Question 1

3.Let $G = (V, E)$ be a directed graph, with source $s \in V$, sink $t \in V$, and nonnegative edge capacities $\{c_e\}$. Give a polynomial-time algorithm to decide whether G has a unique minimum $s - t$ cut(i.e., an $s - t$ of capacity strictly less than that of all other $s - t$ cuts).

We can first invoke "Scaling-Max-Flow" to get the maximum flow, which is also the minimum cut. Then we choose a minimum cut $A_1(s, t)$. For each edge $e \in A_1(s, t)$, we add the capacity by 1 and re-compute the minimum. If for each edge's increasing 1 of its capacity, the value of minimum cut increases, then the original graph has a unique minimum $s - t$ cut, else it has none.

Question 2

4.Given a graph $G = (V, E)$, and a natural number k, we can define a relation $\xrightarrow{G,k}$ on pairs of vertices of G as follows. If $x, y \in V$, we say that $x \xrightarrow{G,k} y$ if there exist k mutually edge-disjoint paths from x to y in G.

Is it true that for every G and every $k \geq 0$, the relation $\xrightarrow{G,k}$ is transitive? That is, is it always the case that if $x \xrightarrow{G,k} y$ and $y \xrightarrow{G,k} z$, then we have $x \xrightarrow{G,k} z$? Give a proof or a counterexample.

It is true. We give the proof as follow.

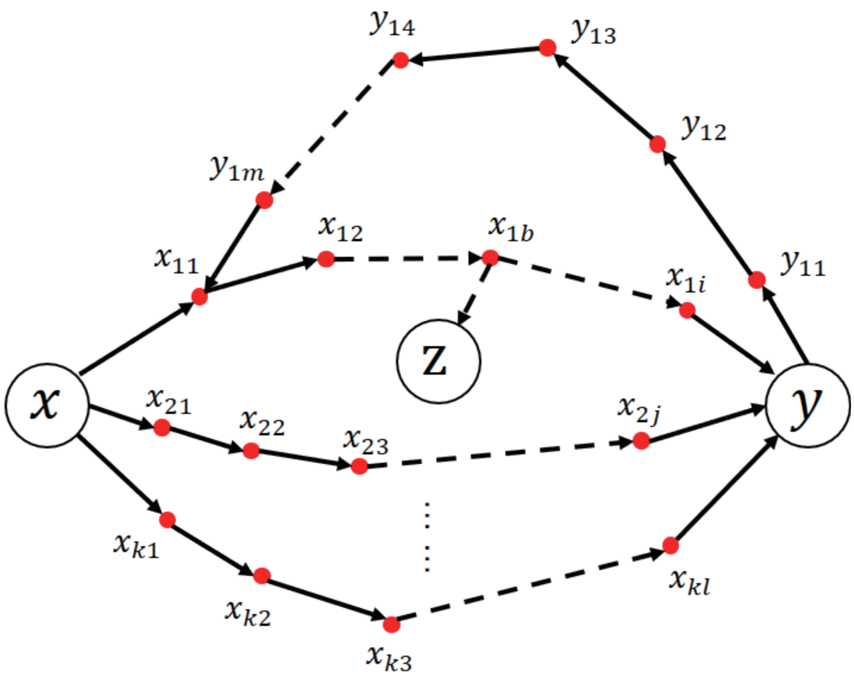

Figure 2.1 Instance

As is shown in Fig 2.1, there are k paths from x to y. If there is none of the paths from y to z, which jointed with those from x to y, then the result is true trivially.

If there are one path from y to z that joint a path from x from y, then we can see that there will appear another path that from x to z without via y. It does not decrease the quantity of paths of the trivial case.

Consider a more general case, as is shown in Fig 2.2, when there is a path from y to z that joint with q paths from x to y. We can see that this path from y to z can be excluded because there is already a path from x to z without via y.

