


# Lectures

*Ng Hui Duc*

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- Non-nodal approximation

- Collocation by points

- Collocation by subdomains

- Galerkin's Method

- Least square method

- Ritz method

25, Numerical integration

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Numerical Analysis Software

# Exercise

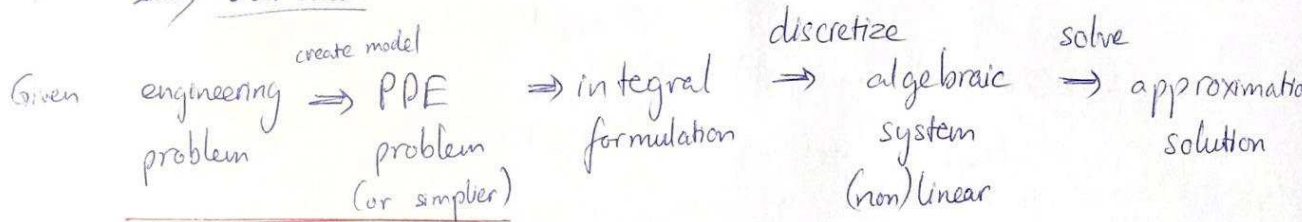
(MATLAB focused)

- 1, Intro to MATLAB
- 2, GUI - Newton-Raphsen Method
- 3, Functions
- 4, Solving PDE, ODE
- 5, PDE Tool box
- 6, More PDE : elliptic, parabolic, hyperbolic
- 7, 2D Flow problem, divergence
- 8, PDE - plate under uniform pressure load
- 9, Lagrange multiplier method
- 10, Nodal approximation
- 11, Collocation method : collocation by points
- 12, Galerkin method
- Least square method
- Ritz method
- 13, Gaussian Quadrature

# Chapter 2. Variational Formulation

L8;

## 2.1, Overview:



⊗ this chapter


## 2.2) Classification of engineering problems:

a) Discrete vs continuous system: no. dof

- Discrete  $\Rightarrow$  leads to finite set of algebraic equations finite dof
- In some cases, the system is linear.

Ex:

- Continuous system  $\Rightarrow$  infinite no of DoF

Ex: bending of a beam 

b) Mechanical meaning:  $\left\{ \begin{array}{l} \text{equilibrium problems (statics)} \\ \text{eigenvalue problems} \\ \text{propagation problems (time-dependent)} \end{array} \right.$

c) Mathematical point of view:  $\left\{ \begin{array}{l} \text{elliptic PDE} \\ \text{parabolic PDE} \\ \text{hyperbolic PDE} \end{array} \right.$



Notation:

$$L(u) + f_V = 0 \quad \text{inside a domain } V$$

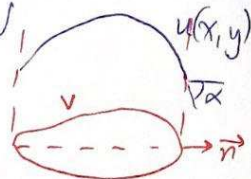
$$C(u) = f_S \quad \text{on the boundary of } V \text{ (} S = \partial V \text{)}$$

## 49, 2.3) Discrete vs. continuous systems

- When the model involves differential equations  $\Rightarrow$  have to satisfy certain boundary / initial conditions

1st  $\rightarrow$  Dirichlet boundary condition: specify the values of  $u(x, y)$  on  $S$   
 unknown func  $u(x, y)$  is a scalar field on  $V$ ,  
 $S = \partial V$  (boundary of  $V$ )

$$u(x, y) = u_s(x, y) \text{ for } (x, y) \in S$$



2nd  $\rightarrow$  Neumann boundary condition: specify the slope  $u(x_n, y_n)$  when  $\rightarrow$  boundary

$$\tan \alpha = \frac{\partial u}{\partial n}(x, y) = \nabla u(x, y) \cdot n(x, y) = f_s(x, y) \text{ for } (x, y) \in S$$

