AST4320 Assignment 1

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1 Exercise 1

1.1 a)

We have the continuity equation

$$\frac{d\rho_0}{dt} + \rho_0 \nabla \cdot \mathbf{v} = \frac{\partial \rho_0}{\partial t} + \mathbf{v} \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \mathbf{v} = 0. \tag{1}$$

Since we have homogeneous universe model, we see that $\nabla \rho \approx 0$, so we get

$$\frac{\partial \rho_0}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} = 0. \tag{2}$$

We are assuming a universe that follows Hubble's law $\mathbf{v} = H\mathbf{r} = \frac{\dot{a}}{a}\mathbf{r}$. Since we are in a spherical universe, with a velocity that only depends on the radial direction, we get a divergence of the following form

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial (r^2 \cdot v_r)}{\partial r} = \frac{1}{r^2}\frac{\partial (r^3)}{\partial r} = 3, \tag{3}$$

which gives us the following for the continuity equation

$$\frac{\partial \rho_0}{\partial t} + 3\frac{\dot{a}}{a}\rho_0 = 0 \Leftrightarrow \frac{d\rho_0}{\rho_0} = -3\frac{da}{a}. \tag{4}$$

Integrating on both sides from $\rho_0(t = t_0) \rightarrow \rho(t)$ and $a_0 = 1 \rightarrow a$ we get

$$\ln \frac{\rho_0(t)}{\rho_0(t = t_0)} = \ln a^{-3} \Leftrightarrow \rho_0(t) = \rho_0(t = t_0) a^{-3}. \tag{5}$$

1.2 b)

We have the Euler and Poisson equation with the permutations

$$\rho = \bar{\rho} + \delta\rho, \quad \phi = \phi_0 + \delta\phi, \quad v = v_0 + \delta v, \quad p = p_0 + \delta p, \quad \delta = \frac{\delta\rho}{\bar{\rho}}. \tag{6}$$

Note that the velocities are vector quantities, but for brevity I have omitted writing them as such here.

1.2.1 i)

We start with the Euler equation

$$\frac{dv}{dt} = -\frac{1}{\rho}\nabla p - \nabla\phi. \tag{7}$$

We start with the LHS:

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + v \cdot \nabla v = \frac{\partial v_0 + \delta v}{\partial t} + (v_0 + \delta v) \cdot \nabla (v_0 + \delta v) \tag{8}$$

$$= \frac{\partial v_0}{\partial t} + \frac{\partial \delta v}{\partial t} + v_0 \cdot \nabla v_0 + \delta v \cdot \nabla v_0 + v_0 \cdot \nabla \delta v + \delta v \cdot \nabla \delta v \tag{9}$$

$$= \frac{d\delta v}{dt} + \frac{dv_0}{dt} + \delta v \cdot \nabla v. \tag{10}$$

In the last step we use that the product of two infinitesimals are more or less zero, and the definition of total derivatives. We can then move on to the RHS:

$$-\frac{1}{\rho}\nabla p-\nabla\phi=-\frac{1}{\bar{\rho}+\delta\rho}\nabla(p_0+\delta p)-\nabla(\phi_0+\delta\phi) \tag{11}$$

$$=-\frac{1}{\bar{\rho}}\frac{1}{1+\frac{\delta\rho}{\rho_0}}\left(\nabla p_0+\nabla\delta p\right)-\nabla\phi_0-\nabla\delta\phi \tag{12}$$

$$=-\frac{1}{\bar{\rho}}\nabla p_0-\frac{1}{\bar{\rho}}\nabla\delta p-\nabla\phi_0-\nabla\delta\phi. \tag{13}$$

In the last step we used that $\delta\rho/\rho_0 << 1$ so that $1/(1+\delta\rho/\rho_0)\approx 1$. Putting all this together, we get

$$\frac{d\delta v}{dt}+\frac{dv_0}{dt}+\delta v\cdot\nabla v.=-\frac{1}{\bar{\rho}}\nabla p_0-\frac{1}{\bar{\rho}}\nabla\delta p-\nabla\phi_0-\nabla\delta\phi. \tag{14}$$

