Digital Communcation Assignment

Surabhi Seetha

Contents

1 Two Dice

1.1 Sum of Independent Random Variables

Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

1.1.1. *The Uniform distribution* Let $X_i \in \{1,2,3,4,5,6\}, i = 1,2,$ be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{1.1.1}$$

The desired outcome is

$$X = X_1 + X_2, \tag{1.1.2}$$
$$\implies X \in \{1,2,\dots,12\} \tag{1.1.3}$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \tag{1.1.4}$$

1.1.2. *Convolution:* From (1.1.2),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{1.1.5}$$
$$= \sum_k \Pr(X_1 = n-k | X_2 = k)\, p_{X_2}(k) \tag{1.1.6}$$

after unconditioning. $\because X_1$ and X_2 are independent,

$$\Pr(X_1 = n-k|X_2 = k)$$
$$= \Pr(X_1 = n-k) = p_{X_1}(n-k) \tag{1.1.7}$$

From (1.1.6) and (1.1.7),

$$p_X(n) = \sum_k p_{X_1}(n-k)p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{1.1.8}$$

where $*$ denotes the convolution operation. Substituting from (1.1.1) in (1.1.8),

$$p_X(n) = \frac{1}{6}\sum_{k=1}^{6} p_{X_1}(n-k) = \frac{1}{6}\sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{1.1.9}$$

$$\because p_{X_1}(k) = 0, \quad k \le 1, k \ge 6. \tag{1.1.10}$$

From (1.1.9),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6}\sum_{k=1}^{n-1} p_{X_1}(k) & 1 \le n-1 \le 6 \\ \frac{1}{6}\sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n-6 \le 6 \\ 0 & n > 12 \end{cases} \tag{1.1.11}$$

Substituting from (1.1.1) in (1.1.11),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \tag{1.1.12}$$

satisfying (1.1.4).

1.1.3. *The Z-transform:* The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \tag{1.1.13}$$

From (1.1.1) and (1.1.13),

$$P_{X_1}(z) = P_{X_2}(z) = \frac{1}{6}\sum_{n=1}^{6} z^{-n} \tag{1.1.14}$$
$$= \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, |z| > 1 \tag{1.1.15}$$

upon summing up the geometric progression.

$$\because p_X(n) = p_{X_1}(n) * p_{X_2}(n), \tag{1.1.16}$$
$$P_X(z) = P_{X_1}(z) P_{X_2}(z) \tag{1.1.17}$$

The above property follows from Fourier analysis and is fundamental to signal processing. From (1.1.15) and (1.1.17),

$$P_X(z) = \left\{ \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)} \right\}^2 \tag{1.1.18}$$

$$= \frac{1}{36} \frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{1.1.19}$$

Using the fact that

$$p_X(n-k) \overset{\mathcal{H}}{\longleftrightarrow} Z P_X(z) z^{-k}, \tag{1.1.20}$$

$$n u(n) \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{1.1.21}$$

after some algebra, it can be shown that

$$\begin{aligned} &\frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) \right. \\ &\qquad \left. + (n-13)\,u(n-13)\right] \\ &\overset{\mathcal{H}}{\longleftrightarrow} Z \frac{1}{36} \frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \end{aligned} \tag{1.1.22}$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{1.1.23}$$

From (1.1.13), (1.1.19) and (1.1.22)

$$\begin{aligned} p_X(n) = &\frac{1}{36}\left[(n-1)\,u(n-1)\right. \\ &\left. -2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \end{aligned} \tag{1.1.24}$$

which is the same as (2.1.4). Note that (2.1.4) can be obtained from (1.1.22) using contour integration as well.

1.1.4. The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure 1.1.1. The theoretical pmf obtained in (2.1.4) is plotted for comparison.

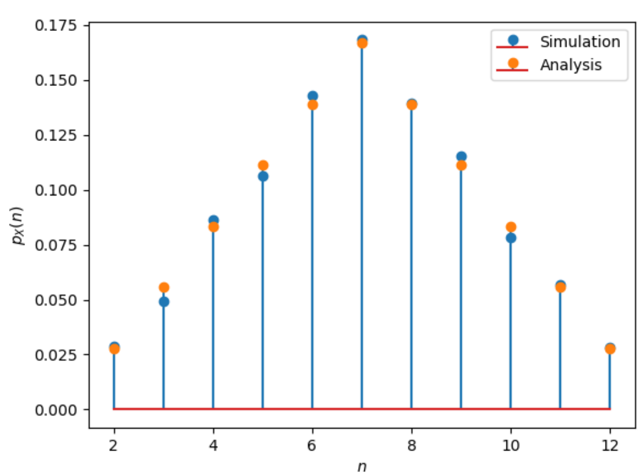

Fig. 1.1.1: Plot of $p_X(n)$ Simulations are close to the Analysis results.

1.1.5. The python code is available in

```
/Codes/Chapter1/Dice.py
```

2 Random Variables

2.1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

2.1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

```
/Codes/Chapter2/coeffs.h
```

```
/Codes/Chapter2/uniformgen.c
```

2.1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr\left(U \leq x\right) \tag{2.1.1}$$

Solution: The below code gives the plot in Fig. 2.1.1

```
/Codes/Chapter2/uni_cdf.py
```

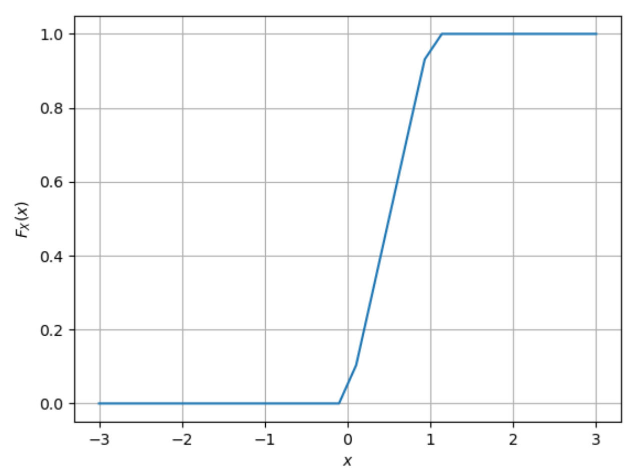

Fig. 2.1.1: The CDF of U

2.1.3 Find a theoretical expression for $F_U(x)$.

Solution: We Know that,

$$F_U(x) = \int_{-\infty}^{x} f_U(x)\,dx \tag{2.1.2}$$

For the uniform random variable U, $f_U(x)$ is

$$f_U(x) = \begin{cases} 1 & a \leq x \leq b \\ 0 & elsewhere \end{cases} \tag{2.1.3}$$

