Advanced Control Systems

1, Introduction, system representation
2, Control-loop requirements & mapping
3, Recap linear algebra / system norms
4, Analysis of MIMO Systems: Gain, Direction, RGA, controllability, observability
5, ______________________ : poles and zeros
6, ______________________ : Internal Stability, Youla parametrization
Small gain theorem
7, Performance Limitations in SISO systems
8, Uncertainty & Robustness: stability & performance of SISO systems
9, ______________________ : ______________________ MIMO systems

⊛ Something about the history:

+) Classical (40's - 50's) : frequency domain methods - for SISO

⊕ can yield insights (phase margin, gain margin, root locus...)

⊕ address uncertainty

⊖ only applicable to SISO system

+) MIMO (60's - 70's): state-space, optimal control
Kalman, etc.

⊕ could address MIMO systems

⊕ could pose control problem as optimization problem

⊖ no clear links to classical methods

⊖ could not address uncertainty explicitly

Thus try to combine with the classical methods ⇒ +) Modern (80's - 90's): robust control ..

⊕ frequency domain based method

⊕ could handle uncertainty explicitly

ACS

Advanced Control System

Exercise

✓ 1, - Differential equation ⇒ linearize ⇒ state space model
- Transfer function matrix ⇒ state space form
- System scaling

2) - Closed Loop performance - MATLAB
? - Classical loop shaping

3) - ✓ Moore - Penrose Pseudo inverse
- ✓ Jordan canonical form
- ? Signal norms

4) - System norms
- Gain & directions
- Relative gain array
- Minimal realisation

$C = [B \ AB \ \cdots \ A^{n-1}B]$

$O = [C \ CA \ \cdots \ CA^{n-1}]^T$

5, - Minimal realisation - MATLAB only
- Poles and zeros of of MIMO

$P(s) = \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}$

[Rosenbrock matrix
[Mac Farlane & Karcanias

- Smith form
- Smith - Mc Millan form

6, Internal Stability
Coprime factorization
Nyquist criterion for MIMO (MATLAB)

7, Root locus & stability MATLAB
Bode Sensitivity integral
Poisson ———

8, Modelling uncertainty
Robust stability

9, MIMO Robust stability
LFT

10, Optimal State-feedback
Linear Quadratic Regulator (MatLAB)
Kalman Filter (MatLAB)

11, LQR margins and $\mathcal{H}_2$-control
LQR margins (MatLAB)
$\mathcal{H}_2$-control design (LQG is a special case of $\mathcal{H}_2$-optimal control)
(MATLAB)

12, Hamiltonians and AREs
$\mathcal{H}_\infty$ control
$\mathcal{H}_2$ and $\mathcal{H}_\infty$ control (MATLAB)

13, Linear Fractional Transformation
$\mathcal{H}_\infty$ loop shaping design (MATLAB)
Hanus anti-windup approach (MATLAB)

14, μ - analysis
Calculation of μ
Robust stability
Robust performance

*) Consider $\dot{x} = Ax$

Solution is: $x(t) = e^{At} \cdot x(0)$

with $e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$

Expand A to eigen values & vector: $A\xi = \lambda\xi$:

$\Rightarrow T = [\xi_1 \dots \xi_n]$ eigen vectors

$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ eigen value

$\Rightarrow AT = TD \Rightarrow T^{-1}AT = D$

- Let z that $x = Tz \Rightarrow \dot{x} = T\dot{z} = Ax = ATz$

$\Rightarrow \dot{z} = T^{-1}ATz = Dz$

$\Rightarrow \dot{z} = Dz$

$z(t) = e^{Dt} z(0)$

Also $e^{At} = T e^{Dt} T^{-1}$

$$\Rightarrow x(t) = e^{At} x(0) = T \cdot e^{Dt} \cdot \underbrace{T^{-1} x(0)}_{z(0)} = T \cdot \underbrace{e^{Dt} \cdot z(0)}_{z(t)}$$

$$= T \cdot z(t)$$

Just like SISO, the eigen value (λ_i of D) has to be < 0 to stable

*) $\dot{x} = Ax + Bu$ is <u>controllable</u> if we can choose $u = -Kx$

and | put the eigen value of A any where we want
| stear x anywhere I want

In practice, we are given fixed A and B

$$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u} \qquad x \in \mathbb{R}^n \qquad A \in \mathbb{R}^{n\times n}$$
$$\underline{y} = \underline{\underline{C}}\,\underline{x} \qquad u \in \mathbb{R}^q \qquad B \in \mathbb{R}^{n\times q}$$
$$y \in \mathbb{R}^p \qquad C \in \mathbb{R}^{p\times n}$$

+) Static gain: assume system in steady-state condition

$$\frac{dx}{dt} = 0$$

$$G(0) = \lim_{s\to 0} G(s) \qquad \Leftrightarrow G(j\omega) \text{ when } \omega \to 0$$

Controllability and Gramians:

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$$

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} \quad \text{controllability matrix}$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

Gramian (controllability):

$$W_t = \int_0^t e^{A\tau}B.B^T e^{A^T\tau}\,d\tau \qquad \in \mathbb{R}^{n\times n}$$

$$W_t\,\xi = \lambda\,\xi \qquad \text{\& } W_t \approx C.C^T$$

$[U, \Sigma, V] =$ svd$(\mathcal{C}$, 'econ') in controlability, there are levels of controllability

there are some directions that are easy to go in

which we can get out from SVD: U

⊗ <u>Popov - Belevitch - Hautus</u> (PBH) test for controllability:

(A, B) is ctrb iff $\quad \text{rank}\left[(A - \lambda I) \;\; B\right] = n \quad \forall \lambda \in \mathbb{C}$

1, $\text{Rank}(A - \lambda I) = n$ except for eigen values λ

$\Rightarrow$ Only need to test at eigen values λ_i

2, B needs to have some component in each eigenvector direction

3, If A has λ_i of power k.

At λ_i, we lost k dof for $(A - \lambda I)$, thus B need at least k columns

But also when we have close values of eigenvalues, we need extra columns in B to prevent that degeneration matrix

⊗ <u>Cayley - Hamilton:</u> $\quad e^{At} = \phi_0(t) . I + \phi_1(t) . A + \dots + \phi_m(t) . A^{n-1}$

$$\mathcal{C} = \left[B \;\; AB \;\; A^2B \;\; \dots \;\; A^{n-1}B\right]$$

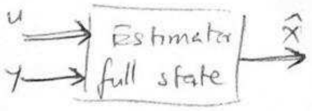
$$\xi = \int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau = \dots$$

$$= \left[B \;\; AB \;\; \dots \;\; A^{n-1}B\right] \left[\; \vdots \; \right]$$

⊗ <u>Observability</u>:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Estimator:



$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu + K_f(y - \hat{y})$$

$$\hat{y} = C\hat{x}$$

$$\Rightarrow \frac{d}{dt}\hat{x} = (A - K_f C)\hat{x} + \left[B \;\; K_f\right]\begin{bmatrix} u \\ y \end{bmatrix}$$

Set $\varepsilon = x - \hat{x}$

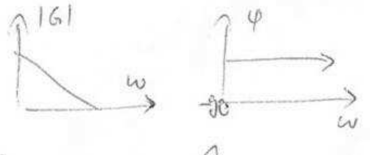
$$\frac{d}{dt}\varepsilon = (A - K_f C)\varepsilon \quad \dots$$

Design K_f in present of disturbance & noise

- Phase margin PM: how robust we are to system delay
  (cause there might be extra delay in control, in system model..
  with these noise, uncertainty.., we still want to assure stability)

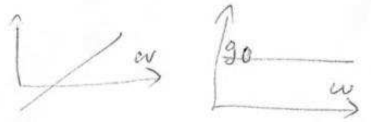
- Gain margin GM: how robust we are to extra gain
  from system model, controller, .. that we still assure stability..