Knowing that v_0, p_0, $\bar{\rho}$ and ϕ_0 have to follow the Euler equation, we can remove half of the term, leaving us with

$$\frac{d\delta v}{dt}+\delta v\cdot\nabla v.=-\frac{1}{\bar{\rho}}\nabla\delta p-\nabla\delta\phi. \tag{15}$$

1.2.2 ii)

We can now look at the Poisson equation:

$$\nabla^2\phi=4\pi G\rho. \tag{16}$$

Inserting the perturbations we get

$$\nabla^2(\phi_0+\delta\phi_0)=4\pi G(\bar{\rho}+\delta\rho) \tag{17}$$

$$\Rightarrow\nabla^2\phi_0+\nabla^2\delta\phi_0=4\pi G\bar{\rho}+4\pi G\delta\rho. \tag{18}$$

And knowing that ϕ_0 and $\bar{\rho}$ have to follow the Poisson equation and can be removed, we get

$$\nabla^2\delta\phi=4\pi G\delta\rho. \tag{19}$$

2 Exercise 2

2.1 a)

We start with the Friedman equation

$$\dot{a}^2=\frac{8\pi G}{3}\rho\cdot a^2-kc^2. \tag{20}$$

We assume a flat universe, so $k=0$. We can then rewrite to get

$$H^2=\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi G}{3}\rho=\frac{8\pi G}{3}\left(\rho_m+\rho_\Lambda\right). \tag{21}$$

We are only looking at universes with matter and a cosmological constants, so the density of the universe is $\rho_m+\rho_\Lambda$. We know from the continuity equation that the densities evolves as follows:

$$\rho=\rho_0 a^{3(1+\omega)}. \tag{22}$$

This gives us that

$$\rho_m=\rho_{0,m}a^{-3},\quad \rho_\Lambda=\rho_{0,\Lambda}. \tag{23}$$

Introducing the critical density

$$\rho_{crit} = \frac{3H_0^2}{8\pi G}, \tag{24}$$

we can rewrite the densities as

$$\rho = \rho_{crit}\Omega a^{3(1+\omega)}. \tag{25}$$

So we then get

$$H^2 = \frac{8\pi G}{3} \cdot \rho_{crit}(\Omega_m a^{-3} + \Omega_\Lambda) = \frac{8\pi G}{3} \cdot \frac{3H_0^2}{8\pi G}(\Omega_m a^{-3} + \Omega_\Lambda). \tag{26}$$

Thus we get our final form

$$H^2 = H_0^2(\Omega_m a^{-3} + \Omega_\Lambda). \tag{27}$$

We now have all we actually need to solve the differential equation for the density contrast. But to be thorough I will find $H = \dot{a}/a$ as a function of time.

We start with the expression when $(\Omega_m, \Omega_\Lambda) = (1, 0)$. We know from the lecture notes that this gives

$$H = 2/3t. \tag{28}$$

For the next two combinations, we need to solve the differential equation

$$H^2 = H_0^2 \left[\Omega_m a^{-3} + \Omega_\Lambda\right]. \tag{29}$$

Knowing that $H = \dot{a}/a$ we get

$$\dot{a} = H_0\sqrt{(\Omega_m a^{-3} + \Omega_\Lambda)}a. \tag{30}$$

$$\Rightarrow \frac{da}{a\sqrt{\Omega_m a^{-3} + \Omega_\Lambda}} = H_0 dt. \tag{31}$$

Integrating from $t = 0$ to $t = t_0$, and $a = 0$ to $a = a_0$,we get

$$H_0 t = \int_0^{a_0} \frac{da}{a\sqrt{\Omega_m a^{-3} + \Omega_\Lambda}} \tag{32}$$

We can either solve this integral numerically or analytically. We can thankfully find the answer from the notes of Cosmology 1. The solution to this integral is

$$a(t) = a_0 \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \left[\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right]^{2/3}, \tag{33}$$

which gives us

$$\frac{\dot{a}}{a} = \sqrt{\Omega_\Lambda}H_0 t \cosh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\left[\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right]^{-1/3}. \tag{34}$$

We can just use $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7)$ and $(\Omega_m, \Omega_\Lambda) = (0.8, 0.2)$ as input here and get the appropriate $H = \dot{a}/a$.

