Modelling and Simulating Opinion Dynamics in Social Networks Using Linear and Non-linear Models

Dulanga Jayawardena

1155088222

Abstract. This paper examines and compares how the processes of the DeGroot and Bounded Confidence models affect networks of different topologies, and then describes how the self-appraisal process described by Friedkin works on certain networks. It proposes an influence-based non-linear self-appraisal process which can be used with both models, and then describes how it can offset certain flaws in the Friedkin self-appraisal process.

Keywords. opinion dynamics, social networks, DeGroot Model, bounded confidence model

1. Introduction and Problem Statement

The internet has connected us on a scale never seen before. And while websites like Wikipedia allow anybody to learn about almost any topic free of charge, and social media sites like Twitter allow regular citizens to instantaneously shoot news out to the entire world, they have also allowed disinformation to spread at unprecedented levels.

When US President Donald Trump mused about injecting disinfectant into the body to exterminate the COVID-19 virus, swarths of people began calling poison control centres to figure out if that was a viable solution—an idea that would have sounded ridiculous to them beforehand. Manipulation of information has had major worldwide implications in recent history, from Cambridge Analytica using Facebook data to help elect Donald Trump, to the UK leaving the European Union. [1]

Clearly, the spread of opinions—especially through means of internet-based social media—has a strong impact on lives, making it an area worth studying. This area of study is called opinion dynamics, and it tries to create models that try to represent how opinions might spread through groups of people. They usually represent a social networks as a graph, where each node represents an individual, or an 'agent'. These models then illustrate interactions between users using mathematics, in an attempt to simulate how their opinions might change based on external influences, opinions of others, or how they might even stay resilient as a result of trusting themselves more than others. [2] A realistic model could potentially allow its user to change a lot of minds by injecting opinions at strategic nodes on the internet; for example by using fake accounts on Twitter to do so. This information could also help the managers of social networks like Twitter and Facebook to learn where and how to look for these people. This is a matter becoming increasingly important to figure out how to deal with.

This project proposes an influence-based self-appraisal process which can be used in conjunction with both the linear DeGroot as well as the non-linear Bounded Confidence models, in which constituents of the social network base their trust in themselves based on their influence on the previous issue. This overcomes a major issue of the Friedkin self-appraisal process, which suggests that agents' self-confidence is dependent solely on the structure of a network, rather than on the outcomes of previous discussions.

1.1. Some basic explanations

Social networks are represented as graphs in opinion dynamics. Each node in a graph represents a person, sometimes referred to as an individual, or an agent. Here are the six major types of graphs mentioned in this study:

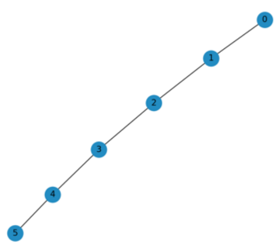

Figure 1 Line graph

(a) Line - a simple line, with each node connected only to 2 other nodes, and the two end-nodes only being connected to one other node each

(b) Ring - The ring graph is similar to a line graph, but with the 2 end-nodes connected to each other

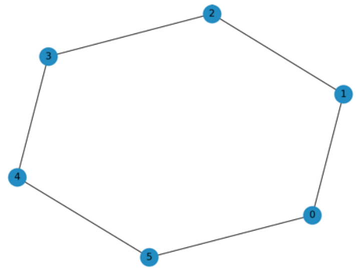

Figure 2 Ring graph

(c) Star – a graph with one central node, connected to all the other nodes, with each of them having the central node as its only connection

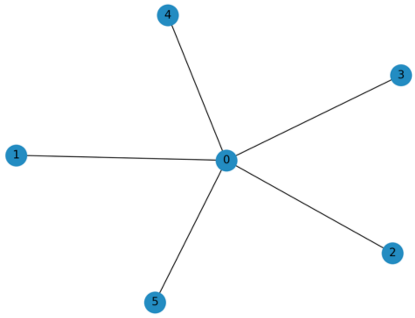

Figure 3 Star graph

(d) Random – The random graph used here is an Erdos Renyi graph where each node is connected to another with probability ***1.1(log n) /n***.