- Integrator $\frac{K}{s}$



pole

Derivative Ks



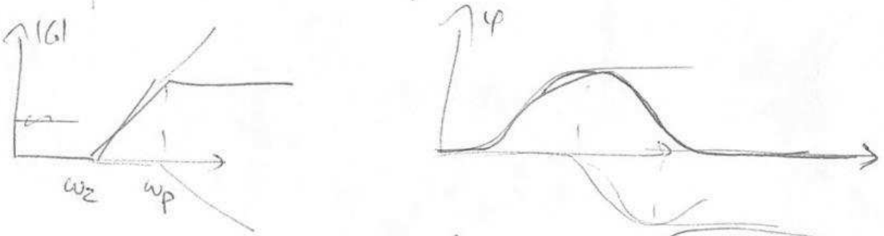
zero

lead $\Rightarrow 1+T_1 s \Rightarrow$ zero $\Rightarrow$ add phase $\qquad \frac{s+w_z}{s+w_p}$

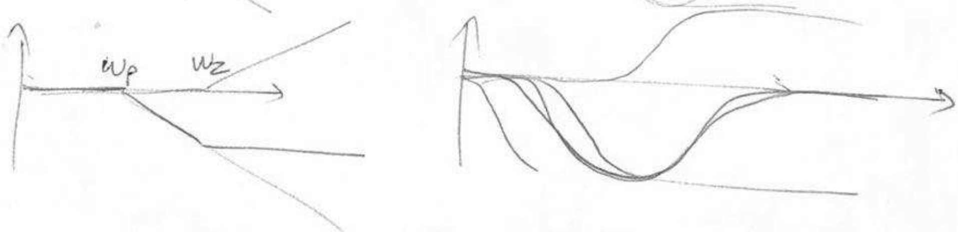
$\Rightarrow$ Compensator

lag $\Rightarrow \frac{1}{1+T_2 s} \Rightarrow$ pole $\Rightarrow$ subtract phase

- $w_z < w_p \Rightarrow$



- $w_z > w_p \Rightarrow$



L1,

- Regulator problem: attenuate effect of d on y
- Servo problem: make y follow r

+) <u>Price & limitations</u> of feedback control:
- Loop might still be unstable
- Noise from measurement
- Control performance depends on both system & controller

+) Classical / Robust Control Design Process

Classical — Process / Plant — Robust

Classical		Robust
System identification & modelling	⟷	Uncertainty modelling
↓		↓
System analysis	⟷	Robustness and performance bounds
↓		↓
Controller design	→	Robust control design

+) Robust control design methodology:

$$G_p = \underbrace{G}_{\text{nominal mode}} + \underbrace{\Delta}_{\text{uncertainty / pertubatten}}$$

(NP) Nominal Performance System statisfies perf. specifications for nominal model	(RP) Robust Performance System satisfies .. under uncertainty
(NS) Nominal Stability System is stable for nominal model	(RS) Robust Stability system is stable under uncertainty

- Linear system: $f(a_1 u_1 + a_2 u_2) = a_1 \cdot f(u_1) + a_2 \cdot f(u_2)$

- Nonlinear system:

$$\frac{d}{dt} x(t) = f(x(t), u(t))$$

$$y(t) = g(x(t), u(t)), \quad u \in \mathbb{R}^m, y \in \mathbb{R}^p, x \in \mathbb{R}^n$$

- Linearization around equilibrium point x_{eq}:

$$f(x_{eq}, u_{eq}) = 0, \qquad g(x_{eq}, u_{eq}) = y_{eq}$$

$$\Delta x = x - x_{eq}$$

$$\Rightarrow \begin{cases} \frac{d}{dt}\Delta x \simeq \left.\frac{\partial f}{\partial x}\right|_{x_{eq}, u_{eq}} \cdot \Delta x + \left.\frac{\partial f}{\partial u}\right|_{x_{eq}, u_{eq}} \cdot \Delta u \\ \Delta y \simeq \left.\frac{\partial g}{\partial x}\right|_{x_{eq}, u_{eq}} \cdot \Delta x + \left.\frac{\partial g}{\partial u}\right|_{x_{eq}, u_{eq}} \cdot \Delta u \end{cases}$$

$$\Rightarrow \begin{cases} \frac{d}{dt}\Delta x = A \cdot \Delta x + B \cdot \Delta u \\ \Delta y = C \cdot \Delta x + D \cdot \Delta y \end{cases} \qquad G \overset{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

+) <u>Transfer function:</u>

$$Y(s) = \left(C(sI-A)^{-1}B + D\right) U(s) \in \mathbb{C}^{p \times m}$$

$$G_{1,1}(s) = \frac{b_{n_n} s^{n_n} + b_{n_n-1} s^{n_n-1} + \dots + b_0 s^0}{a_{n_d} s^{n_d} + a_{n_d-1} s^{n_d-1} + \dots + a_0 s^0}$$

- n_n : order of numerator
  n_d : order of denominator
  $n_d - n_n$: pole excess or relative degree

- Strictly proper : $n_d - n_n > 0$ $\quad \lim_{s \to +\infty} G(s) = 0$ physical system

System is semi proper : $n_d - n_n = 0$ $\quad$ ______ $= D \neq 0$

improper : $n_d - n_n < 0$ $\quad$ ______ $\to \infty$ not exists physical system not causal

+) <u>Irrational transfer function:</u>

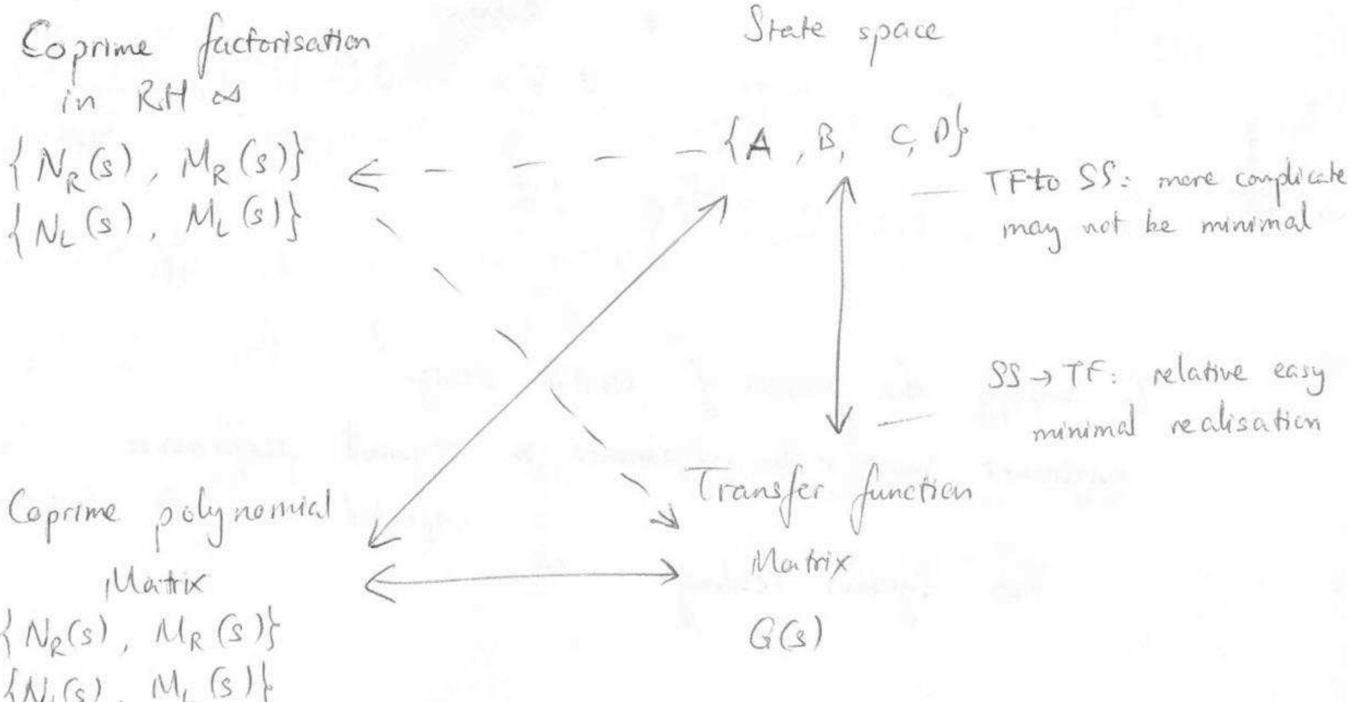
$$G(s) = \begin{bmatrix} G_{1,1}(s).e^{-\theta_{1,1}s} & G_{1,2}(s).e^{-\theta_{1,2}s} & \cdots \\ G_{2,1}(s).e^{-\theta_{2,1}s} & \vdots & \\ \vdots & & \end{bmatrix}$$ due to time delay

- causal if $\theta \geq 0$
- infinite dimensional system (sometimes)
  - ⇒ will not be representable by rational TFs
- Approximation by Pade series

+) <u>Polynomial model:</u>

$$\begin{aligned} \text{left} \quad G(s) &= D_L^{-1}(s) \,.\, N_L(s) \\ \text{right} \quad &= N_R(s) \,.\, D_R^{-1}(s) \end{aligned}$$

+) <u>System conversion:</u>



+) System Scaling

- unscaled system $\hat{y} = \hat{G}\hat{u} + \hat{G}_d \hat{d}$, $\hat{e} = \hat{r} - \hat{y}$

$$d_i = \frac{\hat{d}_i}{\hat{d}_{i,max}}, \quad u_i = \frac{\hat{u}_i}{\hat{u}_{i,max}}, \quad y_i = \frac{\hat{y}_i}{\hat{e}_{i,max}}, \quad r_i = \frac{\hat{r}_i}{\hat{e}_{i,max}}, \quad e_i = \frac{\hat{e}_i}{\hat{e}_{i,max}}$$

