AST5220 Milestone 2

Daniel Heinesen, daniehei

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1 Introduction

In this exercise we are going to look at the density of electrons in the Universe, and their impact on the mean free path, or optical depth of photons. These quantities will contribute on how the distribution of photons evolve, which will be calculated in later milestones.

Due to the large amount of free electrons in the early Universe, photons could not travel far without being scattered by the electrons. This is described by the optical depth. To get a sense of the optical depth of the photons, we need to know the amount of electrons we have in our Universe. This will be done with the Saha and Peebles' equation. Having the density of electrons we can easily find the optical depth and visibility function. From these we can find out at which point the Universe was translucent enough for the photons to travel freely; this is the time of recombination.

As a quality check, the results we get below are given on the same format as Callin [1].

2 Theory

We cannot find the electron density n_e directly. We need to go through the fractional electron density $X_e = n_e/n_H$, where n_H is the density of protons in the Universe. To find X_e we are going to use two different equations. If $X_e \approx 1$ we can use the Saha equation

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b} \left(\frac{m_e T_b k_b}{2\pi\hbar^2} \right)^{3/2} e^{-\epsilon_0/T_b k_b}, \tag{1}$$

where n_b is the density of baryons, m_e the mass of an electron and T_b the temperature of the baryons. If we have $X_e < 0.99$ this equation becomes inaccurate and we need to solve Peebles' equation. This equation describes X_e for all temperatures and n_b, but is more difficult to solve, thus why be use the Saha equations for the earliest times. Peebles' equation is given as

$$\frac{dX_e}{dx} = \frac{C_r}{H} \left[\beta(T_b)(1 - X_e) - n_H \alpha^{(2)}(T_b) X_e^2 \right], \tag{2}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s \to 1s} + \Lambda_\alpha}{\Lambda_{2s \to 1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{3}$$

$$\Lambda_{2s \to 1s} = 8.227 s^{-1}, \tag{4}$$

$$\Lambda_\alpha = \frac{H}{(\hbar c)^3} \frac{27\epsilon_0^3}{(8\pi)^2 n_{1s}}, \tag{5}$$

$$n_{1s} = (1 - X_e)n_H, \tag{6}$$

$$\beta^{(2)}(T_b) = \beta(T_b)e^{3\epsilon_0/4T_b k_b}, \tag{7}$$

$$\beta(T_b) = \alpha^{(2)}(T_b) \left(\frac{m_e T_b k_b}{2\pi\hbar^2} \right)^{3/2} e^{-\epsilon_0/T_b k_b}, \tag{8}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}} \frac{\alpha^2}{m_e^2} \sqrt{\frac{\epsilon_0}{T_b k_b}} \frac{\hbar^2}{c} \phi_2(T_b), \tag{9}$$

$$\phi_2(T_b) = 0.448 \ln(\epsilon_0/T_b k_b). \tag{10}$$

All of these equations are expressed in SI units

From X_e we can find

$$n_e = X_e n_H, \tag{11}$$

where

$$n_H = n_b \approx \frac{\rho_b}{m_H} = \frac{\Omega_b \rho_c}{m_h a^3}. \tag{12}$$

We can use this because we assume that all baryonic matter in the Universe are protons.

With this one could find the optical depth, defined as

$$\tau(\eta) = \int_\eta^{\eta_0} n_e \sigma_T ac d\eta \Leftrightarrow \tau(x) = \int_x^{x_0=0} -\frac{n_e \sigma_T ac}{\mathcal{H}} dx, \tag{13}$$

where σ_T is the Thompson cross section, c is the speed of light, a is the scale factor, x is the logarithm scale factor and $\mathcal{H}$ the scaled Hubble parameter aH. And finally from this we can get the visibility function

$$\tilde{g}(x) = -\tau' e^{-\tau} = -\frac{d\tau}{dx} e^{-\tau}, \tag{14}$$

where $\tau' = \frac{d\tau}{dx}$.

3 Method

To find all the quantities below we need to integrate over a time variable. We are here using the logarithmic scale factor $x = \ln a$. For the integrations we will set up a grid of x-values from $x = \ln 10^{-10} \approx -23$ to 0, with 1000 points.

3.1 Finding X_e and n_e

We mentioned above, we can use two different methods to find X_e for two different epochs: The Saha equation when $X_e \approx 1$, and Peebles' equation when $X_e < 0.99$.