Substituting (2.1.3) in (2.1.2), $F_U(x)$ we get

$$F_U(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x \geq b \end{cases} \tag{2.1.4}$$

were, a=0 and b=1. Hence, (2.1.4) can be written as

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 0 \end{cases} \tag{2.1.5}$$

2.1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{2.1.6}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{2.1.7}$$

Write a C program to find the mean and variance of U.
Solution: The following code prints the mean and variance of U

/Codes/Chapter2/uni_ *mean_var*.c

The output of the program is

```
Uniform stats:
Mean: 0.500007
Variance: 0.083301
```

2.1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{2.1.8}$$

Solution: We Know That, For any given random variable X, the mean μ_X and variance σ_X^2 are formulated as

$$Mean \;\; \mu_X = E[X] = \int_{-\infty}^{\infty} x dF_U(x) \tag{2.1.9}$$

$$Variance \;\; \sigma_X^2 = E\left[X^2\right] - \mu_X^2 = \int_{-\infty}^{\infty} x^2 dF_U(x) - \mu_X^2 \tag{2.1.10}$$

Substituting the CDF of U from (2.1.5) in (2.1.9) and (2.1.10), we get

$$\mu_U = \frac{1}{2} \tag{2.1.11}$$

$$\sigma_U^2 = \frac{1}{12} \tag{2.1.12}$$

which match with the printed values in previous problem.

2.2 *Central Limit Theorem*

2.2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following files and execute the C program.

/Codes/Chapter2/coeffs.h

/Codes/Chapter2/gau_gen.c

2.2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig.2.2.1 using the following code.

/Codes/Chapter2/gau_cdf_plot.py

Properties:

- CDF is non-decreasing function

$$\frac{dF_X(x)}{dx} \geq 0 \tag{2.2.2}$$

- Maximum value of CDF $F(+\infty) = 1$.
- Minimum value of CDF $F(-\infty) = 0$.

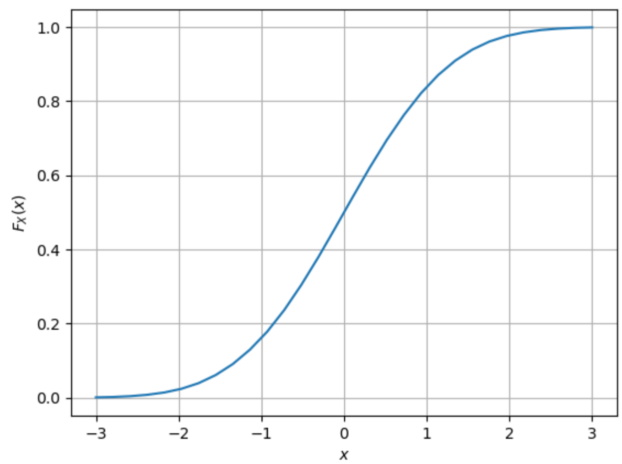

Fig. 2.2.1: The CDF of X

2.2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2.3}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.2.2 using the code below

/Codes/Chapter2/gau_pdf_plot.py

Properties :

- The PDF is always non-negative

$$f_X(x) \geq 0 \tag{2.2.4}$$

- Mean,median and mode are equal.
- The PDF curve is symmetric about mean.
- Area under the curve=1.

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \tag{2.2.5}$$

2.2.4 Find the mean and variance of X by writing a C program.

Solution: The following code gives the mean and variance of X

/Codes/Chapter2/gau_mean_var.c

The outputs are

```
Gaussian stats:
```

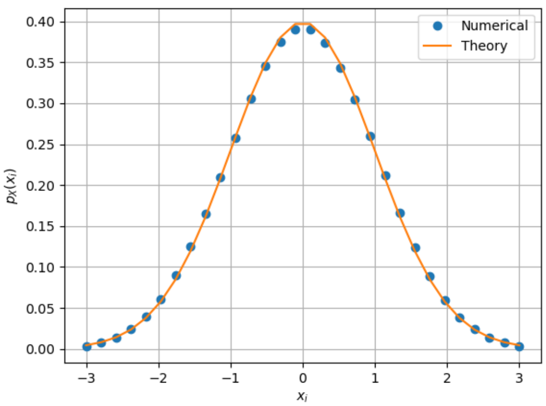

Fig. 2.2.2: The PDF of X

```
Mean: 0.000294
Variance: 0.999562
```

2.2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.2.6}$$

repeat the above exercise theoretically.
Solution: Substituting the PDF from (2.2.6) in (2.1.9),

$$Mean\ (\mu) = E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx \tag{2.2.7}$$

Using

$$\int x \cdot \exp\left(-ax^2\right) \mathrm{d}x = -\frac{1}{2a} \cdot \exp\left(-ax^2\right) \tag{2.2.8}$$

$$\mu_X = \frac{1}{\sqrt{2\pi}} \left[-\exp\left(-\frac{x^2}{2}\right)\right]_{-\infty}^{\infty} \tag{2.2.9}$$

$$\mu_X = 0 \tag{2.2.10}$$

To Find Variance,

$$E\left(X^2\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx \tag{2.2.11}$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} x^2 e^{-\frac{x^2}{2}} dx \tag{2.2.12}$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \sqrt{2t} e^{-t} dt \quad \left(Lett = \frac{x^2}{2}\right) \tag{2.2.13}$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t} t^{\frac{3}{2}-1} dt \tag{2.2.14}$$

$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \tag{2.2.15}$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) \tag{2.2.16}$$

$$= 1 \qquad \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right) \tag{2.2.17}$$

Thus, the variance is

$$\sigma^2 = E\left(X\right)^2 - E^2\left(X\right) = 1 \tag{2.2.18}$$

The values match with the our values printed in the previous problem.

