Multi Body Dynamic

<u>Objective</u>: Deal with 3D kinematics, elasticity, damping...

<u>Aims</u>: Modelling (mathematical description)
Analysis of the system dynamics

<u>Application</u>: Vibration analysis & reduction
Motion design
Forces & torque on system components

System Modelling

1, Specification of the System Components & Physical Effects

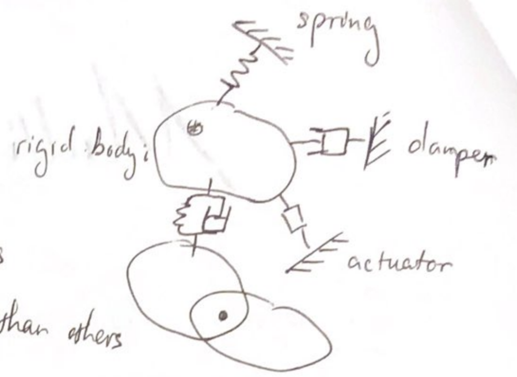
2, Methods of Approach for Equivalent Models

each has different mathematical model

- Continuous Systems: easy to model physically, not mathematically
  elastic bodies with inertia
  actuating forces are distributed evenly — body volume
  pressures ———————— - surface
  mathematical closed-solution only for simple case
- Finite-Element Systems: Separation of a complex geometry into many simple elements ..
- Multi-Body Systems: Rigid bodies with inertia
  connection through joints, springs ...
  forces & torques only on discrete points

3, Multi-Body System:

with rigid body, spring, damper ..



Massless connecting rod: if the kinematics energy of the rod is significantly smaller than others

- **Rigid Bodies**: nodal masses: m (inertia matrix (tensor) ≈ 0)
  geometrically extended rigid bodies: m, J
- **Constraint Elements**: massless bars, bearings, revolute pairs ..
- **Connecting Elements**: springs, dampers, actuators

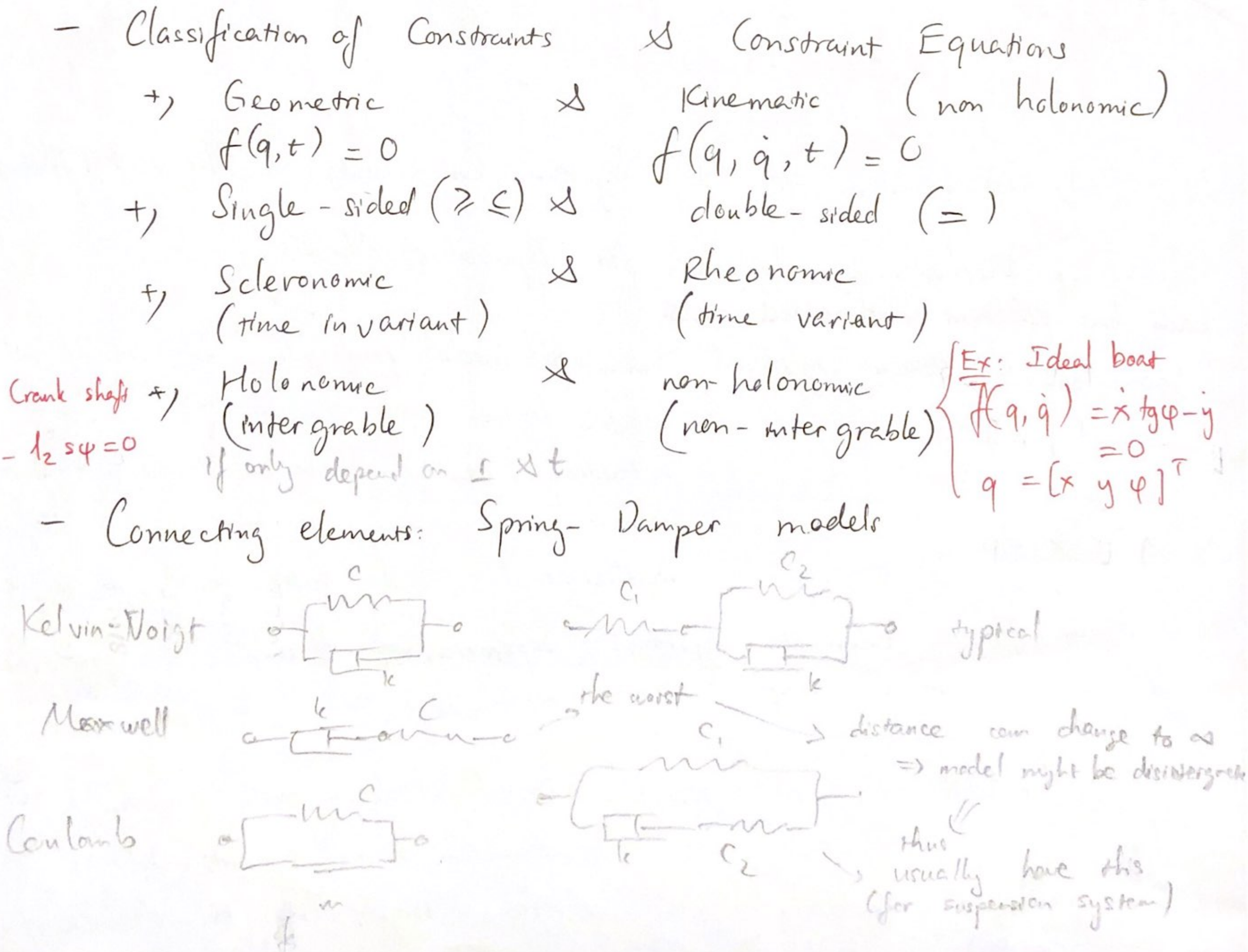
→ affect rigid bodies through kinematics (cause they transfer same ForM
→ affect ______ forces & torques (______ different ___)

- Classification of Constraints & Constraint Equations
  - +) Geometric & Kinematic (non holonomic)
    $f(q,t) = 0$ $\quad$ $f(q,\dot{q},t) = 0$
  - +) Single-sided ($\geq \leq$) & double-sided ($=$)
  - +) Scleronomic & Rheonomic
    (time invariant) $\quad$ (time variant)
  - +) Holonomic & non-holonomic
    (intergrable) $\quad$ (non-intergrable)
    if only depend on r & t

Ex: Crank shaft
$l_1 s\varphi - l_2 s\varphi = 0$

Ex: Ideal boat
$f(q,\dot{q}) = \dot{x}\, tg\varphi - \dot{y} = 0$
$q = [x \;\; y \;\; \varphi]^T$

- Connecting elements: Spring-Damper models



+) Rigid bodies:
- point mass
- geometrically extended rigid bodies

Characteristics:
- mass m
- inertia tensor (for geometrically extended .. only) : J
- centre of gravity

+) Constraint elements: . .

Characteristics:
- reduce the system's DOF
- cause constraint force, constraint equations

+) Connecting / Interlinking elements: . .

Characteristics: cause active forces

DOF of whole system:

P_E : #No. bodies with geometrical extension

P_P : #No. point masses

b : #No. independent constraints

$\Rightarrow$ in 3D Space: $DoF = 6P_E + 3P_P - b$

Surface: $DoF = 3P_E + 2P_P - b$

MBD

<2)

Translational Kinematics

+) Formulation of Arbitrary Position:

$$ {}^c r_{S_i} = \begin{bmatrix} {}^c x_{S_i} \\ {}^c y_{S_i} \\ {}^c z_{S_i} \end{bmatrix} = {}^c\begin{bmatrix} x_{S_i} \\ y_{S_i} \\ z_{S_i} \end{bmatrix} = \begin{bmatrix} {}^c x_{S_i} & {}^c y_{S_i} & {}^c z_{S_i} \end{bmatrix}^T $$

x_0, y_0, z_0 : fixed Cartesian coordinate system

x_{ip}, y_{ip}, z_{ip}: parallelly-shifted Cartesian coor. system

x_{iK}, y_{iK}, z_{iK} : body-fixed Cartesian coor. system

+) Formulation of Arbitrary Orientation:

$$ {}^c r = {}^{cd}S \cdot {}^d r $$

$$ {}^{cd}S = \begin{bmatrix} {}^{cd}s_{11} & \cdots & {}^{cd}s_{13} \\ \vdots & \ddots & \vdots \\ {}^{cd}s_{31} & \cdots & {}^{cd}s_{33} \end{bmatrix} $$

c, d : name of coor. systems (c – new , d old ones)

+) Generalized Coordinates : $q = [q_1 \; .. \; q_f]$ $f \rightarrow$ dof

+) Translational Kinematics

$$ {}^c r_{S_i} = {}^c r_{S_i}(q_1, .., q_f, t) = \begin{bmatrix} {}^c x_{S_i}(q_1 .. q_f, t) \\ {}^c y_{S_i}(q_1 .. q_f, t) \\ {}^c z_{S_i}(q_1 .. q_f, t) \end{bmatrix} $$

Velocity: $$ {}^c v_{S_i} = {}^c \dot{r}_{S_i} = \begin{bmatrix} \dfrac{\partial {}^c r_{S_i}}{\partial q_1} & \cdots & \dfrac{\partial {}^c r_{S_i}}{\partial q_f} \end{bmatrix} \cdot \begin{bmatrix} \dfrac{dq_1}{dt} \\ \vdots \\ \dfrac{dq_f}{dt} \end{bmatrix} + \dfrac{\partial {}^c r_{S_i}}{\partial t} = \boxed{\underbrace{{}^c J_{T_i}}_{\text{3 rows, f columns}} \cdot \dot{q} + \underbrace{{}^c \bar{v}_{S_i}(q,t)}_{\text{local velocity}}} $$

Acceleration: $$ {}^c \dot{v}_{S_i} = \boxed{{}^c J_{T_i} \ddot{q} + {}^c \dot{J}_{T_i} \dot{q} + {}^c \dot{\bar{v}}_{S_i}} $$

$$ {}^c \dot{J}_{T_i} = \frac{d}{dt}\left(\frac{\partial {}^c r_{S_i}}{\partial q^T}\right) $$ (take derivative of each element)

+) <u>Rotational Kinematics</u>

current frame = intrinsic rotations
fixed extrinsic

$$^{dc}S = {^{cd}S}^{-1} = {^{cd}S}^{T}$$

teo quantities
ex / intrinsic

- Cardan : $x \Rightarrow y' \Rightarrow z''$ with α, β, γ => not very intuitive fixed frame (?)
- Euler : $zy'z''$ 12 different set of angles , current frame
- Roll- Pitch- Yaw (RPY): $z \Rightarrow y' \Rightarrow x''$ with ϕ, θ, ψ fixed frame

Location vector $\neq$ Position vector

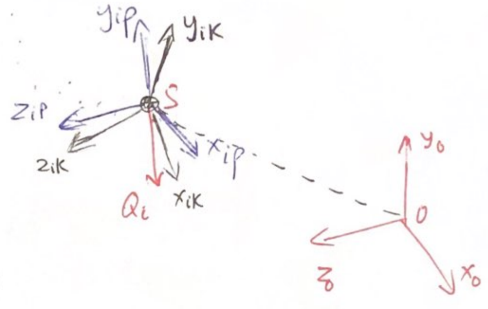
$$q = [q_1 \;..\; q_f]^T$$

$$^{c}r_{S_i} = {^{c}r_{S_i}}(q,t) = \begin{bmatrix} ^{c}x_{S_i}(q,t) \\ ^{c}y_{S_i}(q,t) \\ ^{c}z_{S_i}(q,t) \end{bmatrix}$$

only 1 for whole system

as many as the number of bodies

Rotational Kinematics



L3)

+) $${}^{P}r_{Q_i} = {}^{PK}S_i \, {}^{K}r_{Q_i}$$

Since ${}^{K}\dot{r}_{Q_i} = 0$, $\Rightarrow$ $${}^{P}\dot{r}_{Q_i} = {}^{PK}\dot{S}_i \, {}^{K}r_{Q_i}$$

change in magnitude

With $${}^{K}r_{Q_i} = {}^{PK}S_i^{-1} \cdot {}^{P}r_{Q_i} = {}^{PK}S_i^{T} \cdot {}^{P}r_{Q_i}$$

$$\Rightarrow {}^{P}\dot{r}_{Q_i} = {}^{PK}\dot{S}_i \cdot {}^{PK}S_i^{T} \cdot {}^{P}r_{Q_i}$$

