

AST1100 rapport

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Abstract

In this articles I am interested simulating a complete interplanetary space voyage. I am going to look at the construction of an easy rocket engine, how I can simulate an entire solar system, how to transverse the large empty space between its planets and finally how to land on one of the planets. On the way I am going to try to orient, and try to analyze the atmosphere to the target planet. Last I am going to put these simulation to the test and send a satellite to this planet, and hopefully manage to land safely.

Constants and Variables

k	Boltzmann's constant.
F_b	Force from <i>one</i> box.
n_b	Number of boxes.
m_l	Mass of the satellite.
m_{lander}	Mass of the lander.
m_f	Mass of the fuel.
m_e	Mass of the particles escaping per second per box.
M_s	Mass of the star.
m_{f0}	Mass of fuel when launching.
T_*	Temperature of the Star
T_p and T_0	Temperature of the destination planet.
r_p	Radius of the destination planet.
R_p	Distance from sun to destination planet.
m_h	Mass of H_2 .
g_p	Gravitational acceleration of the destination planet.
μ	The mean molecular mass of the atmosphere of the destination planet.
ρ_0	Density at the surface of the destination planet.
τ_p	Rotational period of the destination planet.

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1 Introduction

In this articles I shall look at many of the aspects of sending and landing a satellite on out neighbor planet Isskji.

I am first going to look at the engine, approximating it by a simple "particle in a box" model, and seeing what kind of force I can expect. I will then you Newton's law of gravity for N-bodies, and simulate our solar system. I also what to see if Isskji is in the habitable zone, and if the trip is worth taking. Having a knowledge about how out solar system will look in the future, I can begin to plan out trip, doing some calculations using Hohmann transfers and seeing if this can get us to Isskji. With list of maneuvers I need to get to our destination, I can look at the fuel usage, and deriving a rocket equation specific for our engine.

Under way to Isskji, the satellite need a way of knowing where it is. By getting data about the distance to the other planets, the spectrum of two distant star and pictures of the starscape, it will be able, though gradient descent, Doppler shift and least square comparisons to find all the information it needs to orient it self.

Finally in orbit around Isskji, I need to make a model of the atmosphere to see how big a parachute the lander will have to have to safely land on the planet. I also have to scout the planet, finding a good place to land, and calculate the trajectory needed to get us to our desired landing point.

I want to acknowledge some of the other space agencies who has been researching the same problems. A special thanks to Gunnar Lange, Fredrik Mellbye, Erlend Liam og Aram Salihi for make this possible.

2 Theory and Methods

2.1 The Engine

To simulate the engine I am going to approximate the complex going-ons in a rocket engine by a box of particles with a hole in the floor. The particles only interact with the walls, thus making them an ideal gas. As some particles leave the hole in the floor, momentum are lost. By pumping in more gas, keeping the pressure constant, momentum is conserved. This means that for every particle escaping, some momentum is lost straight downward, and the box gets a momentum equal to that of the particle, but in the opposite direction. The result is that the engine moves upwards. Since H_2 is one of the most used fuels, out gas will consist of H_2 molecules.

The box is a cube, with sides $L = 10^{-6}m$. The box is simulated with the origin at the center, this gives us complete symmetry, making the code easier. The particles are placed with a uniform distribution

$$p(x) = \frac{1}{b-a} \Theta(x-b)\Theta(a-x) \quad (1)$$

in the box, with a velocity, the components distributed with the *Maxwell-Boltzmann distribution*

$$P(v_i) = \sqrt{\left(\frac{m}{2\pi kT}\right)} e^{-\frac{mv_i^2}{2kT}} \quad (2)$$

This is a Gaussian distribution with $\sigma = \sqrt{\frac{kT}{m}}$ and $\mu = 0$.

To check if the simulation gives us a realistic gas simulation, I check if the analytical and numerical kinetic energy and pressure are the same. I find the numerical pressure by first look at the force on one wall:

$$F_i = \frac{dp}{dt} \approx \frac{\Delta p}{\Delta t} = \frac{2p_i}{\Delta t} \quad (3)$$

This gives us the pressure:

$$P = \frac{F}{A} = \frac{\frac{2p_i}{\Delta t}}{A} = \frac{2p_i}{\Delta t L^2} \quad (4)$$

The analytical expression for the pressure of an ideal gas is:

$$P = nkT \quad (5)$$

The kinetic energy is given by:

$$E_k = \frac{1}{2n} m_{h2} \sum_{i=1}^n (v_x^2 + v_y^2 + v_z^2) \quad (6)$$

numerically and

$$E_k = \frac{3}{2} k T \quad (7)$$

analytically.

A hole with the size $H = \frac{L}{2}$ is opened in floor, and the number of particle escaping is counted, that the momentum gained by box is calculated.

An important choice is what to do when a particle escape. Since I want the pressure to be constant over time, I have to find a method of placing the escaped particles back in the box. One way is to place the particle back in the box with a random position and velocity. This sounds to be the easy and correct way to do this, but one thing I noticed was that both the pressure and kinetic energy has a tendency to decrease over time. I speculate that this is because the particle with higher velocities has the greatest chance of hitting the floor and being placed back in the box with a new random velocity, thereby skewing the velocity distribution towards low velocities. The way I intend for was to just let the particles bounce back just as if they hit the other walls. This guarantees that the kinetic energy and pressure stays the same.

2.2 A simple launch

Before I send the satellite on the full journey, I want to see what it is capable of with a small launch to reach escape velocity. Escape velocity is given by

$$v = \sqrt{\frac{2GM}{r}} \quad (8)$$

As I will see in the results, one box is far from enough to get us anywhere. To find the amount of boxes I need to give us sufficient force, I am going to see how many boxes I need to get escape within a somewhat randomly picked time of 10 min.

$$v(t) = \frac{F}{m}t = \frac{F_{perbox}n_{boxes}}{m_{launcher}}t \quad (9)$$

$$n_{boxes} = \frac{vm}{Ft} \quad (10)$$

I am now going to assume that the fuel does not contribute to the mass of the satellite, and see how much fuel is needed to get to escape velocity. Since this is an unrealistic assumption, I have to have an analytical expression for the needed fuel. In the appendices in section 5 this is shown to be:

$$m_{f0} = m_l(e^{\frac{\Delta v m_e}{F_b}} - 1) \quad (11)$$

I am going to check if this equation gives us the correct amount of fuel.

2.3 Simulation of our Solar System

2.3.1 2-body problem

To get sufficiently accurate information about our solar system during the our trip, I want to simulate it for about 20 years. I am only interested in the 2-body problem exerted between one planet and the sun. And I am only going to look at solar system as flat system, laying in the xy-plain. So to describe the motion of the planets I use Newtonian gravity:

$$\mathbf{F}_G = -\frac{Gm_p M_s}{r^3} \mathbf{r} \quad (12)$$

m_p being the mass of the/a planet. This force will be exerted both on the planet and the sun, but since I want a stationary star, the force will only be exerted from the sun to the planet. Applying Newtons second law, I get the differential equation:

$$\mathbf{a} = \frac{d^2 \mathbf{r}}{dt^2} = -\frac{GM_s}{r^3} \mathbf{r}$$

I want to solve this differential equation numerically. To do this, I am going to use the Leapfrog method 4.1. By knowing the initial position and velocity I can now find the position and velocity for the planets for our whole journey. 20 years are simulated with a time step $\Delta t = 1/20000$.