3rd  $\rightarrow$  Robin / Newton boundary condition: weighted combination of the first two

$$\alpha u(x, y) + \frac{\partial u}{\partial n}(x, y) = f_s(x, y) \text{ for } (x, y) \in S$$

- Defined on whole boundary  $S$  or part of it

Boundary  $\Rightarrow$  problem is  $\left| \begin{array}{l} \text{well-posed or ill-posed} \\ \text{solution exists or not} \end{array} \right.$

$\rightarrow$  If specify both Dirichlet and Neumann / Robin conditions on the same subset  $S_c \subset S$  and none on  $S \setminus S_c$   
 $\Rightarrow$  Cauchy boundary condition

$\rightarrow$  "hidden" b.c. which satisfied by default  $\Rightarrow$  natural bound. cond.

Ex:  $u'(L) = 0$   
 $u'''(L) = 0$

## 40, 2.4) Mechanical classification:

$\rightarrow$  Equilibrium problems: involve ODE or PDE which do not depend on time,  $u = u(x, y, z) \dots$

## +) Eigenvalue Problems

$$Au = \lambda u$$

## +) Propagation Problems

depend on time variable  $t$

→ modeled as Initial Value problem (IVP)

or ——— Boundary Val. ——— (IBVP)

- Lagrange description: approach based on the coordinates in an initial configuration

- Euler ——— approach ——— the actual ———

## 2.5/ Mathematical classification:

Consider 2nd order PDE:

$$a(x,y) \frac{\partial^2 u}{\partial x^2} + b(x,y) \frac{\partial^2 u}{\partial x \partial y} + c(x,y) \frac{\partial^2 u}{\partial y^2} = f_v(x,y)$$

⇒ matrix of coefficients  $\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$  → eigenvalues  $d$  &  $e$

⇒  $\left\{ \begin{array}{ll} d, e \neq 0, \text{ same sign} & \Rightarrow \text{elliptic} \\ \text{—————, opposite sign} & \Rightarrow \text{hyperbolic} \\ d=0 \text{ or } e=0 & \Rightarrow \text{parabolic} \end{array} \right\} \begin{array}{l} \text{depend on } x, y \\ \text{(local, not global)} \end{array}$

## Chapter 3

## Mathematical Tools

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## 3.1, Functions, Functionals and Operators:

+) A function: mapping from subset of  $\mathbb{R}^n \rightarrow$  subset of  $\mathbb{R}^p$

$$f: V \subseteq \mathbb{R}^n \rightarrow W \subseteq \mathbb{R}^p$$

+) An operator: mapping from (a subset of) a func space  $\Rightarrow$  (a subset of) function space

$$L: f(\cdot) \in V \rightarrow (L(f))(\cdot) \in W$$

ex: derivative operator  
Laplace, curl, div ...

+) Some func. space: given a domain  $\Omega \subseteq \mathbb{R}^n$

-  $C^0(\Omega)$ : space of cont. real. func. on  $\Omega$

$C^n(\Omega)$ : \_\_\_\_\_, which cont. differentiable to order  $n$  on  $\Omega$

$C^\infty(\Omega)$ : \_\_\_\_\_, order  $\infty$

$L^1(\Omega)$ : space of integrable func. on  $\Omega$

$L^2(\Omega)$ : \_\_\_\_\_ square \_\_\_\_\_

- for complex func. space:  $C^n(\Omega; \mathbb{C})$

+) Functional: mapping (a subset of) a func. space  $\Rightarrow \mathbb{R}$  (or  $\mathbb{C}$ )

$$F: f(\cdot) \in V \rightarrow F(f) \in W \subseteq \mathbb{R} \text{ (or } W \subseteq \mathbb{C})$$

ex: Lebesgue integral  
 $\int_{-1}^1 f(x) dx$

## +) Properties of operators &amp; functionals

a) linear:  $L(\alpha u + \beta v) = \alpha \cdot L(u) + \beta \cdot L(v) \quad \forall u, v \in \text{domain of } L$   
 $\alpha, \beta \in \mathbb{R}$

b) Homogeneous operators / functionals:

$$L(\alpha u) = \alpha^k L(u) \text{ and } L(0) = 0 \quad \forall u \in \text{domain of } L$$