+) $${}^{P}\dot{r}_{Q_i} = {}^{P}\dot{r}_{S_i} + {}^{P}\dot{r}_{Q_iS}$$ consider ${}^{P}\dot{r}_{S_i} = 0$

$$\begin{bmatrix} {}^{P}\omega_{ix} \\ {}^{P}\omega_{iy} \\ {}^{P}\omega_{iz} \end{bmatrix} = {}^{P}\dot{r}_{Q_iS} = {}^{P}\tilde{\omega}_i \cdot {}^{P}r_{Q_iS} \quad ; \quad {}^{P}\tilde{\omega}_i = \begin{bmatrix} 0 & -{}^{P}\omega_{iz} & {}^{P}\omega_{iy} \\ {}^{P}\omega_{iz} & 0 & -{}^{P}\omega_{ix} \\ -{}^{P}\omega_{iy} & {}^{P}\omega_{ix} & 0 \end{bmatrix}$$ skew matrix

+) $${}^{P}\tilde{\omega}_i = {}^{PK}\dot{S}_i \, {}^{PK}S_i^{T} = \frac{\partial\, {}^{PK}S_i}{\partial q_1} {}^{PK}S_i^{T} \dot{q}_1 + \dots + \frac{\partial\, {}^{PK}S_i}{\partial q_f} {}^{PK}S_i^{T} \dot{q}_f + \frac{\partial\, {}^{PK}S_i}{\partial t} {}^{PK}S_i^{T}$$

$$\Leftrightarrow \begin{bmatrix} 0 & -{}^{P}\omega_{iz} & {}^{P}\omega_{iy} \\ {}^{P}\omega_{iz} & 0 & -{}^{P}\omega_{ix} \\ -{}^{P}\omega_{iy} & {}^{P}\omega_{ix} & 0 \end{bmatrix} = \sum_{j=1}^{f} \begin{bmatrix} 0 & -\frac{\partial s_{iz}}{\partial q_j} & \frac{\partial s_{iy}}{\partial q_j} \\ \frac{\partial s_{iz}}{\partial q_j} & 0 & -\frac{\partial s_{ix}}{\partial q_j} \\ -\frac{\partial s_{iy}}{\partial q_j} & \frac{\partial s_{ix}}{\partial q_j} & 0 \end{bmatrix} \dot{q}_j + \begin{bmatrix} 0 & -\frac{\partial s_{iz}}{\partial t} & \frac{\partial s_{iy}}{\partial t} \\ \frac{\partial s_{iz}}{\partial t} & 0 & -\frac{\partial s_{ix}}{\partial t} \\ -\frac{\partial s_{iy}}{\partial t} & \frac{\partial s_{ix}}{\partial t} & 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} {}^{P}\omega_{ix} = \dfrac{\partial s_{ix}}{\partial q_1} \dot{q}_1 + \dots + \dfrac{\partial s_{ix}}{\partial q_f} \dot{q}_f + \dfrac{\partial s_{ix}}{\partial t} \\ {}^{P}\omega_{iy},\ {}^{P}\omega_{iz} = \dots \end{cases} \Leftrightarrow \boxed{{}^{P}\omega_i = {}^{P}J_{Ri}\, \dot{q} + {}^{P}\bar{\omega}_i}$$

local angular velocity

$$\boxed{{}^{P}\dot{\omega}_i = {}^{P}J_{Ri}(q,t)\,\ddot{q} + {}^{P}\dot{J}_{Ri}(q,t)\,\dot{q} + {}^{P}\dot{\bar{\omega}}_i(q,t)}$$

3 rows, f columns

Newton Euler Equation

L4)

+) Translational Kinetics (Newton)

$$m_i\,{}^C\dot{v}_{S_i} = {}^C f_i, \quad i = 1(1)p$$

$$\Rightarrow m_i\,{}^P\dot{v}_{S_i} = {}^P f_i, \quad i = 1(1)p$$ (parallely shifted coordinate system)

$$\Rightarrow \boxed{m_i\,{}^P J_{T_i}\ddot{q} + m_i\,{}^P\dot{J}_{T_i}\dot{q} + m_i\,{}^P\dot{\bar{v}}_{S_i} = {}^P f_i, \quad i = 1(1)p}$$

+) Rotational Kinetics (Euler)

- Inertia Tensor of Rigid Bodies: In the body fixed system:

independent of time t

$${}^K I_i = \begin{bmatrix} {}^K I_{xx} & -{}^K I_{yx} & -{}^K I_{zx} \\ -{}^K I_{xy} & {}^K I_{yy} & -{}^K I_{zy} \\ -{}^K I_{xz} & -{}^K I_{yz} & {}^K I_{zz} \end{bmatrix} \qquad {}^K I_{xx} = \int ({}^K y^2 + {}^K z^2)\,dm \;\ldots \qquad {}^K I_{xy} = \int {}^K x\,{}^K y\,dm$$

In general $${}^d I_i = {}^{dc}S_i\,{}^c I_i\,{}^{dc}S_i^T \Rightarrow \boxed{{}^P I_i = {}^{PK}S_i\,{}^K I_i\,{}^{PK}S_i^T}$$

- Angular momentum: $${}^P L_i = {}^P I_i\,{}^P\omega_i$$

$\Rightarrow$ Resulting torque m_i around S_i: $$m_i = {}^P\dot{L}_i = \frac{d({}^P I_i\,{}^P\omega_i)}{dt} = {}^P I_i\,\dot{\omega}_i + {}^P\dot{I}_i\,{}^P\omega_i$$

$${}^P\dot{I}_i = {}^{PK}\dot{S}_i\,{}^K I_i\,{}^{PK}S_i^T + {}^{PK}S_i\,{}^K I_i\,{}^{PK}\dot{S}_i^T$$
$$= {}^{PK}\dot{S}_i\,{}^{PK}S_i^T\,{}^P I_i + {}^P I_i\,{}^{PK}S_i\,{}^{PK}\dot{S}_i^T = {}^P\tilde{\omega}_i\,{}^P I_i - {}^P I_i\,{}^P\tilde{\omega}_i$$

$$\Rightarrow m_i = {}^P\dot{L}_i = {}^P I_i\,{}^P\dot{\omega}_i + {}^P\tilde{\omega}_i\,{}^P I_i\,{}^P\omega_i - \underbrace{{}^P I_i\,{}^P\tilde{\omega}_i\,{}^P\omega_i}_{0}$$

$$\Rightarrow {}^P\dot{L}_i = \boxed{{}^P I_i\,{}^P\dot{\omega}_i + {}^P\tilde{\omega}_i\,{}^P I_i\,{}^P\omega_i = {}^P m_i}, \quad i = 1(1)p$$ (Euler equation)

$\Rightarrow$ Euler equations with Rotational Kinematics

In skew matrix form

$$\boxed{{}^P\dot{L}_i = {}^P I_i\,{}^P J_{R_i}\ddot{q} + {}^P I_i\,{}^P\dot{J}_{R_i}\dot{q} + {}^P I_i\,{}^P\dot{\bar{\omega}}_i + ({}^P\tilde{J}_{R_i}\dot{q} + {}^P\tilde{\bar{\omega}}_i)\,{}^P I_i\,({}^P J_{R_i}\dot{q} + {}^P\bar{\omega}_i) \quad i = 1(1)p}$$

11

+) Equations of Motion & Mass Matrix:

- Newton: $m_i \, {}^P J_{Ti} \ddot{q} + m_i \, {}^P \dot{J}_{Ti} \dot{q} + m_i \, {}^P \dot{\bar{v}}_{Si} = {}^P f_i \longrightarrow$ 3p equations (x, y, z) (p no. of bodies)

- Euler: ${}^P I_i \, {}^P J_{Ri} \ddot{q} + {}^P I_i \, {}^P \dot{J}_{Ri} \dot{q} + {}^P I_i \, {}^P \dot{\bar{\omega}}_i + ({}^P \widetilde{J_{Ri} \dot{q}} + {}^P \widetilde{\bar{\omega}}_i) \, {}^P I_i \, ({}^P J_{Ri} \dot{q} + {}^P \bar{\omega}_i) = {}^P m_i$

$\Rightarrow \; {}^P \bar{M}(q,t) \ddot{q}(t) + {}^P \bar{g}(q, \dot{q}, t) = {}^P \bar{d}(q, \dot{q}, t)$ 6p equations

$${}^P \bar{M}(q,t) = \begin{bmatrix} m_1 \, {}^P J_{T1} \\ \vdots \\ m_p \, J_{Tp} \\ {}^P I_1 \, {}^P J_{R1} \\ \vdots \\ {}^P I_p \, {}^P J_{Rp} \end{bmatrix}_{6p \times f} ; \quad {}^P \bar{g}(q, \dot{q}, t) = \begin{bmatrix} m_1 \, {}^P \dot{J}_{Ti} \dot{q} + m_1 \, {}^P \dot{\bar{v}}_{S1} \\ \vdots \\ {}^P I_1 \, {}^P \dot{J}_{Ri} \dot{q} + \dots \\ \vdots \end{bmatrix}$$

vectors of gyroscope forces

Vector of active & constraint Forces:

$${}^P \bar{d}(q, \dot{q}, t) = \begin{bmatrix} {}^P f_1 \\ \vdots \\ {}^P f_p \\ {}^P m_1 \\ \vdots \\ {}^P m_p \end{bmatrix} = \underbrace{{}^P \bar{d}_E}_{\text{active } f, m} + \underbrace{{}^P \bar{d}_Z}_{\text{constraint } f, m} = \begin{bmatrix} {}^P f_{E1} \\ {}^P f_{E2} \\ \vdots \\ {}^P f_{Ep} \\ {}^P m_{E1} \\ {}^P m_{E2} \\ \vdots \\ {}^P m_{Ep} \end{bmatrix} + \begin{bmatrix} {}^P f_{Z1} \\ {}^P f_{Z2} \\ \vdots \\ {}^P f_{Zp} \\ {}^P m_{Z1} \\ \vdots \\ {}^P m_{Zp} \end{bmatrix}$$

- Principle of virtual work:

$$\sum_{i=1}^{p} \left({}^P J_{Ti}^T \, {}^P f_{Zi} + {}^P J_{Ri}^T \, {}^P m_{Zi} \right) = 0 = {}^P J_{ges}^T(q,t) \cdot {}^P \bar{d}_Z$$

$${}^P J_{ges}^T(q,t) = \begin{bmatrix} \frac{\partial {}^P r_{S1}^T}{\partial q_1} & \dots & \frac{\partial {}^P r_{Sp}^T}{\partial q_1} & \frac{\partial {}^P s_1^T}{\partial q_1} & \dots & \frac{\partial {}^P s_p^T}{\partial q_1} \\ \frac{\partial {}^P r_{S1}^T}{\partial q_2} & \ddots & & & \ddots & \vdots \\ \vdots \\ \frac{\partial {}^P r_{S1}^T}{\partial q_f} & \dots & \frac{\partial {}^P r_{Sp}^T}{\partial q_f} & \frac{\partial {}^P s_1^T}{\partial q_f} & \dots & \frac{\partial {}^P s_p^T}{\partial q_f} \end{bmatrix}$$

$f \times 6p$

Size of ${}^P J_{ges}(q,t)$: $6p \times f$

$$= \left[{}^P J_{T1}^T \; \dots \; {}^P J_{Tp}^T \; {}^P J_{R1}^T \; \dots \; {}^P J_{Rp}^T \right]$$

$$\Rightarrow \quad {}^{P}J_{ges}^{T}(q,t)\left({}^{P}\overline{M}(q,t)\ddot{q}(t) + {}^{P}\overline{g}(q,\dot{q},t)\right) = {}^{P}\overline{J}_{ges}^{T}(q,t) \cdot {}^{P}\overline{d}_{E} = d$$

$$\Leftrightarrow \quad \boxed{M(q,t)\ddot{q}(t) + g(q,\dot{q},t) = d(q,\dot{q},t)} \quad \text{~~f~~ equations}$$