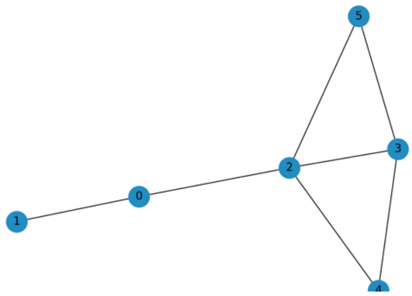

Figure 4 Random graph

(e) Stochastic block model (SBM) – This is a graph that tries to simulate real social networks, by dividing its nodes into groups. Each node is connected to other nodes within its group with probability ***4 (log n) /n***, and to nodes in other graphs with probability ***4 (log n) / 50n***. The number of blocks is set dependent on the number of nodes in the network. A 100-node network has 6 blocks

(f) Real topologies - Real topologies are datasets compiled using real-world social network data. The ones used in this project are derived from KONECT's *Facebook friendships* datasets [3], which contain undirected networks of Facebook users' friendships, which were converted into adjacency matrices for the purpose of this study. These matrices were then, raised to a power of five, and rows with lower sums (i.e., those with fewer connections) were removed to ensure a connected graph.

2. Literature Review

2.1. On The DeGroot Model

In *Reaching a Consensus*, DeGroot proposes an opinion dynamics model which describes how a group of people reach consensus by 'pooling their individual opinions'. Here, a group of people in a connected network start off with different opinions, and then converse with each other repeatedly—with their opinions growing increasingly similar to that of their neighbours', until the entire group reaches a consensus. [4]

While a number of models were built upon the aforementioned model, Friedkin's addition of a self-appraisal mechanism to this was notable. In this model, the group of people on the network interact about multiple topics, reaching a consensus each time. This time, however, the model increases or decreases individual influence on neighbours after the end of a discussion, depending on how well-connected nodes are, which is sometimes proportional to their influence on previous issues. [5] *Opinion Dynamics and the Evolution of Power in Social Networks* uses this model to investigate the properties of equilibrium points of various network structures. [6] This illustrates how social power accumulates at the top of hierarchies, and how different network structures can deterministically result in social structures such as democracies or autocracies in the DeGroot model. [7]

2.2. On The Bounded Confidence Model

Mixing Beliefs among Interacting Agents presents an opinion dynamics model in which individuals converse with each other, as in the DeGroot model, but only adjust their opinions towards the opinions of their neighbours when the differences in their opinions is less than a threshold value. Using a high threshold (like 1) results in opinions converging to a consensus, as in the DeGroot model. Lower threshold values result in "opinion clusters", where members "share the same opinion but are no longer influenced by opinions of members of other clusters".

[8] This model improves on a major drawback of the DeGroot model, which implies that every agent will agree with each other.

3. The DeGroot Model

The DeGroot model is an opinion dynamics model in which individuals in a network repeatedly interact with each other to reach a group-wide consensus. There are four important concepts in the DeGroot model:

(i) network – an $n \times n$ adjacency matrix showing which nodes each node is connected to; in a real social network, this would show each individual's friends

(ii) weight (W) – an $n \times n$ matrix showing how much each node is affected by the opinions of neighboring nodes; that is, how much they trust each friend. Each row of this matrix is stochastic, and adds up to one. The initial weights for matrix W are obtained by calculating how many neighbors a node has and assigning the reciprocal of that number to the individual and each neighboring node. This means that each agent begins with equal weight given to their own opinions, as well as that of others

(iii) opinion (x) – an $n \times 1$ matrix which quantifies each individual's opinion as a probability between 0 and 1. This is initially composed of random values.

(iv) self-weight y(s) – the diagonal of W, which shows how much trust an individual has in themselves; this is a predictor of how much their opinion might sway

3.1. Changing opinion through discussion

During interactions each individual agent interacts with its neighbouring nodes, and shifts their opinion towards that of each neighbour, proportionally to the weight assigned to that neighbour in the initial weight matrix W. The formula for this is as follows:

$$x(t+1) = W\,x(t), \text{ where t = 0, 1, 2...}$$

They influence each other, and finally, create a space where every node is of the same opinion—this is a consensus. Figures 5 and 6 illustrate graphically how the opinions of nodes in a star graph and a random graph with 100 nodes each look at the beginning and end of a simulation.