I also want to you appropriate units. For all the simulations and calculations from now and until I am in orbit around Isskji I am going to use astronomical units, where distance are measured in an astronomical unit $1AU = 1.496 \cdot 10^{11} m$ and mass in solar masses $1M_\odot = 1.988435 \cdot 10^{30} kg$. To use Newton's equation as described over, I have to redefine the gravitational constant G :

I am going to look at the earth-sun-system. For the earth to stay in a circular orbit, the gravitational force has to equal the centripetal force:

$$m_p(\omega)^2 r = \frac{G m_s m_p}{r^2} \quad (13)$$

If I use that $\omega = \frac{2\pi}{T}$, I'll get

$$\left(\frac{2\pi}{1 \text{year}} \right)^2 1 \text{AU} = \frac{G \dot{M}_\odot}{1 \text{AU}^2} \quad (14)$$

$$G = 4\pi^2 [AU^3 M_\odot^{-1} \text{year}^{-2}] \quad (15)$$

2.3.2 N-body problem

If I let all the planets interact with one another, I'll get the N-body problem. I am going to use the same numerical method as above 4.1, but the differential equation looks slightly different:

$$\mathbf{a}_i = \frac{d^2 \mathbf{r}_i}{dt^2} = - \sum_{i \neq j} \frac{G m_j}{r_{ij}^3} \mathbf{r}_{ij} \quad (16)$$

where a_i is the acceleration on planet i , and \mathbf{r}_{ij} the distance from planet i to j . I am going to treat the sun as just another planet. A consequence of this is that the center of mass won't lay in the origin. Not only that, but since the since the momentum of the center of mass is the sum of the momentum of the planets

$$|M \dot{\mathbf{R}}| = \left| \sum_i m_{pi} v_{pi} + m_s v_s \right| > 0 \quad (17)$$

I have to be careful to give the sun an initial velocity such that the momentum of the center of mass is equal to 0. To ensure this, I'll make the sun's momentum to be equal to that of other planets, but in the opposite direction:

$$\sum_i m_{pi} v_{pi} = -m_s v_s \quad (18)$$

Last I want to look at the motion of the sun. If some of the planets orbiting our sun is large enough, their gravitational pull on the sun would give it a large enough velocity, such at if there are observers looking at out solar system far away from a direction parallel to the x-axis, I will see red shift in the light from the sun when sun moves away from them, and a blue shift when it moves towards them, given by:

$$\Delta\lambda = \frac{v}{c} \lambda \quad (19)$$

Given that these observers had sufficient technology to observe very small Doppler shifts, they could plot the speed over time, and deduce that sun. I am interested in making such a plot for our sun, and to see if these observers could conclude that out sun has planets orbiting, if they had the same technology available to them as I have. To make this plot more realistic, I am going to add some noise given by a Gaussian distribution with $\mu = 0$ and $\sigma = \frac{1}{5} \max(v_s)$.

The system was simulated with the 3 biggest planets – the smaller planets will not have a noticeable effect compared with the biggest planets, and are to necessary to take in to account-. The simulation lasted 60 years with a time step $\Delta t = 1/20000$.

2.4 Information about the planet

Before setting out on our journey, there are some things I want to know about where I am going. First is the size of solar panels I need to bring, second is the surface temperature and whether Isskji is inside the habitable zone.

When I have landed on Isskji, I want the lander to be able to operate from the poIr it receives from its solar panels. I therefor need to calculate the poIr due to the sun at Isskji's distance from the sun. Since

$$P = FA \quad (20)$$

I need to find the flux. Flux is given by

$$F = \frac{L}{4\pi R^2} \quad (21)$$

Where R is either the radius of the sun or the distance to the Isskji, dependent on whether I want to calculate F_{in} or F_{out} .

The flux leaving the sun is given by Stefan–Boltzmann law

$$F_{out} = \sigma T^4$$

Meaning that I can rewrite the luminosity, with R being the radius of the sun, as

$$L = 4\pi R_*^2 \sigma T_*^4 \quad (22)$$

I can now find the flux received at the distance of Isskji

$$F_{in} = \frac{L}{4\pi R_p^2} = \frac{R_*^2}{R_p^2} \sigma T_*^4 \quad (23)$$

Giving us the poIr

$$P = A \cdot \frac{R_*^2}{R_p^2} \sigma T_*^4 \quad (24)$$

I can thus calculate the area I need the solar panel to be

$$A = P \cdot \frac{R_p^2}{R_*^2 \sigma T_*^4} \quad (25)$$

T_* can be found by Wien's displacement law

$$T = \frac{b}{\lambda_{max}} \quad (26)$$

$b \approx 2900 \mu m \cdot K$, and λ_{max} is the peak of the wavelength for the sun.

To find the surface temperature of a planet, I can make a simple approximation to its climate system: first that the planet is a black body, and second that the only factor determining the temperature of the planet is the poIr received by the planet and the poIr lost due to black body radiation. PoIr is received on the surface of the planet. Only half of the planet is facing the sun at any given time, and the poles receive less sun light than the equator. But an approximation to

this is that the area receiving the poIr is that of a circle with the same radius of the planet r_p . The area where poIr is lost is the same as the surface area of the planet. The surface temperature is the temperature where there is an equilibrium between the poIr in and the poIr out:

$$P_{in} = P_{out} \quad (27)$$

$$P_{in} = L \frac{\pi r_p^2}{4\pi R_p^2} \quad (28)$$

$$P_{out} = \sigma T_p^4 4\pi r_p^2 \quad (29)$$

using the luminosity from above, I then get the following expression for the surface temperature.

$$T_p = \sqrt{\frac{R_*}{2R_p}} T_* \quad (30)$$

I can see that the surface temperature is not dependent on the size of the planet. This is because its size determines both the poIr in and out, and so cancels out in the equation.

I also want to see if Isskjji is inside the habitable zone. The habitable zone is zone within a distance from the star giving the planets a surface temperature where I can find liquid water and the planet is able to sustain life. This range is $T \in [260, 390]$.

2.5 Planing the journey

In a normal space mission the first step is to get in orbit around the home planet, and then to burn to escape velocity to get to a heliocentric orbit. Then comes the interplanetary trajectory, after getting close to the destination planet a last orbital injection is needed.

To get to the planet I used the so called conic approximation [1], this means that I treated every this as a 2-body problem. I only looked at the satellite and the body of which the satellite was inside the sphere of influence(below).

In our simulation I chose to ignore the first orbit. And instead of going for an escape burn, I chose to calculate a burn giving us zeros velocity at the edge of the so called sphere of influence. This is radius there I am no longer dominated by the gravity of our home planet, but rather that of the sun. This radius is given by [5]

$$r_{soi} \approx a \left(\frac{m_p}{M_*} \right)^{2/5} \quad (31)$$

Where a is the semi major axis of the planet's orbit. Since total energy is conserved, I can say that I want the kinetic energy at r_{soi} to be zeros. I can then solve for the initial speed and find that

$$v_0^2 = 2Gm_p \left(\frac{1}{r} - \frac{1}{r_{soi}} \right) \quad (32)$$

r being the radius of the home planet.

To get to the planet I used Hohmann transfers [1].
The Hohmann transfer is given by

$$v = \sqrt{2\mu \left(\frac{1}{R} - \frac{1}{R + R_p} \right)} \quad (33)$$

where $\mu = GM_*$ and R is the distance from the sun to the home planet. It is important to note that this gives the total speed the satellite, and not the Δv .

After this burn I would like to have an encounter with Isskji at the aphelion of the orbit of the satellite. I can find the time to this encounter:

$$\Delta t = \pi \sqrt{\frac{(R + R_p)^3}{8\mu}} \quad (34)$$

The last thing I need to know is the angle between the home- and destination planet, so that the encounter happens at the aphelion. This angle is given by:

$$\gamma = (\nu_2 - \nu_1) - \omega \Delta t \quad (35)$$

Where ω is the angular speed of Isskji and $(\nu_2 - \nu_1)$ is the angle of the orbit I want the encounter to be at. Since I want this to be at the aphelion, this angle has to be π

Since I am impatient, I want to make the two burns mentioned above in one burn v_{launch} . This is given by

$$v_{\text{launch}}^2 = v_0^2 + v_{\text{transfer}}^2 \quad (36)$$

So the steps for getting an encounter with Isskji:

1. Find Δt
2. Calculate and wait until the angle between the two planets are γ
3. Launch with $v_{\text{launch}}^2 = v_0^2 + v_{\text{transfer}}^2$

After getting an encounter I want to get in orbit with an orbital injection burn. A stable circular orbit happens when the gravitational force is equal to the centripetal force.

$$v_{\text{orb}} = \sqrt{\frac{\mu}{r_{\text{orb}}}} \quad (37)$$

I need to find \mathbf{v}_{orb} . Since the orbit is circular, the velocity has to be perpendicular to the position at all time. And given an angle θ of the position it is easy to find an unit vector perpendicular to it, and \mathbf{v}_{orb} :

$$\mathbf{v}_{\text{orb}} = v_{\text{orb}}(-\sin \theta, \cos \theta) \quad (38)$$

To find the Δv I have to subtract the velocity which I already have, which is the velocity relative to the planet

$$\mathbf{v}_{\text{rel}} = \mathbf{v} - \mathbf{v}_p \quad (39)$$

So the final burn is given by:

$$\Delta\mathbf{v} = v_{orb}(-\sin\theta, \cos\theta) - \mathbf{v}_{rel} \quad (40)$$

This equation can be given more geometrical expression, using an angle $\chi = \pi - \theta$, but in practice the above equation is the one that is implemented in the simulation.

