

Lectures

Ng Huu Duc

1- 8, Introduction

9- 10, Variational formulation - Boundary Conditions
- Mechanical & Mathematical Classification

11- 13, Mathematical Tools - Functions, Functionals & Operators ⊗
- Fundamental Equations

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-

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18- 19, Functionals Associated to Integral Forms - Construction, Linear & Quadratic Functionals
- Reduction to First order systems
- Mixed and Complementary functionals

⊗ 20- 25, Discretization of Integral Forms - Nodal approximation ⊗
- Non-nodal approximation
- Collocation by points
- Collocation by Sub domains
- Galerkin's method
- Least square method
- Ritz method

25, Numerical integration

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Numerical Analysis Software

Exercise

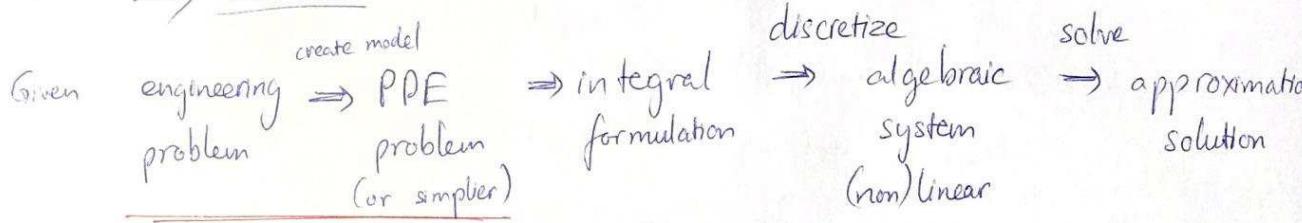
(MATLAB focused)

- 1, Intro to MATLAB
- 2, GUI - Newton-Raphson method
- 3, Functions
- 4, Solving PDE, ODE
- 5, PDE Tool box
- 6, More PDE : elliptic, parabolic, hyperbolic
- 7, 2D Flow problem, divergence
- 8, PDE - plate under uniform pressure load
- 9, Lagrange multiplier method
- 10, Nodal approximation
- 11, Collocation method : collocation by points
- 12, Galerkin method
Least square method
- 13, Ritz method
- 14, Gaussian Quadrature

Chapter 2. Variational Formulation

L8)

2.1, Overview:



⊗ this chapter

2.2, Classification of engineering problems:

a) Discrete vs continuous system: no. dof

- Discrete \Rightarrow leads to finite set of algebraic equations finite dof
In some cases, the system is linear.

Ex:

- Continuous system \Rightarrow infinite no of DoF

Ex: bending of a beam 

b) Mechanical meaning: $\begin{cases} \text{equilibrium problems (statics)} \\ \text{- eigenvalue problems} \\ \text{propagation problems (time-dependent)} \end{cases}$

c) Mathematical point of view: elliptic PDE

$\begin{cases} \text{parabolic PDE} \\ \text{hyperbolic PDE} \end{cases}$



Notation:

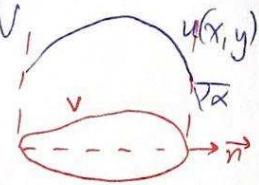
$$\mathcal{L}(u) + f = 0 \quad \text{inside a domain } V$$

$$C(u) = f_s \quad \text{on the boundary of } V \quad f_s = \partial V$$

L9, 2.3) Discrete vs. continuous systems

- When the model involves differential equations \Rightarrow have to satisfy certain boundary / initial conditions

1st \nrightarrow Dirichlet boundary condition: specify the values of $u(x,y)$ on S
 unknown func $u(x,y)$ is a scalar field on V ,
 $S = \partial V$ (boundary of V)
 $u(x,y) = u_s(x,y)$ for $(x,y) \in S$



2nd \nrightarrow Neumann boundary condition: specify the slope $u(x_n, y_n)$ when $\overset{\circ}{\rightarrow}$ boundary
 $\tan \alpha = \frac{\partial u}{\partial n}(x,y) = \nabla u(x,y) \cdot n(x,y) = f_s(x,y)$ for $(x,y) \in S$

3rd \nrightarrow Robin / Newton boundary condition: weighted combination of the first two
 $\alpha u(x,y) + \frac{\partial u}{\partial n}(x,y) = f_s(x,y)$ for $(x,y) \in S$

- Defined on whole boundary S or part of it

Boundary \Rightarrow problem is | well-posed or ill-posed
 solution exists or not

\nrightarrow If specify both Dirichlet and Neumann / Robin conditions on the same subset $S_c \subset S$ and none on $S \setminus S_c$
 \Rightarrow Cauchy boundary condition

\nrightarrow "hidden" b.c. which satisfied by default \Rightarrow natural bound. cond.