- Diagonal scaling matrices D_e, D_u, D_d, D_r

<u>Ex</u>: $D_e = diag(\hat{d}_{e,max})$

=> Scaled variables:

$$d = \hat{D}_d^{-1}\hat{d}, \quad u = \hat{D}_u^{-1}\hat{u}, \quad y = \hat{D}_e^{-1}\hat{y}, \quad r = \hat{D}_e^{-1}\hat{r}, \quad e = \hat{D}_e^{-1}\hat{e}$$

=> Replace back to unscaled system:

$$D_e y = \hat{G} D_u u + \hat{G}_d D_d d \quad ; \quad D_e e = D_e y - D_e . r$$

$$\Rightarrow y = D_e^{-1}\hat{G} D_u . u + D_e^{-1}\hat{G}_d D_d . d \qquad |u, d, e| \in [0, 1]$$

$$\Rightarrow \begin{cases} G = D_e^{-1}\hat{G} D_u \\ G_D = D_e^{-1}\hat{G}_d D_d \end{cases} \Rightarrow y = Gu + G_D . d$$

- Also introduce: $\tilde{r} = \dfrac{\hat{r}}{\hat{r}_{max}} = D_r^{-1}\hat{r}$ $\qquad |\tilde{r}(t)| \leq 1$

$$\Rightarrow r = R . \tilde{r} \quad ; \quad R \triangleq D_e^{-1} D_r = \frac{\hat{r}_{max}}{\hat{e}_{max}}$$

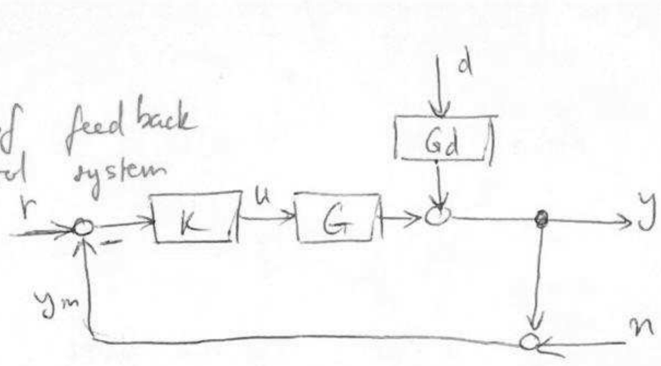
$$\Rightarrow e = y - r = Gu + G_d d - R\tilde{r}$$

To simplify the process of control design
engineers must make judgements on required performance
expected noise & disturbances

=> System scaling

+) Feedback Control.

+) 1 dof feedback control system



$$\underline{u} = \underline{K}\,(\underline{r} - y_m) = \underline{K}\,(\underline{r} - \underline{y} - \underline{n})$$

$$\underline{e} = \underline{y} - \underline{r} \neq \hat{e} = \underline{r} - \underline{y} - \underline{n}$$

$$\Rightarrow \underline{y} = \underline{G}\,\underline{u} + \underline{G_d}\cdot\underline{d}$$

$$\Rightarrow \underline{y} = \underbrace{(I+GK)^{-1}GK}_{T}\, r + \underbrace{(I+GK)^{-1}}_{S}\, G_d\, d - \underbrace{(I+GK)^{-1}GK}_{T}\, n$$

$T \Rightarrow$ want T high for low freq

$S \Rightarrow$ want S small, slow for low freq

$T \Rightarrow$ cause noise is usually high freq $\Rightarrow$ we want T small for high freq

$$= Tr + SG_d d - Tn$$

Since $T - I = -S$: or $S + T = I$

$$\Rightarrow e = y - r = -Sr + SG_d d - Tn$$

and $u = KSr - KSG_d d - KSn$

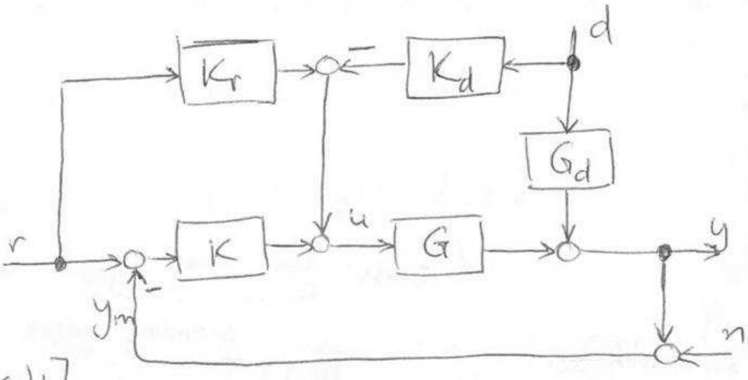
- Loop transfer function: $L = GK$

Sensitivity function: $S = (I + GK)^{-1} = (I+L)^{-1}$

Complementary sensitivity func: $T = (I + GK)^{-1}GK = (I+L)^{-1}L$

$$R = KS$$

+) 2dof controller



$$u = \underbrace{K(r-y-n)}_{\text{feedback}} + \underbrace{K_r r - K_d d}_{\text{feed forward}}$$

$$\Rightarrow y = (I+GK)^{-1}\left[G(K+K_r)r - GKn + (G_d - GK_d)d\right]$$

$$\Rightarrow e = S\left[-S_r r - GKn + S_d G_d d\right]$$

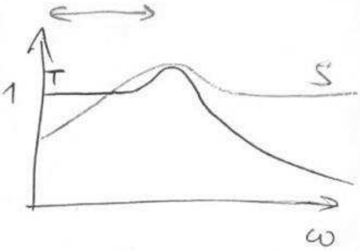
With
$$S = (I + GK)^{-1}$$
$$\left.\begin{aligned} S_r &= I - GK_r \\ S_d &= I - GK_d G_d^{-1} \end{aligned}\right\}$$ can be interpreted as feed-forward sensitivity function

L_2)

+) <u>Bandwidth & system damping</u>

- Fast response time $\Leftrightarrow$ $|T(j\omega)| \simeq 1$ for large freq range
- Maximum magnitude of sens. & comple. sens. functions:

$$M_S = \max_{\omega \in \mathbb{R}} |S(j\omega)| \; ; \quad M_T = \max_{\omega \in \mathbb{R}} |T(j\omega)|$$



$$M_S = \|S\|_\infty \, , \; M_T = \|T\|_\infty$$

$$S + T = 1 \quad \text{thus} \quad \big||S| - |T|\big| \le |S+T| = 1$$

- For specific M_S, M_T, we can derive gain & phase margin:

$$GM \geq \frac{M_S}{M_S - 1} \; ; \quad PM \geq 2\arcsin\left(\frac{1}{2M_S}\right) \geq \frac{1}{M_S} \text{ (rad)}$$

$$GM \geq 1 + \frac{1}{M_T} \; ; \quad PM \geq 2\arcsin\left(\frac{1}{2M_T}\right) > \frac{1}{M_T} \text{ (rad)}$$

⊕ The closed-loop <u>band-width, ω_B</u>, is the freq. where $|S(j\omega)|$ first crosses $1/\sqrt{2}$ ($\simeq -3$dB) from below

- High $\omega_B \Rightarrow$ faster rise time (response time); sensitive to | noise / params variation
- Low $\omega_B \Rightarrow$ slow response time, more robust to noise

(cause $\underbrace{\text{high freq signals}}_{\text{including noise}}$ are easily passed to output)

bandwidth in term of T — ω_{BT} : highest freq at which $|T(j\omega)|$ crosses $1/\sqrt{2}$ from above

The gain crossover freq — ω_c : $|L(j\omega_c)| = 1 \Leftrightarrow |S(j\omega_c)| = |T(j\omega_c)| = 1/\sqrt{2}$

- Good performance: $|S| \simeq 0$; $|T| \simeq 1$, φ_T not too negative

need 2 conditions

Cause $e = S(G_d . d - r)$
$\Rightarrow$ we want $|S| < 1$
With integral, there will be $|S| \simeq 0$

Typically require that $M_S < 2 \, (6 \text{ dB})$

$M_T < 1{,}25 \, (2 \text{ dB})$

⊗ <u>Loop - shaping</u>: shaping $L = GK$ to get desired perf.

$$e = - \underbrace{(I+L)^{-1}}_{S} r + \underbrace{(I+L)^{-1}}_{S} G_d . d - \underbrace{(I+L)^{-1} L}_{T} . n$$

- Assume scaled circuit $|d(j\omega)| \leq 1 \quad \forall \omega$

and we want $|e(j\omega)| \leq 1$

- Consider worst case $|d(j\omega)| = 1$, we need $|S . G_d| < 1 \quad \forall \omega$

$\Leftrightarrow \quad |(I+L)^{-1} G_d| < 1 \Leftrightarrow |I+L| > |G_d|$

At range ω that $|G_d| > 1 \Rightarrow |L| > |G_d|$

P - I - D

$\Rightarrow$ +) <u>1st step</u> $|L_{min}| = |G_d| \Leftrightarrow \boxed{K_{min} \approx |G^{-1} G_d|}$

