



Chapter 6

Dynamic Programming



Algorithmic Paradigms



Greed. Build up a solution incrementally, myopically optimizing some local criterion.

Divide-and-conquer. Break up a problem into two sub-problems, solve each sub-problem independently, and combine solution to sub-problems to form solution to original problem.

Dynamic programming. Break up a problem into a series of overlapping sub-problems, and build up solutions to larger and larger sub-problems.



Dynamic Programming History



Bellman. Pioneered the systematic study of dynamic programming in the 1950s.

Etymology.

Dynamic programming = planning over time.

"programming" referred to the use of the method to find an optimal program, in the sense of a military schedule for training or logistics.

Secretary of Defense was hostile to mathematical research.

Bellman sought an impressive name to avoid confrontation.

- "it's impossible to use dynamic in a pejorative sense"
- "something not even a Congressman could object to"

Reference: Bellman, R. E. *Eye of the Hurricane, An Autobiography*.



Dynamic Programming Applications



Areas.

Bioinformatics.

Control theory.

Information theory.

Operations research.

Computer science: theory, graphics, AI, systems,

Some famous dynamic programming algorithms.

Viterbi for hidden Markov models.

Unix diff for comparing two files.

Smith-Waterman for sequence alignment.

Bellman-Ford for shortest path routing in networks.

Cocke-Kasami-Younger for parsing context free grammars.





6.1 Weighted Interval Scheduling



Weighted Interval Scheduling

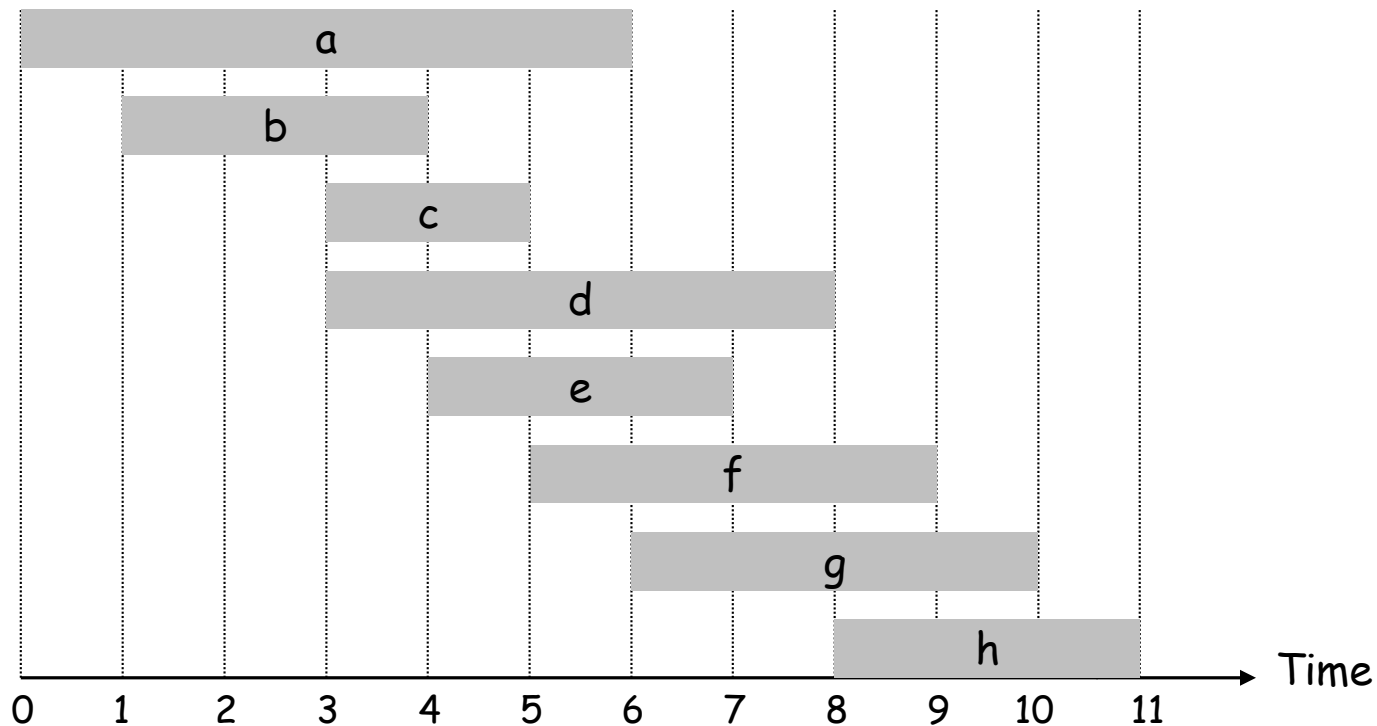


Weighted interval scheduling problem.

Job j starts at s_j , finishes at f_j , and has weight or value v_j .

Two jobs **compatible** if they don't overlap.

Goal: find maximum **weight** subset of mutually compatible jobs.

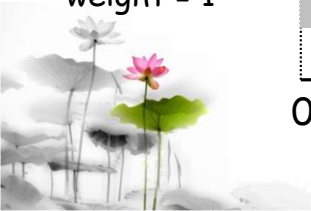
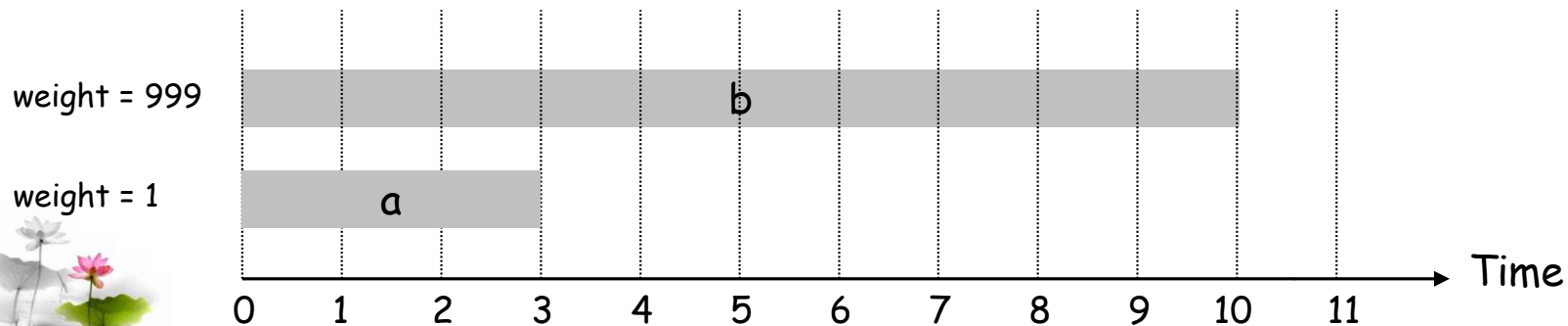


Unweighted Interval Scheduling Review



Recall. Greedy algorithm works if all weights are 1.
Consider jobs in ascending order of finish time.
Add job to subset if it is compatible with previously chosen jobs.

Observation. Greedy algorithm can fail spectacularly if arbitrary weights are allowed.



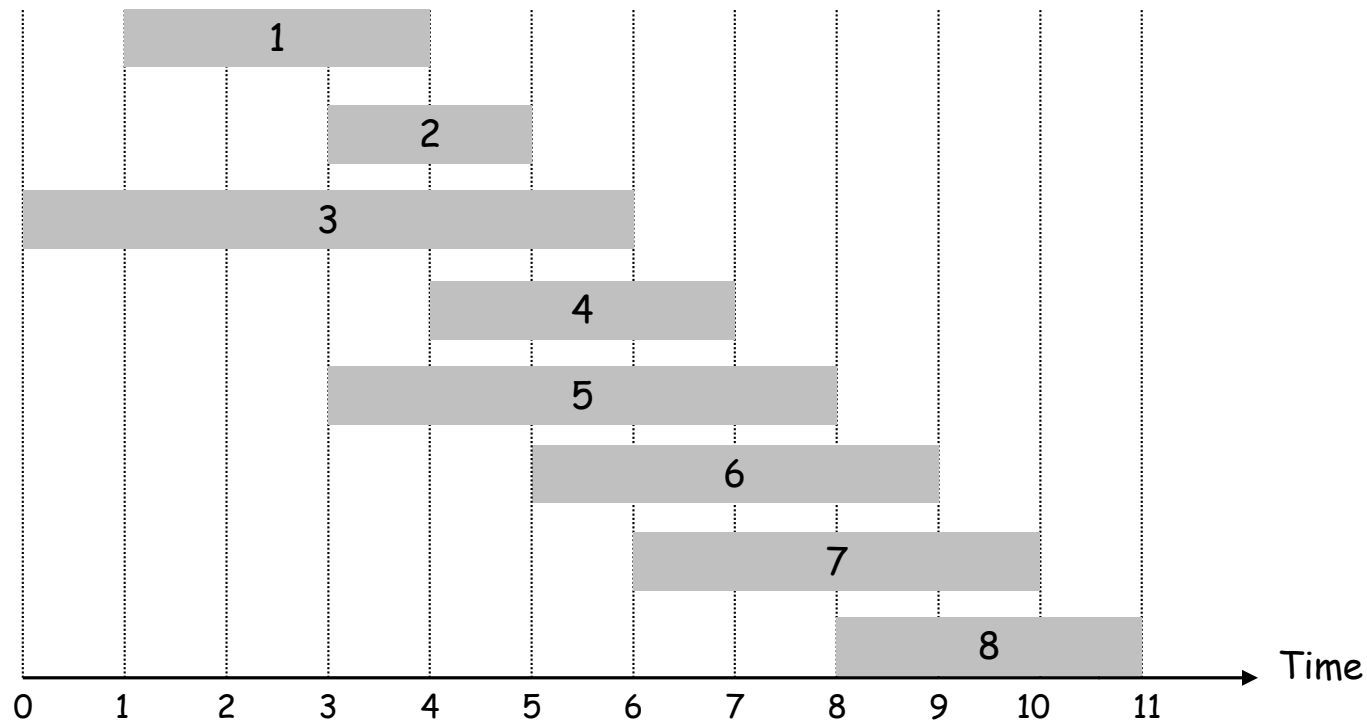
Weighted Interval Scheduling



Notation. Label jobs by finishing time: $f_1 \leq f_2 \leq \dots \leq f_n$.