2.2 b)

We have to solve

$$\frac{d^2\delta}{dt^2} + 2H\frac{d\delta}{dt} = 4\pi G\rho\delta, \tag{35}$$

or in a simpler notation

$$\delta'' + 2H\delta' = \delta 4\pi G\rho\delta = \frac{3}{2}\delta H^2, \tag{36}$$

where the last term comes from the fact that $\rho = 3H^2/8\pi G$. To solve this we need to transform it to a system of first order differential equation. We thus introduce

$$x = \delta', \tag{37}$$

which gives us

$$x' = \frac{3}{2}\delta H^2 - 2Hx. \tag{38}$$

Using a simple *Euler-Chromer* scheme, we can solve this iterative as

$$x_{i+1} = x_i + \left(\frac{3}{2}\delta H^2 - 2Hx\right) \cdot dt \tag{39}$$

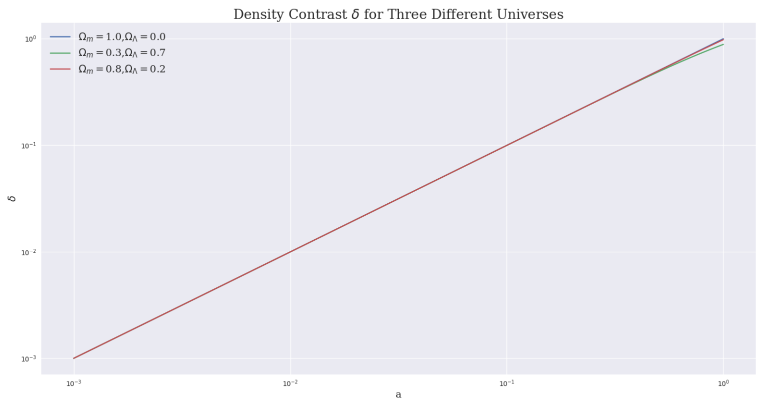
$$\delta_{i+1} = \delta_i + \delta' \cdot dt = \delta_i + x_{i+1} \cdot dt. \tag{40}$$

We also want to change the integration variable to a. We can do this by observing that

$$dt = \frac{da}{\dot{a}}. \tag{41}$$

Using this we can get an integration only dependent on a. The only quantities we need are H and $\dot{a}$. These are simply calculated from (29) and (30), which are dependent only on a as well.

We are given boundary conditions for δ and a, but need one for $x = \delta'$. We are given that $\delta \propto a$ in the early universe, so we simply use $\delta' = \dot{a}$ as the initial condition.



Figur 1: The evolution of the density contrast for three different universe models with only matter and a cosmological constant.

We see from 1 that for $(1, 0)$ we have a linear function, which makes sense since we know that in this universe $\delta \propto a$. For the other two universes we see that $\delta \propto a$ for most of the time. This is due to matter being dominant in the early universe. But later the the cosmological constant takes over, and we get a more rapid expansion. Due to the rapid expansion structures will have a more difficult time forming, and will more likely be torn apart. We therefore see that δ decreases. The larger Ω_Λ is the more expansion there will be, and δ will decrease faster.