The orbital injection is at a distance from the planet where gravitational force from the planet is some constant k larger than that of the sun:

$$F_p = kF_* \Rightarrow \frac{Gm_p}{r^2} = \frac{GM_*}{R_p} \quad (41)$$

$$r = R_p \sqrt{\frac{m_p}{M_*} \frac{1}{k}} \quad (42)$$

In our simulation $k = 10$.

The simulation is done over about 10 years with two different schemes with two different time steps:

The first is in interplanetary space, where I only need a crude, but still accurate simulation. Here the time step is $\Delta t = 1/30000$. The second scheme happens when I am close to planets, where I have to have a more accurate simulation, and therefore use a time step $\Delta t' = \frac{\Delta t}{1000}$. I also use a interpolated function of the positions of the planets. This is so that I have continuous paths.

Any correction burns near Isskji was planned as follow:

1. Find a unit vector \hat{e}_c pointing towards where Isskji was going to be in a day or two
2. Burn with $0.5\hat{e}_c$ to $0.8\hat{e}_c$

This is not the most efficient method, but will make the trajectory of the satellite move more towards the planet. The factors were found by trial and error.

2.6 Navigation

The satellite is equipped with three navigation instruments.

- A camera to take pictures of the stars, this is used to determine orientation.
- A radar to find the distance to all the planets, this is used to determine position.
- A spectrograph to find the doppler shift of two distant stars, this is used to determine the velocity.

The first time the satellite tried to orient I had to use these methods, this was to calibrate the satellites own orientation software, so for the rest of the journey, the satellite was able to use the methods to find its own position.

2.6.1 Orientation

The satellite is able to take pictures of the stars, and compare this picture to a complete atlas of starscape. This picture is then compared using a least square comparison. The orientation which gives smallest error, is determined to be the correct orientation. Since our whole journey is going to be in the solar equatorial plane, I am only interested in knowing the orientation in the ϕ angle,

letting $\theta = \pi/2$.

The starscape as seen by the satellite is a sphere. I'll approximate the stars to be an infinity away from us, this means that the formation of the stars are independent on where in the solar system I am. To get a picture of the stars I have to find a way to translate the curved geometry of the starscape to the flat picture. This is done with *stereographic projection* [4].

Before I start the journey I have to make the atlas, from a place where the orientation always is known. I am going to do this by turning the camera 360° around the equatorial plane, taking a picture for every degree. The place where the pixel $(0, 0)$ is pointing has the coordinate (θ_0, ϕ_0) (as stated above θ_0 is always $\pi/2$). For each pixel of the picture, the camera needs to translate this to the polar coordinates. This is done by:

$$\theta = \frac{\pi}{2} - \sin^{-1} \left(\cos c \cos \theta_0 + \frac{y_{picture} \sin c \sin \theta_0}{\rho} \right) \quad (43)$$

$$\phi = \phi_0 + \tan^{-1} \left(\frac{x_{picture} \sin c}{\rho \sin \theta_0 \cos c - y_{picture} \cos \theta_0 \sin c} \right) \quad (44)$$

where

$$\rho = \sqrt{x_{picture}^2 + y_{picture}^2} \quad (45)$$

$$c = 2 \tan^{-1} \left(\frac{\rho}{2} \right) \quad (46)$$

The camera has a field of view in the vertical direction α_ϕ and the horizontal direction α_θ , both being 70° . I need to find the how large our pictures can be. I am going to use the expressions for $x_{picture}$ and $y_{picture}$ found here [4]. To find how large the picture can be in the x-direction, I set y to be 0, $\theta_0 = 0$ and get

$$y = k(\cos \theta) = 0 \quad (47)$$

$$k = \frac{2}{1 + \sin \theta \cos(\phi - \phi_0)} \quad (48)$$

Solving for θ I find that $\theta = \pi/2$. I can put this in the equation for x and find its max/min value:

$$x = \frac{2 \sin(\phi - \phi_0)}{1 + \cos(\phi - \phi_0)} \quad (49)$$

$$x_{max/min} = \pm \frac{2 \sin(\phi - \phi_0)_{max}}{1 + \cos(\phi - \phi_0)_{max}} \quad (50)$$

I set $(\phi - \phi_0)_{max} = \alpha_\phi/2$. I then get that

$$x_{max/min} = \pm \frac{2 \sin(\alpha_\phi/2)}{1 + \cos(\alpha_\phi/2)} \quad (51)$$

Doing the same for y I find that

$$y_{max/min} = \pm \frac{2 \sin(\alpha_\theta/2)}{1 + \cos(\alpha_\theta/2)} \quad (52)$$

When I now have an atlas of the starscape, the satellite can use this to orientate. To do this, it will take a picture. then compare pixel by pixel with the 360 pictures, finding a total error of

$$E(\phi) = \sum_{ij} (pix_{ij}^\phi - pix_{ij}^{taken})^2 \quad (53)$$

The angle ϕ which gives the loIst error $E(\phi)$ is the orientation of the satellite.

2.6.2 Velocity

A spectrograph on-board the satellite finds the doppler shift of H_α (656.3 nm) of to distant stars. These two stars are again so far away, that the direction of their motion will be observed to be the same at any position in the solar system. From the doppler shift one star it is possible to find the radial velocity relative to the star. I know the relative to the star for the rest frame of star. From these two points of data the velocity of the satellite can be found

$$v_{sat} = v_{refstar} - v_{rel} \quad (54)$$

If I know v_{sat} in the direction both of the two star, I can find the velocity in the xy-plane. If the angle betIen the stars had been $\pi/2$ this would have been trivial, but since this isn't necessarily the case, I have to do a transformation of coordinate system. From [4] I get that:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cos \phi_1 & \sin \phi_1 \\ \cos \phi_2 & \sin \phi_2 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} \quad (55)$$

where v_1, ϕ_1 are the v_{sat} towards planet 1 and the angle for planet 1, and v_2, ϕ_2 the same for planet 2. But I am interested in v_x, v_y , so I have to invert the matrix. For a 2x2 matrix the inverse is given by:

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad (56)$$

So I can find v_x, v_y by the inverted equation

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \frac{1}{\cos \phi_1 \sin \phi_2 - \cos \phi_2 \sin \phi_1} \begin{pmatrix} \sin \phi_2 & -\sin \phi_1 \\ -\cos \phi_2 & \cos \phi_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (57)$$

$$= \frac{1}{\sin(\phi_2 - \phi_1)} \begin{pmatrix} \sin \phi_2 & -\sin \phi_1 \\ -\cos \phi_2 & \cos \phi_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (58)$$

I can see that if $\phi_2 = \phi_1$ the matrix becomes singular and not possible to invert, and it is thus impossible to find the velocity of the satellite. This is because the v_1 and v_2 in this case are parallel, and I haven't enough information to find the velocity in any other direction than this direction.

2.6.3 Position

From the radar the satellite gets the distance to all planets and the sun. It will also know the positions of all the planets from our simulations. From this the satellite has to find its own position. There are two methods for doing this. The first is triangulation: given the distance and positions of three or more of the planets, it is possible to find an analytical expression for the position. Secondly this can be done numerically with a gradient decent^{4.2}. I chose to go for the latter option, mostly for the fun this method is to program.

I used this algorithm to find the position:

1. Start with a guess.
2. Loop over a fixed number of steps (I used 10^5)
3. Update the guess' coordinate x_i with $\omega \frac{\partial E}{\partial x_i}$ for each step.

Where $\frac{\partial E}{\partial x_i}$ is the error function differentiated for x_i

$$\frac{\partial E}{\partial x_i} = \frac{2}{N} \sum_{j=1}^N (d_j^{guess} - d_j^{real}) \frac{(x_{ij}^{guess} - x_{ij}^{planet})}{d_j^{guess}} \quad (59)$$

d_j is the distance to the j^{th} planet j , and x_{ij} is the coordinate component x_i to for that same planet. ω is the *ligh*t, this is to keep the function from diverging 4.2.

To check if this method worked, I wrote a small test function which used this method on random positions, with random guesses.

2.7 Simulating the atmosphere

For getting a successful landing, I have to know what the atmosphere looks like. The atmosphere was modeled as adiabatic if the temperature $T < T_0/2$ and isothermal above this height. The first step for doing this is to find the average molecular light of the atmosphere.