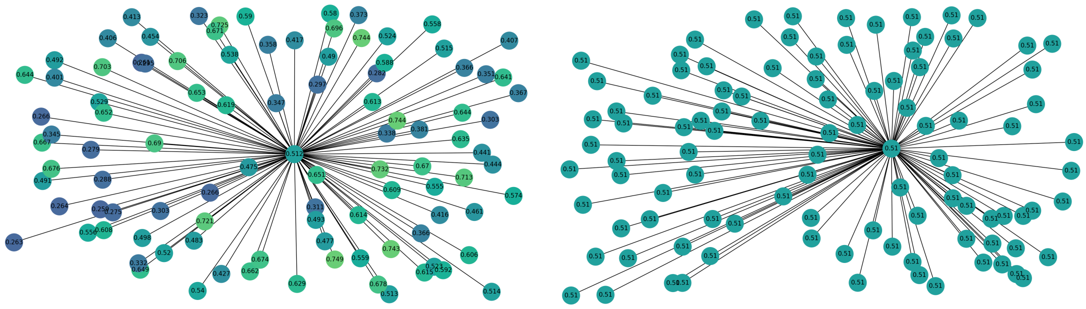

Figure 5 Star graph at initial and at consensus

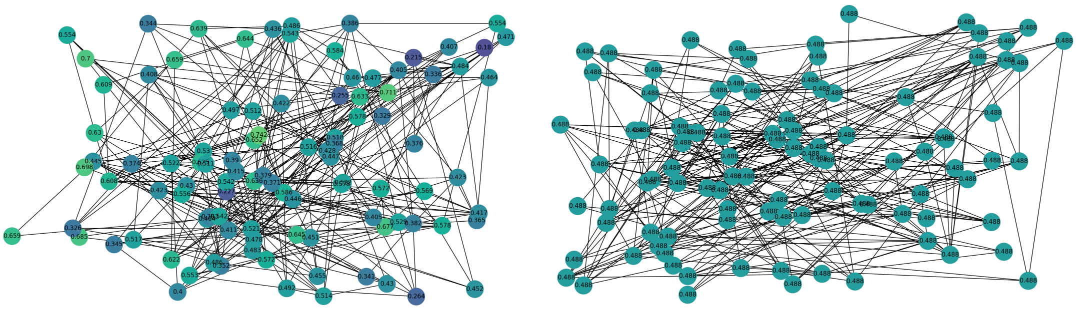

Figure 6 Random graph at initial and at consensus

While opinions converge across all types of networks, they do so at different rates. Table A illustrates the average value of t when all agents in a 100-node network reached the same opinion to 3 significant figures over 100 trials.

Table A: Average time to consensus for 100 nodes over 100 issues

Type of graph	Line	Ring	Star	Random	SBM
Time to consensus	14654.0	4043.2	11.9	33.2	49.7
Standard Deviation	4624	973	1.61	4.95	10.22

The fact that the line graph takes more than thrice the amount of time taken in a ring graph demonstrates how much of a difference a single edge on the graph can make. The the connection between the first and last node that the line graph lacks plays an enormous role in the speed at which opinions spread in the network.

The random graph takes around 20 interactions to reach a consensus, while the SBM takes over 100. This is because the connections between individuals in different clusters acts as a bottleneck to the spread of opinion in SBMs. Meanwhile, the star graph reaches consensus fastest, within 15 interactions.