Def. $p(j)$ = largest index $i < j$ such that job i is compatible with j .

Ex: $p(8) = 5$, $p(7) = 3$, $p(2) = 0$.



Dynamic Programming: Binary Choice



Notation. $OPT(j)$ = value of optimal solution to the problem consisting of job requests $1, 2, \dots, j$.

Case 1: OPT selects job j .

- can't use incompatible jobs $\{ p(j) + 1, p(j) + 2, \dots, j - 1 \}$
- must include optimal solution to problem consisting of remaining compatible jobs $1, 2, \dots, p(j)$

↖
↙
optimal substructure

Case 2: OPT does not select job j .

- must include optimal solution to problem consisting of remaining compatible jobs $1, 2, \dots, j-1$

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max \{ v_j + OPT(p(j)), OPT(j-1) \} & \text{otherwise} \end{cases}$$



Weighted Interval Scheduling: Brute Force



Brute force algorithm.

Input: $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$

Sort jobs by finish times so that $f_1 \leq f_2 \leq \dots \leq f_n$.

Compute $p(1), p(2), \dots, p(n)$

```
Compute-Opt(j) {  
    if (j = 0)  
        return 0  
    else  
        return max( $v_j + \text{Compute-Opt}(p(j))$ ,  $\text{Compute-Opt}(j-1)$ )  
}
```

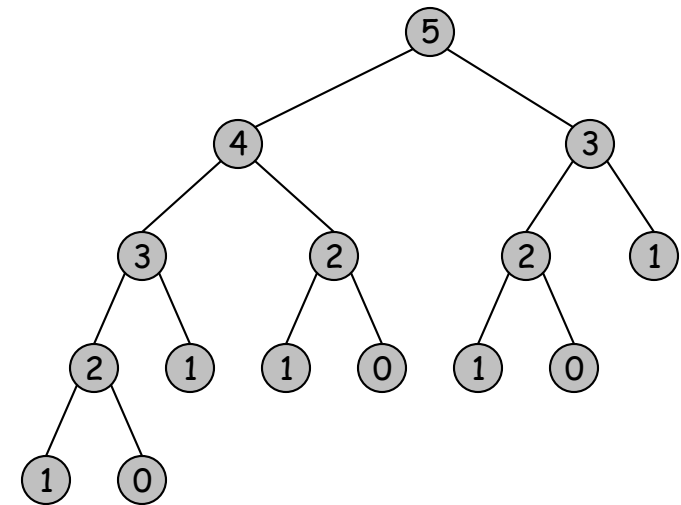
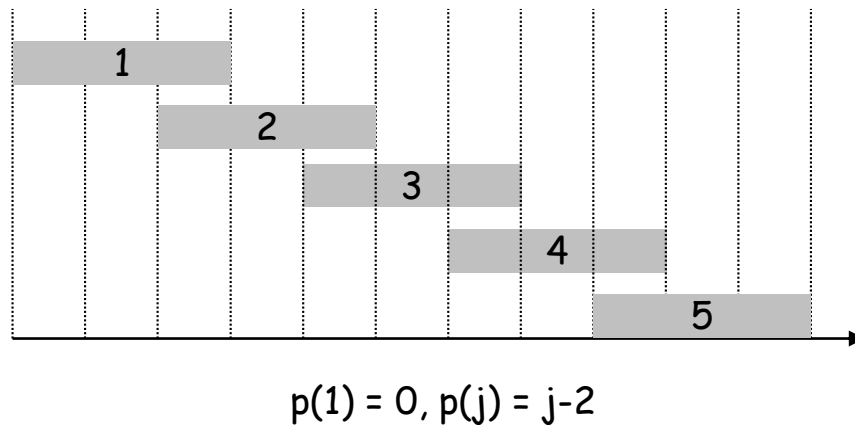


Weighted Interval Scheduling: Brute Force



Observation. Recursive algorithm fails spectacularly because of redundant sub-problems \Rightarrow exponential algorithms.

Ex. Number of recursive calls for family of "layered" instances grows like Fibonacci sequence.



Weighted Interval Scheduling: Memoization



Memoization. Store results of each sub-problem in a cache; lookup as needed.

Input: $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$

Sort jobs by finish times so that $f_1 \leq f_2 \leq \dots \leq f_n$.

Compute $p(1), p(2), \dots, p(n)$

for $j = 1$ to n

$M[j] = \text{empty}$ \leftarrow global array

$M[j] = 0$

M-Compute-Opt(j) {

if ($M[j]$ is empty)

$M[j] = \max(w_j + \text{M-Compute-Opt}(p(j)), \text{M-Compute-Opt}(j-1))$

return $M[j]$

}



Weighted Interval Scheduling: Running Time



Claim. Memoized version of algorithm takes $O(n \log n)$ time.

Sort by finish time: $O(n \log n)$.

Computing $p(\cdot)$: $O(n)$ after sorting by start time.

$M\text{-Compute-Opt}(j)$: each invocation takes $O(1)$ time and either

- (i) returns an existing value $M[j]$
- (ii) fills in one new entry $M[j]$ and makes two recursive calls

Progress measure $\Phi = \#$ nonempty entries of $M[\]$.

- initially $\Phi = 0$, throughout $\Phi \leq n$.
- (ii) increases Φ by 1 \Rightarrow at most $2n$ recursive calls.

Overall running time of $M\text{-Compute-Opt}(n)$ is $O(n)$. ■

Remark. $O(n)$ if jobs are pre-sorted by start and finish times.



Automated Memoization



Automated memoization. Many functional programming languages (e.g., Lisp) have built-in support for memoization.

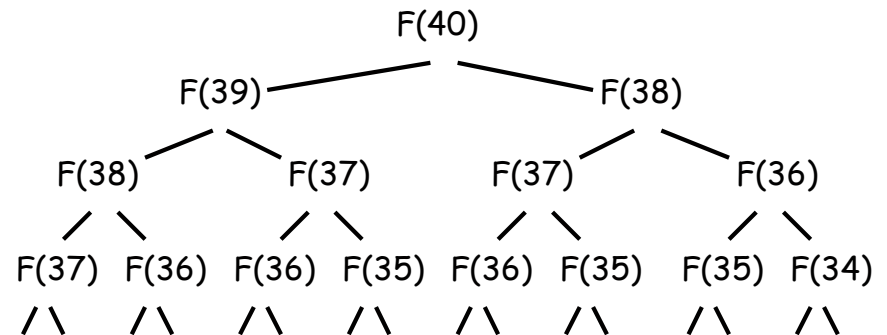
Q. Why not in imperative languages (e.g., Java)?

```
(defun F (n)
  (if
    (<= n 1)
    n
    (+ (F (- n 1)) (F (- n 2))))))
```

Lisp (efficient)

```
static int F(int n) {
  if (n <= 1) return n;
  else return F(n-1) + F(n-2);
}
```

Java (exponential)



Weighted Interval Scheduling: Finding a Solution



Q. Dynamic programming algorithms computes optimal value. What if we want the solution itself?

A. Do some post-processing.

```
Run M-Compute-Opt(n)
Run Find-Solution(n)

Find-Solution(j) {
    if (j = 0)
        output nothing
    else if ( $v_j + M[p(j)] > M[j-1]$ )
        print j
        Find-Solution(p(j))
    else
        Find-Solution(j-1)
}
```

of recursive calls $\leq n \Rightarrow O(n)$.



Weighted Interval Scheduling: Bottom-Up



Bottom-up dynamic programming. Unwind recursion.

Input: $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$

Sort jobs by finish times so that $f_1 \leq f_2 \leq \dots \leq f_n$.

Compute $p(1), p(2), \dots, p(n)$

```
Iterative-Compute-Opt {  
    M[0] = 0  
    for j = 1 to n  
        M[j] = max( $v_j + M[p(j)]$ , M[j-1])  
}
```





6.3 Segmented Least Squares



Segmented Least Squares



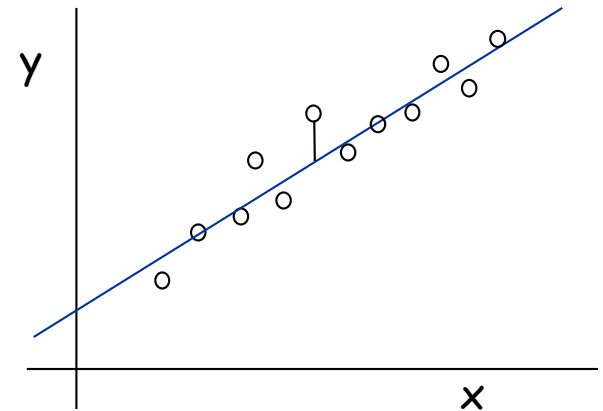
Least squares.

