \rightarrow r)$
4. $(p \rightarrow (q \rightarrow r)) \leftrightarrow ((p \wedge q) \rightarrow r)$

13. Note that "a sufficient condition for s is r" means r is a sufficient condition for s and that "a necessary condition for s is r" means r is a necessary condition for s. Rewrite the statements in a and b in if then form.

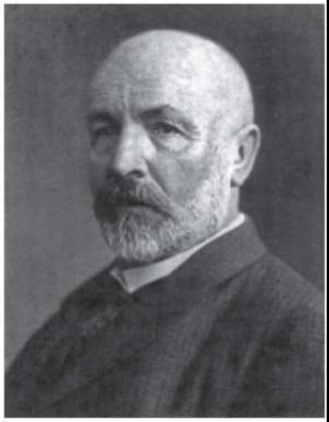
a. A sufficient condition for Jon's team to win the championship is that it win the rest of its games.

b. A necessary condition for this computer program to be correct is that it not produce error messages during translation.

Set (Tập hợp)

1) Definition: **"A set is a collection into a whole,** *M"* (Georg Cantor)

(Tập hợp là một nhóm các đối tượng và nhìn nhận chúng thành một)



Georg Cantor (1845–1918)

2) Describe (mô tả)**:**

a) Roster method: Listing all members which are listed between braces when this is possible. (Liệt kê hết tất cả các phần tử khi có thể. Các phần tử được liệt kê cách nhau bởi dấu phẩy)

Ex: F = {apple, orange, red, unicorn}.

Set

2) Describe:

a) Roster method:

❖ Although sets are usually used to group together elements with common properties, there is nothing that prevents a set from having seemingly unrelated elements.

(Tập hợp thường được dùng để nhóm các đối tượng có cùng chung tính chất. Tuy nhiên vẫn có thể biểu diễn tập hợp gồm những đối tượng không có liên quan với nhau)

Ex: A= {a, 2, Fred, New Jersey} is the set containing the four elements a, 2, Fred, and New Jersey.

Set

2) Describe:

a) Roster method: Sometimes the roster method is used to describe a set without listing all its members. Some members of the set are listed, and then *ellipses* (. . .) are used when the general pattern of the elements is obvious

(Thỉnh thoảng phương pháp này còn được sử dụng để mô tả một tập hợp nhưng không liệt kê hết tất cả các phần tử. Thay vào đó chỉ một vài phần tử được liệt kê và dấu …được dùng để chỉ những phần tử hiển nhiên)

Ex: The set of positive integers less than 100 can be denoted by {1, 2, 3, . . . , 99}

Set

2) Describe:

b) Set builder method:

- We characterize all those elements in the set by stating the property or properties they must have to be members (Chỉ ra tính chất chung của các phần tử)

Ex:

$O = \{x \mid x$ is an odd positive integer less than $10\}$,

$O = \{x \in \mathbf{Z}+ \mid x$ is odd and $x < 10\}$

Set

3) Some important sets in discrete mathematics:

N = {0, 1, 2, 3, . . .}, the set of **natural numbers** (số tự nhiên)

Z = {. . . , -2, -1, 0, 1, 2, . . .}, the set of **integers** (số nguyên)

Z+ = {1, 2, 3, . . .}, the set of **positive integers** (nguyên dương)

Q = {*p/q* | *p* ∈ **Z**, *q* ∈ **Z**, and *q* = 0}, the set of **rational numbers** (số hữu tỉ)

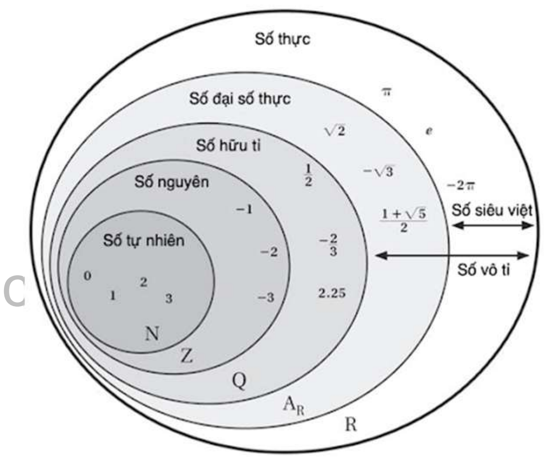
R, the set of **real numbers** (số thực)

R+, the set of **positive real numbers** (số thực

C, the set of **complex numbers** (số phức)

∅: The empty set (null set) is the set has no element (tập rỗng)

(Tập rỗng là tập duy nhất)



Predicate (Vị từ)

A predicate is a sentence that contains a finite number of variables and become a statement when specific values are substituted for the variables (Vị từ là một câu chứa một số hữu hạn các biến và sẽ trở thành mệnh đề khi thay giá trị cụ thể vào các biến trong câu)

Predicate
P(x): $x < -1$

Statement
p(-2): $-2 < -1$ → 1

Statement
p(0): $0 < -1$ → 0

Predicate
Q(x,y): $x + y \neq 0$

Statement
q(1,2): ***1+2*** $\neq 0$ → 1

Statement
q(1,-1) ***1-1*** $\neq 0$ → 0

Predicate(Vị từ)

Predicate
P(x): $x < -1$

- **Statement** ***p(-2):*** $-2 < -1$ → 1
- **Statement** ***p(0):*** $0 < -1$ → 0

Predicate
Q(x,y): $x + y \neq 0$

- **Statement** ***q(1,2):*** ***1+2*** $\neq 0$ → 1
- **Statement** ***q(1,-1)1-1*** $\neq 0$ → 0

Statement variables p,q

Predicate variables P(x), Q(x) Biến vị từ

P,Q are predicate symbols Ký hiệu vị từ

Predicate

1) Predicates → Statement: Assign specific values to all their variables.

2) Statement→ predicates: removing some or all of the nouns from a statement

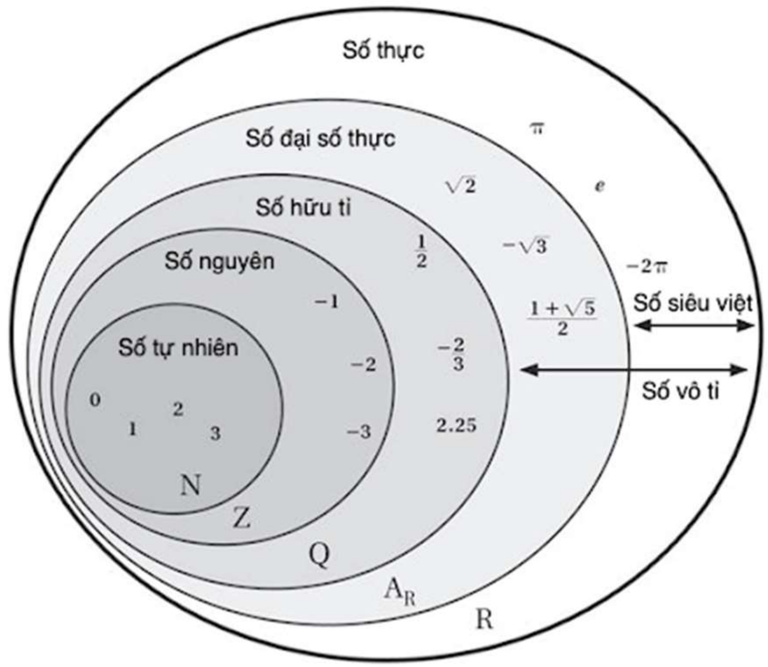
Statement Mệnh đề (Có thể xác định được chân trị 1 hoặc 0)	Nam is a student at TDTU
Predicate variables P(x) Biến vị từ P,Q are predicate symbols ký hiệu vị từ	P(x) = "x is a student at TDTU" Q(*x*, *y*) = "*x* is a student at *y*" R(x,y,z)=

Predicate

- **Definition:** The **domain** of a predicate variable is the set of all values that **may be** substituted in place of the variable

Predicate
P(x): $x < -1$

Statement	**Statement**
p(-2): $-2 < -1$	*p(0):* $0 < -1$
1	0



$\boldsymbol{x \in N\ x < -1}$ $\boldsymbol{x \in Z\ x < -1}$

$\boldsymbol{x \in Q\ x < -1}$ $\boldsymbol{x \in R\ x < -1}$

Predicates Statements

Definition (Truth Set): If P(x) is a predicate and x has domain D, the truth set is the set of **all elements of D that make P(x) true** when they are substituted for x. The truth set of P(x) is denoted $\{x \in D | P(x)\}$

Predicate

P(x): $x < -1$

Statement	**Statement**
***p(-2)*:** $-2 < -1$	***p(0)*:** $0 < -1$
1	0

$x \in Z \; x < -1$

$\{x \in Z \,| x < -1\}$

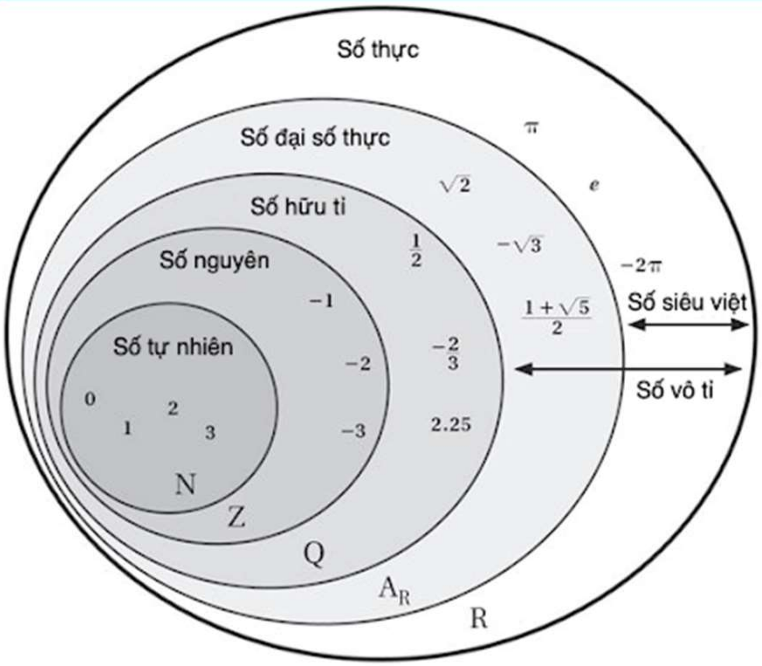
When an element in the domain of the variable of a one-variable predicate is substituted for the variable, the resulting statement is either true or false.

Predicates Statements

Definition (Truth Set): If P(x) is a predicate and x has domain D, the truth set is the set of **all elements of D that make P(x) true** when they are substituted for x. The truth set of P(x) is denoted $\{x \in D | P(x)\}$

$\boldsymbol{x \in Z\ x < -1}$ $\{\boldsymbol{x \in Z\ | x < -1}\}$

$\boldsymbol{x \in N\ x < -1}$ $\{\boldsymbol{x \in N\ | x < -1}\}$



Predicate

- **Definition:** The **domain** of a predicate variable is the set of all values that **may be** substituted in place of the variable

Definition (Truth Set):

If P(x) is a predicate and x has domain D, the truth set is the set of **all elements of D that make P(x) true** when they are substituted for x. The truth set of P(x) is denoted $\{x \in D | P(x)\}$

P(x) = "*x* is a student at TDTU"

Q(*x*, *y*) = "*x* is a student at *y*"

What is D?

Predicates Statements

Definition (Truth Set)

If P(x) is a predicate and x has domain D, the truth set is the set of all elements of D that make P(x) true when they are substituted for x. The truth set of P(x) is denoted $\{\mathrm{x} \in D | P(x)\}$

Ex : **P(n): "n +1 là số nguyên tố".**

P(n) =$\{\mathrm{n} \in Z^+ | P(n)\}$=$\{\mathrm{n} \in Z^+|$n +1 là số nguyên tố}={1,2,4,6,10…}

A(x): "x là số nguyên tố".

A(x) =$\{\mathrm{x} \in Z^+ | A(x)\}$= $\{\mathrm{x} \in Z^+|\mathrm{x}$ là số nguyên tố}={2,3,5,7,11…}

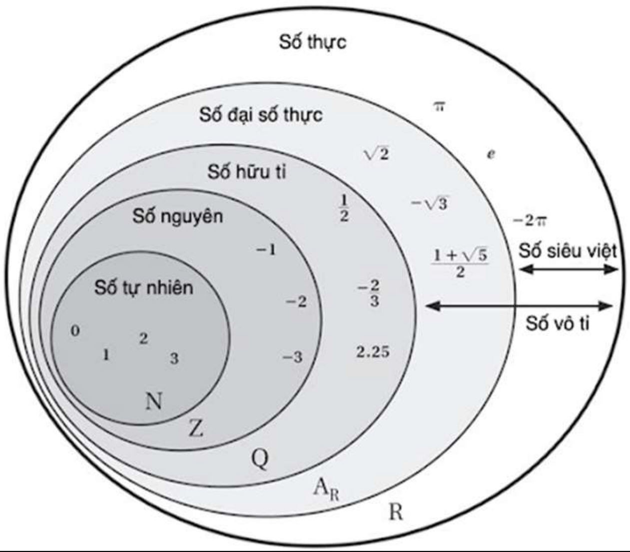
Example

Let Q(n) be the predicate "n is a factor of 8." Find the truth set of Q(n) if

a). the domain of n is the set **Z+** of all positive integers

{1, 2, 4, 8} because these are exactly the positive integers that divide 8 evenly

b. the domain of n is the set **Z** of all integers.

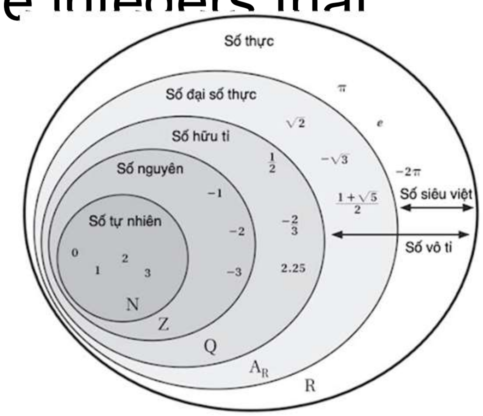


Example

Let *Q*(*n*) be the predicate "*n* is a factor of 8." Find the truth set of *Q*(*n*) if

a. the domain of *n* is the set **Z+** of all positive integers

{1, 2, 4, 8} because these are exactly the positive integers that divide 8 evenly



b. the domain of *n* is the set **Z** of all integers.

the domain of *n* is the set **Z** of all integers

{-8,-4,-2,-1,1, 2, 4, 8} because these are exactly the integers that divide 8 evenly

Predicates Statements

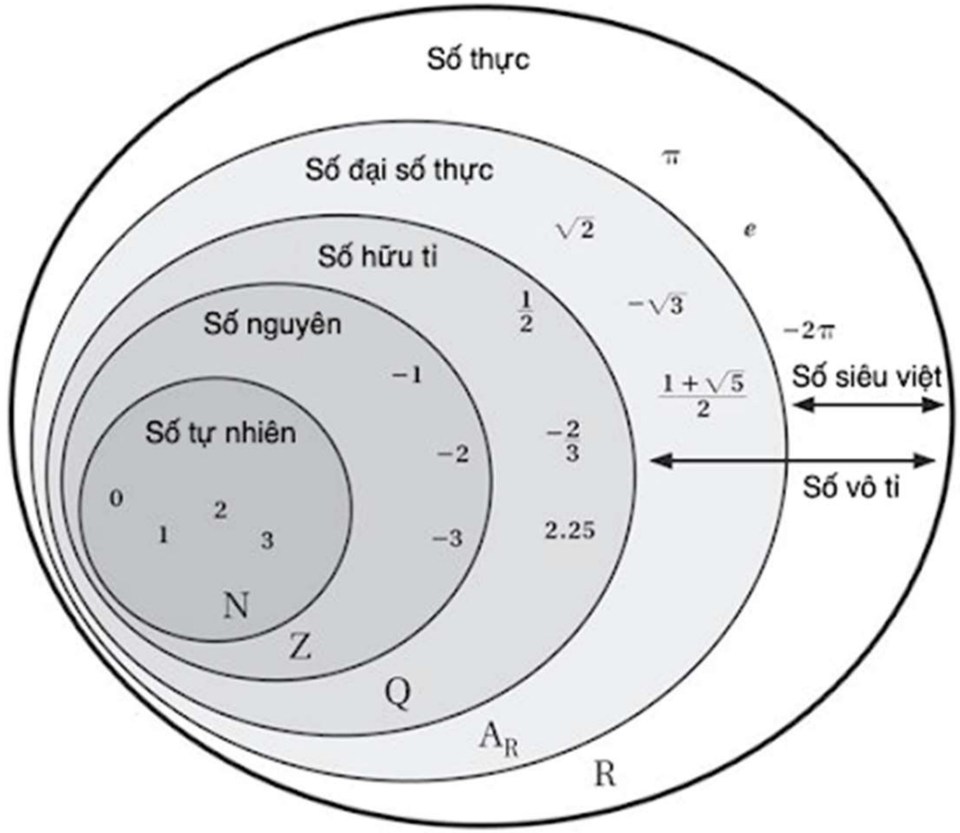
$x < -1$

Predicates Statements

$x < -1$ $\boldsymbol{x \in N \; x < -1}$

$\boldsymbol{x \in @ \; x < -1}$

$\boldsymbol{x \in Z \; x < -1}$



Predicates Statements

$x < -1$ $\boldsymbol{x \in N}$ $\boldsymbol{x < -1}$ $\{\boldsymbol{x \in N} \mid \boldsymbol{x < -1}\}$ **= {} =** $\emptyset$

