Chiral Perturbation Theory

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1 Introduction

Chiral Perturbation Theory (ChPT) is an Effective Field Theory (EFT) that can be used to describe low energy properties of the strong interaction. According to the standard model Quantum Chromodynamics (QCD) the theory of quarks and gluons describes the strong interaction. Since QCD has asymptotic freedom at high energies, it can be analyzed using perturbative techniques. At low energies it's not possible because the larger coupling constant gives non convergent perturbative series expansions. Therefore non-perturbative methods are required to analyze QCD at low energies. Effective field theories and Lattice QCD are well established non-perturbative methods for the low energy end of the spectrum. ChPT is an EFT which is constructed with the Lagrangian consists with the chiral symmetry of QCD.

At low energies, confinement of quarks and gluons make composite particles and the interactions are stronger due to large coupling constant. At high energies, quarks and gluon interactions are weak due to small coupling constant.

2 Spontaneous Symmetry Breaking and Goldstone Bosons

Spontaneous Symmetry breaking is an underling concept of elementary particle physics and many other ares in physics such as superconductivity and ferromagnetism.

Let's assume quarks don't have masses. Then the QCD lagrangian has the perfect chiral symmetry and it also undergoes spontaneous symmetry breaking, which gives us massless Goldstone bosons. Then the pions will not have masses. In reality quarks do have very small masses, therefore the chiral symmetry is not exact. CQD lagrangian undergoes explicit symmetry breaking which gives us non zero masses to pions.

QCD is gauge theory which has continuous U(1) symmetry. QCD lagrangian has perfect chiral symmetry if the quarks don't have masses. In reality quarks have very small masses, therefore the chiral symmetry is not exact.

3 Lowest order Chiral Lagrangian

To lowest order (minimum number of derivatives), the effective chiral Lagrangian:

$$\mathcal{L}_2 = \frac{F^2}{4}\langle \partial_\mu U^\dagger \partial^\mu U \rangle$$

[Pich, 1995]. (page100)

Where, $U = exp(i\Phi/F)$, $\Phi = \lambda^a \phi^a$,(when defining pion nucleon lagrangian,we will see u which is the square roof of this U) and λ's are Gell-Mann matrices. [Scherer, 2003] (page70) $F \approx 93.2MeV$ pion decay constant [Pich, 1995]. (page103)

$$\Phi = \sqrt{2}\begin{bmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & \frac{-\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & \frac{-2\eta_8}{\sqrt{6}} \end{bmatrix}$$

[Scherer, 2003] (page70) (also see My Calculation "Gell Mann Matrices")

The Lagrangian can be simplified to (see my calculations "chiral lagrangian derivations")

$$\mathcal{L}_2 = \frac{1}{4}\langle \partial_\mu \Phi \partial^\mu \Phi \rangle + \frac{1}{48F^2}\langle\, [\partial_\mu \Phi, \Phi]\, [\partial^\mu \Phi, \Phi]\, \rangle + \mathcal{O}\left(\frac{1}{F^4}\right)$$

The first term can be simplified to,[Scherer, 2003] (page71)

$$\mathcal{L}_2 = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \mathcal{O}\left(\frac{1}{F^2}\right)$$

The second term can be expressed with structure constants of SU3(see my calculations "chiral lagrangian derivations")

$$\langle\, [\partial_\mu \Phi, \Phi]\, [\partial^\mu \Phi, \Phi]\, \rangle = -8 \partial_\mu \phi^a \phi^b \partial^\mu \phi^c \phi^d f^{abe} f^{cde}$$

Where $f^{abc} = \frac{1}{4i}\langle \left[\lambda^a \lambda^b\right] \lambda^c \rangle$, [Scherer, 2003] (page 03)

4 Mass Term

Lagrangian with the mass term can be written with the unknown factor B [Kubis, 2007] (page 09) (B is related to the chiral quark condensate Ref: Sherere pg74)

$$\mathcal{L}_2 = \frac{F^2}{4}\langle \partial_\mu U \partial^\mu U^\dagger \rangle + \frac{F^2}{2} B \langle \mathcal{M}\left(U + U^\dagger\right) \rangle$$

where quark mass matrix $\mathcal{M} = \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{bmatrix}$ This mass therm is due to symmetry breaking [Scherer, 2003] (page74)

$$\mathcal{L}_{s.b} = \frac{F^2}{2} B \langle \mathcal{M}\left(U + U^\dagger\right) \rangle = \frac{F^2 B}{2} \langle \mathcal{M} \left(2 - \frac{\Phi^2}{F^2} + \frac{\Phi^4}{12F^4} + ... \right) \rangle$$

"In order to determine the masses of the Goldstone bosons, we identify the terms of second order in the fields in $\mathcal{L}_{s.b}$ [Scherer, 2003] (page74)"

$$\mathcal{L}_{s.b} = \frac{B}{2} \langle \mathcal{M}\left(\Phi^2\right) \rangle + ...$$

$$\begin{aligned} \mathcal{L}_{s.b} &= \frac{B}{2}\{2(m_u + m_d)\pi^+\pi^- + 2(m_u + m_s)K^+K^- + 2(m_d + m_s)K_0\bar{K}_0 \\ &\quad +(m_u + m_d)\pi^0\pi^0 + \frac{2}{\sqrt{3}}(m_u - m_d)\pi^0\eta + \frac{1}{3}(m_u + m_d + 4m_s)\eta\eta \end{aligned}$$

[Scherer, 2003] (page 03) (see Mathematica calculations "Gell Mann Oakes Relations") Gell-Mann, Oakes, and Renner relations, $M^2_{GB} \propto m_q$ can be derived (see my calculations "chiral lagrangian derivations")[Kubis, 2007] (page 10)

$$\begin{aligned} M^2_{\pi^\pm} &= \frac{B}{2}(m_u + m_d) \\ M^2_{K^\pm} &= \frac{B}{2}(m_u + m_s) \\ M^2_{K_0^\pm} &= \frac{B}{2}(m_d + m_s) \end{aligned}$$

Quark mass rations can be calculated using above relations as [Kubis, 2007] (page 10)

$$\begin{aligned} \frac{m_u}{m_d} &= 0.66 \\ \frac{m_s}{m_d} &= 22 \end{aligned}$$

The coupling of the Lagrangian to external vector v_μ and axial vector a_μ can be done by replacing the ordinary derivative by a covariant derivative. [Kubis, 2007] (page 11)

$$\partial_\mu U \to D_\mu U = \partial_\mu U - ir_\mu U + iUl_\mu$$

[Scherer, 2003] (page 80)

$$D_\mu U = \partial_\mu U - i\,[v_\mu, U] - i\{a_\mu, U\}$$

[Kubis, 2007] (page 11)

The most general, locally invariant, effective Lagrangian at lowest chiral order

$$\mathcal{L}_2 = \frac{F^2}{4}\langle D_\mu U (D^\mu U)^\dagger \rangle + \frac{F^2}{4}\langle \chi U^\dagger + U\chi^\dagger \rangle$$

[Scherer, 2003] (page 82)

Where χ , is a linear combination of a scalar and pseudoscalar $\chi = 2B(s+ip)$ [Scherer, 2003] (page 80) In SU(2) $\chi = 2BM = 2B \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$ [Scherer, 2003] (page 88)

At $\mathcal{O}(p^2)$, $M_\pi^2 = 2Bm$ [Scherer, 2003] (page 83)

$$\mathcal{L}_2 = \frac{F^2}{4}\langle D_\mu U (D^\mu U)^\dagger \rangle + \frac{F^2}{4} M_\pi^2 \langle U^\dagger + U \rangle$$

This can be simplified as follows,

$$\mathcal{L}_2 = \frac{1}{4}\langle \partial_\mu \Phi \partial^\mu \Phi \rangle + \frac{1}{48F^2}\langle\, [\partial_\mu \Phi, \Phi]\, [\partial^\mu \Phi, \Phi]\,\rangle + \frac{F^2 M_\pi^2}{4}\langle 2 - \frac{\Phi^2}{F^2} + \frac{\Phi^4}{12F^4}\rangle$$

$$\mathcal{L}_2 = \frac{1}{4}\langle \partial_\mu \Phi \partial^\mu \Phi \rangle + \frac{F^2 M_\pi^2}{2} + \frac{F^2 M_\pi^2}{4}\langle \Phi^2 \rangle + \frac{1}{48F^2}\big(\langle\, [\partial_\mu \Phi, \Phi]\, [\partial^\mu \Phi, \Phi]\,\rangle + M_\pi^2 \langle \Phi^4 \rangle\big)$$

5 The Chiral Lagrangian at Fourth Order

$$\mathcal{L}_4 = L_1 \langle D_\mu U (D^\mu U)^\dagger \rangle +$$

[Scherer, 2003] (page 93) [Pich, 1995]. (page106)

6 Baryonic Effective Lagrangian (remove)

The most general Lagrangian with the smallest number of derivatives is given by

$$\mathcal{L}^{(1)}_{\pi,N} = \bar{\Psi}\left(i\not{D} - \mathring{m}_N + \frac{\mathring{g}_A}{2}\gamma^\mu\gamma_5 u_\mu\right)\Psi$$

Where $\mathring{m}_N = 883MeV$ and $\mathring{g}_A = 1.267MeV$ [Scherer, 2003] (page 122)

With external sources r_μ , l_μ, and $v^{(s)}_\mu$ [Scherer, 2003] (page 121,122)

$$D_\mu = \partial_\mu + \Gamma_\mu - iv^{(s)}_\mu$$

$$\Gamma_\mu = \frac{1}{2}\left[u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger\right]$$

$$u_\mu = i\left[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger\right]$$

Here, $u^2 = U = exp(i\Phi/F)$, $\Phi = \tau^a\phi^a$

Without external sources, $r_\mu = l_\mu = 0$, and $v^{(s)}_\mu = 0$ [Scherer, 2003] (page 123) $D_\mu = \partial_\mu + T_\mu$, $\Gamma_\mu = \frac{1}{2}\left(u^\dagger\partial_\mu u + u\partial_\mu u^\dagger\right)$, and $u_\mu = i\left(u^\dagger\partial_\mu u - u\partial_\mu u^\dagger\right)$ Baryonic Effective Lagrangian

$$\mathcal{L}^{(1)}_{\pi,N} = \bar{\Psi}\left(i\gamma^\mu\left(\partial_\mu + \frac{1}{2}\left(u^\dagger\partial_\mu u + u\partial_\mu u^\dagger\right)\right) - \mathring{m}_N + i\frac{\mathring{g}_A}{2}\gamma^\mu\gamma_5\left(u^\dagger\partial_\mu u - u\partial_\mu u^\dagger\right)\right)\Psi$$

(see my calculations "Baryonic Effective Lagrangian")

Interaction Lagrangian containing one and two pion fields

$$\mathcal{L}^{(1)}_{\pi,N} = \bar{\Psi}\left(i\gamma^\mu\left(\partial_\mu + i\frac{\epsilon_{abc}}{4F^2}\phi^a\partial_\mu\phi^b\sigma_c\right) - \mathring{m}_N - \frac{\mathring{g}_A}{2}\gamma^\mu\gamma_5\partial_\mu\phi^d\sigma_d\right)\Psi$$

(see my calculations "Baryonic Effective Lagrangian")

Pion-nucleon interaction Lagrangian

$$\mathcal{L}_{int} = -\frac{\epsilon_{abc}}{4F^2}\bar{\Psi}\gamma^5\phi^a\partial_\mu\phi^b\sigma_c - \frac{\mathring{g}_A}{2}\bar{\Psi}\gamma^\mu\gamma_5\partial_\mu\phi^a\sigma_a\Psi$$

Here, the first term corresponds to interaction of two pion field with the nucleon at a single point and the second therm is the pseudovector pion-nucleon coupling. [Scherer, 2003] (page 134)

7 Realtivisic Pion Nucleon Lagrangian

$$\mathcal{L}^{(1)}_{\pi,N} = \bar{\Psi}\left(i\not{D} - \mathring{m}_N + \frac{\mathring{g}_A}{2}\gamma^\mu\gamma_5 u_\mu\right)\Psi$$

Where $\mathring{m}_N = 883MeV$ and $\mathring{g}_A = 1.267MeV$ [Scherer, 2003] (page 122)

With external sources r_μ , l_μ, and $v^{(s)}_\mu$ [Scherer, 2003] (page 121,122)

$$D_\mu = \partial_\mu + \Gamma_\mu - iv^{(s)}_\mu$$

$$\Gamma_\mu = \frac{1}{2}\left[u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger\right]$$

$$u_\mu = i\left[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger\right]$$