$L$  is a homogeneous operator of degree  $k$

$\alpha \in \mathbb{R}$

c) Symmetric operators: (self-adjoint)

- Inner product:  $\langle u, v \rangle_w = \int_w w(x) \cdot u(x) \cdot v(x) dx$   
with  $w(x) > 0$  as weight function

= An operator  $L$  is symmetric with respect to a scalar product  $\langle, \rangle$  if  
 $\langle L(u), v \rangle = \langle u, L(v) \rangle$

Ex. Laplace operator  $\Delta$ :  $\langle \Delta u, v \rangle = \langle u, \Delta v \rangle$

d) Positive and positive definite operators:

$\langle L(u), u \rangle \geq 0 \quad \forall u(x) \begin{cases} \in \text{domain of } L \\ \text{satisfy boundary conditions} \end{cases}$

Ex. Laplace operator  $L = -\Delta$  is positive definite  
with geometrical interpretation: curvature

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

⊕ Some Fundamental equations involving Laplace operator

a) Equilibrium equations:  $\Delta u = 0$ : Laplace's equation  
 $\Delta u = f_v$ : Poisson's equation

b) Eigenvalue problems

$\Delta u + \alpha^2 u + f_v = 0$  is the inhomogeneous Helmholtz equation

c) Transient problems

$$\frac{\partial^n u}{\partial t^n} - \Delta u + f_v = 0$$

- Mean Value Theorem: A function  $u$  is harmonic  $\Delta u = 0$  everywhere

$$\text{iff } u(x) = \frac{1}{|B(x, r)|} \int_{B(x, r)} u(y) dy = \frac{1}{|\partial B(x, r)|} \int_{\partial B(x, r)} u(y) dS$$

$$B(x, r) = \{y \in \mathbb{R}^n \mid |y - x| < r\}$$

$\partial B(x, r)$  is boundary of  $B(x, r)$

### 3.2, Fundamental Equations

② Divergence operator.

$$\operatorname{div} u = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

$$\operatorname{div} u = \nabla u$$

⑬ - Lame' - Navier equations

$$\mu \Delta u + (\lambda + \mu) \nabla(\operatorname{div} u) + f_v = 0$$

\* Curl operator

$$\nabla \times u = \begin{bmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{bmatrix}$$

$$= \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) e_1 + \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) e_2 + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) e_3$$

# Chapter 9. The Weighted Residual Method

NM/ME

## 4.1, Introduction:

- WRM helps us find an approximation solution for a well-posed problem:  
$$\begin{aligned} L(u) + f_v &= 0 \text{ in } V \\ C(u) &= f_s \text{ on } S = \partial V \end{aligned} \rightarrow \hat{u}$$

- Residual operator  $R(u) = L(u) + f_v$   
Also could  $R_{bc}(u) = C(u) - f_s$

- Assume  $\hat{u} = \hat{u}(x, a_1, \dots, a_n)$

We will try to choose  $a_1, \dots, a_n$  such that  $R(\hat{u}) \approx 0$

- Choose a set of linear independent weighting functions:  $\psi_1(x), \dots, \psi_n(x)$

We want:

$$\left. \begin{aligned} W_1 &= \int_V \psi_1(x) \cdot R(\hat{u}(x)) \cdot dV = 0 \\ W_2 &= \int_V \psi_2(x) \cdot R(\hat{u}(x)) \cdot dV = 0 \\ \vdots \\ W_n &= \int_V \psi_n(x) \cdot R(\hat{u}(x)) \cdot dV = 0 \end{aligned} \right\} \rightarrow \begin{array}{l} \text{solve a system} \\ \text{of equations} \\ \text{to find } a_1, \dots, a_n \end{array}$$

*n integral forms*

- du Bois-Reymond Lemma:

If a continuous function  $f$  on an open set  $V \subseteq \mathbb{R}^n$  satisfies the equation

$$\int_V f(x) \cdot g(x) \cdot dV = 0$$

for all compactly supported smooth functions  $g$  on  $V$ , then  $f$  is the 0 function

# Chapter 5. Calculus of Variations

NAME

## 45, 5.1, Introduction

- To solve minimization problems for functionals by extending the notion of derivation.

