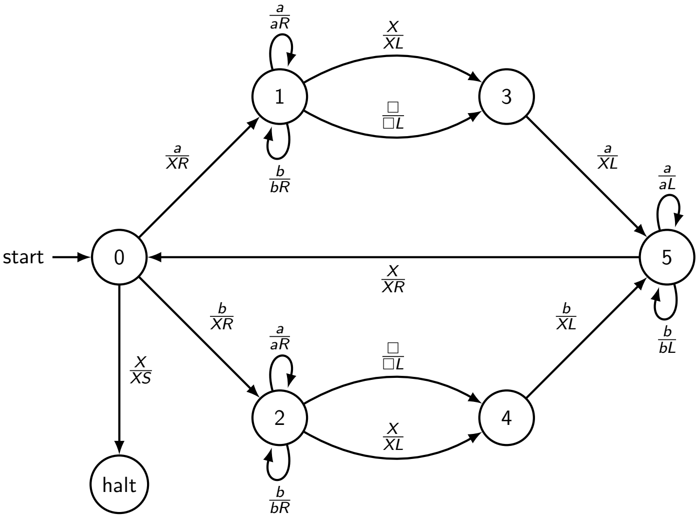
Introduction to Theory of Computation

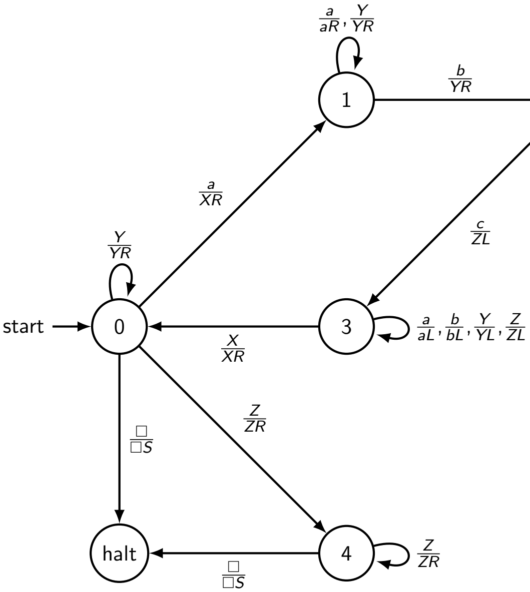
Chapter 4, Turing Machines

March 1, 2017

Even Palindromes



$a^n b^n c^n$



start
0
1
2
3
4
halt
$\frac{a}{aR}, \frac{Y}{YR}$
$\frac{b}{bR}, \frac{Z}{ZR}$
$\frac{b}{YR}$
$\frac{a}{XR}$
$\frac{c}{ZL}$
$\frac{Y}{YR}$
$\frac{a}{aL}, \frac{b}{bL}, \frac{Y}{YL}, \frac{Z}{ZL}$
$\frac{X}{XR}$
$\frac{Z}{ZR}$
$\frac{\square}{\square S}$
$\frac{Z}{ZR}$
$\frac{\square}{\square S}$

Equivalent models:

1. One-tape Turing machines.
2. k-tape Turing machines.
3. Non-deterministic Turing machines.
4. Java programs.
5. Scheme programs.
6. C++ programs.
7. ...

The Church-Turing Thesis

Every computational process that is intuitively considered to be an algorithm can be converted to a Turing machine.

Decidability

A language A over Σ is *decidable* if there exists a Turing machine M such that for every string $w \in \Sigma^*$:

1. If $w \in A$ then M, started on w, halts in the accept state.
2. If $w \notin A$ then M, started on w, halts in the reject state.

Describing machines and problems as strings

- We assume any machine (DFA, PDA, TM) can be described by a string M using some alphabet.
- The input to any machine is a string w using some alphabet.
- We can thus describe both a machine M and its input w, with a pair of strings: (M, w).
- This pair can be converted to a single string $\langle M, w \rangle$.
- For convenience, we assume $\langle M, w \rangle$ is encoded in binary.
- In general, $\langle M \rangle$ means: encode M as a binary string.
- We can now define a language A as the set of all strings $\langle M, w \rangle$ such that w is in the computation model of M.

The language A_{DFA} is decidable

$$A_{DFA} = \{\langle M, w\rangle : M \text{ is a DFA that accepts } w\}$$

Proof?

