AST5220 Milestone4

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1 Introduction

So far we have found out how the Universe expands, found the number density of free electrons and how this impact the mean free path of the photons. We have then used this to calculate how initial perturbations after inflation would have evolved through till today. We are now ready to use all of this to find what we are after: The power spectrum of the Cosmic Microwave Background (CMB).

Our final result will be the power spectrum C_l, which describes the how much power there are between different scales in the Universe, with small ls corresponding to large scales and vice versa.

2 Theory

We are after the power spectrum

$$C_l = \langle |a_{l,m}|^2 \rangle = \int \frac{d^3k}{(2\pi)^3} |a_{l,m}|^2. \tag{1}$$

We need to integrate since we have Fourier transformed all our quantities. $a_{l,m}$ comes from the fact that we observe the temperature of the CMB projected on a sphere, meaning that we can write it as

$$T(\hat{n}) = \sum a_{l,m} Y_{l,m}(\hat{n}), \tag{2}$$

where $Y_{l,m}$ are spherical harmonics and $a_{l,m}$ the coefficients containing the actual physics of the temperature field. Instead of finding $|a_{l,m}|^2$ directly, we can instead define $|a_{l,m}|^2 = P(k)\Theta_l^2$, where $P(k)$ is the primordial power spectrum and Θ_l is the so called *transfer function* which contain all the information about how the power spectrum has evolved from inflation and until today. We can get the primordial power spectrum from the Harrison Zel'dovich spectrum

$$\frac{k^3}{2\pi^2} P(k) = \left(\frac{ck}{H_0}\right)^{n_s - 1}, \tag{3}$$

where n_s is the spectral index. We can now write the power spectrum as

$$C_l = \int_0^\infty \left(\frac{ck}{H_0}\right)^{n_s - 1} \Theta_l^2(k) \frac{dk}{k}. \tag{4}$$

We now need to find the transfer function Θ_l. As the choice of name implies, this is perturbation of the l^{th} momentum we found in milestone 3. The problem is the while we calculated these up to $l = 6$ the CMB spectrum is given for $l > 1200$. Calculating all these perturbations with the use of the method in milestone 3 is slow, so we need a faster way. Thanks to Zaldarriaga and Seljak we can calculate the momenta in minutes and not hours. They found the so called *line of sight integration* method. This means that instead of doing a multipole expansion of the temperature, then solve the differential equations, we instead solve the equations and then expand. This means that we can write, after some calculation,

$$\Theta_l(k, x = 0) = \int_{-\infty}^{0} \tilde{S}(k, x) j_l(k(\eta_0 - \eta)) dx, \tag{5}$$

where $j_l(k(\eta_0 - \eta))$ is the l^{th} spherical Bessel function evaluated at $k(\eta_0 - \eta)$, and $\tilde{S}$ the source function. The source function is defined as

$$\tilde{S}(k,x) = \tilde{g}\left(\Theta_0 + \Psi + \frac{1}{4}\Pi\right) + e^{-\tau}\left(\Psi' - \Phi'\right) - \frac{1}{ck}\frac{d}{dx}\left(\mathcal{H}\tilde{g}v_b\right) + \frac{3}{4c^2k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}\left(\mathcal{H}\tilde{g}\Pi\right)\right], \quad (6)$$

where (here) $\Pi = \Theta_2$. The expression for $\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}\left(\mathcal{H}\tilde{g}\Pi\right)\right]$ is found in Callin [1]. One expression inside this expression, that is not shown in the article is $\frac{d}{dx}\mathcal{H}\mathcal{H}'$. This can be found as

$$\mathcal{H} = e^x H = H_0\sqrt{(\Omega_{b,0} + \Omega_{m,0})\exp(-x) + (\Omega_{r,0} + \Omega_{\nu,0})\exp(-2x) + \Omega_{\Lambda,0}\exp(2x)} \quad (7)$$