Foundational problem in statistic and numerical analysis.

Given n points in the plane: $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$.

Find a line $y = ax + b$ that minimizes the sum of the squared error:

$$SSE = \sum_{i=1}^n (y_i - ax_i - b)^2$$



Solution. Calculus \Rightarrow min error is achieved when

$$a = \frac{n \sum_i x_i y_i - (\sum_i x_i) (\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2}, \quad b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$



Segmented Least Squares



Segmented least squares.

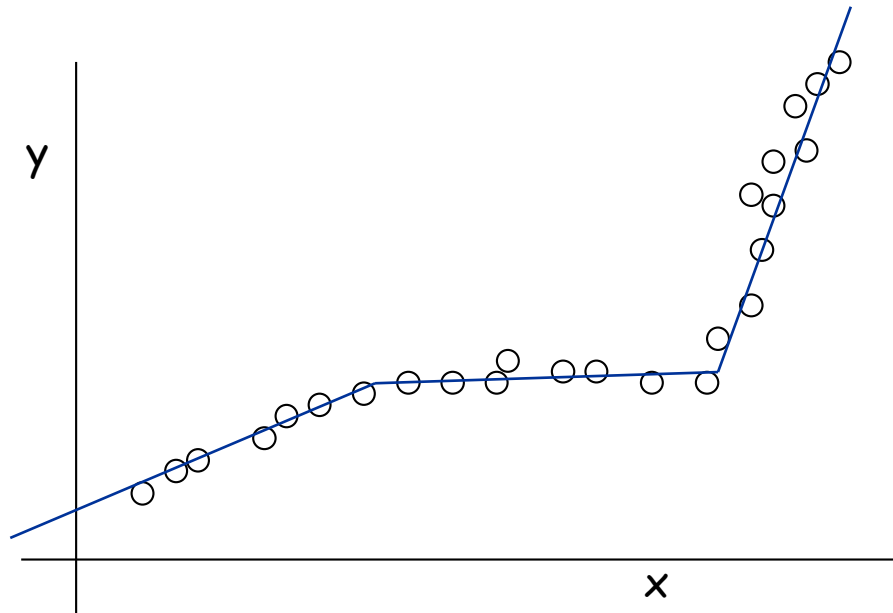
Points lie roughly on a sequence of several line segments.

Given n points in the plane $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ with $x_1 < x_2 < \dots < x_n$, find a sequence of lines that minimizes $f(x)$.

Q. What's a reasonable choice for $f(x)$ to balance accuracy and parsimony?

↑
number of lines

↑
goodness of fit



Segmented Least Squares



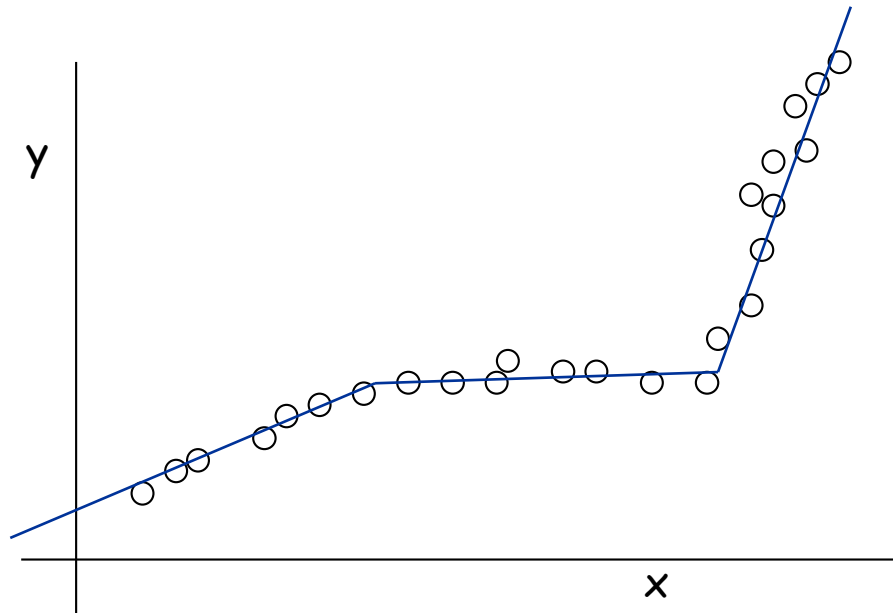
Segmented least squares.

Points lie roughly on a sequence of several line segments.

Given n points in the plane $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ with $x_1 < x_2 < \dots < x_n$, find a sequence of lines that minimizes:

- the sum of the sums of the squared errors E in each segment
- the number of lines L

Tradeoff function: $E + c L$, for some constant $c > 0$.



Dynamic Programming: Multiway Choice



Notation.

$OPT(j)$ = minimum cost for points p_1, p_{i+1}, \dots, p_j .

$e(i, j)$ = minimum sum of squares for points p_i, p_{i+1}, \dots, p_j .

To compute $OPT(j)$:

Last segment uses points p_i, p_{i+1}, \dots, p_j for some i .

Cost = $e(i, j) + c + OPT(i-1)$.

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \min_{1 \leq i \leq j} \{ e(i, j) + c + OPT(i-1) \} & \text{otherwise} \end{cases}$$




Segmented Least Squares: Algorithm



INPUT: n, p_1, \dots, p_N, c

```
Segmented-Least-Squares() {  
    M[0] = 0  
    for j = 1 to n  
        for i = 1 to j  
            compute the least square error  $e_{ij}$  for  
            the segment  $p_i, \dots, p_j$   
  
    for j = 1 to n  
        M[j] = min1 ≤ i ≤ j ( $e_{ij} + c + M[i-1]$ )  
  
    return M[n]  
}
```

Running time. $O(n^3)$.  can be improved to $O(n^2)$ by pre-computing various statistics

Bottleneck = computing $e(i, j)$ for $O(n^2)$ pairs, $O(n)$ per pair using previous formula.





6.4 Knapsack Problem



Knapsack Problem



Knapsack problem.

Given n objects and a "knapsack."

Item i weighs $w_i > 0$ kilograms and has value $v_i > 0$.

Knapsack has capacity of W kilograms.

Goal: fill knapsack so as to maximize total value.

Ex: { 3, 4 } has value 40.

$W = 11$

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Greedy: repeatedly add item with maximum ratio v_i / w_i .

Ex: { 5, 2, 1 } achieves only value = 35 \Rightarrow greedy not optimal.



Dynamic Programming: False Start



Def. $OPT(i)$ = max profit subset of items $1, \dots, i$.

Case 1: OPT does not select item i .

- OPT selects best of $\{1, 2, \dots, i-1\}$

Case 2: OPT selects item i .

- accepting item i does not immediately imply that we will have to reject other items
- without knowing what other items were selected before i , we don't even know if we have enough room for i

Conclusion. Need more sub-problems!



Dynamic Programming: Adding a New Variable



Def. $OPT(i, w)$ = max profit subset of items 1, ..., i with weight limit w.

Case 1: OPT does not select item i.

- OPT selects best of $\{1, 2, \dots, i-1\}$ using weight limit w

Case 2: OPT selects item i.

- new weight limit = $w - w_i$
- OPT selects best of $\{1, 2, \dots, i-1\}$ using this new weight limit

$$OPT(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1, w) & \text{if } w_i > w \\ \max \{ OPT(i-1, w), v_i + OPT(i-1, w - w_i) \} & \text{otherwise} \end{cases}$$



Knapsack Problem: Bottom-Up



Knapsack. Fill up an n -by- W array.

```
Input:  $n, w_1, \dots, w_N, v_1, \dots, v_N$ 

for  $w = 0$  to  $W$ 
     $M[0, w] = 0$ 

for  $i = 1$  to  $n$ 
    for  $w = 1$  to  $W$ 
        if ( $w_i > w$ )
             $M[i, w] = M[i-1, w]$ 
        else
             $M[i, w] = \max \{M[i-1, w], v_i + M[i-1, w-w_i]\}$ 

return  $M[n, W]$ 
```



Knapsack Algorithm



		0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40	40

$n + 1$
 \downarrow

OPT: { 4, 3 }
 value = 22 + 18 = 40

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7



Knapsack Problem: Running Time



Running time. $\Theta(n W)$.

Not polynomial in input size!

"Pseudo-polynomial."

