AST4320 Oblig2

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1 1

1.1 a)

To find the mean molecular weight if the intergalactic medium (IGM) we use that

$$\frac{1}{\mu} = \sum_i \frac{X_i}{A_i} = 2X + 4/3 \cdot Y, \tag{1}$$

where X_i is the fraction of species i and A_i is the atomic mass of said species divided by the total number of particles it ionizes to. For the IGM we have a fraction of hydrogen $X = 0.76$ and a fraction of helium $Y = 0.24$, giving us

$$\mu = \frac{1}{2 \cdot 0.76 + 4/3 \cdot 0.24} = 0.59. \tag{2}$$

1.2 b)

The Jeans length is given as

$$\lambda_j = c_s \left(\frac{\pi}{G\rho}\right)^{1/2}, \tag{3}$$

where the speed of sound c_s for a gas is given as

$$c_s = \sqrt{\frac{k_b T}{\mu m_p}} = 11.83 \text{m/s}. \tag{4}$$

We are using that $T = 10^4$ K. We have to find the pressure of the IGM. For this we assume that the gas only have baryonic matter, so

$$\rho = \rho_b(1+z)^3 = \Omega_b \rho_c (1+z)^3, \tag{5}$$

where $\Omega_b = 0.049$ and $\rho_c = 9.47 \cdot 10^{-27}$ g/cm^3. We then see that get

$$\lambda_j(z) = 3.9 \text{ Mpc} \cdot (1+z)^{-3/2}. \tag{6}$$

This gives k

$$k = \frac{2\pi}{\lambda_j(z)} = 1.61 \text{ Mpc}^{-1}(1+z)^{3/2}. \tag{7}$$

1.3 c)

We can use Hubble's law to find the different in velocity across the Jeans length of the IGM, thereby giving us the velocity width. Through Hubble's law we get

$$\Delta v = H(z)\lambda_j, \tag{8}$$

where

$$H(z) = H_0(\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_\Lambda)^{1/2}. \tag{9}$$

This gives us that

$$\Delta v = H_0(\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_\Lambda)^{1/2} 3.9 \text{ Mpc} \cdot (1+z)^{-3/2} \tag{10}$$

$$= 264.5 \text{ km/s} \cdot (\Omega_m + \Omega_r(1+z) + \Omega_\Lambda(1+z)^{-3})^{1/2} = 264.5 \text{ km/s} \cdot (0.308 + 0.692(1+z)^{-3})^{1/2} \tag{11}$$

1.4 d)

See over this!

1.5 e)

The thermal broadening scale is given as

$$v_{th} = \sqrt{\frac{2k_bT}{m_p}}, \tag{12}$$

where $T = 10^4$ K. This gives us

$$v_{th} = 12.85 \text{ km/s}, \tag{13}$$

which is an order small that all values of z.

2 Exercise 2

We would like to find the optical depth τ_e of the IGM as a function of redshift z. The optical depth is given as

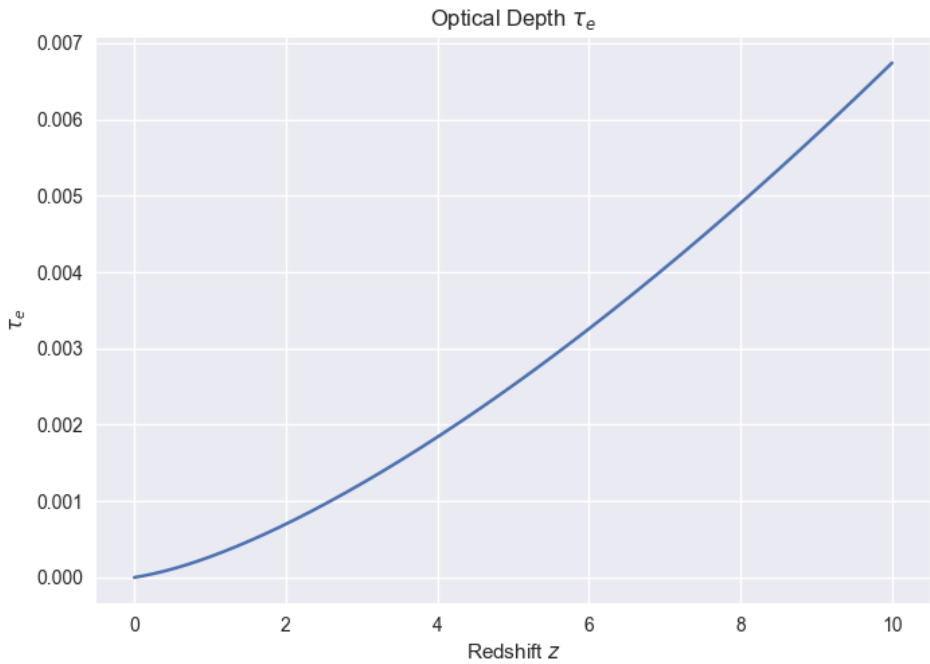
$$\tau_e(z) = c\int_0^z \frac{n_e(z)\sigma_T dz}{H(z)(1+z)}, \tag{14}$$