2.7.1 The Mean Molecular Weight of the Atmosphere

I have the entire spectrum of the atmosphere. The gases I expect I can find are

- Oxygen O_2 has absorption lines at 630nm, 690nm and 760nm
- Water vapour H_2O has absorption lines at 720nm, 820nm and 940nm
- Carbon dioxide CO_2 has absorption lines at 1400nm and 1600nm
- Methane CH_4 has absorption lines at 1660nm, 2200nm
- Carbon monoxide CO has an absorption line at 2340nm
- Nitrous oxide N_2O has an absorption line at 2870nm

The satellite which took the spectrum of the atmosphere had a velocity $< 10km/s$, so there will be a doppler shift in the data. The gases will also have a temperature, which is unknown, and there will be quite a lot of noise in the data. Because of this, it is not easy to see the spectral lines. To check a gas is in the atmosphere I have to try to fit a model for the flux of a spectral data. The model is given by:

$$F^{model}(\lambda) = F_{max} + (F_{min} - F_{max}) e^{\frac{-(\lambda - \lambda_{center})^2}{2\sigma^2}} \quad (60)$$

I know that the $F_{max} = 1$, and I can estimate the $F_{min} > 0.7$, but the exact value is unknown. So the unknown quantities in this model is F_{min} , λ_{center} and σ (*sigma* being the standard deviation of the flux around λ_{center} due to temperature). To find these quantities I am going to use a minimum chi-square estimation. I am going to minimize

$$\chi^2(\lambda_{center}, F_{min}, \sigma) = \sum_{i=1}^N \frac{(F(\lambda) - F^{model}(\lambda))^2}{\sigma_i^2} \quad (61)$$

where σ_i is not the same σ as in the model, but the noise of the data, which varies for each measurement. σ_i is known.

I am going to minimize this by guessing random numbers for the unknown variables, and then pick the numbers which gives the lowest χ^2 . I have some ideas about the values of the variables and will chose random numbers based on this. F_{min} will be between 0.7 and 1. I also knew that the spectral lines at the most can be shifted by

$$\Delta\lambda_{max/min} = \pm \frac{10km/s}{c} \lambda \quad (62)$$

The temperature of the gases will be between 150K and 450K, and the $\sigma_{max/min}$ can be determined from this by:

$$FWHM = \frac{2\lambda_0}{c} \sqrt{\frac{2kT \ln 2}{m}} \quad (63)$$

$$\sigma = \frac{FWHM}{2\sqrt{2 \ln 2}} \quad (64)$$

I chose 300 values for λ_{center} , and 30 for F_{min} and σ . I was not interested in finding χ^2 using the whole spectrum, since the amount of data is huge. Instead I used only a small part of the spectrum around where I knew the spectral lines to be. I then found which values gives the best fitting model for each line in the list above.

Having found the models, I plotted those against the spectrum to see whether the models were real, or they were false positives. This was done visually. Including whether or not the models fitted I looked at the temperature. I already knew what the temperature of the planet was suppose to be, and if temperature of a specific model was to far from the expected, it was thrown out, even if it seemed to fit the data nicely. I then said that for every spectral line of a gas I found, I added one part of this gas in the calculation of the mean molecular weight. The mean molecular weight was then calculated as:

$$\mu = \frac{1}{N} \sum_{i=1}^N m_i \quad (65)$$

m_i being the molecular mass of the gases found.

The spectrum was given in large text documents that was slow to read. So the first time the program runs the data is made to npy-files, which are faster to read. The models are also calculated once, and the parameters are saved. So the next time I want to look at the spectra, the parameters are read from this file.

2.7.2 The atmosphere

I found the expression for density of the atmosphere analytically⁶. The equation for the density of adiabatic part of the part of the atmosphere is

$$\rho(h) = \rho_0 \left(1 - \frac{\gamma - 1}{\gamma} \frac{h}{h_0}\right)^{\frac{1}{\gamma-1}} \quad (66)$$

Where $h_0 = \frac{kT}{m_H g \mu}$. The temperature of the adiabatic atmosphere

$$T(h) = T_0 \left(1 - \frac{\gamma - 1}{\gamma} \frac{h}{h_0}\right) \quad (67)$$

For the isothermal atmosphere the density is

$$\rho(h) = \rho'_0 e^{-\frac{h}{h_0}} \quad (68)$$

and the temperature is alway $T_0/2$. $\rho'_0 = \rho_0 (\frac{1}{2})^{\frac{1}{\gamma-1}}$ is the density at the hight where the atmosphere goes from adiabatic to isothermal.

2.8 Landing

After getting into orbit around Isskji, I Int from astronomical units to SI units, due to the smaller distances and velocities. The origin of the coordinate system was also chanced from the center of the sun to the center of Isskji.

2.8.1 Low orbit

After getting the satellite in orbit around Isskji, I had to fin a way of getting it in a low-Isskji orbit to take pictures and decide where to land.

To get a stable orbit I had to try to stay clear of the atmosphere. I decided to go to a hight where the density of the atmosphere $\rho(h) < 10^{-12}$. This was then simulated to ensure a stable orbit. I then used a Hohmann transfer to get to the desired hight, and when the satellite reach this hight, the orbit was circularized. The satellite when Int in an equatorial orbit to scout the planet for a landing site.

To translate the picture in to a coordinate on the surface of the planet, I asked the satellite to orient it self right after the picture was taken. I then took the position vector of the satellite and turned it around, so it now pointed at the center of the planet. I could now find the angles of the landing site. But this was the angles of the surface of the planet in a global coordinates. since the surface rotates, this global coordinate is not the same as the local coordinate at the surface. One time I Ire sure the global coordinate Ire the same was at the time I entered the first orbit(this time was set to $t = 0$, and I define that the local and global coordinates are the same at this point). To find the global coordinate at t_0 , I used the rotational period (in earth days) of the planet τ_p (since I Ire in an equatorial orbit, $\theta = \pi/2$, so I am only interested in ϕ)

$$\phi = \tan^{-1} \left(\frac{y_{sat_planet}}{x_{sat_planet}} \right) - \frac{1}{\tau} t_{years} \cdot 2\pi \quad (69)$$

(I had to make sure that $\frac{1}{\tau} t_{years} < 1$ and if not I just subtracted the floored value $\frac{1}{\tau} t_{years} - \text{floor}(\frac{1}{\tau} t_{years})$)

I then had the coordinate I wanted to land.

2.8.2 The Landing

After finding a place to land I had to find a way of landing there. Due to the difficulty of solving the landers interaction with the atmosphere in any smart way, this was done with brute force and guessing. The idea was to use another Hohmann transfer to some hight inside the atmosphere. The goal was to find a hight such that I would land on the surface after a half orbit. This hight was a result of trial and error.

I could now simulate this maneuver, and see where I hit the surface. Knowing this position I could use the same method mentioned above to find the local coordinate on the surface. If this coordinate wasn't the desired coordinate the satellite would be asked to way in orbit for some time, then I would simulate the landing again. This procedure would be done until the satellite hit the landing site. This is a very brute force way of doing it, but it should work.

2.8.3 A Safe Landing

The drag force of the atmosphere is given by:

$$F_D = \frac{1}{2} \rho C_D A |\mathbf{v}| \mathbf{v} \quad (70)$$

C_D is a coefficient determined by the geometry of the lander, and is set equal to 1, and A is the surface area of the lander. When this force is equal to the gravitational force, the lander has reach terminal velocity.

$$\frac{GM_p m_l}{R^2} = \frac{1}{2} \rho C_D A |\mathbf{v}| \mathbf{v} \quad (71)$$

If I use the radius of the planet and the density of at the surface, I can find the terminal velocity at close to the surface.

$$v = \sqrt{\frac{2GM_p m_{lander}}{R_p^2 \rho_0 A}} \quad (72)$$

I could use this to find the area I need the parachute to have if I wanted to land safely.

$$A = \frac{2GM_p m_{lander}}{R_p^2 \rho_0 v^2} \quad (73)$$

2.9 Launching the Real Satellite

Having simulated a plan for the whole journey, it was time to launch the actual satellite. The satellite had two stages, first was the interplanetary journey, and the second was the orbit and landing on the planet.

Since I had used the conic approximation to make the plan, I could not expect to use our plan without having to do some corrections under way.

3 Results and Discussion

3.1 The Engine

The fist result is the pressure and kinetic energy inside of the box:

Numerical Pressure: 13756.1188494
 Analytical Pressure: 13800.0
 Analytical energy: 2.07e-19
 Numerical energy: 2.06623799518e-19

As I can see, the numerical and analytical results have relative errors within a factor of 10^{-4} , meaning that our simulation are quite accurate.

