Matrices Assignment - Circle

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Get Python code for the figure from

https://github.com/dukkipativijay/Fwciith2022/tree/main/Assignment%201/Codes/src

Get LaTex code from

https://github.com/dukkipativijay/Fwciith2022/tree/main/Assignment%201%20−%20Assembly/Codes

1 Question

Class 10, Exercise 11.2, Q(1)

Draw a circle of radius 6cm. From a point 10cm away from the centre, construct the pair of tangents to the circle and measure their lengths.

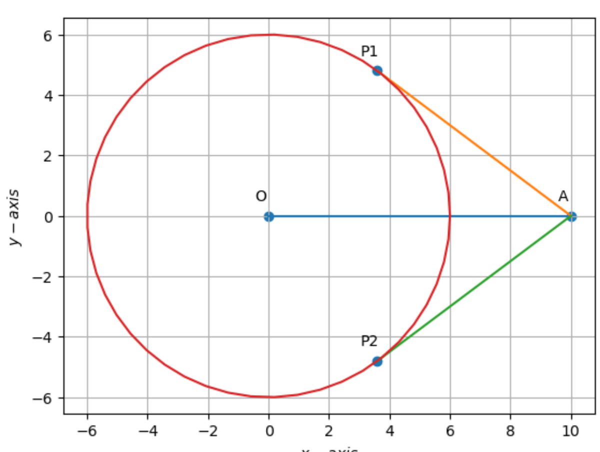

Figure 1 - Circle with Tangents AP_1, AP_2

2 Construction

A circle with centre O, radius 6cm and the tangents AP_1, AP_2 from a point A located 10cm away from the centre were constructed using Python, with the following parameters shown in the table below.

Symbol	Value	Description
O	$\begin{pmatrix}0\\0\end{pmatrix}$	Center
r	6cm	Radius
d	10cm	Distance between O and A
e_1	$\begin{pmatrix}1\\0\end{pmatrix}$	Unit Vector along X-Axis
A	d x e_1	Point A
θ	$\angle P_1OA$	Angle P_1OA
P_1	$r\begin{pmatrix}cos\theta\\sin\theta\end{pmatrix}$	Point of Contact P_1
P_2	$r\begin{pmatrix}cos\theta\\-sin\theta\end{pmatrix}$	Point of Contact P_2

Table 1: Parameters Table

3 Solution

Consider the circle of radius 6cm whose center is at origin and point A at a distance of 10cm away from the center.
Two tangents can be drawn from point A on to the circle and let the point of contacts be P_1 and P_2.

The point of intersection of line

$$L : \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \quad \mu\epsilon\mathbb{R} \tag{3.0.1}$$

With the conic section,

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{3.0.2}$$

is given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i\mathbf{m} \tag{3.0.3}$$

Where,
$\mu_i = \frac{1}{\mathbf{m^T V m}}(-\mathbf{m^T}(\mathbf{Vq} + \mathbf{u})$

$$\pm\sqrt{[\mathbf{m^T}(\mathbf{Vq} + \mathbf{u})]^2 - (\mathbf{q}^T\mathbf{Vq} + 2\mathbf{u^T q} + f)(\mathbf{m^T V m})}) \tag{3.0.4}$$

If the line L touches the conic at exactly one point,the conic intercept has exactly one root. Hence,

$$[\mathbf{m^T}(\mathbf{Vqu})]^2 - (\mathbf{m^T}\mathbf{Vm})(\mathbf{q}^T\mathbf{Vq} + 2\mathbf{u^T}\mathbf{q} + f) = 0 \tag{3.0.5}$$

The equation of our circle is,

$$x^2 + y^2 = 36 \tag{3.0.6}$$

Comparing it with the General Equation of a circle,

$$Ax^2 + Bxy + Cy^2 + Fx + Gy + f = 0 \tag{3.0.7}$$

We Have,

A = 1, B = 0, C = 1, F = 0, G = 0, f = -36

We Know That,

$$V = \begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix} \qquad u = \begin{pmatrix} \frac{F}{2} \\ \frac{G}{2} \end{pmatrix}$$