## 5.2, Definition & Properties

⊗ For function:  $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ ,  $\Delta x = x - x_0$

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0) \cdot \Delta x$$
$$\Delta f(x) = f'(x_0) \cdot \Delta x$$

⊗ Neighborhood of a function:  $F = u(x)$

$$u(x) = u_0(x) + \alpha \cdot v(x), \quad x \in [a, b]$$
$$v(a) = v(b) = 0 \quad \Rightarrow \quad (\delta u(a) = \delta u(b) = 0)$$

Variation of u:  $\alpha \cdot v(x) = (u - u_0)(x) = \delta u(x) = \delta u \in [-\varepsilon, \varepsilon] \quad \forall x \in [a, b]$

+) Weak variation: "zero" norm  $\|u - u_0\|_0 = \sup_{x \in [a, b]} |u(x) - u_0(x)|$

If  $\|u - u_0\|_0$  is small  $\Rightarrow$  weak variation

+) Strong variation: "one" norm  $\|u - u_0\|_1 = \sup_{x \in [a, b]} |u(x) - u_0(x)| + \sup_{x \in [a, b]} |u'(x) - u_0'(x)|$

If  $\|u - u_0\|_1$  is small  $\Rightarrow$  strong variation

+) Properties:  $(\delta u)'(x) = \delta(u')(x) \Rightarrow$  variation of der. is equal der. of variation

$$\delta \left( \int u(x) dx \right) = \int \delta u(x) dx \Rightarrow \text{integral of variation is equal variation of integral}$$

+) Given functional  $F = F(x, u, u')$

We approximate around  $\alpha$

$$F(\alpha) = F(x, u + \alpha v, u' + \alpha v') \quad , \quad F(0) = F(x, u, u')$$

$$\Delta F = F(\alpha) - F(0)$$

$$\Rightarrow \delta F = \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u' \quad \text{First variation of } F$$

$$\delta^2 F = \frac{\partial^2 F}{\partial u^2} (\delta u)^2 + 2 \frac{\partial^2 F}{\partial u \partial u'} \delta u \delta u' + \frac{\partial^2 F}{(\partial u')^2} (\delta u')^2 \quad \text{2nd variation of } F \text{ (twice the 2nd term)}$$

$$\text{from } \Delta F = \frac{dF}{d\alpha}(0) \cdot \alpha + \frac{1}{2!} \frac{d^2 F}{d\alpha^2}(0) \cdot \alpha^2 + O(\alpha^3) \text{ Taylor expansion}$$

$$\delta^2 F = 2 \cdot \left[ \frac{1}{2!} \frac{d^2 F}{d\alpha^2}(0) \cdot \alpha^2 \right]$$

+) Given functional  $F(u)$  in integral form

$$I(u) = \int_a^b F(x, u(x), u'(x)) dx$$

$$\Rightarrow \delta I(u) = \int_a^b \left[ \frac{\partial F}{\partial u} - \frac{d}{dx} \left( \frac{\partial F}{\partial u'} \right) \right] \delta u dx$$

Say that  $I(u)$  is stationary at  $u_0(x)$  if  $\delta I(u_0) = 0$

$\Rightarrow u_0(x)$  is extremal of functional  $I$

$$\delta I(u) = 0 \Leftrightarrow \frac{\partial F}{\partial u} - \frac{d}{dx} \left( \frac{\partial F}{\partial u'} \right) = 0 \quad (\text{Euler-Lagrange equation})$$

+) Energy function(al)  $I(u)$

$$I(u) = \int_{\Omega} \dots dx$$

$$\Rightarrow \delta I = \delta \left( \int_{\Omega} \dots dx \right) = \int_{\Omega} [\dots \delta u \dots \delta u' \dots] dx$$

$$= \int_{\Omega} [ODE] \delta u \cdot dx + [BCs]$$

7) From GDE and BCs.