3.2. The self-appraisal mechanism

After the completion of the discussion on one issue, agents in the network adjust their self-weight $y(s)$, based on how much of an impact they had on the previous issue, and then increase or decrease the amount of trust they have in each neighbor, in an update to W. This is calculated using the equation:

$$W(y(s)) = Diag\ (y(s)) + (I - Diag\ (y(s))\ W$$

The network will then begin interactions on the next issue, using the new weight matrix generated from the above equation, before repeating the process again. Below is the equation used to determine the next opinions of the network, with the self-appraisal mechanism taken into account:

$x(s,t+1) = W(s)\ x(s,t)$, where t = 0, 1, 2; $W(s)$ is static as t changes

3.3 Results and observations

3.3.1. Power of the central node

$$\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1/6 & 5/6 & 0 \\ 1/6 & 0 & 5/6 \end{pmatrix}$$

The above shows the weight matrix of a star graph with three nodes. After the discussion of just three issues, the self-weight of the central node goes to 1, and the final consensus of any succeeding issue will be equal to that of the initial opinion of the central node.

3.3.2. More connections equates to more power

On a random graph with 100 people, after the simulation of 100 issues, the final self-weight of a node is proportional its degree, except in the case where it is only connected to one other node. This is because the self-appraisal process concentrates power in well-connected agents, which results in their opinion being prominent in future issues.

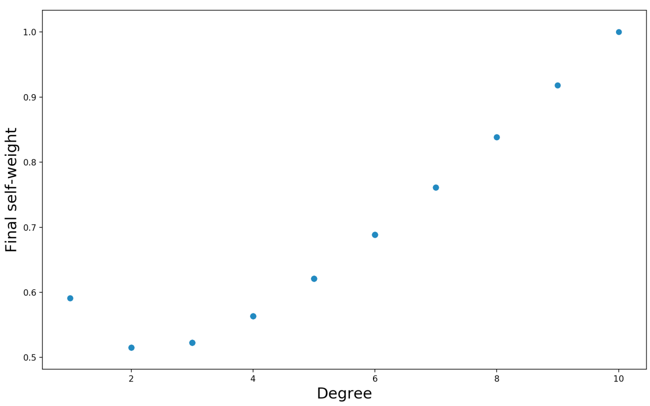

Figure 7 Effect of degree on final self-weight

3.4. Limitations of this DeGroot Model

A major limitation of the DeGroot model is its assumption that all individuals will agree with each other, and all interactions will lead to a consensus. The self-appraisal process proposed by Friedkin is also usually unrealistic when considered in real-world contexts, in that it is wholly dependent on the structure of the weight matrix W, and the number of connections an agent has, rather than their influence on the previous consensus. [6] While this could be used to predict how celebrities or politicians on social media might gather power, or people in an power-based organization like the US senate [6] might act, it is impractical to assume people decipher who has power and who does not before choosing to change their mind about a topic.

This could be improved drastically using a model in which self-appraisal happens whilst multiple issues are being dealt with simultaneously, with these appraisals being affected by both the self-node's as well as the neighbor-nodes' influence on the final opinions of the group.

4. Bounded Confidence Model with Influence-based Self-appraisal

The bounded confidence model shares many similarities with the DeGroot model—it uses the same adjacency matrix as a network, the opinion matrix x, and weight matrix W. It however, does away with the assumption that all agents will agree with each other, and instead sets a threshold value for maximum difference in opinion above which agents will not convince each other (τ). It also introduces a self-appraisal mechanism which depends directly on much influence each node had when discussing the previous issue, rather than it being deterministically based on the weight matrix.

4.1. Changing opinion through discussion

Interaction happens in the bounded confidence model on a 1-on-1 basis. Each node will communicate with adjacent nodes, decide whether the opinion of neighbors is worth considering (if the difference in opinion is less than τ), and then change their opinion depending on how much self-confidence *y(s)* they have:

$$x_i(s, t+1) = (1 - y(s)_i) \sum_{j=1}^{n} \mu \, w_{ij} \, (x_j(s,t) - x_i(s,t)) + x_i(s,t) \, ,$$

$$\textit{where } \mu = 1 \textit{ if } |x_j(s,t) - x_i(s,t)| < \tau, \textit{ and 0 otherwise}$$

μ is a binary variable; it is set to 1 if the absolute difference in opinion is less than the selected bound (τ), and 0 if not. When the bound is set to 1, opinions converge, and the model acts similar to the DeGroot model. The following values were calculated for 100 nodes over 100 issues, with *y(s)* set to 1 for all nodes.