2.3 *From Uniform to Other*

2.3.1 Generate samples of

$$V = -2\ln\left(1 - U\right) \tag{2.3.1}$$

and plot its CDF.
Solution: V is generated using the following code from the uni.dat file generated in problem 2.1.1. The CDF of V is plotted in Fig. 2.3.1 using the code below,

/Codes/Chapter2/coeffs.h

/Codes/Chapter2/v_gen.c

/Codes/Chapter2/v_cdf_plot.py

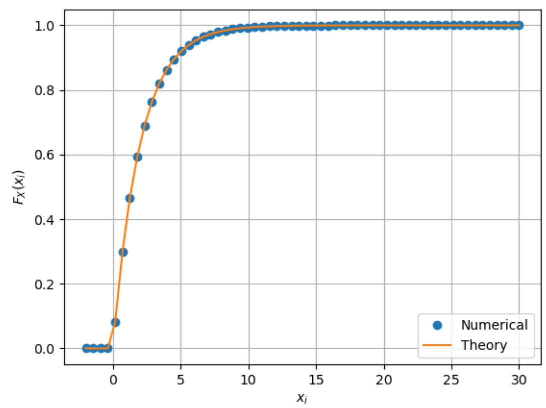

Fig. 2.3.1: The CDF of V

2.3.2 Find a theoretical expression for $F_V(x)$.
Solution: We Know That,

$$F_V(x) = P(V < x) \tag{2.3.2}$$

$$= P(-2\ln\left(1 - U\right) < x) \tag{2.3.3}$$

$$= P(U < 1 - e^{\frac{-x}{2}}) \tag{2.3.4}$$

$$= F_U(1 - e^{\frac{-x}{2}}) \tag{2.3.5}$$

Using $F_U(x)$ from (2.1.5). We finally get

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{\frac{-x}{2}} & x \geq 0 \end{cases} \tag{2.3.6}$$

2.4 *Triangular Distribution*

2.4.1 Generate

$$T = U_1 + U_2 \tag{2.4.1}$$

Solution: T is generated using the following C programs.

/Codes/Chapter2/coeffs.h

/Codes/Chapter2/tri_gen.c

2.4.2 Find the CDF of T.
Solution: The CDF of T is plotted in the Fig.2.4.1 using

the below Python Code

/Codes/Chapter2/tri_cdf_plot.py

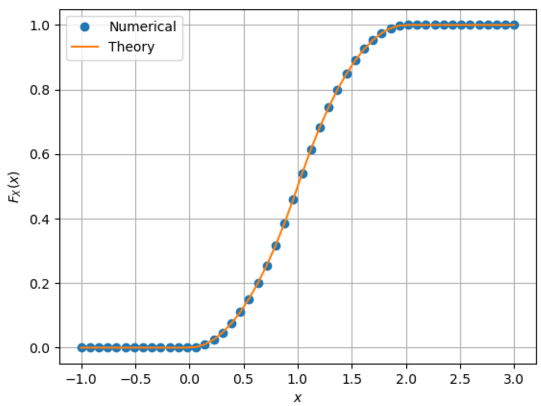

Fig. 2.4.1: The CDF of T

2.4.3 Find the PDF of T.
Solution: : The PDF of T is plotted in the Fig.2.4.2 using the below Python Code

/Codes/Chapter2/tri_pdf_plot.py

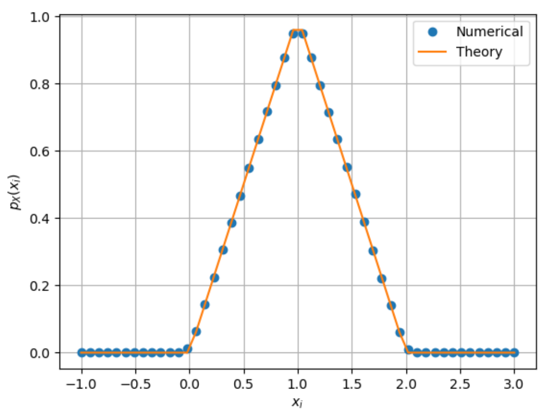

Fig. 2.4.2: The PDF of T

2.4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution: Since T is the sum of the independant random variables $U1$ and $U2$, the PDF of T is given by

$$p_T(x) = p_{U1}(x) * p_{U2}(x) \tag{2.4.2}$$

Using the PDF of U from (2.1.3), we get the answer as

$$p_T(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \\ 0 & x > 2 \end{cases} \tag{2.4.3}$$

The CDF of T is calculated using (2.1.2) where we put T instead of U and solve the integration. We finally arrive at the following result

$$F_T(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{2} & 0 \le x \le 1 \\ 2x - \frac{x^2}{2} - 1 & 1 \le x \le 2 \\ 1 & x > 2 \end{cases} \tag{2.4.4}$$

2.4.5 Verify your results through a plot.
Solution: : The theoretical and simulated results of both CDF and PDF of the Triangular Distribution T have been compared and verified using the plots show in the Fig. 2.4.1and 2.4.2 which were generated using Python.

3 Maximum Likelyhood Detection:BPSK

3.1 Maximum Likelihood

3.1.1 Generate equiprobable $X \in \{1, -1\}$.

Solution: The Random Variable X is generated using the following Python Code

/Codes/Chapter3/eqprob.py

3.1.2 Generate

$$Y = AX + N, \tag{3.1.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.