2.3 c)

We now want to find the grown factor

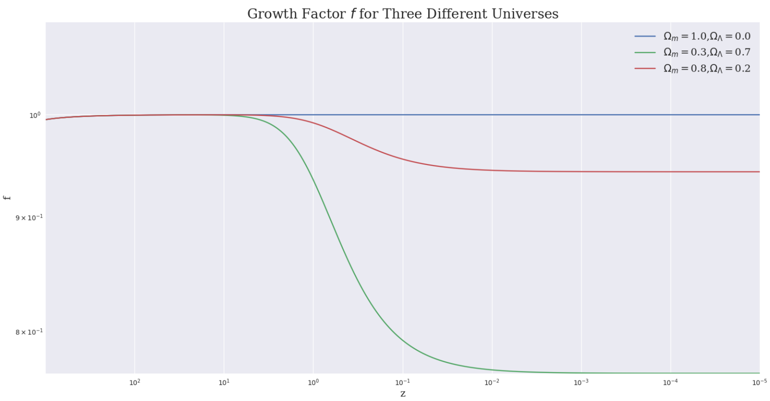
$$f = \frac{d\ln\delta}{d\ln a}. \tag{42}$$

Since we have δ and a, we can approximate this numerically as

$$f_{i+1} \approx \frac{\Delta \ln \delta}{\Delta \ln a} = \frac{\ln \delta_{i+1} - \ln \delta_i}{\ln a_{i+1} - \ln a_i} = \frac{\ln \frac{\delta_{i+1}}{\delta_i}}{\ln \frac{a_{i+1}}{a_i}}. \quad (43)$$

This is straight forward to calculate. We can then plot it against the redshift

$$z = \frac{1}{a} - 1. \quad (44)$$



Figur 2: The evolution of the growth factor for three different universe models with only matter and a cosmological constant.

As we see from 2 the growth factor for the $\Omega_m = 1$ universe is more or less flat at $f = 1$. The two other models again follows this at the early stages, but decreases for lower z, with the model with highest Ω_Λ decreasing most.

3 Exercise 3

3.1 a)

We start with the temperature for the gas. Since we have a adiabatic expansion, we start with an adiabatic expression for the temperature and volume of a gas

$$TV^{\gamma-1} = C, \quad (45)$$

where T is the temperature, V the volume, C a constant and γ is the adiabatic index, given for a monoatomic gas as $\gamma = 5/3$. We also know that the volume has to be $V \propto a^3$, so

$$T_{gas} = \frac{C'}{(a^3)^{5/3-1}} = C' a^{-2}. \quad (46)$$

(C' is some constant).

For the gas we start with Wien's displacement law for a black body

$$T = \frac{b}{\lambda}, \quad (47)$$

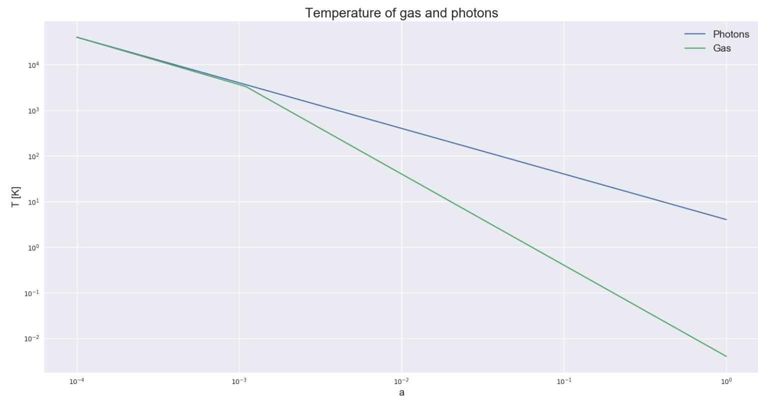
where λ is the wavelength and b a constant (it's value is known, but uninteresting here). We know that $\lambda \propto a$ so

$$T_\gamma = Ba^{-1}. \quad (48)$$

We know that at the point of decoupling, the temperature must have been the same. This happen at $z = 1100 \Rightarrow a \approx 10^{-3}$ with the temperature $T = 4000K$[1]. Using this we find the constants for the temperatures, and find the temperatures evolves like

$$T_{gas} = 4 \cdot 10^{-3} a^{-2}, \quad T_\gamma = 4a^{-1}. \tag{49}$$

The constants are not exact, but of the same order.



