1. For each of the two questions below, decide whether the answer is (i) "Yes," (ii) "No," or (iii) "Unknown, because it would resolve the question of whether $\mathcal{P} = \mathcal{NP}$." Give a brief explanation of your answer.

(a) Let's define the decision version of the Interval Scheduling Problem from Chapter 4 as follows: Given a collection of intervals on a time-line, and a bound k, does the collection contain a subset of nonoverlapping intervals of size at least k?

Question: Is it the case that Interval Scheduling $\leq_P$ Vertex Cover?

(b) Question: Is it the case that Independent Set $\leq_P$ Interval Scheduling?

5. Consider a set $A = \{a_1, \ldots, a_n\}$ and a collection $B_1, B_2, \ldots, B_m$ of subsets of A (i.e., $B_i \subseteq A$ for each i).

We say that a set $H \subseteq A$ is a *hitting set* for the collection $B_1, B_2, \ldots, B_m$ if H contains at least one element from each B_i—that is, if $H \cap B_i$ is not empty for each i (so H "hits" all the sets B_i).

We now define the *Hitting Set Problem* as follows. We are given a set $A = \{a_1, \ldots, a_n\}$, a collection $B_1, B_2, \ldots, B_m$ of subsets of A, and a number k. We are asked: Is there a hitting set $H \subseteq A$ for $B_1, B_2, \ldots, B_m$ so that the size of H is at most k?

Prove that Hitting Set is NP-complete.

3. Suppose you're helping to organize a summer sports camp, and the following problem comes up. The camp is supposed to have at least one counselor who's skilled at each of the n sports covered by the camp (baseball, volleyball, and so on). They have received job applications from m potential counselors. For each of the n sports, there is some subset of the m applicants qualified in that sport. The question is: For a given number $k < m$, is it possible to hire at most k of the counselors and have at least one counselor qualified in each of the n sports? We'll call this the *Efficient Recruiting Problem*.

Show that Efficient Recruiting is NP-complete.

7. Since the 3-Dimensional Matching Problem is NP-complete, it is natural to expect that the corresponding 4-Dimensional Matching Problem is at least as hard. Let us define *4-Dimensional Matching* as follows. Given sets W, X, Y, and Z, each of size n, and a collection C of ordered 4-tuples of the form (w_i, x_j, y_k, z_ℓ), do there exist n 4-tuples from C so that no two have an element in common?

Prove that 4-Dimensional Matching is NP-complete.

16. Consider the problem of reasoning about the identity of a set from the size of its intersections with other sets. You are given a finite set U of size n, and a collection $A_1, \ldots, A_m$ of subsets of U. You are also given numbers $c_1, \ldots, c_m$. The question is: Does there exist a set $X \subset U$ so that for each $i = 1, 2, \ldots, m$, the cardinality of $X \cap A_i$ is equal to c_i? We will call this an instance of the *Intersection Inference Problem*, with input U, $\{A_i\}$, and $\{c_i\}$.

Prove that Intersection Inference is NP-complete.

39. The *Directed Disjoint Paths Problem* is defined as follows. We are given a directed graph G and k pairs of nodes $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$. The problem is to decide whether there exist node-disjoint paths $P_1, P_2, \ldots, P_k$ so that P_i goes from s_i to t_i.

Show that Directed Disjoint Paths is NP-complete.