AST4320 Oblig 2

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1 Exercise 1

We have the 1D window function

$$W(x) = \begin{cases} 1 & \text{if } |x| < R \\ 0 & \text{else} \end{cases} \tag{1}$$

We now want to find the Fourier transform of this function. We do this straight forward from the definition of the Fourier transform

$$\tilde{W}(k) = \int_{-\infty}^{\infty} W(x) e^{-ikx} dx. \tag{2}$$

Inserting our definition of $W(x)$ we find

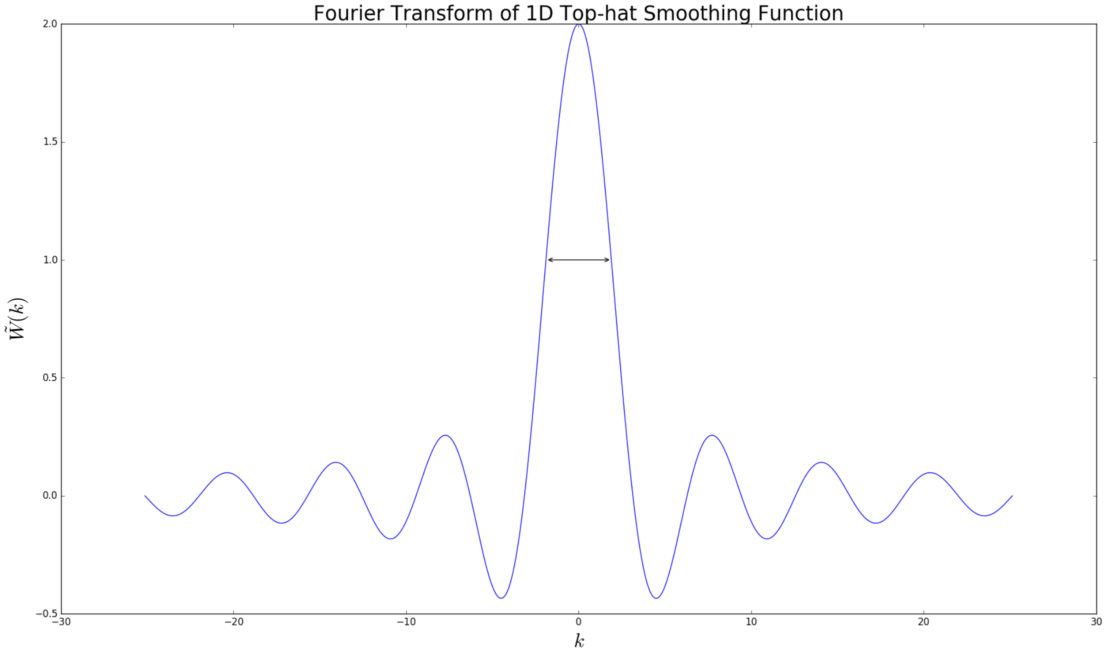
$$\tilde{W}(k) = \int_{-\infty}^{-R} 0 \cdot e^{-ikx} dx + \int_{-R}^{R} 1 \cdot e^{-ikx} dx + \int_{R}^{\infty} 0 \cdot e^{-ikx} dx = \int_{-R}^{R} e^{-ikx} dx \tag{3}$$

$$= \frac{i}{k} e^{-ikx} \Big|_{-R}^{R} = \frac{i}{k}\left(e^{-ikR} - e^{ikR}\right) = \frac{2\sin Rk}{k}. \tag{4}$$

Before we plot this function we need to notice that this is a function that we need to be careful with for $k \to 0$. With the use of L'Hôpitals rule

$$\lim_{k\to 0} \tilde{W}(k) = \lim_{k\to 0} \frac{2\sin Rk}{k} = \lim_{k\to 0} \frac{2R\cos Rk}{1} = 2R. \tag{5}$$

We now see that $\tilde{W}(k)$ is well defined across the whole real line.



Figur 1: The plot of the Fourier Transform of the top-hat smoothing function $W(x)$. This is plotted with $R = 1$. The FWHM is marked with the arrow.