Without external sources, $r_\mu = l_\mu = 0$, and $v^{(s)}_\mu = 0$ [Scherer, 2003] (page 123) $D_\mu = \partial_\mu + T_\mu$, $\Gamma_\mu = \frac{1}{2}\left(u^\dagger\partial_\mu u + u\partial_\mu u^\dagger\right)$, and $u_\mu = i\left(u^\dagger\partial_\mu u - u\partial_\mu u^\dagger\right)$

Baryonic Effective Lagrangian

$$\mathcal{L}^{(1)}_{\pi,N} = \bar{\Psi}\left(i\gamma^\mu\partial_\mu - \mathring{m}_N + i\frac{\gamma^\mu}{2}\left(u^\dagger\partial_\mu u + u\partial_\mu u^\dagger\right) + i\frac{\mathring{g}_A}{2}\gamma^\mu\gamma_5\left(u^\dagger\partial_\mu u - u\partial_\mu u^\dagger\right)\right)\Psi$$

Here, $u^2 = U = exp(i\Pi/F)$, $\Pi = \tau^a\pi^a$

Using following approximations (pi here should be big Pi,this is a mistake,correct it)

$$\left(u^\dagger\partial_\mu u + u\partial_\mu u^\dagger\right) = 2\left(\frac{[\pi,\partial_\mu\pi]}{8F^2} + \frac{[[[\partial_\mu\pi,\pi],\pi],\pi]}{384F^4} + ...\right)$$

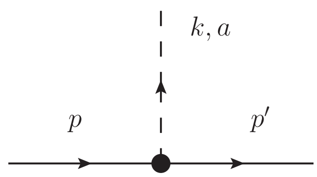
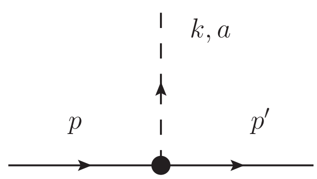
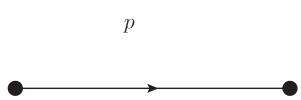
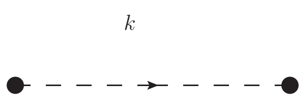
$$\left(u^\dagger\partial_\mu u - u\partial_\mu u^\dagger\right) = -i\left(\frac{-\partial_\mu\pi}{F} + \frac{[[\partial_\mu\pi,\pi],\pi]}{24F^3} + ...\right)$$

Baryonic Effective Lagrangian (pi here should be big Pi,this is a mistake,correct it)

$$\mathcal{L}^{(1)}_{\pi,N} = \bar{\Psi}\left(i\gamma^\mu\partial_\mu - \mathring{m}_N - \frac{\mathring{g}_A\gamma^\mu\gamma_5\partial_\mu\pi}{2F} + \frac{i\gamma^\mu[\pi,\partial_\mu\pi]}{8F^2} + \frac{\mathring{g}_A\gamma^\mu\gamma_5[[\partial_\mu\pi,\pi],\pi]}{48F^3} + \frac{i\gamma^\mu[[[\partial_\mu\pi,\pi],\pi],\pi]}{384F^4} + ...\right)\Psi$$

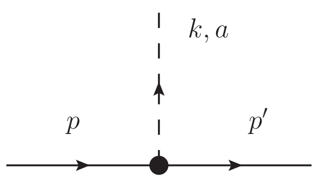
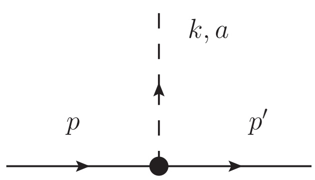
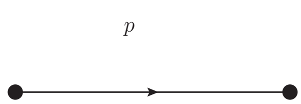
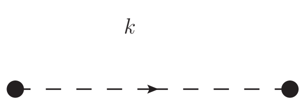
8 Relativistic Feynman Rules

Table 1: Feynman Rules

Graph Element	Mathematical Equivalent
	$\frac{\mathring{g}_A}{2F}\not{k}\gamma_5\tau^a$
	$\frac{\left(\not{k}_1+\not{k}_2\right)}{4F^2}\epsilon_{abc}\tau^c$
	$\frac{i}{11111}$
	$\frac{i}{1111}$

9 HBChPT form Relativistic L , by approximations

Table 2: Feynman Rules

Graph Element	Mathematical Equivalent
	$-\frac{\mathring{g}_A}{F_\pi} k^i G^{ia}$ here $G^{ia} = \frac{\tau^i \tau^a}{2}$
	$\frac{\left(\not{k}_1^0+\not{k}_2^0\right)}{2F^2} \epsilon_{abc} I^c$ here $I^c = \frac{\tau^c}{2}$
	$\frac{iP_n}{p^0-\delta m_n+i\epsilon}$
	$\frac{i}{k^2-M_\pi^2+i\epsilon}$

10 HBChPT and 1/Nc Lagrangian

In the ξ expansion, the Lagrangian for the combined HBChPT and 1/Nc expansions to order ξ

$$\mathcal{L}_B^{(1)} = B^\dagger \left(iD_0 + \mathring{g}_A u^{ia} G^{ia} - \frac{C_{HF}}{N_c} \hat{S}^2 + \frac{c_1}{2\Lambda} \hat{\chi}_+ \right) B$$

[Fernando and Goity, 2018] [Fernando and Goity, 2020] [Cordón and Goity, 2013]
The chiral covariant derivative $D_\mu = \partial_\mu - i\Gamma_\mu$

$$\Gamma_\mu = \frac{1}{2} \left[u^\dagger (i\partial_\mu + r_\mu) u + u(i\partial_\mu + l_\mu) u^\dagger \right]$$

Without external sources, gauge sources become zero $r_\mu = l_\mu = 0$, therefore, $D_0 = \partial_0 + \frac{1}{2} \left(u^\dagger \partial_0 u + u \partial_0 u^\dagger \right)$,

axial Maurer-Cartan

$$u_\mu = \left[u^\dagger (i\partial_\mu + r_\mu) u - u(i\partial_\mu + l_\mu) u^\dagger \right]$$

Without external sources, gauge sources become zero $r_\mu = l_\mu = 0$, therefore, $u_i = i \left(u^\dagger \partial_i u - u \partial_i u^\dagger \right)$and $u^i = i \left(u^\dagger \partial^i u - u \partial^i u^\dagger \right)$.

According to [Calle Cordón et al., 2014]

$$u^{ia} = \frac{1}{2} \left\langle u^i \tau^a \right\rangle$$

Here we use, $u^2 = U = exp(i\Pi/F)$, $\Pi = \tau^a \pi^a$
Approximations

$$\left(u^\dagger \partial_0 u + u \partial_0 u^\dagger \right) = 2 \left(\frac{[\Pi, \partial_0 \Pi]}{8F^2} + \frac{[[[\partial_0 \Pi, \Pi], \Pi], \Pi]}{384 F^4} + ... \right)$$

$$\left(u^\dagger \partial^i u - u \partial^i u^\dagger \right) = -i \left(\frac{-\partial^i \Pi}{F} + \frac{\left[\left[\partial^i \Pi, \Pi \right], \Pi \right]}{24 F^3} + ... \right)$$

Leading order terms of the Lagrangian can be written as fallows.(see my calculations "Feynman Rules from Non relativistic L")

$$\mathcal{L}_B^{(1)} \approx B^\dagger \left(i\partial + \frac{\mathring{g}_A}{2} \frac{\left\langle -\partial^i \Pi \tau^a \right\rangle}{F} G^{ia} + i \frac{[\Pi, \partial_0 \Pi]}{8F^2} + \frac{\mathring{g}_A}{2} \frac{\left\langle \left[\left[\partial^i \Pi, \Pi \right], \Pi \right] \tau^a \right\rangle}{24 F^3} G^{ia} \right.$$
$$\left. + i \frac{[[[\partial_0 \Pi, \Pi], \Pi], \Pi]}{384 F^4} - \frac{C_{HF}}{N_c} \hat{S}^2 + \frac{c_1}{2\Lambda} \hat{\chi}_+ \right) B$$

Where χ , is a linear combination of a scalar and pseudoscalar [Fernando and Goity, 2018] $\chi = 2B_0(s + ip)$ In SU(2) $\chi = 2B_0 M = 2B_0 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$ [Scherer, 2003] (page 88)

$\chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u$
$\chi^0_\pm = \langle \chi_\pm \rangle$
$\chi^a_\pm = \langle \chi_\pm I^a \rangle$
$\tilde{\chi}_\pm = \chi^a_\pm I^a$

$\hat{\chi}_+ = \tilde{\chi}_+ + N_c\, \chi^0_+$

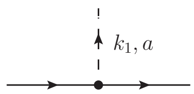
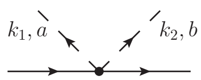
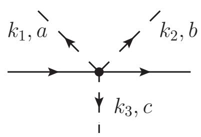
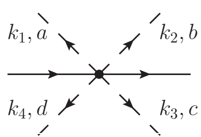
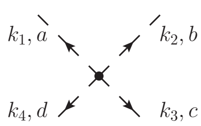
For convenience, a scale Λ is introduced, which can be chosen to be a typical QCD scale, in order to render most of the LECs dimensionless. In the calculations,$\Lambda = m_\rho$ will be chosen.[Fernando and Goity, 2018]

C_{HF} is equal to hyperfine splitting $C_{HF} = M_\Delta - M_N$

11 Feynman Rules

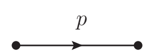
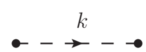
Vertices are given by: i $(factors\ in\ \mathcal{L})$. Outgoing Pion gives ik for $\partial\Pi$. These Feynman Rules can be used to get get $i\mathcal{M}$

Table 3: Feynman Rules for Vertices

Graph Element	Interaction Lagrangian	Feynman Rule
	$B^\dagger \frac{\mathring{g}_A}{2} \frac{\langle -\partial^i \Pi \tau^a \rangle}{F} G^{ia} B$	$\frac{\mathring{g}_A}{F} k^i G^{ia}$
	$B^\dagger i \frac{[\Pi, \partial_0 \Pi]}{8F^2} B$	$\frac{\left(k_2^0 - k_1^0\right)}{2F^2} \epsilon_{abc} I^c$
	$B^\dagger \frac{\mathring{g}_A}{2} \frac{\langle [[\partial^i \Pi, \Pi], \Pi] \tau^a \rangle}{24F^3} G^{ia} B$	
	$B^\dagger i \frac{[[[\partial_0 \Pi, \Pi], \Pi], \Pi]}{384F^4} B$	
	$\frac{1}{48F^2} \left(\langle [\partial_\mu \Pi, \Pi] [\partial^\mu \Pi, \Pi] \rangle + M_\pi^2 \langle \Pi^4 \rangle \right)$	

Here $G^{ia} = \frac{\tau^i \tau^a}{2}$, $I^c = \frac{\tau^c}{2}$
with $\tau^a = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

Table 4: Feynman Rules for Propagators

Graph Element	Feynman Rule
	$\frac{i}{p^0-\delta m_n+i\epsilon}$
	$\frac{i}{k^2-M^2+i\epsilon}$

12 Dimensional Regularization

These integrals usually diverges. Therefore, its impossible to evaluate as a normal integration. There are very profound mathematical techniques to handle those divergences. **Dimensional Regularization** is used here to isolate the divergence of the integral.

introduced in 1972 by ‘t Hooft and Veltman (and by Bollini and Gambiagi) as a method to regularise ultraviolet (UV) divergences in a gauge invariant way

The idea is to work in $D = 4 - 2/\epsilon$ spacetime dimensions. Divergences for D goes to 4 will thus appear as poles in $1/\epsilon$

13 Feynman Parameters

Feynman Parameters is one of the most important mathematical technique which is used to simplify loop integrals. According to this technique, the denominator can be simplified by introducing extra integration with respect to an auxiliary variable. This allows to squeeze **n** number of denominators in to a single quadratic polynomial, raised to the power of **n**. Most general form of the Feynman Parameters is given by,[Peskin and Schroeder, 1995]

$$\frac{1}{A_1 A_2 ... A_n} = \int_0^1 d\alpha_1 \alpha_2 ... \alpha_n \frac{\delta\left(\Sigma\alpha_i - 1\right)(n-1)!}{\left[\alpha_1 A_1 + \alpha_2 A_2 + ... + \alpha_n A_n\right]^n} \tag{1}$$

For two denominators, it simplifies to

$$\frac{1}{AB} = \int_0^1 d\alpha \frac{1}{\left[\alpha A + (1-\alpha)B\right]^2} \tag{2}$$

This formula is used when two meson propagators appears as the denominator. When a meson propagator and baryon propagator appears, following alternative form is used. It can be obtain using change of variable $\lambda = \alpha/(1-\alpha)$

$$\frac{1}{AB} = \int_0^\infty d\lambda \frac{1}{\left[A + \lambda B\right]^2} \tag{3}$$

Combination of these two forms is useful when the denominator has two meson propagators and a baryon propagator.

$$\frac{1}{ABC} = \int_0^1 d\alpha \int_0^\infty d\lambda \frac{2}{\left[\lambda A + \alpha B + (1-\alpha)C\right]^3} \tag{4}$$

After applying Feynman Parameters technique, the denominator simplifies to a quadratic polynomial. Therefore preforming d-dimension integral is much easier, but there are new integrals to evaluate. Some of those new integrals are difficult integrate, specially integrals over λ. They can be evaluated at the very end of the calculation. Following **Feynman parameter Integrals** are very useful for evaluating those λ integrals.