The data from the engine was as follows:

Force: 1.72650965887e-09
 Number of particles colliding with floor: 257317
 Number of particles escaped: 64252
 Number of particles escaped per sec: 6.4252e+13
 Mass lost per sec: 2.120316e-13

I can see that the force exerted by a single box is minuscule, and a lot of boxes are needed to lift the satellite off the ground.

The data above are from one specific simulation, and will vary enough to make results in later parts of this article a bit different. I will therefore use the data given here for all subsequent situations where they are needed.

3.2 A simple launch

Using equation 8 I see that $v_{escape} \approx 16596.9m/s$. And from 10 I get that the amount of boxes I need are $n = 1.76 \cdot 10^{13}$.

The plot I get from launching the satellite without fuel loss:

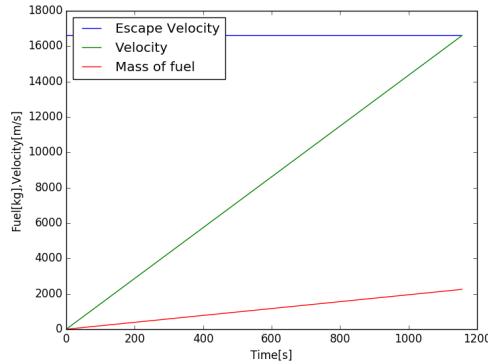


Figure 1: Shows the mass of fuel increasing as the satellite accelerates. The craft ends up using right under 2000 kg of fuel.

The plot of the launch with fuel loss shows how the satellite loses fuel as it ascends. The initial fuel was given by equation 11 as 2250.966kg.

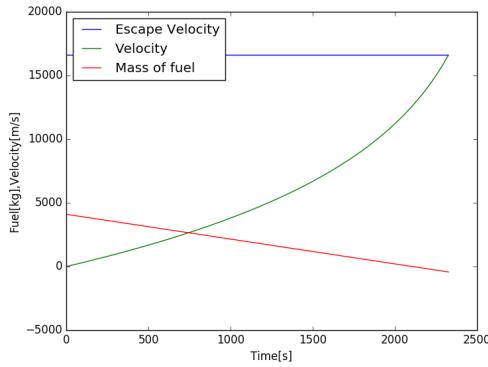


Figure 2: Shows the mass of fuel slowly going to zeros as the satellite reaches escape velocity.

I can see from the plot over that as the satellite reaches escape velocity, the mass of the fuel is more or less zero. This indicates that our expression for fuel gives the correct amount of fuel.

3.3 Simulation of our Solar System

3.3.1 2-body problem

The first result is with one planet and the sun. I choose to just look at the largest planet and the sun

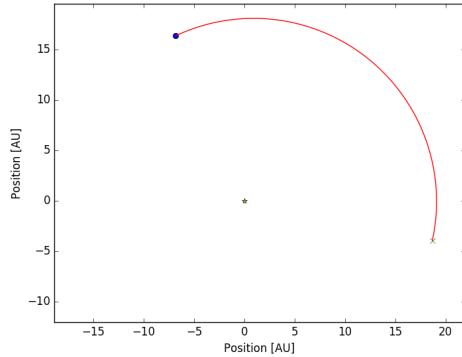
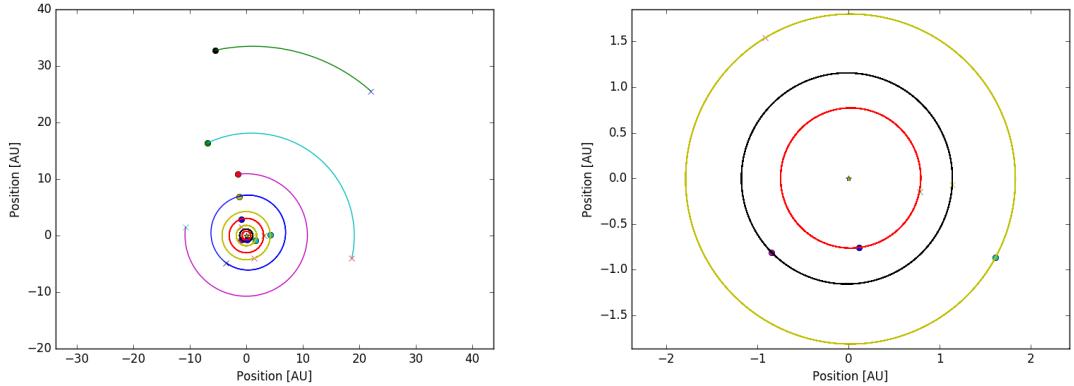


Figure 3: The largest gas giant in the solar system and the sun. Since the planet is quite a distance from the sun, about 16 AU, it only got 1/3 of the way around its orbit during the 20 years.

I can see that I have the circular motion I expect, so it passes the rudimentary test for what I expect from planet-sun-system.

It is more interesting to look at the whole system of 9 planets.



(a) The orbit of the all 9 planets and the sun. x marks the initial position of the planets, while the ball represents the positions at the end of the simulation.

(b) A closer look at the inner planets.

I can see that even for the inner planets the simulation seem stable. Since these planets is closer to the sun, they will have orbited more times than the outer planet. There is an error from the numerical method for each orbit, these planets are expected to have accumulated a greater error than the outer planets. Since the I can see that the orbits are stable for the inner planets, I know that the orbits are stable for the rest of the system as Ill.

The orbits are checked against the analytical solutions, and over the 20 year period, the largest relative error was no more than 0.0036%.

3.3.2 N-body problem

I am interested in seeing how the other planets influence the sun.

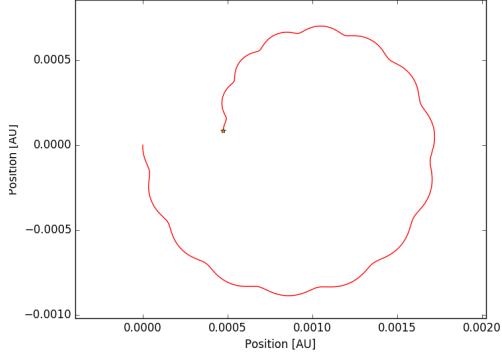
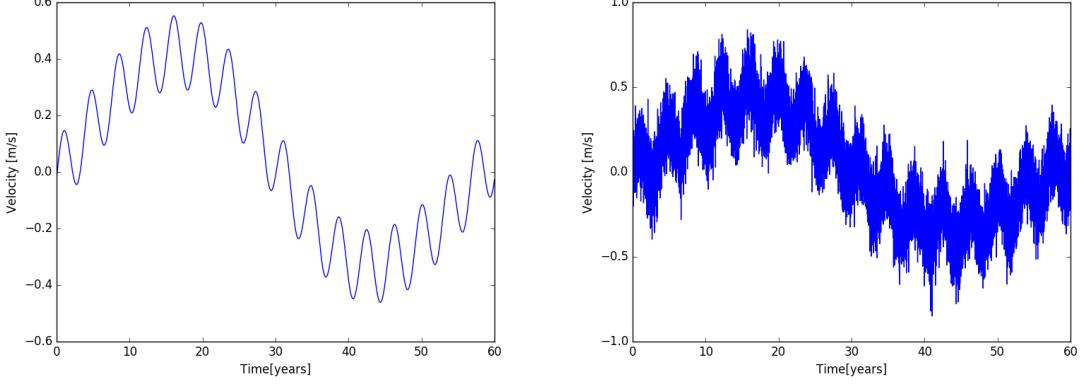


Figure 5: The motion of the sun. The sun has just finished one orbit of the barycenter of itself and the second largest planet. One can see that this is not a exactly circular orbit, this is because the sun is tugged by the largest planet, which is just 1/4 though its orbit.



(a) The velocity of the sun towards and away from an observer, gotten from the red shift of the star light.
(b) The velocity of the sun with noise to emulate the noise one gets in real astronomical observations.

I can see that even with noise added to the velocity, the motion is still clearly visible. The difference in velocity is $\pm 0.6 \text{ m/s}$. With todays technology I am only capable of observing velocity differences of 1 m/s , meaning that despite there being a clear motion in the data, someone with our technology would not be able to observe it.

3.4 Information about the planet

Our star has a temperature of 8284.6 K , and a radius of 1610671.4 km . The solar panels I used have an efficiency of 12%. Using this and 25 I found that the necessary area of the solar panels to have 40 W of power was 17.6 m^2 .