Solution: The Random Variable Y is generated from the Random Variable X from the previous problem using the following Python Code

/Codes/Chapter3/5_1_2.py

3.1.3 Plot Y using a scatter plot.

Solution: The Python Code for the Scatter Plot of Y is given below. The Corresponding plot is shown in the Fig.3.1.1

/Codes/Chapter3/5_1_3.py

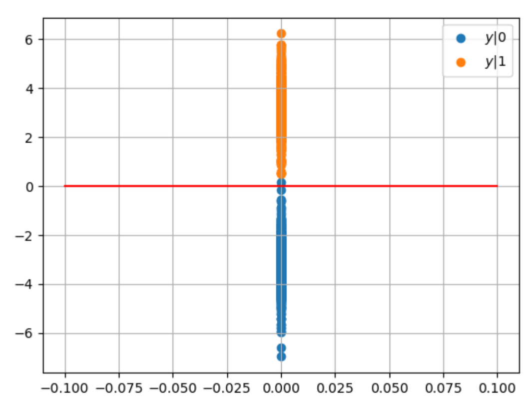

Fig. 3.1.1: The scatter plot of Y

3.1.4 Guess how to estimate X from Y.
Solution: The Given Signal is represented by two signals

'1' for X=1 and
'0' for X=-1. according to decision rule $P(Y > y)$

$$y \underset{-1}{\overset{1}{\gtrless}} 0 \tag{3.1.2}$$

3.1.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{3.1.3}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{3.1.4}$$

Solution: From the previous solution (3.1.2), we can write

$$\begin{aligned}\Pr\left(\hat{X} = -1 | X = 1\right) &= \Pr\left(Y < 0 | X = 1\right) \\ &= \Pr\left(AX + N < 0 | X = 1\right) \\ &= \Pr\left(A + N < 0\right) \\ &= \Pr\left(N < -A\right)\end{aligned}$$

Similarly,

$$\begin{aligned}\Pr\left(\hat{X} = 1 | X = -1\right) &= \Pr\left(Y > 0 | X = -1\right) \\ &= \Pr\left(N > A\right)\end{aligned}$$

Since $N \sim \mathcal{N}(0, 1)$, i.e, it is Symmetric. We can write,

$$\Pr\left(N < -A\right) = \Pr\left(N > A\right) \tag{3.1.5}$$

$$\implies P_{e|0} = P_{e|1} \tag{3.1.6}$$

3.1.6 Find P_e assuming that X has equiprobable symbols.
Solution:

$$P_e = \Pr\left(X = 1\right) P_{e|1} + \Pr\left(X = -1\right) P_{e|0} \tag{3.1.7}$$

assuming that X is equiprobable,

$$P_e = \frac{1}{2} P_{e|1} + \frac{1}{2} P_{e|0} \tag{3.1.8}$$

Substituting from (3.1.6)

$$P_e = \Pr\left(N > A\right) \tag{3.1.9}$$

Given a random varible $X \sim \mathcal{N}(0, 1)$ the Q-function is defined as

$$Q(x) = \Pr\left(X > x\right) \tag{3.1.10}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{u^2}{2}\right) du. \tag{3.1.11}$$

Using the above Q Function, P_e can be rewritten as

$$P_e = Q(A) \tag{3.1.12}$$

3.1.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution: The theoretical and simulated estimations of P_e with respect to A are plotted in the Fig 3.1.2. Below is the python code for the plotting.

/Codes/Chapter3/PevsA.py

3.1.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

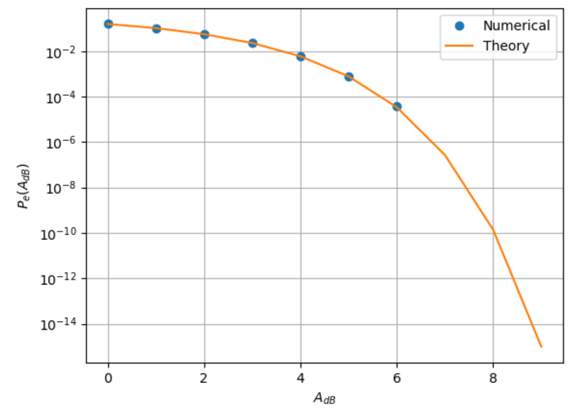

Fig. 3.1.2: P_e of X wrt SNR(A)

Solution: From 3.1.2 the decision rule,

$$y \underset{-1}{\overset{1}{\gtrless}} \delta \tag{3.1.13}$$

$$\begin{aligned}P_{e|0} &= \Pr\left(\hat{X} = -1 | X = 1\right) \\ &= \Pr\left(Y < \delta | X = 1\right) \\ &= \Pr\left(AX + N < \delta | X = 1\right) \\ &= \Pr\left(A + N < \delta\right) \\ &= \Pr\left(N < -A + \delta\right) \\ &= \Pr\left(N > A - \delta\right) \\ &= Q(A - \delta)\end{aligned}$$

$$\begin{aligned}P_{e|1} &= \Pr\left(\hat{X} = 1 | X = -1\right) \\ &= \Pr\left(Y > \delta | X = -1\right) \\ &= \Pr\left(N > A + \delta\right) \\ &= Q(A + \delta)\end{aligned}$$

From the Eq. 3.1.7 P_e can be written as,

$$P_e = \frac{1}{2} Q(A + \delta) + \frac{1}{2} Q(A - \delta) \tag{3.1.14}$$

Using the integral for Q-function from the equation 3.1.11, we get,

$$P_e = k(\int_{A+\delta}^\infty \exp\left(-\frac{u^2}{2}\right) du + \int_{A-\delta}^\infty \exp\left(-\frac{u^2}{2}\right) du) \tag{3.1.15}$$

where k is a constant
Differentiating 3.1.15. wrt δ (using Leibniz's rule) and equating to 0, we get

$$\exp\left(-\frac{(A+\delta)^2}{2}\right) - \exp\left(-\frac{(A-\delta)^2}{2}\right) = 0 \tag{3.1.16}$$

$$\frac{\exp\left(-\frac{(A+\delta)^2}{2}\right)}{\exp\left(-\frac{(A-\delta)^2}{2}\right)} = 1 \tag{3.1.17}$$

$$\exp\left(-\frac{(A+\delta)^2 - (A-\delta)^2}{2}\right) = 1 \tag{3.1.18}$$

$$\exp\left(-2A\delta\right) = 1 \tag{3.1.19}$$

Taking log on both sides

$$\tag{3.1.20}$$

$$-2A\delta = 0 \tag{3.1.21}$$

$$\implies \delta = 0 \tag{3.1.22}$$

$\therefore P_e$ is maximum for $\delta = 0$

3.1.9 Repeat the above exercise when

$$p_X(0) = p \tag{3.1.23}$$

Solution: Given X is not equiprobable, hence P_e is given by,

$$P_e = (1-p)P_{e|1} + pP_{e|0} \tag{3.1.24}$$

$$= (1-p)Q(A+\delta) + pQ(A-\delta) \tag{3.1.25}$$

Using the integral for Q-function from the equation 3.1.11 we get,

$$P_e = k((1-p)\int_{A+\delta}^{\infty} \exp\left(-\frac{u^2}{2}\right) du + p\int_{A-\delta}^{\infty} \exp\left(-\frac{u^2}{2}\right) du) \tag{3.1.26}$$

