

# Fys3110 Hjemmeeksamen

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## 1 Exercise 1:

### 1.1 1.1)

We have a qubit consisting of the two spin-1/2 basis states

$$|0\rangle \equiv |\downarrow\rangle \simeq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1)$$

$$|1\rangle \equiv |\uparrow\rangle \simeq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2)$$

we then introduce the operator  $\hat{\sigma}_x$  represented by the Pauli-matrix

$$\hat{\sigma}_x \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3)$$

We can then look at how the operator acts on the basis states:

$$\hat{\sigma}_x |0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |1\rangle \quad (4)$$

$$\hat{\sigma}_x |1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |0\rangle \quad (5)$$

So the operator  $\hat{\sigma}_x$  inverts the states:

$$\hat{\sigma}_x |0\rangle = |1\rangle, \quad \hat{\sigma}_x |1\rangle = |0\rangle \quad (6)$$

### 1.2 1.2)

We have two states  $|i\rangle$  and  $|o\rangle$ , and the operator  $G$  which acts on the states as follows:

$$|o\rangle = G|i\rangle, \quad |i\rangle = G|o\rangle \quad (7)$$

We can see that

$$G(G|o\rangle) = G|i\rangle = |o\rangle \quad (8)$$

Thus we see that  $GG = I \Rightarrow G = G^{-1}$ . We can check if  $G$  is hermitian looking at the definition of the hermitian conjugate for some operator  $K$ :

$$\langle\psi|K|\phi\rangle = \langle\phi|K^\dagger|\psi\rangle^* \quad (9)$$

And if  $K$  is hermitian then  $K = K^\dagger$  and

$$\langle\psi|K|\phi\rangle = \langle\phi|K|\psi\rangle^* \quad (10)$$

We can check if this holds for  $G$ . For this we calculate

$$\langle o|G|i\rangle = \langle o|o\rangle \quad (11)$$

$$\langle i|G|o\rangle^* = \langle i|i\rangle^* \quad (12)$$

But these qubits are normalized, so  $\langle i|i\rangle^* = \langle i|i\rangle = \langle o|o\rangle = 1$ , and we there for get

$$\langle o|G|i\rangle = 1 = \langle i|G|o\rangle^* \quad (13)$$

And thus we have used (1.2) to show that  $G$  is hermitian,  $G = G^\dagger$ . We have also shown that  $G = G^{-1}$ , but because  $G$  under multiplication is a operator group, we are ensured that the identity is unique, and we can therefore conclude that  $G^{-1} = G^\dagger$ , which means that  $G$  is unitary and hermitian.

### 1.3 1.3)

We want to find the NOT gate that switches the two states. But the operator  $G$  from sec. 1.2 does exactly that: it switches  $|0\rangle$  to  $|i\rangle$  and vice versa. Also, we showed in sec. 1.1 that the operator  $\hat{\sigma}_x$  represented as the Pauli-matrix, took a spin up to down, and down to up. We therefore know that  $G$  is the operator for the NOT gate, represented by a Pauli-Matrix:

$$G = \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (14)$$

And as shown in sec. 1.2, this operator,  $G$ , is unitary hermitian.

### 1.4 1.4)

For the Hadamard gate (H-gate) we have the following operation:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (15)$$

We can look at its properties. First we find the hermitian transform of  $H$ :

$$H^\dagger = (H^T)^* = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H \quad (16)$$

Thus showing that  $H$  is hermitian. If we multiply  $H$  with it self we get

$$H^2 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (17)$$

This means that  $H^2 = I \Rightarrow H = H^{-1}$  and that  $H$  is unitary.

We can now see what  $H$  does to qubit basis states

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (18)$$

$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (19)$$

We can recognize these as the eigenstates for spin in x-direction, so

$$H|0\rangle = |\downarrow_x\rangle \quad (20)$$

$$H|1\rangle = |\uparrow_x\rangle \quad (21)$$

## 1.5 1.5)

We want to find a magnetic field that results in the effect of  $H$  found is eq. (1.4) and (1.4). We look at  $H$  and see that

$$H = \frac{1}{\sqrt{2}} (\sigma_x + \sigma_z) \quad (22)$$

So we make an educated guess that the magnetic field has to be in the  $\hat{i} + \hat{k}$  direction. So our Hamiltonian for the magnetic field will be:

$$\hat{H} = -\mu \cdot \mathbf{B} = -g \frac{\mu_B}{\hbar} \left( \frac{\hbar}{\sqrt{2}} \frac{\sigma_x}{2} + \frac{\hbar}{\sqrt{2}} \frac{\sigma_z}{2} \right) = -g \frac{h\mu_B}{2\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (23)$$

With

$$\mathbf{B} = \frac{h}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad (24)$$

The  $\sqrt{2}$  being there to ensure that the magnitude of  $h$ .