I know that the habitable zone is the zone in the solar system where $T \in [260, 390]$, from 30 I found the habitable zone to be $r \in [8.1765 \cdot 10^8, 3.63402 \cdot 10^8]$ in km . This meant that Isskji, being $6.38 \cdot 10^8 \text{ km}$ from the sun, was in the middle of the habitable zone. The calculated surface temperature of Isskji is 294.4 K , a quite comfortable temperature. This was what I had expected after earlier telescope images had shown Isskji to be a garden planet. This gave us that Isskji was an interesting planet to explore, in the hope of finding life (the fact that it is our neighbor planet also played a role in the decision.)

3.5 Planing the Journey

The correct angle between the two planets, and therefore the launch, happened after 6.71585 years.

There was a lot of problems with this part of the simulation. While the algorithm should have worked, the problem, as I knew but didn't take seriously, was that Hohmann transfers and all the associated expressions only work for circular orbits. While the orbits of our home world and Isskji were almost circular, with eccentricity $e = 0.00526$ and 0.00855 , they are not completely circular. This meant that the transfers missed. To get an encounter with Isskji I therefore had to adjust v_0^2 and v_{transfer}^2 ever so slightly, with what in rocket science terminology is called a *fudge factor*. v_{transfer}^2 is very sensitive to changes [1], so the changes had to be small. Due to an early error in the code, I did not implement that

$$v_{\text{transfer}}^2 = v_0^2 + v_{\text{transfer}}^2 \quad (74)$$

but

$$v_{\text{transfer}} = v_0 + v_{\text{transfer}} \quad (75)$$

This meant that the transfers was even more inaccurate. But after 2 days of tIaking I found that with

$$v_{transfer} = 0.894v_0 + v_{transfer} = 3.525AU/year \quad (76)$$

I where able to get quite close to Isskji. But not close enough. From 42 I find the distance where I could do an orbital injection was $0.000395AU$. The satellite was not able to get with in this distance after one launcher burn. A correction burn was then hard coded in, to burn when the distance to Isskji was $0.003AU$. With a correction burn of $0.5AU/year$, the satellite was able to do an orbital injection. This burn was of size $0.6253AU/year$. This gave the simulated trip a total dv of $4.65AU/year$.

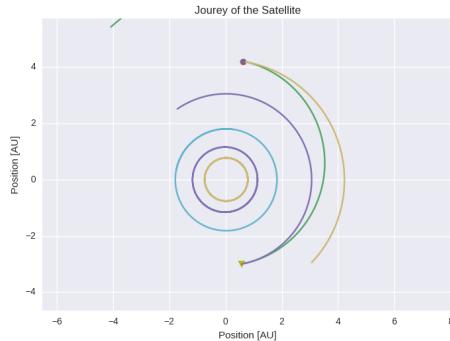
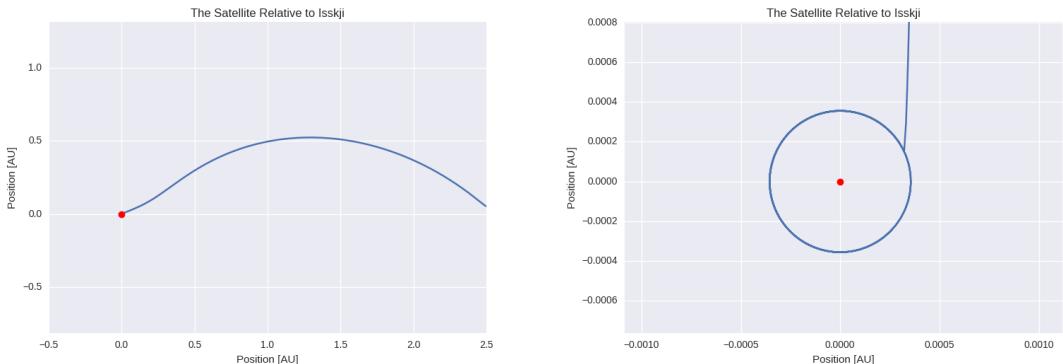


Figure 7: I can see that satellite leaves the home planet and encounters Isskji after almost a half orbit. I can see that the satellite follows the orbit of Isskji, but from this picture it is not clear if it is in orbit.



(a) The relative distance from the satellite to Isskji (b) Same as in (a), but zoomed in to Isskji. I can see that I have a circular orbit.

?? shows that I have simulated a successful journey and orbit around Isskji. It took 2.1 years.

Now I had the total Δv for the journey, and could calculate the fuel the satellite needed to reach orbit around Isskji. I decided to bring 10% extra Δv . The amount of fuel turned out to be $20558.3kg$ fuel.

3.6 Navigation

3.6.1 Velocity

The reference data when in the rest frame of the star

Reference star 1, at phi = 77.518724 degrees,
has delta lambda of -0.017002884383 nm in the home star rest frame.
Reference star 2, at phi = 325.121916 degrees,
has delta lambda of 0.017144686369 nm in the home star rest frame.

With this, the algorithm to find the velocity Int fine, without any problems. I tested it against the trivial case where the satellite got the same doppler shifts as in the rest frame, in other words the satellite is at rest; and where there are no doppler shift, the satellite has no relative velocity. The first test showed $v_{sat} = 0$, and the second gave that $v_{sat} = v_{refstar}$. This showed that the velocity orientation works.

3.6.2 Rotation

There is one problem with this program that became apparent immediately, the pictures have an inverted y-axis. The reason may be that the pictures are made as numpy-arrays. These arrays have the $(0, 0)$ – element, corresponding to the $(0, 0)$ – pixel, in the upper left corner, while a picture has its $(0, 0)$ – pixel in the lower left corner. This causes the picture to be upside down. This is easy to correct, but by the time this bug was discovered, the whole starscape atlas was made. This was a lengthy process, so instead of fixing the bug and remaking the album, I made a function that turns all the incoming pictures upside-down.

Then all this is corrected the program finds the correct angle every time.



Figure 9: A picture taken by the real satellite used for the first calibration of the orientation software.

3.6.3 Position

Everything seemed to work perfectly with this part. I got the correct position on every test and on the orientation on the real satellite. The problem started when I Int through the code while writing this article, the expression for $\partial E / \partial x_i$ was not the one from above 77, but

$$\frac{\partial E}{\partial x_i} = \frac{8}{N} \sum_{j=1}^N (d_j^{guess} - d_j^{real})(x_i j^{guess} - x_i j^{planet}) d_j^{guess} \quad (77)$$

When I changed this to real expression, the success rate of this method Int down from 95–100% to 60 – 90%. I do not have a clue why the wrong expression works while 77 doesn't. There may be something wrong in our math, so that the "wrong" expression is the correct one, but I do not know.

3.7 Simulating the Atmosphere

3.7.1 The Mean Molecular Weight of the Atmosphere

There are 3 sets of pictures I am going to show. The first is the one that looked to fit the model, but had a way to low temperature:

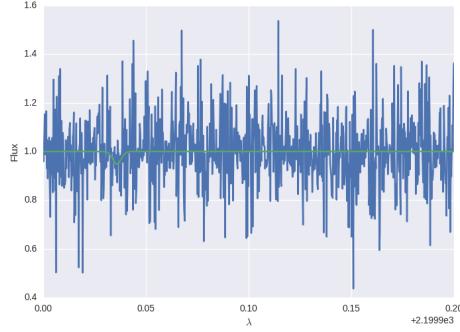
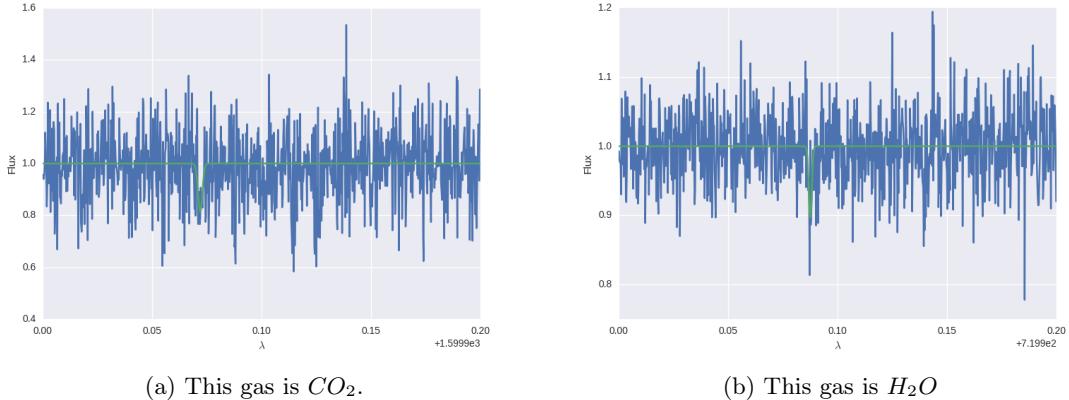
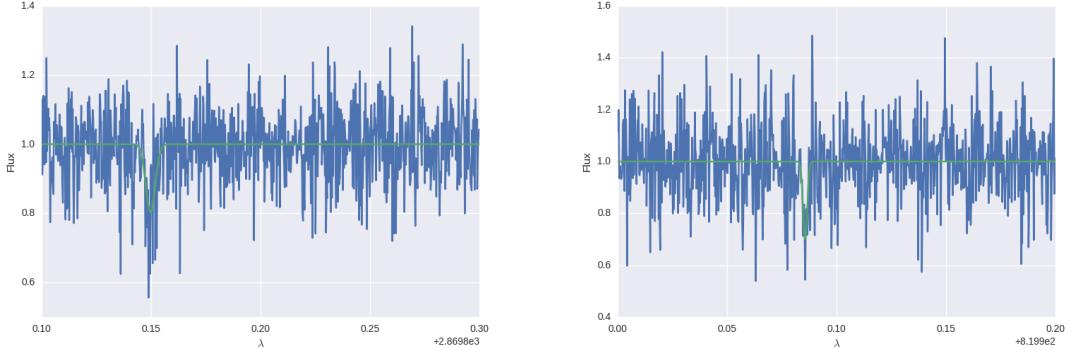