Ex. $u''(L) = 0$
 $u'''(L) = 0$

L9, 2.4) Mechanical classification:

\nrightarrow Equilibrium problems: involve ODE or PDE which do not depend on time, $u = u(x,y,z) \dots$

+ Eigenvalue Problems

$$Au = \lambda u$$

+ Propagation Problems

→ modeled as Initial Value problem (IUP)

or Boundary Val. (IBVP)

- Lagrange description approach based on the coordinates in an initial configuration
- Euler approach the actual

2.5) Mathematical classification:

Consider 2nd order PDE:

$$a(x,y) \frac{\partial^2 u}{\partial x^2} + b(x,y) \frac{\partial^2 u}{\partial x \partial y} + c(x,y) \frac{\partial^2 u}{\partial y^2} = f_v(x,y)$$

→ matrix of coefficients $\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$ → eigenvalues d & e

→ $\left\{ \begin{array}{l} d, e \neq 0, \text{ same sign} \Rightarrow \text{elliptic} \\ \quad , \text{ opposite sign} \Rightarrow \text{hyperbolic} \\ d=0 \text{ or } e=0 \quad \Rightarrow \text{parabolic} \end{array} \right. \right\} \begin{array}{l} \text{depend on } x, y \\ (\text{local, not global}) \end{array}$

Chapter 3Mathematical Tools

4)

3.1 Functions, Functionals and Operators:

+) A function: mapping from subset of $\mathbb{R}^n \rightarrow$ subset of \mathbb{R}^p

$$f: V \subseteq \mathbb{R}^n \rightarrow W \subseteq \mathbb{R}^p$$

+) An operator: mapping from (a subset of) a func space \Rightarrow (a subset of) function space

$$L: f(\cdot) \in V \rightarrow (L(f))(\cdot) \in W$$

ex: derivative operator
Laplace, curl, div ..

+) Some func space: given a domain $\Omega \subseteq \mathbb{R}^n$

- $C^0(\Omega)$: space of cont. real. func. on Ω

$C^n(\Omega)$: _____, which cont. differentiable to order n on Ω

$C^\infty(\Omega)$: _____, order ∞ _____

$L^1(\Omega)$: space of integrable funcs on Ω

$L^2(\Omega)$: _____ square _____

- For complex func space: $C^n(\Omega; \mathbb{C})$

+) Functional: mapping (a subset of) a func space $\Rightarrow \mathbb{R}$ (or \mathbb{C})

$$F: f(\cdot) \in V \rightarrow F(f) \in W \subseteq \mathbb{R}$$

(or $W \subseteq \mathbb{C}$) ex: Lebesgue integral

$$\int_{-1}^1 f(x) dx$$

+) Properties of operators & functionals

a) linear: $L(\alpha u + \beta v) = \alpha \cdot L(u) + \beta \cdot L(v) \quad \forall u, v \in \text{domain of } L$
 $\alpha, \beta \in \mathbb{R}$

b) Homogeneous operators / functionals:

$$L(\alpha u) = \alpha^k L(u) \text{ and } L(0) = 0 \quad \forall u \in \text{domain of } L$$

$\alpha \in \mathbb{R}$
 L is a homogeneous operator of degree k

c) Symmetric operators: (self-adjoint)

- Inner product: $\langle u, v \rangle_w = \int w(x) \cdot u(x) \cdot v(x) dx$

with $w(x) > 0$ as weight function

= An operator L is symmetric with respect to a scalar product $\langle \cdot, \cdot \rangle$ if
 $\langle L(u), v \rangle = \langle u, L(v) \rangle$

Ex. Laplace operator Δ : $\langle \Delta u, v \rangle = \langle u, \Delta v \rangle$

d) Positive and positive definite operators:

$\langle L(u), u \rangle \geq 0 \quad \forall u(\cdot) \begin{cases} \in \text{domain of } L \\ \text{satisfy boundary conditions} \end{cases}$