- Integral form  $W = \int_{\Omega} \psi(x) \cdot R(u)(x) dx$

- Set weighting function  $\psi(x) = \delta u$

$$\Rightarrow W = \int_{\Omega} \delta u \cdot R(u) \cdot dx = \delta \left[ \int_{\Omega} \text{---} dx \right] + [BCs]$$

$$\Rightarrow W = \delta(I(u)) + [BCs]$$

47) 5.9) Constrained extremal problems

# Chapter 7. Discretization of Integral Forms

L19-25,

## 7.1, Introduction:

- Given PDE problem:

$$L(u) + f_V = 0 \quad \text{in } V$$

$$C(u) = f_S \quad \text{on } S = \partial V$$

We have already built:

- Residual:  $R = L(\hat{u}) + f_V$
- Integral form:  $W = \int_V \psi R(\hat{u}) dV$  (+ weak integral forms  $W^*$ ,  $W^0$ )
- Potential  $\pi$  such  $W = \delta\pi$  by choosing  $\psi = \delta\hat{u}$

⊗ Now, how to choose  $\left\{ \begin{array}{l} \text{approximation choices for } \hat{u} \\ \text{weighting functions } \psi \end{array} \right.$

## 7.2, Nodal approximation

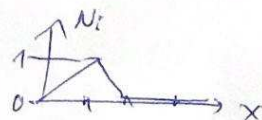
- Given:  $n$  nodes (points):  $x_1 \dots x_n$  in  $V$   
and their nodal values  $u(x_1) \dots u(x_n)$

$\Rightarrow$  We want  $\hat{u}(x)$  to take exact nodal values at corresponding  $n$  nodes

$\hookrightarrow$  Shape functions:  $N_i(x_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

$$\hat{u}(x) = u_1 \cdot N_1(x) + \dots + u_n \cdot N_n(x)$$

- Continuous, but non smooth  $N_i$ :



- For smooth  $N_i$ :

$$N_1(x) = \alpha(x-x_2)(x-x_3)\dots(x-x_n), \quad \alpha \text{ defined from } N_1(x_1) = 1$$

However, this leads to Runge phenomenon...

$\Rightarrow$  Use Chebyshev nodes (not equally spaced)

- Bernstein approximation
- Hermite polynomials

### 7.3) Non-Nodal approximation: (Modal approximation)

$$\hat{u}(x, a_1, \dots, a_n) = a_1 \cdot p_1(x) + \dots + a_n \cdot p_n(x) = \langle p_i(x) \rangle \{a_i\}$$

$p_i(x)$  :- basis / trial functions

$a_i$  :- parameters

- If BCs are homogeneous  $f_s = 0$   
 $\Rightarrow$  simply ask  $C(p_i)(x) = 0$  on  $S = \partial V$
- If BCs are non-homogeneous  $f_s \neq 0$   
 $\Rightarrow \hat{u}(x) = u_0(x) + \langle p_i(x) \rangle \{a_i\}$   
such  $C(u_0)(x) = f_s$  on  $S = \partial V$

### + Basis functions

- Polynomial:

$$\text{ex: } V = [-1, 1] \times [-1, 1] \subseteq \mathbb{R}^2 \Rightarrow \begin{aligned} p_1 &= (x^2 - 1)(y^2 - 1) \\ p_2 &= \text{---} (x^2 + y^2) \\ p_3 &= \text{---} (x^4 + y^4) \end{aligned}$$