Table B: Average time to consensus for 100 nodes over 100 issues

Type of graph	**Line**	**Ring**	**Star**	**Random**	**SBM**
Time to consensus	9861.9	2803.3	11.0	23.6	30.1
Standard Deviation	3343	660.0	1.47	3.04	4.82

When the bound is set to a value below 1, the network forms clusters, which share the same opinion. This is illustrated pictorially using an SBM network of 100 nodes and the bound value, τ set to 0.2 in figure 7, and graphically in figure 8.

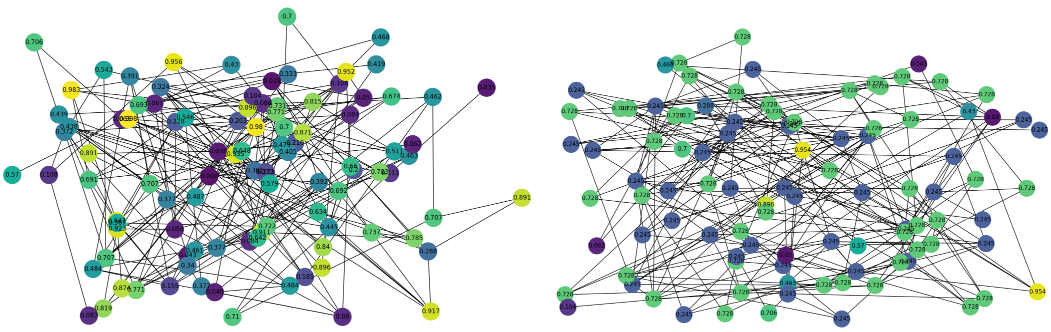

Figure 8 SBM at initial time, and after groups reach consensus

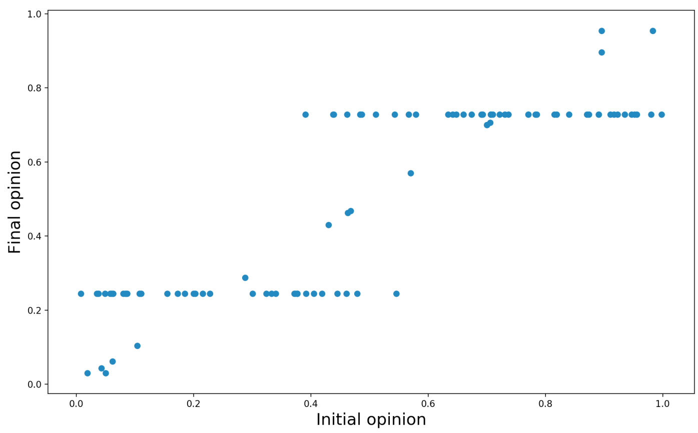

Figure 9 Comparison of initial and final opinions for the above graph

While the initial randomly-assigned opinions (represented on the x-axis) were distributed equally, the final opinions (represented on the y-axis) mostly clustered in two areas, with a few outliers (nodes which had no neighbours with similar opinion), and a couple of smaller clusters.

4.2. The influence-based self-appraisal process

In the influence-based self-appraisal process, self-trust is calculated directly based solely on an agent's perceived contribution to the previous discussion, rather than their number of connections. This is in some ways more similar to reality; as uncovered in a 2007 psychological study which found that a perceived increase in power increases self-esteem. [9] In this model, the agent increases their self-appraisal value in *y(s)* based on how much they were able to stand their ground in the previous discussion—a lower change in initial and final opinion for each individual results in a higher self-appraisal value.