Decision version of Knapsack is NP-complete. [Chapter 8]

Knapsack approximation algorithm. There exists a polynomial algorithm that produces a feasible solution that has value within 0.01% of optimum. [Section 11.8]





6.5 RNA Secondary Structure



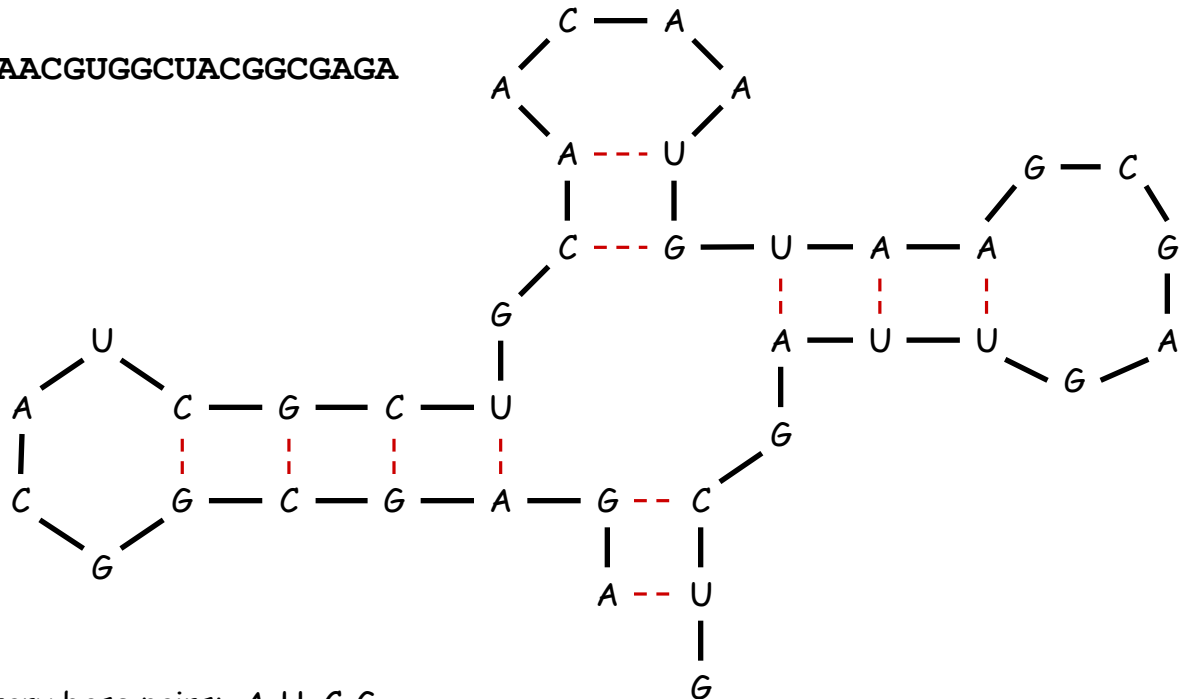
RNA Secondary Structure



RNA. String $B = b_1b_2\dots b_n$ over alphabet $\{A, C, G, U\}$.

Secondary structure. RNA is single-stranded so it tends to loop back and form base pairs with itself. This structure is essential for understanding behavior of molecule.

Ex: GUCGAUUGAGCGAAUGUAACAACGUGGCUACGGCGAGA



complementary base pairs: A-U, C-G



RNA Secondary Structure



Secondary structure. A set of pairs $S = \{ (b_i, b_j) \}$ that satisfy:

- [Watson-Crick.] S is a matching and each pair in S is a Watson-Crick complement: A-U, U-A, C-G, or G-C.
- [No sharp turns.] The ends of each pair are separated by at least 4 intervening bases. If $(b_i, b_j) \in S$, then $i < j - 4$.
- [Non-crossing.] If (b_i, b_j) and (b_k, b_l) are two pairs in S , then we cannot have $i < k < j < l$.

Free energy. Usual hypothesis is that an RNA molecule will form the secondary structure with the optimum total free energy.

↑
approximate by number of base pairs

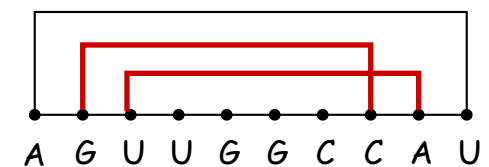
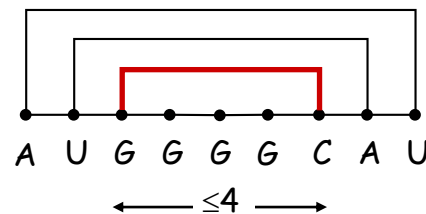
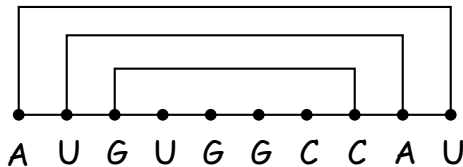
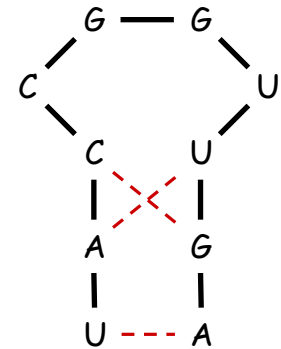
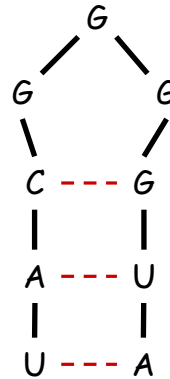
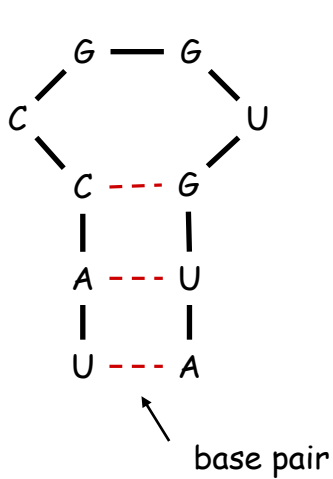
Goal. Given an RNA molecule $B = b_1b_2\dots b_n$, find a secondary structure S that maximizes the number of base pairs.



RNA Secondary Structure: Examples



Examples.



ok

sharp turn

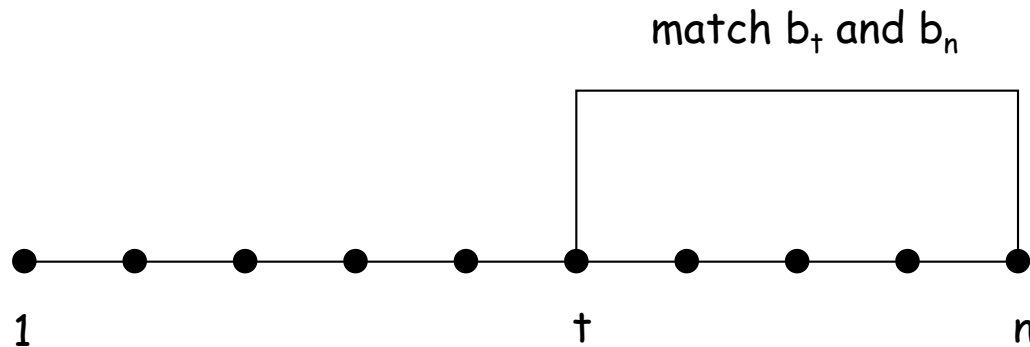
crossing



RNA Secondary Structure: Subproblems



First attempt. $\text{OPT}(j)$ = maximum number of base pairs in a secondary structure of the substring $b_1b_2\dots b_j$.



Difficulty. Results in two sub-problems.

Finding secondary structure in: $b_1b_2\dots b_{t-1}$.

← $\text{OPT}(t-1)$

Finding secondary structure in: $b_{t+1}b_{t+2}\dots b_{n-1}$.

← need more sub-problems



Dynamic Programming Over Intervals



Notation. $\text{OPT}(i, j)$ = maximum number of base pairs in a secondary structure of the substring $b_i b_{i+1} \dots b_j$.

Case 1. If $i \geq j - 4$.

- $\text{OPT}(i, j) = 0$ by no-sharp turns condition.

Case 2. Base b_j is not involved in a pair.

- $\text{OPT}(i, j) = \text{OPT}(i, j-1)$

Case 3. Base b_j pairs with b_t for some $i \leq t < j - 4$.

- non-crossing constraint decouples resulting sub-problems
- $\text{OPT}(i, j) = 1 + \max_t \{ \text{OPT}(i, t-1) + \text{OPT}(t+1, j-1) \}$



take max over t such that $i \leq t < j-4$ and
 b_t and b_j are Watson-Crick complements

Remark. Same core idea in CKY algorithm to parse context-free grammars.





$M(i, j) = 0$ if $i > j - 4$

else

$= 1 + \max_t \{ M(i, t-1) + M(t+1, j-1) \mid i \leq t < j - 4 \}$

where b_j forms a pair with some b_t

or

$= M(i, j-1)$ when b_j does not form a pair with any b_t



Bottom Up Dynamic Programming Over Intervals



Q. What order to solve the sub-problems?

A. Do shortest intervals first.

```
RNA( $b_1, \dots, b_n$ ) {  
    for  $k = 5, 6, \dots, n-1$   
        for  $i = 1, 2, \dots, n-k$   
             $j = i + k$   
            Compute  $M[i, j]$   
  
    return  $M[1, n]$   
}
```

using recurrence

i

4	0	0	0	↗
3	0	0	↗	↗
2	0	↗	↗	↗
1	↗	↗	↗	↗
	6	7	8	9

j

Running time. $O(n^3)$.



Dynamic Programming Summary



Recipe.

- Characterize structure of problem.
- Recursively define value of optimal solution.
- Compute value of optimal solution.
- Construct optimal solution from computed information.

Dynamic programming techniques.

Binary choice: weighted interval scheduling.

Multi-way choice: segmented least squares. ←

Viterbi algorithm for HMM also uses DP to optimize a maximum likelihood tradeoff between parsimony and accuracy

Adding a new variable: knapsack.

Dynamic programming over intervals: RNA secondary structure.

↖ CKY parsing algorithm for context-free grammar has similar structure

Top-down vs. bottom-up: different people have different intuitions.