$x \in @$ $x < -1$

$\boldsymbol{x \in Z}$ $\boldsymbol{x < -1}$ $\{\boldsymbol{x \in Z} \mid \boldsymbol{x < -1}\}$ **= {…,-4,-3,-2}**

Predicates Statements

$x < -1$ $x \in N$ $x < -1$ $\{x \in N \mid x < -1\}$ = {} = ∅ $\forall x \in N, x < -1$

$\exists x \in N, x < -1$

$x \in @$ $x < -1$

$x \in Z$ $x < -1$ $\{x \in Z \mid x < -1\}$ = {…,-4,-3,-2} $\forall x \in Z, x < -1$

$\exists x \in Z, x < -1$

Predicates Statements

$x < -1$ $x \in N$ $x < -1$ $\{x \in N \mid x < -1\}$ = {} = ∅ $\forall x \in N, x < -1$

$\exists x \in N, x < -1$

$x \in @$ $x < -1$

$x \in Z$ $x < -1$ $\{x \in Z \mid x < -1\}$ = {…,-4,-3,-2} $\forall x \in Z, x < -1$

$\exists x \in Z, x < -1$

Predicates Statements

$x < -1$ $\boldsymbol{x \in N\ x < -1}$ $\{\boldsymbol{x} \in \boldsymbol{N}\ |\boldsymbol{x} < -\mathbf{1}\}$ **= {} =** $\emptyset$ $\forall \boldsymbol{x} \in \boldsymbol{N}, \boldsymbol{x} < -\mathbf{1}$

$\exists \boldsymbol{x} \in \boldsymbol{N}, \boldsymbol{x} < -\mathbf{1}$

$x \in @\ x < -1$

$\boldsymbol{x \in Z\ x < -1}$ $\{\boldsymbol{x} \in \boldsymbol{Z}\ |\boldsymbol{x} < -\mathbf{1}\}$ **= {…,-4,-3,-2}** $\forall \boldsymbol{x} \in \boldsymbol{Z}, \boldsymbol{x} < -\mathbf{1}$

$\exists \boldsymbol{x} \in \boldsymbol{Z}, \boldsymbol{x} < -\mathbf{1}$

Predicates Statements

$x < -1$ $x \in N$ $x < -1$ $\{x \in N \,|x < -1\}$ = {} = ∅ $\forall x \in N, x < -1$

$\exists x \in N, x < -1$

$x \in @$ $x < -1$

$x \in Z$ $x < -1$ $\{x \in Z \,|x < -1\}$ = {…,-4,-3,-2} $\forall x \in Z, x < -1$

$\exists x \in Z, x < -1$

Predicates Statements

$x < -1$ $x \in N$ $x < -1$ $\{x \in N \mid x < -1\}$ = {} = ∅ $\forall x \in N, x < -1$

$\exists x \in N, x < -1$

$x \in @$ $x < -1$

$x \in Z$ $x < -1$ $\{x \in Z \mid x < -1\}$ = {...,-4,-3,-2} $\forall x \in Z, x < -1$

$\exists x \in Z, x < -1$

Predicates Statements

$x < -1$ $x \in N$ $x < -1$ $\{x \in N \mid x < -1\}$ = {} = ∅

$\forall x \in N, x < -1$

$\exists x \in N, x < -1$

$x \in @$ $x < -1$

$\forall x \in N, !(x < -1)$

$x \in Z$ $x < -1$ $\{x \in Z \mid x < -1\}$ = {...,-4,-3,-2}

$\forall x \in Z, x < -1$

$\exists x \in Z, x < -1$

$\forall x \leq 2$ và $x \in Z, x < -1$

Predicates Statements

$x < -1$ $x \in N$ $x < -1$ $\{x \in N \mid x < -1\}$ $\forall x \in N, x < -1$

$\exists x \in N, x < -1$

$x \in @$ $x < -1$

$\forall x \in N, !(x < -1)$

$x \in Z$ $x < -1$ $\{x \in Z \mid x < -1\}$ $\forall x \in Z, x < -1$

$\exists x \in Z, x < -1$

Predicate **Domain** **Truth Set** **Quantifier**

Vị từ **Lượng từ**

Logic vị từ

Khi xét một vị từ P(x) với $x \in D$. Ta có các trường hợp sau

TH1. Khi thay x bởi 1 phần tử a tùy ý thuộc D, ta có P(a) đúng.

VD Cho các vị từ P(x) sau với $x \in \mathbf{R}$

$\forall x\ P(x)$

P(x) = "$x^2 + 1 > 0$" đúng với mọi x

TH2. Với một số giá trị $a \in D$, ta có P(a) đúng

$\exists\ x\ P(x)$

P(x) = "$x^2 - 2x + 1 = 0$" chỉ đúng với x = 1.

TH3. Khi thay x bởi 1 phần tử a tùy ý thuộc D, ta có P(a) sai.

P(x) = "$x^2 - 2x + 3 = 0$" sai với mọi x .

$\forall x\ !P(x)$

Quantifiers are words that refer to quantities such as "**some**" or "**all**" and tell for <u>how many elements a given predicate is true</u>.

The Universal Quantifier: ∀ - The Existential Quantifier: ∃

Definition (Universal Quantifier ∀ - *Lượng từ phổ dụng)*

Let P(x) be a predicate and D the domain of x. A **universal statement** is a statement of the form "**"$\forall x \in D, P(x)$"**"

It is defined to be true iff P(x) is **true for every x** in D

It is defined to be false iff P(x) is **false for at least one** x in D

A value for x for which P(x) is false is called a **counterexample**

Cho P(x) là một vị từ theo một biến xác định trên D.

Lượng từ hóa của vị từ P(x): *"Với mọi x thuộc D, P(x)*

"$\forall x \in D, P(x)$"

Lượng từ đúng khi và chỉ khi P(x) luôn đúng với mọi giá trị $x \in D$.

$\forall x \in D$, P(x) đúng $\leftrightarrow$ $P(x_1) \land P(x_2) \land \ldots \land P(x_n)$ đúng

The Universal Quantifier: ∀

Definition (Universal Quantifier).

Let Q(x) be a predicate and D the domain of x. A **universal statement** is a statement of the form "**"∀x ∈ D, Q(x)"**"

It is defined to be true iff Q(x) is true for every x in D

Truth and Falsity of Universal Statements

a. Let *D* = {1, 2, 3, 4, 5}, and consider the statement

$$\forall x \in D, x^2 \geq x$$

Show that this statement is **true**.

Check that "$x^2 \geq x$ " is true for each x in D.

$1^2 \geq 1$, $2^2 \geq 2$, $3^2 \geq 3$, $4^2 \geq 4$, $5^2 \geq 5$ Hence "$\forall x \in D, x^2 \geq x$" is true. This method is called the method of exhaustion. (vét cạn).

The Universal Quantifier: ∀

Truth and Falsity of Universal Statements

a. Let *D* = {1, 2, 3, 4, 5}, and consider the statement

$$\forall x \in D, x^2 \geq x$$

Show that this statement is **true**.

Check that "$x^2 \geq x$ " is true for each x in D.

$1^2 \geq 1$, $2^2 \geq 2$, $3^2 \geq 3$, $4^2 \geq 4$, $5^2 \geq 5$ Hence "$\forall x \in D, x^2 \geq x$" is true. This method is called the method of exhaustion (vét cạn).

Ex2: Consider the statement $\forall x \in R, x^2 \geq x$

Find a counterexample to show that this statement is **false**.

Definition (Existential Statement ∃ : *lượng từ tồn tại* **).**

Let P(x) be a predicate and D the domain of x. A **existential statement** is a statement of the form "**"∃x ∈ D such that Q(x)"**"

It is defined to be true iff P(x) is true for at least one x in D

It is defined to be false iff P(x) is false for all x in D

Cho P(x) là một vị từ theo một biến xác định trên A. Lượng từ hóa của P(x):

"Tồn tại (ít nhất) một x thuộc A

Có (ít nhất) một x thuộc A

Đối với một x nào đó" làm cho P(x) đúng

"∃x ∈ A such that P(x)"

Là mệnh đề đúng khi và chỉ khi có ít nhất một giá trị $x = a_0$ nào đó sao cho mệnh đề $P(a_0)$ đúng.

Definition (Universal Quantifier).

Let P(x) be a predicate and D the domain of x. A **universal statement** is a statement of the form "**"$\forall x \in D, P(x)$"**"

It is defined to be true iff P(x) is true for every x in D

It is defined to be false iff P(x) is false for at least one x in D

Definition (Existential Statement).

Let P(x) be a predicate and D the domain of x. A **existential statement** is a statement of the form "**"$\exists x \in D$ such that $P(x)$"**"

It is defined to be true iff P(x) is true for at least one x in D

It is defined to be false iff P(x) is false for all x in D

Lượng từ	**Khi nào đúng**	**Khi nào sai**
$\forall x\ P(x)$	P(x) đúng với mọi x	Có một giá trị x để P(x) là sai
$\exists x\ P(x)$	Có một x sao cho P(x) là đúng	P(x) sai với mọi x

Lượng từ	Khi nào đúng	Khi nào sai
$\forall x$ P(x)	P(x) đúng với mọi x	Có một giá trị x để P (x) là sai
$\exists x$ P(x)	Có một x sao cho P (x) là đúng	P(x) sai với mọi x

a. Show that the following statement is true.

$$\exists \mathbf{m} \in Z^{+} \text{ such that } m^2 = \mathbf{m}$$

Observe that $1^2 = 1$. Thus "$m^2 = m$" is true for at least one integer m. Hence "$\exists m \in Z^{+}$ *such that* $m^2 = m$" is true

b. Let E = {5, 6, 7, 8}. Show that the following statement is false. $\exists \mathbf{m} \in E$ **such that** $m^2 = \mathbf{m}$

Note that $m^2 = m$ is not true for any integer m from 5 through 8:

$5^2=25 \neq 5$, $6^2=36 \neq 6$, $7^2=49 \neq 7$, $8^2=64 \neq 8$,

Hence "$\exists m \in Z^{+}$ *such that* $m^2 = m$" is false

Lượng từ hóa vị từ một biến

Lượng từ	**Khi nào đúng**	**Khi nào sai**
$\forall x\ P(x)$	P(x) đúng với mọi x	Có một giá trị x để P (x) là sai
$\exists x\ P(x)$	Có một x sao cho P (x) là đúng	P(x) sai với mọi x

Tìm giá trị chân lý của các lượng từ hóa sau

- "$\forall x \in \mathbf{R}, x^2 + 3x + 1 \leq 0$"
- "$\exists x \in \mathbf{R}, x^2 + 3x + 1 \leq 0$"
- "$\forall x \in \mathbf{R}, x^2 + 1 \geq 2x$"
- "$\exists x \in \mathbf{R}, x^2 + 1 < 0$"

Valid and Invalid Arguments

Proof by Division into Cases	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$

It often happens that you know one thing or another is true. If you can show that in either case a certain conclusion follows, then this conclusion must also be true.

Ex: Suppose you know that x is a nonzero real number.

The trichotomy property of the real numbers says that any number is positive, negative, or zero. Thus (by elimination) you know that x is positive or negative.

You can deduce that $x^2 > 0$ by arguing as follows:

x is positive or x is negative.

If x is positive, then $x^2 > 0$.

If x is negative, then $x^2 > 0$.

● $x^2 > 0$.

Formal Versus Informal Language

The symbol ∀ denotes: F**or all, for any, for every, for each**

Rewrite the following **formal statements** in a variety of equivalent but more **informal ways**. Do not use the symbol ∀,∃.

a. $\forall x \in \mathbf{R}, x^2 \geq 0$.

All real numbers have non-negative squares.

Every/Any real number has a non-negative square.

The square of each real number is non-negative.

b. $\forall x \in \mathbf{R}, x^2 \neq -1$.

All real numbers have squares that are not -1.

No real numbers have squares equal to -1.

Formal Versus Informal Language

Some alternative expressions for "**there exists**" are "**there is a**", "**we can find a**", "**there is at least one**", "**for some**", and "**for at least one**".

c. $\exists\, m \in \mathbf{Z}^+$ **such that** $m^2 = m$.

There is a positive integer whose square is itself.

We can find at least one positive integer equal to its own square.

Some positive integer equals its own square.

The words ***such that*** are inserted just before the predicate.

"There is a student in CTRR" can be written as

$\exists$ a person *p* **such that** *p* is a student in CTRR.

$\exists\, p \in P$ **such that** *p* is a student in CTRR.

where *P* is the set of all people.

Các phép toán trên vị từ

Cho trước các vị từ P(x), Q(x) theo một biến $x \in A$. Khi ấy, ta cũng có các phép toán tương ứng như trên mệnh đề

- *Phủ định* $\neg P(x)$
- *Phép nối liền* $P(x) \wedge Q(x)$
- *Phép nối rời* $P(x) \vee Q(x)$
- *Phép kéo theo* $P(x) \rightarrow Q(x)$
- *Phép kéo theo hai chiều* $P(x) \leftrightarrow Q(x)$

Universal Conditional Statements

Ex: Rewrite the following statement informally, without quantifiers or variables: $\forall x \in R$, if $x > 2$ then $x^2 \geq 4$

If a real number is greater than 2, **then** its square is greater than 4.

Whenever a real number is greater than 2, its square is greater than 4.

The squares of all real numbers greater than 2 are greater than 4

Equivalent Forms of Universal Statements

By narrowing U to be the domain D consisting of all values of the variable x that make P(x) true,

$\forall x \in U$, if $P(x)$ then $Q(x)$ → $\forall x \in D, Q(x)$

Ex: Tất cả sinh viên khoa CNTT đều đã học cấu trúc rời rạc”

Đặt: D là tập các sinh viên. S(x) là câu “x là sinh viên khoa CNTT”

P(x) là câu “x đã học cấu trúc rời rạc”

Tất cả <u>sinh viên</u> khoa CNTT đều đã học <u>cấu trúc rời rạc</u>”

$$\forall x, S(x) \rightarrow P(x)$$

Hoặc $\forall x \in D, P(x)$

Equivalent Forms of Universal Statements

By narrowing U to be the domain D consisting of all values of the variable x that make P(x) true,

$\forall x \in U$, if $P(x)$ then $Q(x)$ → $\forall x \in D, Q(x)$

Ex: Rewrite the statement “All squares are rectangles” in the two forms:

∀ x, if_______________then_________________

∀ ____________x, ________________

Equivalent Forms of Existential Statements

By narrowing U to be the domain D is the set of all x for which P(x) is true.

$\exists\, x$ such that $P(x)$ and $Q(x)$ $\Rightarrow$ $\exists\, x \in D$ such that $Q(x)$

VD: Diễn đạt câu "Chỉ có một sinh viên khoa CNTT đã học cấu trúc rời rạc"

P(x) là câu "x đã học cấu trúc rời rạc"

D: tập sinh viên khoa CNTT

$$\exists x \in D \; such\ that\ P(x)$$

Equivalent Forms of Existential Statements

By narrowing U to be the domain D is the set of all x for which P(x) is true.

$\exists\, x$ such that $P(x)$ and $Q(x)$ → $\exists\, x \in D$ such that $Q(x)$

Ex: A prime number is an integer whose only positive integer factors are itself and 1. Consider the statement "There is an integer that is both prime and even."

Let Prime(n) be "n is prime" and Even(n) be "n is even". Use the notation Prime(n) and Even(n) to rewrite this statement in the following two forms:

∃n such that_______________^_________________

∃____________ n such that ____________________

Implicit Quantification

Notation: Let P(x) Q(x) be a predicate and suppose the common domain of x is D.

