Optimization Assignment - Advanced

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Get Python code for the figure from

https://github.com/dukkipativijay/Fwciith2022/tree/main/Assignment%201/Codes/src

Get LaTex code from

https://github.com/dukkipativijay/Fwciith2022/tree/main/Assignment%201%20−%20Assembly/Codes

1 Question

Q(26), Class - 12, CBSE Paper, 2015

The sum of the surface areas of a cuboid with sides x, 2x, $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if x is equal to three times the radius of sphere. Also find the minimum value of the sum of their volumes.

2 Solution

Let x, 2x, $\frac{x}{3}$ be the lengths of the sides of the Cuboid and r be the radius of the Sphere

Surface Area of the cuboid is,

$$A_1 = 2(lb + bh + hl)$$
$$A_1 = 2x^2 + \frac{2x^2}{3} + \frac{x^2}{3}$$

$$A_1 = 6x^2 \tag{2.0.1}$$

Surface Area of the Sphere is,

$$A_2 = 4\pi r^2 \tag{2.0.2}$$

By the given problem from Eq. (2.0.1) and (2.0.2) we can write,

$$6x^2 + 4\pi r^2 = k\,(constant) \tag{2.0.3}$$

Now, the volumes of the cuboid and the sphere can combined will be,

$$V = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3 \tag{2.0.4}$$

Now putting the value of r from Eq. (2.0.3) in the Eq. (2.0.4) we get,

$$V = \frac{2}{3}x^3 + \frac{4}{3}\left(\frac{k - 6x^2}{4\pi}\right)^{\frac{3}{2}} \tag{2.0.5}$$

Given to prove that x = 3r for V_{min}. And to find the V_{min}

Hence, differentiating Eq. (2.0.5) w.r.t x we get,

$$\frac{dV}{dr} = \frac{d}{dr}\left(\frac{2}{3}x^3 + \frac{4}{3}\left(\frac{k - 6x^2}{4\pi}\right)^{\frac{3}{2}}\right)$$

By simplification we get,

$$\frac{dV}{dr} = 2x^2 - 6x\left(\frac{k - 6x^2}{4\pi}\right)^{\frac{1}{2}} \tag{2.0.6}$$

Now Substitute k from Eq. (2.0.1) and then solving for x by equating the (2.0.6) to zero,

$$x = 3r \tag{2.0.7}$$

Differentiating Eq. (2.0.5) once again w.r.t x and substituting (2.0.7) it is observed that,

$$\frac{d^2V}{dr^2} > 0 \tag{2.0.8}$$

Hence it is a point of Minima.

Therefore, the sum of their volumes is minimum, if x = 3r.

Hence Proved.

Now to find V_{min}, By Substituting Eq. (2.0.7) in Eq. (2.0.4) we get,

$$V_{min} = \frac{2}{3}(3r)^3 + \frac{4}{3}\pi r^3$$

$$\therefore V_{min} = r^3\left(18 + \frac{4}{3}\pi\right)$$

3 Verification Using Geomtric Programming

Disciplined geometric programming, DGP is a subset of log-log-convex program (LLCP). An LLCP is defined as,

$$\begin{aligned} &\text{minimize } f_0(x) \\ &\text{subject to } f_i(x) \leq \tilde{f}_i, \text{ i } = 1,2,...,\text{m}. \\ &\qquad\qquad g_i(x) = \tilde{g}_i, \text{ i } = 1,2,...,\text{p}. \end{aligned} \tag{3.0.1}$$

where the functions f_i are log-log convex, $\tilde{f}_i$ are log-log concave, and the functions g_i and $\tilde{g}_i$ are log-log affine. An optimization problem with constraints of the above form in which the goal is to maximize or minimize a log-log concave function is also an LLCP. These LLCPs generalize geometric programming.

The given problem can be formulated as a DGP as,

$$V = \min_{r,h} \frac{2}{3}x^3 + \frac{4}{3}\pi r^3 \tag{3.0.2}$$

$$s.t \quad 6x^2 + 4\pi r^2 = k \ \ (constant) \tag{3.0.3}$$

By assuming k = 40 as input, and solving the above DGP Equations using Cvxpy we get,

$$V_{min} = 136.11 \tag{3.0.4}$$

$$x = 3.92 \tag{3.0.5}$$

$$r = 1.30 \tag{3.0.6}$$

Hence it is proved that $x = 3r$ for V_{min}.