Most general form of λ integrals (J integrals) is following

$$\begin{aligned} J(\nu, n) \equiv & \quad J(c_0, c_1, \lambda_0, d, \nu, n) \\ = & \int_0^\infty (\lambda - \lambda_0)^n \left[c_0 + c_1 (\lambda - \lambda_0)^2\right]^{d/2-\nu} d\lambda \end{aligned} \tag{5}$$

Here $J(\nu, 1)$ can be calculated explicitly.

$$J(\nu, 1) = \frac{-\left(c_0 + c_1 \lambda_0^2\right)^{d/2-\nu+1}}{2c_1 (d/2 - \nu + 1)}$$

$J\left(3,0\right)$ can be calculated with $d=4-2\epsilon$

$$
\begin{aligned}
(3,0) = & \quad \frac{1}{\sqrt{c_0 c_1}}\left(\frac{\pi}{2}+arctan\left(\lambda_0\sqrt{\frac{c_1}{c_0}}\right)\right)-\epsilon\int_0^{\infty}\frac{log\left(c_0+c_1\lambda_0^2\right)}{\left(c_0+c_1\lambda_0^2\right)}d\lambda \\
= & \quad J_{30}^0+\epsilon J_{30}^1 \qquad (6)
\end{aligned}
$$

Following recurrence relationships are useful to evaluate remaining integrals.

$$
J\left(\nu,0\right)=\frac{c_0\left(d-2\nu\right)}{\left(d-2\nu+1\right)}J\left(\nu+1,0\right)+\frac{\lambda_0\left(c_0+c_1\lambda_0^2\right)^{d/2-\nu}}{\left(d-2\nu+1\right)} \qquad (7)
$$

$$
J\left(\nu,n\right)=\frac{1}{c_1}\left(J\left(\nu-1,n-2\right)-c_0 J\left(\nu,n-2\right)\right) \qquad (8)
$$

Some commonly used J integrals are given below

$$
J\left(2,0\right) = \lambda_0+\epsilon\left(\lambda_0\left[2-log\left(c_0+c_1\lambda_0^2\right)\right]-2c_0 J_{30}^0\right)+\mathcal{O}\left(\epsilon^2\right) \qquad (9)
$$

$$
\begin{aligned}
J\left(1,0\right) &= c_0\lambda_0+\frac{c_1\lambda_0^3}{3} \qquad (10) \\
&+\epsilon\left(\frac{-4c_0^2 J_{30}^0}{3}+\frac{c_1\lambda_0^3}{9}\left[2-3log\left(c_0+c_1\lambda_0^2\right)\right]+\frac{c_0\lambda_0}{3}\left[4-3log\left(c_0+c_1\lambda_0^2\right)\right]\right)+\mathcal{O}\left(\epsilon^2\right)
\end{aligned}
$$

$$
J\left(2,1\right) = \frac{-\left(c_0+c_1\lambda_0^2\right)}{2c_1}+\epsilon\frac{\left(c_0+c_1\lambda_0^2\right)}{2c_1}\left(log\left(c_0+c_1\lambda_0^2\right)-1\right)+\mathcal{O}\left(\epsilon^2\right) \qquad (11)
$$

$$
J\left(2,2\right) = \frac{\lambda_0^3}{3}+\epsilon\left(\frac{2c_0^2 J_{30}^0}{3c_1}-\frac{2c_0\lambda_0}{3c_1}+\frac{\lambda_0^3}{9}\left[2-3log\left(c_0+c_1\lambda_0^2\right)\right]\right)+\mathcal{O}\left(\epsilon^2\right) \qquad (12)
$$

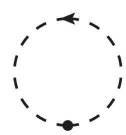
14 Loop Integrals

Loop integrals appear, when calculating Feynman diagrams with loops. They are 4-dimensional integral over the four momentum of a virtual particle. Depending on the diagram, couple of different types on integrals appear. They were categorized based on their denominator.

14.1 Types of Loop Integrals

14.1.1 H integral

Figure 1: single meson loop

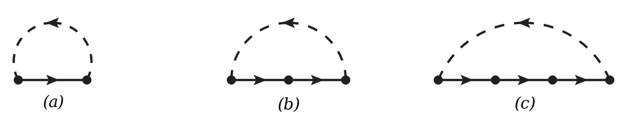


When a single meson make a loop, following type of integral appears

$$H = \int d^{\tilde{d}}k \frac{1}{[k^2 - M_\pi^2]} \tag{13}$$

14.1.2 I integral

Figure 2: Meson and baryon loop



When a single meson and a single baryon make a loop (a), following type of integral appears

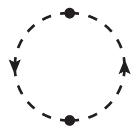
$$I = \int d^{\tilde{d}}k \frac{1}{\left[(p-k)^0 - \delta m_n\right][k^2 - M_\pi^2]} \tag{14}$$

Loop integrals with two or more baryons and a meson (b,c) can also be simplified to above type by decomposing baryon propagators into partial fractions.

14.1.3 J integral

When two mesons make a loop, following type of integral appears

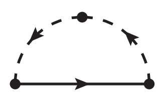
Figure 3: Two meson loop



$$H = \int d^{\tilde{d}}k \frac{1}{[k^2 - M_\pi^2]\,[(k+q)^2 - M_\pi^2]} \tag{15}$$

14.1.4 JΔ integral

Figure 4: Baryon and two meson loop



When a baryon and two mesons make a loop, following type of integral appears

$$H = \int d^{\tilde{d}}k \frac{1}{\left[(p-k)^0 - \delta m_n\right]\,[k^2 - M_\pi^2]\,[(k+q)^2 - M_\pi^2]} \tag{16}$$

14.2 Evaluating Loop Integrals

Following steps were used to evaluate loop integrals.

1. Feynman parameter integrals

$$\frac{1}{AB} = \int_0^\infty \frac{d\lambda}{[A + \lambda B]^2}$$

2. Wick rotation to convert Minkowski vector into a Euclidian vector

3. Standard integral for Euclidian vectors [Ramond,]

$$\int \frac{d^d k}{[k^2 + \Lambda^2]^n} = \pi^{d/2} \frac{\Gamma(n - d/2)}{\Gamma(n)} \frac{1}{[\Lambda^2]^{(n-d/2)}}$$

$$
\begin{aligned}
\int d^{\tilde{d}}k \frac{\{1; k^0; k^i; k^0 k^i; (k^0)^2; k^i k^j\}}{\left[(p-k)^0 - \delta m_n\right]\left[k^2 - M_\pi^2\right]} = \frac{2i}{(4\pi)^{d/2}} \int_0^\infty d\lambda \{ \Gamma\left(2 - d/2\right)\left[\Lambda^2\right]^{(d/2-2)} &\qquad ; \\
\Gamma\left(2 - d/2\right)\lambda\left[\Lambda^2\right]^{(d/2-2)} &\qquad ; \\
0 &\qquad ; \\
0 &\qquad ; \\
\Gamma\left(2 - d/2\right)\lambda^2\left[\Lambda^2\right]^{(d/2-2)} - \frac{1}{2}\Gamma\left(1 - d/2\right)\left[\Lambda^2\right]^{(d/2-1)} &\qquad ; \\
\frac{\delta^{i,j}}{2}\Gamma\left(1 - d/2\right)\left[\Lambda^2\right]^{(d/2-1)} &\qquad \}
\end{aligned}
$$

14.2.1 I integrals

$$\begin{aligned}
I\left(n,p^0,M_\pi\right) &\equiv \int d\tilde{^d}k \frac{1}{\left[p^0-k^0-\delta m_n\right]\left[k^2-M_\pi^2\right]} \\
&= \frac{2i}{(4\pi)^{d/2}} \int_0^\infty d\lambda \Gamma\left(2-d/2\right)\left[\Lambda^2\right]^{(d/2-2)} \\
&= \frac{2i}{(4\pi)^{d/2}} \Gamma\left(2-d/2\right) J\left(2,0\right)
\end{aligned}$$

$$\begin{aligned}
I_0\left(n,p^0,M_\pi\right) &\equiv \int d\tilde{^d}k \frac{k^0}{\left[p^0-k^0-\delta m_n\right]\left[k^2-M_\pi^2\right]} \\
&= \frac{2i}{(4\pi)^{d/2}} \int_0^\infty d\lambda \Gamma\left(2-d/2\right)\lambda\left[\Lambda^2\right]^{(d/2-2)} \\
&= \frac{2i}{(4\pi)^{d/2}} \Gamma\left(2-d/2\right)\left[J\left(2,1\right)+\lambda_0 J\left(2,0\right)\right]
\end{aligned}$$

$$\begin{aligned}
I_i\left(n,p^0,M_\pi\right) &\equiv \int d\tilde{^d}k \frac{k^i}{\left[p^0-k^0-\delta m_n\right]\left[k^2-M_\pi^2\right]} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
I_{0i}\left(n,p^0,M_\pi\right) &\equiv \int d\tilde{^d}k \frac{k^0k^i}{\left[p^0-k^0-\delta m_n\right]\left[k^2-M_\pi^2\right]} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
I_{00}\left(n,p^0,M_\pi\right) &\equiv \int d\tilde{^d}k \frac{(k^0)^2}{\left[p^0-k^0-\delta m_n\right]\left[k^2-M_\pi^2\right]} \\
&= \frac{2i}{(4\pi)^{d/2}} \int_0^\infty d\lambda \{\Gamma\left(2-d/2\right)\lambda^2\left[\Lambda^2\right]^{(d/2-2)} - \frac{1}{2}\Gamma\left(1-d/2\right)\left[\Lambda^2\right]^{(d/2-1)}\} \\
&= \frac{i}{(4\pi)^{d/2}} \{2\Gamma\left(2-d/2\right)\left[J\left(2,2\right)+2\lambda_0 J\left(2,1\right)+\lambda_0^2 J\left(2,0\right)\right] - \Gamma\left(1-d/2\right) J\left(1,0\right)\}
\end{aligned}$$

$$\begin{aligned}
I_{ij}\left(n,p^0,M_\pi\right) &\equiv \int d\tilde{^d}k \frac{k^ik^j}{\left[p^0-k^0-\delta m_n\right]\left[k^2-M_\pi^2\right]} \\
&= \frac{2i}{(4\pi)^{d/2}} \int_0^\infty d\lambda \frac{\delta^{i,j}}{2}\Gamma\left(1-d/2\right)\left[\Lambda^2\right]^{(d/2-1)} \\
&= \frac{i\delta^{i,j}}{(4\pi)^{d/2}} \Gamma\left(1-d/2\right) J\left(1,0\right)
\end{aligned}$$

Here $\int d\tilde{^d}k = \int \frac{d^dk}{(2\pi)^d}$, $\Lambda^2 = \lambda^2 - 2\lambda\left(p^0-\delta m_n\right) + M_\pi^2$,$c_0 = M_\pi^2 - \left(p^0-\delta m_n\right)^2$, $c_1 = 1$, and $\lambda_0 = p^0 - \delta m_n$

14.2.2 JΔ integrals

$$J\Delta_1(p,q,M) = \quad \equiv \int d^{\tilde{d}}k \frac{1}{\left[(p-k)^0\right]\left[(k+q)^2 - M^2\right][k^2-M^2]}$$

$$= \frac{-2i}{(4\pi)^{d/2}} \Gamma(3-d/2) \int_0^\infty d\lambda \int_0^1 d\alpha \left(\frac{1}{\Lambda}\right)^{3-d/2}$$

$$J\Delta_{k_\mu}(p,q,M) = \quad \equiv \int d^{\tilde{d}}k \frac{k_\mu}{\left[(p-k)^0\right]\left[(k+q)^2 - M^2\right][k^2-M^2]}$$

$$= \frac{-2i}{(4\pi)^{d/2}} \Gamma(3-d/2) \int_0^\infty d\lambda \int_0^1 d\alpha \left(\frac{1}{\Lambda}\right)^{3-d/2} \delta k_\mu$$

$$J\Delta_{k_\mu k_\nu}(p,q,M) = \quad \equiv \int d^{\tilde{d}}k \frac{k_\mu k_\nu}{\left[(p-k)^0\right]\left[(k+q)^2 - M^2\right][k^2-M^2]}$$

$$= \frac{2i}{(4\pi)^{d/2}} \Gamma(2-d/2) \int_0^\infty d\lambda \int_0^1 d\alpha \left(\frac{1}{\Lambda}\right)^{2-d/2} \left(\frac{g_{\mu\nu}}{2} - \frac{(2-d/2)}{\Lambda}\delta k_\mu \delta k_\nu\right)$$

$$J\Delta_{k_\mu k_\nu k_\sigma}(p,q,M) = \quad \equiv \int d^{\tilde{d}}k \frac{k_\mu k_\nu k_\sigma}{\left[(p-k)^0\right]\left[(k+q)^2 - M^2\right][k^2-M^2]}$$

$$= \frac{2i}{(4\pi)^{d/2}} \Gamma(2-d/2) \int_0^\infty d\lambda \int_0^1 d\alpha \left(\frac{1}{\Lambda}\right)^{2-d/2}$$

$$\left(\frac{\delta k_\mu g_{\nu\sigma} + \delta k_\nu g_{\mu\sigma} + \delta k_\sigma g_{\mu\nu}}{2} - \frac{(2-d/2)}{\Lambda}\delta k_\mu \delta k_\nu \delta k_\sigma\right)$$