$$\mathcal{H}' = \frac{H_0^2}{2\mathcal{H}}\left[-(\Omega_{b,0} + \Omega_{m,0})\exp(-x) - 2(\Omega_{r,0} + \Omega_{\nu,0})\exp(-2x) + 2\Omega_{\Lambda,0}\exp(2x)\right] \quad (8)$$

$$\Rightarrow \mathcal{H}\mathcal{H}' = \frac{H_0^2}{2}\left[-(\Omega_{b,0} + \Omega_{m,0})\exp(-x) - 2(\Omega_{r,0} + \Omega_{\nu,0})\exp(-2x) + 2\Omega_{\Lambda,0}\exp(2x)\right] \quad (9)$$

$$\Rightarrow \frac{d}{dx}\mathcal{H}\mathcal{H}' = \frac{H_0^2}{2}\left[(\Omega_{b,0} + \Omega_{m,0})\exp(-x) + 4(\Omega_{r,0} + \Omega_{\nu,0})\exp(-2x) + 4\Omega_{\Lambda,0}\exp(2x)\right] \quad (10)$$

The source function can be thought of a function describing all the different things a photon encounters on our way to us that decreases or increases its energy. The first term describes the Sachs-Wolfe effect – the loss of energy as a photon climbs out of a gravitational potential –, the second term describes the integrated Sachs-Wolfe effect – if the gravitational potential have change from when a photon enters to it leaves, this affects its energy –, the third term is the Doppler term, and the fourth a term that corrects a bit for polarization. Notice that the source function only needs temperature momenta up to Θ_2, so we do not need to solve the differential equations for l higher than 2. In reality, since we use an approximation to find Θ'_{lmax}, there is a small error in the highest momentum. If we solve for l up to 6 this error is small enough for Θ_2 and below, that the error can be ignored.

3 Method

The method consists of two parts: Calculating the source function and integrating everything to get the final power spectrum.

3.1 Source Function

We want the source function from some initial start time until today. From milestone 3 we have all the quantities we need to calculate $\tilde{S}$, but not with the precision we want. So with our quadratic k grid with 100 ks and our x grid with 500 points, we can find an initial grid of $\tilde{S}(k,x)$. We can then spline this 2D array. With new, finer k and x grids, both with 5000 points in the same intervals, we can use the spline to get a new $\tilde{S}$ array with a finer grid. It is these grid we are going to use for the rest of the calculations.

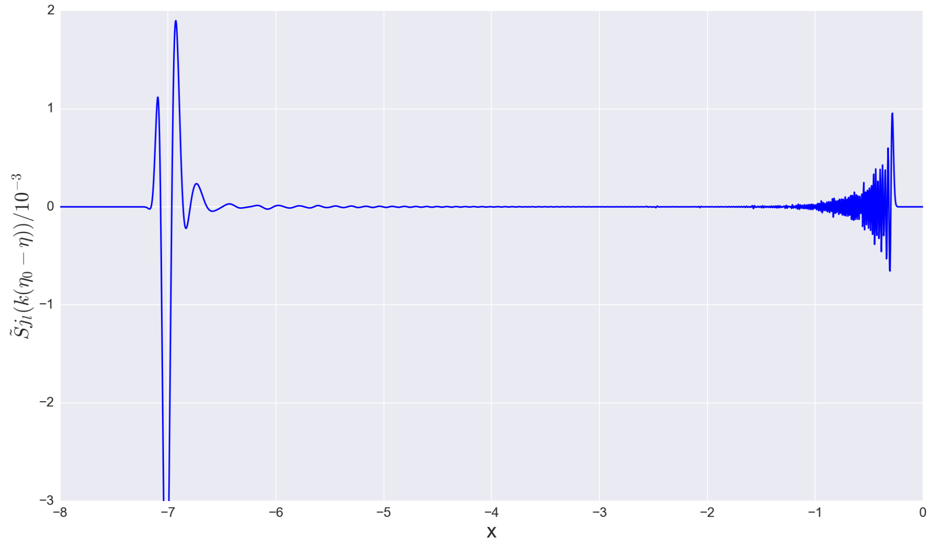
Both $\tilde{S}$ and j_l are saved in binary files, so that if all the calculations below needs to be run multiple times – for any reason –, we don't need to solve all the differential equations and calculate the Bessel functions again.

