Chapter 6

11. Suppose you're consulting for a company that manufactures PC equipment and ships it to distributors all over the country. For each of the next n weeks, they have a projected *supply* s_i of equipment (measured in pounds), which has to be shipped by an air freight carrier.

Each week's supply can be carried by one of two air freight companies, A or B.

- Company A charges a fixed rate r per pound (so it costs $r \cdot s_i$ to ship a week's supply s_i).
- Company B makes contracts for a fixed amount c per week, independent of the weight. However, contracts with company B must be made in blocks of four consecutive weeks at a time.

A *schedule*, for the PC company, is a choice of air freight company (A or B) for each of the n weeks, with the restriction that company B, whenever it is chosen, must be chosen for blocks of four contiguous weeks at a time. The *cost* of the schedule is the total amount paid to company A and B, according to the description above.

Give a polynomial-time algorithm that takes a sequence of supply values $s_1, s_2, \ldots, s_n$ and returns a *schedule* of minimum cost.

Example. Suppose $r = 1$, $c = 10$, and the sequence of values is

$$11, 9, 9, 12, 12, 12, 12, 9, 9, 11.$$

Then the optimal schedule would be to choose company A for the first three weeks, then company B for a block of four consecutive weeks, and then company A for the final three weeks.

12. Suppose we want to replicate a file over a collection of n servers, labeled $S_1, S_2, \ldots, S_n$. To place a copy of the file at server S_i results in a *placement cost* of c_i, for an integer $c_i > 0$.

Now, if a user requests the file from server S_i, and no copy of the file is present at S_i, then the servers $S_{i+1}, S_{i+2}, S_{i+3} \ldots$ are searched in order until a copy of the file is finally found, say at server S_j, where $j > i$. This results in an *access cost* of $j - i$. (Note that the lower-indexed servers $S_{i-1}, S_{i-2}, \ldots$ are not consulted in this search.) The access cost is 0 if S_i holds a copy of the file. We will require that a copy of the file be placed at server S_n, so that all such searches will terminate, at the latest, at S_n.

We'd like to place copies of the files at the servers so as to minimize the sum of placement and access costs. Formally, we say that a *configuration* is a choice, for each server S_i with $i = 1, 2, \ldots, n-1$, of whether to place a copy of the file at S_i or not. (Recall that a copy is always placed at S_n.) The *total cost* of a configuration is the sum of all placement costs for servers with a copy of the file, plus the sum of all access costs associated with all n servers.

Give a polynomial-time algorithm to find a configuration of minimum total cost.

20. Suppose it's nearing the end of the semester and you're taking n courses, each with a final project that still has to be done. Each project will be graded on the following scale: It will be assigned an integer number on a scale of 1 to $g > 1$, higher numbers being better grades. Your goal, of course, is to maximize your average grade on the n projects.

You have a total of $H > n$ hours in which to work on the n projects cumulatively, and you want to decide how to divide up this time. For simplicity, assume H is a positive integer, and you'll spend an integer number of hours on each project. To figure out how best to divide up your time, you've come up with a set of functions $\{f_i : i = 1, 2, \ldots, n\}$ (rough estimates, of course) for each of your n courses; if you spend $h \le H$ hours on the project for course i, you'll get a grade of $f_i(h)$. (You may assume that the functions f_i are *nondecreasing*: if $h < h'$, then $f_i(h) \le f_i(h')$.)

So the problem is: Given these functions $\{f_i\}$, decide how many hours to spend on each project (in integer values only) so that your average grade, as computed according to the f_i, is as large as possible. In order to be efficient, the running time of your algorithm should be polynomial in n, g, and H; none of these quantities should appear as an exponent in your running time.

23. Suppose you are given a directed graph $G = (V, E)$ with costs on the edges c_e for $e \in E$ and a sink t (costs may be negative). Assume that you also have finite values $d(v)$ for $v \in V$. Someone claims that, for each node $v \in V$, the quantity $d(v)$ is the cost of the minimum-cost path from node v to the sink t.

(a) Give a linear-time algorithm (time $O(m)$ if the graph has m edges) that verifies whether this claim is correct.

(b) Assume that the distances are correct, and $d(v)$ is finite for all $v \in V$. Now you need to compute distances to a different sink t'. Give an $O(m \log n)$ algorithm for computing distances $d'(v)$ for all nodes $v \in V$ to the sink node t'. (*Hint:* It is useful to consider a new cost function defined as follows: for edge $e = (v, w)$, let $c'_e = c_e - d(v) + d(w)$. Is there a relation between costs of paths for the two different costs c and c'?)