+) <u>2nd step</u>: <u>add intergral</u> to deal with disturbance:

$K = k . \dfrac{s + \omega_I}{s}$ $\qquad k \dfrac{1 + T_1 s}{T_1 s}$

$$\frac{s + \omega_I}{s} = \begin{cases} \text{meaningful value} & \Rightarrow \text{active at small freq} \\ 1 \quad \omega > \omega_I & \Rightarrow \text{want } \omega_I \text{ high} \end{cases}$$

but if ω_I too high, can't maintain acceptable PM

$\Rightarrow$ Choose $\boxed{\omega_I = 0{,}2 \, \omega_c}$

+) <u>3rd step</u>: High freq correction. to increase PM ..

lead - lag term

$$\frac{k_{D_I} . s + 1}{k_{D_{II}} . s + 1} = \begin{cases} 1 \text{ for } \omega < \dfrac{1}{k_{D_I}} \\ \text{meaningful value} \Rightarrow \text{only active after } \omega > \dfrac{1}{k_{D_I}} \end{cases}$$

$k_{D_I} > k_{D_{II}}$ $\Rightarrow$ effective over $\dfrac{k_{D_I}}{10 . k_{D_{II}}}$ decade, starting at $\omega = \dfrac{1}{k_{D_I}}$ rad/s

+) 2dof - design

- Let call desired tf. for ref. tracking: T_{ref}
- Design K_y first, for disturbance rejection

$$T = L(I+L)^{-1} \quad ; \qquad L = G K_y$$

Then $T K_r = T_{ref}$

$$\Rightarrow \quad K_r = T^{-1} T_{ref}$$

Usually $K_r(s) = \dfrac{\tau_{lead}\, s + 1}{\tau_{lag} \cdot s + 1}$ $\begin{cases} \tau_{lead} > \tau_{lag} & \text{if we want speed up the response} \\ \tau_{lead} < \tau_{lag} & \text{if we want slow down} \end{cases}$

+) $\mathcal{H}_\infty$ and $\mathcal{H}_2$

- $\mathcal{H}_\infty$ norm : the peak value

$$\|f(s)\|_\infty \triangleq \max_{\omega} |f(j\omega)| = \lim_{p \to \infty} \left(\int_{-\infty}^{\infty} |f(j\omega)|^p \, d\omega \right)^{1/p}$$

- $\mathcal{H}_\infty$: Hardy space = set of tf. with bounded ∞-norm
  $\Rightarrow$ set of stable & proper tf.

- $\mathcal{H}_2$ norm:

$$\|f(s)\|_2 \triangleq \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |f(j\omega)|^2 \, d\omega \right)^{1/2}$$

- Semiproper tf: $\mathcal{H}_2$ norm = ∞
  $\mathcal{H}_\infty$ norm is finite

Example: S is semi-proper

AES

3)

Linear Algebra
Signal & ~~System norms~~

1) Definitions

- Vector space

- Sub spaces

- Linear dependence
  & independence

- Basis & dimension
  - Orthogonal vs orthonormal basis

- Null space: $Ker(A) = \mathcal{N}(A) = \{\underline{x} \in \mathbb{F}^n : \underline{A}\underline{x} = \underline{0}\}$

  Image: $Im(A) = \mathcal{R}(A) = \{y \in \mathbb{F}^m : y = \underline{A}\underline{x},\ \underline{x} \in \mathbb{F}^n\}$

- Eigen values & eigen vectors: $Ax_i = \lambda_i x_i$; $x_i \in \mathbb{C}$

  $AT = T\Lambda$ (T is called Modal matrix)

- Jordan normal form (Jordan canonical form)

  $J = T^{-1}AT$ (or Λ)

  $J_k(\lambda) \in \mathbb{C}^{k \times k}$; $J_k(\lambda) = \begin{bmatrix} \lambda & 1 & \cdots & 0 \\ \vdots & \ddots & & 0 \\ \vdots & & \lambda & 1 \\ 0 & \cdots & \cdots & \lambda \end{bmatrix} \Rightarrow J = \begin{bmatrix} J_{k_1}(\lambda_1) & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & J_{k_m}(\lambda_m) \end{bmatrix}$

- Singular value decomposition:

  $A = U \begin{bmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{bmatrix} . V^H$

  Moore-Penrose inverse: $A^+ \triangleq V \begin{bmatrix} \Sigma_0^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^H$

2) Invariant Subspaces:

- Subspace S is called $\begin{cases} A\text{- invariant} \\ \text{invariant to transformation } A \end{cases}$ if $Ax \in S \; \forall x \in S$

- Generally A has eigenvalues: $\lambda_1 = \lambda_2 = \ldots = \lambda_l$
  eigenvectors: x_1

generalized eigenvectors: $\begin{cases} (A-\lambda_1 I)x_1 = 0 \\ (A-\lambda_1 I).x_2 = x_1 \\ \vdots \\ (A-\lambda_1 I).x_l = x_{l-1} \Leftrightarrow (A-\lambda_I I)^l x_l = 0 \end{cases}$

$\Rightarrow$ Subspace S with $x_t \in S$ for some $t \leq l$ is A-invariant
iff all $\begin{cases} \text{lower rank eigenvectors} \\ \text{generalized eigenvectors} \end{cases}$ of $x_t \in S$

- ~~S~~ $S \subset \mathbb{F}^n$ is stable invariant subspace
if all eigenvalues of A has negative real parts

- Singular Value Decomposition:

$A = U.\Sigma.V^H$

$\Rightarrow A^+ = V.\Sigma^H.U^H$ (Moore-Penrose inverse)

+) Norm: $\begin{cases} \text{non-negative: } \|e\| \geq 0 \\ \text{- positive } \|e\| = 0 \Leftrightarrow e = 0 \\ \text{- homogeneous } \|\alpha.e\| = |\alpha|.\|e\| \text{ all complex scalar } \alpha \\ \text{triangle inequality } \|e_1 + e_2\| \leq \|e_1\| + \|e_2\| \end{cases}$

+) Conjugate Transpose - <u>Hermitian transpose</u>

$$M = \begin{bmatrix} 2+3i & i & 6-9i \\ 7 & 2-3i & -i \end{bmatrix} \Rightarrow M^H = M^+ = \begin{bmatrix} 2-3i & 7 \\ -i & 2+3i \\ 6+9i & i \end{bmatrix}$$

$\Rightarrow$ +) Hermitian matrix: $M^H = M$

+) Orthogonal <u>(real)</u> $\Longleftrightarrow$ unitary (<u>complex</u>)

columns form orthonormal basis — columns form orthonormal vectors

$A^T = A^{-1}$ — $U^H = U^{-1}$

$A.A^T = I$ But $A.A^T \neq I$ $\Leftrightarrow$ $U.U^H = I$

+) <u>Vector norms:</u> $\|a\|_p = \left(\sum_i |a_i|^p\right)^{1/p}$

- Vector 1-norm (sum norm)

$$\|a\|_1 \triangleq \sum_i |a_i|$$

Ex: $d = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} \Rightarrow \|d\|_1 = 9$

- Vector 2-norm (Euclidean norm)

$$\|a\|_2 \triangleq \sqrt{\sum_i |a_i|^2}$$

$\|d\|_2 = \sqrt{35}$

- Vector ∞-norm (max norm)

$$\|a\|_\infty \equiv \|a\|_{max} \triangleq \max_i |a_i|$$

$\|d\|_{max} = 5 = |-5|$

+) <u>Matrix norms:</u> $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$

- Multiplicative property: $\|A.B\| \leq \|A\|.\|B\|$

- Sum matrix norm

$$\|A\|_{sum} = \sum_{i,j} |a_{i,j}|$$

$\|A\|_{sum} = 10$

- Frobenius matrix norm:

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{tr(AA^H)}$$

$\|A\|_F = \sqrt{30}$

- Induced matrix norms

$$\|A\|_{ip} \triangleq \max_{\omega \neq 0} \frac{\|A\omega\|_p}{\|\omega\|_p}$$

$$\|A\|_{i1} = \max_j \left(\sum_i |a_{ij}| \right) \quad \text{max column sum}$$

$$\|A\|_{i2} = \max_{1 \le i \le n} \sqrt{\lambda_i(A^T A)} = \bar{\sigma}(A) = \sqrt{\rho(A^H A)} \quad \text{singular value/spectral norm}$$

$$\|A\|_{i\infty} = \max_i \left(\sum_j |a_{ij}| \right) \quad \text{max row sum}$$

+) Signal norms : $e(t)$

l_p norm: $$\left(\|e(t)\|_p\right)^p = \int_{-\infty}^{\infty} \underbrace{\sum_i |e_i(\tau)|^p}_{\|e(\tau)\|_p^p} d\tau$$