Integrals	Results after the d-dimensional integral
$J\Delta_1$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^\infty d\lambda\int_0^1 d\alpha\quad\Gamma\left(3-d/2\right)\left(\frac{1}{\Lambda}\right)^{3-d/2}$$
$J\Delta_{k_\mu}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^\infty d\lambda\int_0^1 d\alpha\quad\Gamma\left(3-d/2\right)\left(\frac{1}{\Lambda}\right)^{3-d/2}\delta k_\mu$$
$J\Delta_{k_\mu k_\nu}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^\infty d\lambda\int_0^1 d\alpha\Big(-\Gamma\left(2-d/2\right)\left(\frac{1}{\Lambda}\right)^{2-d/2}\frac{g_{\mu\nu}}{2} +\Gamma\left(3-d/2\right)\left(\frac{1}{\Lambda}\right)^{3-d/2}\delta k_\mu\delta k_\nu\Big)$$
$J\Delta_{k_\mu k_\nu k_\rho}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^\infty d\lambda\int_0^1 d\alpha\Big(-\Gamma\left(2-d/2\right)\left(\frac{1}{\Lambda}\right)^{2-d/2}\frac{1}{2}\left(\delta k_\mu g_{\nu\rho}+\delta k_\nu g_{\mu\rho}+\delta k_\rho g_{\mu\nu}\right) +\Gamma\left(3-d/2\right)\left(\frac{1}{\Lambda}\right)^{3-d/2}\delta k_\mu\delta k_\nu\delta k_\rho\Big)$$
$J\Delta_{k_\mu k_\nu k_\rho k_\sigma}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^\infty d\lambda\int_0^1 d\alpha\Big(\Gamma\left(1-d/2\right)\left(\frac{1}{\Lambda}\right)^{1-d/2}\frac{1}{4}\left(g_{\mu\nu}g_{\rho\sigma}+g_{\mu\rho}g_{\nu\sigma}+g_{\mu\sigma}g_{\nu\rho}\right) +\Gamma\left(3-d/2\right)\left(\frac{1}{\Lambda}\right)^{3-d/2}\delta k_\mu\delta k_\nu\delta k_\rho\delta k_\sigma -\Gamma\left(2-d/2\right)\left(\frac{1}{\Lambda}\right)^{2-d/2}\frac{1}{2}(g_{\mu\nu}\delta k_\rho\delta k_\sigma+g_{\mu\rho}\delta k_\nu\delta k_\sigma+g_{\mu\sigma}\delta k_\rho\delta k_\nu +g_{\nu\rho}\delta k_\mu\delta k_\sigma+g_{\sigma\nu}\delta k_\rho\delta k_\mu+g_{\rho\sigma}\delta k_\mu\delta k_\nu)\Big)$$

$$\begin{aligned}
\Lambda^2 &= \left(\lambda-\lambda_0\right)^2+c_0 & & \\
c_0 &= M^2-\lambda_0^2-\alpha\left(1-\alpha\right)q^2; & \delta k_0 &= \lambda-\left(1-\alpha\right)q_0\\
\lambda_0 &= p_0+\left(1-\alpha\right)q_0; & \delta k_i &= -\left(1-\alpha\right)q_i
\end{aligned}$$

Integrals	Result
$J\Delta_1$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha \quad \Gamma\left(3-d/2\right)J\left(3,0\right)$$
$J\Delta_{k_i}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha \quad \Gamma\left(3-d/2\right)J(3,0)\delta k_i$$
$J\Delta_{k_0}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha \quad \Gamma\left(3-d/2\right)\left(J(3,1)+J(3,0)p_0\right)$$
$J\Delta_{k_ik_j}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(-\Gamma\left(2-d/2\right)J(2,0)\frac{g_{ij}}{2}+\Gamma\left(3-d/2\right)J(3,0)\delta k_i\delta k_j\Big)$$
$J\Delta_{k_0k_j}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha \quad \Gamma\left(3-d/2\right)\left(J(3,1)+J(3,0)p_0\right)\delta k_j$$
$J\Delta_{k_0k_0}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(-\frac{1}{2}\Gamma\left(2-d/2\right)J(2,0) +\Gamma\left(3-d/2\right)\left(J(3,2)+2J(3,1)p_0+J(3,0)p_0^2\right)\Big)$$
$J\Delta_{k_ik_jk_l}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(-\frac{1}{2}\Gamma\left(2-d/2\right)J(2,0)\left(\delta k_i g_{jl}+\delta k_j g_{il}+\delta k_l g_{ij}\right) +\Gamma\left(3-d/2\right)J(3,0)\delta k_i\delta k_j\delta k_l\Big)$$

$J\Delta_{k_0k_jk_l}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(-\frac{1}{2}\Gamma\left(2-d/2\right)\left(J(2,1)+J(2,0)p_0\right)g_{jl} \\ +\Gamma\left(3-d/2\right)\left(J(3,1)+J(3,0)p_0\right)\Big)\delta k_j\delta k_l$$
$J\Delta_{k_0k_0k_l}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(-\frac{1}{2}\Gamma\left(2-d/2\right)J(2,0)\delta k_l \\ +\Gamma\left(3-d/2\right)\left(J(3,2)+2J(3,1)p_0+J(3,0)p_0^2\right)\delta k_l\Big)$$
$J\Delta_{k_0k_0k_0}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(-\frac{3}{2}\Gamma\left(2-d/2\right)\left(J(2,1)+J(2,0)p_0\right) \\ +\Gamma\left(3-d/2\right)\left(J(3,3)+3J(3,2)p_0+3J(3,1)p_0^2+J(3,0)p_0^3\right)\Big)$$
$J\Delta_{k_ik_jk_lk_m}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(\frac{1}{4}\Gamma\left(1-d/2\right)J(1,0)\left(g_{ij}g_{lm}+g_{il}g_{jm}+g_{im}g_{jl}\right) \\ +\Gamma\left(3-d/2\right)J(3,0)\delta k_i\delta k_j\delta k_l\delta k_m \\ -\frac{1}{2}\Gamma\left(2-d/2\right)J(2,0)(g_{ij}\delta k_l\delta k_m+g_{il}\delta k_j\delta k_m+g_{im}\delta k_l\delta k_j \\ +g_{jl}\delta k_i\delta k_m+g_{mj}\delta k_l\delta k_i+g_{lm}\delta k_i\delta k_j)\Big)$$
$J\Delta_{k_0k_jk_lk_m}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(\Gamma\left(3-d/2\right)\left(J(3,1)+J(3,0)p_0\right)\delta k_j\delta k_l\delta k_m \\ -\frac{1}{2}\Gamma\left(2-d/2\right)\left(J(2,1)+J(2,0)p_0\right)\left(g_{jl}\delta k_m+g_{mj}\delta k_l+g_{lm}\delta k_j\right)\Big)$$

$J\Delta_{k_0k_0k_lk_m}$	$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(\frac{1}{4}\Gamma\left(1-d/2\right)J(1,0)g_{lm} +\Gamma\left(3-d/2\right)\left(J(3,2)+2J(3,1)p_0+J(3,0)p_0^2\right)\delta k_l\delta k_m -\frac{1}{2}\Gamma\left(2-d/2\right)\left(J(2,0)\delta k_l\delta k_m+\left(J(2,2)+2J(2,1)p_0+J(2,0)p_0^2\right)g_{lm}\right)\Big)$
$J\Delta_{k_0k_0k_0k_m}$	$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(\Gamma\left(3-d/2\right)\left(J(3,3)+3J(3,2)p_0+3J(3,1)p_0^2+J(3,0)p_0^3\right)\delta k_m -\frac{3}{2}\Gamma\left(2-d/2\right)\left(J(2,1)+J(2,0)p_0\right)\delta k_m\Big)$
$J\Delta_{k_0k_0k_0k_0}$	$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(\frac{3}{4}\Gamma\left(1-d/2\right)J(1,0) +\Gamma\left(3-d/2\right)\left(J(3,4)+4J(3,3)p_0+6J(3,2)p_0^2+4J(3,1)p_0^3+J(3,0)p_0^4\right) -3\Gamma\left(2-d/2\right)\left(J(2,2)+2J(2,1)p_0+J(2,0)p_0^2\right)\Big)$

Where $\delta k_i = -\left(1-\alpha\right)q_i$

$$J\left(\nu,n\right)=\int_0^\infty\left(\lambda-\lambda_0\right)^n\left[c_0+c_1\left(\lambda-\lambda_0\right)^2\right]^{d/2-\nu}d\lambda$$

Integrals	$1/\epsilon$ terms
$J\Delta_1$	0
$J\Delta_{k_i}$	0
$J\Delta_{k_0}$	$-\frac{i}{16\pi^2}$
$J\Delta_{k_ik_j}$	$-\frac{i}{32\pi^2}\left(2p_0+q_0\right)\delta_{ij}$
$J\Delta_{k_0k_j}$	$\frac{i}{32\pi^2}q_j$
$J\Delta_{k_0k_0}$	$-\frac{i}{32\pi^2}(2p_0-q_0)$
$J\Delta_{k_ik_jk_l}$	$\frac{i}{96\pi^2}\left(3p_0+2q_0\right)\left(q_i\delta_{jl}+q_j\delta_{il}+q_l\delta_{ij}\right)$
$J\Delta_{k_0k_jk_l}$	$-\frac{i}{192\pi^2}\left(4q_jq_l+(q^2+6p_0q_0+12p_0^2-6M^2)\delta_{jl}\right)$
$J\Delta_{k_0k_0k_l}$	$\frac{i}{96\pi^2}\left(3p_0-2q_0\right)q_l$
$J\Delta_{k_0k_0k_0}$	$-\frac{i}{192\pi^2}\left(6M^2+12p_0^2-q^2-6p_0q_0+4q_0^2\right)$

Integrals	$1/\epsilon$ terms
$J\Delta_{k_i k_j k_l k_m}$	$-\frac{i}{384\pi^2}\Big((2p_0+q_0)\left(q^2+4p_0q_0+4p_0^2+2q_0^2-6M^2\right)(\delta_{ij}\delta_{lm}+\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl})$ $+2\,(4p_0+3q_0)\,(\delta_{ij}q_lq_m+\delta_{il}q_jq_m+\delta_{im}q_jq_l+\delta_{jl}q_iq_m+\delta_{jm}q_iq_l+\delta_{lm}q_iq_j)\Big)$
$J\Delta_{k_0 k_j k_l k_m}$	$-\frac{i}{384\pi^2}\Big((6M^2-12p_0^2-q^2-8p_0q_0)(q_j\delta_{lm}+q_l\delta_{jm}+q_m\delta_{jl})-6q_jq_lq_m\Big)$
$J\Delta_{k_0 k_0 k_l k_m}$	$\frac{i}{384\pi^2}\Big(2(3q_0-4p_0)q_lq_m+(6M^2(2p_0-q_0)-12p_0^2q_0+q^2q_0-24p_0^3-2p_0q^2)\delta_{lm}\Big)$
$J\Delta_{k_0 k_0 k_0 k_m}$	$\frac{i}{384\pi^2}\Big(6M^2+12p_0^2-q^2-8p_0q_0+6q_0^2\Big)q_m$
$J\Delta_{k_0 k_0 k_0 k_0}$	$-\frac{i}{384\pi^2}\Big(24p_0^3-2p_0q^2+6M^2(2p_0-3q_0)-12p_0^2q_0+3q^2q_0+8p_0q_0^2-6q_0^3\Big)$

