AST4320 Oblig2

Daniel Heinesen, daniehei

27. oktober 2019

1 Exercise 2

We would like to find the optical depth τ_e of the IGM as a function of redshift z. The optical depth is given as

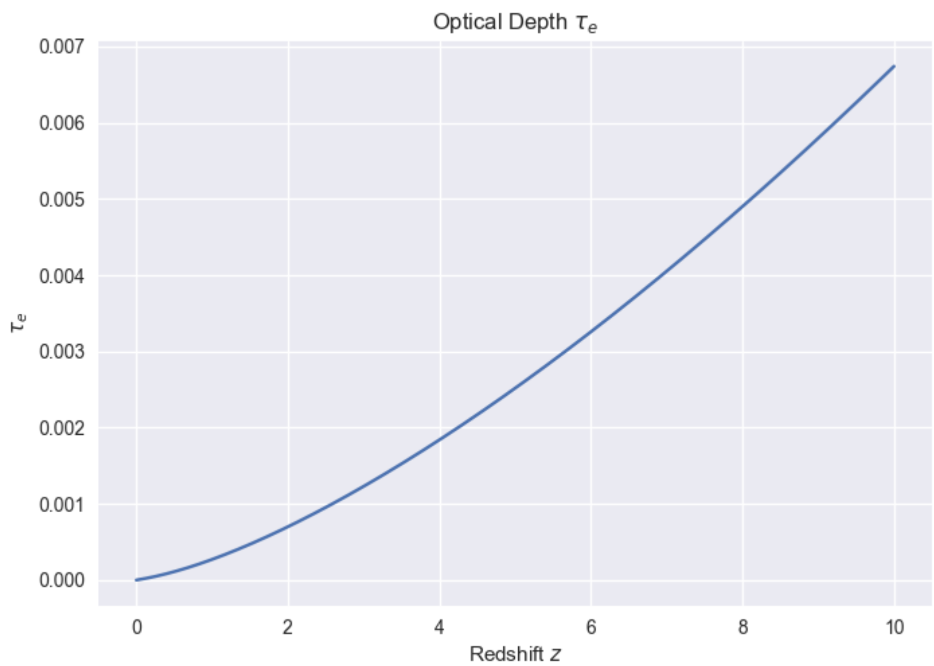
$$\tau_e(z) = c\int_0^z \frac{n_e(z)\sigma_T dz}{H(z)(1+z)}, \tag{1}$$

where σ_t is the Thompson cross section, n_e the electron density and $H(z)$ the Hubble parameter. Since the Universe is taken to be completely ionized, he assume that $n_e \approx \bar{n_H} = 1.9 \cdot 10^{-7}(1+z)^3$ cm^{-3}. We get the Hubble parameter from the Friedmann equation

$$H(z) = H_0\sqrt{\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_\Lambda}, \tag{2}$$

where $\Omega_\Lambda = 0.692$, $\Omega_m = 0.308$ and $\Omega_r = 0$.

We will integrate (1) from $z = 0$ to 10. We assume that $\tau_e(0) \approx 0$.



Figur 1: The optical depth of the intergalactic medium. We see that for larger redshift the optical depth increases.

2 Exercise 3

2.1 a)

We have the differential equation for an isothermal halo

$$-\frac{k_b T}{m_{DM} r^2}\frac{d}{dr}r^2\frac{d}{dr}\ln\rho = 4\pi G\rho. \tag{3}$$

We have an ansatz that

$$\rho(r) = \frac{A}{r^2}, \qquad A = \frac{k_b T}{2\pi G m_{DM}}. \tag{4}$$

To see that this is a solution, we put (4) into (3). Looking at the RHS of (3) we get

$$-\frac{k_b T}{m_{DM} r^2} \frac{d}{dr} r^2 \frac{d}{dr} \ln \rho = -\frac{k_b T}{m_{DM} r^2} \frac{d}{dr} r^2 \frac{r^2}{A} \cdot \left(-\frac{2A}{r^3}\right) \tag{5}$$

$$2\frac{k_b T}{m_{DM} r^2} \frac{d}{dr} r = 2\frac{k_b T}{m_{DM} r^2}. \tag{6}$$

Thus we get

$$2\frac{k_b T}{m_{DM} r^2} = 4\pi G \frac{A}{r^2} \Rightarrow A = \frac{k_b T}{2\pi G m_{DM}}. \tag{7}$$

Thus (4) solves (3).