The notation **P(x) $\Rightarrow$ Q(x)** means that every element in the truth set of P(x) is **in** the truth set of Q(x), or, equivalently, $\forall x,\ P(x) \rightarrow Q(x)$

The notation **P(x) $\Leftrightarrow$ Q(x)** means P(x) and Q(x) have **identical** truth sets, or, equivalently, $\forall x,\ P(x) \leftrightarrow Q(x)$

Let Q(n) be "n is a factor of 8", R(n) be "n is a factor of 4", S(n) be "$n < 5$ and $n \neq 3$", and suppose the domain of n is Z^+, the set of positive integers. Use the $\Rightarrow$ and $\Leftrightarrow$ symbols to indicate true relationship among Q(n), R(n), and S(n).

Implicit Quantification

The notation **P(x)** **Q(x)** means that every element in the truth set of P(x) is in the truth set of Q(x), or, equivalently,

∀x, P(x)→Q(x)

The notation **P(x)⇔Q(x)** means P(x) and Q(x) have identical truth sets, or, equivalently, ∀x, P(x) Q(x)

$Q(n)$ be "n is a factor of 8",
$R(n)$ be "n is a factor of 4",
$S(n)$ be "$n < 5$ and $n \neq 3$"

1. The truth set of Q(n) is {1,2,4,8} and the truth set of R(n) is {1,2,4}.

Thus it is true that every element in the truth set of R(n) is in the truth set of Q(n), or, ∀ n in Z^+, R(n) → Q(n) so $R(n) \Rightarrow Q(n)$

Or, equivalently,

$$n \text{ is a factor of } 4 \Rightarrow n \text{ is a factor of } 8$$

Implicit Quantification

The notation **P(x) Q(x)** means that every element in the truth set of P(x) is in the truth set of Q(x), or, equivalently,

$\forall x$, P(x)$\rightarrow$Q(x)

The notation **P(x)$\Leftrightarrow$Q(x)** means P(x) and Q(x) have identical truth sets, or, equivalently, $\forall x$, P(x) Q(x)

$Q(n)$ be "n is a factor of 8",

$R(n)$ be "n is a factor of 4",

$S(n)$ be "$n < 5$ and $n \neq 3$"

2. The truth set of S(n) is {1,2,4} which is the same truth set of R(n)

or, $\forall$ n in Z^+, R(n) $\leftrightarrow$ Q(n) so $R(n) \Leftrightarrow S(n)$

Or, equivalently,

$$n \text{ is a factor of } 4 \Leftrightarrow n < 5 \text{ and } n \neq 3.$$

Tarski's World

Tarski's World is a computer program developed by information scientists Jon Barwise and John Etchemendy to help teach the principles of logic.

It is described in their book *The Language of FirstOrder Logic*, which is accompanied by a CD-ROM containing the program.

Tarski's World, named after the great logician Alfred Tarski.

The program for Tarski's World provides pictures of blocks of various sizes, shapes, and colors, which are located on a grid.

Shown in Figure 3.1.1 is a picture of an arrangement of objects in a two-dimensional Tarski world

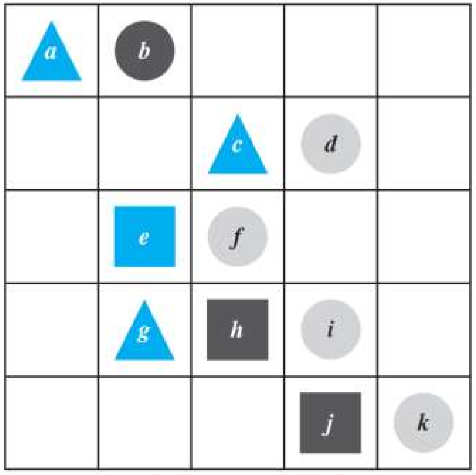


Figure 3.1.1

Tarski's World

The configuration can be described using logical operators and — for the two-dimensional version — notation such as:

Triangle(x), meaning "x is a triangle,"

Blue(y), meaning "y is blue," and

RightOf(x, y), meaning "x is to the right of y (but possibly in a different row)."

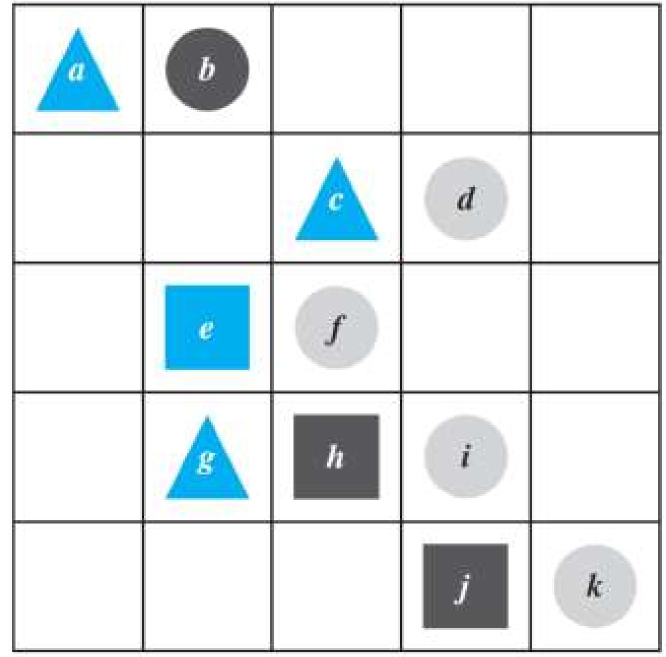


Figure 3.1.1

Individual objects can be given names such as a, b, or c

Tarski's World

Determine the truth or falsity of the following statements. The domain for all variables is the set of objects in the Tarski's world shown on the right.

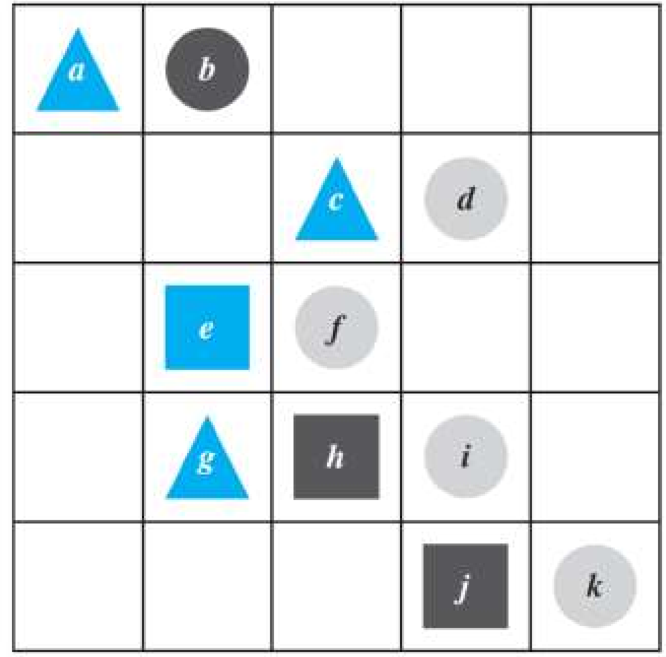


Figure 3.1.1

a. $\forall t$, Triangle(t) → Blue(t).

b. $\forall x$, Blue(x) → Triangle(x).

c. $\exists y$ such that Square(y) ∧ RightOf(d, y).

d. $\exists z$ such that Square(z) ∧ Gray(z).

Predicates Statements

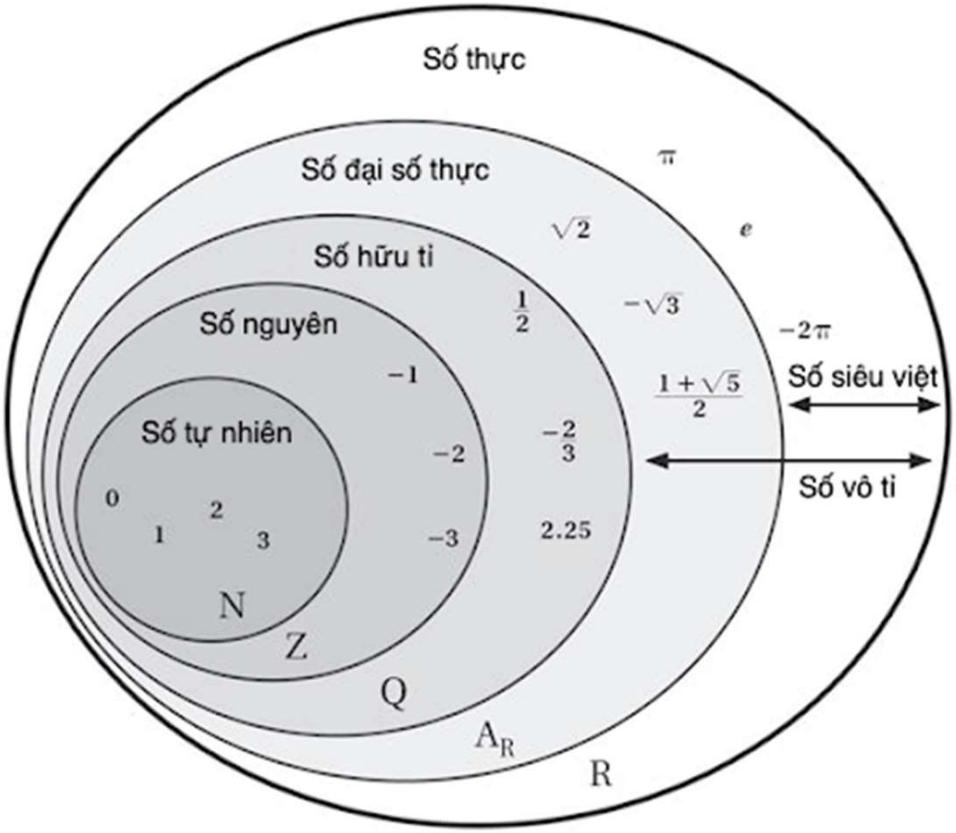
$x < -1$

Predicates Statements

$x < -1$ $\boldsymbol{x \in N\ x < -1}$

$\boldsymbol{x \in @\ x < -1}$

$\boldsymbol{x \in Z\ x < -1}$



Predicates Statements

$x < -1$ $\boldsymbol{x} \in \boldsymbol{N}\ \boldsymbol{x} < -\mathbf{1}$ $\{\boldsymbol{x} \in \boldsymbol{N}\ |\boldsymbol{x} < -\mathbf{1}\}$ **= {} =** $\emptyset$

$x \in @\ x < -1$

$\boldsymbol{x} \in \boldsymbol{Z}\ \boldsymbol{x} < -\mathbf{1}$ $\{\boldsymbol{x} \in \boldsymbol{Z}\ |\boldsymbol{x} < -\mathbf{1}\}$ **= {…,-4,-3,-2}**

Predicates Statements

$x < -1$ $x \in N$ $x < -1$ $\{x \in N \mid x < -1\}$ = {} = ∅ $\forall x \in N, x < -1$

$\exists x \in N, x < -1$

$x \in @$ $x < -1$

$x \in Z$ $x < -1$ $\{x \in Z \mid x < -1\}$ = {...,-4,-3,-2} $\forall x \in Z, x < -1$

$\exists x \in Z, x < -1$

Predicates Statements

$x < -1$ $x \in N$ $x < -1$ $\{x \in N \mid x < -1\}$ = {} = ∅ $\forall x \in N, x < -1$

$\exists x \in N, x < -1$

$x \in @$ $x < -1$

$x \in Z$ $x < -1$ $\{x \in Z \mid x < -1\}$ = {…,-4,-3,-2} $\forall x \in Z, x < -1$

$\exists x \in Z, x < -1$

Predicates Statements

$x < -1$ $x \in N$ $x < -1$ $\{x \in N \mid x < -1\}$ = {} = ∅ $\forall x \in N, x < -1$

$\exists x \in N, x < -1$

$x \in @$ $x < -1$

$x \in Z$ $x < -1$ $\{x \in Z \mid x < -1\}$ = {…,-4,-3,-2} $\forall x \in Z, x < -1$

$\exists x \in Z, x < -1$

Predicates Statements

$x < -1$ $x \in N$ $x < -1$ $\{x \in N \mid x < -1\}$ = {} = ∅ $\forall x \in N, x < -1$

$\exists x \in N, x < -1$

$x \in @$ $x < -1$

$x \in Z$ $x < -1$ $\{x \in Z \mid x < -1\}$ = {...,-4,-3,-2} $\forall x \in Z, x < -1$

$\exists x \in Z, x < -1$

Predicates Statements

$x < -1$ $x \in N$ $x < -1$ $\{x \in N \mid x < -1\}$ = {} = ∅ $\forall x \in N, x < -1$

$\exists x \in N, x < -1$

$x \in @$ $x < -1$

$x \in Z$ $x < -1$ $\{x \in Z \mid x < -1\}$ = {…,-4,-3,-2} $\forall x \in Z, x < -1$

$\exists x \in Z, x < -1$

Predicates Statements

$x < -1$ $x \in N$ $x < -1$ $\{x \in N \mid x < -1\}$ = {} = ∅ $\forall x \in N, x < -1$

$\exists x \in N, x < -1$

$x \in @$ $x < -1$

$\forall x \in N, !(x < -1)$

$x \in Z$ $x < -1$ $\{x \in Z \mid x < -1\}$ = {…,-4,-3,-2} $\forall x \in Z, x < -1$

$\exists x \in Z, x < -1$

$\forall x \leq -2$ và $x \in Z, x < -1$

Predicates Statements

$x < -1$ $x \in N$ $x < -1$ $\{x \in N \mid x < -1\}$ $\forall x \in N, x < -1$

$\exists x \in N, x < -1$

$x \in @$ $x < -1$

$\forall x \in N, !(x < -1)$

$x \in Z$ $x < -1$ $\{x \in Z \mid x < -1\}$ $\forall x \in Z, x < -1$

$\exists x \in Z, x < -1$

Predicate **Domain** **Truth Set** **Quantifier**

Vị từ

Lượng từ

Negations of Quantified Statements

Phủ định của mệnh đề lượng từ hóa vị từ p(x,y,..) có được bằng các thay $\forall$ thành $\exists$, thay $\exists$ thành $\forall$ và vị từ p(x,y,..) thành $\neg$ p(x,y,..).

Với vị từ theo 1 biến ta có : $\overline{\forall x \in A, p(x)} \Leftrightarrow \exists x \in A, \overline{p(x)}$

$$\overline{\exists x \in A, p(x)} \Leftrightarrow \forall x \in A, \overline{p(x)}$$

VD: Hãy phủ định các câu

a)“Tất cả sinh viên khoa CNTT đều đã học cấu trúc rời rạc”

Gọi P(x) là câu “x đã học cấu trúc rời rạc”, D: tập sv khoa CNTT

a) $\forall x \in D, P(x) \Leftrightarrow \exists x \in \mathrm{D}, \sim P(x)$

→Không phải tất cả các sinh viên khoa CNTT đều đã học môn CTRR”

→ Có một sinh viên khoa CNTT chưa học môn CTRR

b)“Có 1 sinh viên khoa CNTT đã học cấu trúc rời rạc”

$$\exists x \in D, P(x) \Leftrightarrow \forall x \in D, \sim P(x)$$

→ “Không có một sinh viên nào của khoa CNTT đã học môn CTRR”

→ “Mọi sinh viên của khoa CNTT đều chưa học môn CTRR”

Negations of Quantified Statements

Theorem: Negation of a Universal Statement

The negation of a statement of the form

$$\forall x \text{ in } D, P(x)$$

Is logically equivalent to a statement of the form

$$\exists x \text{ in } D \textbf{ such that } {\sim}P(x)$$

Symbolically, $\sim(\forall x \text{ in } D, P(x)) \equiv \exists x \text{ in } D$ **such that** $\sim P(x)$

The negation of a universal statement ("all are") is logically equivalent to an existential statement ("some are not" or "there is at least one that is not").

Negations of Quantified Statements

Theorem: Negation of an Existential Statement

The negation of a statement of the form

$$\exists x \text{ in } D \text{ such that } P(x)$$

Is logically equivalent to a statement of the form

$$\forall x \text{ in } D, \sim P(x)$$

Symbolically, $\sim(\exists x \text{ in } D \text{ such that } P(x)) \equiv \forall x \text{ in } D, \sim P(x)$

That is, the negation of an existential statement (“some are”) is logically equivalent to a universal statement (“none are” or “all are not”).

Write formal negations for the following statements:

a. $\forall$primes p, p is odd.

b. $\exists$a triangle T such that the sum of the angles of T equals 200^o.

Negations of Quantified Statements

Phủ định của mệnh đề lượng từ hóa vị từ p(x,y,..) có được bằng cách thay ∀ thành ∃, thay ∃ thành ∀ và vị từ p(x,y,..) thành p(x,y,..).