6.6 Sequence Alignment



String Similarity



How similar are two strings?

ocurance

occurrence

o	c	u	r	r	a	n	c	e	-
---	---	---	---	---	---	---	---	---	---

o	c	c	u	r	r	e	n	c	e
---	---	---	---	---	---	---	---	---	---

5 mismatches, 1 gap

o	c	-	u	r	r	a	n	c	e
---	---	---	---	---	---	---	---	---	---

o	c	c	u	r	r	e	n	c	e
---	---	---	---	---	---	---	---	---	---

1 mismatch, 1 gap

o	c	-	u	r	r	-	a	n	c	e
---	---	---	---	---	---	---	---	---	---	---

o	c	c	u	r	r	e	-	n	c	e
---	---	---	---	---	---	---	---	---	---	---

0 mismatches, 3 gaps



Edit Distance



Applications.

Basis for Unix diff.

Speech recognition.

Computational biology.

Edit distance. [Levenshtein 1966, Needleman-Wunsch 1970]

Gap penalty δ ; mismatch penalty α_{pq} .

Cost = sum of gap and mismatch penalties.

C	T	G	A	C	C	T	A	C	C	T
---	---	---	---	---	---	---	---	---	---	---

-	C	T	G	A	C	C	T	A	C	C	T
---	---	---	---	---	---	---	---	---	---	---	---

C	C	T	G	A	C	T	A	C	A	T
---	---	---	---	---	---	---	---	---	---	---

C	C	T	G	A	C	-	T	A	C	A	T
---	---	---	---	---	---	---	---	---	---	---	---

$$\alpha_{TC} + \alpha_{GT} + \alpha_{AG} + 2\alpha_{CA}$$

$$2\delta + \alpha_{CA}$$



Sequence Alignment



Goal: Given two strings $X = x_1 x_2 \dots x_m$ and $Y = y_1 y_2 \dots y_n$ find alignment of minimum cost.

Def. An **alignment** M is a set of ordered pairs $x_i - y_j$ such that each item occurs in at most one pair and no crossings.

Def. The pair $x_i - y_j$ and $x_{i'} - y_{j'}$ **cross** if $i < i'$, but $j > j'$.

$$\text{cost}(M) = \underbrace{\sum_{(x_i, y_j) \in M} \alpha_{x_i y_j}}_{\text{mismatch}} + \underbrace{\sum_{i: x_i \text{ unmatched}} \delta + \sum_{j: y_j \text{ unmatched}} \delta}_{\text{gap}}$$

Ex: CTACCG **vs.** TACATG.

Sol: $M = x_2 - y_1, x_3 - y_2, x_4 - y_3, x_5 - y_4, x_6 - y_6$.

x_1	x_2	x_3	x_4	x_5		x_6
C	T	A	C	C	-	G

	y_1	y_2	y_3	y_4	y_5	y_6
-	T	A	C	A	T	G



Sequence Alignment: Problem Structure



Def. $OPT(i, j)$ = min cost of aligning strings $x_1 x_2 \dots x_i$ and $y_1 y_2 \dots y_j$.

Case 1: OPT matches x_i - y_j .

- pay mismatch for x_i - y_j + min cost of aligning two strings

$x_1 x_2 \dots x_{i-1}$ and $y_1 y_2 \dots y_{j-1}$

Case 2a: OPT leaves x_i unmatched.

- pay gap for x_i and min cost of aligning $x_1 x_2 \dots x_{i-1}$ and $y_1 y_2 \dots y_j$

Case 2b: OPT leaves y_j unmatched.

- pay gap for y_j and min cost of aligning $x_1 x_2 \dots x_i$ and $y_1 y_2 \dots y_{j-1}$

$$OPT(i, j) = \begin{cases} j\delta & \text{if } i = 0 \\ \min \begin{cases} \alpha_{x_i y_j} + OPT(i-1, j-1) \\ \delta + OPT(i-1, j) \\ \delta + OPT(i, j-1) \end{cases} & \text{otherwise} \\ i\delta & \text{if } j = 0 \end{cases}$$



Sequence Alignment: Algorithm



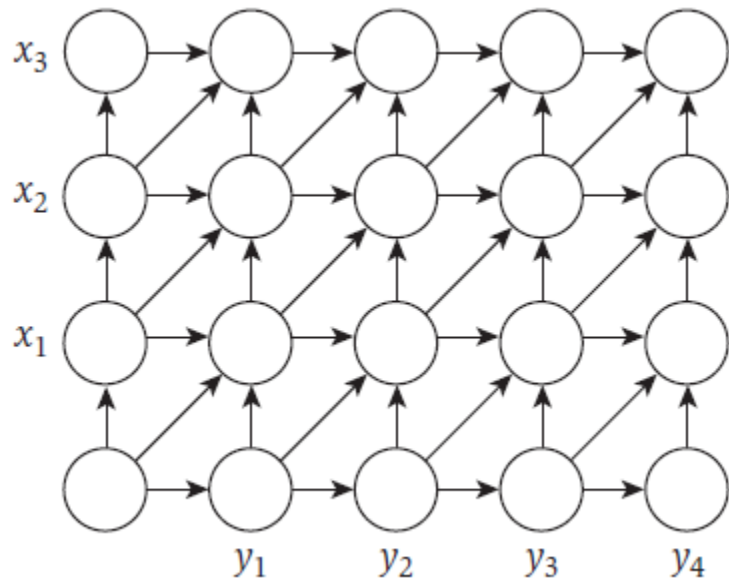
```
Sequence-Alignment( $m, n, x_1x_2\dots x_m, y_1y_2\dots y_n, \delta, \alpha$ ) {  
  for  $i = 0$  to  $m$   
     $M[0, i] = i\delta$   
  for  $j = 0$  to  $n$   
     $M[j, 0] = j\delta$   
  
  for  $i = 1$  to  $m$   
    for  $j = 1$  to  $n$   
       $M[i, j] = \min(\alpha[x_i, y_j] + M[i-1, j-1],$   
                     $\delta + M[i-1, j],$   
                     $\delta + M[i, j-1])$   
  
  return  $M[m, n]$   
}
```

Analysis. $\Theta(mn)$ time and space.

English words or sentences: $m, n \leq 10$.

Computational biology: $m = n = 100,000$. 10 billions ops OK, but 10GB array?





n	8	6	5	4	6
a	6	5	3	5	5
e	4	3	2	4	4
m	2	1	3	4	6
—	0	2	4	6	8
	—	n	a	m	e

Figure 6.18 The OPT values for the problem of aligning the words *mean* to *name*.



网格的最短路径



设有一个网格图， $(m+1)*(n+1)$ 个点，连成方格，每条边带有权，其水平或垂直边权为 δ ，斜边权为 $\alpha_{\{xi,yj\}}=weight((i-1,j-1),(i,j))$. 于是前页的计算正好就是从点 $(0,0)$ 到 (m,n) 的最短距离。当得到最短距离之后，反过来，可以追踪得到最短路径。





6.7 Sequence Alignment in Linear Space



Sequence Alignment: Linear Space



Q. Can we avoid using quadratic **space**?

Easy. Optimal **value** in $O(m + n)$ space and $O(mn)$ time.

Compute $\text{OPT}(i, \cdot)$ from $\text{OPT}(i-1, \cdot)$.

No longer a simple way to recover alignment itself.

Theorem. [Hirschberg 1975] Optimal **alignment** in $O(m + n)$ space and $O(mn)$ time.

Clever combination of divide-and-conquer and dynamic programming.

Inspired by idea of Savitch from complexity theory.



Space-Efficient-Alignment(X,Y)



Space-Efficient-Alignment(X,Y)

Array $B[0 \dots m, 0 \dots 1]$

Initialize $B[i, 0] = i\delta$ for each i (just as in column 0 of A)

For $j = 1, \dots, n$

$B[0, 1] = j\delta$ (since this corresponds to entry $A[0, j]$)

For $i = 1, \dots, m$

$$B[i, 1] = \min[\alpha_{x_i y_j} + B[i - 1, 0], \\ \delta + B[i - 1, 1], \delta + B[i, 0]]$$

Endfor

Move column 1 of B to column 0 to make room for next iteration:

Update $B[i, 0] = B[i, 1]$ for each i

Endfor



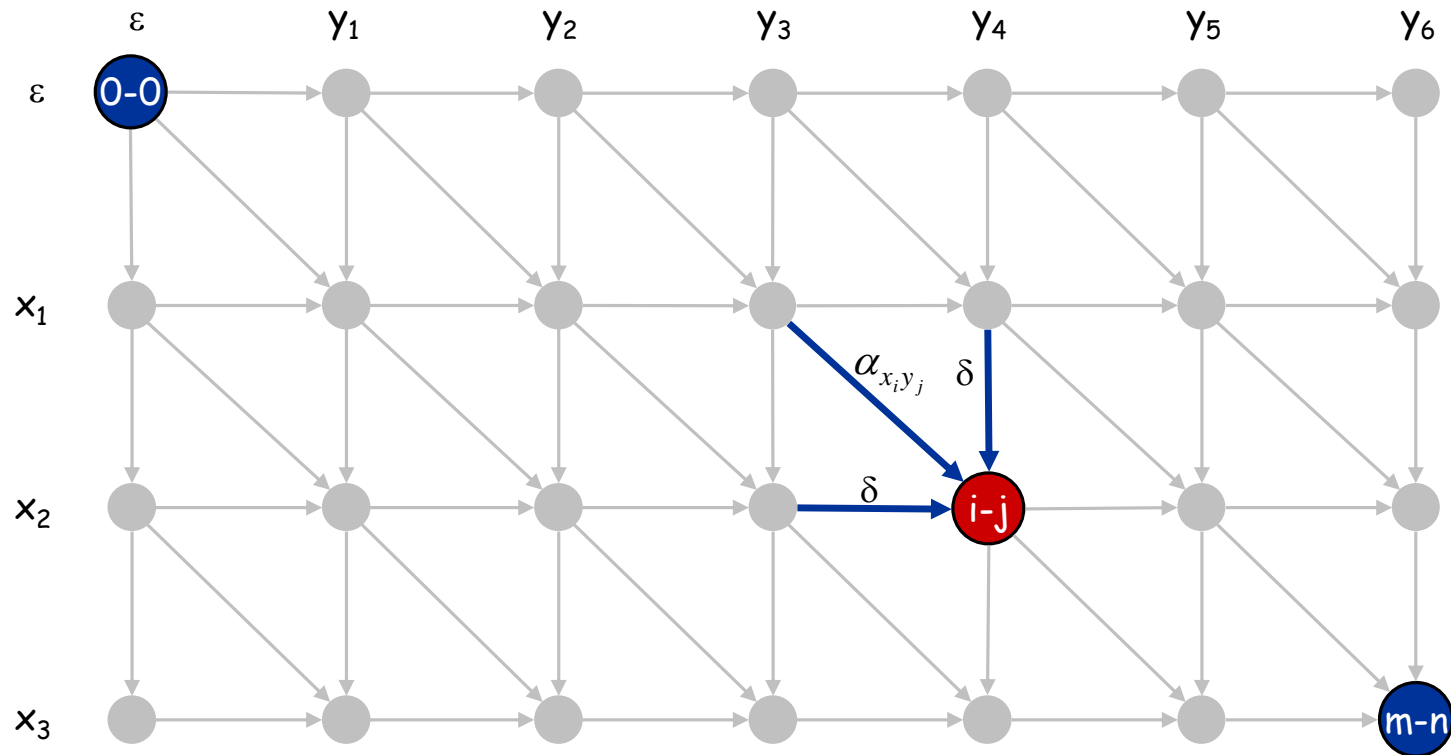
Sequence Alignment: Linear Space



Edit distance graph.

Let $f(i, j)$ be shortest path from $(0,0)$ to (i, j) .

Observation: $f(i, j) = \text{OPT}(i, j)$.



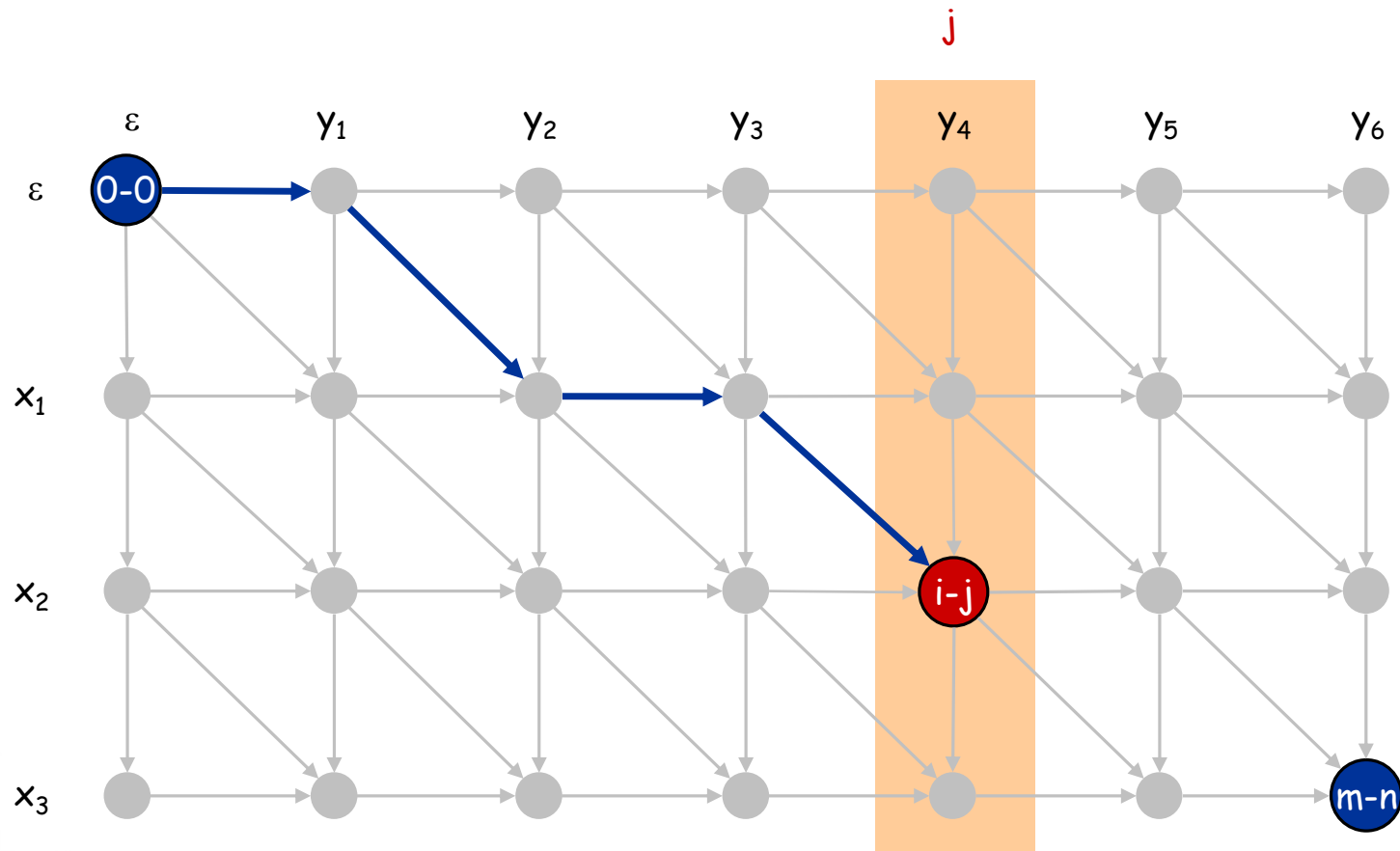
Sequence Alignment: Linear Space



Edit distance graph.

Let $f(i, j)$ be shortest path from $(0,0)$ to (i, j) .

Can compute $f(\cdot, j)$ for any j in $O(mn)$ time and $O(m + n)$ space.



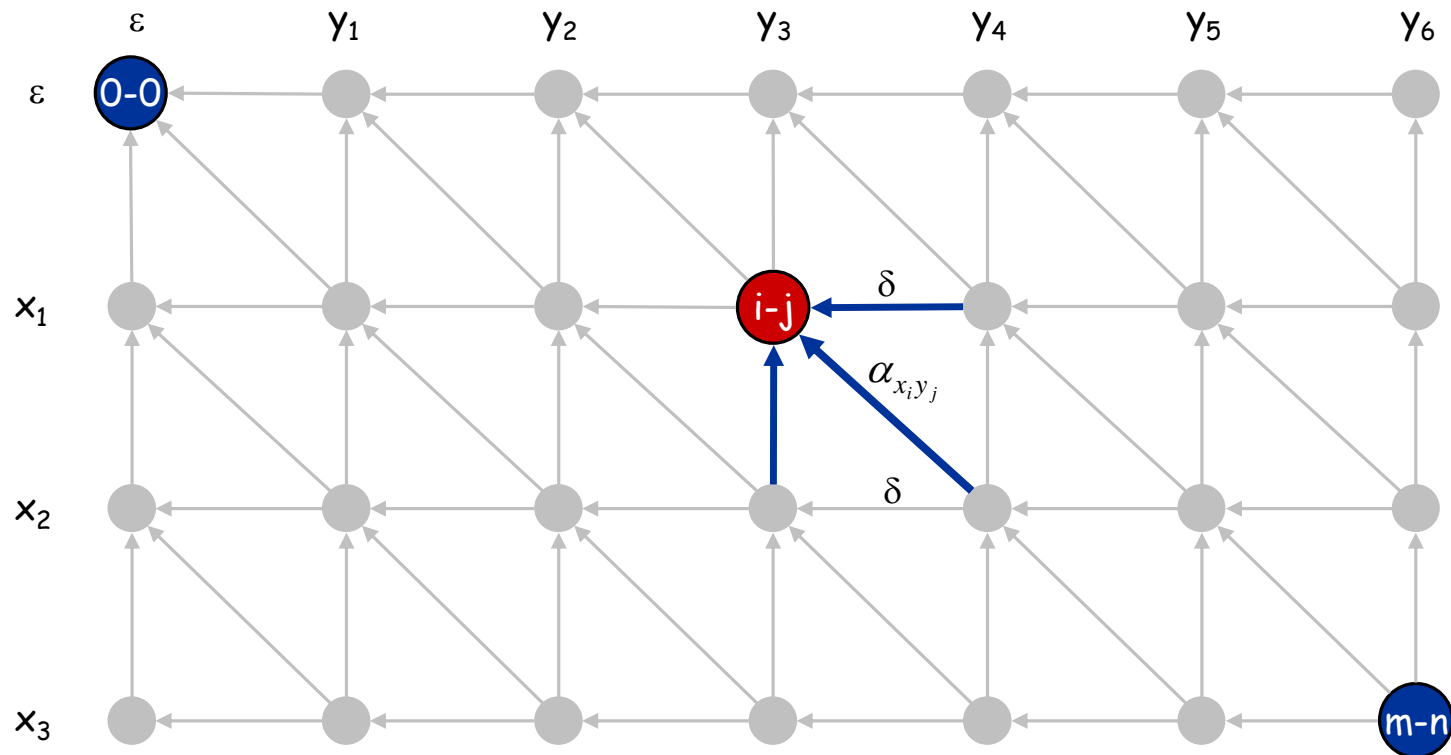
Sequence Alignment: Linear Space



Edit distance graph.