- 1-norm in time

$$\|e(t)\|_1 = \int_{-\infty}^{\infty} \sum_i |e_i(\tau)| d\tau$$

- 2-norm in time : "energy" of signal

$$\|e(t)\|_2 = \sqrt{\int_{-\infty}^{\infty} \sum_i |e_i(\tau)|^2 d\tau}$$

- ∞ norm in time: peak value in time

$$\|e(t)\|_\infty = \max_\tau \left(\max_i |e_i(\tau)| \right)$$

- minimizing $\mathcal{H}_\infty$ norm $\Leftrightarrow$ minimizing peak of largest singular value (worst direction worst freq)
- minimizing $\mathcal{H}_2$ norm $\Leftrightarrow$ minimize average direction, average freq

+) System norms $\quad e = Gd$

- $\mathcal{H}_\infty$ norm: $$\|G(s)\|_\infty = \max_{d(t)} \frac{\|e(t)\|_2}{\|d(t)\|_2}$$

- l_1 - norm: $$\|g(t)\|_1 = \max_{d(t)} \frac{\|e(t)\|_\infty}{\|d(t)\|_\infty}$$

- $\mathcal{H}_2$ - norm: $$\|G(s)\|_2 \triangleq \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} tr\left(G(j\omega)^H G(j\omega)\right) d\omega}$$
$$= \|g(t)\|_2 = \sqrt{\int_0^\infty tr\left(g(\tau)^T g(\tau)\right) d\tau}$$

As if we use unit impulse $\delta_i(t)$ to each input $\cancel{t}$

Analysis of MIMO system

L4,

1, Gain & directions of MIMO System

- Gain of a matrix $G \in \mathbb{R}^{p \times m}$

$$y = G.u$$

Apply SVD: $G = U.\Sigma.V^T$ $\quad U \in \mathbb{R}^{p \times p}$

$= \sigma_1 u_1 v_1^T + \ldots + \sigma_r u_r v_r^T$ $\quad V \in \mathbb{R}^{m \times m}$

$\bar{\sigma} = \max_i \sigma_i$; $\underline{\sigma} = \min_i \sigma_i$

with input direction v_1 ⇒ output dir. u_1 and gain σ_1

⇒ Interpretation: U, V as unitary matrix ⇒ rotation only, not scaling

Σ is scaling with σ_i

⇒ $\bar{\sigma}$ is max gain, $\underline{\sigma}$ is min gain

- Condition number of a matrix: $\gamma(G) = \dfrac{\bar{\sigma}(G)}{\underline{\sigma}(G)}$

Need lots of change in u to compensate small change in A

If cond. number is large ⇒ difficult to control.

(Ill-conditioned matrix: small change in A → large change in C or X $\quad Ax = C$

+) Frequency response matrix:

$$G(j\omega) = U(j\omega)\underbrace{\Sigma(\omega)}_{\text{freq dependent gain}}V(j\omega)^H$$

+) Relative gain array: (RGA)

2x TITO

$$\frac{\text{gain when others open loop}}{\text{gain when others close loop}} \quad \frac{\left.\dfrac{y_1(s)}{u_1(s)}\right|_{OL}}{\left.\dfrac{y_1(s)}{u_2(s)}\right|_{CL}} \triangleq \lambda_{1,1}(s)$$

- we desired $\lambda_{ij} = 1$
  desirable pairing, because it is not effected by others loop
- if $\lambda_{ij} = 0$ ⇒ avoid pairing

⇒ For general non-singular square complex matrix.

$$\boxed{RGA\,(G(s)) = \Lambda(s) \triangleq G(s) \times \left(G^{-1}(s)\right)^T}$$

element multiplication

RGA = G.* pinv(G).'

- <u>Usage of RGA</u>: for MIMO system, we want to pair which output is controlled by which input.
And we decide this by the RGA: $\Lambda(G)$

- Good pairing: close to 1
- If RGA is great ⇒ bad for control

+) Note: for 2×2 matrix $G(s)$

$$\Lambda(G) = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} = \begin{bmatrix} \lambda_{11} & 1-\lambda_{11} \\ 1-\lambda_{11} & \lambda_{11} \end{bmatrix} \quad \text{with } \lambda_{11} = \frac{1}{1 - \frac{g_{12}g_{21}}{g_{11}g_{22}}}$$

2) <u>State controllability and observability</u>:

$$\begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix} t_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

⇒ System is observable if rank $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n \quad \forall \lambda$ Hautus test

controllable if rank $\left[\lambda I - A \mid B\right] = n \quad \forall \lambda$

- When transferring from transfer function model ⇒ State space might not lead to minimal realisation (deleting all uncontrollable &/or unobservable states)

- Uncontrollable states do not need to be considered if not unstable

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Observability matrix

$$\mathcal{C}(A,B) = \left[B \;\; AB \;\; \ldots \;\; A^{n-1}B\right]$$

controllability matrix

5) +, Minimal realisation given A, B, C

1, Compute $\mathcal{C}(A,B)$ & $\mathcal{O}(A,C)$

2, Calculate SVD of $\mathcal{C}(A,B)$

$$\mathcal{C}(A,B) = \begin{bmatrix} U_c & U_{\bar{c}} \end{bmatrix} \cdot \begin{bmatrix} \Sigma_c & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} V_c^T \\ V_{\bar{c}}^T \end{bmatrix} = U_c \Sigma_c V_c^T$$

3, Get rank: $r_c = \operatorname{rank} \mathcal{C}(A,B) = \dim \Sigma_c$

4, Calculate SVD of:

$$\mathcal{O}(A,C) U_c \Sigma_c^{1/2} = \begin{bmatrix} U_{co} & U_{c\bar{o}} \end{bmatrix} \begin{bmatrix} \Sigma_{co} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{co}^T \\ V_{c\bar{o}}^T \end{bmatrix} = U_{co} \Sigma_{co} V_{co}^T$$

5, For which the rank is given by

$$r_{co} = \operatorname{rank} \mathcal{O}(A,C) U_c \Sigma_c^{1/2} = \dim \Sigma_{co}$$

6, And the tranformation matrices:

$$R_1 = U_c \Sigma_c^{1/2} V_{co} \Sigma_{co}^{1/2} \qquad T_1 = U_c \Sigma_c^{1/2} V_{co} \Sigma_{co}^{-1/2}$$

$$\Rightarrow (\bar{A}, \bar{B}, \bar{C}) = \left[R_1^T A T_1,\ R_1^T B,\ C T_1 \right]$$

3, Poles and zeros in MIMO systems

- Non-minimal system will leads to poles (eigenvalues) that correspond to uncontrollable / unobservable states..

+, How to find poles & zeros of MIMO:

- MacFarlane and Karcanias

  Matrix minor: Determinant of smaller matrix (by removing 1 or more col, rows)

  $\Rightarrow$ Pole polynomial $\phi(s)$: least common denominator of all non-identically-zero minors of all orders of $G(s)$

and Zero polynomial $z(s)$: greatest common divisor of all numerators of all order-r minors of $G(s)$, r = normal rank of $G(s)$

+) Zeros from state-space realizations:

- Rosenbrock matrix:

$$P(s) = \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}$$

Zeros all z such: $s = z \Rightarrow P(s)$ loose rank

+) TF2SS

SIMO $\Rightarrow$ controllable canonical form

MISO $\rightarrow$ observable canonical form

+) Gilbert's realisation: to find poles & zeros

$$G(s) = D + \sum_{i=1}^{r} \frac{W_i}{s - \lambda_i}$$

Ex. Given TF matrix:

$$G(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} 2s+1 & s \\ s+1 & 1 \end{bmatrix} = D + \frac{1}{s+2} \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} + \frac{1}{s+1} \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ SS: $G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \ldots$

+) Smith form to find poles & zeros

- Given polynomial matrix $P(s)$, find $S(s)$ such

$$\underline{S(s)} = \begin{bmatrix} \gamma_1'(s) & & & \\ & \gamma_2'(s) & & \\ & & \ddots \; \gamma_r'(s) & \\ & & & 0 \end{bmatrix} = U(s) . P(s) . V(s)$$

- Step: Find minor of $P(s)$

Find greatest common divider (gcd) & $D_0(s)$, $D_1(s)$, $D_2(s)$, ...

