Control Engineering

L1, Introduction, Linearalization ✓

L2, Differential Equation, Linear ODEs Solution * ✓

L3, ~~Zeros of the~~ characteristic polynomial ✓

L4, Unit step, impulse, response, Laplace transformation ✓

L5, Frequency response, Bode diagram (Amplitude & Phase Plot) *

L6, Control loop elements ✓

L7, Controller Setting (PID) *

L8, Stability of Control loops * ✓ Hurwitz, Routh

L9, Controller Setting & Stability of control loops,

Nyquist criterion, ✓ Gain, phase margins *

L10, Controller design in Bode diagram *

Ex1, Linearization ✓

Ex2, Differential Equations
Solve ODEs ✓

Ex4, Laplace ✓

Ex5, Bode Diagram ✓

✓ Ex6, Functional Diagram

* Ex7, Control loop + Bode Diagram

✓ Ex8, Stability. Routh, Hurwitz

✓ Ex9, ——. Nyquist

Ex10, —— Nyquist, Bode

Ex11, —— Controller Design

Ng Huu Duc

Control Engineering

Lecture 2

Differential Equation

- ODE - Ordinary differential equation $\boxed{1 \text{ variable}}$

$$\frac{d^2y}{dt^2} + a_1 \cdot \frac{dy}{dt} + a_0 y = u(t)$$

- PDE - Partial differential equation >1 variable

$$a\frac{d^2y}{dt^2} + b\frac{d^2y}{dt\,dx} + c\frac{d^2y}{dx^2} = u(x,t)$$

Non linear ODEs:

$$a_2(t)\,\ddot{y}^2 + a_1(t)\,\dot{y}\,y + \frac{a_0(t)}{y} = f(t)$$

Linear ODEs:

$$a_n(t)\,y^{(n)} + \dots\, a_1(t)\dot{y} + a_0(t)\,y = f(t)$$

Linear Time Invariant ODEs (LTI Systems)

$$a_n\,y^{(n)} + \dots + a_1\dot{y} + a_0 y = f(t)$$

Solution Linear ODEs $\quad a_n(t)\,y^{(n)} + \dots\, a_0(t)\,y = f(t)$

→ if $f(t) = 0 \Rightarrow$ ODE homogeneous $\Rightarrow y_h$ = homogeneous solution

→ if $f(t) \neq 0 \Rightarrow$ — inhomogeneous $\Rightarrow$ + y_p = + particular solution

y = solution

n-th order LTI system $y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0 y = 0$

$\lambda_1 \dots \lambda_n$ are solutions characteristic polynomial:

$$\lambda^n + a_{n-1}\lambda^{(n-1)} + \dots + a_1\lambda + a_0 = 0$$

$$\Rightarrow y_h = c_1 e^{\lambda_1 t} + \dots + c_n e^{\lambda_n t}$$

$$z = e^{\alpha + i\beta} = e^{\alpha}(\cos\beta + i\sin\beta)$$
$$= e^{\alpha}\cos\beta + i\, e^{\alpha}\sin\beta$$

Define $c_1, \dots c_2$ from initial conditions

* if $\lambda_1 = \lambda_2 \Rightarrow$ solution $\begin{cases} y_1 = c_1 e^{\lambda_1 t} \\ y_2 = c_2 t e^{\lambda_1 t} \end{cases}$

$\lambda_1 = \dots = \lambda_m \Rightarrow$ solution $\begin{cases} y_1 = c_1 e^{\lambda_1 t} \\ \vdots \\ y_m = c_m t^{(m-1)} e^{\lambda_1 t} \end{cases}$

$\lambda_1 = \alpha + i\beta,\ \lambda_2 = \bar{\lambda}_1,\ \beta \neq 0 \Rightarrow \begin{cases} y_1 = c_1 e^{\alpha t}\cos(\beta t) \\ y_2 = c_2 e^{\alpha t}\sin(\beta t) \end{cases}$

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for LTI: $f(t) = p_m(t)\, e^{kt},\ k \in \mathbb{C}$

with $p_m(t) = p_0 + p_1 t + \dots + p_m t^m$

reuse the $C(\lambda)$ $\Rightarrow C(\lambda) = \lambda^n + \dots$

$$y_p(t) = \begin{cases} q_m(t)\, e^{kt} & \text{if } C(k) \neq 0 \\ t\, q_m(t)\, e^{kt} & C(k) = 0 \quad C'(k) \neq 0 \\ t^2 q_m(t)\, e^{kt} & C(k) = C'(k) = 0 \quad C''(k) \neq 0 \end{cases}$$