Figur 3: The evolution of the temperature of the gas and photons found in the universe.

We see from 3 that the temperatures were the same before decoupling[2]. After decoupling the temperature of the gas decrease much faster due to its dependence on a^{-2}, with the photons decreasing less to their dependence on a^{-1}.

3.2 b)

We now want to look at the Jeans length and mass. They are generally given as

$$\lambda_J = c_s \sqrt{\frac{\pi}{G\rho}}, \quad M_J = \frac{\pi^{5/2}}{6G^{3/2}\rho^{1/2}} \cdot c_s^3. \tag{50}$$

The difference in these quantities before and after decoupling comes from the sound speed c_s. Before decoupling the gas is coupled with the photons, thus the sound speed is just $c/\sqrt{3}$, which is constant. For a gas we can find the sound speed as a function of redshift

$$c_s = \sqrt{\frac{k_b T}{\mu m_p}} = \sqrt{\frac{k_b}{\mu m_p}} T^{1/2} = \sqrt{\frac{k_b}{\mu m_p}} \left(4 \cdot 10^{-3} a^{-2}\right)^{1/2} = 4 \cdot 10^{-3/2} (1+z) \sqrt{\frac{k_b}{\mu m_p}}, \tag{51}$$

where k_b is Boltzmann's constant, m_p is the proton mass and $a = 1/(1+z)$. We are going to use that $\rho = \rho_0 a^{-3}$, we thus get that

$$\rho = \rho_0 (1+z)^3. \tag{52}$$

We can then write the Jeans mass and length as functions of redshift

$$\lambda_{J,after} = 4 \cdot 10^{-3/2} (1+z)(1+z)^{-3/2} \sqrt{\frac{k_b}{\mu m_p}} \sqrt{\frac{\pi}{G\rho_0^{1/2}}} = 4 \cdot 10^{-3/2} (1+z)^{-1/2} \sqrt{\frac{k_b}{\mu m_p}} \sqrt{\frac{\pi}{G\rho_0^{1/2}}} \tag{53}$$

[1] I've used this temperature here, which seems that this is for the start of decoupling. 3000 K is also used alot, since this is the temperature at the end.

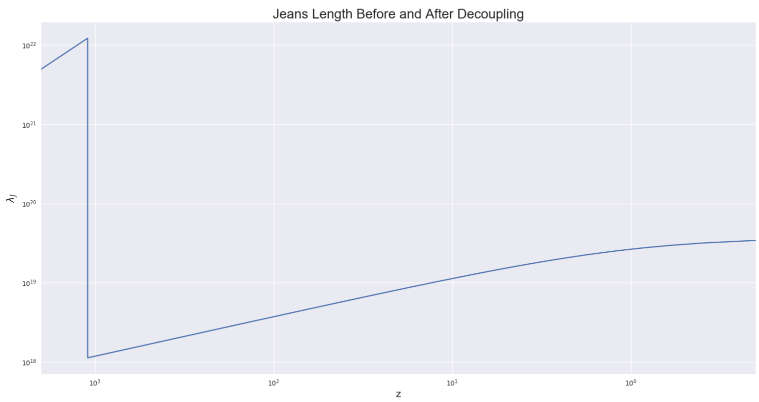
[2] Hard coded to be that way...

$$M_{J,after} = \frac{\pi^{5/2}}{6G^{3/2}\rho_0^{1/2}} \cdot \left(4 \cdot 10^{-3/2}(1+z)\right)^3 \cdot (1+z)^{-3/2} = 12 \cdot 10^{-1/2} \frac{\pi^{5/2}}{6G^{3/2}\rho_0^{1/2}}(1+z)^{3/2}. \quad (54)$$