(*) with
$$\begin{cases} M(q,t) = {}^{P}\overline{J}_{ges}^{T}(q,t) \cdot {}^{P}\overline{M}(q,t) & f \times f \\ g(q,\dot{q},t) = {}^{P}\overline{J}_{ges}^{T}(q,t) \cdot {}^{P}\overline{g}(q,\dot{q},t) & f \times 1 \\ d(q,\dot{q},t) = {}^{P}\overline{J}_{ges}^{T}(q,t) \cdot {}^{P}\overline{d}_{E}(q,\dot{q},t) & f \times 1 \end{cases}$$

dpcm: Q.E.D. = quod erat demonstrandum (Latin)

Step to solve exercise:

1) Kinematics: ${}^0r_{Si} \Rightarrow {}^0J_{Ti}$ & ${}^0\bar{v}_{Si} \Rightarrow {}^0v_{Si}$

$\Rightarrow {}^0\dot{J}_{Ti}$ & ${}^0\dot{\bar{v}}_{Si} \Rightarrow {}^0\dot{v}_{Si} = {}^0a_{Si}$

(Can skip this, just take $\frac{d}{dt}{}^0v_{Si} = {}^0a_{Si}$)

2) Rotational Kinematics:

We are given ${}^KI_i$ (the inertia tensor regards to fix-frame body)

Find ${}^0I_i$ by: ${}^0I_i = {}^{0K}S_i \cdot {}^KI_i \cdot {}^{0K}S_i^T$

Find rotation matrix ${}^0S_i, {}^0\tilde{\omega}_i \Rightarrow$ skew matrix ${}^0\tilde{\omega}_i = {}^0\dot{S}_i \, {}^0S_i^T$

$\Rightarrow {}^0\omega_i \Rightarrow {}^0\dot{\omega}_i$ (Just take ${}^0\dot{\omega}_i = \frac{d}{dt}{}^0\omega_i$ as well)

$\Rightarrow {}^0m_i = {}^0I_i \, {}^0\dot{\omega}_i + {}^0\tilde{\omega}_i \, {}^0I_i \, {}^0\omega_i$ (Euler - Rotational Kinetics)

${}^0f_i = m_i \, {}^0\dot{v}_i$ (Newton - Translational Kinetics)

3) $m_i \cdot {}^0J_{Ti}^T \cdot {}^0\dot{v}_{Si}$

${}^0J_{Ri}^T \cdot {}^0m_i$

The compact part
By multiply with the corresponding Jacobian J
$m\dot{v} = f$ with J_{Tr} , ${}^0m_i$ with J_{Rot}

4) Identify force & torque (active , not reaction ones)

Lagrangian Equa. of 2nd kind

L5,

- Principle of Virtual Work:

$$\delta W_i = \underbrace{\delta W_i^E}_{\text{active force}} + \underbrace{\delta W_i^z}_{\text{reaction force}} + \underbrace{\delta W_i^H}_{\text{accessory forces}} \Rightarrow \text{Scalar value}$$

$$= f_i^E \cdot \delta r_i + f_i^z \cdot \delta r_i - \dot{p}_i \cdot \delta r_i$$

$$= f_i^E \cdot \delta r_i + 0 - m_i \frac{dv_i}{dt} \delta r_i$$

- D'Alembert's Principle in the Lagrangian form:

$$\sum_p m_i \frac{dv_i}{dt} \delta r_i = \sum_p f_i^E \delta r_i$$

.. very long transformation ..

$$\sum_p \left(m_i \frac{dv_i}{dt} \cdot \delta r_i \right) = \left(\frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}} E_{kin,ges} \right) - \frac{\partial}{\partial q} E_{kin,ges} \right) \cdot \delta q = \sum_p f_i^E \delta r_i$$

Mà: $\sum_p f_i^E \delta r_i + \sum_p m_i^E \delta s_i = \sum_p \left(f_i^E J_{T_i} \delta q \right) + \sum_p \left(m_i^E J_{R_i} \delta q \right) + \sum_p m_i^E \delta s_i$

$$\Leftrightarrow \left(\frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}} E_{kin,ges} \right) - \frac{\partial E_{kin,ges}}{\partial q} \right) \cdot \delta q = \sum_p \left({}^{c}\bar{J}_{T,i}^{T} \, {}^{c}f_i^E + {}^{c}\bar{J}_{R_i}^{T} \, {}^{c}m_i^E \right) \cdot \delta q \quad \text{scalar}$$

$$\Leftrightarrow \boxed{\frac{d}{dt} \left(\frac{\partial E_{kin}}{\partial \dot{q}} \right) - \frac{\partial E_{kin}}{\partial q} = \sum_{i=1}^{p} \left({}^{c}\bar{J}_{T,i}^{T} \, {}^{c}f_i^E + {}^{c}\bar{J}_{R_i}^{T} \, {}^{c}m_i^E \right) = d} \quad f \text{ equati[ons]}$$

we go directly to the minimum set of equations

But if we want to know the reaction forces, we must go back to Newton-Euler equation

⊛ Algorithm to set-up the equation:

- Formulation of the kinematics: ${}^{c}r_{S_i}$, ${}^{c}S_i$, ${}^{c}v_{S_i}$, ${}^{c}\omega_{S_i}$
- Calculation of the kinetic energy: $E_k = \frac{1}{2}\sum_p \left({}^{c}v_{S_i}^T m_i \, {}^{c}v_{S_i} + {}^{c}\omega_i^T \, {}^{c}I_i \, {}^{c}\omega_i \right)$
- Lagrange equation: $\frac{d}{dt}\left(\frac{\partial E_{kin}}{\partial \dot q}\right) - \frac{\partial E_{kin}}{\partial q} = d$
- Formulation of the kinetics $\left({}^{c}f_i^E, \, {}^{c}m_i^E\right)$
- Calculation of the generalised forces: $d = \sum_p {}^{c}J_{T,i}^T \left({}^{c}f_i^E\right)^T + \sum_p {}^{c}J_{R_i}^T \left({}^{c}m_i^E\right)^T$
  $= {}^{c}J_{ges}^T \, \overline{d}_E$
- Equation of motion: $M(q,t)\ddot q(t) + g(q,\dot q,t) = d(q,\dot q,t)$

connecting elements → active forces → $F_c = k\,\Delta x$, $F_{damper} = k\,\Delta \dot x$

constraint → reaction forces

⊛ For system with only conservative forces (gravitational, spring)..
no damper, no actuator..

$$\delta E_{kin} + \delta E_{pot} = 0$$

$$\Leftrightarrow \left(\frac{d}{dt}\left(\frac{\partial E_{kin}}{\partial \dot q}\right) - \frac{\partial E_{kin}}{\partial q}\right)\delta q = -\frac{\partial E_{pot}}{\partial q}\delta q$$

spring $E_{pot} = \frac{kx^2}{2} = \frac{cx^2}{2}$

gravi $E_{pot} = mgh$

$F_{spr} = kx$ or cy

⟺ Lagrange equations for conservative system:

$$\frac{d}{dt}\left(\frac{\partial E_{kin}}{\partial \dot q}\right) - \frac{\partial E_{kin}}{\partial q} = -\frac{\partial E_{pot}}{\partial q}$$

$$\Rightarrow d = -\frac{\partial E_{pot}}{\partial q} = \left(-\frac{\partial E_{pot}}{\partial q_1} \quad -\frac{\partial E_{pot}}{\partial q_2} \quad \cdots \quad \frac{\partial E_{pot}}{\partial q_f}\right)^T$$

Since the Lagrangian $L = E_{kin} - E_{pot}$

⇒ We can also write as: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = 0$

⊛ For non-conservative systems:

$$\frac{d}{dt}\left(\frac{\partial E_{kin}}{\partial \dot q}\right) - \frac{\partial E_{kin}}{\partial q} = -\frac{\partial E_{pot}}{\partial q} + \tau(q,\dot q,t) \quad \text{or} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = \tau(q,\dot q,t)$$

MBD

Lagrangian Equa. of 1st kind

L6, Very computer-oriented

$$r_i = \begin{bmatrix} r_{i,tr} \\ r_{i,rot} \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \\ z_i \\ \alpha_i \\ \beta_i \\ \gamma_i \end{bmatrix}$$

Holonomic constraints

$$\begin{bmatrix} g_1(r_1,\dots,r_p,t) \\ \vdots \\ g_b(r_1,\dots,r_p,t) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \Leftrightarrow g(r,t) = 0$$

$$\underline{r = (r_1 \,..\, r_p)^T} = (x_1 \,..\, \gamma_1 \,..\, x_p \,..\, \gamma_p)^T \qquad f = {}^3_6|\, p - b$$

- Newton & Euler's Theorem:

identity matrix

$$\begin{bmatrix} m_i E & 0 \\ 0 & {}^0I_i \end{bmatrix} \cdot \begin{bmatrix} {}^0\dot{v}_{si} \\ {}^0\dot{\omega}_i \end{bmatrix} = \begin{bmatrix} {}^0f_i^E \\ {}^0m_i^E \end{bmatrix} - \begin{bmatrix} 0 \\ {}^0\tilde{\omega} \cdot {}^0I \cdot {}^0\omega \end{bmatrix} + \begin{bmatrix} {}^0f_i^z \\ {}^0m_i^z \end{bmatrix}$$

check L4)

$$\begin{cases} m_i \cdot {}^0\dot{v}_{si} = {}^0f_i^E + {}^0f_i^z \\ {}^0\dot{L}_i = {}^0m_i^E + {}^0m_i^z \\ \quad = {}^0I_i \, {}^0\dot{\omega}_i + {}^{0K}\dot{S}_i \cdot {}^{0K}S_i^T \cdot {}^0I_i \cdot {}^0\omega_i \\ \quad = {}^0I_i \, {}^0\dot{\omega}_i + {}^0\tilde{\omega}_i \cdot {}^0I_i \cdot {}^0\omega_i \end{cases} \Leftrightarrow$$

$$\Rightarrow T_i \cdot \ddot{r}_i = e_i + z_i$$

$\Rightarrow$ For all bodies:

$$T \cdot \ddot{r} = e + z$$

diagonal matrix

$$z_i(r_i,t) = \begin{bmatrix} f_i^z \\ m_i^z \end{bmatrix} = \sum_{k=1}^{b} \left[\text{grad } g_k(r_i,t) \cdot \lambda_k(t) \right] = \begin{bmatrix} \frac{\partial g_1}{\partial x_i} \\ \vdots \\ \frac{\partial g_1}{\partial \gamma_i} \end{bmatrix} \cdot \lambda_1(t) + \dots + \begin{bmatrix} \frac{\partial g_b}{\partial x_i} \\ \vdots \\ \frac{\partial g_b}{\partial \gamma_i} \end{bmatrix} \lambda_b(t)$$

reaction ~~forces~~ torque

$$= \sum_{k=1}^{b} \frac{\partial g_k}{\partial r_i} \cdot \lambda_k = \begin{bmatrix} \frac{\partial g_1}{\partial r_i} & \dots & \frac{\partial g_b}{\partial r_i} \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_b \end{bmatrix}$$

$G_{i\,[b\times 6]}$

$$= \begin{bmatrix} G_{i,tr} & G_{i,rot} \end{bmatrix}^T \cdot \lambda = G_i^T \cdot \lambda$$

$$G_{i,tr} = \frac{\partial g(r,t)}{\partial r_{i,tr}^T} = \begin{bmatrix} \frac{\partial g_1}{\partial x_i} & \frac{\partial g_1}{\partial y_i} & \frac{\partial g_1}{\partial z_1} \\ \vdots & \vdots & \vdots \\ \frac{\partial g_b}{\partial x_i} & \frac{\partial g_b}{\partial y_i} & \frac{\partial g_b}{\partial z_i} \end{bmatrix}; \quad G_{i,rot} = \frac{\partial g(r,t)}{\partial r_{i,rot}^T} = \begin{bmatrix} \frac{\partial g_1}{\partial \alpha_i} & \frac{\partial g_1}{\partial \beta_i} & \frac{\partial g_1}{\partial \gamma_i} \\ \vdots & \vdots & \vdots \\ \frac{\partial g_b}{\partial \alpha_i} & \frac{\partial g_b}{\partial \beta_i} & \frac{\partial g_b}{\partial \gamma_i} \end{bmatrix}$$

$$\hookrightarrow \quad G(r,t) = \begin{bmatrix} G_1 & \dots & G_p \end{bmatrix} = \begin{bmatrix} \frac{\partial g_1}{\partial r_{1,tr}^T} & \frac{\partial g_1}{\partial r_{1,rot}^T} & \dots & \frac{\partial g_1}{\partial r_{p,tr}^T} & \frac{\partial g_1}{\partial r_{p,rot}^T} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial g_b}{\partial r_{1,tr}^T} & \frac{\partial g_b}{\partial r_{1,rot}^T} & \dots & \frac{\partial g_b}{\partial r_{p,tr}^T} & \frac{\partial g_b}{\partial r_{p,rot}^T} \end{bmatrix}_{[b \times 6p]}$$

$$\boxed{z = G^T \cdot \lambda}$$

$\Rightarrow$ Lagrangian Equations of 1st kind:

$$\begin{bmatrix} T_1 & & 0 \\ & \ddots & \\ 0 & & T_p \end{bmatrix} \cdot \begin{bmatrix} \ddot{r}_1 \\ \vdots \\ \ddot{r}_p \end{bmatrix} = \begin{bmatrix} e_1 \\ \vdots \\ e_p \end{bmatrix} + G^T \lambda \quad \& \quad g(r,t) = 0$$

$$\boxed{T \cdot \ddot{r} = e + z} \quad \Leftrightarrow \quad \boxed{T\ddot{r} - G^T \lambda = e} \quad (1)$$