The language A_{DFA} is decidable

$$A_{DFA} = \{\langle M, w \rangle : M \text{ is a DFA that accepts } w\}$$

Proof?

- Given input $\langle M, w \rangle$:
  - Run M on w.
  - It must terminate.
  - If it accepts, accept, else reject.

The language A_{NFA} is decidable

$$A_{NFA} = \{\langle M, w\rangle : M \text{ is a NFA that accepts } w\}$$

Proof?

The language A_{NFA} is decidable

$$A_{NFA} = \{\langle M, w \rangle : M \text{ is a NFA that accepts } w\}$$

Proof?

- Given input $\langle M, w \rangle$:
  - Convert NFA M to DFA N.
  - This algorithm terminates.
  - Run N on w.
  - It must terminate.
  - If it accepts, accept, else reject.

The language A_{CGF} is decidable

$$A_{NFA} = \{\langle M, w \rangle : M \text{ is a CFG that accepts } w\}$$

Proof?

The language A_{CGF} is decidable

$$A_{NFA} = \{\langle M, w\rangle : M \text{ is a CFG that accepts } w\}$$

Proof?

- Given input $\langle M, w\rangle$:
  - Convert CFG M to Chomsky normal form CFG N.
  - This algorithm terminates.
  - Run N on w for all derivations up to length $2|w|$.
  - There are a finite number of these, so it must terminate.
  - If any derivation accepts, accept, else reject.

The language A_{TM} is not decidable.

$$A_{TM} = \{\langle M, w\rangle : M \text{ is a TM that accepts } w\}$$

Proof?

The language A_{TM} is not decidable.

$$A_{TM} = \{\langle M, w\rangle : M \text{ is a TM that accepts } w\}$$

Proof? By contradiction.

- Assume there is a TM H that decides this language.
- Construct the following TM, D:

D: On input $\langle M\rangle$:
Step 1: Run H on $\langle M, \langle M\rangle\rangle$.
Step 2: If H accepts, reject, else accept.

- If H accepts $\langle D, \langle D\rangle\rangle$, then D rejects $\langle D\rangle$.
- If H rejects $\langle D, \langle D\rangle\rangle$, then D accepts $\langle D\rangle$.
- H does not recognize A_{TM}.

Diagonal argument

- Machine H that decides A_{TM} can fill in this table:

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	...
M_0	**accept**	*accept*	*accept*	*reject*	*accept*	*reject*	...
M_1	*accept*	**reject**	*accept*	*accept*	*accept*	*reject*	...
M_2	*accept*	*reject*	**accept**	*accept*	*accept*	*reject*	...
M_3	*accept*	*accept*	*reject*	**reject**	*accept*	*accept*	...
M_4	*reject*	*accept*	*accept*	*reject*	**accept**	*accept*	...
M_5	*reject*	*reject*	*accept*	*accept*	*accept*	**reject**	...
...	...	...	...	...	...	...	...

- D uses H to give the opposite answer on the diagonal.
- H must give the wrong answer somewhere on machine D.

The language *Halt* is not decidable.

$$Halt = \{\langle M, w\rangle : M \text{ is a TM that terminates on } w\}$$

Proof?

The language *Halt* is not decidable.

$$Halt = \{\langle M, w\rangle : M \text{ is a TM that terminates on } w\}$$

Proof?

By contradiction. Assume there is a TM H that decides this language. Construct the following TM, Q:

$$\boxed{\begin{array}{l} Q\text{: On input } \langle M\rangle\text{:} \\ \qquad\qquad\qquad \text{While } H(\langle M, \langle M\rangle\rangle) \text{ do endwhile} \end{array}}$$

- What happens if we run Q on itself?
- $Q(\langle Q\rangle)$ terminates iff $Q(\langle Q\rangle)$ does not terminate.

The language M_a is not decidable.

$$M_a = \{\langle M \rangle \mid \mathcal{L}(M) = \{a\}\}$$

Proof?

The language M_a is not decidable.

$$M_a = \{\langle M \rangle \mid \mathcal{L}(M) = \{a\}\}$$

Proof?

By contradiction.

- Suppose TM A decides M_a.
- Construct the following TM, H:

> H: On input $\langle M, w \rangle$:
>
> - Construct TM D:
>
>   > D: On input $\langle s \rangle$:
>   >
>   > - Run M on w.
>   > - If $s = a$ accept, else reject.
>
> - Run A on D.