Figure 10: This is CH_4 , and had a very low temperature 147K. If this is a real hit, that we may expect life on Isskji.

The next pictures had the same temperature 184.7K, but it was lower then expected on our planet.



The last group also have the same temperature, 271.4, which is very close to the surface temperature of 294.4K



(a) This gas is N_2O , which indicates that there might we life on Isskji.
(b) This gas is H_2O , water. Also important for life.

I am going to break the fourth wall here!

There has been a lot of confusion on what the criteria for accepting a gas is. I was not sure if the gases in the atmosphere could have more than one temperature. The first case is quite weak, since the trend is minor. But I was not completely sure if I were to accept the gases at $184.7K$. If I accepted these and the gases at $294.4K$ I got a mean molecular weight of $31.013215u$, and with only the gases at $294.4K$ I got a weight of $31.01404u$. Since I accepted 2 extra gases which weighed about the same as those at $294.4K$, the mean changed only slightly.

For the rest of the simulations I went with $\mu = 31.01404u$

3.7.2 The Atmosphere

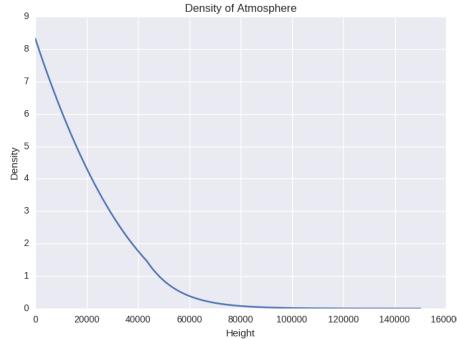


Figure 13: The density of the atmosphere as a function of hight over the surface.

We can see that the adiabatic density decreases fast. Near $40km$ the atmosphere becomes isothermal, and the decrease of the density begins to go slower. Since the atmosphere is found using an analytical expression, there are few things that went wrong the simulation.

3.8 Landing

3.8.1 Low orbit

As will be describes below, the Hohmann transfer to get to low orbit in the simulation did not correspond well with reality, so I had to update the position in the simulation to ensure a correct, circular low orbit.

The atmosphere of Isskji is quite thick, with $\rho_0 \approx 8$, this means that the safe distance for a stable low orbit is 390km, which is quite high compared with earth where a low orbit occurs at 160km above the surface.

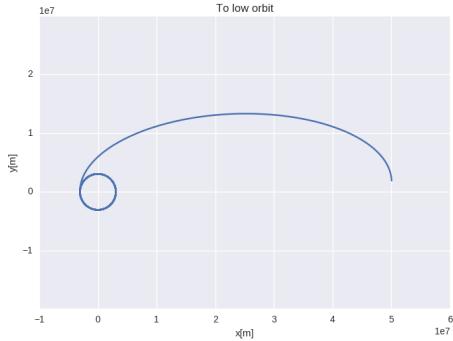


Figure 14: Shows the Hohmann transfer from high to low orbit.

In its equatorial orbit at 390km 20 pictures were taken, between 90461 – 107461sec, to find a suitable landing site. Four are shown here (due to the fact that 20 pictures would take a lot of space).

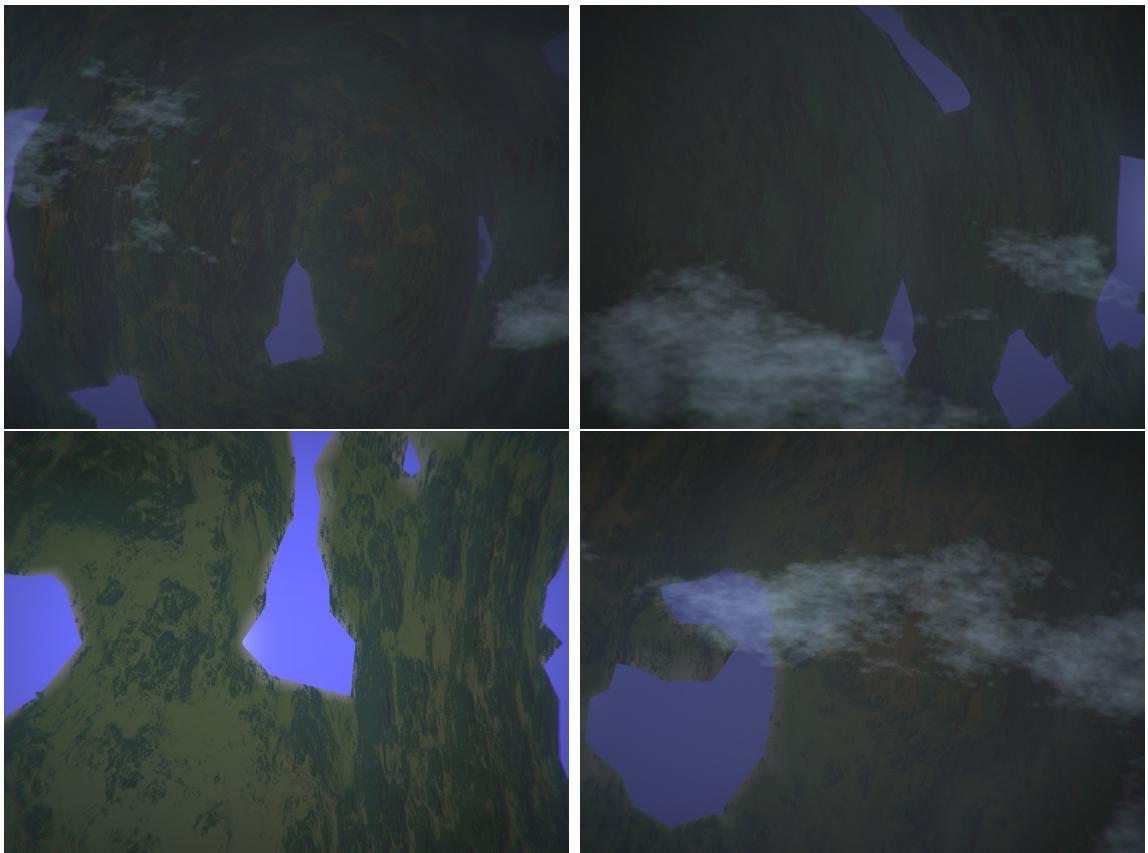


Figure 15: Pictures of the surface taken over a couple of orbits.(The subplots are mashed together because a couldn't find a way to make a good looking figure with 4 subfigures...)

Neither in these nor the rest of the pictures showed any place that sticks out. But I chose to land at the site showed in the upper left picture, due to the small lake surrounded by, what looks like, two mountains. I found that the local coordinates were $(180^\circ, 96.97^\circ)$.

I could have orbited the planet in an polar orbit if I wanted to find a better place to land. In an polar orbit I would have been able to take pictures of the whole surface. The reason I chose not to go into polar orbit was simply time constrains.