$\hookrightarrow$ Since $g(r,t)$ is <u>holonomic</u>; we can take derivative of it.

$$\dot{g} = \frac{\partial g}{\partial r_1^T} \cdot \dot{r}_1 + \dots + \frac{\partial g}{\partial r_p^T} \cdot \dot{r}_p + \frac{\partial g}{\partial t} = G(r,t) \cdot \dot{r} + \bar{\gamma}(r,t) = 0$$

$$\frac{\partial g}{\partial t} = \begin{bmatrix} \frac{\partial g_1}{\partial t} \\ \vdots \\ \frac{\partial g_b}{\partial t} \end{bmatrix}$$

Partial temporal ~~differ~~derivation

Continue to acceleration level:

$$\ddot{g} = G(r,t)\,\ddot{r} + \underbrace{\frac{dG}{dt} \cdot \dot{r} + \frac{d\bar{\gamma}}{dt}} = G(r,t) \cdot \ddot{r} + \bar{\bar{\gamma}}(r,v,t) = 0$$

$$\Rightarrow \boxed{-G(r,t)\,\ddot{r} = \bar{\bar{\gamma}}(r,v,t)} \quad (2)$$

$\Leftrightarrow$ Combine (1) & (2), differential-algebraic-equation system (DAE):

$$\begin{bmatrix} T & -G^T \\ -G & 0 \end{bmatrix} \begin{bmatrix} \ddot{r} \\ \lambda \end{bmatrix} = \begin{bmatrix} e \\ \bar{\bar{\gamma}} \end{bmatrix} \quad \begin{matrix} - \ 6p \text{ equations} \\ - \ b \text{ equations} \end{matrix}$$

$(6p+b) \times (6p+b)$

It is not possible to solve DAE numerically.

$$\ddot{r} = T^{-1}(e + G^T\lambda)$$

$$\bar{\bar{\gamma}} = -G.\ddot{r} = -GT^{-1}e - \underbrace{GT^{-1}G^T}_{Q}\lambda$$

$$\Leftrightarrow \quad Q\lambda = -GT^{-1}e - \bar{\bar{\gamma}}$$

when G has full rank (all constraint funcs are independent)

$$\Leftrightarrow \quad \lambda(r,\dot{r},t) = -Q^{-1}(G.T^{-1}.e + \bar{\bar{\gamma}}) \quad \text{Lagrangian multipliers}$$

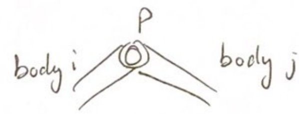
$$z = G^T\lambda = -G^TQ^{-1}(G.T^{-1}e + \bar{\bar{\gamma}})$$

$$\ddot{r} = T^{-1}(e+z)$$

Constraints:

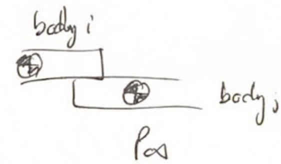
+) Revolute joint: between body i and j

$$g_{k1}(r,t) = g_{k1}(r) = x_{iP} - x_{jP} = 0$$
$$g_{k2}(r,t) = g_{k2}(r) = y_{iP} - y_{jP} = 0$$



+) Prismatic joint: x-direction

$$g_{k1}(r,t) = g_{k1}(r) = y_{iP} - y_{jP} = 0$$
$$g_{k2}(r,t) = g_{k2}(r) = \gamma_i - \gamma_j = 0$$



Exercise step:

1, Find T, T^{-1}

2, Find $g(r,t)$ $\Rightarrow$ G $\Rightarrow$ $\bar{\bar{\gamma}} = \dot{G}.\dot{r} + \frac{d\bar{g}(r,t)}{dt}$ $\Rightarrow$ $Q = G T^{-1} G^T$

3, Lagrangian multipliers λ: $\lambda = -Q^{-1}(G T^{-1} e + \bar{\bar{\gamma}})$

Linearization

<7>

- General idea: Use Taylor series to linearize

$$\varphi(z) = \varphi(x,y) = \varphi(x_0+h, y_0+k) = \varphi(x_0,y_0) + \left[\frac{\partial\varphi(x,y)}{\partial x} \quad \frac{\partial\varphi(x,y)}{\partial y}\right]_{z=z_0} \cdot \begin{bmatrix} h \\ k \end{bmatrix}$$

$$z = [x \quad y]^T$$

$q_s(t)$: reference motion
$x(t)$: deviation

- Linearize the Equations of Motion (in minimal form)

either from Newton-Euler or Lagrangian 2nd kind

$$M(q,t)\,\ddot{q}(t) + g(q,\dot{q},t) = d(q,\dot{q},t)$$

$$\ddot{q}(t) = \ddot{q}_s(t) + \ddot{x}(t)$$

$$M(q,t) \approx M_0(q_s,t) + M_1(q_s,x,t)$$

$\neq M_1(q_s,t).\,x(t)$ we can't seperate like this cause M is a matrix unlike G, d are column vector

$$g(q,\dot{q},t) \approx g_0(t) + G_{1x}(t).x(t) + G_{1\dot{x}}(t).\dot{x}(t)\,; \qquad g_0(t) = g(q_s, \dot{q}_s, t)$$

$$d(q,\dot{q},t) \approx d_0(t) + D_{1x}(t).x(t) + D_{1\dot{x}}(t).\dot{x}(t)\,; \qquad d_0(t) = \,..$$

$$\Rightarrow \left[M_0(t) + M_1(x,t)\right].\ddot{x}(t) + \left[G_{1\dot{x}}(t) - D_{1\dot{x}}(t)\right].\dot{x}(t) + \left[G_{1x}(t) - D_{1x}(t)\right].x(t)$$

$$= d_0(t) - g_0(t) - \left[M_0(t) + M_1(x,t)\right].\ddot{q}_s(t)$$

$M_1(x,t).\,\ddot{x}(t)$
every matrix elements have $x(t)$ or $\dot{x}(t)$, which are the very small deviation $\Rightarrow$ final result is 0 matrix, can be neglected

With $M_1(x,t).\ddot{q}_s(t) = \bar{M}_1(t).x(t)$

$$\Rightarrow M_0(t).\ddot{x}(t) + \underbrace{(G_{1\dot{x}} - D_{1\dot{x}})}.\dot{x}(t) + \underbrace{(\bar{M}_1 + G_{1x} - D_{1x})}.x(t) = \underbrace{d_0 - g_0 - M_0.\ddot{q}_s(t)}$$

$$\Rightarrow M(t).\ddot{x}(t) + P(t).\dot{x}(t) + Q(t).x(t) = h(t)$$

+) Time-invariant ~~system~~ vibration system: M, P, Q do not depend on time
luckily, most of the time, they are so

$$M.\ddot{x}(t) + P.\dot{x}(t) + Q.x(t) = h(t)$$

$$\Rightarrow M.\ddot{x}(t) + (D+G).\dot{x}(t) + (C+N).x(t) = h(t)$$

$h(t)$ → excitation forces

$M.\ddot{x}(t)$ → Generalized inertia forces

$D = \frac{1}{2}(P+P^T)$

$D.\dot{x}(t)$: velocity dependent damping forces

$G = \frac{1}{2}(P-P^T)$

$G.\dot{x}(t)$: gyroscopic forces

$C = \frac{1}{2}(Q+Q^T)$

$C.x(t)$

position dependent restoring | forces
conservative

$N = \frac{1}{2}(Q-Q^T)$

$N.x(t)$

position dependent damping forces

D, G, C, N tell us if the system is conservative or not
If we have D, N ⇒ have damping forces ⇒ non-conservative system

+) Energy consideration: left-sided multiply the above equation with $\dot{x}(t)^T$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{x}^T M\dot{x} + \frac{1}{2}x^T C x\right) = -\dot{x}^T D\dot{x} - \dot{x}^T G\dot{x} - \dot{x}^T N x + \dot{x}^T h$$

kinetic energy | potential energy | dissipative power | power of conservative gyroscopic forces | fanning power | excitation power

+) <u>Exercise step:</u>

- Given $M(q).\ddot{q}(t) + g(q,\dot{q}) = d(q)$

  $q_s(t),\ \dot{q}_s(t),\ \ddot{q}_s(t),\ \ x(t),\ \dot{x}(t),\ \ddot{x}(t)$

- Calculate $M_0(t),\ g_0(t),\ d_0(t)$ simply put values in

- From $M(q) \Rightarrow \bar{M}_1(t)$ $\qquad M(t) = M_0(t)$

See next page to know how

$$\left.\begin{array}{l} g(q,\dot{q},t) \Rightarrow G_{1x}(t),\ G_{1\dot{x}}(t) \\ d(q,\dot{q},t) \Rightarrow D_{1x}(t),\ D_{1\dot{x}}(t) \end{array}\right\} \Rightarrow \begin{array}{l} P = G_{1\dot{x}} - D_{1\dot{x}} \\ Q = \bar{M}_1 + G_{1x} - D_{1x} \end{array}$$

- $h(t) = d_0 - g_0 - M_0\,\ddot{q}_s(t)$

$\Rightarrow$ Final linear equation:

$$M(t).\ddot{x}(t) + P(t).\dot{x}(t) + Q(t).x(t) = h(t)$$

- Check $D, G, C, N \Rightarrow$ if system is conservative or not sum of derivative of whole row to ∂q_1

row 1 to row f of M

$$\overline{M}_1(t) = \begin{bmatrix} \sum_{k=1}^{f} \ddot{q}_{ks} \frac{\partial m_{1k}(q,t)}{\partial q_1} & \cdots & \sum_{k=1}^{f} \ddot{q}_{ks} \frac{\partial m_{1k}(q,t)}{\partial q_f} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{f} \ddot{q}_{ks} \frac{\partial m_{fk}(q,t)}{\partial q_1} & \cdots & \sum_{k=1}^{f} \ddot{q}_{ks} \frac{\partial m_{fk}(q,t)}{\partial q_f} \end{bmatrix}_{q=q_s}$$

q_1 to q_f

g_1 to g_f

$\dot{q}_1$ to $\dot{q}_f$

$$G_{1x}(t) = \begin{bmatrix} \frac{\partial g_1(q,\dot{q},t)}{\partial q_1} & \cdots & \frac{\partial g_1(q,\dot{q},t)}{\partial q_f} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_f(q,\dot{q},t)}{\partial q_1} & \cdots & \frac{\partial g_f(q,\dot{q},t)}{\partial q_f} \end{bmatrix}_{q=q_s} ; \quad G_{1\dot{x}}(t) = \begin{bmatrix} \frac{\cdots}{\partial \dot{q}_1} & \cdots \\ \vdots & \ddots & \vdots \\ \cdots & \cdots \end{bmatrix}_{q=q_s}$$

$$D_{1x}(t) = \begin{bmatrix} \frac{\partial d_1(q,\dot{q},t)}{\partial q_1} & \cdots & \frac{\partial d_1}{\partial q_f} \\ \vdots & \ddots & \vdots \\ \frac{\partial d_f}{\partial q_1} & \cdots & \frac{\partial d_f}{\partial q_f} \end{bmatrix}_{q=q_s} ; \quad D_{1\dot{x}}(t) = \ldots$$