The language M_a is not decidable.

$$M_a = \{\langle M \rangle \mid \mathcal{L}(M) = \{a\}\}$$

Proof?

By contradiction.

- Suppose TM A decides M_a.
- Construct the following TM, H:

  H: On input $\langle M, w \rangle$:
  - Construct TM D:

    D: On input $\langle s \rangle$:
    - Run M on w.
    - If $s = a$ accept, else reject.
  - Run A on D.
- $\mathcal{L}(D) = \{a\}$ iff M halts on w.

The language M_a is not decidable.

$$M_a = \{\langle M \rangle \mid \mathcal{L}(M) = \{a\}\}$$

Proof?

By contradiction.

- Suppose TM A decides M_a.
- Construct the following TM, H:

  H: On input $\langle M, w \rangle$:
  - Construct TM D:

    D: On input $\langle s \rangle$:
    - Run M on w.
    - If $s = a$ accept, else reject.
  - Run A on D.
- $\mathcal{L}(D) = \{a\}$ iff M halts on w.
- H decides the language *Halt*.
- But that's impossible!

Rice's Theorem

Let $\mathcal{T}$ be the set of all binary encoded TMs.
Let $\mathcal{P}$ be a subset of $\mathcal{T}$ such that

1. $\mathcal{P} \neq \emptyset$
2. $\mathcal{P} \neq \mathcal{T}$
3. If $L(M_1) = L(M_2)$, then either both or neither is in $\mathcal{P}$.

Then $\mathcal{P}$ is undecidable.

Rice's Theorem Examples

1. $\{\langle M \rangle \mid M$ accepts only inputs in the language $a^*b^*\}$
2. $\{\langle M \rangle \mid M$ accepts input of length $n^2\}$
3. $\{\langle M \rangle \mid M$ accepts input of length $k\}$
4. $\{\langle M \rangle \mid M$ accepts $\epsilon\}$
5. $\{\langle M \rangle \mid M$ accepts all inputs$\}$
6. $\{\langle M \rangle \mid M$ does not accept all inputs$\}$
7. $\{\langle M \rangle \mid M$ accepts some input$\}$
8. $\{\langle M \rangle \mid M$ does not accept any input$\}$

None of these is decideable, or even enumerable.

Enumerability

A language A over Σ is *enumerable* if there exists a Turing machine M such that for every string $w \in \Sigma^*$:

1. If $w \in A$ then M, started on w, halts in the accept state.
2. If $w \notin A$ then M, started on w, either halts in the reject state or loops forever.

Hilbert's problem is enumerable but not decidable

$$\mathit{Hilbert} = \{\langle p \rangle \quad : \quad p \text{ is a polynomial with integer coefficients that has an integral root}\}$$

The language A_{TM} is enumerable but not decidable.

$$A_{TM} = \{\langle M, w\rangle : M \text{ is a TM that accepts } w\}$$

Proof?

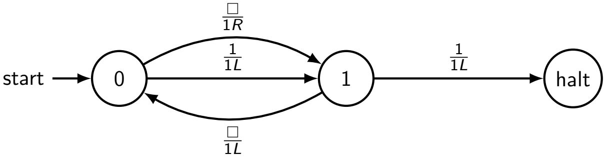
Why do we call it enumerable?

- If we can enumerate the elements with an algorithm, then we can create an algorithm to correctly identify elements of the set.
- If we can correctly identify elements of the set, then we can build an algorithm to enumerate them.

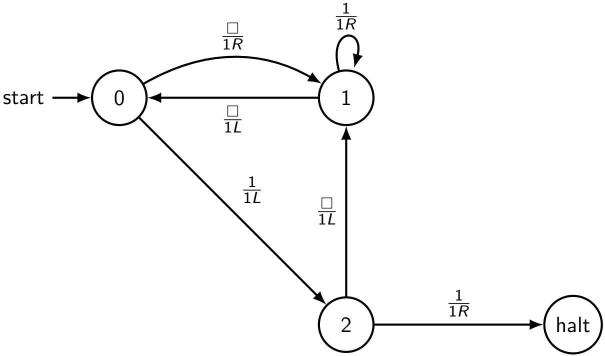
Busy beavers are not enumerable

The nth busy beaver number is the largest (finite) number of 1s that can be output by a Turing machine with n states.