So we get,

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence, the Eq.(3.0.6) of our circle can be written in the form of Conic Eq.(3.0.2) as,

$$\mathbf{x^T}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} 0 & 0 \end{pmatrix}\mathbf{x} - 36 = 0 \tag{3.0.8}$$

Let us consider the direction vector of **L** as m,

$$\mathbf{m} = \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \tag{3.0.9}$$

and **q** be the point A,

$$\mathbf{q} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} \tag{3.0.10}$$

Substituting Eq.(3.0.8), (3.0.10) and (3.0.10) in

Eq.(3.0.5), we get

$$[\mathbf{m^T}(\mathbf{Iq})]^2 - (\mathbf{m^T}\mathbf{Im})(\mathbf{q}^T\mathbf{Iq} + (-36) = 0$$

$$[\begin{pmatrix} 1 & \lambda \end{pmatrix}\begin{pmatrix} 10 \\ 0 \end{pmatrix}]^2 - (\begin{pmatrix} 1 & \lambda \end{pmatrix}\begin{pmatrix} 1 \\ \lambda \end{pmatrix})(\begin{pmatrix} 10 & 0 \end{pmatrix}\begin{pmatrix} 10 \\ 0 \end{pmatrix} - 36) = 0$$

$$(10)^2 - (1 + \lambda^2)(100 - 36) = 0$$

$$100 - (1 + \lambda^2)(64) = 0$$

$$(1 + \lambda^2)(64) = 100$$

$$1 + \lambda^2 = \frac{100}{64}$$

$$\lambda^2 = \frac{36}{64}$$

$$\lambda = \pm\frac{6}{8}$$

$$\therefore \lambda = \pm\frac{3}{4} = \pm 0.75$$

Hence,

$$m = \begin{pmatrix} 1 \\ \pm\frac{3}{4} \end{pmatrix}$$

From Eq.(3.0.4) and (3.0.5)

$$\mu_i = \frac{1}{\mathbf{m^T}\mathbf{Vm}}(-\mathbf{m^T}(\mathbf{Vq} + \mathbf{u}))$$

$$\mu_i = \frac{1}{\begin{pmatrix} 1 & \frac{3}{4} \end{pmatrix}\mathbf{I}\begin{pmatrix} 1 \\ \frac{3}{4} \end{pmatrix}}(-\begin{pmatrix} 1 & \frac{3}{4} \end{pmatrix}(\mathbf{I}\begin{pmatrix} 10 \\ 0 \end{pmatrix})$$

$$\mu_i = \frac{1}{1+\frac{9}{16}}(-(10 + 0))$$

$$\mu_i = \frac{-10}{\frac{25}{16}}$$

$$\mu_i = \frac{-32}{5}$$

$$\mu_i = -6.4$$

Now Eq.(3.0.3) becomes,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} + (-6.4)\begin{pmatrix} 1 \\ \pm 0.75 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} + \begin{pmatrix} -6.4 \\ \mp 4.8 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3.6 \\ \mp 4.8 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3.6 \\ 4.8 \end{pmatrix} or \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3.6 \\ -4.8 \end{pmatrix}$$

Therefore,

$$P_1 = \begin{pmatrix} 3.6 \\ 4.8 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 3.6 \\ -4.8 \end{pmatrix}$$

Now,

$$\text{Length of Tangent 1} = \|\mathbf{P_1} - \mathbf{A}\|$$

$$= \|\begin{pmatrix} 3.6 \\ 4.8 \end{pmatrix} - \begin{pmatrix} 8 \\ 0 \end{pmatrix}\|$$

$$= \|\begin{pmatrix} -4.4 \\ 4.8 \end{pmatrix}\|$$

$$= 8cm$$

Similarly,

$$\text{Length of Tangent 2} = \|\mathbf{P_2} - \mathbf{A}\|$$

$$= \|\begin{pmatrix} 3.6 \\ -4.8 \end{pmatrix} - \begin{pmatrix} 8 \\ 0 \end{pmatrix}\|$$

$$= \|\begin{pmatrix} -4.4 \\ -4.8 \end{pmatrix}\|$$

$$= 8cm$$