Let $g(i, j)$ be shortest path from (i, j) to (m, n) .

Can compute by reversing the edge orientations and inverting the roles of $(0, 0)$ and (m, n)



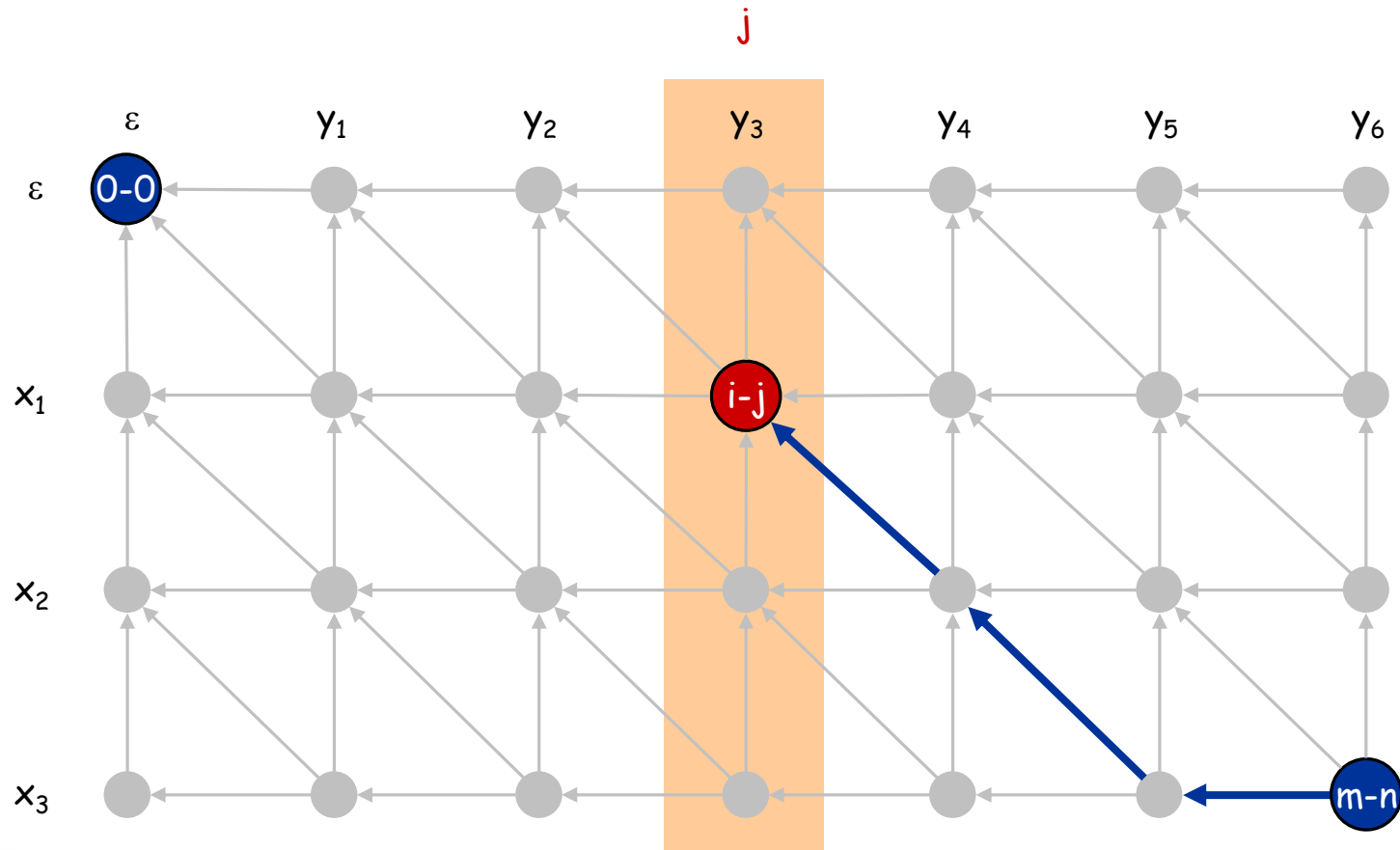
Sequence Alignment: Linear Space



Edit distance graph.

Let $g(i, j)$ be shortest path from (i, j) to (m, n) .

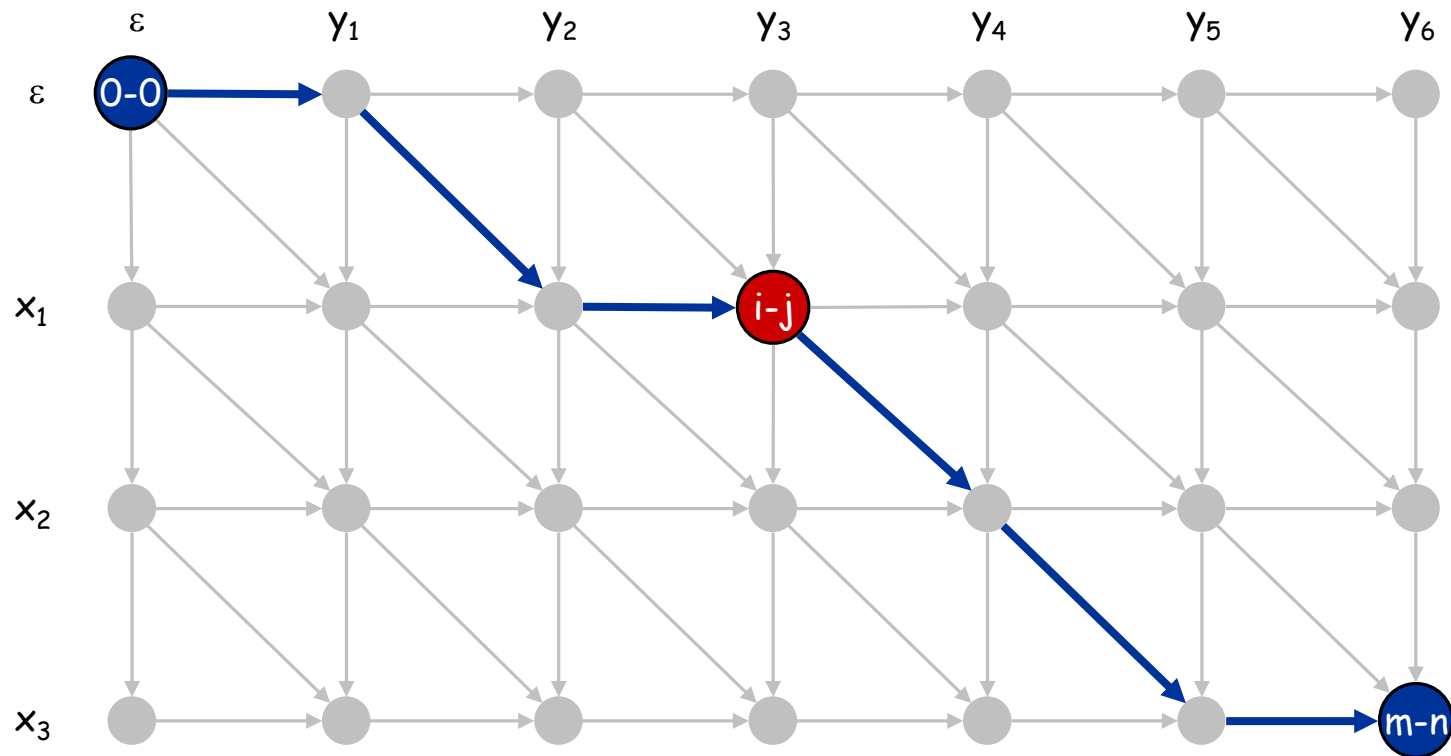
Can compute $g(\cdot, j)$ for any j in $O(mn)$ time and $O(m + n)$ space.