Với vị từ theo 1 biến ta có :

$$\sim\forall x \in A, p(x) \Leftrightarrow \exists x \in A, \sim p(x)$$

$$\sim\exists x \in A, p(x) \Leftrightarrow \forall x \in A, \sim p(x)$$

Phủ định các lượng từ			
Phủ định	Mệnh đề tương đương	Khi nào phủ định là đúng	Khi nào là sai
$\sim\exists x$ P(x)	$\forall x$ ~P(x)	P(x) sai với mọi x	Có một x để P(x) là đúng
$\sim\forall x$ P(x)	$\exists x$ ~P(x)	Có một x để P(x) sai	P(x) đúng với mọi x

Lượng từ	**Khi nào đúng**	**Khi nào sai**
∀x P(x)	P(x) đúng với mọi x	Có một giá trị x để P (x) là sai
∃x P(x)	Có một x sao cho P (x) là đúng	P(x) sai với mọi x

Negations of Universal Conditional Statements

Of special importance in mathematics.

12. **If – then law**

$p \to q \Leftrightarrow \sim p \vee q$ Representation of If-Then as Or

$\Leftrightarrow \sim q \to \sim p$ Contrapositive

$$\sim(\forall x, P(x) \to Q(x)) \equiv \exists x \text{ such that } \sim(P(x) \to Q(x)) \quad \ldots (A)$$

$$\sim(P(x) \to Q(x)) \equiv P(x) \wedge \sim Q(x) \quad \ldots (B)$$

Substituting (B) into (A):

$$\sim(\forall x, P(x) \to Q(x)) \equiv \exists x \text{ such that } P(x) \wedge \sim Q(x)$$

Ex: Write a formal negation for statement (a) and an informal negation for statement (b):

a. ∀ people p, if p is blond then p has blue eyes.

b. If a computer program has more than 100,000 lines, then it contains a bug

Negations of Quantified Statements

Theorem: Negation of a Universal Statement

The negation of a statement of the form

$$\forall x \text{ in } D, P(x)$$

Is logically equivalent to a statement of the form

$$\exists x \text{ in } D \textbf{ such that } {\sim}P(x)$$

Symbolically, $\sim(\forall x \text{ in } D, P(x)) \equiv \exists x \text{ in } D$ **such that** $\sim P(x)$

The negation of a universal statement ("all are") is logically equivalent to an existential statement ("some are not" or "there is at least one that is not").

Negations of Universal Conditional Statements

Of special importance in mathematics.

$$\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } \sim(P(x) \rightarrow Q(x)) \quad \text{... (A)}$$

$$\sim(P(x) \rightarrow Q(x)) \equiv P(x) \wedge \sim Q(x) \quad \text{... (B)}$$

Substituting (B) into (A):

$$\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } P(x) \wedge \sim Q(x)$$

Ex: Write a formal negation for statement (a) and an informal negation for statement (b):

b. If a computer program has more than 100,000 lines, then it contains a bug

There is at least one computer program that has more than 100,000 lines and does not contain a bug.

The Relation among ∀, ∃, ∧ and ∨

$$\overline{\forall x \in A, p(x)} \Leftrightarrow \exists x \in A, \overline{p(x)}$$

$$\overline{\exists x \in A, p(x)} \Leftrightarrow \forall x \in A, \overline{p(x)}$$

-The negation of a "**for all**" statement is a "**there exists**" statement

-The negation of a "**there exists**" statement is a "**for all**" statement

The Relation among ∀, ∃, ∧ and ∨

These facts are analogous to De Morgan's laws, which state that the negation of an "**and**" statement is an "**or**"statement and that the negation of an "**or**" statement is an "**and**" statement.

If Q(x) is a predicate and the domain D of x is the set $\{x_1, x_2, \ldots, x_n\}$,

then $$\forall x \in D, Q(x) \equiv Q(x_1) \wedge Q(x_2) \wedge \ldots \wedge Q(x_n)$$

Similarly, if Q(x) is a predicate and $D = \{x_1, x_2, \ldots, x_n\}$, then

$$\exists x \in D \text{ such that } Q(x) \equiv Q(x_1) \vee Q(x_2) \vee \ldots \vee Q(x_n)$$

Ex: Let Q(x) be "x.x = x" and suppose D = {0, 1}.

Then $\forall x \in D$, Q(x) can be rewritten as ∀ binary digits x, x.x = x.

This is equivalent to 0.0=0 and 1.1=1, which can be rewritten in symbols as Q(0)∧Q(1).

The Relation among ∀, ∃, ∧ and ∨

If Q(x) is a predicate and the domain D of x is the set $\{x_1, x_2, \ldots, x_n\}$,

then $\forall x \in D, Q(x) \equiv Q(x_1) \wedge Q(x_2) \wedge \ldots \wedge Q(x_n)$

Similarly, if Q(x) is a predicate and $D = \{x_1, x_2, \ldots, x_n\}$, then

$$\exists x \in D \text{ such that } Q(x) \equiv Q(x_1) \vee Q(x_2) \vee \ldots \vee Q(x_n)$$

Ex: Let Q(x) be "x+x = x" and suppose D = {0, 1}.

Then $\exists x \in D$ such that Q(x)

can be rewritten as

∃ a binary digits x, such that x+x = x.

This is equivalent to

0+0=0 and 1+1=1,

which can be rewritten in symbols as Q(0) ∨ Q(1).

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

The Relation among ∀, ∃, ∧ and ∨

If Q(x) is a predicate and the domain D of x is the set $\{x_1, x_2, \ldots, x_n\}$,

then $$\forall x \in D,\ Q(x) \equiv Q(x_1) \wedge Q(x_2) \wedge \ldots \wedge Q(x_n)$$

Similarly, if Q(x) is a predicate and $D = \{x_1, x_2, \ldots, x_n\}$, then

$$\exists x \in D \text{ such that } Q(x) \equiv Q(x_1) \vee Q(x_2) \vee \ldots \vee Q(x_n)$$

-These facts are analogous to De Morgan's laws, which state that the negation of an "**and**" statement is an "**or**"statement and that the negation of an "**or**" statement is an "**and**" statement.

-This similarity is not accidental. In a sense, universal statements are generalizations of "**and**" statements, and existential statements are generalizations of "**or**" statements.

Vacuous Truth of Universal Statements

All the balls in the bowl are blue.

Vacuous Truth of Universal Statements

Suppose a bowl sits on a table and next to the bowl is a pile of five blue and five gray balls, any of which may be placed in the bowl.

If three blue balls and one gray ball are placed in the bowl, as shown in Figure 3.2.1(a), the statement "All the balls in the bowl are blue" would be false (since one of the balls in the bowl is gray).

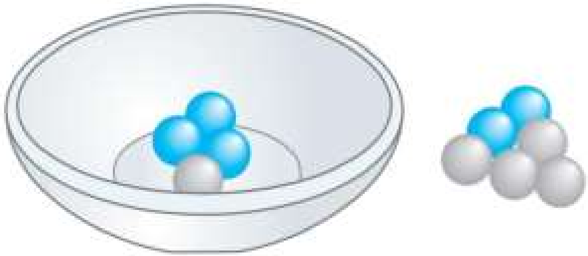


Figure 3.2.1(a)

Now suppose that no balls at all are placed in the bowl, as shown in Figure 3.2.1(b).

Consider the statement:

All the balls in the bowl are blue.

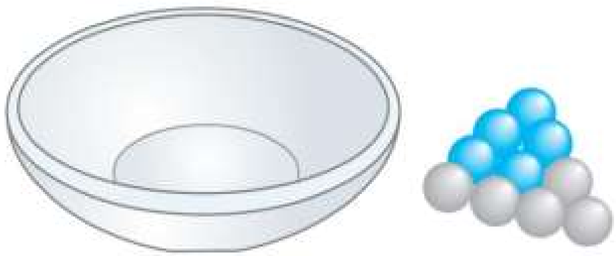


Figure 3.2.1(b)

Vacuous Truth of Universal Statements

p = All the balls in the bowl are blue.

Is the statement true or false? The statement is false if, and only if, its negation is true.

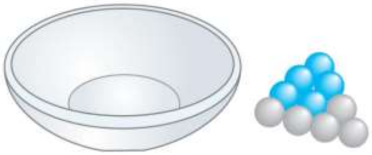


Figure 3.2.1(b)

And its negation is $\overline{\forall x \in A, p(x)} \Leftrightarrow \exists x \in A, \overline{p(x)}$

~p = There exists a ball in the bowl that is not blue.

But the only way this negation can be true is for there actually to be <u>a non-blue ball in the bowl</u>.

And there is not! Hence the negation is false, and so the statement is true "by default".

p	$\bar{p}$
1	0
0	1

In general, a statement of the form

$\forall x$ in D, if $P(x)$ then $Q(x)$

is called vacuously true or true by default if, and only if, $P(x)$ is false for every x in D.

p	q	p→q
0	0	1
0	1	1
1	0	0
1	1	1

Variants of Universal Conditional Statements

Definition 3.2.1 (Contrapositive, converse, inverse)

Consider a statement of the form: $\forall x \in D$, if $P(x)$ then $Q(x)$.

1. Its **contrapositive** is: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
2. Its **converse** is: $\forall x \in D$, if $Q(x)$ then $P(x)$.
3. Its **inverse** is: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

Write a formal and an informal **contrapositive**, **converse**, and **inverse** for the following statement:

If a real number is greater than 2, then its square is greater than 4.

The formal version: $\forall x \in R$, if $x > 2$ then $x^2 > 4$

- Contrapositive: $\forall x \in R$, if $x^2 \leq 4$ then $x \leq 2$

If the square of a real number is less than or equal to 4, then the number is less than or equal to 2.

Variants of Universal Conditional Statements

1. Its **contrapositive** is: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
2. Its **converse** is: $\forall x \in D$, if $Q(x)$ then $P(x)$.
3. Its **inverse** is: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

If a real number is greater than 2, then its square is greater than 4. The formal version: $\forall x \in R$, if $x > 2$ then $x^2 > 4$

Contrapositive: $\forall x \in R$, if $x^2 \leq 4$ then $x \leq 2$

If the square of a real number is less than or equal to 4, then the number is less than or equal to 2.

Converse: $\forall x \in R$, if $x^2 > 4$ then $x > 2$

If the square of a real number is greater than 4, then the number is greater than 2.

Inverse: $\forall x \in R$, if $x \leq 2$ then $x^2 \leq 4$

If a real number is less than or equal to 2, then the square of the number is less than or equal to 4.

Variants of Universal Conditional Statements

Let $P(x)$ and $Q(x)$ be any predicates, let D be the domain of x, and consider the statement: $\forall x \in D$, if $P(x)$ then $Q(x)$

and its contrapositive

$$\forall x \in D, \text{ if } {\sim}Q(x) \text{ then } {\sim}P(x)$$

Any particular x in D that makes "if P(x) then Q(x)" true also makes "if ~Q(x) then ~P(x)" true (by the logical equivalence between p → q and ~q → ~p).

It follows that "if P(x) then Q(x)" is true for all x in D iff "if ~Q(x) then ~P(x)" is true for all x in D.

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x) \equiv \forall x \in D, \text{ if } {\sim}Q(x) \text{ then } {\sim}P(x)$$

If not q then not p		if p then q		if q then p		If not p then not q
${\sim}q \rightarrow {\sim}p$	$\equiv$	$p \rightarrow q$	$\not\equiv$	$q \rightarrow p$	$\equiv$	${\sim}p \rightarrow {\sim}q$
contrapositive (Phản đảo)		*conditional statement*		*converse* (Đảo)		*inverse* (Nghịch Đảo)

Variants of Universal Conditional Statements

Consider the statement:

$\forall x \in \mathbf{R}$, if $x > 2$ then $x^2 > 4$ — True

and its converse

$\forall x \in \mathbf{R}$, if $x^2 > 4$ then $x > 2$ — False

A universal conditional statement is not logically equivalent to its converse.

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x) \not\equiv \forall x \in D, \text{ if } Q(x) \text{ then } P(x)$$

If not q then not p		if p then q		if q then p		If not p then not q
$\sim q \rightarrow \sim p$	$\equiv$	$p \rightarrow q$	$\not\equiv$	$q \rightarrow p$	$\equiv$	$\sim p \rightarrow \sim q$
contrapositive (Phản đảo)		*conditional statement*		*converse* (Đảo)		*inverse* (Nghịch Đảo)

Necessary and Sufficient Conditions, Only if

Definition 3.2.2 (Necessary and Sufficient conditions, Only if)

- "$\forall x$, $r(x)$ is a **sufficient condition** for $s(x)$" means "$\forall x$, if $r(x)$ then $s(x)$".
- "$\forall x$, $r(x)$ is a **necessary condition** for $s(x)$" means "$\forall x$, if $\sim r(x)$ then $\sim s(x)$" or, equivalently, "$\forall x$, if $s(x)$ then $r(x)$".
- "$\forall x$, $r(x)$ **only if** $s(x)$" means "$\forall x$, if $\sim s(x)$ then $\sim r(x)$" or, equivalently, "$\forall x$, if $r(x)$ then $s(x)$" .

Rewrite the following statements as quantified conditional statements. Do not use the word *necessary* or *sufficient*:

a. Squareness is a sufficient condition for rectangularity.

b. Being at least 35 years old is a necessary condition for being President of the United States

Necessary and Sufficient Conditions, Only if

- "$\forall x$, $r(x)$ is a **sufficient condition** for $s(x)$" means "$\forall x$, if $r(x)$ then $s(x)$".
- "$\forall x$, $r(x)$ is a **necessary condition** for $s(x)$" means "$\forall x$, if $\sim r(x)$ then $\sim s(x)$" or, equivalently, "$\forall x$, if $s(x)$ then $r(x)$".
- "$\forall x$, $r(x)$ **only if** $s(x)$" means "$\forall x$, if $\sim s(x)$ then $\sim r(x)$" or, equivalently, "$\forall x$, if $r(x)$ then $s(x)$" .

Rewrite the following statements as quantified conditional statements. Do not use the word *necessary* or *sufficient*:

a. Squareness is a sufficient condition for rectangularity.

∀x, if x is a square, then x is a rectangle.

Informal: If a figure is a square, then it is a rectangle.

Necessary and Sufficient Conditions, Only if

- "$\forall x$, $r(x)$ is a **sufficient condition** for $s(x)$" means "$\forall x$, if $r(x)$ then $s(x)$".
- "$\forall x$, $r(x)$ is a **necessary condition** for $s(x)$" means "$\forall x$, if $\sim r(x)$ then $\sim s(x)$" or, equivalently, "$\forall x$, if $s(x)$ then $r(x)$".
- "$\forall x$, $r(x)$ **only if** $s(x)$" means "$\forall x$, if $\sim s(x)$ then $\sim r(x)$" or, equivalently, "$\forall x$, if $r(x)$ then $s(x)$" .

b. Being at least 35 years old is a necessary condition for being President of the United States.

$\forall$ people x, if x is younger than 35, then x cannot be president of the United States.

$\forall$ people x, if x is president of the United States, then x is at least 35 years old.

Necessary and Sufficient Conditions, Only if

- "$\forall x$, $r(x)$ is a **sufficient condition** for $s(x)$" means "$\forall x$, if $r(x)$ then $s(x)$".
- "$\forall x$, $r(x)$ is a **necessary condition** for $s(x)$" means "$\forall x$, if $\sim r(x)$ then $\sim s(x)$" or, equivalently, "$\forall x$, if $s(x)$ then $r(x)$".
- "$\forall x$, $r(x)$ **only if** $s(x)$" means "$\forall x$, if $\sim s(x)$ then $\sim r(x)$" or, equivalently, "$\forall x$, if $r(x)$ then $s(x)$" .

c. A product of two numbers is 0 only if one of the numbers is 0.

If it is not the case that one of two numbers is 0, then the product of the numbers is not 0

If neither of two numbers is 0, then the product of the numbers is not 0

If a product of two numbers is 0, then one of the numbers is 0

Lượng từ hóa vị từ hai biến
Statements with Multiple Quantifiers

Consider the Tarski's world again.

Show that the following statement is true:

For all triangles *x*, there is a square *y* such that *x* and *y* have the same color.

The statement says that no matter which triangle someone gives you, you will be able to find a square of the same color.

There are only 3 triangles *d*, *f*, and *i*.

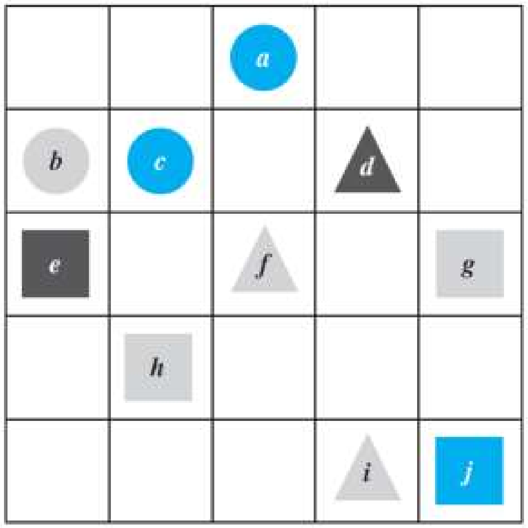


Figure 3.3.1

Given $x =$	choose $y =$	and check that y is the same color as x.
d	e	yes •
f or i	h or g	yes •

Interpreting Multiply-Quantified Statements

If you want to establish the truth of a statement of the form:

$$\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y)$$

your challenge is to allow someone else to pick whatever element *x* in *D* they wish and then you must find an element *y* in *E* that "works" for that particular *x*.