Integrals	Results when all indices are spatial
$J\Delta_1$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha \quad \Gamma\left(3-d/2\right)J\left(3,0\right)$$
$J\Delta_{k_i}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha \quad \Gamma\left(3-d/2\right)J(3,0)\delta k_i$$
$J\Delta_{k_ik_j}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(-\Gamma\left(2-d/2\right)J(2,0)\frac{g_{ij}}{2}+\Gamma\left(3-d/2\right)J(3,0)\delta k_i\delta k_j\Big)$$
$J\Delta_{k_ik_jk_l}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(-\frac{1}{2}\Gamma\left(2-d/2\right)J(2,0)\left(\delta k_ig_{jl}+\delta k_jg_{il}+\delta k_lg_{ij}\right) +\Gamma\left(3-d/2\right)J(3,0)\delta k_i\delta k_j\delta k_l\Big)$$
$J\Delta_{k_ik_jk_lk_m}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(\frac{1}{4}\Gamma\left(1-d/2\right)J(1,0)\left(g_{ij}g_{lm}+g_{il}g_{jm}+g_{im}g_{jl}\right) +\Gamma\left(3-d/2\right)J(3,0)\delta k_i\delta k_j\delta k_l\delta k_m -\frac{1}{2}\Gamma\left(2-d/2\right)J(2,0)(g_{ij}\delta k_l\delta k_m+g_{il}\delta k_j\delta k_m+g_{im}\delta k_l\delta k_j +g_{jl}\delta k_i\delta k_m+g_{mj}\delta k_l\delta k_i+g_{lm}\delta k_i\delta k_j)\Big)$$

Where $\delta k_i = -\left(1-\alpha\right)q_i$

$$J\left(\nu,n\right)=\int_0^\infty \left(\lambda-\lambda_0\right)^n\left[c_0+c_1\left(\lambda-\lambda_0\right)^2\right]^{d/2-\nu}d\lambda$$

Integrals	Results when index $\mu = 0$ and all the other indices are spatial
$J\Delta_1$	$$\frac{-2i}{(4\pi)^{d/2}} \int_0^1 d\alpha \quad \Gamma\left(3 - d/2\right) J\left(3, 0\right)$$
$J\Delta_{k_0}$	$$\frac{-2i}{(4\pi)^{d/2}} \int_0^1 d\alpha \quad \Gamma\left(3 - d/2\right) \left(J(3,1) + J(3,0)p_0\right)$$
$J\Delta_{k_0 k_j}$	$$\frac{-2i}{(4\pi)^{d/2}} \int_0^1 d\alpha \quad \Gamma\left(3 - d/2\right) \left(J(3,1) + J(3,0)p_0\right) \delta k_j$$
$J\Delta_{k_0 k_j k_l}$	$$\frac{-2i}{(4\pi)^{d/2}} \int_0^1 d\alpha \Big(-\frac{1}{2}\Gamma\left(2 - d/2\right) \left(J(2,1) + J(2,0)p_0\right) g_{jl} + \Gamma\left(3 - d/2\right) \left(J(3,1) + J(3,0)p_0\right) \Big) \delta k_j \delta k_l$$
$J\Delta_{k_0 k_j k_l k_m}$	$$\frac{-2i}{(4\pi)^{d/2}} \int_0^1 d\alpha \Big(\Gamma\left(3 - d/2\right) \left(J(3,1) + J(3,0)p_0\right) \delta k_j \delta k_l \delta k_m - \frac{1}{2}\Gamma\left(2 - d/2\right) \left(J(2,1) + J(2,0)p_0\right) \left(g_{jl}\delta k_m + g_{mj}\delta k_l + g_{lm}\delta k_j\right) \Big)$$

Integrals	Results when $\mu=\nu=0$ and ρ and σ are spatial
$J\Delta_1$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha \quad \Gamma\left(3-d/2\right)J\left(3,0\right)$$
$J\Delta_{k_0}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha \quad \Gamma\left(3-d/2\right)\left(J(3,1)+J(3,0)p_0\right)$$
$J\Delta_{k_0k_0}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(-\frac{1}{2}\Gamma\left(2-d/2\right)J(2,0) +\Gamma\left(3-d/2\right)\left(J(3,2)+2J(3,1)p_0+J(3,0)p_0^2\right)\Big)$$
$J\Delta_{k_0k_0k_l}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(-\frac{1}{2}\Gamma\left(2-d/2\right)J(2,0)\delta k_l +\Gamma\left(3-d/2\right)\left(J(3,2)+2J(3,1)p_0+J(3,0)p_0^2\right)\delta k_l\Big)$$
$J\Delta_{k_0k_0k_lk_m}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(\frac{1}{4}\Gamma\left(1-d/2\right)J(1,0)g_{lm} +\Gamma\left(3-d/2\right)\left(J(3,2)+2J(3,1)p_0+J(3,0)p_0^2\right)\delta k_l\delta k_m -\frac{1}{2}\Gamma\left(2-d/2\right)\left(J(2,0)\delta k_l\delta k_m+\left(J(2,2)+2J(2,1)p_0+J(2,0)p_0^2\right)g_{lm}\right)\Big)$$

Integrals	Results when $\mu=\nu=\rho=0$ and σ is spatial
$J\Delta_1$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha \quad \Gamma\left(3-d/2\right)J\left(3,0\right)$$
$J\Delta_{k_0}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha \quad \Gamma\left(3-d/2\right)\left(J(3,1)+J(3,0)p_0\right)$$
$J\Delta_{k_0k_0}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(-\frac{1}{2}\Gamma\left(2-d/2\right)J(2,0) + \Gamma\left(3-d/2\right)\left(J(3,2)+2J(3,1)p_0+J(3,0)p_0^2\right)\Big)$$
$J\Delta_{k_0k_0k_0}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(-\frac{3}{2}\Gamma\left(2-d/2\right)\left(J(2,1)+J(2,0)p_0\right) + \Gamma\left(3-d/2\right)\left(J(3,3)+3J(3,2)p_0+3J(3,1)p_0^2+J(3,0)p_0^3\right)\Big)$$
$J\Delta_{k_0k_0k_0k_m}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(\Gamma\left(3-d/2\right)\left(J(3,3)+3J(3,2)p_0+3J(3,1)p_0^2+J(3,0)p_0^3\right)\delta k_m - \frac{3}{2}\Gamma\left(2-d/2\right)\left(J(2,1)+J(2,0)p_0\right)\delta k_m\Big)$$

Integrals	Results when $\mu=\nu=\rho=\sigma=0$
$J\Delta_1$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha \quad \Gamma\left(3-d/2\right)J\left(3,0\right)$$
$J\Delta_{k_0}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha \quad \Gamma\left(3-d/2\right)\left(J(3,1)+J(3,0)p_0\right)$$
$J\Delta_{k_0k_0}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(-\frac{1}{2}\Gamma\left(2-d/2\right)J(2,0) +\Gamma\left(3-d/2\right)\left(J(3,2)+2J(3,1)p_0+J(3,0)p_0^2\right)\Big)$$
$J\Delta_{k_0k_0k_0}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(-\frac{3}{2}\Gamma\left(2-d/2\right)\left(J(2,1)+J(2,0)p_0\right) +\Gamma\left(3-d/2\right)\left(J(3,3)+3J(3,2)p_0+3J(3,1)p_0^2+J(3,0)p_0^3\right)\Big)$$
$J\Delta_{k_0k_0k_0k_0}$	$$\frac{-2i}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(\frac{3}{4}\Gamma\left(1-d/2\right)J(1,0) +\Gamma\left(3-d/2\right)\left(J(3,4)+4J(3,3)p_0+6J(3,2)p_0^2+4J(3,1)p_0^3+J(3,0)p_0^4\right) -3\Gamma\left(2-d/2\right)\left(J(2,2)+2J(2,1)p_0+J(2,0)p_0^2\right)\Big)$$

Integrals	$1/\epsilon$ terms when all indices are spatial
$J\Delta_1$	0
$J\Delta_{k_i}$	0
$J\Delta_{k_ik_j}$	$-\frac{i}{32\pi^2}\left(2p_0+q_0\right)\delta_{ij}$
$J\Delta_{k_ik_jk_l}$	$\frac{i}{96\pi^2}\left(3p_0+2q_0\right)\left(q_i\delta_{jl}+q_j\delta_{il}+q_l\delta_{ij}\right)$
$J\Delta_{k_ik_jk_lk_m}$	$-\frac{i}{384\pi^2}\left(2p_0+q_0\right)\left(q^2+4p_0q_0+4p_0^2+2q_0^2-6M^2\right)\left(\delta_{ij}\delta_{lm}+\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl}\right)$ $-\frac{i}{192\pi^2}\left(4p_0+3q_0\right)\left(\delta_{ij}q_lq_m+\delta_{il}q_jq_m+\delta_{im}q_jq_l+\delta_{jl}q_iq_m+\delta_{jm}q_iq_l+\delta_{lm}q_iq_j\right)$

Integrals	$1/\epsilon$ terms $\mu = 0$ and ν, ρ, σ is spatial
$J\Delta_1$	0
$J\Delta_{k_0}$	$-\frac{i}{16\pi^2}$
$J\Delta_{k_0 k_j}$	$-\frac{i}{32\pi^2} q_j$
$J\Delta_{k_0 k_j k_l}$	$-\frac{i}{192\pi^2}\left(4q_j q_l + (q^2 + 6p_0 q_0 + 12p_0^2 - 6M^2)\delta_{jl}\right)$
$J\Delta_{k_0 k_j k_l k_m}$	$-\frac{i}{384\pi^2}\Big((6M^2 - 12p_0^2 - q^2 - 8p_0 q_0)(q_j \delta_{lm} + q_l \delta_{jm} + q_m \delta_{jl}) - 6q_j q_l q_m\Big)$

Integrals	$1/\epsilon$ terms $\mu = \nu = 0$ and ρ, σ is spatial
$J\Delta_1$	0
$J\Delta_{k_0}$	$-\frac{i}{16\pi^2}$
$J\Delta_{k_0 k_0}$	$-\frac{i}{32\pi^2}(2p_0 - q_0)$
$J\Delta_{k_0 k_0 k_l}$	$-\frac{i}{96\pi^2}\left(3p_0 - 2q_0\right) q_l$
$J\Delta_{k_0 k_0 k_l k_m}$	$\frac{i}{384\pi^2}\Big(2(3q_0 - 4p_0)q_l q_m + (6M^2(2p_0 - q_0) - 12p_0^2 q_0 + q^2 q_0 - 24p_0^3 - 2p_0 q^2)\delta_{lm}\Big)$

Integrals	$1/\epsilon$ terms $\mu = \nu = \rho = 0$ and σ is spatial
$J\Delta_1$	0
$J\Delta_{k_0}$	$-\frac{i}{16\pi^2}$
$J\Delta_{k_0 k_0}$	$-\frac{i}{32\pi^2}(2p_0 - q_0)$
$J\Delta_{k_0 k_0 k_0}$	$-\frac{i}{192\pi^2}\left(6M^2 + 12p_0^2 - q^2 - 6p_0 q_0 + 4q_0^2\right)$
$J\Delta_{k_0 k_0 k_0 k_m}$	$\frac{i}{384\pi^2}\Big(6M^2 + 12p_0^2 - q^2 - 8p_0 q_0 + 6q_0^2\Big)q_m$

Integrals	$1/\epsilon$ terms $\mu=\nu=\rho=\sigma=0$
$J\Delta_1$	0
$J\Delta_{k_0}$	$-\frac{i}{16\pi^2}$
$J\Delta_{k_0k_0}$	$-\frac{i}{32\pi^2}(2p_0-q_0)$
$J\Delta_{k_0k_0k_0}$	$-\frac{i}{192\pi^2}\left(6M^2+12p_0^2-q^2-6p_0q_0+4q_0^2\right)$
$J\Delta_{k_0k_0k_0k_0}$	$-\frac{i}{384\pi^2}\Big(24p_0^3-2p_0q^2+6M^2(2p_0-3q_0)-12p_0^2q_0+3q^2q_0+8p_0q_0^2-6q_0^3\Big)$