Ex. Laplace operator $L = -\Delta$ is positive definite
with geometrical interpretation: curvature

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

(*) Some Fundamental equations involving Laplace operator

a) Equilibrium equations : $\Delta u = 0$: Laplace's equation

b) Eigenvalue problems

$\Delta u + \alpha^2 u + f_r = 0$ is the inhomogeneous Helmholtz equation

c) Transient problems

$$\frac{\partial^n u}{\partial t^n} - \Delta u + f_v = 0$$

- Mean Value Theorem: A function u is harmonic $\Delta u = 0$ everywhere

iff $u(x) = \frac{1}{B(x, r)} \int_{B(x, r)} u(y) dy = \frac{1}{|B(x, r)|} \int_{\partial B(x, r)} u(y) dS$

$$B(x, r) = \{y \in \mathbb{R}^n \mid |y-x| < r\}$$

$\partial B(x, r)$ is boundary of $B(x, r)$

L₂) 3.2, Fundamental Equations

Q) Divergence operator.

$$\operatorname{div} u = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

$$\operatorname{div} u = \nabla \cdot u$$

L₃) - Lame' - Navier equations

$$\mu \Delta u + (\lambda + \mu) \nabla(\operatorname{div} u) + f_v = 0$$

* Curl operator

$$\begin{aligned}\nabla \times u &= \begin{bmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{bmatrix} \\ &= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) e_1 + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) e_2 + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) e_3\end{aligned}$$

Chapter 9. The Weighted Residual Method

4.1) Introduction:

- WRM helps us find an approximation solution for a well-posed problem.

$$L(u) + f_v = 0 \text{ in } V \quad \Rightarrow \hat{u}$$

$$C(u) = f_s \text{ on } S = \partial V$$

- Residual operator $R(u) = L(u) + f_v$

Also could $R_B(u) = C(\hat{u}) - f_s$

- Assume $\hat{u} = \hat{u}(x, a_1, \dots, a_n)$

We will try to choose a_1, \dots, a_n such that $R(\hat{u}) \approx 0$

- choose a set of linear independent weighting functions: $\psi_1(x), \dots, \psi_n(x)$

We want:

$$W_1 = \int_V \psi_1(x) \cdot R(\hat{u}(x)) \cdot dV = 0$$

$$W_2 = \int_V \psi_2(x) \cdot R(\hat{u}(x)) \cdot dV = 0$$

$$\dots \quad W_n = \int_V \psi_n(x) \cdot R(\hat{u}(x)) \cdot dV = 0$$

} solve a system
of equation
to find a_1, \dots, a_n

- du Bois-Reymond Lemma:

If a continuous function f on an open set $V \subseteq \mathbb{R}^n$ satisfies the equation

$$\int_V f(x) \cdot g(x) \cdot dV = 0$$

for all compactly supported smooth functions g on V , then f is the 0 function

Chapter 5. Calculus of Variations

45, 5.1, Introduction

- To solve minimization problems for functionals by extending the notion of derivation.

5.2) Definition & Properties

④ For function: $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$, $\Delta x = x - x_0$

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0) \cdot \Delta x$$

$$\Delta f(x) = f'(x_0) \cdot \Delta x$$

④ Neighborhood of a function: $F = u(x)$

$$u(x) = u_0(x) + \alpha \cdot v(x), \quad x \in [a, b]$$

$$v(a) = v(b) = 0 \Rightarrow (\delta u)(a) = (\delta u)(b) = 0$$

$$\text{Variation of } u: \alpha \cdot v(x) = (u - u_0)(x) = \delta u(x) = \delta u \in (-\varepsilon; \varepsilon) \quad \forall x \in [a, b]$$

⇒ Weak variation: "zero" norm $\|u - u_0\|_0 = \sup_{x \in [a, b]} |u(x) - u_0(x)|$

If $\|u - u_0\|_0$ is small \Rightarrow weak variation

⇒ Strong variation: "one" norm $\|u - u_0\|_1 = \sup_{x \in [a, b]} |u(x) - u_0(x)| + \sup_{x \in [a, b]} |u'(x) - u'_0(x)|$

If $\|u - u_0\|_1$ is small \Rightarrow strong variation

⇒ Properties: $(\delta u)'(x) = \delta(u')(x) \Rightarrow$ variation of der. is equal der. of variation