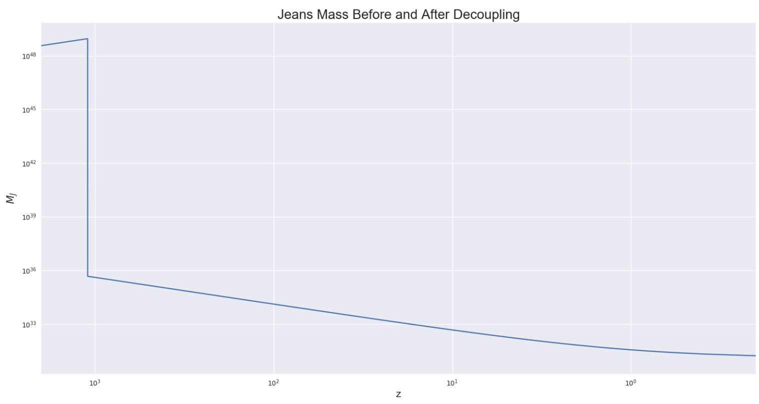
This is after the decoupling. Before decoupling the only dependence on redshift comes from the density, so we get

$$\lambda_{J,before} = \frac{c}{\sqrt{3}}\sqrt{\frac{\pi}{G\rho_0}}(1+z)^{-3/2} \quad (55)$$

$$M_{J,before} = \frac{\pi^{5/2}}{6G^{3/2}\rho_0^{1/2}} \cdot \left(\frac{c}{\sqrt{3}}\right)^3 \cdot (1+z)^{-3/2}. \quad (56)$$



Figur 4: The Jeans Length before and after the decoupling is mark by a significant drop before it starts increasing.



Figur 5: The Jeans mass before and after the decoupling is mark by a significant drop before it starts decreasing.

We can see from 4 and 5 that for both the Jeans length and mass there is a significant drop at recombination. This is because the speed of sound goes from that of a photon gas to that of a normal (ideal) gas. The length and mass then evolves with z (almost inverse of one another), with the length increasing and mass decreasing.

4 Exercise 4

We have the differential equation

$$\ddot{R} = -\frac{GM}{R^2}, \tag{57}$$

and the parametric solutions

$$R = A(1-\cos\theta), \quad t = B(\theta - \sin\theta), \quad A^3 = GMB^2. \tag{58}$$

We start with eq. (57), and multiply both sides with dR/dt

$$\frac{dR}{dt}\cdot\frac{d^2R}{dt^2} = -\frac{dR}{dt}\frac{GM}{R^2} \Leftrightarrow \frac{1}{2}\frac{d}{dt}\left(\frac{dR}{dt}\right) = \frac{d}{dt}\left(\frac{GM}{R}\right). \tag{59}$$

Integrating on both sides gives us

$$\frac{1}{2}\left(\frac{dR}{dt}\right)^2 - \frac{GM}{R} = E, \tag{60}$$

where E is a constant (total energy). We are going to assume that R and t from (58) are partially solutions to this differential equation. If this is correct, we should retrieve the relation between A and B. Let us start be noting that

$$\frac{dR}{dt} = \frac{dR}{d\theta}\frac{d\theta}{dt} = \frac{dR}{d\theta}\left(\frac{dt}{d\theta}\right)^{-1} = \frac{A\sin\theta}{B(1-\cos\theta)}. \tag{61}$$

We then see that (60) becomes

$$\frac{1}{2}\frac{A^2}{B^2}\frac{\sin^2\theta}{(1-\cos\theta)^2} - \frac{GM}{A(1-\cos\theta)} = E. \tag{62}$$

E is constant, so it has the same value for all values of θ. So to find its value, we choose to check it at $\theta = \pi$, with gives $\sin\theta = 0$ and $\cos\theta = -1$

$$E = 0 - \frac{GM}{A(1+1)} = -\frac{GM}{2A}. \tag{63}$$

We can now input this into eq. (60) to get

$$\frac{1}{2}\frac{A^2}{B^2}\frac{\sin^2\theta}{(1-\cos\theta)^2} - \frac{GM}{A(1-\cos\theta)} = -\frac{GM}{2A} \tag{64}$$

$$\Rightarrow \frac{1}{2}\frac{A^2}{B^2}\frac{\sin^2\theta}{(1-\cos\theta)^2} = -\frac{GM}{2A}\left(1 - \frac{2}{1-\cos\theta)}\right). \tag{65}$$