Check whether following is same as before

$$
\begin{aligned}
I_1\left(n, p^0, M_\pi\right) \quad & \equiv \int d^{\tilde{d}}k \frac{1}{\left[p^0-k^0-\delta m_n\right]\left[k^2-M_\pi^2\right]} \\
& = \frac{2i}{(4\pi)^{d/2}} \int_0^\infty d\lambda \Gamma\left(2-d/2\right)\left[\Lambda^2\right]^{(d/2-2)} \\
& = \frac{2i}{(4\pi)^{d/2}} \Gamma\left(2-d/2\right) J\left(2,0\right)
\end{aligned}
$$

$$
\begin{aligned}
I_{k^0}\left(n, p^0, M_\pi\right) \quad & \equiv \int d^{\tilde{d}}k \frac{k^0}{\left[p^0-k^0-\delta m_n\right]\left[k^2-M_\pi^2\right]} \\
& = \frac{2i}{(4\pi)^{d/2}} \int_0^\infty d\lambda \Gamma\left(2-d/2\right)\lambda\left[\Lambda^2\right]^{(d/2-2)} \\
& = \frac{2i}{(4\pi)^{d/2}} \Gamma\left(2-d/2\right)\left[J\left(2,1\right)+\lambda_0 J\left(2,0\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
I_{k^i}\left(n, p^0, M_\pi\right) \quad & \equiv \int d^{\tilde{d}}k \frac{k^i}{\left[p^0-k^0-\delta m_n\right]\left[k^2-M_\pi^2\right]} \\
& = 0
\end{aligned}
$$

$$
\begin{aligned}
I_{k^0 k^i}\left(n, p^0, M_\pi\right) \quad & \equiv \int d^{\tilde{d}}k \frac{k^0 k^i}{\left[p^0-k^0-\delta m_n\right]\left[k^2-M_\pi^2\right]} \\
& = 0
\end{aligned}
$$

$$
\begin{aligned}
I_{(k^0)^2}\left(n, p^0, M_\pi\right) \quad & \equiv \int d^{\tilde{d}}k \frac{(k^0)^2}{\left[p^0-k^0-\delta m_n\right]\left[k^2-M_\pi^2\right]} \\
& = \frac{2i}{(4\pi)^{d/2}} \int_0^\infty d\lambda \{\Gamma\left(2-d/2\right)\lambda^2\left[\Lambda^2\right]^{(d/2-2)} - \frac{1}{2}\Gamma\left(1-d/2\right)\left[\Lambda^2\right]^{(d/2-1)}\} \\
& = \frac{i}{(4\pi)^{d/2}} \{2\Gamma\left(2-d/2\right)\left[J\left(2,2\right)+2\lambda_0 J\left(2,1\right)+\lambda_0^2 J\left(2,0\right)\right] - \Gamma\left(1-d/2\right) J\left(1,0\right)\}
\end{aligned}
$$

$$
\begin{aligned}
I_{k^i k^j}\left(n, p^0, M_\pi\right) \quad & \equiv \int d^{\tilde{d}}k \frac{k^i k^j}{\left[p^0-k^0-\delta m_n\right]\left[k^2-M_\pi^2\right]} \\
& = \frac{2i}{(4\pi)^{d/2}} \int_0^\infty d\lambda \frac{\delta^{i,j}}{2}\Gamma\left(1-d/2\right)\left[\Lambda^2\right]^{(d/2-1)} \\
& = \frac{i\delta^{i,j}}{(4\pi)^{d/2}} \Gamma\left(1-d/2\right) J\left(1,0\right)
\end{aligned}
$$

15 Matrix Element Calculation

The basis states of the symmetric representation are denoted by $|SS_3II_3\rangle$ where S is the Spin, I is the Iso-spin, S_3 is projection of the Spin, and I_3 is projection of the Iso-spin. Since always $I = S$, basis states can be written as $|SS_3I_3\rangle$.
Matrix elements of Spin generators S^i, Iso-spin generators(flavor generators) I^a, and Spin-flavor generators G^{ia} are given by [Cordón and Goity, 2013]

$$\begin{aligned}
\langle S'S'_3I'_3|\, S^i\, |SS_3I_3\rangle &= \sqrt{S(S+1)}\delta_{SS'}\delta_{I_3I'_3}\, \langle SS_3, 1i|S'S'_3\rangle \\
\langle S'S'_3I'_3|\, I^a\, |SS_3I_3\rangle &= \sqrt{S(S+1)}\delta_{SS'}\delta_{S_3S'_3}\, \langle SI_3, 1a|S'I_3'\rangle \\
\langle S'S'_3I'_3|\, G^{ia}\, |SS_3I_3\rangle &= \frac{1}{4}\sqrt{\frac{2S+1}{2S'+1}}\xi(N_c, S, S')\, \langle SS_3, 1i|S'S'_3\rangle\, \langle SI_3, 1a|S'I'_3\rangle
\end{aligned} \tag{17}$$

Here, $\xi(N_c, S, S') = \sqrt{(2+N_c)^2 - (S-S')^2(S+S'+1)^2}$

The following table shows operator identities in the totally symmetric irreducible representation of SU(4). The second column shows the operator's quantum numbers (J, I) [Cordón and Goity, 2013]

Table 5: Operator Identities

Operator Identities	Quantum numbers
$\{S^i, S^i\} - \{I^a, I^a\} = 0$	$(0, 0)$
$\{S^i, S^i\} + \{I^a, I^a\} + 4\{G^{ia}, G^{ia}\} = (3/4)N_c(4+N_c)$	$(0, 0)$
$2\{S^i, G^{ia}\} = (2+N_c)I^a$	$(0, 1)$
$2\{I^a, G^{ia}\} = (2+N_c)S^i$	$(1, 0)$
$(1/2)\{S^k, I^c\} - \epsilon^{ijk}\epsilon^{abc}\{G^{ia}, G^{jb}\} = (2+N_c)G^{kc}$	(1,1)
$\epsilon^{ijk}\{S^i, G^{jc}\} = \epsilon^{abc}\{I^a, G^{kb}\}$	(1,1)
$4\{G^{ia}, G^{ib}\}\mid_{I=2} = \{I^a, I^b\}$	(0,2)
$4\{G^{ia}, G^{ja}\}\mid_{J=2} = \{S^i, S^j\}$	(2,0)

15.1 Other Useful Operator Identities

$$S^2 = I^2 \tag{18}$$

$$G^2 = \frac{3}{16}N_c(4+N_c) - \frac{1}{2}S^2 \tag{19}$$

$$G^{ia}G^{jb}G^{ia} = \frac{1}{2}\left(G^2G^{jb} + G^{jb}G^2 - \frac{3}{8}N_c(4+N_c)G^{jb}\right) \tag{20}$$

$$G^{ia}G^{jb}G^{ia} = \frac{1}{2}\left(G^2G^{jb} + G^{jb}G^2 - \frac{3}{8}N_c(4+N_c)G^{jb}\right) \tag{21}$$

15.2 Spherical basis

Spherical basis (angular momentum basis) is more convenient to work with spherical tensors such as spin and iso-spin. Therefore, spherical basis is used to project operators. Transformation from vector $V^i; i = 1, 2, 3$ in cartesian basis to a vector $V^m; m = -1, 0, +1$ in spherical basis is given by,

$$\begin{aligned} V^{+1} &= -(1/\sqrt{2})(V^1 + iV^2) \\ V^0 &= V^3 \\ V^{-1} &= (1/\sqrt{2})(V^1 - iV^2) \end{aligned} \tag{22}$$

In Spherical basis Delta function and totally anti-symmetric Levi-Civita tensor are given by

$$\begin{aligned} \delta_S[a, b] &= (-1)^b \delta[a, -b] \\ \epsilon_S[abc] &= i\sqrt{2}(-1)^c \langle 1a, 1b|1 - c\rangle \end{aligned} \tag{23}$$

15.3 Projectors

In cartesian basis, Tensor can be decomposed into it's angular momentum values. $T^{ij} = T_0^{ij} + T_1^{ij} + T_2^{ij}$. Following projection cartesian operators are used to decompose the tensor.

$$\begin{aligned} T_0^{ij} &= \left(\tfrac{1}{3}\delta^{ij}\delta^{kl}\right) T^{kl} \\ T_1^{ij} &= \left(\tfrac{1}{2}\epsilon^{ijm}\epsilon^{mkl}\right) T^{kl} \\ T_2^{ij} &= \left(\tfrac{1}{2}(\delta^{ik}\delta^{jl} + \delta^{jk}\delta^{il}) - \tfrac{1}{3}\delta^{ij}\delta^{kl}\right) T^{kl} \end{aligned} \tag{24}$$

In Spherical basis, Clebsch-Gordan coefficients are used to decompose states into a given angular momentum. Total angular momentum eigen state(coupled states) can be written as linear combination of uncoupled product basis as follows.

$$|JM\rangle = \sum_{m_1,m_2} \langle j_1m_1, j_2m_2|JM\rangle \, |j_1m_1, j_2m_2\rangle$$

Product states can be written as linear combination of coupled states as follows.

$$|j_1n_1, j_2n_2\rangle = \sum_{J,M} \langle j_1n_1, j_2n_2|JM\rangle \, |JM\rangle$$

Product states with a particular angular momentum j can be written as

$$\begin{aligned} |j_1n_1, j_2n_2\rangle_{J=j} &= \sum_M \langle j_1n_1, j_2n_2|jM\rangle \, |jM\rangle \\ |j_1n_1, j_2n_2\rangle_{J=j} &= \sum_M \sum_{m_1,m_2} \langle j_1n_1, j_2n_2|jM\rangle \, \langle j_1m_1, j_2m_2|jM\rangle \, |j_1m_1, j_2m_2\rangle \end{aligned}$$

Similarly, they are used to decompose tensors in to given angular momentum.

$$(O_1O_2)_{J=j}^{n_1n_2} = \sum_{M}\sum_{m_1,m_2} \langle j_1n_1, j_2n_2|jM\rangle \langle j_1m_1, j_2m_2|jM\rangle \left(O_1^{m_1}O_2^{m_2}\right) \tag{25}$$

Here, J_1 and J_2 are the spins of operator O_1 and O_2 respectively. This equation 25 can be extended to iso-spin decomposition as follows

$$\begin{aligned}(O_1O_2)_{J=j,I=i}^{n_1n_2b_1b_2} = \quad & \sum_{j3}\sum_{m_1,m_2} \langle j_1n_1, j_2n_2|jj_3\rangle \langle j_1m_1, j_2m_2|jj_3\rangle \\ & \sum_{i3}\sum_{a_1,a_2} \langle i_1b_1, i_2b_2|ii_3\rangle \langle i_1a_1, i_2a_2|ii_3\rangle \\ & \left(O_1^{m_1a_1}O_2^{m_2a_2}\right)\end{aligned} \tag{26}$$

Following projector operator is defined in Mathematica program.

$$P\left(J, \{j_1, j_2\}, m_1, m_2, n_1, n_2\right) = \sum_{M=-J}^{J} \langle j_1n_1, j_2n_2|JM\rangle \langle j_1m_1, j_2m_2|JM\rangle \tag{27}$$

$$\begin{aligned}(O_1O_2)_{J=j,I=i}^{n_1n_2b_1b_2} = \quad & \sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2} P\left(j, \{j_1, j_2\}, m_1, m_2, n_1, n_2\right) \\ & \sum_{a_1=-i_1}^{i_1}\sum_{a_2=-i_2}^{i_2} P\left(i, \{i_1, i_2\}, a_1, a_2, b_1, b_2\right) \\ & \left(O_1^{m_1a_1}O_2^{m_2a_2}\right)\end{aligned} \tag{28}$$

15.4 Mathematica program to decompose operators

The table 6 shows all possible operator combinations which are relevant for the calculation. The first column shows the assigned number for the operator combination, the second column shows operator combination, and the last column shows spin and iso-spin values for the operator combination ($m_1m_2a_1a_2$).