Then $\gamma_i'(s) = \dfrac{D_i(s)}{D_{i-1}(s)}$ $\left(\gamma_1'(s) = \dfrac{D_1(s)}{D_0(s)}, \; \gamma_2'(s) = \dfrac{D_2(s)}{D_1(s)} \right.$

+) <u>Smith - Mc Millan form</u> also just to find poles & zeros

$$\underline{M(s)} = \frac{1}{d(s)} S(s)$$

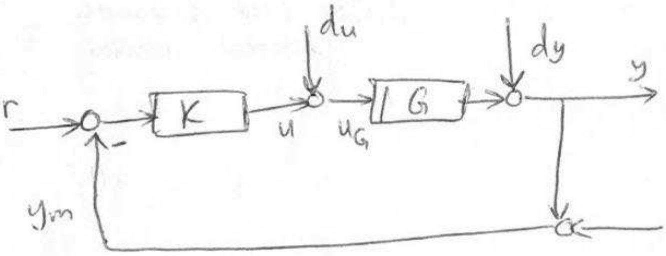
with $d(s)$ from $G(s) = \frac{D(s)}{d(s)}$

- Poles : determine stability
  I/O system behavior
  Location to be influenced by feedback
- Zeros: stability of feedback
  I/O system behavior
  location not manipulable by feedback

L6) Analysis of MIMO system

1) Internal Stability

- When K & G are stable, doesn't ensure that feedback is realisable



$\Rightarrow$ also want to check if $M(s)$ is stable

$$\begin{bmatrix} u \\ y \end{bmatrix} = M(s) \cdot \begin{bmatrix} d_u \\ d_y \end{bmatrix} \quad ; \quad M(s) = \begin{bmatrix} I & K \\ -G & I \end{bmatrix}^{-1}$$

$$\Leftrightarrow \begin{cases} u = (I+KG)^{-1} d_u - K(I+GK)^{-1} d_y \\ y = G(I+KG)^{-1} d_u + (I+GK)^{-1} d_y \end{cases}$$

Internally stable iff ~~these~~ 4 of 4 of funcs are stable

2) We have:

$$K(I+GK)^{-1} = Q$$
$$(I+GK)^{-1} = I - GQ$$
$$(I+KG)^{-1} = I - QG$$
$$G(I+KG)^{-1} = G(I-QG)$$

If Q & G are stable $\Rightarrow$ all are (internally) stable

- Q is stable if : $K = Q(I-GQ)^{-1}$ with a proper Q
  (Youla parametrisation)

+) Coprime factorisation

- Given tf $G(s)$, find $G(s) = \frac{N(s)}{M(s)}$

  such that $\exists$ ~~N(s)~~ $U(s)$ & $V(s)$ that

  $$U(s)\,N(s) + V(s)\,M(s) = 1 \quad \text{(Bezout's identity)}$$

- Step: - Given $G(s)$
  - Replace $s = \frac{1-\lambda}{\lambda} \Rightarrow \widetilde{G}(\lambda) = \frac{n(\lambda)}{m(\lambda)}$
  - Find quotient & remainder: q_1 & r_1

  $$n = m.q_1 + r_1$$
  $$m = r_1 q_2 + r_2$$
  $$r_1 = r_2 q_3 + r_3$$
  $$\dots$$

$\Rightarrow$ Choose controller $K = \frac{X(s)}{Y(s)}$ $\left(M(s)Y(s) + N(s)U(s) = 1 \right)$

a controller that guarantee internal stability

3, Nyquist criterion for MIMO systems

$$\det\left(I + L(s)\right) = \Gamma(s)$$

- Plot Nyquist graph of
- The closed-loop control circuit is assyptotically stable if the plot contour makes $\Delta\varphi(\Gamma) = 2m_0\pi$ counter-clockwise encirclements of the point $(0,0)$ & does not pass through the point.

  $m_0 = \#$ No of open-loop RHP-poles

4, Small gain theorem (basically closed-loop stability condition)

- Spectral radius: $\rho(L(j\omega)) = \max_i |\lambda_i(L(j\omega))|$ maximum eig value

$\Rightarrow$ Spectral radius stability condition: Closed loop system is stable if $\rho(L(j\omega)) < 1\ \forall\omega$ + L is stable

In most case, this is too much (conservative), since we only need for $\omega = -\pi \pm k2\pi$

- Small gain theorem: L is stable, closed ________ $\|L(j\omega)\| < 1\ \forall\omega$

  any induced norm (not $\mathcal{H}_2$), also applied for nonlinear systems

  independent of sign for feedback, more conservative than spectral rad. stab.

Performance Limitations

4, SISO systems

1, Phase-gain relation

- As feedback gain K increase $\to \infty$, closed loop poles go ~~from~~ to open-loop zeros
$\Rightarrow$ If there is RHP zeros, implies high-gain instability

2) Bode's sensitivity integral

- Open loop tf $L(s)$ is rational & at least 2 more poles than zeros
$L(s)$ has N_p RHP poles at p_i

$$\Rightarrow \int_0^{\infty} \ln |S(j\omega)| \, d\omega = \pi \sum_{i=1}^{N_p} Re(p_i)$$

- If plant is stable $\Rightarrow$ no RHP poles $(=0)$

$$\Rightarrow \int_0^{\infty} \ln |S(j\omega)| \, d\omega = \int_0^{\infty} \ln \left| \frac{1}{1+G(j\omega)K(j\omega)} \right| d\omega = 0$$

- In practice, we try to limit to a certain freq range

$$\int_0^{\omega_c} \ln |S(j\omega)| \, d\omega = 0$$

3) Poissen's sensitivity integral

- $G(s)$ has N_z RHP zeros z_j $\qquad N_p - N_z \geq 1$

  N_p RHP poles p_i

$$\int_0^{\infty} \ln |S(j\omega)|\, w(\omega, z_k)\, d\omega = \pi \ln \prod_{i=1}^{N_p} \left| \frac{p_i + z_k}{\bar{p}_i - z_k} \right|$$

- Real zero: $w(\omega, z_k) = \dfrac{2z_k}{z_k^2 + \omega^2}$

  Complex zero: $w(\omega, z_k) = \dfrac{x}{x^2 + (y-\omega)^2} + \dfrac{x}{x^2 + (y+\omega)^2}$

- Weighting func: $d\theta(\omega, z_k) = \dfrac{d}{d\omega} \arctan\left[\dfrac{\omega - y}{x}\right] d\omega$

4) Bounds on peaks.

- S is primarily limited by RHP zeros $\qquad$ T is primarily limited by RHP poles

- Stability: no RHP poles, $\qquad$ no RHP zeros & delay

  $\| w_S S \|_\infty \geq |w_S(z)|$ $\qquad$ $\| w_T T \|_\infty \geq |w_T(p)|$

- For closed loop stability,

  for each RHP zeros of $G(s)$ $\qquad$ for RHP poles of $G(s)$

  $\| w_S S \|_\infty \geq |w_S(z)| \cdot \prod_{i=1}^{N_p} \left| \dfrac{z + p_i}{z - p_i} \right|$ $\qquad$ $\| w_T T \|_\infty \geq |w_T(p)| \cdot \prod_{j=1}^{N_z} \dfrac{|z_j + p|}{|z_j - p|} |e^{p\theta}|$

- With out weight w_S $\qquad$ w_T

$$\Rightarrow \quad \|S\|_\infty = M_S \geq \prod_{i=1}^{N_p} \frac{|z + p_i|}{|z - p_i|} \qquad \|T\|_\infty \geq M_T \geq \prod_{j=1}^{N_z} \frac{|z_j + p|}{|z_j - p|} \cdot |e^{p\theta}|$$

ACS

Uncertainty & Robustness

↳ RS: Robust stability

RP: Robust performance

Π – set of possible pertubed models

$G(s) \in \Pi$ – a nominal plant model (no uncertainty)

$G_p(s) \in \Pi$ – a particular pertubed plant model

1) Uncertainty

- Parametric uncertainty – model structure is known, some params are unknown
- Dynamic uncertainty – missing or neglected dynamic – frequency dependent

$\bar{\sigma}(\Delta_i(j\omega)) \leq 1 \quad \forall \omega \ \forall i, \quad \|\Delta\| \leq 1$

+) Additive Uncertainty

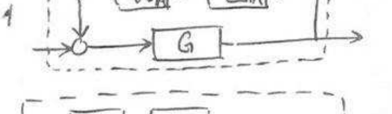
$\Pi_A: \ G_p(s) = G(s) + w_A(s) . \Delta_A(s)$

– Additive Inverse Uncertainty

$\Pi_{iA}: \ G_p(s) = G(s)\left[I - w_{iA}(s) . \Delta_{iA}(s)\, G(s)\right]^{-1}$

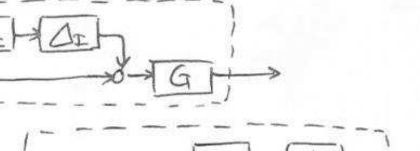
– Multiplicative Input Uncertainty

$\Pi_I: \ G_p(s) = G(s)\left(I + w_I(s) . \Delta_I(s)\right)$



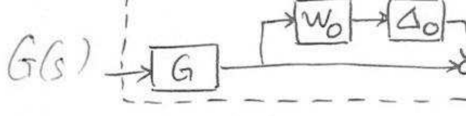
– Multiplicative Output Uncertainty

$\Pi_O: \ G_p(s) = \cancel{G(s}\left(I + w_O(s) . \Delta_O(s)\right) . G(s)$



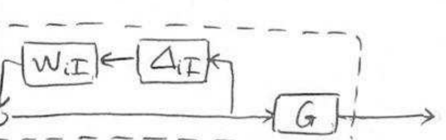
– Multiplicative inverse input uncertainty