This self-appraisal—indicated by the $n\times 1$ matrix *y(s)* formed after the discussion of issue *s*—is calculated using:

$$y_i(s+1) = e^{-\delta\,(|x_i(s,t)-x_j(s,t)|)}$$

4.3. Results, observations, and limitations

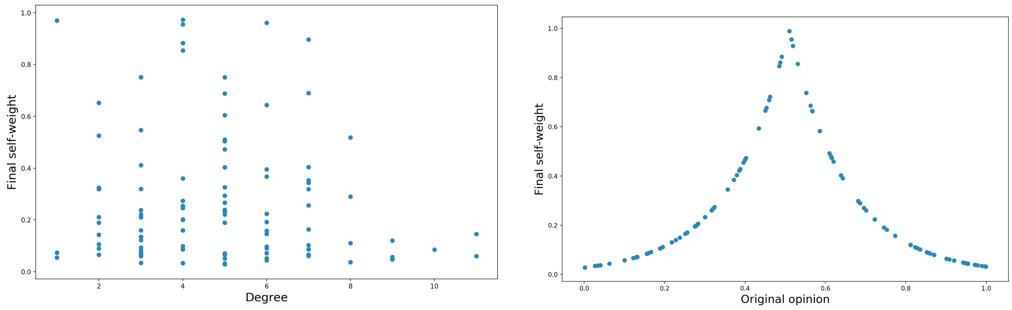

Figure 10 Self-weight is dependant on original opinion, rather than degree using a random network of 100 nodes

While the influence-based self-appraisal process gets rid of the issue of the Friedkin self-appraisal process's degree-based self-weight, it creates a situation where final self-weight is dependent on the agent's original opinion, as this is where the average tends to lie. This could be improved upon in the future by taking other factors, including degree into account in calculating self-weights.

5. Conclusion

The way that opinions take hold in the real world is haphazard. As the coronavirus, COVID-19 spread across the world in 2020, people in Hong Kong rapidly began to take precautions, and came to an almost immediate consensus to don masks, while people in America fought over a what their response should be. While this situation alone could cause one to write off America as a non-consensus society, and Hong Kong as a consensus-reaching one, the Hong Kong protests during the latter half of 2019 deeply divided [10] the same society that had very little resistance to all wearing masks. This is just one illustration of the fact that people are capable of changing their ways of being influenced fast, and based on the situation. Therefore, it is very likely that no single-dimensional model can will be able to reflect the real world well.

This paper outlined two linear and non-linear models which describe how individuals might base their decisions based on sentiments of the people around them, and discussed their implications on various network topologies. It described two forms of self-appraisal which reflect different factors in their calculation. In order to create more realistic models, these must be combined, and their merits in differing situations must be understood better.

Acknowledgements. I am grateful to Prof. Hoi To Wai for assisting me with understanding the concepts behind this paper and guiding me through the process of completing it.

Works Cited

[1] D. Jayawardena, "SEEM4998 Planning Report," 2019.

[2] "Opinion dynamics and social influence," Dec 2019. [Online]. Available: https://www.sg.ethz.ch/research/opinion-dynamics-and-social-influence/.

[3] KONECT, "Facebook friendships," [Online]. Available: http://konect.uni-koblenz.de/networks/facebook-wosn-links. [Accessed April 2020].

[4] M. H. Degroot, "Reaching a consensus," *Journal of the American Statistical Association,* 1974.

[5] N. E. Friedkin, "A Formal Theory of Reflected Appraisals in the Evolution of Power," *Administrative Science Quarterly,* vol. 56, pp. 501-529, 2011.

[6] P. Jia, A. Miratabatabaei, N. E. Friedkin and F. Bullo, "Opinion Dynamics and the Evolution of Social Power in Influence Networks," *Siam Review,* 2015.

[7] D. Jayawardena, "Modelling and Simulating Opinion Dynamics in Social Networks," 2019.

[8] D. N. F. A. a. G. W. Guillaume Deffuant, "Mixing Beliefs Among Interacting Agents," *Advances in Complex Systems,* pp. 87-98, 2000.

[9] B. Wojciszke and A. Struzynska–Kujalowicz, "Power Influences Self-Esteem," *Social Cognition,* vol. 25, no. 4, pp. 472-494, 2007.

[10] M. Rowse, "Hong Kong protests: to bridge city's 'blue' and 'yellow' divide, police watchdog must walk the talk," 20 Apr 2020. [Online]. Available: https://www.scmp.com/comment/opinion/article/3080422/hong-kong-protests-bridge-citys-blue-and-yellow-divide-police.