3.2 Integrating Everything

To find Θ_l we see from eq. (5) that we need to find the spherical Bessel functions. Fortunately we are given functions to find these. But we need to save the Bessel function in a grid for each l and for 5400 values from 0 to 3500. For each l we will spline the Bessel function for use later.

We can now use eq. (5) to find the transfer functions Θ_l. This integration was done with a simple euler integration.

We then integrate the Θ_ls with the primordial power spectrum to get the C_ls, eq. (4). This again is done via an Euler Scheme. We start by finding C_l for 44 ls from 1 to 1200. We then spline



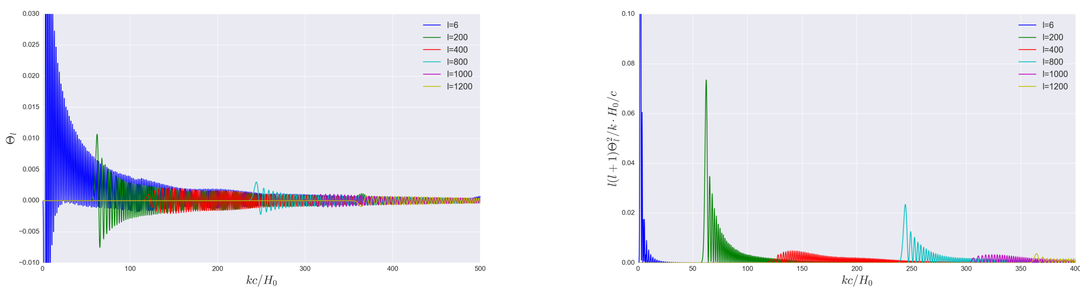
Figur 1: This shows the source function multiplied with the spherical Bessel function for $l = 100$ and $k = 340 \cdot H_0/c$.

and retrieve a C_l for each l in the same interval. We then normalize as $C_l \cdot l(l+1)/2\pi$, and we have our long sought after power spectrum.

All the calculation are in the first instance done with a default set of parameters Ω_b, Ω_m, Ω_r, n_s and h. With these parameters, the resulting power spectrum are not a good fit for the one observed by the Planck Satellite, so we need to find the correct values for the parameters. This is done by adjusting the parameters one by one and seeing who this affect the power spectrum, this making it possible to infer how we can change the parameters to fit the observed power spectrum.

4 Results

The first plots we are going to look at source function and the transfer function.



(a) The figure shows a completly unscaled transfer function $\Theta_l(k)$.

(b) Plot showing $\Theta_l^2(k)/k$. The plots are scaled with $l(l+1)H_0/c$ to make the functions for the different ls more similar.

Figur 2: Plots showing the transfer function $\Theta_l(k)$ for a selection of ls.

Looking at figure 1 we see $\tilde{S}j_l(k(\eta_0 - \eta))$. This is plotted as a sanity check to compare with figure 3 in Callin [1]. The plot is for $l = 100$ and the closes we can get to $k = 340 \cdot H_0/c$. This is more or less the same plot as Callin got, meaning that we are on the right track. If we look at the transfer function $\Theta_l(k)$ in figure 2 we see that the transfer function on all scales are oscillation as damped harmonic oscillators.