State Equation

LS,

1, Common Mechanical Systems:

- From Equations of motion of a linear oscillation system

$$M(t).\ddot{x}(t) + P(t).\dot{x}(t) + Q(t).x(t) = h(t)$$

$$\Rightarrow \ddot{x}(t) = M^{-1}(t)\left[-P(t).\dot{x}(t) - Q(t)x(t) + h(t)\right]$$

- With $w(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$ we will have

$$\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & E \\ -M^{-1}(t)Q(t) & -M^{-1}(t)P(t) \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}(t)h(t) \end{bmatrix}$$

$$\Leftrightarrow \underset{2f\times 1}{\dot{w}(t)} = \underbrace{A(t)}_{\text{system matrix}} \cdot \underbrace{w(t)}_{\text{state vector}} + \underbrace{b(t)}_{\text{excitation vector}}$$

Only when we have invertable M (common mechanical system)

2, General Mechanical Systems:

can't be completely described by equations of motion, need extra equations

equations of motions $\rightarrow f_1(q,\dot{q},\ddot{q},z,\dot{z}) = 0 \Rightarrow \ddot{q} = f_{1,\ddot{q}}(q,\dot{q},z,\dot{z})$ 2nd order

non-holonomic constraint or balance equations $\rightarrow f_2(q,\dot{q},z,\dot{z}) = 0 \Rightarrow \dot{z} = f_{2,\dot{z}}(q,\dot{q},z)$ 1st order

$$\Rightarrow \ddot{q} = f_{1,\ddot{q}}(q,\dot{q},z,f_{2,\dot{z}}(q,\dot{q},z)) = f_3(q,\dot{q},z)$$

$$\Rightarrow \text{Set } w = \begin{bmatrix} q \\ z \\ \dot{q} \end{bmatrix}, \quad \dim(w) = 2.\dim(q) + \dim(z)$$

If we have:

$$\text{and: } \begin{cases} M\ddot{q} + P\dot{q} + Qq + P^{*}\dot{z} + Q^{*}z = h & f_1 \\ P'\dot{q} + Q'q + P^{*\prime}\dot{z} + Q^{*\prime}z = h' & f_2 \end{cases}$$

$$\Rightarrow \quad \dot{z} = (P^{*\prime})^{-1}\left[-P'\dot{q} - Q'q - Q^{*\prime}z + h'\right]$$ Find $\dot{z}$ from f_2

Substitute back to f_1 to get f_3:

$$\ddot{q} = M^{-1}\left[-P\dot{q} - Qq - P^{*}(P^{*\prime})^{-1}(-P'\dot{q} - Q'q - Q^{*\prime}z + h') - Q^{*}z + h\right]$$

$$\Rightarrow \dot{w} = \begin{bmatrix} \dot{q} \\ \dot{z} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & 0 & E \\ -(P^{*\prime})^{-1}Q' & -(P^{*\prime})^{-1}Q^{*\prime} & -(P^{*\prime})^{-1}P' \\ -M^{-1}\left[-P^{*}(P^{*\prime})^{-1}Q' + Q\right] & -M^{-1}\left[-P^{*}(P^{*\prime})^{-1}Q^{*\prime} + Q^{*}\right] & -M^{-1}\left[-P^{*}(P^{*\prime})^{-1}P' + P\right] \end{bmatrix} \cdot w$$

$$+ \begin{bmatrix} 0 \\ (P^{*\prime})^{-1}h' \\ M^{-1}\, h - P^{*}(P^{*\prime})^{-1}h' \end{bmatrix}$$

Eigen value

<g>

- Eigen value Problem, no excitation: $h(t) = 0$, $b(t) = 0$

General E.P.

- From: Homogeneous equa. of motion

$$M\ddot{x}(t) + P\dot{x}(t) + Q\,x(t) = \underline{0}$$

- Solution statement:

$$x(t) = \tilde{x}\cdot e^{\lambda t}$$

$$\Rightarrow \underline{(M\lambda^2 + P\lambda + Q)\,\tilde{x} = 0}$$

- It would be meaning less / trivial if:

$$\tilde{x} = 0$$

$\Rightarrow$ Condition for non-trivial solution:

$$\det(M\lambda^2 + P\lambda + Q) = 0$$

$$\Leftrightarrow a_n\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$$

$\underline{n = 2f}$

Normalize it:

$$\det\left(M^{-1}(M\lambda^2 + P\lambda + Q)\right) = 0$$

$$\Leftrightarrow \frac{1}{\det(M)}\cdot\det(M\lambda^2 + P\lambda + Q) = 0$$

$$\Leftrightarrow \lambda^n + b_{n-1}\lambda^{n-1} + \dots + b_1\lambda + b_0 = 0$$

Special Matrix E.P.

- State space representation.

$$\dot{w}(t) = A\,w(t) + b(t) \quad \text{"0"}$$

$$w(t) = \tilde{w}\cdot e^{\lambda t}$$

$$\Rightarrow \cancel{w(t) = \tilde{w}\cdot e^{\lambda t}}\ \underline{(\lambda E - A)\,\tilde{w} = 0}$$

$$\tilde{w} = 0$$

$$\det(\lambda E - A) = 0$$

From <s>, $A = \begin{bmatrix} 0 & E \\ -M^{-1}Q & -M^{-1}P \end{bmatrix}$

$$\Rightarrow \det(\lambda E - A) = \det\left[\lambda E(M^{-1}P + \lambda E) - (-E)(M^{-1}Q)\right]$$

$$= \det(\lambda^2 E + \lambda M^{-1}P + M^{-1}Q)$$

$$= \frac{1}{\det M}\det(M\lambda^2 + P\lambda + Q)$$

We end up with same problem equation

Eigenvectors

$$(M\lambda_k^2 + P\lambda_k + G)\tilde{x}_k = 0$$

$$(\lambda_k E - A) \cdot \tilde{\omega}_k = 0$$

$$(\bar{\lambda}_k E - A) \cdot \bar{\tilde{\omega}}_k = 0$$

conjugate complex root

- We can normalize : $\left[\begin{array}{l} \dfrac{\tilde{x}_k(\lambda_k)}{\tilde{x}_i} \quad \text{with } K = 1(1)(2f) \\ \text{or } \sqrt{\tilde{x}_k^T \tilde{x}_k} = 1 \text{ (Euclidian norm)} \end{array} \right.$

+) Eigen vectors tell us about eigen oscillation ?

+) Solution for conservative, non-gyroscopic systems:
(no damped force)

$\begin{cases} D, G, N = 0 \\ C \neq 0 \end{cases} \Leftrightarrow \begin{cases} P = 0 \\ Q \text{ is diagonal matrix} \end{cases}$

- Eigenvalues have only imaginary part
$\lambda_k = \pm i\omega_k \quad ; \quad \omega_k^2 = -\lambda_k^2$

$$\Rightarrow x(t) = \sum_{k=1}^{f} \left(c_k \tilde{x}_k e^{\lambda_k t} + \bar{c}_k \bar{\tilde{x}}_k e^{\lambda_k t} \right)$$

- For conjugate complex eigen values, the eigen vectors will be the same
$\tilde{x}_k = \bar{\tilde{x}}_k$

$$\Rightarrow x(t) = \sum_{k=1}^{f} \tilde{x}_k \left(c_k e^{i\omega_k t} + \bar{c}_k e^{-i\omega_k t} \right)$$

- c_k & $\bar{c}_k$ are also conjugate complexes : $c_k = \frac{1}{2}(A_k + iB_k)$, $k = 1(1)f$

Find c_k from initial conditions of x_0 & $\dot{x}_0$

- From $e^{\pm i\omega_k t} = \cos(\omega_k t) \pm i.\sin(\omega_k t)$

$$\Rightarrow \boxed{x(t) = \sum_{k=1}^{f} \tilde{x}_k \left(A_k \cos(\omega_k t) + B_k \sin(\omega_k t) \right)} \quad \text{or} \quad x(t) = \sum_{k=1}^{f} \tilde{x}_k \, p_k \cos(\omega_k t - \varphi_k)$$

with $\begin{cases} p_k = \sqrt{A_k^2 + B_k^2} \\ \tan \varphi_k = \dfrac{B_k}{A_k} \end{cases}$

this is the vibration stuff..

+) Solution for weakly damped, gyroscopic Systems $D, G \neq 0$

- Eigenvalues have real part:

$$\lambda_k = -\delta_k \pm i \cdot \omega_k$$

$\Rightarrow$ Eigenvectors: $\begin{cases} \tilde{x}_k = \tilde{x}_{k,Re} + \tilde{x}_{k,Im} \\ \bar{\tilde{x}}_k = \tilde{x}_{k,Re} - \tilde{x}_{k,Im} \end{cases}$ $k = 1(1)f$

$$\Rightarrow x(t) = \sum_{k=1}^{f} \underline{e^{-\delta_k t}} \cdot \rho_k \cdot \left[\tilde{x}_{k,Re} \cdot \cos(\omega_k t - \varphi_k) - \tilde{x}_{k,Im} \cdot \sin(\omega_k t - \varphi_k)\right]$$

with $\rho_k = \sqrt{A_k^2 + B_k^2}$ and $\tan \varphi_k = \dfrac{B_k}{A_k}$

$\rightarrow$ The oscillation applitude changes over time.

$\Rightarrow$ Stability depends on δ_k

$\Rightarrow$ +) Eigenvalue Stability Criteria:

- Asymptotically stable: $Re(\lambda_k) < 0 \quad \forall k = 1(1)n$

- Marginally stable: $Re(\lambda_k) \leq 0 \quad \forall_k = 1(1)n$

  & at least $\exists\ \lambda_i$ that: $Re(\lambda_i) = 0$

- Unstable: $\exists\ \lambda_i$ that $Re(\lambda_i) > 0$

L10) +) Homogeneous Solution of the State equations:

the eigen vector

$\dot{w}(t) = A \cdot w(t)$ Modal matrix $W = [\tilde{w}_1 | \tilde{w}_2 | .. | \tilde{w}_n]$

$w_h(t) = \sum_{k=1}^{n} (c_k \tilde{w}_k e^{\lambda_k t}) = W \cdot e^{\Lambda t} \cdot c$ with $\Lambda = diag(\lambda_k)$

$c = [c_1 \ .. \ c_n]^T$

- From $t = 0$: $w_0 = Wc \Rightarrow c = W^{-1} \cdot w_0$

Spectral matrix ~~Spectral~~

$\Rightarrow \underline{w_h(t)} = W \cdot e^{\Lambda t} \cdot W^{-1} \cdot w_0 = \phi(t) \cdot w_0$

Homogeneous solution

Fundamental matrix

State Equations & Fundamental Matrix

To solve State Equations, is to find the speeds & accelerations of state, given .. everything else

(9, 10)

- Homogeneous, time invariant state equation:

$b(t) = 0$ no excitation

$$\dot{w}(t) = A \cdot w(t) \quad ; \quad A(t) = \begin{bmatrix} 0 & E \\ -M^{-1}(t)\,Q(t) & -M^{-1}(t)\,P(t) \end{bmatrix}$$

- New solution statement:

$$w_h(t) = e^{At} \cdot w_0$$

Make sense cause : $\dot{w}_h(t) = A \cdot e^{At} \cdot w_0$

$\Rightarrow$ Satisfy state equation

- what we already have: $w_h(t) = \Phi(t) \cdot w_0$

$\Rightarrow$ Thus : $w_h(t) = \Phi(t) \cdot w_0 = e^{At} \cdot w_0 \iff \Phi(t) = e^{At}$

?