where k is a constant.
differentiate P_e wrt δ and equate to zero,

$$(1-p)\exp\left(-\frac{(A+\delta)^2}{2}\right) - p\exp\left(-\frac{(A-\delta)^2}{2}\right) = 0 \tag{3.1.27}$$

$$\exp\left(-\frac{(A+\delta)^2 - (A-\delta)^2}{2}\right) = \frac{p}{(1-p)} \tag{3.1.28}$$

$$\exp(-2A\delta) = \frac{p}{(1-p)} \tag{3.1.29}$$

Taking log on both sides we finally get,

$$\delta = \frac{1}{2A}\log\left(\frac{1}{p} - 1\right) \tag{3.1.30}$$

3.1.10 Repeat the above exercise using the MAP criterion.
Solution: The MAP rule can be stated as

$$\text{Set } \hat{x} = x_i \text{ if} p_X(x_k)p_Y(y|x_k) \text{ is maximum for } k = i \tag{3.1.31}$$

But in the case of BPSK, the point of equality between $p_X(x=1)p_Y(y|x=1)$ and $p_X(x=-1)p_Y(y|x=-1)$ is the optimum threshold.
Assuming that this threshold is δ, then we can formulate the below equations

$$pp_Y(y|x=1) > (1-p)p_Y(y|x=-1) \text{ when } y > \delta \tag{3.1.32}$$

$$pp_Y(y|x=1) < (1-p)p_Y(y|x=-1) \text{ when } y < \delta \tag{3.1.33}$$

The Python code for visualizing the above set of inequalities is given below.

/Codes/Chapter3/bpsk.py

The plot of the above inequalities for $p = 0.3$ and $A = 3$ is show in the Fig Fig. 3.1.3
Given $Y = AX + N$ where $N \sim \mathcal{N}(0,1)$, the optimum threshold is found to be,

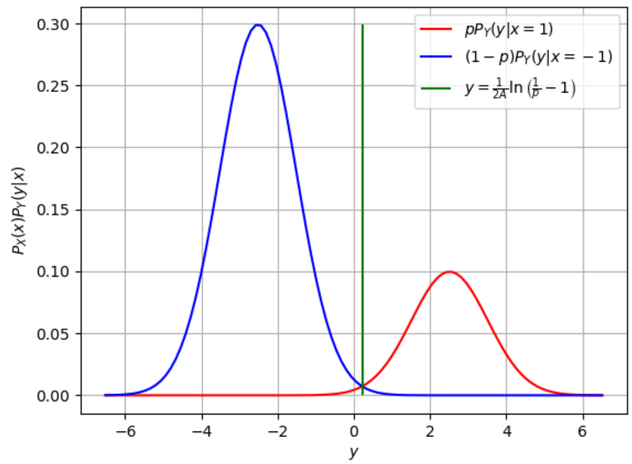

Fig. 3.1.3: $p_X(X = x_i)p_Y(y|x = x_i)$ versus y plot for $X \in \{-1, 1\}$

$$y_{eq} = \delta = \frac{1}{2A}\ln\left(\frac{1}{p} - 1\right) \tag{3.1.34}$$

which is same as δ obtained in problem 3.1.30

4 Transformation of random variables

4.1 Gaussian to Other

4.1.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{4.1.1}$$

Solution: The CDF and PDF of V are plotted using the following codes respectively. The corresponding plots are show in the Figures 4.1.1 , 4.1.2 respectively.

/Codes/Chapter4/sq_sum_cdf.py

/Codes/Chapter4/sq_sum_pdf.py

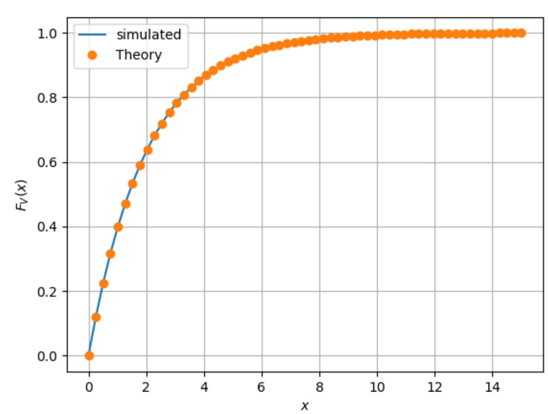

Fig. 4.1.1: The CDF of V

4.1.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{4.1.2}$$

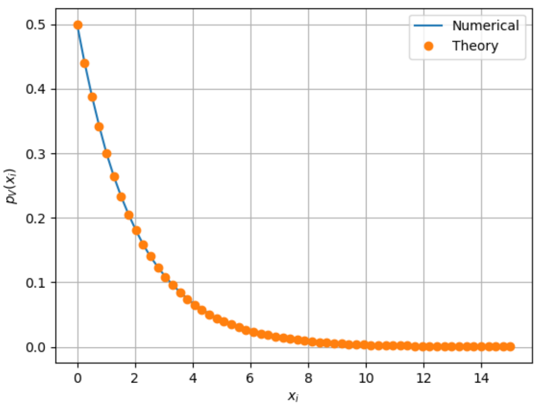

Fig. 4.1.2: The PDF of V

find α.
Solution: Let $Z = X^2$ where $X \sim \mathcal{N}(0,1)$. The CDF of Z can eb defined as,

$$
\begin{aligned}
P_Z(z) &= \Pr(Z < z) \\
&= \Pr\left(X^2 < z\right) \\
&= \Pr\left(-\sqrt{z} < X < \sqrt{z}\right) \\
&= \int_{-\sqrt{z}}^{\sqrt{z}} p_X(x)\,dx
\end{aligned}
$$