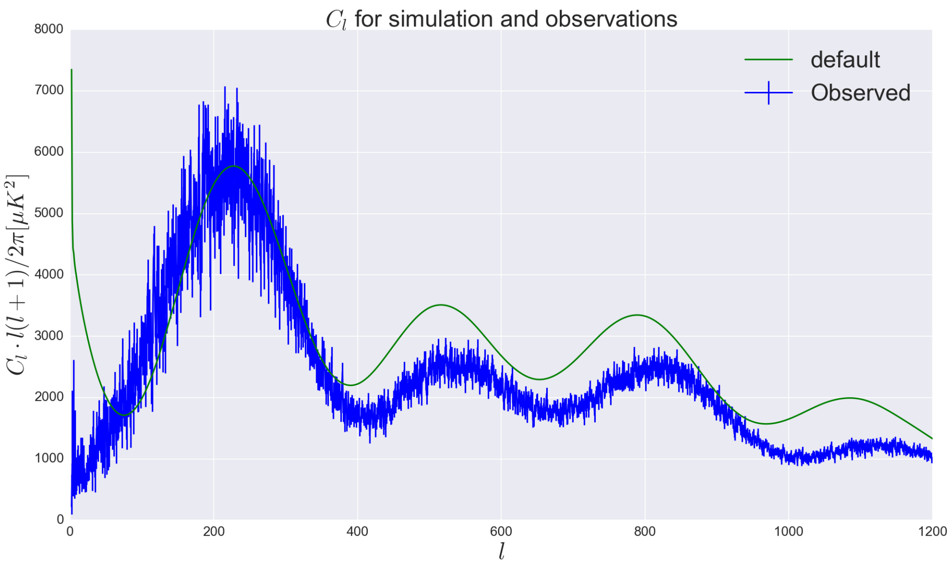
In fig. 2b we see that generally the scaled $l(l+1) \cdot \Theta_l/k$ seems to decrease for larger l, and have shapes similar to the damped harmonic oscillator of Θ_l.

4.1 Power Spectrum C_l

The first power spectrum calculated is with the parameters found in the *Default* row of table 1.

	Ω_b	Ω_m	Ω_r	n_s	h
Default	0.046	0.224	$8.3 \cdot 10^{-5}$	0.967	0.7
Best Fit	0.065	0.20	$8.3 \cdot 10^{-5}$	0.80	0.7

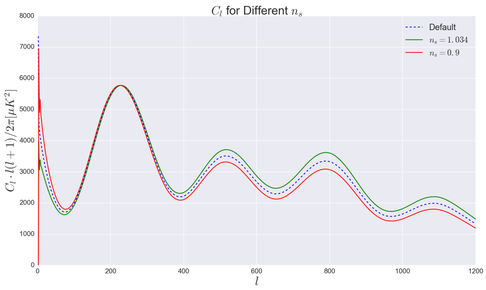
Tabell 1: Parameters used for the default power spectrum and the best fit.



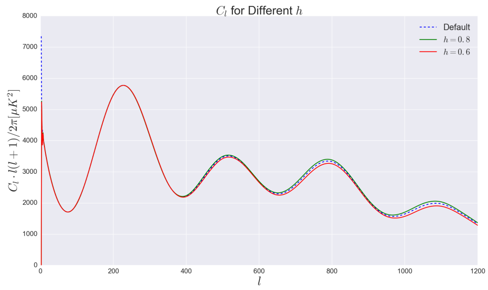
Figur 3: The power spectrum $C_l l(l+1)/2\pi$ in μK^2. We see that thoug it has the same shape as the Planck data, the values after the acoustic peak is a bit to low. We can also see that for low ls the power spectrum as a way to large value.

The resulting power spectrum can be seen in figure 3. Since the top of the acoustic peak in both the simulated data and in the Planck data is normalized to be at $5775\mu K^2$, they will always agree here. After the peak there is disagreement between the simulated result and the observations, with the simulated C_l having a higher value. Before the peak there is agreement above $l \approx 90$; below this the simulated data shoots upwards, and at $l \approx 2$ there is only mess... This problem is not removed by adjusting the parameters – have not test by changing Ω_k and Ω_Λ, even though they have alot to say for small ls –, and will in some instances make it worse. There is most likely some expression somewhere in the code that is wrong, but I've yet to find it.