$\iff$ The ~~tranformation matrix A~~ or fundamental matrix Φ transforms the initial state w_0 to the final state $w_h(t)$

- 3 Methods to calculate the fundamental matrix

1) Using Modal Matrix: W

$$\Phi(t) = W \cdot e^{\Lambda t} \cdot W^{-1}$$

- Algorithm: [W, lambda] = eig(A)

W_invers $= W^{-1}$

for t in $0 \to T$:

for J in $1 \to n$:

$e^{\Lambda t} = [e^{\lambda_i t} \ldots]$

$\Phi(t) = W \cdot e^{\Lambda t} \cdot W^{-1}$

$w_h(t) = \Phi(t) \cdot w_0$

deltaT = Tcalc / Timesteps

for I = 1 : Timesteps

PHIexact

But it is not really $\Phi(t)$, because when finding lambda, we have numerical approximation

2) Using Truncated Finite Series:

PHI series as approximation

- $\Phi(t) = e^{At} = E + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \dots = \sum_{k=0}^{m} \frac{(At)^k}{k!}$

- Algorithm:

after every Δt recalculate $\Phi(t)$ from $t=0$ => when $t >> \cancel{k}$ will overflow

```
for t in 0→T:
    PHI series = E(n)
    for k in 1 → order:
        └ PHIseries = PHIseries + (At)^k /(k!)
    w_h(t) = PHIseries . w_0
```

overflow??

order k as hyperparameter the higher k the more accurate but with computation cost

? taken out of loop

3) Cayley - Hamilton Theorem:

- From the characteristic polynomials of system matrix A:

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0 \; ; \quad a_n = 1$$

(1) $\Rightarrow \lambda^n \tilde{w} + a_{n-1}\lambda^{n-1}\tilde{w} + \dots + a_1\lambda\tilde{w} + a_0\tilde{w} = 0$; $\tilde{w}$ is an abitrary eigenvector

- Given: $A\tilde{w} = \lambda\tilde{w} \Rightarrow A^2\tilde{w} = \lambda A\tilde{w} = \lambda^2\tilde{w} \Rightarrow A^k\tilde{w} = \lambda^k\tilde{w}$

$\Rightarrow$ Replace back to (1):

$$A^n\tilde{w} + a_{n-1}A^{n-1}\tilde{w} + \dots + a_0 E\tilde{w} = 0$$ correct for all eigenvectors

$$\Leftrightarrow A^n + a_{n-1}A^{n-1} + \dots + a_0 E = 0$$

$$\Leftrightarrow A^n = -a_{n-1}A^{n-1} - a_{n-2}A^{n-1} - \dots - a_0 E$$

$n = 2.f$

f: DoF

$\Rightarrow$ Replace into infinite series of e^{At}:

$$\Phi(t) = e^{At} = \alpha_0(t)E + \alpha_1(t)A + \dots + \alpha_{n-1}(t)A^{n-1}$$

from an infinite series of e^{At} we get a finite series

+) To find $\alpha_i(t)$:

$$e^{\lambda_k t} = \alpha_0(t) + \alpha_1(t)\lambda_k + .. + \alpha_{n-1}(t) . \lambda_k^{n-1} \qquad k = 1(1)n$$

$$\Rightarrow \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & .. & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & .. & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \lambda_n^2 & .. & \lambda_n^{n-1} \end{bmatrix} . \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$$

$\Rightarrow$ Algorithm:

λ = eig (system A)

λ_{mat} = ..

evek = ..

alpha = λ_{mat}^{-1} . evek^T

PHIexactCH = .. alpha ..

$w_h(t) = \Phi(t) . w_0$

(4) Characteristics of Fundamental Matrix $\Phi(t)$

+) $\Phi(t=0) = E$

+) $\dot{\Phi}(t) = A.\Phi(t) = \Phi(t).A$

+) $\Phi(t).A^{-1} = A^{-1}.\Phi(t)$

+) $\int_0^t \Phi(\tau)d\tau = (\Phi(t) - E).A^{-1}$

+) $\Phi^{-1}(t) = \Phi(-t)$

+) $\Phi(t_1 + t_2) = \Phi(t_1).\Phi(t_2) = \Phi(t_2).\Phi(t_1)$

+) Solution of the Inhomogeneous State Equation:

- Inhomogeneous, time invariant State Equation:

(1) $$\dot{w}(t) = A \cdot w(t) + b(t)$$

- Particular solution statement with yet unknown funcs: $a_i^*(t)$

$$w_p(t) = \Phi(t) \cdot a^*(t) = e^{At} \cdot a^*(t)$$

$$\Rightarrow \dot{w}_p(t) = A \cdot e^{At} \cdot a^*(t) + e^{At} \cdot \dot{a}^*(t)$$

$$= A \cdot w_p(t) + \Phi(t) \cdot \dot{a}^*(t)$$

- Compared with (1):

$$\Rightarrow \dot{a}^*(t) = \Phi^{-1}(t) \cdot b(t)$$

$$\Leftrightarrow \int_{t_0=0}^{t} \dot{a}^*(\tau)\, d\tau = \int_0^t \Phi^{-1}(\tau)\, b(\tau)\, d\tau \quad ; \text{ choose } a^*(0) = 0$$

$$\Leftrightarrow a^*(t) = \int_0^t \Phi(-\tau)\, b(\tau)\, d\tau$$

$$\Leftrightarrow \boxed{w_p(t) = \Phi(t) \cdot \int_0^t \Phi(-\tau)\, b(\tau)\, d\tau = \int_0^t \Phi(t-\tau)\, b(\tau)\, d\tau}$$

- General solution: $w_t = w_p(t) + w_h(t)$

$$= \Phi(t) \cdot w_0 + \Phi(t) \cdot \int_0^t \Phi(-\tau)\, b(\tau)\, d\tau$$

$$= \Phi(t) \left[w_0 + \int_0^{t-\Delta t} \Phi(-\tau) b(\tau) d\tau + \int_{t-\Delta t}^{t} \ldots \right]$$

- With $\Phi(t) = \Phi(\Delta t)\, \Phi(t-\Delta t)$

$$\Rightarrow w(t) = \Phi(\Delta t) \left[w(t-\Delta t) + \Phi(t-\Delta t) \int_{t-\Delta t}^{t} \Phi(-\tau)\, b(\tau)\, d\tau \right]$$

- Use integral approximation for scalar func:

$$\int_{t-\Delta t}^{t} \Phi(-\tau)\, b(\tau)\, d\tau = \frac{\Delta t}{2} \left[\Phi(-(t-\Delta t))\, b(t-\Delta t) + \Phi(-t)\, b(t) \right]$$

$\Rightarrow \dots$

$$\Rightarrow w(t) = \Phi(\Delta t)\left[w(t-\Delta t) + \frac{\Delta t}{2}\cdot b(t-\Delta t)\right] + \frac{\Delta t}{2} b(t)$$

$$\Leftrightarrow \boxed{\begin{aligned} & w(t+\Delta t) = \Phi(\Delta t)\left[w(t) + \frac{1}{2} b(t)\,\Delta t\right] + \frac{1}{2} b(t+\Delta t)\,.\,\Delta t \\ & \begin{cases} w_h(t+\Delta t) = \Phi(\Delta t)\,.\, w_h(t) \\ w_p(t+\Delta t) = \Phi(\Delta t)\left[w_p(t) + \frac{1}{2} b(t)\,.\,\Delta t\right] + \frac{1}{2} b(t+\Delta t)\,.\,\Delta t \end{cases} \end{aligned}}$$

↳ Recursion algorithm.

- Linearized Equation of Motion: $M\ddot{x} + P\dot{x} + Q = h(t)$
- Excitation profile $\Rightarrow h(t) \Rightarrow b(t) = [0 \quad M^{-1}h]^T$
- → State Equations $\dot{w} = Aw + b$ → Find fundamental matrix Φ from Modal matrix W or (in)finite series..
- Recursion on $w(t)$

⊕ This is for general excitation profile

With special excitation (step, harmonic, periodic) we will have specific approaches to each.

MBD

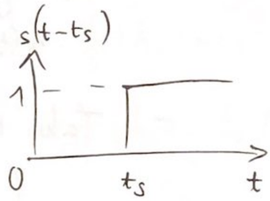
4.1) State Equation of Linear Oscillation Systems

⊗ System with Step Excitation

$$\dot{w}(t) = A \cdot w(t) + b(t)$$

$$w(t) = \Phi(t)\, w_0 + \int_0^t \Phi(t-\tau)\, b(\tau)\, d\tau$$

- Step function: $s(t-t_s) = \begin{cases} 0 & \text{for } t < t_s \\ 1 & \text{for } t \geq t_s \end{cases}$



$\Rightarrow \; b(t) = b_s \cdot s(t-t_s)$

b_s ↳ constant scaling vector

- Step response:

$0 \leq t \leq t_s \Rightarrow w(t) = \Phi(t) \cdot w_0$

$$t > t_s \Rightarrow w(t) = \Phi(t) \cdot w_0 + \int_0^t \Phi(t-\tau) \cdot b_s\, d\tau$$

$$= \Phi(t) \cdot w_0 + \left[E - \Phi(t-t_s)\right](-A^{-1} b_s)$$

$$= \Phi(t) \cdot w_0 + \left[E - \Phi(t-t_s)\right] \cdot w_\infty$$

$w_\infty = -A^{-1} \cdot b_s$ is the new static equilibrium

$$w(t) = \Phi(t) \cdot w_0 + w_\infty - \Phi(t-t_s) \cdot w_\infty$$

$$= w_h(t) + w_p(t)$$

$$w_p(t) = -\left[E - \Phi(t)\right](A^{-1} b)$$

- If system is stable, $\lim_{t \to \infty} \Phi(t) = 0$

$$w_\infty = 0 \cdot w_0 + (E - 0)(-A^{-1} b_s) = -A^{-1} b_s$$

Equations of Motion of Linear Oscillation Systems

System with Harmonic Excitation:

- Inhomogeneous equation of motion with harmonic excitation

$$\underline{M.\ddot{x}(t) + P.\dot{x}(t) + Q.x(t)} = \underline{h_c.\cos\Omega t + h_s.\sin\Omega t}$$

- Steady state solution statement

$$x(t) = x_c.\cos\Omega t + x_s.\sin\Omega t$$

Real Notation

- Take the derivatives & replace back, we will get:

$$\begin{bmatrix} Q-\Omega^2 M & \Omega P \\ -\Omega.P & Q-\Omega^2 M \end{bmatrix} \cdot \begin{bmatrix} x_c \\ x_s \end{bmatrix} = \begin{bmatrix} h_c \\ h_s \end{bmatrix}$$

Dynamic Stiffness Matrix

- Solution only for $\det(F^{-1}) \neq 0$; $F = \begin{bmatrix} Q-\Omega^2 M & \Omega P \\ -\Omega P & Q-\Omega^2 M \end{bmatrix}^{-1}$

Real frequency response matrix

$$\Rightarrow \begin{bmatrix} x_c \\ x_s \end{bmatrix} = F.\begin{bmatrix} h_c \\ h_s \end{bmatrix}$$

shows how the system oscillation behave under given load $\begin{bmatrix} h_c \\ h_s \end{bmatrix}$

- For the k^{th} oscillation coordinate:

$$x_k(t) = \hat{x}_k.\cos(\Omega t - \psi_k) \quad \text{with} \quad \begin{cases} \hat{x}_k = \sqrt{x_{k,c}^2 + x_{k,s}^2} \\ \tan\psi_k = \dfrac{x_{k,s}}{x_{k,c}} \end{cases} \quad k = 1(1)f$$

q=

+) Harmonic Excitation without Damping & Gyroscopic Influence:

$$P = 0 \Rightarrow D = G = 0$$

$$\Rightarrow \begin{cases} (Q-\Omega^2 M)\,x_c = h_c \\ (Q-\Omega^2 M)\,x_s = h_s \end{cases} \Rightarrow \text{Solution using the rule of Cramer} \begin{cases} x_{ck} = \dfrac{D_{ck}}{N} \\ x_{sk} = \dfrac{D_{sk}}{N} \end{cases}$$

$$N = \det(Q - \Omega^2 M) \qquad k = 1(1)f$$

$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \neq q_s + x$; $x = \begin{bmatrix} \bar{q}_1 \\ \bar{q}_1 \end{bmatrix}$ as deviation (after linearization)