where σ_t is the Thompson cross section, n_e the electron density and $H(z)$ the Hubble parameter. Since the Universe is taken to be completely ionized, he assume that $n_e \approx \bar{n_H} = 1.9 \cdot 10^{-7}(1+z)^3$ cm^{-3}. We get the Hubble parameter from the Friedmann equation

$$H(z) = H_0\sqrt{\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_\Lambda}, \tag{15}$$

where $\Omega_\Lambda = 0.692$, $\Omega_m = 0.308$ and $\Omega_r = 0$.

We will integrate (14) from $z = 0$ to 10. We assume that $\tau_e(0) \approx 0$.



Figur 1: The optical depth of the intergalactic medium. We see that for larger redshift the optical depth increases.

3 Exercise 3

3.1 a)

We have the differential equation for an isothermal halo

$$-\frac{k_b T}{m_{DM} r^2}\frac{d}{dr}r^2\frac{d}{dr}\ln\rho = 4\pi G\rho. \tag{16}$$

We have an ansatz that

$$\rho(r) = \frac{A}{r^2}, \qquad A = \frac{k_b T}{2\pi G m_{DM}}. \tag{17}$$

To see that this is a solution, we put (17) into (16). Looking at the RHS of (16) we get

$$-\frac{k_b T}{m_{DM} r^2}\frac{d}{dr}r^2\frac{d}{dr}\ln\rho = -\frac{k_b T}{m_{DM} r^2}\frac{d}{dr}r^2\frac{r^2}{A}\cdot\left(-\frac{2A}{r^3}\right) \tag{18}$$

$$2\frac{k_b T}{m_{DM} r^2}\frac{d}{dr}r = 2\frac{k_b T}{m_{DM} r^2}. \tag{19}$$

Thus we get

$$2\frac{k_b T}{m_{DM} r^2} = 4\pi G\frac{A}{r^2} \Rightarrow A = \frac{k_b T}{2\pi G m_{DM}}. \tag{20}$$

Thus (17) solves (16).

3.2 b)

We have that a gas in hydrostatic equilibrium

$$\frac{dp}{dr} = -\frac{GM(<r)\rho}{r^2}, \tag{21}$$

where $M(<r)$ is the mass within a radius r and p is the pressure. We can see that our isothermal gas, with the density defined in (17), behaves in the similar way. We start by finding the mass, which for a spherical symmetric mass is given as

$$M(<r) = 4\pi \int_0^r \rho(r')r'^2 dr' = 4\pi \int_0^r \frac{A}{r'^2} r'^2 dr' = 4\pi \int_0^r A dr' = 4\pi Ar. \tag{22}$$

In out expression of A we have the dark matter mass m_{DM}. Since we now have a gas, we let $m_{MD} \to m_p$, which is the proton mass. Thus we have

$$\rho = \frac{A}{r^2} = \frac{k_b T}{2\pi G m_p r^2}. \tag{23}$$

We when use that for an isothermal gas the pressure is given as

$$p = \frac{k_b T}{m_p}\rho = \frac{k_b T A}{m_p r^2}. \tag{24}$$

We can now take the differentiation of this with respect to r and use (22) and (17) to find

$$\frac{dp}{dr} = -\frac{2k_b T A}{m_p r^3} = -\frac{2k_b T}{m_p r^3} \cdot \frac{M(<r)}{4\pi r} = -\frac{GM(<r)}{r^2}\frac{k_b T}{2\pi G m_p r^2} = -\frac{GM(<r)\rho}{r^2}. \tag{25}$$

This is the same as for the gas in hydrostatic equilibrium from (21).