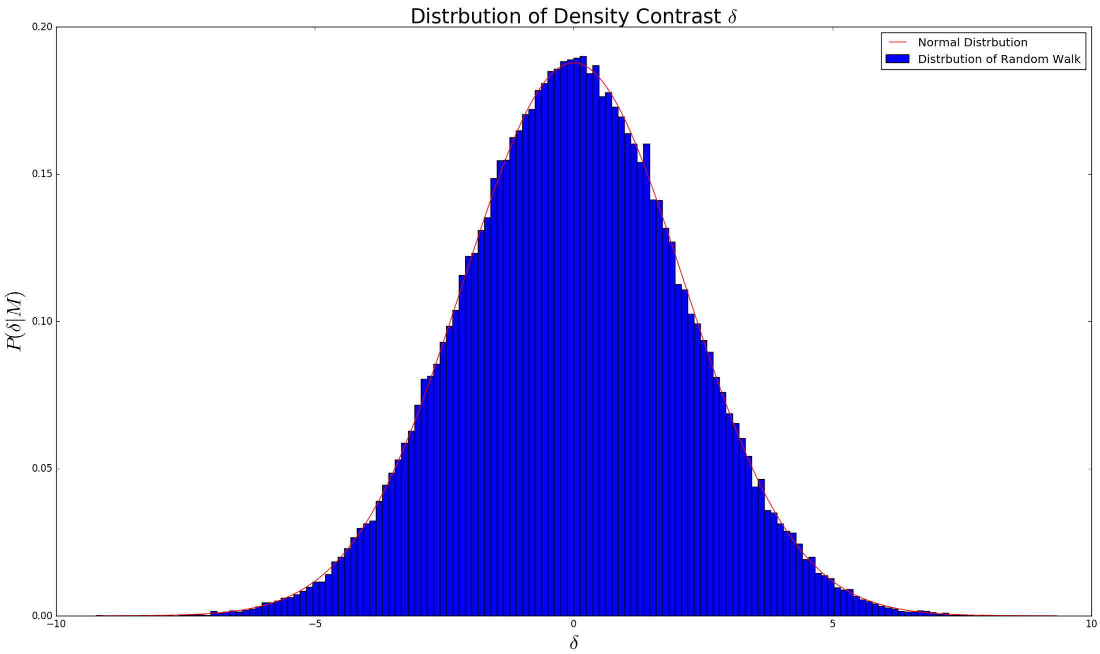
We now want to find the *full width at half maximum*, FWHM, for this function. We first need to find the half maximum. From fig.2 we see that maximum is at 0. We know from (5) that here $\tilde{W}(k=0) = 2R$, so half maximum is R. For the width we just need to find for which k we have $\tilde{W} = R$ and multiply it by 2 (this is because $\tilde{W}$ is symmetric around 0). So we first need to solve

$$R = \frac{2}{k_{\text{half max}}} \sin R k_{\text{half max}}. \tag{6}$$

This is difficult to solve analytical, but easy numerically. From our program we easily find that

$$\text{FWHM} = 2 \cdot k_{\text{half max}} = 3.77. \tag{7}$$

2 Exercise 2



Figur 2: Distribution of the endstep in the random walk. This is plotted with $\epsilon = 0.1$ and $\sigma^2(S_c) = 10^{-4}$. We see that this follows a Gaussian distribution with $\sigma^2 = \pi$.

3 Exercise 3

3.1 a)

We have that the probability of δ provided that δ was never larger than δ_{crit} at some scale $M' > M$ is

$$P_{nc}(\delta|M) = \frac{1}{\sqrt{2\pi}\sigma(M)} \left[\exp\left(-\frac{\delta^2}{2\sigma^2(M)} \right) - \exp\left(-\frac{(2\delta_{crit} - \delta)^2}{2\sigma^2(M)} \right) \right]. \tag{8}$$

Since δ never has crossed δ_{crit} it is safe to assume that collapse has never happened here. This means that the probability of a mass at $\mathbf{x}$ is therefore embedded within a *non*-collapsed object of mass $> M$ is therefore

$$\int_{-\infty}^{\delta_{crit}} P_{nc}(\delta|M) d\delta. \tag{9}$$

This means that the probability that this mass is within a collapsed mass is 1 minus this probability. So

$$P(>M) = 1 - \int_{-\infty}^{\delta_{crit}} P_{nc}(\delta|M) d\delta. \tag{10}$$