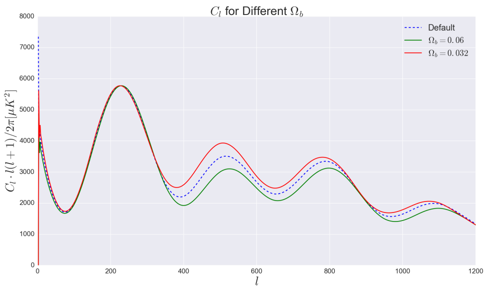
Figure 4 we see that happens when each of the parameters are changed one by one. Below I describe what happens when the parameters are increased, but the opposite should happen if they are decreased (as the plots show).



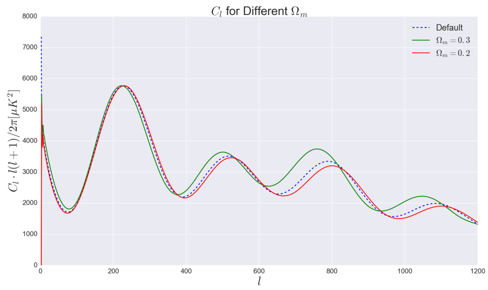
(a) C_l when the spectral index n_s is changed. We see that with a lower n_s, C_l decreases. The increase for smaller ls is due to the normalization at the peak of the acoustic peak.



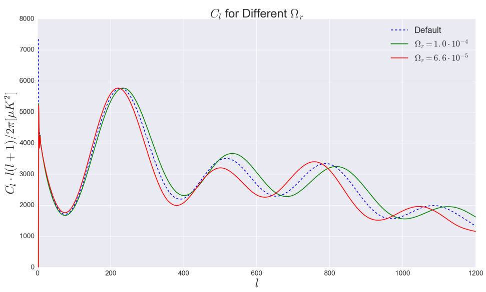
(b) C_l when the dimensionless Hubble parameter h is changed. We see that there is not much difference when the change in the range that would expect to see.



(c) C_l when the density parameter of baryons Ω_b is changed. We see that for when Ω_b is higher, there are more baryons, meaning they are able to compress more, and decompress less, meaning that the second compression peak now are larger than the first decompression peak. The opposite is true when decreases Ω_b.



(d) C_l when the density parameter of cold dark matter Ω_m is changed. With a higher Ω_m, the main effect here is that since dark matter is more prevalent, recombination will happen later, moving the first compression peak towards smaller ls. The second effect is to lower all peaks, this is not seen here, since we normalize the first compression peak. This makes it look like higher Ω_m increases the peaks. The opposite is true when decreases Ω_m.



(e) C_l when the density parameter of radiation Ω_r is changed. With a higher Ω_r recombination seems to happen earlier, this pushing the first compression peak to higher ls. It also streches the distance between the peaks. Since we now have more radiation the pressure in the compressed baryon-photon fluid is higher, and thus the first decompression peak is higher than the second compression peak. The opposite is true when decreases Ω_r.

Figur 4: Plots showing the differences in the the power spectrum C_l when changing the five parameters one by one.

4.1.1 n_s

The result is showed in fig. 4a. Since we have the primordial power spectrum found in eq. 3 there is a strong dependence on n_s. If we have a non-unity n_s the power spectrum will depend on scale. With a $n_s < 1$ the primordial power spectrum decreases with scale, and the same would be true for C_l. This is what we see from fig. 4a. But in the figure we see that C_l increases if $l < 220$. This is because we have renormalized so that all the first acoustic peaks have the same hight.

4.1.2 h

Since many of the equation used in our calculation is dependent on H and thus the dimensionless Hubble parameter h, we would expect to see differences when changing it. But from fig. 4b we see that the changes are very small.

4.1.3 Ω_b

Ω_b, fig. 4c, determines the amount of baryons in the Universe. With more baryon the gravitation wells the baryon-photon fluid collapses into are deeper, and the compression is larger and decompression smaller. This is why we see that the ratio between the compression/decompression peaks are larger. The first compression peak should also higher, and the ratio between this and the rest of the peaks should be larger, but due to the normalization it stays the same, while the later peaks decrease. More baryons also changes the sound speed(and thus the sound horizon), and the first peak shifts to higher ls. This effect can be seen, but is very small.