Sequence Alignment: Linear Space



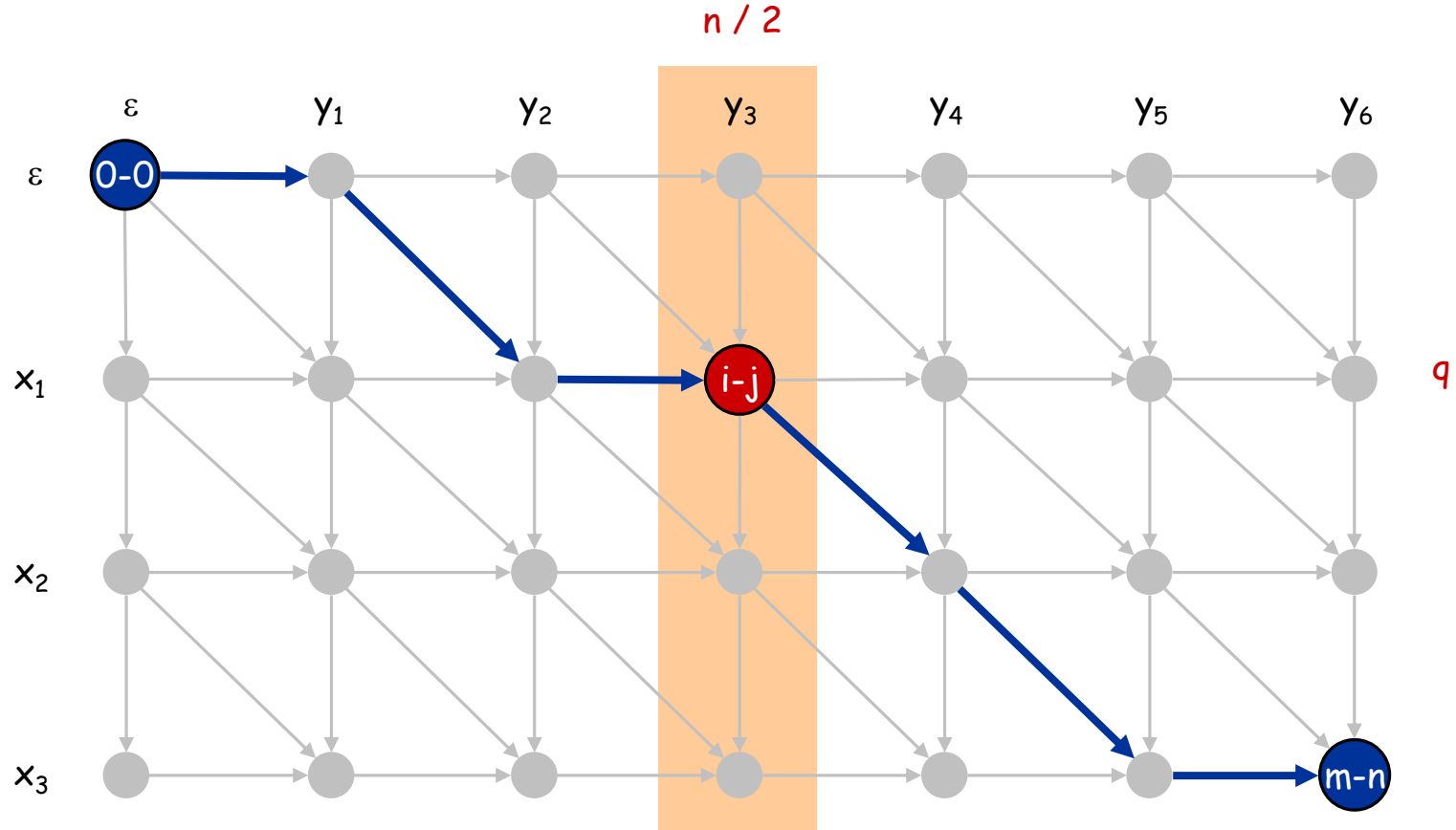
Observation 1. The cost of the shortest path that uses (i, j) is $f(i, j) + g(i, j)$.



Sequence Alignment: Linear Space



Observation 2. let q be an index that minimizes $f(q, n/2) + g(q, n/2)$. Then, the shortest path from $(0, 0)$ to (m, n) uses $(q, n/2)$.



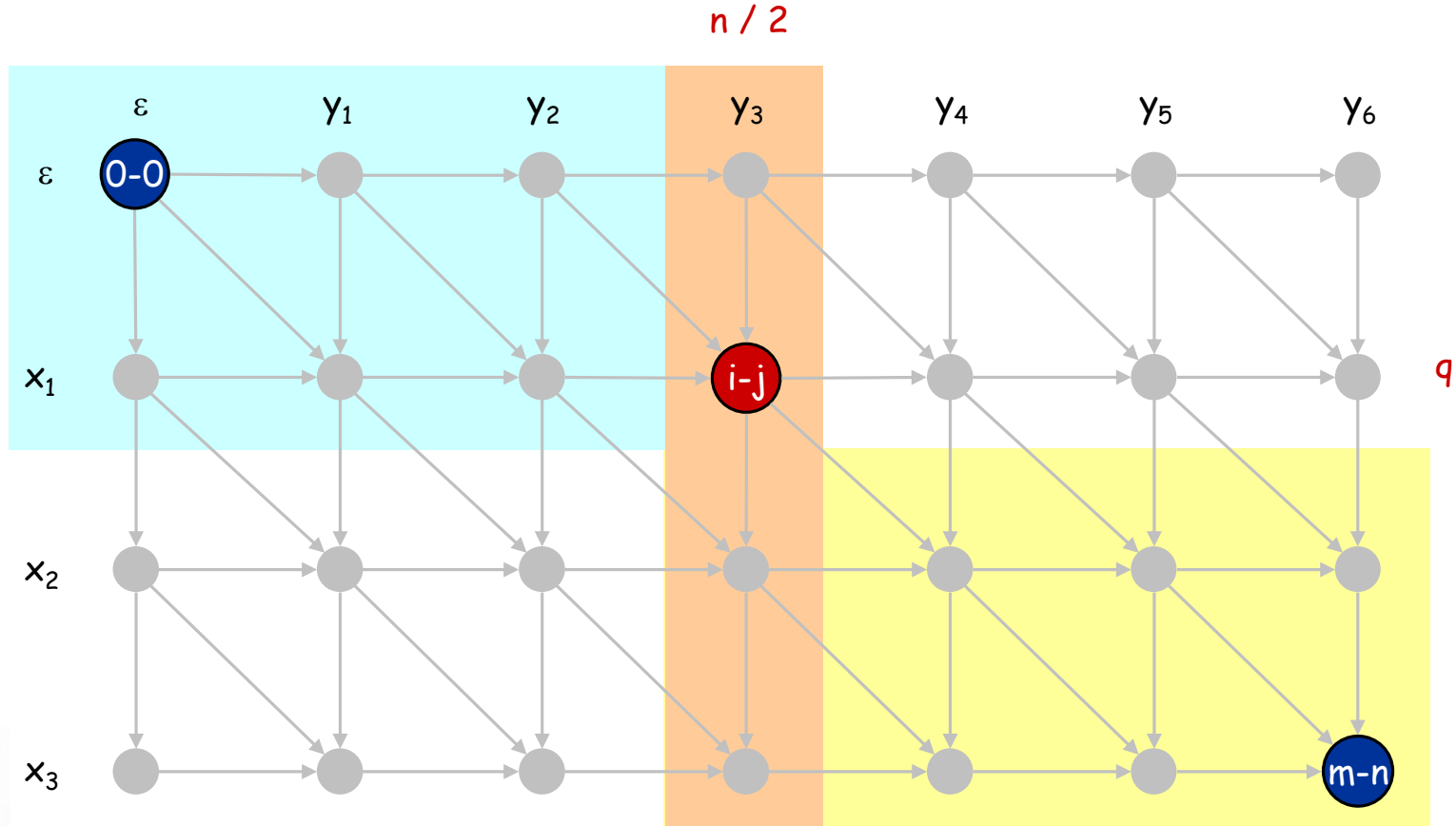
Sequence Alignment: Linear Space



Divide: find index q that minimizes $f(q, n/2) + g(q, n/2)$ using DP.

Align x_q and $y_{n/2}$.

Conquer: recursively compute optimal alignment in each piece.



Divide-and-Conquer-Alignment(X, Y)



Divide-and-Conquer-Alignment(X, Y)

Let m be the number of symbols in X

Let n be the number of symbols in Y

If $m \leq 2$ or $n \leq 2$ then

 Compute optimal alignment using Alignment(X, Y)

 Call Space-Efficient-Alignment($X, Y[1:n/2]$)

 Call Backward-Space-Efficient-Alignment($X, Y[n/2 + 1:n]$)

 Let q be the index minimizing $f(q, n/2) + g(q, n/2)$

 Add $(q, n/2)$ to global list P

 Divide-and-Conquer-Alignment($X[1:q], Y[1:n/2]$)

 Divide-and-Conquer-Alignment($X[q + 1:n], Y[n/2 + 1:n]$)

Return P





算法1 Space-Efficient-Alignment(X, Y)

算法2 Divide-and-Conquer-Alignment (X, Y) 的比较

	算法1		算法2	
	求最优值	求最优方案	求最优值	求最优方案
时间复杂度	$O(m \cdot n)$	$O(m \cdot n)$	$O(m \cdot n)$	$O(m \cdot n)$
空间复杂度	$O(2 \cdot \min(m, n))$	$O(m \cdot n)$	$O(m+n)$	$O(m+n)$



Sequence Alignment: Running Time Analysis Warmup



Theorem. Let $T(m, n)$ = max running time of algorithm on strings of length at most m and n . $T(m, n) = O(mn \log n)$.

$$T(m, n) \leq 2T(m, n/2) + O(mn) \Rightarrow T(m, n) = O(mn \log n)$$

Remark. Analysis is not tight because two sub-problems are of size $(q, n/2)$ and $(m - q, n/2)$. In next slide, we save $\log n$ factor.



Sequence Alignment: Running Time Analysis



Theorem. Let $T(m, n)$ = max running time of algorithm on strings of length m and n . $T(m, n) = O(mn)$.

Pf. (by induction on n)

$O(mn)$ time to compute $f(\cdot, n/2)$ and $g(\cdot, n/2)$ and find index q .

$T(q, n/2) + T(m - q, n/2)$ time for two recursive calls.

Choose constant c so that:

$$T(m, 2) \leq cm$$

$$T(2, n) \leq cn$$

$$T(m, n) \leq cmn + T(q, n/2) + T(m - q, n/2)$$

Base cases: $m = 2$ or $n = 2$.

Inductive hypothesis: $T(m, n) \leq 2cmn$.

$$\begin{aligned} T(m, n) &\leq T(q, n/2) + T(m - q, n/2) + cmn \\ &\leq 2cq(n/2) + 2c(m - q)(n/2) + cmn \\ &= cq(n/2) + cmn - cq(n/2) + cmn \\ &= 2cmn \end{aligned}$$