If you want to establish the truth of a statement of the form:

$$\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y)$$

your job is to find one particular *x* in *D* that will "work" no matter what *y* in *E* anyone might choose to challenge you with.

Statements with Multiple Quantifiers

Định nghĩa. Cho p(x, y) là một vị từ theo hai biến x, y xác định trên $A \times B$. Ta định nghĩa các mệnh đề lượng từ hóa của p(x, y) như sau:

"$\forall x \in A, \forall y \in B, p(x, y)$" = "$\forall x \in A, (\forall y \in B, p(x, y))$"

"$\forall x \in A, \exists y \in B, p(x, y)$" = "$\forall x \in A, (\exists y \in B, p(x, y))$"

"$\exists x \in A, \forall y \in B, p(x, y)$" = "$\exists x \in A, (\forall y \in B, p(x, y))$"

"$\exists x \in A, \exists y \in B, p(x, y)$" = "$\exists x \in A, (\exists y \in B, p(x, y))$"

VD: Diễn đạt câu:

"Mọi người đều có chính xác một người bạn tốt nhất"

Đặt B(x,y) là câu "y là bạn tốt nhất của x"

$$\forall x\ \exists y\ \forall z\ [B(x,y) \land ((z \neq y) \rightarrow \neg B(x,z))]$$

Statements with Multiple Quantifiers

Mệnh đề	**Khi nào đúng**	**Khi nào sai**
$\forall x\ \forall y\ p(x,y)$ $\forall y\ \forall x\ p(x,y)$	p(x,y) đúng với mọi cặp (x,y)	Có một cặp (x,y) đối với nó p(x,y) là sai
$\forall x\ \exists y\ p(x,y)$	Với mọi x, có một y sao cho p(x,y) là đúng	Có một x sao cho p(x,y) là sai với mọi y
$\exists x\ \forall y\ p(x,y)$	Có một x sao cho p(x,y) đúng với mọi y	Với mọi x, có một y sao cho p(x,y) là sai
$\exists x\ \exists y\ p(x,y)$ $\exists y\ \exists x\ p(x,y)$	Có một cặp (x,y) sao cho p(x,y) là đúng	p(x,y) đúng với mọi cặp (x,y)

VD

Lượng từ "**$\forall x \in R, \forall y \in R, x + 2y < 1$**" đúng hay sai?

Lượng từ sai vì tồn tại $x_0 = 0$, $y_0 = 1 \in R$ mà $0 + 2.1 = 2 < 1$ (Sai).

Vậy lượng từ trên là sai

Order of Quantifiers

Định lý. Cho p(x, y) là một vị từ theo hai biến x, y xác định trên $A \times B$.

Khi đó: 1) "$\forall x \in A, \forall y \in B, p(x, y)$" $\Leftrightarrow$ "$\forall y \in B, \forall x \in A, p(x, y)$"

2) "$\exists x \in A, \exists y \in B, p(x, y)$" $\Leftrightarrow$ "$\exists y \in B, \exists x \in A, p(x, y)$"

3) "$\exists x \in A, \forall y \in B, p(x, y)$" $\Rightarrow$ "$\forall y \in B, \exists x \in A, p(x, y)$"

Chiều đảo của 3) nói chung không đúng.

In a statement containing both ∀ and ∃, changing the order of the quantifiers usually changes the meaning of the statement. However, if one quantifier immediately follows another quantifier *of the same type*, then the order of the quantifiers does not affect the meaning.

Order of Quantifiers

Định lý. Cho p(x, y) là một vị từ theo hai biến x, y xác định trên $A \times B$.

Khi đó: 1) "$\forall x \in A, \forall y \in B, p(x, y)$" $\Leftrightarrow$ "$\forall y \in B, \forall x \in A, p(x, y)$"

2) "$\exists x \in A, \exists y \in B, p(x, y)$" $\Leftrightarrow$ "$\exists y \in B, \exists x \in A, p(x, y)$"

3) "$\exists x \in A, \forall y \in B, p(x, y)$" $\Rightarrow$ "$\forall y \in B, \exists x \in A, p(x, y)$"

Chiều đảo của 3) nói chung không đúng.

∀ people *x*, ∃ a person *y* such that *x* loves *y*.

∃ a person *y* such that ∀ people *x*, *x* loves *y*.

Except for the order of the quantifiers, these statements are identical.

Given any person, it is possible to find someone whom that person loves.

They are not logically equivalent!

There is one amazing individual who is loved by all people.

Interpreting Multiply-Quantified Statements

A college cafeteria line has four stations:

The **salad** station offers a choice of green salad or fruit salad;

The **main course** station offers spaghetti or fish;

The **dessert station** offers pie or cake;

The **beverage station** offers milk, soda, or coffee.

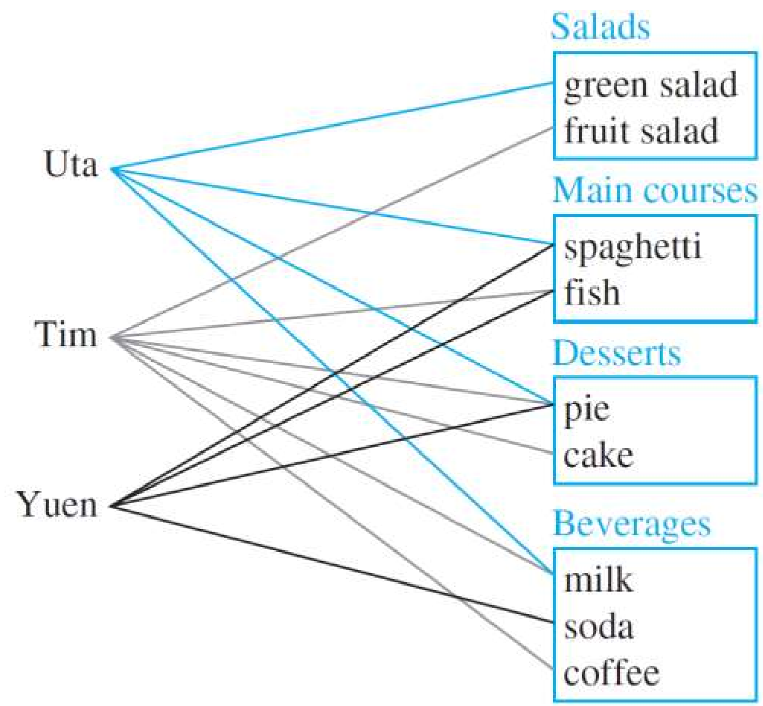


Figure 3.3.2

Three students, Uta, Tim, and Yuen, go through the line and make the following choices:

- Uta: green salad, spaghetti, pie, milk
- Tim: fruit salad, fish, pie, milk, coffee
- Yuen: spaghetti, fish, pie, soda

Interpreting Multiply-Quantified Statements

Write each of following statements informally and find its truth value.

a. ∃ an item *I* such that ∀ students *S*, *S* chose *I*.

b. ∃ a student *S* such that ∀ items *I*, *S* chose *I*.

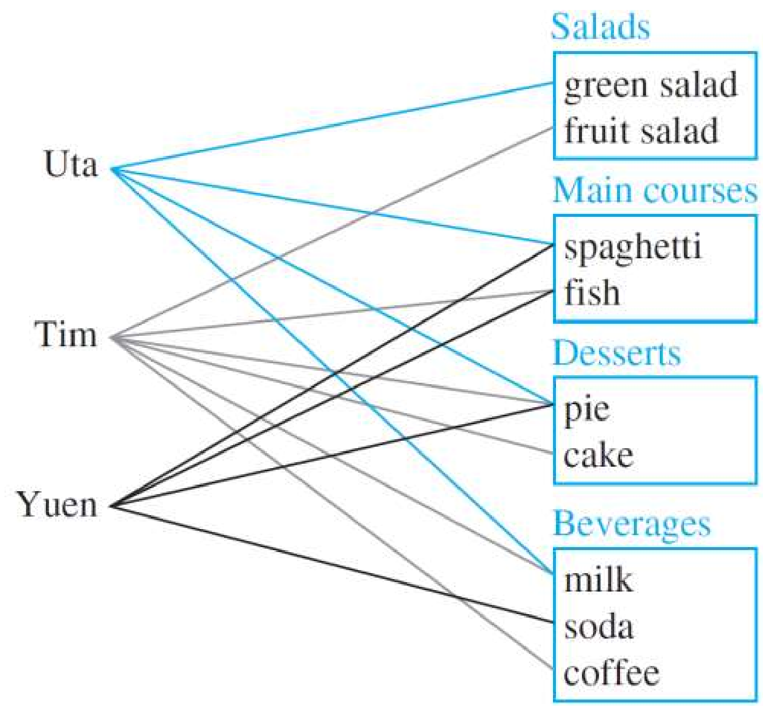


Figure 3.3.2

c. ∃ a student S such that ∀ stations Z, ∃ an item I in Z such that S chose I.

d. ∀ students S and ∀ stations Z, ∃ an item I in Z such that S chose I

Interpreting Multiply-Quantified Statements

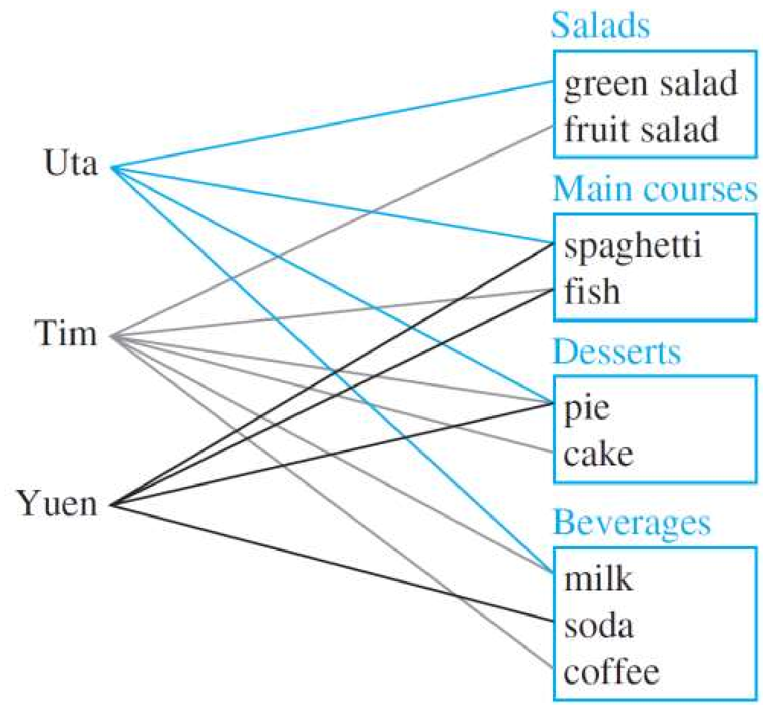


Figure 3.3.2

a. ∃ an item *I* such that ∀ students *S*, *S* chose *I*.

There is an item that was chosen by every student. This is true; every student chose pie.

b. **∃ a student *S* such that ∀ items *I*, *S* chose *I*.**

There is a student who chose every available item. This is false; no student chose all nine items.

Interpreting Multiply-Quantified Statements

c. ∃ a student S such that ∀ stations Z, ∃ an item I in Z such that S chose I.

There is a student who chose at least one item from every station. This is true; both Uta and Tim chose at least one item from every station

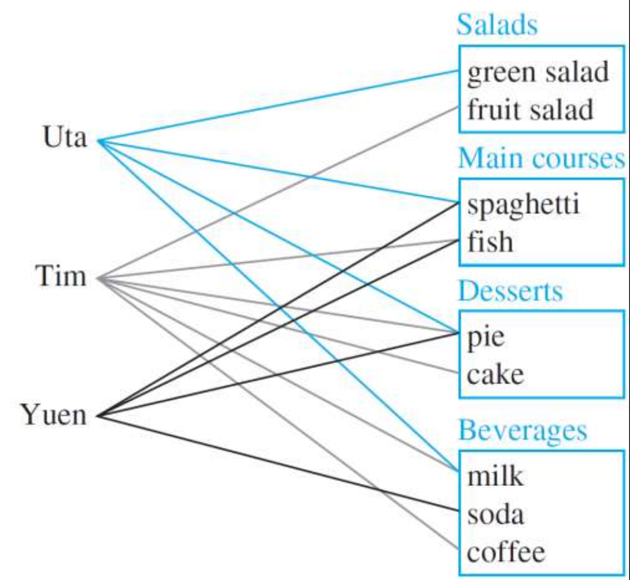


Figure 3.3.2

d. ∀ students S and ∀ stations Z, ∃ an item I in Z such that S chose I

Every student chose at least one item from every station. This is false; Yuen did not choose a salad.

Translating from Informal to Formal Language

Most problems are stated in informal language, but solving them often requires translating them into more formal terms.

Example: The reciprocal (số nghịch đảo) of a real number a is a real number b such that ab = 1. The following 2 statements are true. Rewrite them formally using quantifiers and variables:

a. Every nonzero real number has a reciprocal.

∀ nonzero real numbers u, ∃ a real number v such that uv = 1.

b. There is a real number with no reciprocal.

∃ a real numbers c such that ∀ real number d, cd ≠ 1

Ambiguous Language

You are visiting a computer microchips factory. The factory guide tells you:

There is a person supervising every detail of the production process.

"there is" – existential quantifier; "every" – universal quantifier.

Which of the following best describes its meaning?

- ☐ There is one single person who supervises all the details of the production process.
- ☐ For any particular production detail, there is a person who supervises the detail, but there might be different supervisors for different details.

Ambiguous Language

You are visiting a computer microchips factory. The factory guide tells you:

There is a person supervising every detail of the production process.

“there is” – existential quantifier; “every” – universal quantifier.

Which of the following best describes its meaning?

- ☐ There is one single person who supervises all the details of the production process.
- ☐ For any particular production detail, there is a person who supervises the detail, but there might be different supervisors for different details.

Ambiguous Language

Once you interpreted the sentence in one way, it may have been hard for you to see that it could be understood in the other way.

Perhaps you had difficulty even though the two possible meanings were explained.

Although statements written informally may be open to multiple interpretations, we cannot determine their truth or falsity without interpreting them one way or another

Therefore, we have to use context to try to ascertain their meaning as best we can.

Negations of Multiply-Quantified Statements

Recall in 3.2.1:

$\sim(\forall x \text{ in } D, P(x)) \equiv \exists x \text{ in } D \text{ such that } \sim P(x)$

$\sim(\exists x \text{ in } D \text{ such that } P(x)) \equiv \forall x \text{ in } D, \sim P(x)$

(A) So, to find: $\sim(\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y))$

➔ $\exists x \text{ in } D \text{ such that } \sim(\exists y \text{ in } E \text{ such that } P(x, y))$

➔ $\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, \sim P(x, y)$.

$\sim(\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y)) \equiv \exists x \text{ in } D \text{ such that } \forall y \text{ in } E, \sim P(x, y)$

(B) Similarly, to find: $\sim(\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y))$

➔ $\forall x \text{ in } D, \sim(\forall y \text{ in } E, P(x, y))$

➔ $\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } \sim P(x, y)$.

$\sim(\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y)) \equiv \forall x \text{ in } D, \exists y \text{ in } E \text{ such that } \sim P(x, y)$

Phủ định của mệnh đề lượng từ

Vị từ theo 2 biến. $\overline{\forall x \in A, \forall y \in B, p(x,y)} \Leftrightarrow \exists x \in A, \exists y \in B, \overline{p(x,y)}$

$$\overline{\forall x \in A, \exists y \in B, p(x,y)} \Leftrightarrow \exists x \in A, \forall y \in B, \overline{p(x,y)}$$

$$\overline{\exists x \in A, \forall y \in B, p(x,y)} \Leftrightarrow \forall x \in A, \exists y \in B, \overline{p(x,y)}$$

$$\overline{\exists x \in A, \exists y \in B, p(x,y)} \Leftrightarrow \forall x \in A, \forall y \in B, \overline{p(x,y)}$$

Ví dụ phủ định các mệnh đề sau

a) "$\forall x \in A, 2x + 1 \le 0$"

"$\exists x \in A, 2x + 1 > 0$"

b) "$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in R, |x - a| < \delta \to |f(x) - f(a)| < \varepsilon$".