The Saha equation (1) is a normal quadratic equation, which we can write as

$$X_e^2 + X_e \cdot C - C = 0, \tag{15}$$

where $C = \frac{1}{n_b} \left(\frac{m_e T_b k_b}{2\pi \hbar^2} \right)^{3/2} e^{-\epsilon_0/T_b k_b}$. This can be solved by the normal quadratic formula, but since this is an equation which can be a bit unstable [1], we will instead use a more stable solution [2]

$$X_e = \frac{2c}{-b \mp \sqrt{b^2 - 4ac}} = \frac{-2C}{-C - \sqrt{C^2 + 4C}}. \tag{16}$$

We only used the negative sign in the $\mp$ because this ensures us a positive X_e.

When X_e goes below 0.99 we will switch to Peebles' equation (2). This is a differential equation, so here we will use the same solver as in milestone1. This is a integration method, using the *rkqs* integration scheme. There can be some numerical instability in the integrand, with the two exponents in (7) and (8) causing some problems [3]. To ensure that nothing goes wrong, we will combine them, so they become stable. Note that we will need to find H for a given x, this is done with a function from milestone 1.

From X_e we can now use (11) to get n_e. Later we will have to get values of n_e for any values of x. Since we found n_e for only 1000 specific x-values, we need to spline n_e. This is done with an algorithm from *Numerical Recipes*.

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[1]By this I mean that we can have a subtraction of two small numbers, which is numerically unstable

[2]Known as the *citardauq* formula, which is just *quadratic* backwards.

[3]Or at least I had some problems with them.

3.2 Finding τ and $\tilde{g}$

Next step is to calculate $\tau(x)$ from (13). This is straight forward except from the fact that we do not have an initial condition for the earliest x, instead we know that the optical depth today is $\tau(x_0) = 0$. This means that we will integrate (13) in reverse, starting at today, and integrating backwards in time. As with n_e, we will also make a spline of τ. But we will also use a spline function to get the first derivatives τ', which are calculated with *splint_deriv*. Since the spline gives us an array with the second derivatives of τ, we will also make a spline function of the second derivatives τ''.

Having τ' we can calculate $\tilde{g}$ from (14). We will also use splines to get a continuous function for $\tilde{g}$, as well as $\tilde{g}'$ and $\tilde{g}''$.
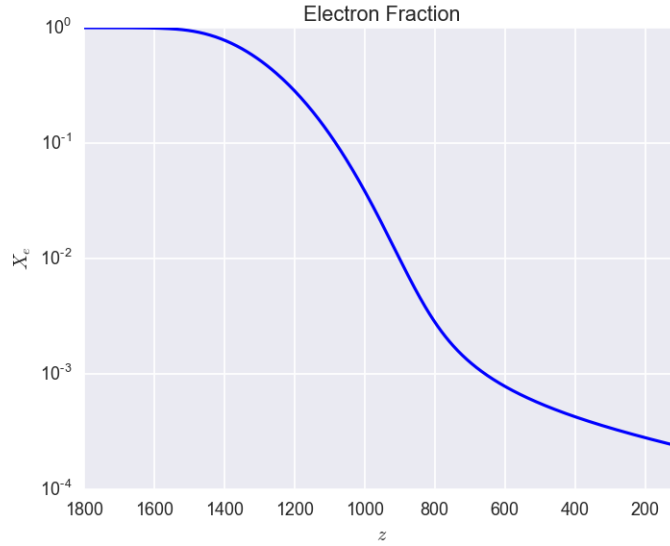
4 Results

Figur 1: Electron fraction X_e as a function of red-shift z.

Figure 1 show the evolution of X_e. This figure starts at red-shift $z = 1800$ which corresponds to $x \approx -7.5$. At about $z \approx 1500$, $x \approx -7.3$, we see that X_e becomes smaller than 0.99, meaning that we used the Saha equation for over half of the values X_e. Had we used Peebles' equation for every value of x, this calculation would had taken over double the time.

In the period $z = 1400$ to $z = 800$ we see a sharp decrease in the fraction of electrons in the Universe. During this period we say that recombination occurs.

Figure 2 shows that the optical depth steady decreases due to the expansion of the Universe. But around $x \approx -7$, there is a sharp drop, which is not unlike the drop in X_e. Is, again, is the recombination.

Figure 3 show the visibility function. Again we see that $\tilde{g}$, and its derivatives, are more or less constant throughout time, but around $x \approx -7$ there is a sudden change in all of the quantities. Once more indicating recombination.