Then note that

$$\frac{\sin^2\theta}{(1-\cos\theta)^2} = -\left(1 - \frac{2}{1-\cos\theta)}\right) = \cot^2\left(\frac{\theta}{2}\right), \tag{66}$$

which gives us

$$\frac{A^2}{B^2} = \frac{GM}{A} \Leftrightarrow A^3 = GMB^2. \tag{67}$$

This retrieves the relation from eq. (58). This means that the parameterized equations are solutions to (60) which again means they are solutions to (57).

5 Exercise 5

From the definition of v and what we calculated in (61), we see that

$$v = \frac{dR}{dt} = \frac{A\sin\theta}{B(1-\cos\theta)} \Rightarrow v^2 = \frac{A^2}{B^2}\frac{\sin^2\theta}{(1-\cos\theta)^2}. \tag{68}$$

The last step is just to make it easier later. We want to find to find the velocity at the virial radius, which we know is $R_{vir} = 0.5R_{max}$. For the parameterized solution, we know that $R_{max} = 2A$, which gives $R_{vir} = A$. From the definition of R we get that

$$R = A(1-\cos\theta) = A \Rightarrow \theta = \pi \text{ or } \frac{3\pi}{2}. \tag{69}$$

Since the virialization don't happen before R_{max} we must have that

$$\theta_{vir} = \frac{3\pi}{2}. \tag{70}$$

This gives us $\sin^2\theta = 1$, so we get

$$v^2 = \frac{A^2}{B^2} = \frac{A^2}{\frac{A^3}{GM}} = \frac{GM}{A}. \tag{71}$$

We know that $R_{vir} = A$, so we get out answer

$$v_{infall} = \sqrt{\frac{GM}{R_{vir}}} \tag{72}$$

6 Exercise 6

The gravitational binding energy is found in the following way[3]:

We can imagine having a mass – a sphere – consisting of many infinitesimal layers of thickness dr. The gravitational potential between a given shell and the mass found inside the shell is then given as

$$dU = -G\frac{m_{shell}m}{r}, \tag{73}$$

where m_{shell} is the mass of the shell, while m is the mass if the matter inside the shell. m is just given by the radius from the center to the shell r and the density ρ of the matter

$$m = \frac{4}{3}\pi r^3\rho. \tag{74}$$

The shell itself is of infinitesimal size, and the mass can therefore be given as the density times the area of the shell, times the infinitesimal thickness of the shell

$$m_{shell} = 4\pi r^2\rho dr. \tag{75}$$

This gives us

$$dU = -G\frac{\frac{4}{3}\pi r^3\rho \cdot 4\pi r^2\rho dr}{r} = -G\frac{16}{3}\pi^2\rho^2 r^4 dr. \tag{76}$$

Now we have the potential energy of one such shell, so to get the total gravitational binding energy we need to integrate over all the shell, from the center of the sphere to the total radius $r = R$

[3] To not be called out on plagiarism, this derivation is **heavily** inspired by `https://en.wikipedia.org/wiki/Gravitational_binding_energy` and `http://scienceworld.wolfram.com/physics/SphereGravitationalPotentialEnergy.html`

$$U = -\frac{16}{3} G\pi^2\rho^2 \int_0^R r^4 dr = -\frac{16}{15} G\pi^2\rho^2 R^5. \tag{77}$$

For a sphere with radius R, the density ρ is simply given as

$$\rho = \frac{M}{4/3\pi R^3}. \tag{78}$$

This gives us the binding energy

$$U = -\frac{3GM^2}{5R}. \tag{79}$$