"$\exists \varepsilon > 0, \forall \delta > 0, \exists x \in R, |x - a| < \delta \wedge (|f(x) - f(a)| \ge \varepsilon)$".

Negations of Multiply-Quantified Statements

Refer to the Tarski's world of Figure 3.3.1 again.

Write a negation for each of the following statements, and determine which is true, the given statement or its negation.

a. For all squares *x*, there is a circle *y* such that *x* and *y* have the same color.

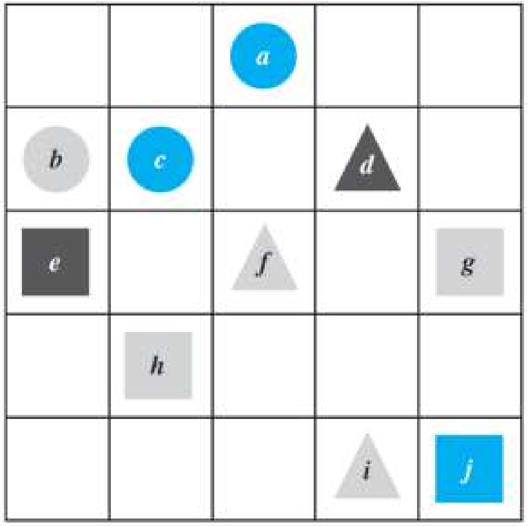


Figure 3.3.1

Negation:

∃ a square *x* such that ~(∃ a circle *y* such that *x* and *y* have the same color)

➔ ∃ a square *x* such that ∀ circles *y*, *x* and *y* do not have the same color.

TRUE (Square *e* is black and no circle is black).

Lượng từ hóa vị từ hai biến

Mệnh đề	Khi nào đúng	Khi nào sai
$\forall x \forall y\, p(x,y)$ $\forall y \forall x\, p(x,y)$	p(x,y) đúng với mọi cặp (x,y)	Có một cặp (x,y) đối với nó p(x,y) là sai
$\forall x \exists y\, p(x,y)$	Với mọi x, có một y sao cho p(x,y) là đúng	Có một x sao cho p(x,y) là sai với mọi y
$\exists x \forall y\, p(x,y)$	Có một x sao cho p(x,y) đúng với mọi y	Với mọi x, có một y sao cho p(x,y) là sai
$\exists x \exists y\, p(x,y)$ $\exists y \exists x\, p(x,y)$	Có một cặp (x,y) sao cho p(x,y) là đúng	p(x,y) đúng với mọi cặp (x,y)

Negations of Multiply-Quantified Statements

Refer to the Tarski's world of Figure 3.3.1 again.

Write a negation for each of the following statements, and determine which is true, the given statement or its negation.

b. There is a triangle *x* such that for all squares *y*, *x* is to the right of *y*.

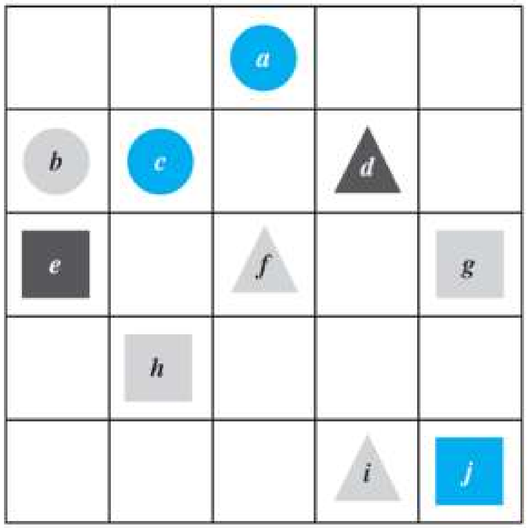


Figure 3.3.1

Lượng từ hóa vị từ hai biến

Mệnh đề	Khi nào đúng	Khi nào sai
$\forall x\ \forall y\ p(x,y)$ $\forall y\ \forall x\ p(x,y)$	p(x,y) đúng với mọi cặp (x,y)	Có một cặp (x,y) đối với nó p(x,y) là sai
$\forall x\ \exists y\ p(x,y)$	Với mọi x, có một y sao cho p(x,y) là đúng	Có một x sao cho p(x,y) là sai với mọi y
$\exists x\ \forall y\ p(x,y)$	Có một x sao cho p(x,y) đúng với mọi y	Với mọi x, có một y sao cho p(x,y) là sai
$\exists x\ \exists y\ p(x,y)$ $\exists y\ \exists x\ p(x,y)$	Có một cặp (x,y) sao cho p(x,y) là đúng	p(x,y) đúng với mọi cặp (x,y)

Negations of Multiply-Quantified Statements

Refer to the Tarski's world of Figure 3.3.1 again.

Write a negation for each of the following statements, and determine which is true, the given statement or its negation.

b. There is a triangle *x* such that for all squares *y*, *x* is to the right of *y*.

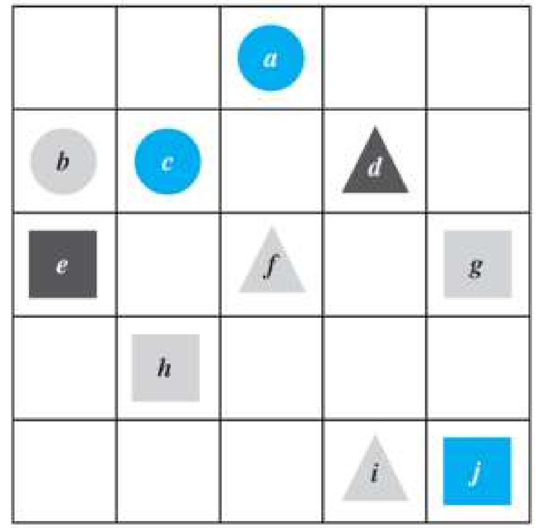


Figure 3.3.1

∀ triangle *x*, ~(∀ square *y*, *x* is to the right of *y*).

∀ triangle *x*, ∃ a square *y* such that *x* is not to the right of *y*.

The negation is true because no matter what triangle is chosen, it is not to the right of square *g* or square *j*, which are the only squares in this Tarski world.

Order of Quantifiers

Refer to the Tarski's world of Figure 3.3.1. What are the truth values of the following two statements?

a. For every square *x*, there is a triangle *y* such that *x* and *y* have different colors.

b. There exists a triangle *y* such that for every square *x*, *x* and *y* have different colors.

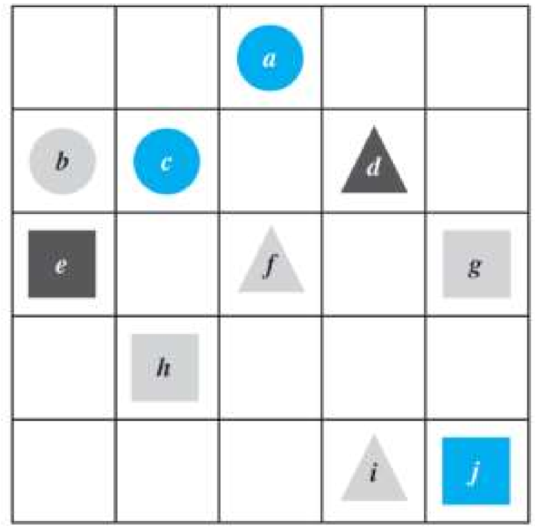


Figure 3.3.1

Order of Quantifiers

Refer to the Tarski's world of Figure 3.3.1. What are the truth values of the following two statements?

a. For every square *x*, there is a triangle *y* such that *x* and *y* have different colors.

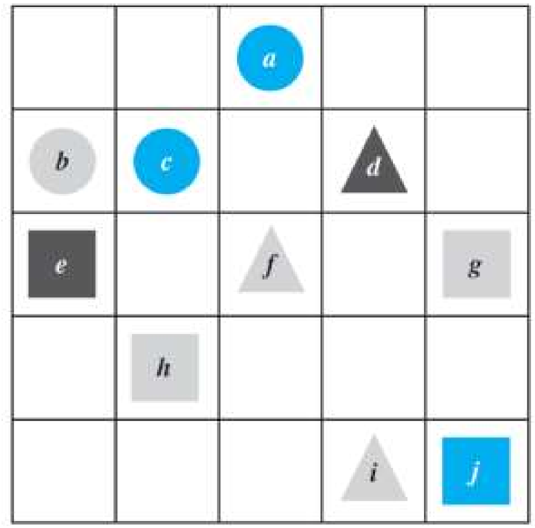


Figure 3.3.1

Statement (a) says that if someone gives you one of the squares from the Tarski world, you can find a triangle that has a different color. This is true.

If someone gives you square *g* or *h* (which are gray), you can use triangle *d* (which is black);

if someone gives you square e (which is black), you can use either triangle f or i (which are gray);

And if someone gives you square j (which is blue), you can use triangle d (which is black) or triangle f or I (which are gray).

Order of Quantifiers

Refer to the Tarski's world of Figure 3.3.1. What are the truth values of the following two statements?

b. There exists a triangle *y* such that for every square *x*, *x* and *y* have different colors.

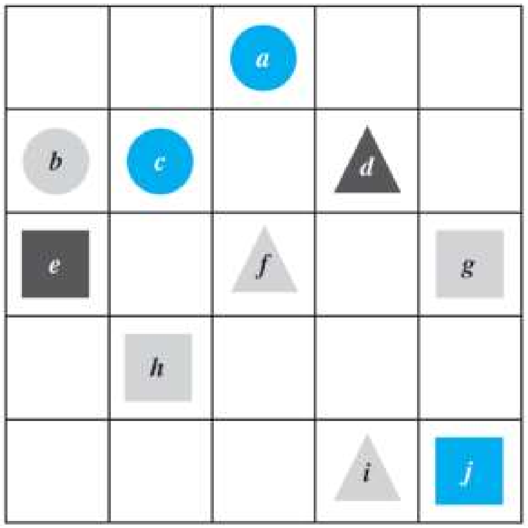


Figure 3.3.1

(b) says that there is one particular triangle in the Tarski world that has a different color from every one of the squares in the world. This is false.

Two of the triangles are gray, but they cannot be used to show the truth of the statement because the Tarski world contains gray squares.

The only other triangle is black, but it cannot be used either because there is a black square in the Tarski world.

Thus one of the statements is true and the other is false, and so they have opposite truth values.

Formal Logical Notation

In some areas of computer science, logical statements are expressed in purely symbolic notation.

The notation involves using predicates to describe all properties of variables and omitting the words *such as* in existential statements

"$\forall x$ in D, $P(x)$" written as $\forall x\ (x \text{ in } D \rightarrow P(x))$

"$\exists x$ in D such that $P(x)$" written as $\exists x\ (x \text{ in } D \wedge P(x))$

Ví dụ.

"$\forall x$ in D, $P(x)$" written as $\forall x\ (x \text{ in } D \rightarrow P(x))$

"$\exists x$ in D such that $P(x)$" written as $\exists x\ (x \text{ in } D \wedge P(x))$

Xét các câu sau, trong đó hai câu đầu là tiền đề và câu thứ ba là kết luận đúng. Toàn bộ tập hợp ba câu này được gọi là một **suy lí**

a) "Tất cả sư tử đều hung dữ" b) "Một số sư tử không uống cafe"

c) "Một số sinh vật hung dữ không uống cafe"

Giả sử rằng không gian là tập hợp toàn bộ các sinh vật, hãy diễn đạt các câu trong suy lý trên bằng cách dung P(x), Q(x),R(x) và các lượng từ

Gọi

P(x): x là sư tử ;

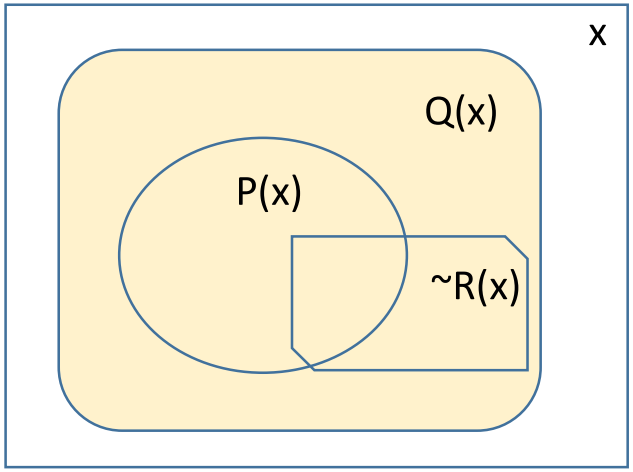
Q(x): x hung dữ ;

R(x): x uống café

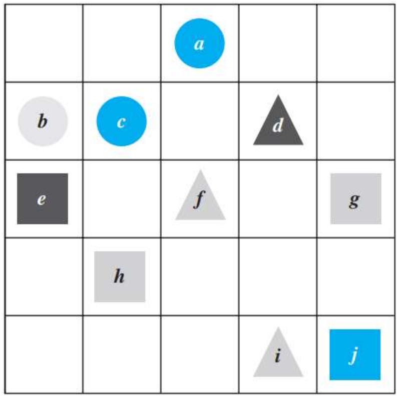
a) $\forall x$, $(P(x) \rightarrow Q(x))$

b) $\exists x$, $(P(x) \wedge \sim R(x))$

c) $\exists x$, $(Q(x) \wedge \sim R(x))$



Formalizing Statements in a Tarski's World



Example:

- Tarski's world.
- Let the common domain *D* of all variables be the set of all the objects in the Tarski's world.

Triangle(x): "x is a triangle"
Circle(x): "x is a circle"
Square(x): "x is a square"

Blue(x): "x is blue"
Gray(x): "x is gray"
Black(x): "x is black"

RightOf(x,y): "x is to the right of y"
Above(x,y): "x is above y"
SameColorAs(x,y): "x has the same color as y"
$x = y$: "x is equal to y"

Formalizing Statements in a Tarski's World

"$\forall x$ in D, $P(x)$" written as $\forall x\,(x \text{ in } D \to P(x))$

"$\exists x$ in D such that $P(x)$" written as $\exists x\,(x \text{ in } D \wedge P(x))$

Representation of If-Then as Or

$p \to q \Leftrightarrow \sim p \vee q$

Negation of Conditional Statement

$\sim(p \to q) \equiv \sim(\sim p \vee q) \equiv p \wedge \sim q$

Use formal, logical notation to write the following statements, and write a formal negation for each statement.

a. For all circles x, x is above f.

Statement: $\forall x\,(\text{Circle}(x) \to \text{Above}(x, f))$

Negation: $\sim(\forall x\,(\text{Circle}(x) \to \text{Above}(x, f)))$

$\equiv \exists x \sim(\text{Circle}(x) \to \text{Above}(x, f)) \equiv \exists x\,(\text{Circle}(x) \wedge \sim\text{Above}(x, f))$

b. There is a square x such that x is black.

Statement: $\exists x\,(\text{Square}(x) \wedge \text{Black}(x))$

Negation: $\sim(\exists x\,(\text{Square}(x) \wedge \text{Black}(x)))$

$\equiv \forall x \sim(\text{Square}(x) \wedge \text{Black}(x)) \equiv \forall x\,(\sim\text{Square}(x) \vee \sim\text{Black}(x))$

Formalizing Statements in a Tarski's World

"$\forall x$ in D, $P(x)$" written as $\forall x\,(x \text{ in } D \rightarrow P(x))$

"$\exists x$ in D such that $P(x)$" written as $\exists x\,(x \text{ in } D \wedge P(x))$

Use formal, logical notation to write the following statements, and write a formal negation for each statement.

c. For all circles x, there is a square y such that x and y have the same color.

Statement: $\forall x(\text{Circle}(x) \rightarrow \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y)))$

Negation:

d. There is a square x such that for all triangles y, x is to right of y.