4 Exercise 3

4.1 a)

We have that for cusp density profile

$$\rho^{cusp}(r) = \begin{cases} \rho_0 \left(\frac{r}{r_s}\right)^{-1} & \text{if } r < r_s \\ \rho_0 \left(\frac{r}{r_s}\right)^{-3} & \text{if } r \geq r_s \end{cases} \tag{26}$$

And for the cored profile, we have

$$\rho^{cusp}(r) = \begin{cases} \rho_0 & \text{if } r < r_s \\ \rho_0 \left(\frac{r}{r_s}\right)^{-3} & \text{if } r \geq r_s \end{cases} \tag{27}$$

The mass of a spherical symmetric system can be written as

$$M(<r) = 4\pi \int_0^r \rho(r')r'^2 dr'. \tag{28}$$

So for $r < r_s$ we get

$$M^{cusp}(<r) = 4\pi \int_0^r \rho_0 \left(\frac{r'}{r_s}\right)^{-1} r'^2 dr' = 4\pi r_s \rho_0 \int_0^r r' dr' = 2\pi r_s \rho_0 r^2, \tag{29}$$

and

$$M^{core}(<r) = 4\pi \int_0^r \rho_0 r'^2 dr' = \frac{4}{3}\pi\rho_0 r^3. \tag{30}$$

For $r \geq r_s$ we integrate from r_s to r and include all the mass that was within r_s, so

$$M^{cusp}(<r) = M(<r_s) + 4\pi \int_{r_s}^r \rho_0 \left(\frac{r'}{r_s}\right)^{-3} r'^2 dr' = M(<r_s) + 4\pi r_s^3 \rho_0 \int_0^r r'^{-1} dr' \tag{31}$$

$$= M(<r_s) + 4\pi r_s^3 \rho_0 \ln r \Big|_{r_s}^r = 2\pi r_s \rho_0 r_s^2 + 4\pi r_s^3 \rho_0 (\ln r - \ln r_s), \tag{32}$$

and similarly for the core mass

$$M^{core}(<r) = M^{core}(<r_s) + 4\pi \int_{r_s}^r \rho_0 \left(\frac{r'}{r_s}\right)^{-3} r'^2 dr' = \frac{4}{3}\pi\rho_0 r_s^3 + 4\pi r_s^3 \rho_0 (\ln r - \ln r_s). \tag{33}$$

So we get that

$$M^{cusp}(<r) = \begin{cases} 2\pi r_s \rho_0 r^2 & \text{if } r < r_s \\ 2\pi \rho_0 r_s^3 + 4\pi r_s^3 \rho_0 (\ln r - \ln r_s) & \text{if } r \geq r_s \end{cases} \tag{34}$$

and

$$M^{core}(<r) = \begin{cases} \frac{4}{3}\pi\rho_0 r^3 & \text{if } r < r_s \\ \frac{4}{3}\pi\rho_0 r_s^3 + 4\pi r_s^3 \rho_0 (\ln r - \ln r_s) & \text{if } r \geq r_s \end{cases} \tag{35}$$

4.2 b)

We get the gravitation potential energy by integrating over the energies felt by a small shall of mass from the mass inside it. First we get the energy of one such shell

$$dW(r) = -\frac{GM(<r)m_{shell}}{r_{shell}} = -4\pi GM(<r)\frac{\rho r_{shell}^2 dr}{r_{shell}} = -4\pi GM(<r)\rho r_{shell} dr, \tag{36}$$

where we have used that

$$m_{shell} = \rho V_{shell} = 4\pi\rho r_{shell}^2 dr. \tag{37}$$

Thus the gravitational potential energy is

$$W = \int_0^{r_{vir}} dW = -4\pi G \int_0^{r_{vir}} M(<r)\rho(r) r dr. \tag{38}$$