$q_m(t) = q_0 + \dots q_m t^m$

Particular Solution for ODEs: y_p

$f(t)$	$\rightarrow$ y_p
$\alpha e^{\beta t}$	$A e^{\beta t}$
$a\cos(\beta t) + b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
n^{th} order polynomial	$A_n t^n + A_{n-1} t^{n-1} + \dots + A_1 t + A_0$

Control Engineering

Linearization: $\frac{\partial F}{\partial x_i} x_i + \dots = \dots$

- $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{j\omega t} dt$

Partial Fraction Decomposition

- $1(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \Rightarrow \dot{1}(t) = \delta(t) \Rightarrow h(t) = \int_{-\infty}^{t} g(\tau) d\tau \Rightarrow g(t) = \frac{d\, h(t)}{dt}$

$$y(t) = \sum_{i=1}^{n} u(t_i) g(t - t_i) \Delta t$$
$$= \int_{0}^{t} u(\tau) g(t-\tau) d\tau$$

$$y(t) = \int_{-\infty}^{+\infty} u(\tau) g(t-\tau) d\tau = \int_{-\infty}^{+\infty} g(\tau) u(t-\tau) d\tau$$

$$\Rightarrow y(t) = g(t) * u(t) = u(t) * g(t)$$

convolution operator

- $F(s) = \int_{-0}^{\infty} f(t) e^{-st} dt = \mathcal{L}\{f(t)\}, \quad s = \sigma + j\omega, \quad \sigma > \alpha > 0$ $\qquad F(s) \circ\!\!-\!\!\bigcirc f(t)$

$$f(t) = \left\{ \begin{array}{ll} \frac{1}{2\pi j} \int_{\alpha - j\infty}^{\alpha + j\infty} F(s) e^{st} ds & t \geq 0 \\ 0 & t < 0 \end{array} \right\} = \mathcal{L}^{-1}\{F(s)\}$$

$$\mathcal{L}\{\dot{f}(t)\} = -f(-0) + s \cdot \mathcal{L}\{f(t)\}$$

- $x = x_0 \cdot e^{j\varphi} \in \mathbb{C}$ $\qquad |x| = \sqrt{Re^2(x) + Im^2(x)} = X_0$

$= x_0 (\cos\varphi + j \sin\varphi)$ $\qquad \varphi = \arctan \frac{Im(x)}{Re(x)}$

$= x_0 \cos\varphi + j\, x_0 \sin\varphi$

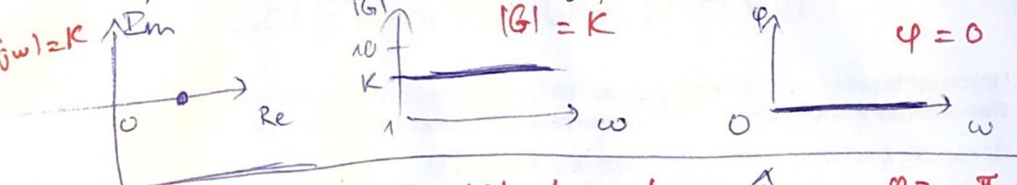
$= Re(x) + j\, Im(x)$ $\qquad \Rightarrow e^{\alpha + j\beta} = e^{\alpha} \cdot \cos\beta + j e^{\alpha} \cdot \sin\beta$

- $G(j\omega) = \frac{Z(j\omega)}{N(j\omega)} \Rightarrow \begin{cases} |G| = \frac{|Z|}{|N|} & \text{amplitude} \\ \varphi_G = \varphi_Z - \varphi_N & \text{phase} \end{cases}$

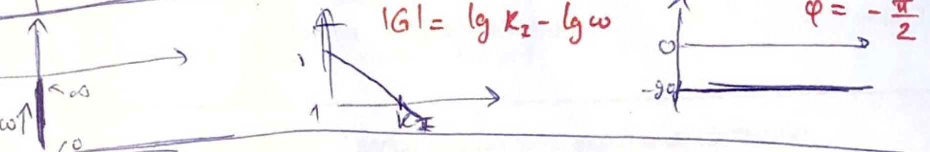
$$G = G_1 G_2 \Rightarrow \lg|G| = \lg|G_1| + \lg|G_2| ; \quad \varphi = \varphi_1 + \varphi_2$$