From the Eq. (2.2.3), the PDF of Z is given by

$$
\begin{aligned}
\frac{d}{dz}P_Z(z) &= p_Z(z) \\
= \frac{p_X(\sqrt{z}) + p_X(-\sqrt{z})}{2\sqrt{z}} &\text{ [By Lebniz's rule]}
\end{aligned} \tag{4.1.3}
$$

By Substituting the gaussian function $p_X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ in (4.1.3) we get,

$$
p_Z(z) = \begin{cases} \frac{1}{\sqrt{2\pi z}}e^{-\frac{z}{2}} & z \geq 0 \\ 0 & z < 0 \end{cases} \tag{4.1.4}
$$

The PDF of X_1^2 and X_2^2 are given by (4.1.4). Since V is the sum of two independant random variables we can write,

$$
\begin{aligned}
p_V(v) &= p_{X_1^2}(x_1) * p_{X_2^2}(x_2) \\
&= \frac{1}{2\pi}\int_0^v \frac{e^{-\frac{x}{2}}}{\sqrt{x}} \frac{e^{-\frac{v-x}{2}}}{\sqrt{v-x}}\,dx \\
&= \frac{e^{-\frac{v}{2}}}{2\pi}\int_0^v \frac{1}{\sqrt{x(v-x)}}\,dx \\
&= \frac{e^{-\frac{v}{2}}}{2\pi}\left[-\arcsin\left(\frac{v-2x}{v}\right)\right]_0^v \\
&= \frac{e^{-\frac{v}{2}}}{2\pi}\pi \\
&= \frac{e^{-\frac{v}{2}}}{2} \text{ for } v \geq 0
\end{aligned}
$$

$F_V(v)$ can be obtained from $p_V(v)$ using the equation (2.1.2)

$$
\begin{aligned}
F_V(v) &= \frac{1}{2}\int_0^v \exp\left(-\frac{v}{2}\right) \\
&= 1 - \exp\left(-\frac{v}{2}\right) \text{ for } v \geq 0
\end{aligned} \tag{4.1.5}
$$

Comparing (4.1.5) with (4.1.2), $\alpha = \frac{1}{2}$

4.1.3 Plot the CDF and PDf of

$$
A = \sqrt{V} \tag{4.1.6}
$$

Solution: The CDF and PDF of A are plotted using the following codes respectively. The corresponding plots are show in the Figures 4.1.3,4.1.4 respectively.

```
/Codes/Chapter4/sqroot_cdf.py
```

```
/Codes/Chapter4/sqroot_pdf.py
```

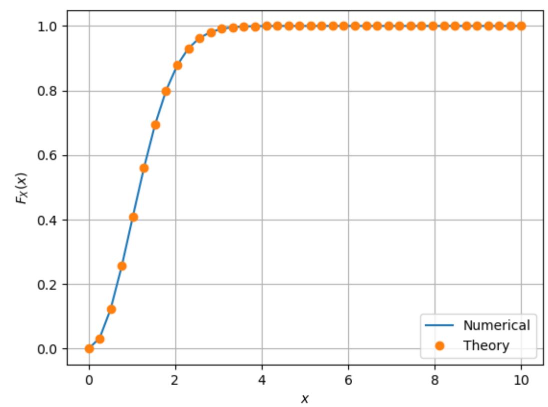

Fig. 4.1.3: The CDF of A

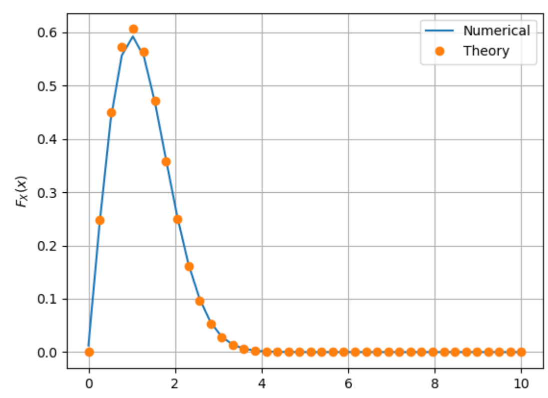

Fig. 4.1.4: The PDF of A

4.2 *Conditional Probability*

4.2.1 Plot

$$
P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{4.2.1}
$$

for

$$
Y = AX + N, \tag{4.2.2}
$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in (-1,1)$ for $0 \le \gamma \le 10$ dB.

4.2.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

Solution: Assuming the decision rule in (3.1.2), when N is constant, P_e can be calculated by the following steps,

$$\hat{X} = \begin{cases} +1 & Y > 0 \\ -1 & Y < 0 \end{cases} \tag{4.2.3}$$

For $X = 1$,

$$Y = A + N \tag{4.2.4}$$

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{4.2.5}$$

$$= \Pr\left(Y < 0 | X = 1\right) \tag{4.2.6}$$

$$= \Pr\left(AX + N < 0 | X = 1\right) \tag{4.2.7}$$

$$= \Pr\left(A + N < 0\right) \tag{4.2.8}$$

$$= \Pr\left(N < -A\right) \tag{4.2.9}$$

$$= \begin{cases} F_A(-N) & N \ge 0 \\ 0 & N < 0 \end{cases} \tag{4.2.10}$$

By definition For a Rayleigh random variable X with $E\left[X^2\right] = \gamma$, the PDF and CDF are given by,

$$f_A(x) = \frac{x}{\gamma}\exp\left(-\frac{x^2}{\gamma}\right) \quad \text{for } x \ge 0 \tag{4.2.11}$$

$$F_A(X) = 1 - \exp\left(-\frac{x^2}{\gamma}\right) \quad \text{for } x \ge 0 \tag{4.2.12}$$

If $N < 0$, $f_A(x) = 0$. Then,

$$P_e = 0 \tag{4.2.13}$$

If $N \ge 0$. Then,

$$P_e(N) = \int_{-\infty}^{-N} f_A(x)dx \tag{4.2.14}$$

$$= \int_{-\infty}^{0} 0dx + \int_{0}^{-N} f_A(x)dx \tag{4.2.15}$$

$$= \int_{0}^{-N} \frac{x}{\gamma}\exp\left(-\frac{x^2}{\gamma}\right)dx \tag{4.2.16}$$

$$= 1 - \exp\left(-\frac{N^2}{\gamma}\right) \tag{4.2.17}$$

Therefore,

$$P_e(N) = \begin{cases} 1 - \exp\left(-\frac{N^2}{\gamma}\right) & N \ge 0 \\ 0 & otherwise \end{cases} \tag{4.2.18}$$

4.2.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x)p_X(x)\,dx \tag{4.2.19}$$

Find $P_e = E\left[P_e(N)\right]$.