Statement:

Negation:

Formalizing Statements in a Tarski's World

For all circles x, there is a square y such that x and y have the same color

For all circles x, there is a square y such that x and y have the same color

$\underbrace{\forall\text{circles } x}_{\forall x \text{ in } D}$, $\underbrace{\text{there is a square y such that x and y have the same color}}_{P(x)}$

"∀*x* in *D*, *P*(*x*)" written as $\forall x\,(x \text{ in } D \rightarrow P(x))$

∀x(Circles (x) → there is a square y such that x and y have the same color)

"∃*x* in *D* such that *P*(*x*)" written as $\exists x\,(x \text{ in } D \wedge P(x))$

∀x(Circles (x) → **there is** a square y **such that** x and y have the same color)

$\forall x(\text{Circles }(x) \rightarrow \exists \underbrace{\text{a square y}}_{\text{x in D}}$ **such that** $\underbrace{\text{x and y have the same color}}_{P(x)})$

∀x(Circles (x) → ∃**y(Square(y)∧** x and y have the same color))

∀x(Circles (x) → ∃y(Square(y)∧ **Samecolor(x,y)**))

Formalizing Statements in a Tarski's World

Representation of If-Then as Or

$p \to q \Leftrightarrow \sim p \vee q$

Negation of Conditional Statement

$\sim(p \to q) \equiv \sim(\sim p \vee q) \equiv p \wedge \sim q$

For all circles x, there is a square y such that x and y have the same color

$\forall x(\text{Circles}(x) \to \exists y(\text{Square}(y) \wedge \text{Samecolor}(x,y)))$

Negation: $\sim(\forall x(\text{Circle}(x) \to \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y))))$

$\equiv \exists x \sim(\text{Circle}(x) \to \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y)))$

by the law for negating a $\forall$ statement

$\equiv \exists x(\text{Circle}(x) \wedge \sim(\exists y(\text{Square}(y) \wedge \text{SameColor}(x, y))))$

by the law for negating an if-then statement

$\equiv \exists x(\text{Circle}(x) \wedge \forall y(\sim(\text{Square}(y) \wedge \text{SameColor}(x, y))))$

by the law for negating a $\exists$ statement

$\equiv \exists x(\text{Circle}(x) \wedge \forall y(\sim \text{Square}(y) \vee \sim \text{SameColor}(x, y)))$

by De Morgan's law

Formalizing Statements in a Tarski's World

"$\forall x$ in D, $P(x)$" written as $\forall x\ (x \text{ in } D \rightarrow P(x))$

"$\exists x$ in D such that $P(x)$" written as $\exists x\ (x \text{ in } D \wedge P(x))$

There is a square x such that for all triangles y, x is to right of x

There is a square x **such that** for all triangles y, x is to right of x

Formal Logical Notation

Formal logical notation is used in branches of computer science such as *artificial intelligence*, *program verification*, and *automata theory and formal languages*.

Taken together, the symbols for quantifiers, variables, predicates, and logical connectives make up what is known as the **language of first-order logic**.

Even though this language is simpler in many respects than the language we use every day, learning it requires the same kind of practice needed to acquire any foreign language.

Prolog

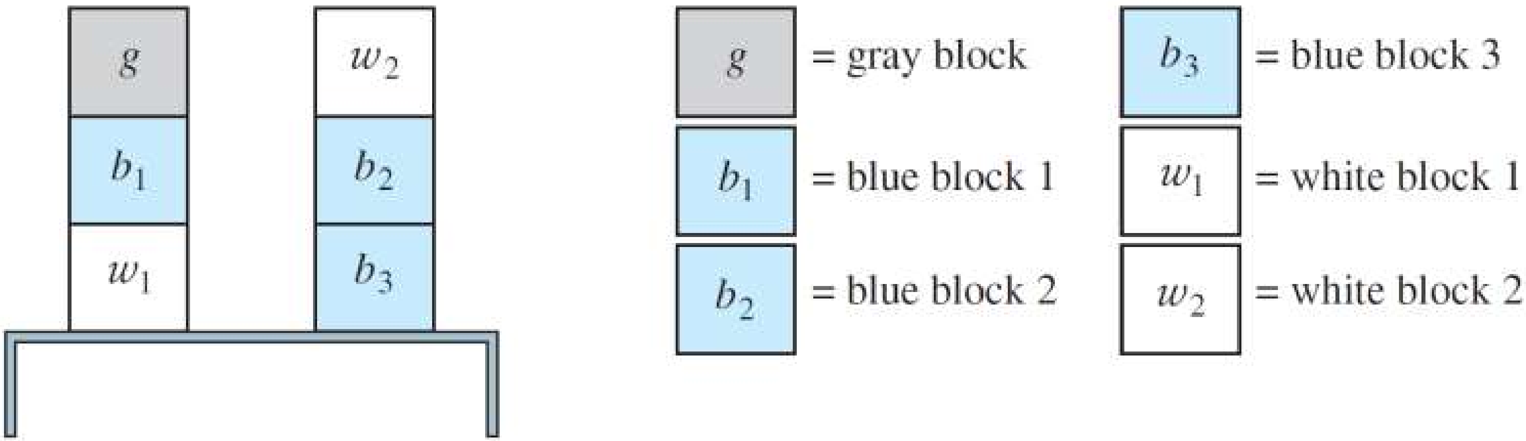
The programming language **Prolog** (short for *pro*gramming in *log*ic) was developed in France in the 1970s by A. Colmerauer and P. Roussel to help programmers working in the field of artificial intelligence.

A simple Prolog program consists of a set of statements describing some situation together with questions about the situation. Built into the language are search and inference techniques needed to answer the questions by deriving the answers from the given statements.

This frees the programmer from the necessity of having to write separate programs to answer each type of question.

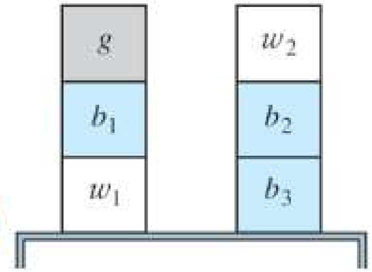
Prolog

Consider the following picture, which shows colored blocks stacked on a table.



The following are statements in Prolog that describe this picture and ask two questions about it.

Prolog



isabove(g, b_1) color(g, gray) color(b_3, blue)

isabove(b_1, w_1) color(b_1, blue) color(w_1, white)

isabove(w_2, b_2) color(b_2, blue) color(w_2, white)

isabove(b_2, b_3) isabove(X, Z) if isabove(X, Y) and isabove(Y, Z)

?color(b_1, blue) ?isabove(X, w_1)

The statements "isabove(g, b_1)" and "color(g, gray)" are to be interpreted as "g is above b_1" and "g is colored gray".

The statement "isabove(X, Z) if isabove(X, Y) and isabove(Y, Z)" is to be interpreted as "For all X, Y, and Z, if X is above Y and Y is above Z, then X is above Z."

Prolog

The program statement

?color(b_1, blue)

is a question asking whether block b_1 is colored blue.

Prolog answers this by writing

Yes

The program statement

?isabove(X, w_1)

is a question asking for which blocks X the predicate "X is above w_1" is true..

Prolog answers this by giving a list of all such blocks. In this case, the answer is

$X = b_1$, $X = g$.

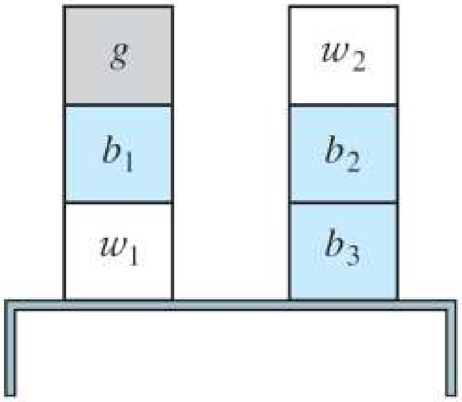
Infer the solution $X = g$ from the following statements:

- isabove(g, b_1)
- isabove(b_1, w_1)
- isabove(X, Z) if isabove(X, Y) and isabove(Y, Z)

70

Prolog

Write the answers Prolog would give if the following questions were added to the program above.



a. ?isabove(b_2, w_1) “No”

b. ?color(w_1, X)

c. ?color(X, blue)

isabove(g, b_1)	color(g, gray)	color(b_3, blue)
isabove(b_1, w_1)	color(b_1, blue)	color(w_1, white)
isabove(w_2, b_2)	color(b_2, blue)	color(w_2, white)
isabove(b_2, b_3)	isabove(X, Z) if isabove(X, Y) and isabove(Y, Z)	

Arguments with Quantified Statements

The rule of universal instantiation:

If some property is true of *everything* in the set, then it is true of *any particular* thing in the set.

Universal instantiation is the fundamental tool of deductive reasoning

The rule of universal instantiation can be combined with modus ponens to obtain the valid form of argument called universal modus ponens.

Modus Ponens Qui tắc khẳng định	$p \to q$ p $\therefore q$
Modus Tollens Qui tắc phủ định	$p \to q$ $\sim q$ $\therefore \sim p$

Universal Modus Ponens

Formal version	*Informal version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$P(a)$ for a particular a.	a makes $P(x)$ true.
• $Q(a)$.	• a makes $Q(x)$ true.

Recognizing Universal Modus Ponens

Rewrite the following argument using quantifiers, variables, and predicate symbols. Is this argument valid? Why?

> If an integer is even, then its square is even.
> *k* is a particular integer that is even.
> • k^2 is even.

Modus Ponens Qui tắc khẳng định	$p \rightarrow q$ p $\therefore q$
Modus Tollens Qui tắc phủ định	$p \rightarrow q$ $\sim q$ $\therefore \sim p$

Solution:

Premise: $\forall x$, if x is an even integer then x^2 is even.

Let $E(x)$ be "x is an even integer", let $S(x)$ be "x^2 is even", and let k stand for a particular integer that is even.

> $\forall x$, if $E(x)$ then $S(x)$.
> $E(x)$, for a particular k.
> • $S(k)$.

This argument has the form of universal modus ponens and is therefore valid.

Use of Universal Modus Ponens in a Proof

Proof: The sum of any two even integers is even.

$\forall$integers x, x is even iff $\exists$ an integer k such that $x = 2k$.

Suppose m and n are particular but arbitrarily chosen even integers, then $m = 2r$ for some integer r(1), and $n = 2s$ for some integer s(2).

Hence

$$m + n = 2r + 2s = 2(r + s) \quad (3)$$

Now $(r + s)$ is an integer(4), and so $2(r + s)$ is even(5).

Thus $m + n$ is even.

Use of Universal Modus Ponens in a Proof

How universal modus ponens is used in the proof.

Suppose *m* and *n* are particular but arbitrarily chosen even integers, then $m = 2r$ for some integer r[(1)], and $n = 2s$ for some integer s[(2)].

(1) If an integer is even, then it equals twice some integer.
m is a particular even integer.
• *m* equals twice some integer *r*.

(2) Similar to (1).

Modus Ponens Qui tắc khẳng định	$p \rightarrow q$ p $\therefore q$
Modus Tollens Qui tắc phủ định	$p \rightarrow q$ $\sim q$ $\therefore \sim p$

Use of Universal Modus Ponens in a Proof

How universal modus ponens is used in the proof.

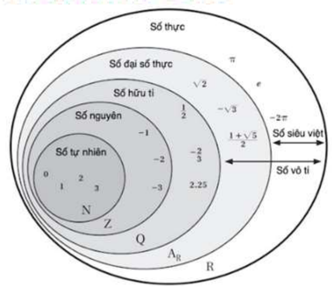
Hence

$$m + n = 2r + 2s = 2(r + s)^{(3)}$$

(3) If a quantity is an integer, then it is a real number.

r and *s* are particular integers.

• *r* and *s* are real numbers.



For all *a*, *b*, and *c*, if *a*, *b*, and *c* are real numbers, then $ab + ac = a(b + c)$.

2, *r*, and *s* are particular real numbers.

• $2r + 2s = 2(r + s)$.

Modus Ponens Qui tắc khẳng định	$p \rightarrow q$ p $\therefore q$
Modus Tollens Qui tắc phủ định	$p \rightarrow q$ $\sim q$ $\therefore \sim p$

Use of Universal Modus Ponens in a Proof

How universal modus ponens is used in the proof.

Now $(r + s)$ is an integer[4], and so $2(r + s)$ is even[5].

Thus $m + n$ is even.

(4) For all u and v, if u and v are integers, then $(u + v)$ is an integer.

r and s are two particular integers.

- $(r + s)$ is an integer.

(5) If a number equals twice some integer, then that number is even.

$2(r + s)$ equals twice the integer $(r + s)$.

- $2(r + s)$ is even.

Modus Ponens Qui tắc khẳng định	$p \rightarrow q$ p $\therefore q$
Modus Tollens Qui tắc phủ định	$p \rightarrow q$ $\sim q$ $\therefore \sim p$

Universal Modus Tollens

Another crucially important rule of inference is universal modus tollens. Its validity results from combining universal instantiation with modus tollens.

Universal modus tollens is the heart of proof of contradiction.

Universal Modus Tollens

Formal version	*Informal version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
~$Q(a)$ for a particular a.	a does not make $Q(x)$ true.
• ~$P(a)$.	• a does not makes $P(x)$ true.

Modus Ponens Qui tắc khẳng định	$p \to q$ p $\therefore q$
Modus Tollens Qui tắc phủ định	$p \to q$ $\sim q$ $\therefore \sim p$

Recognizing Universal Modus Tollens

Rewrite the following argument using quantifiers, variables, and predicate symbols. Write the major premise in conditional form. Is this argument valid? Why?

All human beings are mortal.
Zeus is not mortal.
• Zeus is not human.

Modus Ponens Qui tắc khẳng định	$p \to q$ p $\therefore q$
Modus Tollens Qui tắc phủ định	$p \to q$ $\sim q$ $\therefore \sim p$

Solution:

Premise: ∀*x*, if *x* is human then *x* is mortal.

Let *H*(*x*) be "*x* is human", let *M*(*x*) be "*x* is mortal", and let *Z* stand for Zeus.

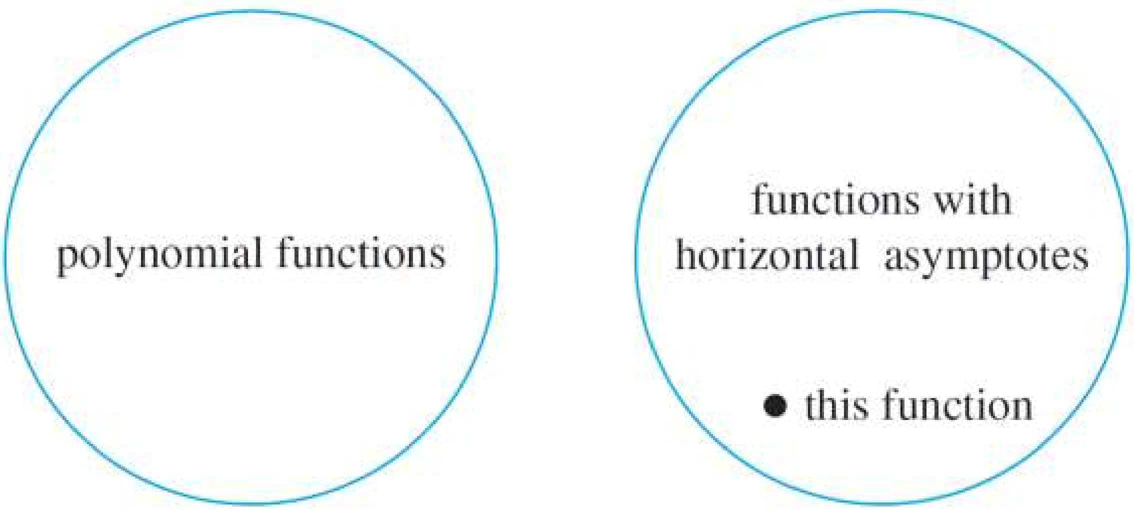
∀*x*, if *H*(*x*) then *M*(*x*) .
~*M*(*Z*).
• ~*H*(*Z*).

This argument has the form of universal modus tollens and is therefore valid.

An Argument with “No

Use diagrams to test the following argument for validity:

No polynomial functions have horizontal asymptotes.
This function has a horizontal asymptote.
• This function is not a polynomial function.



Hence argument is valid.

Figure 3.4.6

Modus Ponens Qui tắc khẳng định	$p \to q$ p $\therefore q$
Modus Tollens Qui tắc phủ định	$p \to q$ $\sim q$ $\therefore \sim p$

An Argument with “No

No polynomial functions have horizontal asymptotes.
This function has a horizontal asymptote.
- This function is not a polynomial function.

Alternatively, transform the first statement into:

$\forall x$, if x is a polynomial function, then x does not have a horizontal asymptote.

Modus Ponens Qui tắc khẳng định	$p \to q$ p $\therefore q$
Modus Tollens Qui tắc phủ định	$p \to q$ $\sim q$ $\therefore \sim p$

Then the argument has the form:

$\forall x$, if $P(x)$ then $Q(x)$.
$\sim Q(a)$, for a particular a.
- $\sim P(a)$.

This is valid by universal modus tollens.

Creating Additional Forms of Argument

Modus Ponens Qui tắc khẳng định	$p \to q$ p $\therefore q$
Modus Tollens Qui tắc phủ định	$p \to q$ $\sim q$ $\therefore \sim p$

We have seen:

Modus ponens + Universal instantiation ➔ Universal modus ponens

Modus tollens + Universal instantiation ➔ Universal modus tollens

In the same way, additional forms of arguments involving universally quantified statements can be obtained by combining universal instantiation with other of the valid argument forms discussed earlier.