$u = 0$  on  $S = \partial V$

- Trigonometric basis functions:

$$\text{ex } V = [0, 2] \times [0, 2] \Rightarrow T_k(x, y) = \sin\left(\frac{k\pi}{2}x\right) \cdot \sin\left(\frac{l\pi}{2}y\right)$$

7.4) Collocation by Points:  $n$  points / nodes  $x_i \in V$

- Set  $R(\hat{u})(x_i) = 0$

$\Rightarrow$  System of  $n$  equations  $\Rightarrow$  solve  $a_1 \dots a_n$

7.5) Collocation by Subdomains:

- Split  $V$  into  $n$  subdomains  $V_1 \dots V_n$ :

$$V = V_1 \cup V_2 \cup \dots \cup V_n$$

- Ask  $\int_{V_i} R(\hat{u}) dV = 0$  for  $i = 1 \dots n$

Also leads to system of  $n$  equations to solve  $a_1 \dots a_n$

7.6) Galerkin's Method:

- Choose  $\psi_i = p_i$

Then ask  $W_i = \int_V p_i(x) \cdot R(\hat{u})(x) dV = 0$  for  $i = 1 \dots n$

Again  $\Rightarrow$  system of  $n$  equations  $\Rightarrow$  find  $a_1 \dots a_n$

7.7) The Least Squares Method

- Try to minimize  $R^2(\hat{u}) \rightarrow 0$  on  $V$

$$\Leftrightarrow \text{minimize } \pi = \pi(a_1 \dots a_n) = \int_V R^2(\hat{u}) dV$$

$$\Leftrightarrow \frac{\partial \pi}{\partial a_i} = 0 \text{ for } i = 1 \dots n$$

$$\frac{\partial \pi}{\partial a_i} = \frac{\partial}{\partial a_i} \left[ \int_V R^2(\hat{u}) dV \right] = 2 \int_V \frac{\partial R(\hat{u})}{\partial a_i} R(\hat{u}) dV = 0$$

Thus ask for  $\int_V \frac{\partial R}{\partial a_i} R(\hat{u}) dV = 0$  for  $i = 1 \dots n$

7.8, the Ritz method

Ask for  $\boxed{\frac{\partial \pi}{\partial a_i} = 0} \quad i = 1 \dots n$

which is exactly  $\int_V P_i R(u) dV = 0$  from Galerkin's method

  
Ng Hui Duc

# NMIME exam prep

## ⊗ Set CALC in Radian mode

+ Type of boundary conditions: Chapter 2

- Dirichlet BC
- Neumann BC
- Robin/Newton BC
- Cauchy BC
- Natural BC

+ Operators: Chapter 3

- Symmetric operators - Self adjoint
- Positive & positive definite operators

+ Integration by parts:

$$\int u dv = \cancel{uv} - \int v du$$

Weak integral form by apply this to original integral form to reduce the order of  $\frac{\partial u}{\partial x}$

+  $F(u) = \pi(u)$  energy / potential equation

$$\begin{aligned} W &= \delta I + BCs = \delta \pi + BCs \\ &= \int [ODE] \delta u dx + BCs \end{aligned}$$

*[Handwritten signatures]*

1) Given energy func.  $I(u) = \int_{\Omega} \dots dx$

Asked to find  $\left\{ \begin{array}{l} \text{governing differential equation} \\ \text{natural boundary conditions} \end{array} \right.$

Solution: Transform variation  $\delta I$ :

$$\begin{aligned} \delta I &= \delta \left( \int_{\Omega} \dots dx \right) = \int_{\Omega} [\dots \delta u \dots \delta u' \dots (\delta u')^2 \dots] dx \\ &= \int_{\Omega} [ODE] \delta u \cdot dx + [BCs] \end{aligned}$$