Frequency response and their Nyquist plots & Bode diagram

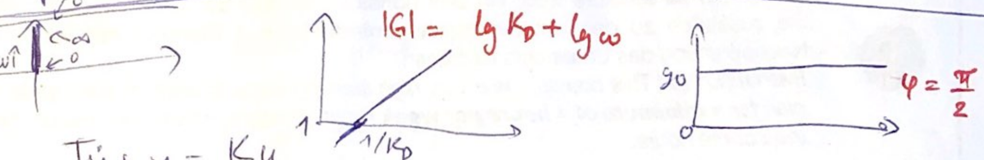
P: $G(j\omega) = K$, $|G| = K$, $\varphi = 0$



I: $G = \frac{K_I}{j\omega}$, $|G| = \lg K_I - \lg \omega$, $\varphi = -\frac{\pi}{2}$

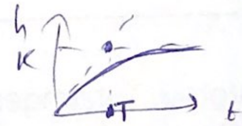


D: $G = j\omega K_D$, $|G| = \lg K_D + \lg \omega$, $\varphi = \frac{\pi}{2}$

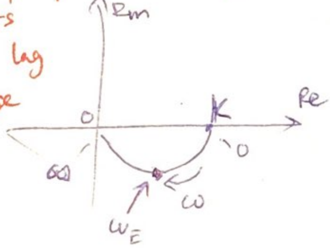


First order lag $\quad T\dot{y} + y = Ku$

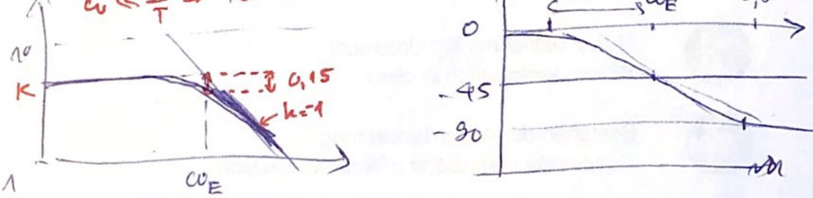
PT_1 element $\Rightarrow G(j\omega) = \frac{K}{1 + j\omega T} \Rightarrow \omega_E = \frac{1}{T} \Rightarrow$



output is proportional to input but reacts with a lag of response



$\omega \ll \frac{1}{T} \Rightarrow |G| = K$

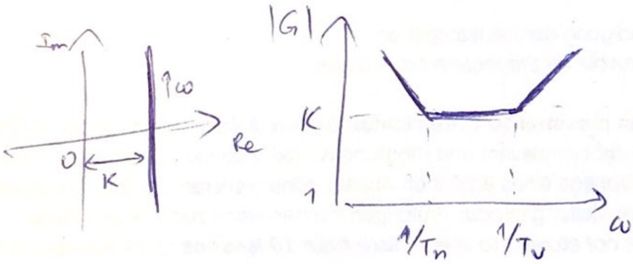


$\omega \gg \frac{1}{T} \Rightarrow |G| = \lg K - \lg \omega T$

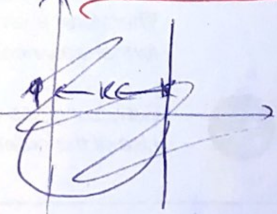
PID element

$G = K_R \left(1 + \frac{1}{T_n s} + T_v s\right)$

$T_v \ll T_n$



Lecture 6



PT_2 2nd order lag $\quad G = \frac{K\omega_0^2}{s^2 + 2D\omega_0 s + \omega_0^2}$

poles: $s_{1,2} = -\omega_0 \left(D \pm \sqrt{D^2 - 1}\right) \Rightarrow$

$\Rightarrow$ Depends on damping ratio

$D > 1 \Rightarrow s_{1,2} \in \mathbb{R}$

$D = 1 \Rightarrow s_1 = s_2 \in \mathbb{R} \quad (s_1 = s_2 = -\omega_0)$

$D < 1$

PT_n element can be splitted into PT_1 & PT_2 elements

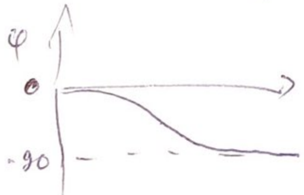
Minimal phase system $\Rightarrow$ no poles or zeros with positive real part