2.2 b)

We have that a gas in hydrostatic equilibrium

$$\frac{dp}{dr} = -\frac{GM(< r)\rho}{r^2}, \tag{8}$$

where $M(< r)$ is the mass within a radius r and p is the pressure. We can see that our isothermal gas, with the density defined in (4), behaves in the similar way. We start by finding the mass, which for a spherical symmetric mass is given as

$$M(< r) = 4\pi \int_0^r \rho(r') r'^2 dr' = 4\pi \int_0^r \frac{A}{r'^2} r'^2 dr' = 4\pi \int_0^r A dr' = 4\pi A r. \tag{9}$$

In out expression of A we have the dark matter mass m_{DM}. Since we now have a gas, we let $m_{MD} \to m_p$, which is the proton mass. Thus we have

$$\rho = \frac{A}{r^2} = \frac{k_b T}{2\pi G m_p r^2}. \tag{10}$$

We when use that for an isothermal gas the pressure is given as

$$p = \frac{k_b T}{m_p} \rho = \frac{k_b T A}{m_p r^2}. \tag{11}$$

We can now take the differentiation of this with respect to r and use (9) and (4) to find

$$\frac{dp}{dr} = -\frac{2k_b T A}{m_p r^3} = -\frac{2k_b T}{m_p r^3} \cdot \frac{M(< r)}{4\pi r} = -\frac{GM(< r)}{r^2} \frac{k_b T}{2\pi G m_p r^2} = -\frac{GM(< r)\rho}{r^2}. \tag{12}$$

This is the same as for the gas in hydrostatic equilibrium from (8).

3 Exercise 3

3.1 a)

We have that for cusp density profile

$$\rho^{cusp}(r) = \begin{cases} \rho_0 \left(\dfrac{r}{r_s}\right)^{-1} & \text{if } r < r_s \\ \rho_0 \left(\dfrac{r}{r_s}\right)^{-3} & \text{if } r \geq r_s \end{cases} \tag{13}$$

And for the cored profile, we have

$$\rho^{cusp}(r) = \begin{cases} \rho_0 & \text{if } r < r_s \\ \rho_0 \left(\frac{r}{r_s}\right)^{-3} & \text{if } r \geq r_s \end{cases} \tag{14}$$

The mass of a spherical symmetric system can be written as

$$M(< r) = 4\pi \int_0^r \rho(r')r'^2 dr'. \tag{15}$$

So for $r < r_s$ we get

$$M^{cusp}(< r) = 4\pi \int_0^r \rho_0 \left(\frac{r'}{r_s}\right)^{-1} r'^2 dr' = 4\pi r_s \rho_0 \int_0^r r' dr' = 2\pi r_s \rho_0 r^2, \tag{16}$$

and

$$M^{core}(< r) = 4\pi \int_0^r \rho_0 r'^2 dr' = \frac{4}{3}\pi \rho_0 r^3. \tag{17}$$

For $r \geq r_s$ we integrate from r_s to r and include all the mass that was within r_s, so

$$M^{cusp}(< r) = M(< r_s) + 4\pi \int_{r_s}^r \rho_0 \left(\frac{r'}{r_s}\right)^{-3} r'^2 dr' = M(< r_s) + 4\pi r_s^3 \rho_0 \int_0^r r'^{-1} dr' \tag{18}$$

$$= M(< r_s) + 4\pi r_s^3 \rho_0 \ln r\Big|_{r_s}^r = 2\pi r_s \rho_0 r_s^2 + 4\pi r_s^3 \rho_0 (\ln r - \ln r_s), \tag{19}$$

and similarly for the core mass

$$M^{core}(< r) = M^{core}(< r_s) + 4\pi \int_{r_s}^r \rho_0 \left(\frac{r'}{r_s}\right)^{-3} r'^2 dr' = \frac{4}{3}\pi \rho_0 r_s^3 + 4\pi r_s^3 \rho_0 (\ln r - \ln r_s). \tag{20}$$

So we get that

$$M^{cusp}(< r) = \begin{cases} 2\pi r_s \rho_0 r^2 & \text{if } r < r_s \\ 2\pi \rho_0 r_s^3 + 4\pi r_s^3 \rho_0 (\ln r - \ln r_s) & \text{if } r \geq r_s \end{cases} \tag{21}$$

and

$$M^{core}(< r) = \begin{cases} \frac{4}{3}\pi \rho_0 r^3 & \text{if } r < r_s \\ \frac{4}{3}\pi \rho_0 r_s^3 + 4\pi r_s^3 \rho_0 (\ln r - \ln r_s) & \text{if } r \geq r_s \end{cases} \tag{22}$$

3.2 b)