Solution: Since $N \sim \mathcal{N}(0,1)$,

$$p_N(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) \tag{4.2.20}$$

And from (4.2.18)

$$P_e(x) = \begin{cases} 1 - \exp\left(-\frac{x^2}{\gamma}\right) & x \ge 0 \\ 0 & otherwise \end{cases} \tag{4.2.21}$$

$$P_e = E\left[P_e(N)\right] = \int_{-\infty}^{\infty} P_e(x)p_N(x)\,dx \tag{4.2.22}$$

We know that for any even function, we can write

$$\int_{-\infty}^{\infty} f(x) = 2\int_{-\infty}^{0} f(x) \tag{4.2.23}$$

we get

$$P_e = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\exp\left(-\frac{x^2}{2}\right)\left(1 - \exp\left(-\frac{x^2}{\gamma}\right)\right)dx \tag{4.2.24}$$

$$P_e = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-\frac{x^2}{2}}\,dx - \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\exp\left(-x^2\left(\frac{1}{\gamma} + \frac{1}{2}\right)\right)dx \tag{4.2.25}$$

$$\therefore P_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{2+\gamma}} \tag{4.2.26}$$

4.2.4 Plot P_e in problems 4.2.1 and 4.2.19 on the same graph w.r.t γ. Comment.

Solution: P_e w.r.t γ plot is show in the Fig. 4.2.1. The value of P_e is much higher when the channel gain A is Rayleigh distributed than the case where A is a constant (compare with Fig. 3.1.2).

```
/Codes/Chapter4/pe_vs_gamma.py
```

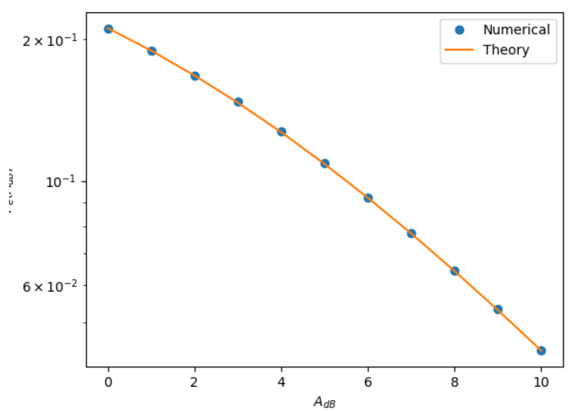

Fig. 4.2.1: The P_e wrt γ

5 Bivariate Random Variables:FSK

5.1 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n},$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0,1).$$

5.1.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{5.1.1}$$

on the same graph using a scatter plot.
Solution: The scatter plot for $\mathbf{x} = \mathbf{s}_0$ and $\mathbf{x} = \mathbf{s}_1$ shown in Fig. 5.1.1 is generated by the following Python code.

/Codes/Chapter5/bfsk_scatter.py

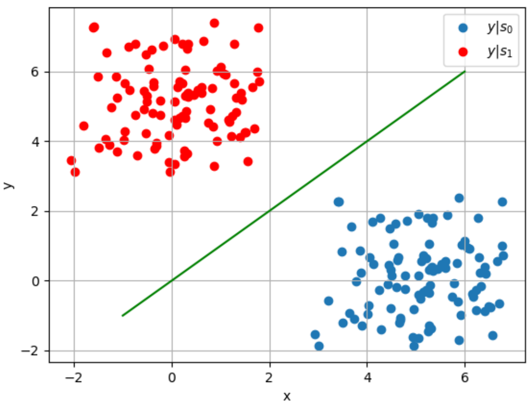

Fig. 5.1.1: Y scatter plot

5.1.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.
Solution:
For a bivariate gaussian distribution,

$$p(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2\left(1-\rho^2\right)} \times \left\{\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{\left(y-\mu_y\right)^2}{\sigma_y^2} - \frac{2\rho\left(x-\mu_x\right)\left(y-\mu_y\right)}{\sigma_x\sigma_y}\right\}\right] \tag{5.1.2}$$

where

$$\mu = \begin{pmatrix}\mu_{\mathbf{x}} \\ \mu_{\mathbf{y}}\end{pmatrix}, \mathbf{\Sigma} = \begin{pmatrix}\sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2\end{pmatrix} \tag{5.1.3}$$

$$\rho = \frac{E\left[\left(x-\mu_x\right)\left(y-\mu_y\right)\right]}{\sigma_x\sigma_y}. \tag{5.1.4}$$

$$\mathbf{y}|s_0 = \begin{pmatrix}A+n_1 \\ n_2\end{pmatrix} \tag{5.1.5}$$

$$\mathbf{y}|s_1 = \begin{pmatrix}n_1 \\ A+n_2\end{pmatrix} \tag{5.1.6}$$

$\mathbf{y}|\mathbf{s}_i$ is a random vector with each of its components normally distributed. The PDF of $\mathbf{y}|\mathbf{s}_i$ is given by,

$$p_{\mathbf{y}|\mathbf{s}_i}(\mathbf{y}) = \frac{1}{2\pi\sqrt{|\mathbf{\Sigma}|}} \exp\left(-\frac{1}{2}\left(\mathbf{y}-\mathbf{s}_i\right)^\top \mathbf{\Sigma}^{-1}\left(\mathbf{y}-\mathbf{s}_i\right)\right) \tag{5.1.7}$$