$x_k = \bar{q}_k$ each has its own oscillation

+) Description in Complex Notation

$$e^{\pm i\Omega t} = \cos\Omega t \pm i \sin\Omega t$$

$$\Rightarrow \quad h(t) = h_c \cdot \cos\Omega t + h_s \cdot \sin\Omega t$$

$$= h_c \cdot \frac{1}{2}\left(e^{i\Omega t} + e^{-i\Omega t}\right) + h_s \cdot \frac{1}{2} i \left(e^{-i\Omega t} - e^{i\Omega t}\right)$$

$$= e^{i\Omega t}\frac{1}{2}(h_c - i h_s) + e^{-i\Omega t}\frac{1}{2}(h_c + i h_s)$$

$$= h^* \cdot e^{i\Omega t} + \bar{h}^* \cdot e^{-i\Omega t} \quad ; \quad \begin{cases} h^* = \frac{1}{2}(h_c - i \cdot h_s) \\ \bar{h}^* = \frac{1}{2}(h_c + i \cdot h_s) \end{cases}$$

- Equation of Motion with harmonic excitation in complex notation:

$$M \cdot \ddot{x}(t) + P \cdot \dot{x}(t) + Q \cdot x(t) = h^* \cdot e^{i\Omega t} + \bar{h}^* \cdot e^{-i\Omega t}$$

- Steady state solution statement in complex notation:

$$x(t) = x^* \cdot e^{i\Omega t} + \bar{x}^* \cdot e^{-i\Omega t} \quad ; \quad x^* = \frac{1}{2} x_c - i \cdot x_s$$

- Replace back: $\begin{cases} (Q - \Omega^2 M + i\Omega P)\, x^* = h^* \\ (Q - \Omega^2 M - i\Omega P)\, \bar{x}^* = \bar{h}^* \end{cases}$

$$\Leftrightarrow \quad F^* = \left(Q - \Omega^2 M + i\Omega P\right)^{-1}$$

$$\Leftrightarrow \quad \begin{cases} x^* = F^* h^* \\ \bar{x}^* = \bar{F}^* \cdot \bar{h}^* \end{cases} \quad \text{then} \quad \begin{cases} x_c = x^* + \bar{x}^* \\ x_s = i(x^* - \bar{x}^*) \end{cases} \quad \text{or} \quad \begin{cases} x_c = 2 \cdot Re(x^*) \\ x_s = -2 Im(x^*) \end{cases}$$

When $\Omega = 0$, we get a static ~~dyn~~ stiffness matrix

$$F^{-1} = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}$$

⑧ State Equation of Linear Oscillation Systems

- State equation with harmonic excitation:

$$\dot{w}(t) = A.w(t) + b_c.\cos\Omega t + b_s.\sin\Omega t$$

$$b(t) = b_c.\cos\Omega t + b_s.\sin\Omega t$$

Revisit harmonic excitation but with state equation (before was Equation of Motion L_{11})

- As from Euler equation: $e^{\pm i\Omega t} = \cos\Omega t \pm i.\sin\Omega t$

$$\Rightarrow \quad b(t) = e^{i\Omega t}\frac{1}{2}(b_c - i.b_s) + e^{-i\Omega t}\frac{1}{2}(b_c + i.b_s)$$

$$= b^*.e^{i\Omega t} + \overline{b^*}.e^{-i\Omega t}$$

$$\Rightarrow \quad \dot{w}(t) = A.w(t) + b^* e^{i\Omega t} + \overline{b^*}e^{-i\Omega t}$$

From $L_{9,10}$, we already have a way to find homogeneous solution of the state equation : $w_h(t) = \Phi(t).w_0$

$\Rightarrow$ Only need to focus on Particular Solution:

check L_{10}/

- Particular solution using Fundamental Matrix:

$$w_p(t) = \int_0^t \Phi(t-\tau)\,b(\tau)\,d\tau \quad \text{with} \quad \Phi(t-\tau) = e^{A(t-\tau)}, \quad b(\tau) = b^* e^{i\Omega\tau} + \overline{b^*}e^{-i\Omega\tau}$$

$$= e^{At}\left(\int_0^t e^{(i\Omega E - A)\tau}d\tau\, b^* + \int_0^t e^{(-i\Omega E - A)\tau}d\tau\, \overline{b^*}\right)$$

$$= e^{At}\left[\left(e^{(i\Omega E - A)t} - E\right)(i\Omega E - A)^{-1}b^* + \left(e^{(-i\Omega E - A)t} - E\right)(-i\Omega E - A)^{-1}\overline{b^*}\right]$$

$$= e^{i\Omega t}(i\Omega E - A)^{-1}b^* + e^{-i\Omega t}(-i\Omega E - A)^{-1}\overline{b^*}$$

$$- e^{At}\left[(i\Omega E - A)^{-1}b^* + (-i\Omega E - A)^{-1}\overline{b^*}\right]$$

- Set $(i\Omega E - A)^{-1}b^* = w^* = \frac{1}{2}(w_c - i.w_s)$ — w^*: complex amplitude vector

$(-i\Omega E - A)^{-1}\overline{b^*} = \overline{w^*} = \frac{1}{2}(w_c + i.w_s)$ — $\overline{w^*}$: conjugate c.a.v.

$F_z^* = (i\Omega E - A)^{-1} \Rightarrow w^* = F_z^*.b^*$ — F_z^*: complex frequency response matrix

$\overline{F_z^*} = (-i\Omega E - A)^{-1} \Rightarrow \overline{w^*} = \overline{F_z^*}.\overline{b^*}$ — $\overline{F_z^*}$: conjugate c.f.r.m.

$$\Rightarrow w_p(t) = w^* e^{i\Omega t} + \overline{w}^* e^{-i\Omega t} - e^{At}(w^* + \overline{w}^*)$$

$$= w_c \cdot \cos\Omega t + w_s \cdot \sin\Omega t - \Phi(t) \cdot w_c$$

Also: $w_c = 2 \cdot Re(w^*) = 2 \cdot Re(\overline{w}^*) = 2 \cdot Re(F_2^* b^*) = 2 \cdot Re(\overline{F}_2^* \cdot \overline{b}^*)$

$w_s = -2 \cdot Im(w^*) = 2 \cdot Im(\overline{w}^*) = -2 \cdot Im(F_2^* \cdot b^*) = 2 \cdot Im(\overline{F}_2^* \cdot \overline{b}^*)$

⊛ <u>Algorithm</u>: From given $M\ddot{x} + P\dot{x} + Qx = h(t)$ and $h(t)$

$\Rightarrow$ State Equation: $\dot{w}(t) = A \cdot w(t) + b$

$$= \begin{bmatrix} 0 & E \\ -M^{-1}Q & -M^{-1}P \end{bmatrix} \cdot w(t) + \begin{bmatrix} 0 \\ M^{-1} \cdot h \end{bmatrix}$$

① Calculate complex frequency response matrix:

$$F_2^* = (i\Omega E - A)^{-1}$$

② Compilation of the real excitation vector:

$h_c =$ $\qquad$ $h_s =$

③ Calculation of complex excitation vector of state equation:

$$b^* = \frac{1}{2}\left(\begin{bmatrix} 0 \\ M^{-1} \cdot h_c \end{bmatrix} - i \cdot \begin{bmatrix} 0 \\ M^{-1} \cdot h_s \end{bmatrix}\right)$$

$W = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$

④ Calculate complex state vector:

$W_c = \begin{bmatrix} x_c \\ \dot{x}_c \end{bmatrix}$; $W_s = \begin{bmatrix} x_s \\ \dot{x}_s \end{bmatrix}$

$$w^* = F_2^* \cdot b^*$$

⑤ Calculate real state vectors:

$$w_c = 2 \cdot Re(w^*) \quad ; \quad w_s = -2 \cdot Im(w^*)$$

Front wheel ⑥

⑦

Rear ⑧

wheel ⑨

+) Linear Mechanical Systems with Periodic Excitation:

- By Fourier theorem, a periodic func can be expressed as a sum of harmonic func.s

$$\Rightarrow b_p(t) = \frac{1}{2} b_{p0} + \sum_{\nu=1}^{\infty} \left(b_{c\nu} \cos\nu\Omega t + b_{s\nu} \sin\nu\Omega t \right)$$

$$= \frac{1}{2} b_{p0} \frac{e^{0\Omega t} + e^{-0\Omega t}}{2} + \sum_{\nu=1}^{\infty} \left[\frac{1}{2}(b_{c\nu} - i b_{s\nu}) e^{i\nu\Omega t} + \frac{1}{2}(b_{c\nu} + i.b_{s\nu}) e^{-i\nu\Omega t} \right]$$

$$= \sum_{\nu=0}^{\infty} \left[\frac{1}{2}(b_{c\nu} - i.b_{s\nu}).e^{i\nu\Omega t} + \frac{1}{2}(b_{c\nu} + i.b_{s\nu}) e^{-i\nu\Omega t} \right] \quad \text{with } b_{c0} = \frac{1}{2} b_{p0}$$

$$= \sum_{\nu=-\infty}^{\infty} c_\nu e^{i\nu\Omega t} \quad \text{with } c_\nu = \frac{1}{2}(b_{c\nu} - i.b_{s\nu})$$

$$c_{-\nu} = \frac{1}{2}(b_{c\nu} + i.b_{s\nu})$$

$$c_0 = \frac{1}{2} b_{p0}$$

$$b_{c\nu} = \frac{2}{T} \int_0^T b_p(t).\cos\nu\Omega t.\,dt, \quad \nu = 0(1)\infty$$

$$b_{s\nu} = \frac{2}{T} \int_0^T b_p(t).\sin\nu\Omega t.\,dt, \quad \nu = 0(1)\infty$$

$b_{s\nu} =$

- Periodic excitation as a combination of step excitation and harmonic excitation:

$$b_p(t) = \underbrace{\frac{1}{2} b_{p0}}_{\text{Step excitation}} + \underbrace{\sum_{\nu=1}^{\infty} \left(b_{c\nu} \cos \nu\Omega t + b_{s\nu} \sin \nu \Omega t \right)}_{\text{harmonic excitation}}$$

$\Downarrow$ General stationary solution:

$$w_{\infty} = -A^{-1} b_s$$

$$\Rightarrow w_{p0\infty} = -\frac{1}{2} A^{-1} b_{p0}$$

$\Downarrow$ Particular solution:

$$w_p(t) = w_c \cdot \cos \Omega t + w_s \cdot \sin \Omega t - \Phi(t) \cdot w_c$$

If the system is asymptotically stable (all real part of $\lambda_k < 0$)

$$\Rightarrow \lim_{t \to \infty} \Phi(t) = \lim_{t \to \infty} W e^{\Lambda t} W^{-1} = 0$$

$\Rightarrow$ Thus, stationary solution is given by:

$$w_{p\infty}(t) = w_c \cdot \cos \Omega t + w_s \cdot \sin \Omega t$$

$$= w^* e^{i\Omega t} + \overline{w}^* e^{-i\Omega t}$$

$$w^* = \frac{1}{2} (w_c - i \cdot w_s)$$

$\Rightarrow$ Combine all: Stationary solution for general periodic excitation:

$$w_{p\nu\infty}(t) = w_{p0\infty} + \sum_{\nu=1}^{\infty} \left(w_{c\nu} \cos \nu \Omega t + w_{s\nu} \sin \nu \Omega t \right)$$

with

$$w_{p0\infty} = -\frac{1}{2} A^{-1} b_{p0}$$

$$w_{c\nu} = 2 \cdot \mathrm{Re}(w_\nu^*) = 2 \cdot \mathrm{Re}(F_{z\nu}^* b_\nu^*)$$

$$w_{s\nu} = -2 \cdot \mathrm{Im}(w_\nu^*) = -2 \cdot \mathrm{Im}(F_{z\nu}^* b_\nu^*)$$

$$F_{z\nu}^* = (i\nu\Omega E - A)^{-1}; \quad b_\nu^* = \frac{1}{2}(b_{c\nu} - i \cdot b_{s\nu}); \quad w_\nu^* = F_{z\nu}^* \cdot b_\nu^*$$

/ Ⓐ Pay attention to unit (N, Nm, ...) from 1.0 → 2.0

MBD

Formulas

Ng Huu Duc

1, **DoF**: 3D Space $f = 6.P_E + 3.P_b - b$
Planar System $f = 3P_E + 2P_b - b$

2, Generalized Coordinates: $q = [q_1 \,..\, q_f]^T$ f: dof
aka Location vector

Position vector ${}^c r_{Si} = {}^c r_{Si}(q,t) = [{}^c x_{Si} \;\; {}^c y_{Si} \;\; {}^c z_{Si} \;\; \alpha \;\; \beta \;\; \gamma]^T$