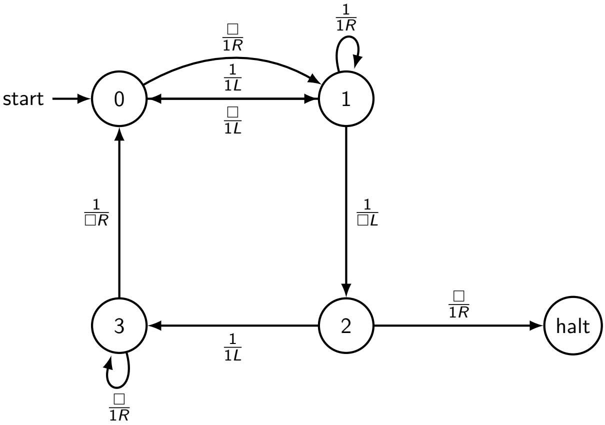
2 State Busy Beaver: four 1s



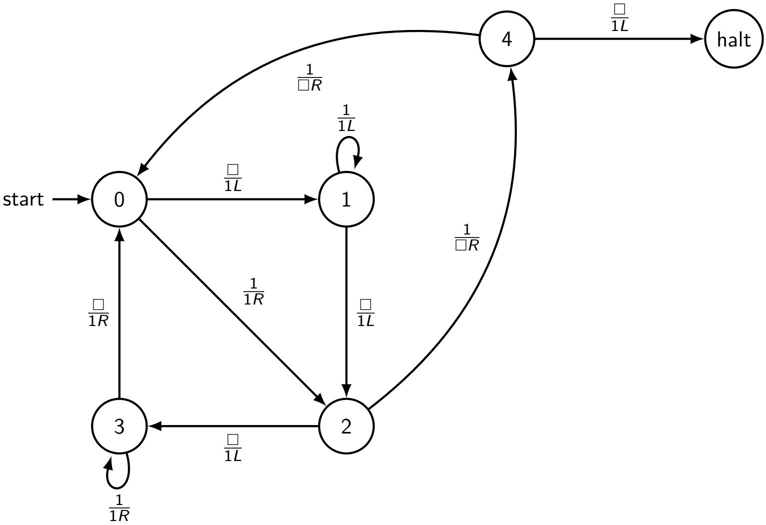
3 State Busy Beaver: six 1s



4 State Busy Beaver: thirteen 1s



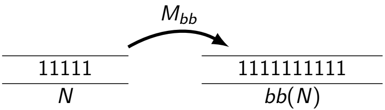
5 State Busy Beaver (?): 4098 1s



Proof Busy Beaver function is uncomputable

Proof by contradiction.

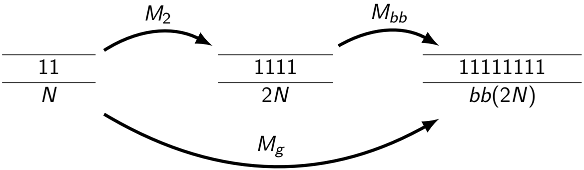
- Let $bb(n)$ be the largest (finite) number of 1's output by a Turing Machine with n states.
- Suppose there is a Turing Machine M_{bb} that computes $bb(n)$, that is, starting with n on the tape, the machine halts with $bb(n)$ on the tape.



- Note: this is a new use of TMs, computing a function from input to output, not recognizing a language.

Busy Beaver proof

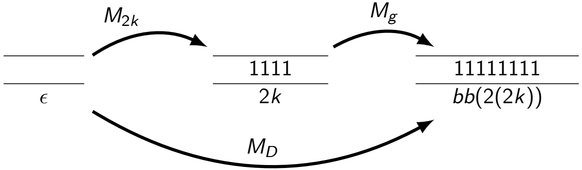
- Let $g(n) = bb(2n)$. We can build a TM for g by starting with a machine that doubles the input, and then runs the machine M_{bb}.



- Suppose the machine for g, M_g has k states.

Busy Beaver proof

- Build a machine M_{2k} with $2k$ states that does nothing but put $2k$ 1s on a blank tape.
- Now build a machine M_D that starts by putting $2k$ 1's on the tape, and then runs the M_g machine.



- M_D can be built with $3k$ states.
- The output of M_D is $g(2k) = bb(2(2k)) = bb(4k)$ 1s.
- Do you see the problem?