$\delta \left(\int u(x) dx \right) = \int \delta u(x) dx \Rightarrow$ integral of variation is equal variation of integral

Given functional $F = F(x, u, u')$

We approximate around α

$$F(\alpha) = F(x, u + \alpha v, u' + \alpha v'), \quad F(0) = F(x, u, u')$$

$$\Delta F = F(\alpha) - F(0)$$

$$\Rightarrow \delta F = \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u' \quad \text{First variation of } F$$

$$\delta^2 F = \frac{\partial^2 F}{\partial u^2} (\delta u)^2 + 2 \frac{\partial^2 F}{\partial u \partial u'} \delta u \delta u' + \frac{\partial^2 F}{\partial u'^2} (\delta u')^2 \quad \text{2nd variation of } F \\ (\text{twice the 2nd term})$$

$$\text{from } \Delta F = \frac{dF}{dx}(0). \alpha + \frac{1}{2!} \frac{d^2 F}{d\alpha^2}(0). \alpha^2 + O(\alpha^3) \text{ Taylor expansion}$$

$$\delta^2 F = 2 \left[\frac{1}{2!} \frac{d^2 F}{d\alpha^2}(0). \alpha^2 \right]$$

Given functional $F(u)$ in integral form

$$I(u) = \int_a^b F(x, u(x), u'(x)) dx$$

$$\Rightarrow \delta I(u) = \int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx} \left(\frac{\partial F}{\partial u'} \right) \right] \delta u dx$$

Say that $I(u)$ is stationary at $u_0(x)$ if $\delta I(u_0) = 0$

$\Rightarrow u_0(x)$ is extremal of functional I

$$\delta I(u) = 0 \Leftrightarrow \frac{\partial F}{\partial u} - \frac{d}{dx} \left(\frac{\partial F}{\partial u'} \right) = 0 \quad (\text{Euler-Lagrange equation})$$

Given functional $I(u)$.

$$I(u) = \int_{\Omega} \underline{\dots} dx$$

$$\Rightarrow \delta I = \delta \left(\int_{\Omega} \underline{\dots} dx \right) = \int_{\Omega} [\dots \delta u \dots \delta u' \dots] dx$$

$$= \int_{\Omega} [ODE] \delta u \cdot dx + [BCs]$$

From ODE and BCs:

- Integral form $W = \int_{\Omega} \psi(x) \cdot R(u)(x) dx$

- Set weighting function $\psi(x) = \delta u$

$$\Rightarrow W = \int_{\Omega} \delta u \cdot R(u) dx = \delta \left[\int_{\Omega} \dots dx \right] + [BCs]$$

$$\Rightarrow W = \delta(I(u)) + [BCs]$$

4.7) 5.9) Constrained extremal problems

Chapter 7. Discretization of Integral Forms

L19-25)

7.1 Introduction:

- Given PDE problem:

$$L(u) + f_v = 0 \quad \text{in } V$$

$$\ell(u) = f_s \quad \text{on } S = \partial V$$

We have already built:

- Residual: $R = L(\hat{u}) + f_v$
- Integral form: $W = \int_V \psi R(\hat{u}) dV$ (+ weak integral forms $W^*, W^{\#}$)
- Potential π such $W = \delta\pi$ by choosing $\psi = \delta\hat{u}$
- ⊗ Now, how to choose approximation choices for \hat{u}
weighting functions ψ

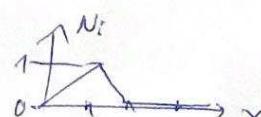
7.2 Nodal approximation

- Given n nodes (points): $x_1 \dots x_n$ in V and their nodal values $u(x_1) \dots u(x_n)$
- ⇒ We want $\hat{u}(x)$ to take exact nodal values at corresponding n nodes

⇒ Shape func: $N_i(x_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

$$\hat{u}(x) = u_1 \cdot N_1(x) + \dots + u_n \cdot N_n(x)$$

- Continuous, but non smooth N_i :



- For smooth N_i :

$$N_i(x) = \alpha(x-x_2)(x-x_3)\dots(x-x_n), \quad \alpha \text{ defined from } N_i(x_i)=1$$

However, this leads to Runge phenomenon ...