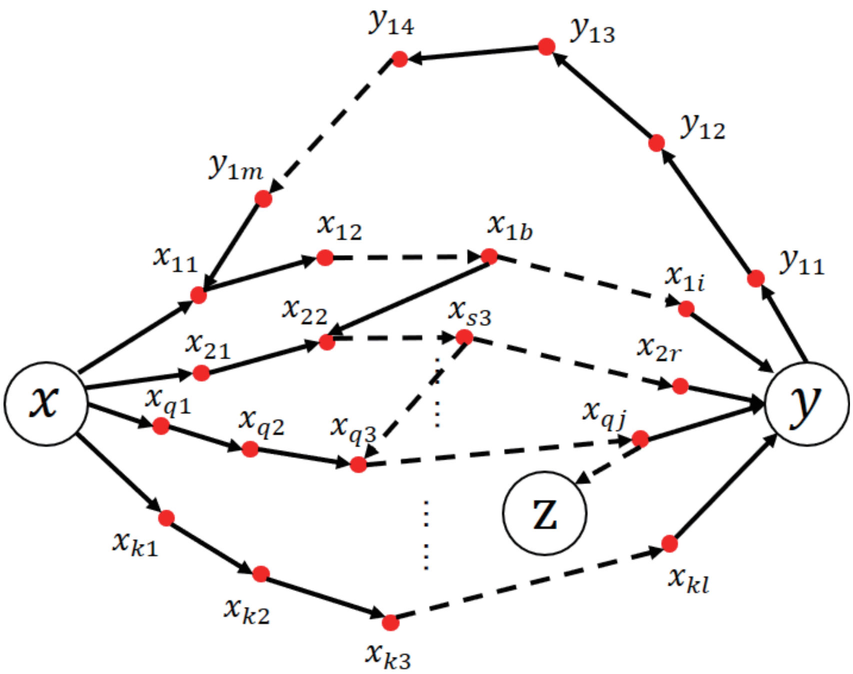

Figure 2.2 Instance

Now consider the general case, as is shown in Fig 2.3,when there is t paths from y to z, all of which joint with q paths from x to y. We can see that these t paths make sure that the sub path joint with the q paths from x to y disjoint. We reconstruct the path by choosing $x \to x_{q1} \to x_{q2} \to x_{q3} \ldots \to x_{qu} \to \ldots \to z$, $x \to x_{q-1\,1} \to x_{q-1\,2} \to x_{q-1\,3} \ldots \to x_{q-1\,f} \to x_{qg} \to \ldots \to z \ldots x \to x_{11} \to x_{12} \to x_{13} \ldots \to x_{1h} \to x_{2o} \to \ldots \to z$. It is obviously that the quantity paths will not decrease.

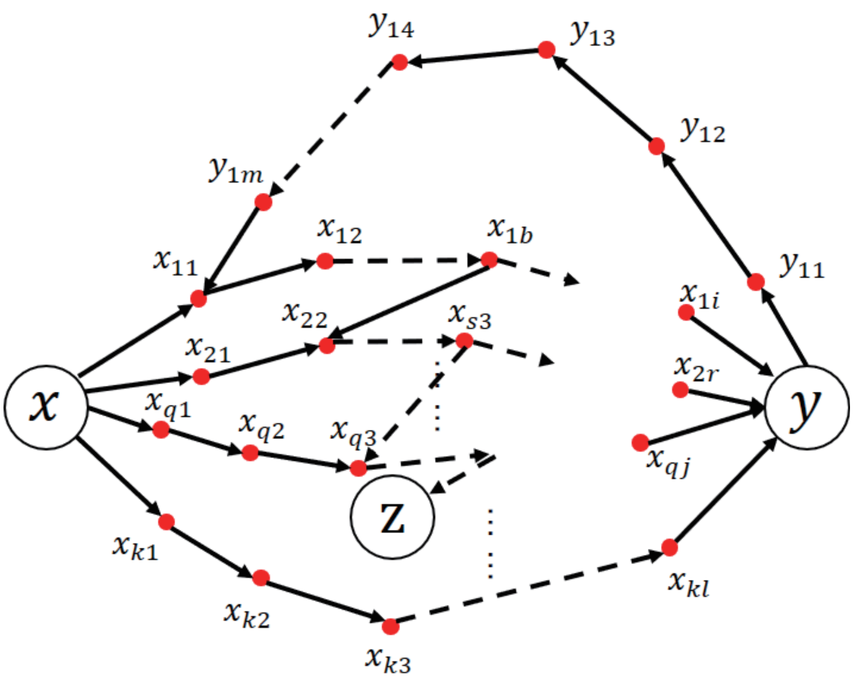

Figure 2.3 Instance

Therefore the conclusion is "Yes".

Question 3

5.Let $G = (V, E)$ be a directed graph, and suppose that for each node v, the number of edges into v is equal to the number of edges out of v. That is for all v,

$$|\{(u, v) : (u, v) \in E\}| = |\{(v, w) : (v, w) \in E\}|$$

Let x, y be two nodes of G, and suppose that there exist k mutually edge-disjoint paths from y to x? Give a proof or a counterexample with explanation.