Due to stationary condition  $\delta I = 0 \quad \forall \delta u(x)$

2) Given ODE & BCs

Asked to find governing energy function  $I(u)$

Solution: From ODE  $\Rightarrow$  get residual  $R(u)$

$$\text{Integral form: } W = \int_{\Omega} \psi(x) R(u)(x) dx$$

Set weighting func:  $\psi(x) = \delta u$

$$\begin{aligned} \Rightarrow W &= \int_{\Omega} \delta u \cdot R(u) \cdot dx = \delta \left[ \int_{\Omega} \dots dx \right] + [BCs] \\ &= \delta I + [BCs] \end{aligned}$$

3) Given PDE of  $u(x)$   
Boundary conditions

ex:  $u''(x) + \frac{1}{2}(u'(x))^2 + 1 = 0$   
 $u(0) = 0$   
 $u(1) = 0$   $x \in [0, 1]$

Asked to find  $n^{\text{th}}$  term approximation  $\hat{u}(x)$

Solution:  $\hat{u}(x) = u_0(x) + P_1(x) \cdot a_1 + \dots + P_n(x) \cdot a_n$

+ If polynomial approximation:  
From BCs  $\Rightarrow P_i(x)$

ex:  $\begin{cases} u(0)=0 \\ u(1)=0 \end{cases} \Rightarrow \begin{cases} \text{choose } P_1(x) = x(1-x) \\ P_2(x) = x^2(1-x) \\ \dots \end{cases}$

+ If trigonometry approximation:  
ex:  $\begin{cases} \psi(0)=0 \\ \psi(3)=0 \end{cases} \Rightarrow \begin{cases} P_1(x) = \sin\left(\frac{\pi x}{3}\right) \\ \dots \end{cases}$

+ Find  $u_0(x)$  also from BCs (non-homogeneous case)

+ Substitute  $P_1 \dots P_n \rightarrow \hat{u}(x)$   
then  $\hat{u}(x) \rightarrow \text{PDE} \dots$

+ Using Collocation by Points: Given points:  $x_1 \dots x_n$   
Set  $R(\hat{u})(x_i) = 0 \Rightarrow \text{solve } a_1 \dots a_n$

+ Using Collocation by Subdomain: Given subdomain  $\Omega_1 \dots \Omega_n$   
Set  $\int_{\Omega_i} R(\hat{u}) \cdot dx = 0 \rightarrow \text{solve } a_1 \dots a_n$

+ Galerkin's Method:  
Set  $\int_{\Omega} P_i(x) R(\hat{u}) \cdot dx = 0$

+ The Least Squares Method:  
Set  $\int_{\Omega} \frac{\partial R}{\partial a_i} R(\hat{u}) \cdot dx = 0$

+ The Ritz Method:  
Set  $\frac{\partial \pi}{\partial a_i} = 0$

## ⊗ Operator:

+ Gradient:  $\text{grad} / \nabla f$

ex: -  $p = (x_1, \dots, x_n) \Rightarrow \nabla f(p) = \left[ \frac{\partial f}{\partial x_1}(p) \dots \frac{\partial f}{\partial x_n}(p) \right]^T$

$$- \nabla f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

+ Divergence:  $\text{div } F = \nabla \cdot F$

ex: -  $\text{div } \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

+ Curl:  $\text{curl } F = \nabla \times F$

ex:  $\nabla \times F = \begin{bmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{bmatrix}$

+ Laplace:  $\Delta u = \text{div } \nabla u = \nabla \cdot \nabla u$

ex:  $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

+  $(\delta u)' = \delta(u')$   $\delta\left(\frac{dw}{dx}\right) = \delta w'$   $\downarrow$  Properties of variation

$$\delta\left(\int u(x) dx\right) = \int \delta u(x) dx$$

$$\delta F(x, u, u') = \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u'$$

$$\Rightarrow \delta\left(\int_0^1 x u dx\right) = \int_0^1 x \delta u dx$$