- ⇒ Use Chebychev nodes (not equally spaced)

- Bernstein approximation
- Hermite polynomials

7.3) Non-Nodal approximation: (Modal approximation)

$$\hat{u}(x, a_1, \dots, a_n) = a_1 \cdot P_1(x) + \dots + a_n \cdot P_n(x) = \langle P_i(x) \rangle \{a_i\}$$

$P_i(x)$:- basis / trial functions
 a_i :- parameters

- If BCs are homogeneous $f_s = 0$
 \Rightarrow simply ask $C(P_i)(x) = 0$ on $S = \partial V$
- If BCs are non-homogeneous $f_s \neq 0$
 $\Rightarrow \hat{u}(x) = u_0(x) + \langle P_i(x) \rangle \{a_i\}$
such $\boxed{C(u_0)(x) = f_s \text{ on } S = \partial V}$

+/Basis functions:

- Polynomial:

ex: $V = [-1, 1] \times [-1, 1] \subseteq \mathbb{R}^2 \Rightarrow P_1 = (x^2 - 1)(y^2 - 1)$
 $u = 0 \text{ on } S = \partial V \quad P_2 = \underline{\hspace{2cm}} (x^2 + y^2)$
 $P_3 = \underline{\hspace{2cm}} (x^4 + y^4)$

- Trigonometric basis funs.

$$T_k(x, y) = \cos\left(\frac{(2k-1)\pi}{2}x\right) \cdot \cos\left(\frac{(2k-1)\pi}{2}y\right)$$

ex $V = [0, 2] \times [0, 2] \Rightarrow T_k(x, y) = \sin\left(\frac{k\pi}{2}x\right) \cdot \sin\left(\frac{k\pi}{2}y\right)$

7.4) Collocation by Points: n points / nodes $x_i \in V$

- Set $R(\hat{u})(x_i) = 0$
- \Rightarrow System of n equations \Rightarrow solve $a_1 \dots a_n$

7.5) Collocation by Subdomains:

- Split V into n subdomains $V_1 \dots V_n$:

$$V = V_1 \cup V_2 \cup \dots \cup V_n$$

- Ask $\int_{V_i} R(\hat{u}) dV = 0$ for $i = 1 \dots n$
Also leads to system of n equations to solve $a_1 \dots a_n$

7.6) Galerkin's Method:

- Choose $\psi_i = p_i$
then ask $\int_V \psi_i(x) \cdot R(\hat{u})(x) dV = 0$ for $i = 1 \dots n$
- Again \Rightarrow system of n equations \Rightarrow find $a_1 \dots a_n$

7.7) The Least Squares Method

- Try to minimize $R^2(\hat{u}) \rightarrow 0$ on V

$$\Leftrightarrow \text{minimize } \pi = \pi(a_1 \dots a_n) = \int_V R^2(\hat{u}) dV$$

$$\Leftrightarrow \frac{\partial \pi}{\partial a_i} = 0 \text{ for } i = 1 \dots n$$

$$\frac{\partial \pi}{\partial a_i} = \frac{\partial}{\partial a_i} \left[\int_V R^2(\hat{u}) dV \right] = 2 \int_V \frac{\partial R(\hat{u})}{\partial a_i} R(\hat{u}) dV = 0$$

Thus ask for $\int_V \frac{\partial R}{\partial a_i} R(\hat{u}) dV = 0$ for $i = 1 \dots n$

7.8, The Ritz method

Ask for $\boxed{\frac{\partial \pi}{\partial a_i} = 0} \quad i = 1 \dots n$

which is exactly $\int_V P_i R(u) dV = 0$ from Galerkin's method


Ng Hieu Due

NM111E exam prep

⊗ Set CALC in Radian mode

- + Type of boundary conditions: Chapter 2
- Dirichlet BC
 - Neumann BC
 - Robin/Newton BC
 - Cauchy BC
 - Natural BC

+ Operators: Chapter 3

- Symmetric operators - Self adjoint
- Positive & positive definite operators

+ Integration by parts:

$$\int u dv = \cancel{uv} - \cancel{\int v du}$$

Weak integral form by apply this to original integral form
to reduce the order of $\frac{\partial u}{\partial x}$