3.2 b)

Using (11) and (8) we get

$$P(>M) = 1 - \int_{-\infty}^{\delta_{crit}} \frac{1}{\sqrt{2\pi}\sigma(M)} \left[\exp\left(-\frac{\delta^2}{2\sigma^2(M)}\right) - \exp\left(-\frac{(2\delta_{crit}-\delta)^2}{2\sigma^2(M)}\right)\right] d\delta \tag{11}$$

$$= 1 - \underbrace{\int_{-\infty}^{\delta_{crit}} \frac{1}{\sqrt{2\pi}\sigma(M)} \left[\exp\left(-\frac{\delta^2}{2\sigma^2(M)}\right)\right] d\delta}_{I_1} + \underbrace{\int_{-\infty}^{\delta_{crit}} \frac{1}{\sqrt{2\pi}\sigma(M)} \left[\exp\left(-\frac{(2\delta_{crit}-\delta)^2}{2\sigma^2(M)}\right)\right] d\delta}_{I_2}. \tag{12}$$

We then look at this term by term:

I_1:

We know that we want something with an error function. To get this we see that we need to introduce the variable

$$u = \frac{\delta}{\sqrt{2}\sigma}, \tag{13}$$

which gives

$$\Rightarrow \frac{du}{d\delta} = \frac{1}{\sqrt{2}\sigma} \Rightarrow d\delta = \sqrt{2}\sigma du. \tag{14}$$

We need to take care of the upper limit $u_\uparrow$. We see that this becomes

$$\delta_\uparrow = \delta_{crit} = \sqrt{2}\sigma u_\uparrow \Rightarrow u_\uparrow = \frac{\delta_{crit}}{\sqrt{2}\sigma} \equiv \frac{\nu}{\sqrt{2}}. \tag{15}$$

We can then solve I_1 as

$$I_1 = \frac{1}{2} \int_{-\infty}^{\nu/\sqrt{2}} \frac{2}{\sqrt{\pi}} e^{-u^2} du. \tag{16}$$

Looking this up on integration tables we find that this is

$$I_1 = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{\nu}{\sqrt{2}}\right)\right] \tag{17}$$

I_2:

This integral is quite similar to I_1, so we use a similar variable change

$$u = \frac{2\delta_{crit} - \delta}{\sqrt{2}\sigma}, \tag{18}$$

which gives us

$$\Rightarrow \frac{du}{d\delta} = -\frac{1}{\sqrt{2}\sigma} \Rightarrow d\delta = -\sqrt{2}\sigma du. \tag{19}$$

We now need to look at the limits. While the upper limit stays the same, we need to take a look at the lower limit

$$\delta_\downarrow = x = 2\delta_{crit} - \sqrt{2}u_\downarrow\sigma \Rightarrow= u_\downarrow = \frac{2\delta_{crit} - x}{\sqrt{2}\sigma}, \tag{20}$$

then taking the limit of x we get

$$u_{\downarrow} = \lim_{x \to -\infty} \frac{2\delta_{crit} - x}{\sqrt{2}\sigma} = \infty. \tag{21}$$

We can then use the minus sign from (19) to turn the limits around and get

$$I_2 = \frac{1}{2} \int_{\nu/\sqrt{2}}^{\infty} \frac{2}{\sqrt{\pi}} e^{-u^2} du = \frac{1}{2} \mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right) = \frac{1}{2}\left[1 - \mathrm{erf}\left(\frac{\nu}{\sqrt{2}}\right)\right] \tag{22}$$

We thus get the probability

$$P(>M) = 1 - I_1 + I_2 = 1 - \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{\nu}{\sqrt{2}}\right)\right] + \frac{1}{2}\left[1 - \mathrm{erf}\left(\frac{\nu}{\sqrt{2}}\right)\right] = 1 - \mathrm{erf}\left(\frac{\nu}{\sqrt{2}}\right) = 2P(\delta > \delta_{crit}|M). \tag{23}$$

And there is no need for the fudge factor!