$\Rightarrow$ have the minimal possible phase shift

$\Rightarrow$ the phase responce (φ graph) can be derived from amplitude response ($|G|$ graph)

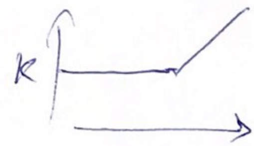
$$G(s) = K(1+Ts)$$

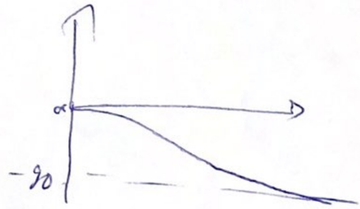
$$G(s) = \frac{K}{1+Ts}$$



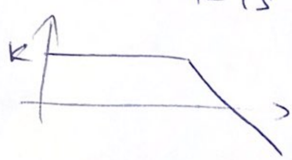
Non minimal phase system

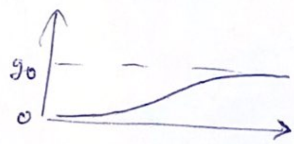
$$G(s) = K(1-Ts)$$

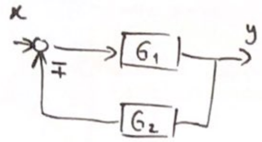




$$G(s) = \frac{K}{1-Ts}$$







$$G = \frac{G_1}{1 \pm G_1 G_2}$$

Lecture 8: Stability

- Overshoot, transient tolerance band, response time, settling time, max deviation, steady state deviation

- Hurwitz / Routh : $a_i > 0$ $i = 0, \dots n$ (1st condition)

Hurwitz:
$$\begin{matrix} a_3 & a_1 & 0 \\ a_4 & a_2 & a_0 \\ a_5 & a_3 & a_1 \end{matrix} = H$$

$\boxed{H_i > 0}$

Hurwitz determinants > 0

Routh
$$\begin{array}{c|cc} a_n & a_{n-2} & a_{n-4} \; \dots \\ a_{n-1} & a_{n-3} & a_{n-5} \; \dots \\ \hline \multicolumn{3}{l}{a_{n-2} - a_{n-3}\frac{a_n}{a_{n-1}} \quad \dots} \end{array}$$

$\boxed{R_n > 0}$ (1st column)

Routh test funcs > 0

number of sign changes
⇒ no of roots of characteristics polynomial with positive real part.

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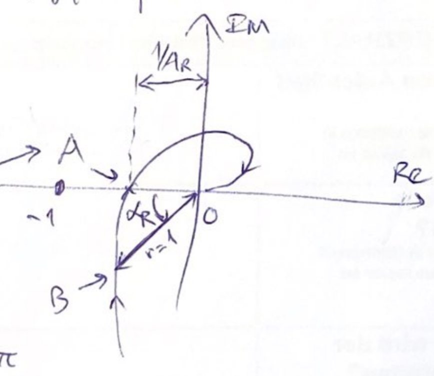
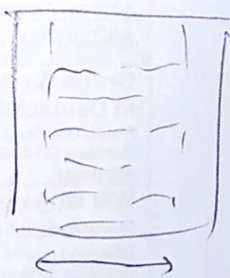
gain margin A_R

phase margin ~~ω_π~~ α_R

Nyquist plot





like SUM margin: is where u are safe, secured

+) Gain margin: point A

$$A_R = \frac{1}{G_o(j\omega_\pi)} \quad \text{with } \omega_\pi = -(2n+1)\pi$$

when the graph intersect "neg Real axis"

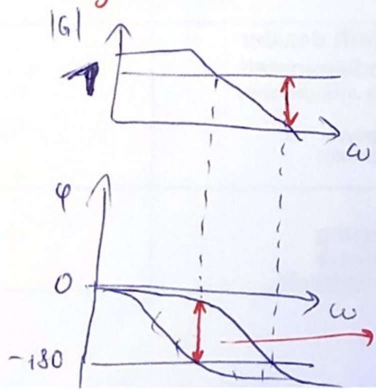
$A_R > 1 \Rightarrow$ stability

+) Phase margin: point B

$\alpha_R = \varphi_o(\omega_d) - \varphi_o(\omega_\pi)$

ω_d : $|G_o(j\omega_d)| = 1$

$\alpha_R > 0 \Rightarrow$ stability



gain margin: { distance to 0 at $\varphi = -\pi$

phase margin { distance to $-\pi$ at ~~$|G| = 0$~~ $|G| = 1 \Rightarrow \log|G| = 0$