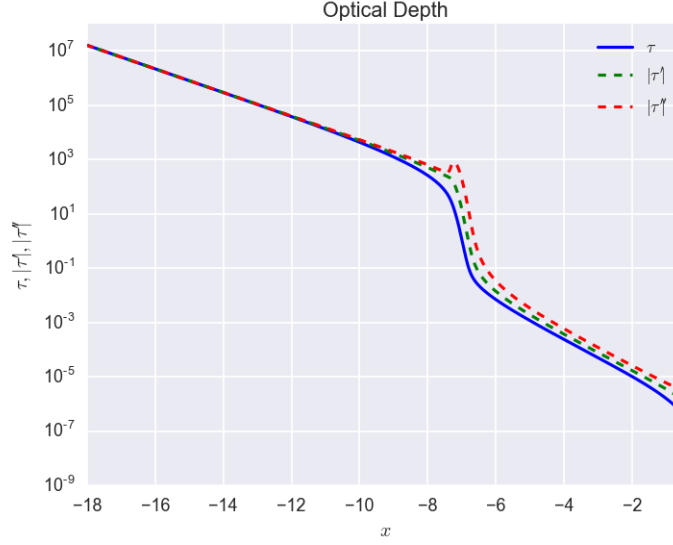We can see that the above figures are the same as found by Callin [1]

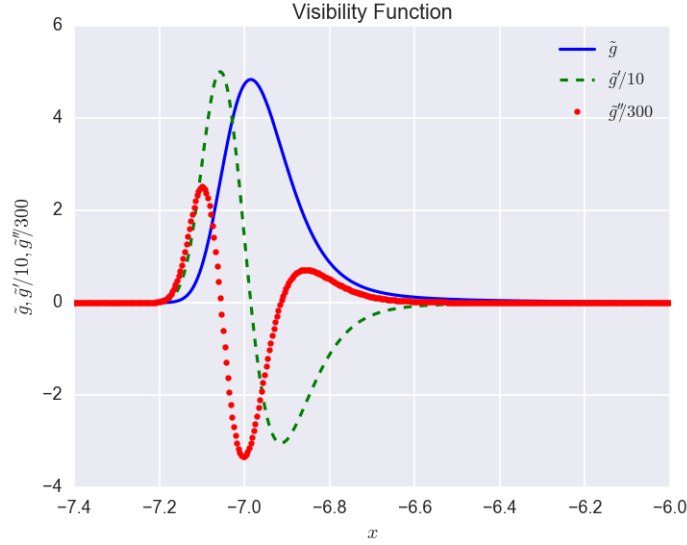Figur 2: Optical depth τ, and its derivatives, as a function of the logarithmic scale factor x.

Figur 3: The visibility function $\tilde{g}$, and its scaled derivatives, as a function of x.

5 Discussion and Conclusion

We have seen that for X_e, τ and $\tilde{g}$ there is something happening around $x \approx -7$. What is happening is that as the temperature in the Universe decreases, the primordial plasma and photons becomes colder. At a certain temperature the electrons and protons were able to form neutral hydrogen: They recombined. Rapidly all the free electrons becomes apart of the neutral hydrogen atoms, leading n_e to become smaller. This causes the rapid fall in the electron fraction X_e we see in fig. 1.

With the high density of free electrons in the early Universe, photons were unable to travel far due to their (Thomson) scattering with the free electrons, and the Universe was opaque. But as X_e decreased, the photons became more free to travel further. This ability to travel freely is

what the optical depth τ measures: the bigger τ is, the thicker the medium is, thus more opaque, meaning that the photons have a lower free mean path, and travels shorter before being absorbed or scattered. This is why we see the sudden, 5-fold decrease in τ near $x \approx -7$: As the density of free electron fall, the photons can travel more freely and the opaqueness decreases. Note that even before, and long after recombination, τ is decreasing. This is because the Universe is expanding, which lowers the density of electron.

Finally, the visibility function $\tilde{g}(x)$ describes the probability that an observed photon was last scattered at x. Before recombination the Universe was opaque, so we do not expect the photon to be from this epoch. After recombination the Universe is more or less empty at larger scales, so the photons won't scatter of anything. So we expect that the last time the photons scattered was as recombination happened – to exaggerate: In one instance the photons scattered of a bunch of free electrons, and in the next instance, the Universe was empty –. Looking at fig. 3 we see that it is most likely that an observed photon was scattered at $x \approx -7$, which means, once again, that this is where recombination happened.

So we now have continuous functions of n_e, τ and $\tilde{g}$ – and the derivatives of the last two –, which will be used later when calculating the Einstein-Boltzmann equations. We also saw that we could use these quantities to show that recombination happened around $x \approx -7$ or $z \approx 1100$.

Referanser

[1] Callin, Peter, *How to calculate the CMB spectrum*, astro-ph/0606683 2006.