Position vector of a system: $r = [{}^c r_{S1} \,..\, {}^c r_{Sp}]^T$; p bodies

Example: $r = [x_1 \;\; y_1 \;\; \gamma_1 \;\; x_2 \;\; y_2 \;\; \gamma_2]^T$

3, Kinematics

+, Translational: $${}^c r_{Si} = \begin{bmatrix} {}^c x_{Si}(q_1, .. q_f, t) \\ {}^c y_{Si}(q_1, .. q_f, t) \\ {}^c z_{Si}(q_1, .. q_f, t) \end{bmatrix}$$

$$\underset{3\times1}{{}^c v_{Si}} = {}^c \dot{r}_{Si} = \underset{3\times f}{{}^c \bar{J}_{Ti}} \cdot \underset{f\times1}{\dot{q}} + \underset{3\times1}{\underbrace{{}^c \bar{v}_{Si}}} \longrightarrow \text{local velocity}$$

$${}^c a_{Si} = {}^c \dot{v}_{Si} = {}^c J_{Ti} \cdot \ddot{q} + {}^c \dot{J}_{Ti} \cdot \dot{q} + {}^c \dot{\bar{v}}_{Si}$$

+) Rotational

$$ {}^{P}\omega_i = {}^{P}J_{Ri} \cdot \dot{q} + {}^{P}\overline{\omega}_i $$

$$ {}^{P}\dot{\omega}_i = {}^{P}J_{Ri} \cdot \ddot{q} + {}^{P}\dot{J}_{Ri} \cdot \dot{q} + {}^{P}\overline{\omega}_i $$

$$ \frac{\partial\, {}^{PK}S_i}{\partial q_i} \cdot {}^{PK}S_i^{T} = \begin{bmatrix} \square_z & & \square_y \\ & \square_x & \end{bmatrix} \Rightarrow {}^{P}J_{Ri} = \begin{array}{c} i^{th} \text{ column} \\ \begin{bmatrix} & \square & \end{bmatrix}_{3\times f} \end{array} $$

4) Kinetics

- Inertia Tensor: $\quad {}^{P}I_i = {}^{PK}S_i \cdot {}^{K}I_i \cdot {}^{PK}S_i^{T}$

+) Newton - Euler Equation:

p bodies

- Newton: $\quad {}^{P}f_i = m_i \cdot {}^{P}\dot{v}_{Si}$ $\qquad$ $3p$ funcs
- Euler: $\quad {}^{P}m_i = {}^{P}I_i \cdot {}^{P}\dot{\omega}_i + {}^{P}\tilde{\omega}_i \cdot {}^{P}I_i \cdot {}^{P}\omega_i$ $\qquad$ $3p$ funcs

$$ \Rightarrow \underbrace{{}^{P}\overline{M}(q,t)}_{6p\times f} \cdot \underbrace{\ddot{q}(t)}_{f\times 1} + \underbrace{{}^{P}\overline{g}(q,\dot{q},t)}_{6p\times 1} = \underbrace{{}^{P}\overline{d}(q,\dot{q},t)}_{6p\times 1} \qquad 6p \text{ equations} $$

$$ \underbrace{{}^{P}\overline{J}_{ges}^{T}(q,t)}_{f\times 6p} \Rightarrow \underbrace{M(q,t)}_{f\times f} \cdot \ddot{q}(t) + \underbrace{g(q,\dot{q},t)}_{f\times 1} = \underbrace{d(q,\dot{q},t)}_{f\times 1} \qquad f \text{ equations} $$

- The compact form: multiply Newton equation with $J_{Tr,i}^{T}$ left-side

Euler ———— $J_{Ro,i}^{T}$

+) Active force: $\quad d = \underbrace{{}^{P}\overline{J}_{ges}^{T}(q,t)}_{f\times 3} \cdot \underbrace{{}^{P}\overline{d}_E}_{3\times 1}$

- Gravitational force: $\quad mg$
- Spring: $\quad F_C = c \cdot \Delta x = c\Delta\gamma$
- Damping: $\quad F_D = k \cdot \Delta\dot{x}$

+) Planar System Tips

- $M = \begin{bmatrix} m_i & & \\ & m_i & \\ & & I_{zz} \end{bmatrix}$; ${}^{C}I_i = {}^{K}I_{zz}$; $d^E = \begin{bmatrix} f_x \\ f_y \\ m_z \end{bmatrix}$

+) Lagrangian Equation of 2nd kind:

$$\frac{d}{dt}\left(\frac{\partial E_{kin}}{\partial \dot{q}}\right) - \left(\frac{\partial E_{kin}}{\partial q}\right) = \underset{f\times 1}{d} = \sum_i^p \left(\underset{f\times 3}{{}^{C}J^T_{T,i}} \cdot \underset{3\times 1}{{}^{C}f^E_i} + \underset{f\times 3}{{}^{C}J^T_{R,i}} \cdot \underset{3\times 1}{{}^{C}m^E_i} \right)$$

- Conservative System:

$$\frac{d}{dt}\left(\frac{\partial E_{kin}}{\partial \dot{q}}\right) - \left(\frac{\partial E_{kin}}{\partial q}\right) = \frac{-\partial E_{pot}}{\partial q}$$

$L = E_{kin} - E_{pot}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

- Non-conservative System:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tau(q, \dot{q}, t)$$

+) Lagrangian Equation of 1st kind:

$$\begin{bmatrix} m_i E & \\ & {}^{0}I_i \end{bmatrix} \cdot \begin{bmatrix} {}^{0}\dot{v}_{Si} \\ {}^{0}\dot{\omega}_i \end{bmatrix} = \begin{bmatrix} {}^{0}f^E_i \\ {}^{0}m^E_i \end{bmatrix} - \begin{bmatrix} 0 \\ {}^{0}\tilde{\omega} \cdot {}^{0}I \cdot {}^{0}\omega \end{bmatrix} + \begin{bmatrix} {}^{0}f^z_i \\ {}^{0}m^z_i \end{bmatrix}$$

$$\Rightarrow \underset{6\times 6}{T_i} \cdot \underset{6\times 1}{\ddot{r}_i} = \underset{6\times 1}{e_i} + \underset{6\times 1}{z_i}$$

$$\Rightarrow \underset{6p\times 6p}{T} \cdot \underset{6p\times 1}{\ddot{r}} = \underset{6p\times 1}{e} + \underset{6p\times 1}{z}$$

p bodies

$$\underset{6p\times 1}{z} = \underset{6p\times b}{G^T} \cdot \underset{b\times 1}{\lambda}$$

b boundaries

$$\Leftrightarrow T \cdot \ddot{r} - G^T\lambda = e$$

$$\underset{b\times 6p}{G} = \left[G_1 \;\; \cdots \;\; G_p \right] \downarrow b \text{ boundary}$$

$\xrightarrow{\hspace{2em}}$ p bodies

- $g = 0$

$\Leftrightarrow G(r,t) \cdot \dot{r} + \bar{\gamma}(r,t) = 0$ with $\bar{\gamma}(r,t) = \frac{\partial g}{\partial t}$

$\Leftrightarrow G(r,t) \cdot \ddot{r} + \frac{dG}{dt} \cdot \dot{r} + \frac{d\bar{\gamma}}{dt} = 0 = G(r,t) \cdot \ddot{r} + \bar{\bar{\gamma}}(r,v,t)$

$\Leftrightarrow -G(r,t)\,\ddot{r} = \bar{\bar{\gamma}}(r,v,t)$ with $\bar{\bar{\gamma}} = \frac{dG}{dt}\dot{r} + \frac{d}{dt}\left(\frac{\partial g}{\partial t}\right)$

⇒ The Differential Algebraic Equation System:

$$\begin{bmatrix} T & -G^T \\ -G & 0 \end{bmatrix} \cdot \begin{bmatrix} \ddot{r} \\ \lambda \end{bmatrix} = \begin{bmatrix} e \\ \bar{\bar{\gamma}} \end{bmatrix} \begin{array}{l} 6p \\ b \end{array}$$

$(6p+b) \times 1$

$$\bar{\bar{\gamma}} = -G\ddot{r} = -GT^{-1}e - GT^{-1}G^T\lambda$$
$$= -GT^{-1}e - Q\lambda$$
$$\Leftrightarrow \quad Q\lambda = -GT^{-1}e - \bar{\bar{\gamma}}$$

- For planar system: $3p + b$

5) Linearization:

$$(M_0 + \tilde{M}_1)(\ddot{q}_s + \ddot{x}) + (g_0 + G_{1x} \cdot x + G_{1\dot{x}} \cdot \dot{x}) = d_0 + D_{1x} \cdot x + D_{1\dot{x}} \cdot \dot{x}$$

$$\Leftrightarrow \quad M_0 \cdot \ddot{x} + (G_{1\dot{x}} - D_{1\dot{x}}) \cdot \dot{x} + (\bar{M}_1 + G_{1x} - D_{1x}) \cdot x = d_0 - g_0 - M_0 \cdot \ddot{q}_s$$

$$M \cdot \ddot{x} + P \cdot \dot{x} + Q \cdot x = h$$

$M_1 \cdot \ddot{q}_s = \bar{M}_1 \cdot x$

- G, D: $\begin{bmatrix} \; \end{bmatrix}_{f \times f}$ $\quad \begin{array}{cc} g_1 & d_1 \\ \downarrow & \\ g_f & d_f \end{array}$

$\partial q_1 \longrightarrow \partial q_f$

$$M = \begin{bmatrix} \sum \ddot{q}_{ks} \cdot \frac{\text{wholerow}}{\partial q_i} & & \\ & & \\ & & \end{bmatrix}$$

- DGCN: P. velocity dependent   Q: position dependent

Damping $\quad D = \frac{1}{2}(P + P^T)$ Conservative $\quad C = \frac{1}{2}(Q + Q^T)$

Gyroscope $\quad G = \frac{1}{2}(P - P^T)$ Damping $\quad N = \frac{1}{2}(Q - Q^T)$

6) State Equation:

Conservative system
$P = 0$

+) Common Mechanical System:

$$M\ddot{x} + P\dot{x} + Qx = h$$

$$\Leftrightarrow \quad \ddot{x} = M^{-1}(-P\dot{x} - Qx + h)$$

$$\Leftrightarrow \quad \dot{w} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & E \\ -M^{-1}Q & -M^{-1}P \end{bmatrix} \cdot \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}h \end{bmatrix} = A.w + b$$

+) General mechanical System:

7) Eigen value problem: (E.P.)

General E. P.	Special Matrix. E. P.
$(M.\lambda^2 + P\lambda + Q).\tilde{x} = 0$	$(\lambda E - A).\tilde{w} = 0$
$\frac{1}{\det M} \cdot \det(M\lambda^2 + P\lambda + Q) = 0$	$\det(\lambda E - A) = 0$

- After all λ are found, replace back λ_k to find eigen vector $\tilde{x}, \tilde{w}$

$$[\ ?\ ]\,\tilde{x} = 0 \Rightarrow \tilde{x} \qquad\qquad [\ ?\ ]\,\tilde{w} = 0 \Rightarrow \tilde{w}$$

- For non-damped system: $P = 0$. $\lambda_k = \pm\, i\omega_k$ (no real part)

$A = W.\Lambda.W^{-1}$?

$$W(t) = \Phi(t).w_0 + w_c . \cos \Omega t + w_s . \sin \Omega t - \Phi(t). w_c$$

$$w_p(t) = e^{i\Omega t}(i\Omega E - A)^{-1} b^* + e^{-i\Omega t}(-i\Omega E - A)^{-1} \bar{b}^* - e^{At}\left[(i\Omega E - A)^{-1} b^* + (-i\Omega E - A)^{-1} \bar{b}^*\right]$$

$$= e^{i\Omega t} . w^* + e^{-i\Omega t} \bar{w}^* - e^{At}(w^* + \bar{w}^*)$$

$$= w_c . \cos \Omega t + w_s . \sin \Omega t - \Phi(t). w_c$$

$$\cancel{= e^{i\Omega t}(i\Omega E - A)^{-1} b^*}$$

$$= (e^{i\Omega t} - e^{At}).(i\Omega E - A)^{-1} b^* + (e^{-i\Omega t} - e^{At}).(-i\Omega E - A)^{-1} \bar{b}^*$$