$\Pi_{iI}: \ G_p(s) = G(s)\left[I - w_{iI}(s)\Delta_{iI}(s)\right]^{-1}$



– Multiplicative inverse output uncertainty

$\Pi_{iO}: \ G_p(s) = \left[I - w_{iO}(s)\Delta_{iO}(s)\right]^{-1} G(s)$



- Scalar uncertainty weight:

$$\Pi_O : \; G_p(s) = \left(I + w_O(s).\,\Delta_O(s)\right).\,G(s)$$

where $$l_O(\omega) = \max_{G_p(j\omega)\in\Pi} \bar{\sigma}\left[\left(G_p(j\omega) - G(j\omega)\right) G^{-1}(j\omega)\right]$$

with $|w_O(j\omega)| \geq l_O(j\omega) \;\; \forall\omega$

or $$G_p(s) = G(s)\left[I + w_I(s).\,\Delta_I(s)\right], \; \|\Delta_I(s)\|_\infty \leq 1$$

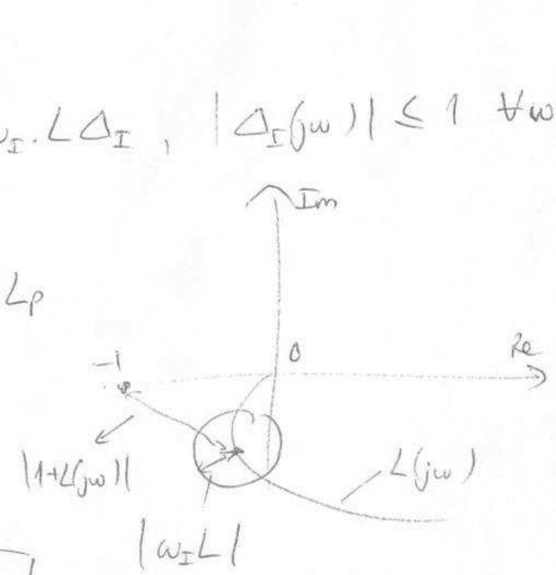
where $$l_I(\omega) = \max_{G_p(j\omega)\in\Pi} \bar{\sigma}\left[G^{-1}(j\omega)\left(G_p(j\omega) - G(j\omega)\right)\right], \; |w_I(j\omega)| \geq l_I(j\omega) \; \forall\omega$$

$\Rightarrow$ Lumping uncertainty into a single perturbation

2) SISO Robust Stability

+) For multiplicative uncertainty:

$$L_p(s) = GK\left(1 + w_I\Delta_I\right) = L + w_I.\,L\Delta_I, \quad |\Delta_I(j\omega)| \leq 1 \;\; \forall\omega$$



RS $\Leftrightarrow$ System stable with $\forall L_p$

$\Leftrightarrow$ L_p should not encircle -1 $\forall L_p$

$\Leftrightarrow |w_I L| < |1+L| \;\; \forall\omega$

$\Leftrightarrow \left|\dfrac{w_I L}{1+L}\right| < 1 \;\; \forall\omega$

$\Leftrightarrow |w_I T| < 1 \;\; \forall\omega \Leftrightarrow \boxed{\|w_I T\|_\infty < 1}$

+) For inverse multiplicative uncertainty $\quad |w_{iI} S| < 1 \;\; \forall\omega$

⊗ Nominal performance:

- Weighting func: $w_p(s)$

$\Leftrightarrow |w_p(s).\,S(s)| < 1 \;\; \forall\omega \;\; \Leftrightarrow \;\; |w_p(s)| < |1 + L(s)| \;\; \forall\omega$

3) Robust performance:

$$RP \overset{def}{\Leftrightarrow} |w_p S_p| < 1 \quad \forall S_p \; \forall \omega$$
$$\Leftrightarrow |w_p| < |1 + L_p| \; \forall L_p \; \forall \omega$$

And if we consider multiplicative uncertainty $\Rightarrow L_p = L + w_I L \Delta_I$

$$\Leftrightarrow RP \Leftrightarrow |w_p| + |w_I L| < |1+L| \quad \forall \omega$$
$$\Leftrightarrow \max_\omega \left(|w_p S| + |w_I T| \right) < 1$$

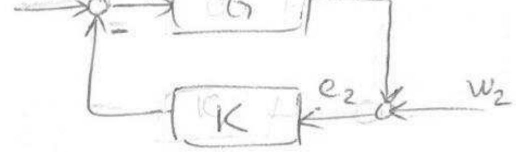
$$\text{also} \Leftrightarrow \max_{S_p} |w_p S_p| < 1 \quad \forall \omega$$

⊗ Comparison:

$$NP \Leftrightarrow |w_p S| < 1 \quad \forall \omega$$
$$RS \Leftrightarrow |w_I T| < 1 \quad \forall \omega$$
$$RP \Leftrightarrow |w_p S| + |w_I T| < 1 \quad \forall \omega$$

2g) Robust Stability & Performance

1) MIMO Robust Stability



+) Form control loop
for considering internal stability

$$\begin{bmatrix} I & K \\ -G & I \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$\Rightarrow$ Sensitivity & complementary sensitivity funcs regarding I/O direction

$$S_o = (I + GK)^{-1}, \qquad T_o = I - S_o$$
$$S_i = (I + KG)^{-1}, \qquad T_i = I - S_i$$

$\Rightarrow$ Closed loop is well-posed and internally stable if $\begin{bmatrix} I & K \\ -G & I \end{bmatrix}^{-1} = \ldots \in RH_\infty$ ↯

+) Small Gain Theorem extension for pertubed system:

Assume $M(s) \in \mathcal{RH}_\infty$ and $\gamma > 0$. The system is well-posed and internally stable for $\forall\, \Delta(s) \in \mathcal{RH}_\infty$ if

$\|\Delta(s)\|_\infty \leq \frac{1}{\gamma}$ if $\|M(s)\|_\infty < \gamma$

or $\|\Delta(s)\|_\infty < \frac{1}{\gamma}$ if $\|M(s)\|_\infty \leq \gamma$

$\Rightarrow$ Using Small gain theorem extension, we then can derive Robust stability condition for each type of dynamic uncertainties and also coprime factorisation:

Example:

- $G_p = (I + W_1 \Delta W_2) G$ (Multiplicative uncertainty) $\Rightarrow$ RS: $\|W_2 T_0 W_1\|_\infty \leq 1$
- $G_p = G + W_1 \Delta W_2$ (Additive uncertainty) $\Rightarrow$ RS: $\|W_2 K S_0 W_1\|_\infty \leq 1$
- $G(s) = M_l^{-1}(s) . N_l(s)$

  $G_p(s) = (M_l(s) + \Delta M(s))^{-1} (N_l(s) + \Delta_N(s))$

  $\Rightarrow$ RS: $\| \Delta_M(s) . \Delta_N(s) \| \leq \varepsilon$

...

2) Robust performance: MIMO

⊗ Keep the t/f from $\tilde{d}$ to e small:

$$\sup_{\|\tilde{d}\|_2 \leq 1} \|e\|_2 \leq \epsilon$$

+) TF: $T_{e\tilde{d}} = W_e (I + G_p K)^{-1} W_d$

$\Rightarrow$ RP: $\underset{\text{conditions}}{RS}$ & $\|T_{e\tilde{d}}\|_\infty \leq 1$

Linear Quadratic Regulator / Gaussian
LQR / LQG

<10>

1) Formulation:

$$J = \int_0^T [x^T Q x(t) + u^T(t).R.u(t)]\,dt + x^T(T).Q_T\,x(T)$$

$x(0) = x_0$

$\frac{dx}{dt} = Ax + Bu$

$y = Cx + Du$

2) LQR: $T \to \infty$ $\lim_{T\to\infty} x(T) = 0$

$$\Rightarrow J = \int_0^\infty [x^T(t).Q.x(t) + u^T(t).R.u(t)]\,dt$$

Q is non-negative definite (semi-definite)

R is positive definite

+) Optimal control:

$u(t) = -K.x(t)$, $K = R^{-1}.B^T.P$

With P is the unique positive definite solution to the matrix Algebraic Riccati Equation

$$A^TP + PA - PBR^{-1}B^TP + Q = 0$$

+) LQR		stable systems	unstable systems
	GM	∞	6 dB
	PM	$\geq 60°$	$\geq 60°$

3) <u>LQG:</u> is LQR with Gaussian noise

combine optimal state estimation & optimal state feedback

+) $\dot{x} = Ax + Bu + \underline{v}$

$y = Cx + \underline{w}$

+) $v(t)$ & $w(t)$ as white Gaussian noise, independent

$\mathbb{E}[v(t)] = 0, \quad \mathbb{E}[w(t)] = 0, \quad \mathbb{E}[v^T(t) w(t)] = 0$

$\mathbb{E}[v(t) v^T(t)] = \underline{V}, \quad \mathbb{E}[w(t) w^T(t)] = \underline{W}$