Lecture 9, Stability: Nyquist criterion

Pre conditions

- single control loop
- linear system (including delay)
- graphical representation of $G_0(j\omega)$ (Bode, Nyquist)

$\Longrightarrow$ Properties of open loop / Nyquist plot $\Rightarrow$ stability of closed loop

$$n = m + p \qquad G_z(s) = \frac{G_s(s)}{1 + G_0(s)}$$

m: no. of revolutions of Nyquist plot of $G_0(j\omega)$ around -1 clock wise

p: number of poles of $G_0(s)$ in right half plane

n: number of poles of $G_z(s)$ in righ half plane

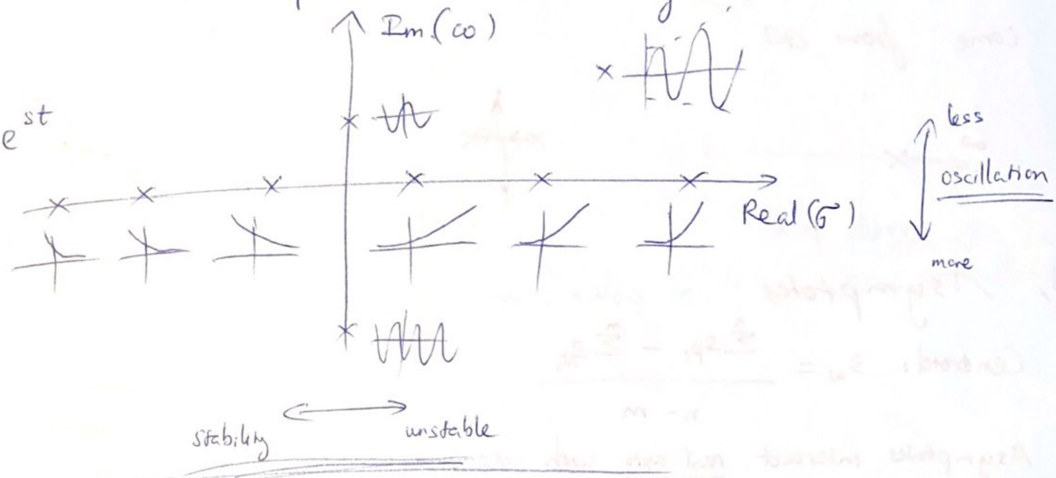
Note: Draw finished Nyquist plot: from $\omega: -\infty \to +\infty$, closed the loop in away, the arrow turn right $\uparrow \rightarrow$ go right

- Simple Nyquist version?

Root locus

<u>why?</u> Because poles positions ⇒ stability

S·Plane
wave form $:= e^{st}$



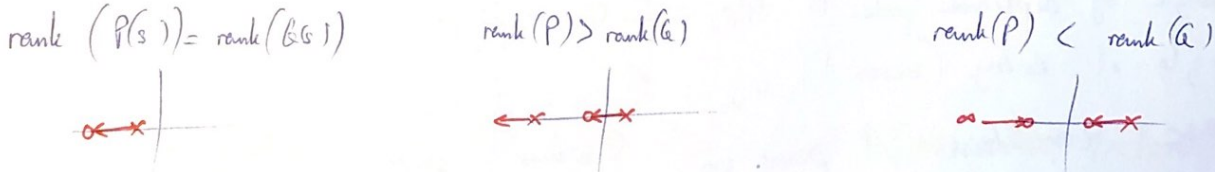
Rules

$$1 + K \cdot G(s) = 0$$

$$1 + K \cdot \frac{Q(s)}{P(s)} = 0$$

1, There are n lines (loci) $n = \max$ (degree of P & Q) rank

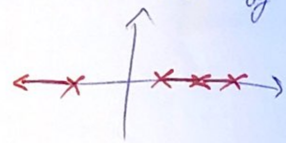
2, As $k: 0 \to \infty$, the roots move from poles to zeros of $G(s)$