Creating Additional Forms of Argument

Consider the following argument:

$$p \to q$$
$$q \to r$$
$$\bullet\ p \to r$$

This can be combined with universal instantiation to obtain a valid argument form.

Universal Transitivity

Formal version	*Informal version*
$\forall x, P(x) \to Q(x)$.	Any x that makes $P(x)$ true makes $Q(x)$ true.
$\forall x, Q(x) \to R(x)$.	Any x that makes $Q(x)$ true makes $R(x)$ true.
$\bullet\ \forall x, P(x) \to R(x)$.	• Any x that makes $P(x)$ true makes $R(x)$ true.

Evaluating an Argument for Tarski's World

Consider the Tarski's world:

Reorder and rewrite the premises to show that the conclusion follows as a valid consequence from the premises

1. All the triangles are blue.
2. If an object is to the right of all the squares, then it is above all the circles.
3. If an object is not to the right of all the squares, then it is not blue.

- All the triangles are above all the circles.

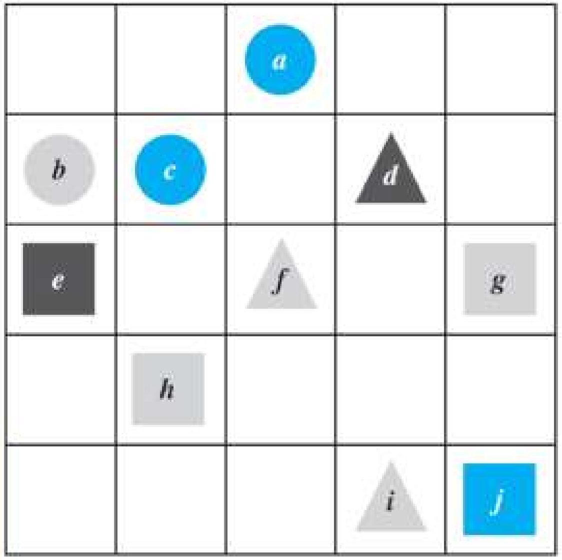


Figure 3.3.1

Evaluating an Argument for Tarski's World

1. All the triangles are blue.
2. If an object is to the right of all the squares, then it is above all the circles.
3. If an object is not to the right of all the squares, then it is not blue.

- All the triangles are above all the circles.

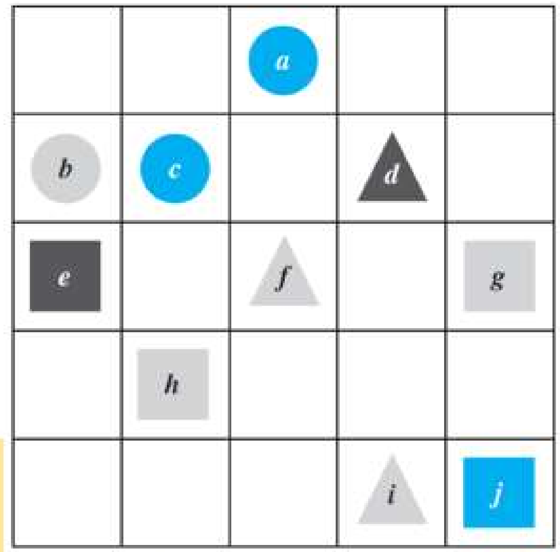


Figure 3.3.1

1. $\forall x$, if x is a triangle, then x is blue.
2. $\forall x$, if x is to the right of all the squares, then x is above all the circles.
3. $\forall x$, if x is not to the right of all the squares, then x is not blue.

- $\forall x$, if x is a triangle then x is above all the circles.

Should be same as conclusion of the last premise.

Should be same as hypothesis of the first premise.

93

Evaluating an Argument for Tarski's World

Consider the Tarski's world:

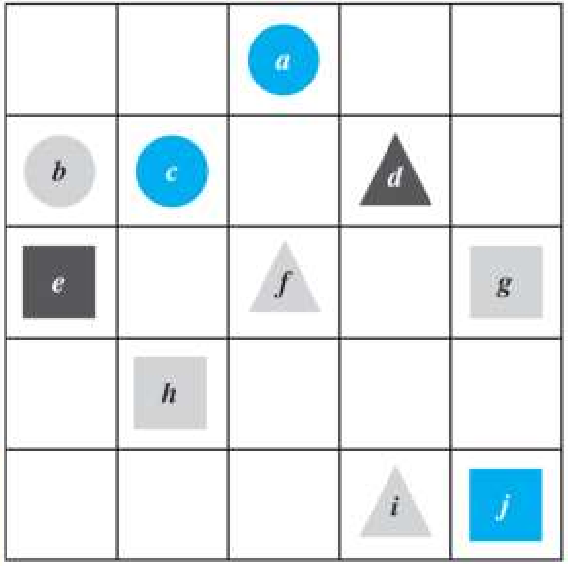


Figure 3.3.1

Reorder and rewrite the premises to show that the conclusion follows as a valid consequence from the premises

Step 2:

1. ∀*x*, if *x* is a triangle, then *x* is blue.
2. ∀*x*, if *x* is not to the right of all the squares, then *x* is not blue.
3. ∀*x*, if *x* is to the right of all the squares, then *x* is above all the circles.

- ∀*x*, if *x* is a triangle, then *x* is above all the circles.

Rewrite it in contrapositive form.

Evaluating an Argument for Tarski's World

Consider the Tarski's world:

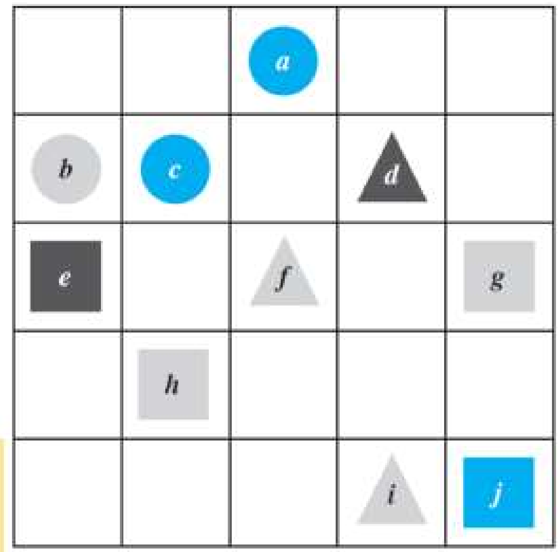


Figure 3.3.1

Reorder and rewrite the premises to show that the conclusion follows as a valid consequence from the premises

Step 3:

1. $\forall x$, if x is a triangle, then x is blue.
2. $\forall x$, if x is blue, then x is to the right of all the squares.
3. $\forall x$, if x is to the right of all the squares, then x is above all the circles.

- $\forall x$, if x is a triangle, then x is above all the circles.

Transitivity Qui tắc tam đoạn luận	$p \to q$ $q \to r$ $\therefore p \to r$

Proving Validity of Arguments with Quantified Statements

The intuitive definition of validity for arguments with quantified statements is the same as for arguments with compound statements.

An argument is valid if, and only if, the truth of its conclusion follows *necessarily* from the truth of its premises.

Definition 3.4.1 (Valid Argument Form)

To say that **an argument form is valid** means the following: No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true.

An **argument is called valid** if, and only if, its form is valid.

Using Diagrams to Test for Validity

Consider the statement: All integers are rational numbers.

$\forall$integers *n*, *n* is a rational number.

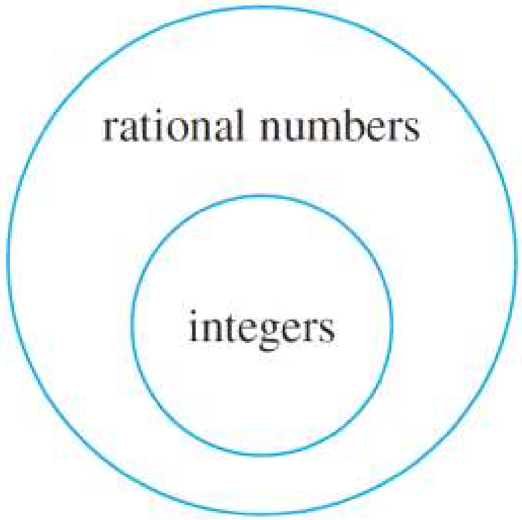
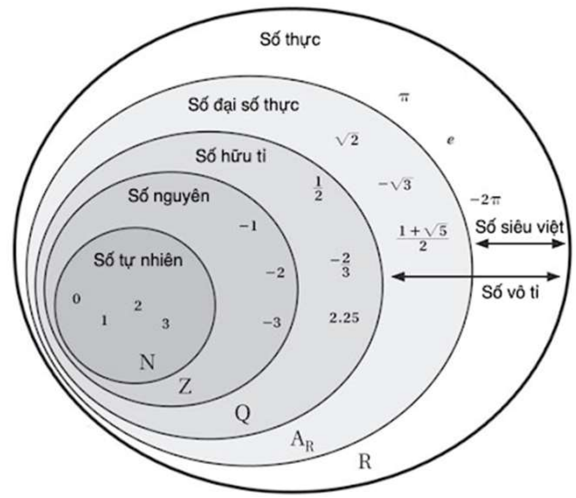


Figure 3.4.1



Using Diagrams to Show Invalidity

Use a diagram to show the invalidity of the following argument:

Modus Ponens Qui tắc khẳng định	$p \to q$ p $\therefore q$
Modus Tollens Qui tắc phủ định	$p \to q$ $\sim q$ $\therefore \sim p$

All human beings are mortal.
Felix is mortal.
- Felix is a human being.

Major premise

Minor premise

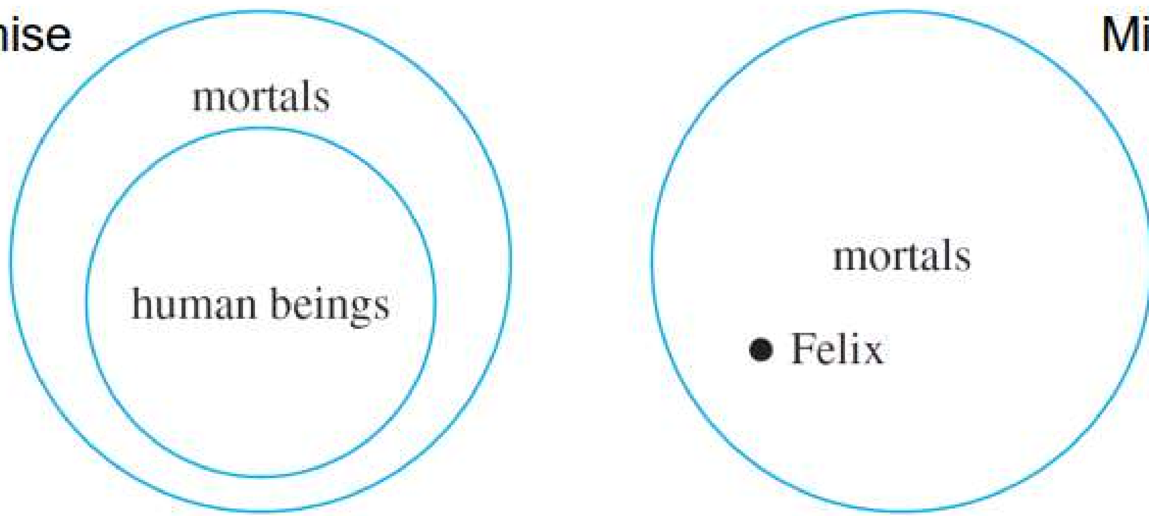


Figure 3.4.4

Using Diagrams to Show Invalidity

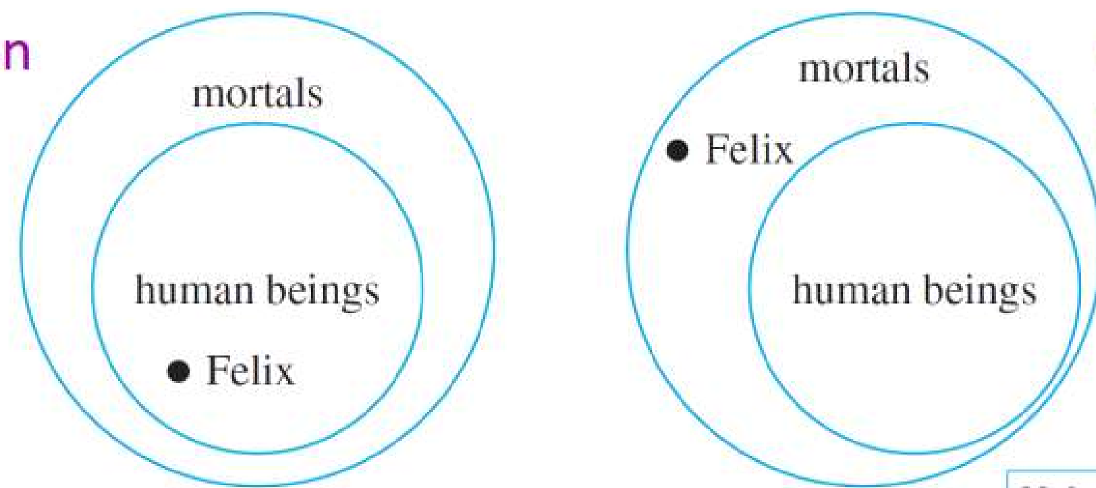
Use a diagram to show the invalidity of the following argument:

All human beings are mortal.
Felix is mortal.
• Felix is a human being.

Hence, argument is invalid.

Conclusion is true.

Conclusion is false.



(a) (b)

Figure 3.4.5

Modus Ponens Qui tắc khẳng định	$p \to q$ p $\therefore q$
Modus Tollens Qui tắc phủ định	$p \to q$ $\sim q$ $\therefore \sim p$

Using Diagrams to Test for Validity

The argument of previous example would be valid if the major premise were replaced by its converse. But since a universal conditional statement is not logically equivalent to its converse, such a replacement cannot, in general, be made.

Modus Ponens Qui tắc khẳng định	$p \to q$ p $\therefore q$
Modus Tollens Qui tắc phủ định	$p \to q$ $\sim q$ $\therefore \sim p$

We say that this argument exhibit the converse error.

Converse Error (Quantified Form)

Formal version	*Informal version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$Q(a)$ for a particular a.	a makes $Q(x)$ true.
• $P(a)$.	• a makes $P(x)$ true.

If not q then not p		if p the q		if q the p		If not p then not q
$\sim q \to \sim p$	$\equiv$	$p \to q$	$\not\equiv$	$q \to p$	$\equiv$	$\sim p \to \sim q$
contrapositive (Phản đảo)		*conditional statement*		*converse* (Đảo)		*inverse* (Nghịch Đảo)

Using Diagrams to Test for Validity

The following form of argument would be valid if a conditional statement were logically equivalent to its inverse. But it is not, and the argument form is invalid.

We say that this argument exhibit the inverse error.

Modus Ponens Qui tắc khẳng định	$p \to q$ p $\therefore q$
Modus Tollens Qui tắc phủ định	$p \to q$ $\sim q$ $\therefore \sim p$

Inverse Error (Quantified Form)

Formal version	*Informal version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim P(a)$ for a particular a.	a does not make $P(x)$ true.
• $\sim Q(a)$.	• a does not make $Q(x)$ true.

If not q then not p		if p the q		if q the p		If not p then not q
$\sim q \to \sim p$	$\equiv$	$p \to q$	$\not\equiv$	$q \to p$	$\equiv$	$\sim p \to \sim q$
contrapositive (Phản đảo)		*conditional statement*		*converse* (Đảo)		*inverse* (Nghịch Đảo)

Remark on the Converse and Inverse Errors

Only for reading.

A variation of the converse error is a very useful reasoning tool, provided that it is used with caution.

It is the type of reasoning that is used by doctors to make medical diagnoses and by auto mechanics to repair cars.

It is the type of reasoning used to generate explanations for phenomena. It goes like this: If a statement of the form

For all x, if $P(x)$ then $Q(x)$

is true, and if

$Q(a)$ is true, for a particular a,

then check out the statement $P(a)$; it just might be true.

Remark on the Converse and Inverse Errors

Only for reading.

For instance, suppose a doctor knows that

> For all *x*, if *x* has pneumonia, then *x* has a fever and chills, coughs deeply, and feels exceptionally tired and miserable.

Modus Ponens Qui tắc khẳng định	$p \to q$ p $\therefore q$
Modus Tollens Qui tắc phủ định	$p \to q$ $\sim q$ $\therefore \sim p$

And suppose the doctor also knows that

> John has a fever and chills, coughs deeply, and feels exceptionally tired and miserable.

On the basis of these data, the doctor concludes that a diagnosis of pneumonia is a strong possibility, though not a certainty.

This form of reasoning has been named abduction by researchers working in artificial intelligence.

End of File