3.8.2 The Landing

The height to which I had to do a Hohmann transfer was found, by trial and error, to be 290km below the current orbit. This gave me a nice half orbit before landing:

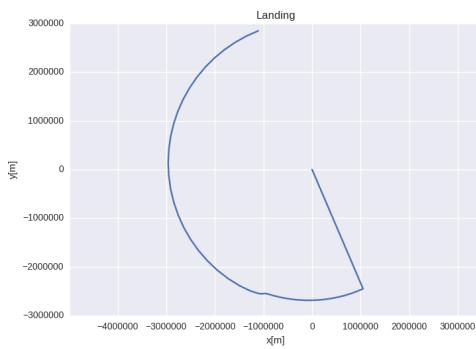


Figure 16: Shows the half orbit before hitting the ground. The simulation was stopped when hitting the surface, but due to a bug where the last positions in the array is not filled, and thus only contains zeros, it looks like the lander continues to the center of the planet.

And after more trial and error I found that if I let the satellite loiter in orbit for 40950sec after entering the low orbit, before launching the lander. The simulation then gave me the information

```
You landed at time 145815 with velocity -2.93852
ParachuteSize: 7.64794319168
Landing Angle: 92.002628546
Landing Angle in Rad: 1.60574878862
```

The goal was 96.97° (being in an equatorial orbit θ is always 180°). The simulation said I landed at 92° , which I found satisfactory, and decided to go for it.

3.8.3 A Safe Landing

As we have seen Isskjis atmosphere is quite thick, $\rho_0 \approx 8$, so I didn't expect that the size of the parachute had to be very large to reach a safe landing velocity of 3m/s . This turned out to be correct: the parachute need only be 7.65m^2 , which is smaller than a standard skydiving parachute.

3.9 Launching the Real Satellite

Then it was time to launch the satellite for real. We waited 6.72 years and launched. The launch was the same as in the simulation. I asked the satellite to orient it self after 1 and 2 years. The first orientation was the calibration orientation, and went without any hitch. Both orientations showed the satellite to be a little off course, but not by very much.

Orient 2:

Expected Position: [1.07228774 4.08232315]

Real Position: [1.07602195366 , 4.0834412672]

This was so close that no correction burn was done yet. When the satellite came close to the point of the correction burn, it got the command to do multiple orientations. When the point closest to Isskji was found after 2.05 years, I decided to do the correction burn, even though it wasn't the same place it the simulation. It was only as small burn ($-0.49152321, 0.26571668$). One and a half days later the satellite was inside the safe distance from Isskji and was able to do an injection burn of $(0.11537089, -0.32227829)$. The total Δv needed on the real journey was $3.145AU/year$, which is lower than the expected Δv from the simulations.

4 Appendix A: Numerical Methods

4.1 Leapfrog

4.2 Gradient descend

- F_b : Force from *one* box.
- n_b : Number of boxes.
- m_l : Mass of the satellite.
- m_f : Mass of the fuel.
- m_e : Mass of the particles escaping per second per box.
- m_{f0} : Mass of fuel when launching.

5 Appendix B: Fuel Equation

To find an analytical expression for the fuel use, I am going to derive the rocket equation using the parameters I get from out engine. I am going to start with Newton's 2. law to find the acceleration for our system:

$$a = \frac{F}{m} = \frac{F_b n_b}{m_f(t) + m_l} \quad (78)$$

Since the system are losing mass:

$$a = \frac{F_b n_b}{(m_{f0} - n_b m_e t) + m_l} = \frac{F_b n_b}{(m_{f0} - \Delta m t) + m_l} \quad (79)$$

Where m_{f0} is the initial mass of the fuel, and Δm is the mass loss per second. I separate the variables and solve the differential equation:

$$v(t) = \int \frac{F_b n_b}{(m_{f0} - \Delta m t) + m_l} dt = K - \frac{F_b}{m_e} \ln(m_{tot} - \Delta m t) \quad (80)$$

Where $m_{tot} = m_l + m_{f0}$. Using the boundary expression I find

$$K = v_0 + \frac{F_b}{m_e} \ln(m_{tot}) \quad (81)$$

This gives us the rocket equation for our satellite:

$$v(t) = v_0 + \frac{F_b}{m_e} \ln \left(\frac{m_{tot}}{m_{tot} - \Delta m t} \right) \quad (82)$$

I can see that $\frac{F_b}{m_e}$ has the dimension of velocity. This expression corresponds to the velocity of particles escaping (exhaust velocity). If I call this v_e I have the original rocket equation. I want to have use all the fuel when the desired Δv is reached. The only mass left then is the mass of the satellite, so $m_{tot} - \Delta m t = m_l$. From this I can find the equation for fuel needed:

$$m_{f0} = m_l (e^{\frac{\Delta v m_e}{F_b}} - 1) \quad (83)$$

6 Appendix C: The Expression for the Atmosphere

6.1 The isothermal atmosphere

To find the analytical expression for the isothermal part of atmosphere I start with the equations for an ideal gas [3]

$$P = \frac{\rho k T}{\mu m_H} \quad (84)$$

and for hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho \frac{GM(r)}{r^2} \quad (85)$$

I am going to make the approximation that the gravitational acceleration is the same in the whole atmosphere:

$$\frac{GM(r)}{r^2} = g_r \quad (86)$$

I solve 84 for ρ and sat it in to 85, I get that

$$\frac{dP}{dr} = -Pgh_0 \quad (87)$$

$$h_0 = \frac{kT}{m_H g \mu} \quad (88)$$

This differential equation can easily be solved by separating variables, and I get that

$$P(h) = C e^{-\frac{h}{h_0}} = P_0 e^{-\frac{h}{h_0}} \quad (89)$$

And using this in the ideal gas law I find

$$\rho(h) = \frac{P_0}{h_0} e^{-\frac{h}{h_0}} = \rho_0 e^{-\frac{h}{h_0}} \quad (90)$$

6.2 The adiabatic atmosphere

Including equations 84 and 85 there is one more equation for an adiabatic system

$$P^{1-\gamma}T^\gamma = \text{constant} \quad (91)$$

$\gamma = 1.4$. This constant is not known but by differentiating the expression we can get rid of it (this first step was inspired by [2], but the rest was done by us)

$$\frac{d}{dh}(P^{1-\gamma}T^\gamma) = \frac{dP}{dh}(1-\gamma)\left(\frac{T}{P}\right)^\gamma + \frac{dT}{dh}\gamma\left(\frac{T}{P}\right)^{\gamma-1} = 0 \quad (92)$$

$$\Rightarrow \frac{dP}{P} = \frac{\gamma}{\gamma-1} \frac{dT}{T} \quad (93)$$

rewriting the hydrostatic equilibrium we get that

$$\frac{dP}{P} = -(gh_0)dh = \frac{\gamma}{\gamma-1} \frac{dT}{T} \quad (94)$$

And solving this differential equation gives us the temperature

$$T(h) = -gh_0 \frac{\gamma-1}{\gamma} h + C \quad (95)$$

$$T(0) = C = T_0$$

$$T(h) = T_0 \left(1 - \frac{\gamma-1}{\gamma} \frac{h}{h_0} \right) \quad (96)$$

Setting this back into 91 we get that

$$P(h) = cT(h)^{\frac{\gamma}{\gamma-1}} = cT_0 \left(1 - \frac{\gamma-1}{\gamma} \frac{h}{h_0} \right)^{\frac{\gamma}{\gamma-1}} = P_0 \left(1 - \frac{\gamma-1}{\gamma} \frac{h}{h_0} \right)^{\frac{\gamma}{\gamma-1}} \quad (97)$$

Now using the ideal gas law we get that

$$\rho(h) = \frac{P(h)}{T(h)} \frac{\mu m_H}{k} = \frac{P_0}{T_0} \frac{\mu m_H}{k} \left(1 - \frac{\gamma-1}{\gamma} \frac{h}{h_0} \right)^{\frac{\gamma}{\gamma-1}-1} \quad (98)$$

$$\rho(h) = \rho_0 \left(1 - \frac{\gamma-1}{\gamma} \frac{h}{h_0} \right)^{\frac{1}{\gamma-1}} \quad (99)$$

6.3 The boundary

I was interested in finding ρ_0 for the isothermal atmosphere, which is the pressure at the height where $T(h) = T_0/2$. If I use 96 with $T(h) = T_0/2$ and solve for h , I found that this height is

$$h_{\text{boundary}} = \frac{\gamma}{2(\gamma-1)} h_0 \quad (100)$$

and using this in 99 I got that

$$\rho_0^{\text{isothermal}} = \rho(h_{\text{boundary}}) = \rho_0 \left(\frac{1}{2} \right)^{\frac{1}{\gamma-1}} \quad (101)$$

References

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