3, when roots are complex they occur in conjugate pairs

4, Never loci crosses over its path

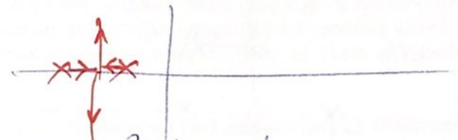
5, The portion of real axis to the left of an odd number of open loop poles & zeros are path of the loci ($K>0$, for $K<0 \Rightarrow$ even number of poles, zeros)



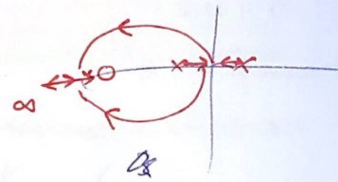
6, Loci leave / enter real axis at $90°$

7, Not enough poles / zeros to make a pair $\Rightarrow$ extra lines go to or come from ∞

1 extra pole



2 extra poles

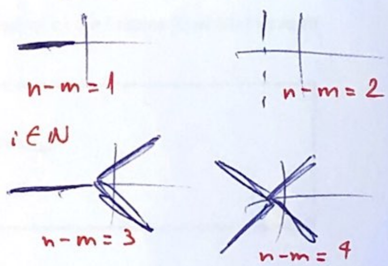


&

8, Asymptotes: n poles, m zeroz $\Rightarrow$ $n-m$ asymptotes / á zym p thot/

Centroid: $s_w = \dfrac{\sum^{n} s_{P_i} - \sum^{m} s_{N_i}}{n-m}$

Asymptotes intersect real axis with α_i: $\alpha_i = \begin{cases} \frac{2i-1}{n-m}\pi, & K>0 \\ \frac{2i}{n-m}\pi, & K<0 \end{cases}$ $i \in N$, $i = 1, \dots, (n-m)$



9, If there are at least 2 lines $\rightarrow \infty$, the sum of all roots is CONSTANT

- Break away point: ...
- Angle of departure (poles): $\varphi_{P_1} = \sum_{i=1}^{m} \varphi_{N_i} - \sum_{i=2}^{n} \varphi_{P_i} - \begin{Bmatrix} \pi + q2\pi, & K>0 \\ 0 + q2\pi, & K<0 \end{Bmatrix}$ $q \in \mathbb{Z}$
- Angle of entry (zeros): $\varphi_{N_1} = \sum_{i=1}^{n} \varphi_{P_i} - \sum_{i=1}^{m} \varphi_{N_i} +$
- Phase condition: A point on S plane, satisfy $\rightarrow$ lies on root locus

$$\sum_{i=1}^{m} \varphi_{N_i} - \sum_{i=1}^{n} \varphi_{P_i} = \varphi_0(s) = \begin{cases} \pi + q2\pi & K>0 \\ 0 + q2\pi & K<0 \end{cases}$$

- Magnitude condition: $|K| \cdot |G_0(s)| = 1$

Point satisfies both conditions $\Rightarrow$ pole location of $G_w(s)$, $G_z(s)$ with K

- Gain of a point on root locus $|K| = \dfrac{\prod_{i=1}^{n} |s - s_{P_i}|}{\prod_{i=1}^{m} |s - s_{N_i}|}$

State Space

$$G(s) = \frac{b_0 + b_1 s + \dots + b_{n-1} s^{n-1}}{a_0 + a_1 s + \dots + a_n s^n} \qquad (a_n = 1)$$

$$\Rightarrow \quad \dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & & -a_{n-1} \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

Canonical form

$$y = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix} x$$

Set $G(s) = \frac{Y}{U} = \frac{Y}{X} \cdot \frac{X}{U}$

Numerator   Denominator

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

A: system matrix   only A is a part of closed loop (also called Frobenius matrix)

B: input ————   ⇒ Stability, damping... depend only on A

C: output ————

D: in-output ————

Stability:

Characteristic polynomial: $\det(sI - A)$

eigenvalues with negative real parts

If $u = -Kx$

$$\Rightarrow \dot{x} = (A - BK)x = A_k x \Longrightarrow \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0-k_1 & -a_1-k_2 & -a_2-k_3 & \cdots & -a_{n-1}-k_n \end{bmatrix}$$

Change K $\Rightarrow$ change A_k $\Rightarrow$ change poles positions (even number of poles)