+ $F(u) = \pi(u)$ energy / potential equation

$$W = \delta I + BC_s = \delta \pi + BC_s$$

$$= \int [ODE] \delta u dx + BC_s$$

2018 2019

1) Given energy func. $I(u) = \int_{\Omega} - dx$

Asked to find governing differential equation
natural boundary conditions

Solution: Transform variation δI :

$$\begin{aligned}\delta I &= \delta \left(\int_{\Omega} - dx \right) = \int_{\Omega} [- \delta u .. \delta u' .. (\delta u')^2 ..] dx \\ &= \int_{\Omega} [\text{ODE}] \delta u. dx + [\text{BCs}]\end{aligned}$$

Due to stationary condition $\delta I = 0 \quad \forall \delta u(x)$

2) Given ODE & BCs

Asked to find governing energy function $J(u)$

Solution: From ODE \rightarrow get residual $R(u)$

$$\text{Integral form: } W = \int_{\Omega} \psi(x) R(u)(x) dx$$

Set weighting func: $\psi(x) = \delta u$

$$\begin{aligned}\Rightarrow W &= \int_{\Omega} \delta u \cdot R(u). dx = \delta \left[\int_{\Omega} - dx \right] + [\text{BCs}] \\ &= \delta I + [\text{BCs}]\end{aligned}$$

3) Given PDE of $u(x)$

$$ex: u''(x) + \frac{1}{2} (u'(x))^2 + 1 = 0$$

Boundary conditions

$$\begin{cases} u(0) = 0 \\ u(1) = 0 \end{cases} \quad x \in [0, 1]$$

Asked to find n^{th} term approximation $\hat{u}(x)$

Solution: $\hat{u}(x) = u_0(x) + P_1(x) \cdot a_1 + \dots + P_n(x) \cdot a_n$

→ If polynomial approximation:
From BCs $\Rightarrow P_i(x)$

$$ex: \begin{cases} u(0) = 0 \\ u(1) = 0 \end{cases} \Rightarrow \begin{cases} P_1(x) = x(1-x) \\ P_2(x) = x^2(1-x) \\ \dots \end{cases}$$

→ If trigonometry approximation:
 $ex: \begin{cases} y(0) = 0 \\ y(\pi) = 0 \end{cases} \Rightarrow \begin{cases} P_1(x) = \sin\left(\frac{\pi x}{3}\right) \\ P_2(x) = \dots \end{cases}$

→ Final $u_0(x)$ also from BCs (non-homogeneous case)

→ Substitute $P_1 \dots P_n \rightarrow \hat{u}(x)$
then $\hat{u}(x) \rightarrow \text{PDE} \dots$

→ Using Collocation by Points: Given points: $x_1 \dots x_n$
Set $R(\hat{u})(x_i) = 0 \Rightarrow \text{solve } a_1 \dots a_n$

→ Using Collocation by Subdomain: Given subdomain $\Omega_1 \dots \Omega_n$
Set $\int_{\Omega_i} R(\hat{u}) \cdot dx = 0 \Rightarrow \text{solve } a_1 \dots a_n$

→ Galerkin's Method:

Set $\int_{\Omega} P_i(x) R(\hat{u}) \cdot dx = 0$

→ The Least Squares Method:

Set $\int_{\Omega} \frac{\partial R}{\partial a_i} R(\hat{u}) \cdot dx = 0$

→ The Ritz Method:

Set $\frac{\partial \pi}{\partial a_i} = 0$

② Operator:

→ Gradient: grad / ∇f
 ex: - $p = (x_1, \dots, x_n) \Rightarrow \nabla f(p) = \left[\frac{\partial f(p)}{\partial x_1}, \dots, \frac{\partial f(p)}{\partial x_n} \right]^T$
 - $\nabla f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$

→ Divergence: $\operatorname{div} F = \nabla \cdot F$
 ex: - $\operatorname{div} \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

→ Curl: $\operatorname{curl} F = \nabla \times F$
 ex: $\nabla \times F = \begin{bmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{bmatrix}$

→ Laplace: $\Delta u = \operatorname{div} \nabla u = \nabla \cdot \nabla u$
 ex: $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

→ $(\delta u)' = \delta(u')$ $\delta \left(\frac{dw}{dx} \right) = \delta w' \quad \downarrow \text{Properties of variation}$

$$\delta \left(\int u(x) dx \right) = \int \delta u(x) dx$$

$$\delta F(x, u, u') = \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u'$$

$$\Rightarrow \delta \left(\int_0^1 x u dx \right) = \int_0^1 x \delta u dx$$