+) <u>Assume</u> also $A, C, V^{1/2}$ is observable (need to check)

$A, B, Q^{1/2}$ is stabilisable

$\Rightarrow J_e = \mathbb{E}\left[\{x(t) - \hat{x}\}^T . \{x(t) - \hat{x}\}\right] = 0$ Kalman filter problem

$\dot{\hat{x}} = A\hat{x} + Bu + K_e(y - C\hat{x}), \quad \hat{x}(0) = \hat{x}_0$

$\Rightarrow$ Optimal $K_e = P_e . C^T W^{-1}$

With P_e as positive definite solution of the ARE

$P_e A^T + A.P_e - P_e C^T W^{-1} C P_e + V = 0$

+) Usually W & V as design parameters

E.g. $W = B.B^T, \quad V = m.I$

1) $\mathcal{H}_2$ optimal control:

- Find a stabilizing controller K which minimizes

$$\|F(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\, F(j\omega)^T \, d\omega} \quad , \quad F = F_l(P, K)$$

- LQG : a special case of $\mathcal{H}_2$ optimal controller

2) $\mathcal{H}_\infty$ optimal control:

- Find all stabilizing controllers K which minimize:

$$\| F_l(P, K)\|_\infty = \max_{\omega} \bar{\sigma}\left(F_l(P, K)(j\omega)\right)$$

→ $\mathcal{H}_\infty$ sub optimal control, $\| F_l(P, K)\|_\infty < \gamma$

- $\mathcal{H}_2$ norm is not an induced norm

- Induced norm will satisfy: multiplicative properties

$$\|AB\| \leq \|A\| \cdot \|B\|$$

which later on $\|z\| \leq \|P\| \cdot \|w\|$

*) μ as structured singular value

μ synthesis $=$ $\mathcal{H}_\infty$ - synthesis $+$ μ - analysis

⊛ Linear Fractional Transformation:

- Consider P of dimension $(n_1+n_2)\times(m_1+m_2)$ and partition as:

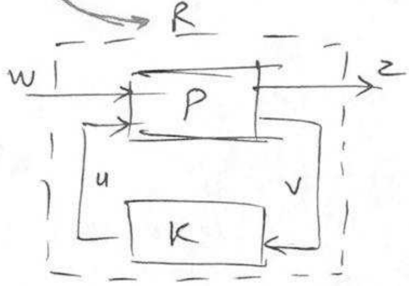
$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

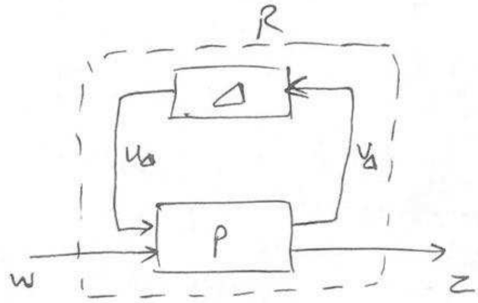
Let $\Delta \in \mathbb{R}^{m_1\times n_1}$ and $K \in \mathbb{C}^{m_2\times n_2}$

$\Rightarrow$ Lower and upper LFT:

$$F_\ell(P,K) \triangleq P_{11} + P_{12}K(I-P_{22}K)^{-1}P_{21}$$

$$F_u(P,\Delta) \triangleq P_{22} + P_{21}\Delta(I-P_{11}K)^{-1}P_{12}$$





$z = P_{11}w + P_{12}u$

$v = P_{21}w + P_{22}u$

$u = Kv$

$\Rightarrow z = Rw$

$= \left[P_{11} + P_{12}K(I-P_{22}K)^{-1}P_{21}\right]w$

+) Relation between F_ℓ and F_u

$$R = F_\ell(M,K) \quad \Rightarrow \quad F_u(\tilde{M},K) = F_\ell(M,K)$$

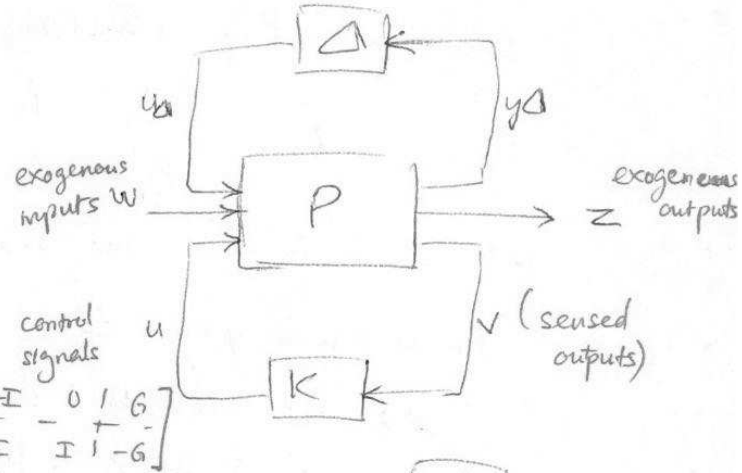
with $\tilde{M} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \cdot M \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$

+) Inverse of LFT: assumption all relevant inverses exist

$$\left(F_\ell(M,K)\right)^{-1} = F_\ell(\tilde{M},K)$$

where $\tilde{M} = \begin{bmatrix} M_{11}^{-1} & -M_{11}^{-1}M_{12} \\ M_{21}M_{11}^{-1} & M_{22}-M_{21}M_{11}^{-1}M_{12} \end{bmatrix}$

⊗ The 3 structure formulation:



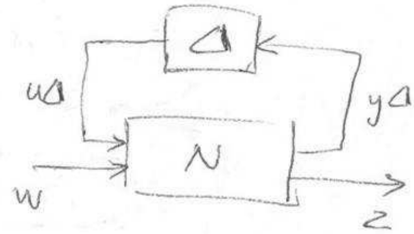
1) "Pull out" the controller K

P is the generalized plant

⇒ useful ~~for~~ controller synthesis

$$w = \begin{bmatrix} d \\ r \\ n \end{bmatrix}; \quad z = e = y - r; \quad P = \left[\begin{array}{ccc|c} I & -I & 0 & G \\ \hline -I & I & I & -G \end{array}\right]$$

2) $N\Delta$ structure: if the controller is given



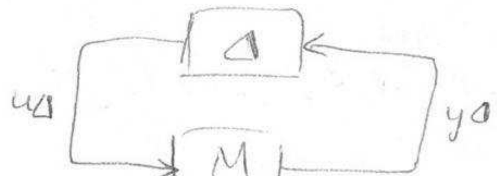
⇒ useful when we want to analyze the uncertain system

- $\boxed{N = F_\ell(P, K)} = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}$ (N as lower LFT of P & K)
- The uncertain closed loop transfer function from w to z:

$$z = F(\Delta)\, w$$

$$\boxed{F = F_u(N, \Delta)} = N_{22} + N_{21} \Delta (I - N_{11} \Delta)^{-1} N_{12} \quad (F \text{ as upper LFT of } N \,\&\, \Delta)$$

3) $M\Delta$ structure:



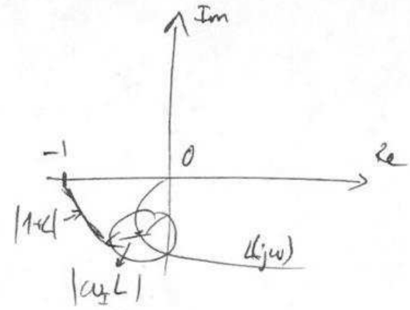
⇒ when we only need to analyze robust stability of F

$$\boxed{M = N_{11}}$$

↑ All of these relate to Linear Fractional Transformations (LFT)

SISO: + RS $\Leftrightarrow$ $\|w_I T\|_\infty < 1$

$$\left(\left\| \frac{w_I L}{1+L} \right\|_\infty < 1 \quad \forall \omega \right)$$



(Multiplicative uncertainty)

+ NP: $|w_p(s)| < |1 + L(s)| \quad \forall \omega \quad \Leftrightarrow \quad |w_p S| < 1$

+ RP: $|w_p S| + |w_I T| < 1 \quad \forall \omega$

For: $L_p(s) = GK(1 + w_I \Delta_I) = L + w_I L \Delta_I$ (multiplicative uncertainty
only

$\underline{w_p}$ is a upper bound for the performance of S

$$|w_p S_p| < 1 \quad \forall S_p \; \forall \omega$$
$$\Leftrightarrow |w_p| < |1 + L_p| \quad \forall L_p \; \forall \omega$$

MIMO: + RS: first: LFT $\Rightarrow P \Rightarrow N \Rightarrow M$

RS $\Leftrightarrow$ $\|M(s)\|_\infty < \gamma$ with $M(s) \in RH_\infty$ and $\gamma > 0$

Use small gain theorem extension for each type of dynamic uncertainty

+ RP = RS + $\|T_{e\tilde{d}}\|_\infty \leq 1$