Where $\mathbf{\Sigma}$ is the covariance matrix. Substituting $\mathbf{\Sigma} = \sigma\mathbf{I}$,

$$p_{\mathbf{y}|\mathbf{s}_i}(\mathbf{y}) = \frac{1}{2\pi\sigma} \exp\left(-\frac{1}{2\sigma}\left(\mathbf{y}-\mathbf{s}_i\right)^\top \mathbf{I}\left(\mathbf{y}-\mathbf{s}_i\right)\right) \tag{5.1.8}$$

$$= \frac{1}{2\pi\sigma} \exp\left(-\frac{1}{2\sigma}\left(\mathbf{y}-\mathbf{s}_i\right)^\top\left(\mathbf{y}-\mathbf{s}_i\right)\right) \tag{5.1.9}$$

For equiprobably symbols, the MAP criterion is defined as

$$p\left(\mathbf{y}|s_0\right) \underset{s_1}{\overset{s_0}{\gtrless}} p\left(\mathbf{y}|s_1\right) \tag{5.1.10}$$

Since there are only two possible symbols $\mathbf{s}_0$ and $\mathbf{s}_1$, the optimal decision criterion is found by equating $p_{\mathbf{y}|\mathbf{s}_0}$ and $p_{\mathbf{y}|\mathbf{s}_1}$.

$$p_{\mathbf{y}|\mathbf{s}_0} = p_{\mathbf{y}|\mathbf{s}_1} \tag{5.1.11}$$

$$\implies \exp\left(-\frac{1}{2\sigma}\left(\mathbf{y}-\mathbf{s}_0\right)^\top\left(\mathbf{y}-\mathbf{s}_0\right)\right) = \exp\left(-\frac{1}{2\sigma}\left(\mathbf{y}-\mathbf{s}_1\right)^\top\left(\mathbf{y}-\mathbf{s}_1\right)\right) \tag{5.1.12}$$

$$\implies \left(\mathbf{y}-\mathbf{s}_0\right)^\top\left(\mathbf{y}-\mathbf{s}_0\right) = \left(\mathbf{y}-\mathbf{s}_1\right)^\top\left(\mathbf{y}-\mathbf{s}_1\right) \tag{5.1.13}$$

$$\implies \mathbf{y}^\top\mathbf{y} - 2\mathbf{s}_0^\top\mathbf{y} + \mathbf{s}_0^T\mathbf{s}_0 = \mathbf{y}^\top\mathbf{y} - 2\mathbf{s}_1^\top\mathbf{y} + \mathbf{s}_1^T\mathbf{s}_1 \tag{5.1.14}$$

$$\implies 2\left(\mathbf{s}_1-\mathbf{s}_0\right)^\top\mathbf{y} = \|\mathbf{s}_1\|^2 - \|\mathbf{s}_0\|^2 \tag{5.1.15}$$

$$\implies \left(\mathbf{s}_1-\mathbf{s}_0\right)^\top\mathbf{y} = 0 \tag{5.1.16}$$

$$\implies \begin{pmatrix}-1 \\ 1\end{pmatrix}^\top \mathbf{y} = 0 \tag{5.1.17}$$

On simplifying, we get the decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{5.1.18}$$

5.1.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1|\mathbf{x} = \mathbf{s}_0\right) \tag{5.1.19}$$

with respect to the SNR from 0 to 10 dB.
Solution: The Fig. 5.1.2 Shows the P_e vs SNR plot for Theoretical and Numerically Estimated values generated using the Python code give below

/Codes/Chapter5/pe_vs_snr.py

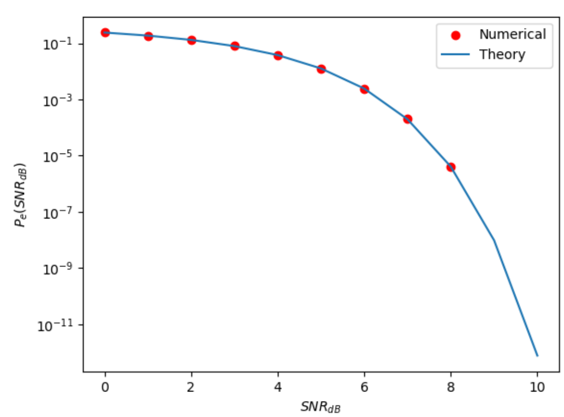

Fig. 5.1.2: Pe vs SNR

5.1.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.

Solution: Using the decision rule from (5.1.18),

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{5.1.20}$$
$$= \Pr\left(y_1 < y_2 | \mathbf{x} = \mathbf{s}_0\right)$$
$$= \Pr\left(A + n_1 < n_2\right)$$
$$= \Pr\left(n_1 - n_2 < -A\right) \tag{5.1.21}$$
$$= \Pr\left(n_2 - n_1 > A\right) \tag{5.1.22}$$

Let $Z = n_1 - n_2$ where $n_1, n_2 \sim \mathcal{N}\left(0, \sigma^2\right)$. The PDF of X is given by,

$$p_Z(z) = p_{n_1}(n_1) * p_{-n_2}(n_2) \tag{5.1.23}$$
$$= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} e^{-\frac{(t-z)^2}{2\sigma^2}}\, dt \tag{5.1.24}$$
$$= \frac{e^{-\frac{z^2}{2(\sqrt{2}\sigma)^2}}}{\sqrt{2\pi}\sqrt{2}\sigma} \tag{5.1.25}$$

Hence we have, $Z \sim \mathcal{N}\left(0, 2\sigma^2\right)$.
From (5.1.21) we can write,

$$P_e = \Pr\left(Z > A\right) \tag{5.1.26}$$
$$= \Pr\left(\sqrt{2}w > A\right) \tag{5.1.27}$$
$$= \Pr\left(w > \frac{A}{\sqrt{2}}\right) \tag{5.1.28}$$

Where w $\sim \mathcal{N}(0, 1)$,
Now, Substituting $\sigma = 1$, $Z \sim \mathcal{N}(0, 2)$ in (5.1.25) we get,

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{A}{\sqrt{2}}}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \tag{5.1.29}$$
$$\Rightarrow P_e = Q\left(\frac{A}{\sqrt{2}}\right) \tag{5.1.30}$$

The comparison of theoretical and simulated values can be observed in the P_e vs SNR Graph shown in the Fig.5.1.2.