4.3 c)

We can now find the minimal energy needed to form the cored profile, given as

$$\Delta E = \frac{W^{core} - W^{cusp}}{2} = -2\pi G \int_0^{r_{vir}} \left(\rho^{core} M^{core}(<r) - \rho^{cusp} M^{cusp}(<r)\right) r dr \tag{39}$$

We split this into two parts. First we look at $r = 0$ to r_s

$$-2\pi G \int_0^{r_s} \left(\rho^{core} M^{core}(<r) - \rho^{cusp} M^{cusp}(<r)\right) r dr = -2\pi G \int_0^{r_s} \left(\rho_0 \frac{4}{3}\pi\rho_0 r^3 - \rho_0 \left(\frac{r}{r_s}\right)^{-1} 2\pi r_s \rho_0 r^2\right) r dr \tag{40}$$

$$= -2\pi^2 G\rho_0^2 \int_0^{r_s} (\frac{4}{3}r^4 - 2r^2 r_s^2) dr = -2\pi^2 G\rho_0^2 (\frac{4}{15}r_s^5 - \frac{2}{3}r_s^5) = \frac{4}{5}\pi^2 G\rho_0^2 r_s^5. \tag{41}$$

We then look from r_s to r_{vir}

$$-2\pi G\int_{r_s}^{r_{vir}}\left(\rho^{core}M^{core}(<r)-\rho^{cusp}M^{cusp}(<r)\right)rdr \tag{42}$$

$$=-2\pi G\int_{r_s}^{r_{vir}}\left(\rho_0\left(\frac{r}{r_s}\right)^{-3}\left(\frac{4}{3}\pi\rho_0 r_s^3+4\pi r_s^3\rho_0(\ln r-\ln r_s)\right)-\rho_0\left(\frac{r}{r_s}\right)^{-3}\left(2\pi\rho_0 r_s^3+4\pi r_s^3\rho_0(\ln r-\ln r_s)\right)\right)rdr \tag{43}$$

$$=-2\pi G\int_{r_s}^{r_{vir}}\left(\rho_0\left(\frac{r}{r_s}\right)^{-3}\left(\frac{4}{3}\pi\rho_0 r_s^3\right)-\rho_0\left(\frac{r}{r_s}\right)^{-3}\left(2\pi\rho_0 r_s^3\right)\right)rdr \tag{44}$$

$$=2\pi^2 G\rho_0^2 r_s^6\int_{r_s}^{r_{vir}}\frac{1}{r^2}\frac{2}{3}dr=\frac{4}{3}\pi^2 G\rho_0^2 r_s^6\frac{-1}{1}\left(r_{vir}^{-1}-r_s^{-1}\right)\approx\frac{4}{3}\pi^2 G\rho_0^2 r_s^5, \tag{45}$$

where the last step is because $r_s << r_{vir}$. We then get

$$\Delta E=\frac{4}{5}\pi^2\rho_0^2 G r_s^5+\frac{4}{3}\pi^2 G\rho_0^2 r_s^5=\frac{32}{15}\pi^2 G\rho_0^2 r_s^5. \tag{46}$$

4.4 d)

We are now looking at a dwarf galaxy at $z=0$ with $M_{vir}=3\cdot 10^{10}M_\odot$ and $R_{vir}=45$ kpc, with $r_s=1$ kpc. With this mass and radius, we can find

$$\rho_0=\frac{M_{vir}}{4/3r_s^3\pi+4\pi r_s^3(\ln R_{vir}-\ln r_s)}=6.13\cdot 10^{-23}\text{g cm}^{-3}. \tag{47}$$

This gives us that

$$\Delta E=\frac{32}{15}\pi^2 G\rho_0^2 r_s^5=1.477\cdot 10^{57}\text{ergs}. \tag{48}$$

Check this!!!

4.5 e)