As an example operator $G^{m_1a_1}S^{2n}G^{m_2a_2}$ can be decompose using equation.

$$\begin{aligned}\left(GS^{2n}G\right)_{J=j,I=i}^{n_1n_2b_1b_2} &= \sum_{j_3=-j}^{j}\sum_{i_3=-i}^{i}\sum_{m_1=-1}^{1}\sum_{m_2=-1}^{1}\sum_{a_1=-1}^{1}\sum_{a_2=-1}^{1} \langle 1n_1, 1n_2|jj_3\rangle \\ & \langle 1m_1, 1m_2|jj_3\rangle \langle 1b_1, 1b_2|ii_3\rangle \langle 1a_1, 1a_2|ii_3\rangle \left(G^{m_1a_1}S^{2n}G^{m_2a_2}\right)\end{aligned}$$

Table 6: Operator Combinations

Referance Number(N)	Operator Combination	Spin and Iso-spin
1	1	$(0,0,0,0)$
2	S^{m_1}	$(1,0,0,0)$
3	I^{a_1}	$(0,0,1,0)$
4	$G^{m_1a_1}$	$(1,0,1,0)$
5	$S^{m_1}S^{2n}S^{m_2}$	$(1,1,0,0)$
6	$S^{m_2}S^{2n}S^{m_1}$	$(1,1,0,0)$
7	$I^{a_1}S^{2n}I^{a_2}$	$(0,0,1,1)$
8	$I^{a_2}S^{2n}I^{a_1}$	$(0,0,1,1)$
9	$G^{m_1a_1}S^{2n}G^{m_2a_2}$	$(1,1,1,1)$
10	$G^{m_2a_2}S^{2n}G^{m_1a_1}$	$(1,1,1,1)$
11	$S^{m_1}S^{2n}I^{a_2}$	$(1,0,0,1)$
12	$S^{m_1}S^{2n}G^{m_2a_2}$	$(1,1,0,1)$
13	$I^{a_1}S^{2n}S^{m_2}$	$(0,1,1,0)$
14	$I^{a_1}S^{2n}G^{m_2a_2}$	$(0,1,1,1)$
15	$G^{m_1a_1}S^{2n}S^{m_2}$	$(1,1,1,0)$
16	$G^{m_1a_1}S^{2n}I^{a_2}$	$(1,0,1,1)$

Following equation is used in Mathematica program to decompose any operator combinations in table 6 is given by,

$$\begin{aligned}
\left(O_1S^{2n}O_2\right)^{n_1n_2b_1b_2}_{J=j,I=i} \quad &= \sum_{j_3=-j}^{j}\ \sum_{i_3=-i}^{i}\ \sum_{m_1=-m_1m_2a_1a_2(N)[1]}^{m_1m_2a_1a_2(N)[1]}\ \sum_{m_2=-m_1m_2a_1a_2(N)[2]}^{m_1m_2a_1a_2(N)[2]} \\
&\sum_{a_1=-m_1m_2a_1a_2(N)[3]}^{m_1m_2a_1a_2(N)[3]}\ \sum_{a_2=-m_1m_2a_1a_2(N)[4]}^{m_1m_2a_1a_2(N)[4]} \\
&\langle m_1m_2a_1a_2(N)\,[1]\ \ n_1, m_1m_2a_1a_2(N)\,[2]\ \ n_2|jj_3\rangle \\
&\langle m_1m_2a_1a_2(N)\,[1]\ \ m_1, m_1m_2a_1a_2(N)\,[2]\ \ m_2|jj_3\rangle \\
&\langle m_1m_2a_1a_2(N)\,[3]\ \ b_1, m_1m_2a_1a_2(N)\,[4]\ \ b_2|ii_3\rangle \\
&\langle m_1m_2a_1a_2(N)\,[3]\ \ a_1, m_1m_2a_1a_2(N)\,[4]\ \ a_2|ii_3\rangle \\
&\left(O_1^{m_1a_1}S^{2n}O_2^{m_2a_2}\right) \qquad (29)
\end{aligned}$$

Equation 29 Can be simplified to

$$\begin{aligned}\left(O_1 S^{2n} O_2\right)^{n_1 n_2 b_1 b_2}_{J=j,I=i} \quad &= \sum_{m_1=-m_1 m_2 a_1 a_2(N)[1]}^{m_1 m_2 a_1 a_2(N)[1]} \sum_{m_2=n_1+n_2-m_1}^{n_1+n_2-m_1} \sum_{a_1=-m_1 m_2 a_1 a_2(N)[3]}^{m_1 m_2 a_1 a_2(N)[3]} \sum_{a_2=b_1+b_2-a_1}^{b_1+b_2-a_1} \\ &\langle m_1 m_2 a_1 a_2(N)\,[1]\ \ n_1, m_1 m_2 a_1 a_2(N)\,[2]\ \ n_2 | j\ n_1+n_2\rangle \\ &\langle m_1 m_2 a_1 a_2(N)\,[1]\ \ m_1, m_1 m_2 a_1 a_2(N)\,[2]\ \ m_2 | j\ n_1+n_2\rangle \\ &\langle m_1 m_2 a_1 a_2(N)\,[3]\ \ b_1, m_1 m_2 a_1 a_2(N)\,[4]\ \ b_2 | i\ b_1+b_2\rangle \\ &\langle m_1 m_2 a_1 a_2(N)\,[3]\ \ a_1, m_1 m_2 a_1 a_2(N)\,[4]\ \ a_2 | i\ b_1+b_2\rangle \\ &\left(O_1^{m_1 a_1} S^{2n} O_2^{m_2 a_2}\right) \end{aligned} \tag{30}$$

Equation 30 is used in my Mathematica program.

$$\langle S'S'_3I'_3| S^2 |SS_3I_3\rangle = S(S+1)\delta_{SS'}\delta_{S_3S'_3}\delta_{I_3I'_3} \tag{31}$$

$$\langle S'S'_3I'_3| I^2 |SS_3I_3\rangle = S(S+1)\delta_{SS'}\delta_{S_3S'_3}\delta_{I_3I'_3} \tag{32}$$

$$\langle S'S'_3I'_3| G^2 |SS_3I_3\rangle = ((3/16)N_c(N_c+4) - (1/2)S(S+1)) \delta_{SS'}\delta_{S_3S'_3}\delta_{I_3I'_3} \tag{33}$$

$$\langle S'S'_3I'_3| S^{2n} |SS_3I_3\rangle = (S(S+1))^n \delta_{SS'}\delta_{S_3S'_3}\delta_{I_3I'_3} \tag{34}$$

$$\langle S'S'_3I'_3| S^iI^a |SS_3I_3\rangle = S(S+1)\delta_{SS'} \langle SS_3, 1i|S'S'_3\rangle \langle SI_3, 1a|S'I'_3\rangle \tag{35}$$

15.5 Operator Basis

Table 7: Operator Basis

Projections $(J;I)$	Operator	Bodyness
(0;0)	1	1
(0;0)	S^2, I^2, G^2	2
(1;0)	$S^i, \{I^a, G^{jb}\}_{10}$	1
(0;1)	$I^a, \{S^i, G^{jb}\}_{01}$	1
(1;1)	G^{ia}	1
(1;1)	S^iI^a	2
(1;1)	$\{S^i, G^{jb}\}_{11}$	2
(1;1)	$\{I^a, G^{jb}\}_{11}$	2
(2;0)	$(S^iS^j)_{20}$	2
(0;2)	$(I^aI^b)_{02}$	2
(2;1)	$\{S^i, G^{jb}\}_{21}$	2
(1;2)	$\{I^a, G^{jb}\}_{12}$	2
(2;2)	$\{G^{jb}, G^{jb}\}_{22}$	2

$$
\begin{aligned}
\left(O_1^{m_1a_1}S^{2n}O_2^{m_2a_2}\right)_{J=2,I=2}^{l_1l_2c_1c_2} &= \sum_{n_1,n_2,b_1,b_2=-1}^{1} \left(O_1^{m_1a_1}S^{2n}O_2^{m_2a_2}\right)_{J=2,I=2}^{n_1n_2b_1b_2} \delta_{l_1,n_1}\delta_{l_2,n_2}\delta_{c_1,b_1}\delta_{c_2,b_2} \\
\left(O_1^{m_1a_1}S^{2n}O_2^{m_2a_2}\right)_{J=2,I=1}^{l_1l_2c_1} &= \sum_{n_1,n_2,b_1,b_2=-1}^{1} \left(O_1^{m_1a_1}S^{2n}O_2^{m_2a_2}\right)_{J=2,I=1}^{n_1n_2b_1b_2} \delta_{l_1,n_1}\delta_{l_2,n_2} \langle 1b_1,1b_2|1c_1\rangle \\
\left(O_1^{m_1a_1}S^{2n}O_2^{m_2a_2}\right)_{J=2,I=0}^{l_1l_2} &= \sum_{n_1,n_2,b_1,b_2=-1}^{1} \left(O_1^{m_1a_1}S^{2n}O_2^{m_2a_2}\right)_{J=2,I=0}^{n_1n_2b_1b_2} \delta_{l_1,n_1}\delta_{l_2,n_2} \langle 1b_1,1b_2|00\rangle \\
\left(O_1^{m_1a_1}S^{2n}O_2^{m_2a_2}\right)_{J=1,I=2}^{l_1c_1c_2} &= \sum_{n_1,n_2,b_1,b_2=-1}^{1} \left(O_1^{m_1a_1}S^{2n}O_2^{m_2a_2}\right)_{J=1,I=2}^{n_1n_2b_1b_2} \langle 1n_1,1n_2|1l_1\rangle\, \delta_{c_1,b_1}\delta_{c_2,b_2} \\
\left(O_1^{m_1a_1}S^{2n}O_2^{m_2a_2}\right)_{J=1,I=1}^{l_1c_1} &= \sum_{n_1,n_2,b_1,b_2=-1}^{1} \left(O_1^{m_1a_1}S^{2n}O_2^{m_2a_2}\right)_{J=1,I=1}^{n_1n_2b_1b_2} \langle 1n_1,1n_2|1l_1\rangle \langle 1b_1,1b_2|1c_1\rangle \\
\left(O_1^{m_1a_1}S^{2n}O_2^{m_2a_2}\right)_{J=1,I=0}^{l_1} &= \sum_{n_1,n_2,b_1,b_2=-1}^{1} \left(O_1^{m_1a_1}S^{2n}O_2^{m_2a_2}\right)_{J=1,I=0}^{n_1n_2b_1b_2} \langle 1n_1,1n_2|1l_1\rangle \langle 1b_1,1b_2|00\rangle \\
\left(O_1^{m_1a_1}S^{2n}O_2^{m_2a_2}\right)_{J=0,I=2}^{c_1c_2} &= \sum_{n_1,n_2,b_1,b_2=-1}^{1} \left(O_1^{m_1a_1}S^{2n}O_2^{m_2a_2}\right)_{J=0,I=2}^{n_1n_2b_1b_2} \langle 1n_1,1n_2|00\rangle\, \delta_{c_1,b_1}\delta_{c_2,b_2} \\
\left(O_1^{m_1a_1}S^{2n}O_2^{m_2a_2}\right)_{J=0,I=1}^{c_1} &= \sum_{n_1,n_2,b_1,b_2=-1}^{1} \left(O_1^{m_1a_1}S^{2n}O_2^{m_2a_2}\right)_{J=0,I=1}^{n_1n_2b_1b_2} \langle 1n_1,1n_2|00\rangle \langle 1b_1,1b_2|1c_1\rangle \\
\left(O_1^{m_1a_1}S^{2n}O_2^{m_2a_2}\right)_{J=0,I=0} &= \sum_{n_1,n_2,b_1,b_2=-1}^{1} \left(O_1^{m_1a_1}S^{2n}O_2^{m_2a_2}\right)_{J=0,I=0}^{n_1n_2b_1b_2} \langle 1n_1,1n_2|00\rangle \langle 1b_1,1b_2|00\rangle
\end{aligned}
$$

$$
\begin{aligned}
(O)^{l_1 l_2 c_1 c_2}_{J=2,I=2} &= \sum_{n_1,n_2,b_1,b_2=-1}^{1} (O)^{n_1 n_2 b_1 b_2}_{J=2,I=2}\, \delta_{l_1,n_1}\delta_{l_2,n_2}\delta_{c_1,b_1}\delta_{c_2,b_2} \\
(O)^{l_1 l_2 c_1}_{J=2,I=1} &= \sum_{n_1,n_2,b_1,b_2=-1}^{1} (O)^{n_1 n_2 b_1 b_2}_{J=2,I=1}\, \delta_{l_1,n_1}\delta_{l_2,n_2}\, \langle 1b_1, 1b_2 | 1c_1 \rangle \\
(O)^{l_1 l_2}_{J=2,I=0} &= \sum_{n_1,n_2,b_1,b_2=-1}^{1} (O)^{n_1 n_2 b_1 b_2}_{J=2,I=0}\, \delta_{l_1,n_1}\delta_{l_2,n_2}\, \langle 1b_1, 1b_2 | 00 \rangle \\
(O)^{l_1 c_1 c_2}_{J=1,I=2} &= \sum_{n_1,n_2,b_1,b_2=-1}^{1} (O)^{n_1 n_2 b_1 b_2}_{J=1,I=2}\, \langle 1n_1, 1n_2 | 1l_1 \rangle\, \delta_{c_1,b_1}\delta_{c_2,b_2} \\
(O)^{l_1 c_1}_{J=1,I=1} &= \sum_{n_1,n_2,b_1,b_2=-1}^{1} (O)^{n_1 n_2 b_1 b_2}_{J=1,I=1}\, \langle 1n_1, 1n_2 | 1l_1 \rangle\, \langle 1b_1, 1b_2 | 1c_1 \rangle \\
(O)^{l_1}_{J=1,I=0} &= \sum_{n_1,n_2,b_1,b_2=-1}^{1} (O)^{n_1 n_2 b_1 b_2}_{J=1,I=0}\, \langle 1n_1, 1n_2 | 1l_1 \rangle\, \langle 1b_1, 1b_2 | 00 \rangle \\
(O)^{c_1 c_2}_{J=0,I=2} &= \sum_{n_1,n_2,b_1,b_2=-1}^{1} (O)^{n_1 n_2 b_1 b_2}_{J=0,I=2}\, \langle 1n_1, 1n_2 | 00 \rangle\, \delta_{c_1,b_1}\delta_{c_2,b_2} \\
(O)^{c_1}_{J=0,I=1} &= \sum_{n_1,n_2,b_1,b_2=-1}^{1} (O)^{n_1 n_2 b_1 b_2}_{J=0,I=1}\, \langle 1n_1, 1n_2 | 00 \rangle\, \langle 1b_1, 1b_2 | 1c_1 \rangle \\
(O)_{J=0,I=0} &= \sum_{n_1,n_2,b_1,b_2=-1}^{1} (O)^{n_1 n_2 b_1 b_2}_{J=0,I=0}\, \langle 1n_1, 1n_2 | 00 \rangle\, \langle 1b_1, 1b_2 | 00 \rangle
\end{aligned}
$$

$$
\begin{aligned}
(O)^{l_1 l_2 c_1 c_2}_{JI} &= \sum_{n_1,n_2,b_1,b_2=-1}^{1} \left(\delta_{J2}\ \delta_{n_1 l_1}\delta_{n_2 l_2} + \delta_{J1}\, \langle 1n_1, 1n_2 | 1l_1 \rangle + \delta_{J0}\, \langle 1n_1, 1n_2 | 00 \rangle\right) \\
&\quad \left(\delta_{I2}\ \delta_{b_1 c_1}\delta_{b_2 c_2} + \delta_{I1}\, \langle 1b_1, 1b_2 | 1c_1 \rangle + \delta_{I,0}\, \langle 1b_1, 1b_2 | 00 \rangle\right) (O)^{n_1 n_2 b_1 b_2}_{J=2,I=1}
\end{aligned}
$$

