AST4320 Oblig 2

Daniel Heinesen, daniehei

26. september 2018

1 Exercise 1

We have the 1D window function

$$W(x) = \begin{cases} 1 & \text{if } |x| < R \\ 0 & \text{else} \end{cases} \tag{1}$$

We now want to find the Fourier transform of this function. We do this straight forward from the definition of the Fourier transform

$$\tilde{W}(k) = \int_{-\infty}^{\infty} W(x) e^{-ikx} dx. \tag{2}$$

Inserting our definition of $W(x)$ we find

$$\tilde{W}(k) = \int_{-\infty}^{-R} 0 \cdot e^{-ikx} dx + \int_{-R}^{R} 1 \cdot e^{-ikx} dx + \int_{R}^{\infty} 0 \cdot e^{-ikx} dx = \int_{-R}^{R} e^{-ikx} dx \tag{3}$$

$$= \frac{i}{k} e^{-ikx} \Big|_{-R}^{R} = \frac{i}{k} \left(e^{-ikR} - e^{ikR} \right) = \frac{2 \sin Rk}{k}. \tag{4}$$

Before we plot this function we need to notice that this is a function that we need to be careful with for $k \to 0$. With the use of L'Hôpitals rule

$$\lim_{k\to 0} \tilde{W}(k) = \lim_{k\to 0} \frac{2 \sin Rk}{k} = \lim_{k\to 0} \frac{2R \cos Rk}{1} = 2R. \tag{5}$$

We now see that $\tilde{W}(k)$ is well defined across the whole real line.

Figur 1: The plot of the Fourier Transform of the top-hat smoothing function $W(x)$. This is plotted with $R = 1$.

We now want to find the *full width at half maximum*, FWHM, for this function. We first need to find the half maximum. From fig.1 we see that maximum is at 0. We know from (5) that here $\tilde{W}(k = 0) = 2R$, so half maximum is R. For the width we just need to find for which k we have $\tilde{W} = R$ and multiply it by 2 (this is because $\tilde{W}$ is symmetric around 0). So we first need to solve

$$R = \frac{2}{k_{\text{half max}}} \sin Rk_{\text{half max}}. \tag{6}$$

This is difficult to solve analytical, but easy numerically. From our program we easily find that

$$\text{FWHM} = 2 \cdot k_{\text{half max}} = 3.77. \tag{7}$$