4.1.4 Ω_m

Ω_m, fig. 4d, determines the amount of cold dark matter in the Universe. A larger amount of dark matter would make a deeper gravitational well, which will dampen the acoustic oscillations, so the peaks decay faster. It would also decrease all peaks, but due to the normalization the first compression peak is always the same hight, while the later peaks decreases. Since dark matter makes up the larger part of matter, an increase in dark matter starts matter domination earlier and thus recombination happens later. As we see in fig. 4d the first peak is towards lower ls thus happening later.

4.1.5 Ω_r

Ω_r, fig. 4e, determines the amount of radiation – relativistic particles – in the Universe. When the the baryon-photon fluid oscillates it is the radiation which makes the pressure. So with more radiation the compression should be weaker and the decompression stronger. This is what we see in fig. 4e. We also see that the peaks are farther from each other.

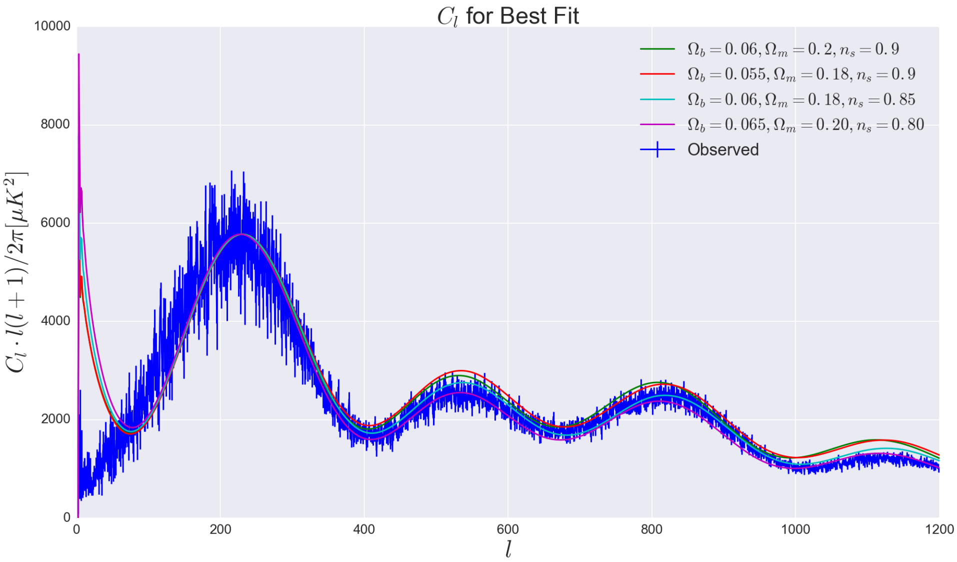
4.2 The Best Fit Parameters

So since we have to decrease the later peaks, the easiest parameters to change is Ω_b, Ω_m and n_s. We are going to increase Ω_b, and decrease Ω_m and n_s. Figure 5 shows the result for four different combinations. I chose the purple as the best fit, but a better fit would be somewhere between the light blue and purple. The parameters for this power spectrum is found in table 1. We see that we have a quite good fit, but there is still problems for smaller ls.

It is possible to use Ω_r and h and try to make better fit, but was not done due to time restrictions.

5 Conclusion

We have finally managed to take the theory and make a power spectrum with can be compared with real observations,fig. 5! We were so able to find estimations of the cosmological parameters



Figur 5: The power spectrum $C_l l(l+1)/2\pi$ in μK^2. I chose the purple as the best fit, but a better fit would be somewhere between the light blue and purple.

based on the observations, tab. 1. The estimations fitted the observations quite well, except for at low ls. There are some problems here which I did not have the time to debug.

Even though the best fit curve correlated well with the data, it was not the same parameters found by the Planck team. This is most likely due to the fact that polarization is ignored in this analysis[1]!

But all in all the result was very satisfying, and the way there very educational!

Write more?

Referanser

[1] Callin, Peter, *How to calculate the CMB spectrum*, astro-ph/0606683 2006.

[1] And that the Planck team is way better at parameter estimation than me...