15.6 Ambiguities raised in the Calculation

15.6.1 Order of indices in projectors when taking hermitian-conjugate(this idea was wrong, later I realized it)

As an example lets consider the $J=1$ and $I=0$ projection of $G^{ia}G^{lc}G^{jb}G^{lc}$

$$G^{ia}G^{lc}G^{jb}G^{lc}\Big|_{10} = \frac{1}{6}\epsilon^{ijk}\delta^{ab}\epsilon^{ki'j'}\delta^{a'b'}G^{i'a'}G^{lc}G^{j'b'}G^{lc}$$

Here the order of indices are important. In this example i' and j' are in the corresponding positions of i and j.

Calculating the $J=1$ and $I=0$ projection of $G^{lc}G^{ia}G^{lc}G^{jb}$ was done as follows.

$$G^{lc}G^{ia}G^{lc}G^{jb}\Big|_{10} =$$

15.6.2 selcting no pole terms

When selecting no pole terms from diagrams, following two methods were used. Let's assume that the loop integrals gives z as $z = a + b\mathfrak{p}^{\circ} + c{\mathfrak{p}^{\circ}}^{2} + ...$

- $nople - term = \frac{\partial z}{\partial \mathfrak{p}^{\circ}} = b + 2c\mathfrak{p}^{\circ} + ...$
- $nople - term = \frac{z}{\mathfrak{p}^{\circ}} = b + c\mathfrak{p}^{\circ} + ...$

They are correct only up to $\mathcal{O}\left({\mathfrak{p}^{\circ}}^{0}\right)$.

16 $1/\epsilon$ terms of g_A^4 diagrams

Table 8: $1/\epsilon$ terms of g_A^4 diagrams

Projection	$1/\epsilon$ term
$J=0, I=0$	0
$J=1, I=0$	$-i\alpha^{ij}\left(k_1^0+k_2^0\right)\delta^{ab}\epsilon^{ijk}S^k$
$J=0, I=1$	$-i\alpha^{ij}\left(k_1^0+k_2^0\right)\delta^{ij}\epsilon^{abc}I^c$
$J=1, I=1$	$\alpha^{ij}\epsilon^{ijk}\epsilon^{abc}\Big(\left(k_1^0-k_2^0\right)(2+N_c)\left[G^{kc},S^2\right]+$ $\frac{C_{HF}}{N_c}\Big((2+N_c)\left(8G^{kc}+\left[\left[G^{kc},S^2\right],S^2\right]\right)-24S^kI^c\Big)\Big)$
$J=2, I=0$	$16\alpha^{ij}\delta^{ab}\frac{C_{HF}}{N_c}(SS)\mid_{J=2}^{ij}$
$J=0, I=2$	$16\alpha^{ij}\delta^{ij}\frac{C_{HF}}{N_c}(II)\mid_{I=2}^{ab}$
$J=2, I=1$	0
$J=1, I=2$	0
$J=2, I=2$	$8\alpha^{ij}\Big(\left(k_1^0-k_2^0\right)\left[(GG)\mid_{J=I=2}^{ijab},S^2\right]+$ $\frac{C_{HF}}{N_c}\Big(\left[\left[(GG)\mid_{J=I=2}^{ijab},S^2\right],S^2\right]-24(GG)\mid_{J=I=2}^{ijab}\Big)\Big)$

Here, $\alpha^{ij}=\frac{ig_A^4}{768\pi^2F_\pi^4}k_1^ik_2^j$

17 $1/\epsilon$ terms of g_A^2 diagrams

Table 9: $1/\epsilon$ terms of g_A^2 diagrams

Projection	$1/\epsilon$ term
$J=0, I=0$	$2\alpha\,\delta^{ab}\frac{C_{HF}}{N_c}\left(3N_c(4+Nc)-20S^2\right)\left(4k_1.k_2-3M^2\right)$
$J=1, I=0$	0
$J=0, I=1$	$-i\alpha\Big(\epsilon^{abc}I^c\Big(6\frac{C_{HF}}{N_c}\left({k_2^0}^2-{k_1^0}^2\right)(4-5N_c(4+Nc))+ \left(k_1^0+k_2^0\right)\left(3\left(k_1-k_2\right)^2+3\left(k_1^0-k_2^0\right)^2+4\vec{k_1}.\vec{k_2}\right)- 4\left(k_1^0\vec{k_2}^{\,2}+k_2^0\vec{k_1}^{\,2}\right)\Big)+\epsilon^{abc}I^cS^2\left(168\frac{C_{HF}}{N_c}\left({k_2^0}^2-{k_1^0}^2\right)\right)\Big)$
$J=1, I=1$	$24\alpha\,\epsilon^{abc}\epsilon^{ijk}\frac{C_{HF}}{N_c}\left((2+N_c)G^{kc}-3S^kI^c\right)k_1^ik_2^j$
$J=2, I=0$	0
$J=0, I=2$	$-48\alpha\,\frac{C_{HF}}{N_c}k_1.k_2\,(II)\,\|^{ab}_{I=2}$
$J=2, I=1$	0
$J=1, I=2$	0
$J=2, I=2$	0

Here, $\alpha=\frac{ig_A^2}{768\pi^2F_\pi^4}$

18 $1/\epsilon$ terms of g_A^0 diagrams

Table 10: $1/\epsilon$ terms of g_A^0 diagrams

Projection	$1/\epsilon$ term
$J=0, I=0$	0
$J=1, I=0$	0
$J=0, I=1$	$i\alpha\ \epsilon^{abc}I^c \left(k_1^0+k_2^0\right)\Big(40M^2+16k_1.k_2-3(3k_1^0+k_2^0)(k_1^0+3k_2^0)\Big)$
$J=1, I=1$	0
$J=2, I=0$	0
$J=0, I=2$	0
$J=2, I=1$	0
$J=1, I=2$	0
$J=2, I=2$	0

Here, $\alpha = \frac{i}{768\pi^2 F_\pi^4}$

19 Construction of the Effective Lagrangian

Leading order Lagrangian

$$
\begin{aligned}
\mathcal{L}^{(1)}_{\pi N} = \quad & B^\dagger \left(iD_0 + \mathring{g}_A u^{ia} G^{ia} - \frac{C_{HF}}{N_c}\hat{S}^2 + \frac{c_1}{2\Lambda}\hat{\chi}_+ \right) B \\
\mathcal{L}^{(2)}_{\pi\pi} = \quad & \frac{F^2}{4}\langle D_\mu U (D^\mu U)^\dagger \rangle + \frac{F^2}{4}\langle \chi U^\dagger + U \chi^\dagger \rangle
\end{aligned}
$$

Effective Lagrangian

$$
\mathcal{L}_{eff} = \mathcal{L}^{(2)}_{\pi\pi} + \mathcal{L}^{(4)}_{\pi\pi} + ... + \mathcal{L}^{(1)}_{\pi N} + \mathcal{L}^{(2)}_{\pi N} + \mathcal{L}^{(3)}_{\pi N} + ...
$$

Building blocks of the effective Lagrangian

$$
\begin{aligned}
D_\mu = \quad & \partial_\mu + \frac{1}{2}\left(u^\dagger \partial_\mu u + u \partial_\mu u^\dagger \right) \\
u_\mu = \quad & i\left(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger \right) \\
\chi_\pm = \quad & u^\dagger \chi u^\dagger \pm u \chi^\dagger u
\end{aligned}
$$

Where χ , is a linear combination of a scalar and pseudoscalar $\chi = 2B(s+ip)$ [Scherer, 2003] (page 80) In SU(2) $\chi = 2BM = 2B \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$ [Scherer, 2003] (page 88) At $\mathcal{O}(p^2)$, $M_\pi^2 = 2Bm$ [Scherer, 2003] (page 83)

$\chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u$
$\chi^0_\pm = \langle \chi_\pm \rangle$
$\chi^a_\pm = \langle \chi_\pm I^a \rangle$
$\tilde{\chi}_\pm = \chi^a_\pm I^a$

$\hat{\chi}_+ = \tilde{\chi}_+ + N_c\, \chi^0_+$

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type equations

$$-i\Sigma = \sum_n \frac{\mathring{g}_A}{F^2} G^{ia} P_n G^{ja} \int d^{\tilde{d}}k \frac{k^i k^j}{[p^0 - k^0 - \delta m_n]\,[k^2 - M_\pi^2]} \tag{36}$$

$$\Sigma = \sum_n \frac{\mathring{g}_A}{F^2} G^{ia} P_n G^{ja} \frac{\delta_{ij}}{48\pi^2} \left[(3c_0\lambda_0 + \lambda_0^3)\lambda_\epsilon + polynomial + nonanalitic \right] \tag{37}$$

$$\delta Z = \frac{\partial}{\partial \mathfrak{p}^\circ} (\Sigma) \Big|_{\mathfrak{p}^\circ \to 0} \tag{38}$$

$$\Sigma = \sum_n \frac{\mathring{g}_A}{48\pi^2 F^2} G^{ia} P_n G^{ia} \left((3c_0\lambda_0 + \lambda_0^3)\lambda_\epsilon + \left(7c_0\lambda_0 + \frac{3\lambda_0^3}{3} \right) - (3c_0\lambda_0 + \lambda_0^3) log(c_0 + \lambda_0^2) + 4c_0^2 J(3,0) \right) \tag{39}$$

$$\Sigma_{UV} = \lambda_\epsilon \sum_n \frac{\mathring{g}_A}{48\pi^2 F^2} G^{ia} P_n G^{ia} \left(3M^2(\delta m_{in} - \delta m_n) - 2(\delta m_{in} - \delta m_n)^3 + \mathfrak{p}^\circ \left(3M^2 - 12(\delta m_{in} - \delta m_n)^2 \right) + \mathcal{O} \left(\right.\right.$$

$$\begin{aligned} \Sigma_{UV} = \lambda_\epsilon \frac{\mathring{g}_A}{48\pi^2 F^2} & \left[\frac{C_{HF}}{24N_c} \left(-3M^2(3N_c(4+N_c) - 20\hat{S}^2) \right) + 8\frac{C_{HF}^2}{N_c^2} (N_c(4+N_c)(3+5\hat{S}^2) - 4\hat{S}^2(6+7\hat{S}^2)) \right] \\ & \left[+ \mathfrak{p}^\circ \left(\frac{M^2}{16}(3N_c(4+N_c) - \hat{S}2) - \frac{C_{HF}^2}{4N_c^2}(N_c(4+N_c)(3+2\hat{S}^2)) - 8\hat{S}^2((3+\hat{S}^2) \right) + \mathcal{O}\left({\mathfrak{p}^\circ}^2 \right) \right] \end{aligned}$$

$$\begin{aligned} F = \{ F_x \in F_c : & (|S| > |C|) \\ & \cap (\text{minPixels} < |S| < \text{maxPixels}) \\ & \cap (|S_{\text{connected}}| > |S| - \epsilon) \} \end{aligned} \tag{42}$$

$$\mathfrak{p}^\circ = \delta m_{in} - p^0 \tag{43}$$

$$\Delta^{++}\,(uuu) \qquad \Delta^{+}\,(uud) \qquad \Delta^{0}\,(udd) \qquad \Delta^{-}\,(ddd) \tag{44}$$

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