CE exam prep

1) Linearization

$x = X - X_0$

$Y = f(X) \Rightarrow y = f'(X_0) x$

Linearize each element

$Y = f(X, u) \Rightarrow y = \frac{\partial f}{\partial X}\Big|_{X_0, u_0} \cdot x + \frac{\partial f}{\partial u}\Big|_{X_0, u_0} \cdot u$

$\dot{Y} + f(Y) = g(X) \Rightarrow \dot{y} + f'(Y_0) y = g'(X_0) x$

2) Solve ODEs

$a_n(t) y^{(n)} + \dots + a_0(t) y = f(t)$

→ Homogeneous solution:

Characteristic polynomial:

$C(\lambda) = \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0$

$\Rightarrow$ ηo: $\lambda_1 \dots \lambda_n$

$\Rightarrow y_h = c_1 e^{\lambda_1 t} + \dots + c_n e^{\lambda_n t}$

if: $\lambda_1 = \lambda_2 = \dots = \lambda_m \Rightarrow \begin{cases} y_1 = c_1 e^{\lambda_1 t} \\ y_2 = c_2 t e^{\lambda_2 t} \end{cases}$

if: $\lambda_1 = \bar{\lambda}_2$, $\lambda_1 = \alpha + i\beta \Rightarrow \begin{cases} y_1 = c_1 e^{\alpha t} \cos\beta t \\ y_2 \; c_2 e^{\alpha t} \sin\beta t \end{cases}$

→ Particular solution: $y_P = p(t) y_H(t)$; $p(t) = \int \frac{1}{y_H(t)} f(t) \, dt + c_P$

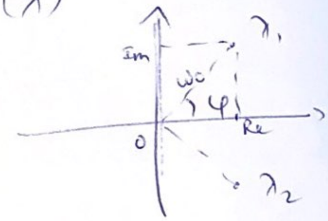
→ $y_G = y_H + y_P$

$\boxed{3, \text{Laplace}}$

Damping Ratio from 2nd Characteristic polynomial

$$C(\lambda) \Rightarrow \lambda_{1,2} \Rightarrow \omega_0 = \sqrt{Re(\lambda)^2 + Im(\lambda)^2}$$

$$D = \frac{|Re(\lambda)|}{\omega_0} = \cos(\varphi)$$



4, - Root Locus

+, Phase condition: A point on S-plane satisfy

$$\sum_{i=1}^{m} \varphi_{N_i} - \sum_{i=1}^{n} \varphi_{P_i} = \varphi_0(s) = \begin{cases} \pi + q2\pi & K>0 \\ 0 + q2\pi & K<0 \end{cases}$$

=> lies on root locus

+, Magnitude condition $|K| \cdot |G_0'(s)| = 1$ (?)

Satisfies both => pole location

5, Routh, Hurwitz?

Only for closed loop func (?)

6, State Space: Canonical form

7, Linear Sampling: $y = \int u\,dt$

$$\Rightarrow y_k - y_{k-1} = \int_{t_{k-1}}^{t_k} u\,dt$$

8, Bode diagram

+) Correction table: for PT_1 , PT_2

from $\omega_E = \omega_0$ (intersection point of 2 asymptotes)

PT_2: $D < 1$

$$(\lg|G| - \lg|\text{Asymptote}|) * (1 \text{ unit length})$$

$$\frac{\log(\text{Value}) - \log(\text{left})}{\log(\text{Right}) - \log(\text{Left})} = \frac{A}{S}$$

S → usually unit length ($\lg 1 - \lg 0{,}1 = 1$)

+) $\varphi = \varphi_N - \varphi_D$

$$\varphi_D = \arctan \frac{Im}{Re}$$

Note: it's $\frac{1}{T}$ not T

9, Functional Diagram: use the unit step response graph to represent the block

sign "−" on the right

10, Gain margin A_R from ω_π $-(2n+1)\pi = \varphi_\pi$

Phase margin α_R from $|G_0(j\omega_d)| = 1$

11) Time discrete difference equation:

Example: $u = -2y - 5\dot{y}$

$$\Rightarrow u_k = -2y_k - \frac{5}{T}(y_k - y_{k-1})$$

$$\dot{y} \Rightarrow \frac{1}{T}(y_k - y_{k-1})$$

12) Nyquist Criterion:

close the loop going right

Ng Huu Duc