It is true. Consider the k paths from x to y. We can see that these k path generate a flow. Its source x's outer degree and sink y's inner degree are both k, the other vertexes' outer degrees are equal to the inner degrees. Because the condition says the number of edges into each vertex's is equal to the number of those out of the vertex, y has to find vertex to make its outer degree equal to its inner degree, which will remedy the inner degree of x.

Question 4

6.考虑到地震救灾场景，n个伤员需要被尽快总往医院。在这个地区有k所医院，这n个人中每个人需要被送到距离他们目前的地点半小时车程以内的医院（因此不同的人将对医院有不同的选择，依赖于他们当前所在的地方）。同时，人们不想由于太多的病人送来而使得任何一个医院超负荷。医护人员通过移动电话联系，他们想共同解决是否可以为每个受伤的人选择一所医院，这种选择方式要求医院负荷是均衡的，即每个医院至多接受$\lceil n/k \rceil$的人。给出一个多项式时间的算法，它以关于这些人所在位置的给定信息作为输入并且确定这是不可能的。

我们将伤员(w_i)和医院(h_j)表示成点，根据题目条件，加上源点s和汇点t可以构成Fig 4.1。

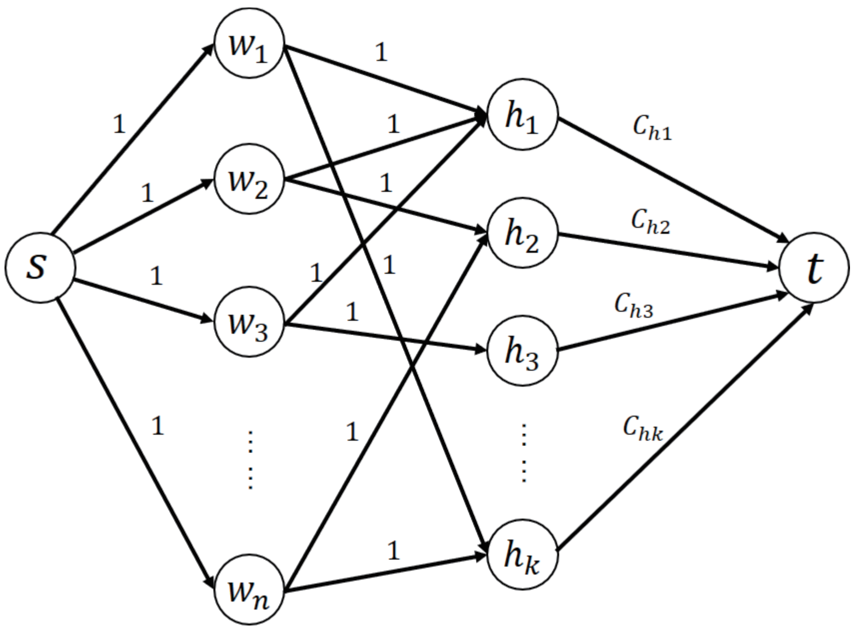

Figure 4.1 Instance

图中C_{hi} $(1 \le i \le k)$ 表示各个医院的最大容量。因此只要执行最大流算法，就可以计算出结果。最大流算法的时间复杂度是多项式时间的。

Question 5

7.考虑郊野环境中的移动通信场景，给定n个基站的位置，它们由平面上的点$b_1, b_2, ..., b_n$来指定，以及n个手机用户的位置，它们也指定为平面上的点$p_1, p_2, ..., p_n$，最后，给定一个域参数$\Delta > 0$。如果能以下述这样的方式把每个电话分配给一个基站，我们就说这组便携式电话是完全联通的。

每个手机被分到不同的基站，且如果位于p_i的电话被分配到位于b_j的基站，那么在点p_i和b_j之间的直线距离至多是Δ。

假设在点p_i的用户决定开车向东经过z距离，需要修改手机对基站的分配（可能要几次）以便保持完全连通，给出一个多项式时间算法。（假定在此期间其他手机固定。）如果可能，报告一个电话到基站的分配序列；如果不可能，报告使得完全连通性不可能在维持的一个点。算法运行时间在$O(n^3)$时间。

首先以n个基站位置为圆心，半径Δ到b_j运行线段所在直线的交点，并计算出各个交点到p_i的距离，进行排序。遍历所有交点，如果是和圆第一次相交，就是进入点，如果第二次相交就是离开点，离开点的后继还是离开点，那么改点就是一个不可维持点。

该过程对某一个p_i，执行时间是$O(nlogn)$.