We get the gravitation potential energy by integrating over the energies felt by a small shall of mass from the mass inside it. First we get the energy of one such shell

$$dW(r) = -\frac{GM(< r)m_{shell}}{r_{shell}} = -4\pi GM(< r)\frac{\rho r_{shell}^2 dr}{r_{shell}} = -4\pi GM(< r)\rho r_{shell} dr, \tag{23}$$

where we have used that

$$m_{shell} = \rho V_{shell} = 4\pi \rho r_{shell}^2 dr. \tag{24}$$

Thus the gravitational potential energy is

$$W = \int_0^{r_{vir}} dW = -4\pi G \int_0^{r_{vir}} M(< r)\rho(r) r dr. \tag{25}$$

3.3 c)

We can now find the minimal energy needed to form the cored profile, given as

$$\Delta E = \frac{W^{core} - W^{cusp}}{2} = -2\pi G \int_0^{r_{vir}} \left(\rho^{core} M^{core}(< r) - \rho^{cusp} M^{cusp}(< r)\right) r dr \tag{26}$$

We split this into two parts. First we look at $r = 0$ to r_s

$$-2\pi G \int_0^{r_s} \left(\rho^{core} M^{core}(< r) - \rho^{cusp} M^{cusp}(< r)\right) r dr = -2\pi G \int_0^{r_s} \left(\rho_0 \frac{4}{3}\pi \rho_0 r^3 - \rho_0 \left(\frac{r}{r_s}\right)^{-1} 2\pi r_s \rho_0 r^2\right) r dr \tag{27}$$

$$= -2\pi^2 G \rho_0^2 \int_0^{r_s} (\frac{4}{3} r^4 - 2r^2 r_s^2) dr = -2\pi^2 G \rho_0^2 (\frac{4}{15} r_s^5 - \frac{2}{3} r_s^5) = \frac{4}{5}\pi^2 G \rho_0^2 r_s^5. \tag{28}$$

We then look from r_s to r_{vir}

$$-2\pi G \int_{r_s}^{r_{vir}} \left(\rho^{core} M^{core}(< r) - \rho^{cusp} M^{cusp}(< r)\right) r dr \tag{29}$$

$$= -2\pi G \int_{r_s}^{r_{vir}} \left(\rho_0 \left(\frac{r}{r_s}\right)^{-3} \left(\frac{4}{3}\pi \rho_0 r_s^3 + 4\pi r_s^3 \rho_0 (\ln r - \ln r_s)\right) - \rho_0 \left(\frac{r}{r_s}\right)^{-3} \left(2\pi \rho_0 r_s^3 + 4\pi r_s^3 \rho_0 (\ln r - \ln r_s)\right)\right) r dr \tag{30}$$

$$= -2\pi G \int_{r_s}^{r_{vir}} \left(\rho_0 \left(\frac{r}{r_s}\right)^{-3} \left(\frac{4}{3}\pi \rho_0 r_s^3\right) - \rho_0 \left(\frac{r}{r_s}\right)^{-3} \left(2\pi \rho_0 r_s^3\right)\right) r dr \tag{31}$$

$$= 2\pi^2 G \rho_0^2 r_s^6 \int_{r_s}^{r_{vir}} \frac{1}{r^2} \frac{2}{3} dr = \frac{4}{3}\pi^2 G \rho_0^2 r_s^6 \frac{-1}{1} \left(r_{vir}^{-1} - r_s^{-1}\right) \approx \frac{4}{3}\pi^2 G \rho_0^2 r_s^5, \tag{32}$$

where the last step is because $r_s << r_{vir}$. We then get

$$\Delta E = \frac{4}{5}\pi^2 \rho_0^2 G r_s^5 + \frac{4}{3}\pi^2 G \rho_0^2 r_s^5 = \frac{32}{15}\pi^2 G \rho_0^2 r_s^5. \tag{33}$$

3.4 d)

We are now looking at a dwarf galaxy at $z = 0$ with $M_{vir} = 3 \cdot 10^{10} M_\odot$ and $R_{vir} = 45$ kpc, with $r_s = 1$ kpc. With this mass and radius, we can find

$$\rho_0 = \frac{M_{vir}}{4/3 r_s^3 \pi + 4\pi r_s^3 (\ln R_{vir} - \ln r_s)} = 6.13 \cdot 10^{-23} \text{g cm}^{-3}. \tag{34}$$

This gives us that

$$\Delta E = \frac{32}{15}\pi^2 G \rho_0^2 r_s^5 = 1.477 \cdot 10^{57} \text{ergs}. \tag{35}$$

Check this!!!

3.5 e)