We now want to find the eigenstates of the Hamiltonian so we can express the time evolution of  $|0\rangle$  and  $|1\rangle$ . This is simply done by using your favourite linear algebra program, in my case wolfram alpha:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \rightarrow |h_1\rangle = \begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix}, |h_2\rangle = \begin{pmatrix} 1 - \sqrt{2} \\ 1 \end{pmatrix} \quad (25)$$

with eigenvalues

$$h_1 = \sqrt{2}, \quad h_2 = -\sqrt{2} \quad (26)$$

These eigenvectors are not normalized, but when we use them below to represent states, we get coefficients that normalize the states.

This gives us the energies for the Hamiltonian

$$E_1 = -g \frac{h\mu_B}{2}, \quad E_2 = g \frac{h\mu_B}{2} \quad (27)$$

We can now express our qubit states as linear combinations of the eigenvectors of the Hamiltonian:

$$|1\rangle = a |h_1\rangle + b |h_2\rangle, \quad |0\rangle = c |h_1\rangle + d |h_2\rangle \quad (28)$$

Inverting  $2 \times 2$  matrix given as  $(h_1 h_2)$  we get the coefficients. This was also done with wolfram alpha:

$$|1\rangle = \frac{\sqrt{2}}{4} \begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix} - \frac{\sqrt{2}}{4} \begin{pmatrix} 1 - \sqrt{2} \\ 1 \end{pmatrix} \quad (29)$$

$$|0\rangle = \frac{2 - \sqrt{2}}{4} \begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix} + \frac{2 + \sqrt{2}}{4} \begin{pmatrix} 1 - \sqrt{2} \\ 1 \end{pmatrix} \quad (30)$$

We can then use the energy to add time-evolution to these states under that Hamiltonian:

$$|1(t)\rangle = \frac{\sqrt{2}}{4} \begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix} e^{iEt/\hbar} - \frac{\sqrt{2}}{4} \begin{pmatrix} 1 - \sqrt{2} \\ 1 \end{pmatrix} e^{-iEt/\hbar} \quad (31)$$

$$|0(t)\rangle = \frac{2 - \sqrt{2}}{4} \begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix} e^{iEt/\hbar} + \frac{2 + \sqrt{2}}{4} \begin{pmatrix} 1 - \sqrt{2} \\ 1 \end{pmatrix} e^{-iEt/\hbar} \quad (32)$$

Where  $E = g \frac{\hbar \mu_B}{2}$ . We now want to find the time  $t = T$  where these states have evolved into the states given by the H-gate, (1.4) and (1.4). We can do this two different ways: The first involves just comparing the components of the vectors, and then solving for  $t$ , but this leaves us with four equations for a single unknown. We can solve just one of these, but we want to be sure that all of the equations gives the same time  $t = T$ . The other way is to write these as linear combinations of  $|\uparrow_x\rangle$  and  $|\downarrow_x\rangle$ . This will leave us with only two equations, one for  $|0(t)\rangle$  and one for  $|1(t)\rangle$ . So lets do this. First:

$$\begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix} = (1 + \sqrt{2}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (33)$$

$$= (1 + \sqrt{2}) \frac{1}{2} \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] + \frac{1}{2} \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \quad (34)$$

We recognize that  $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \sqrt{2} |\uparrow_x\rangle$  and  $\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \sqrt{2} |\downarrow_x\rangle$

$$= (1 + \sqrt{2}) \frac{1}{2} \left[ \sqrt{2} |\uparrow_x\rangle + \sqrt{2} |\downarrow_x\rangle \right] + \frac{1}{2} \left[ \sqrt{2} |\uparrow_x\rangle - \sqrt{2} |\downarrow_x\rangle \right] \quad (35)$$

$$= \sqrt{2} |\uparrow_x\rangle + |\uparrow_x\rangle + |\downarrow_x\rangle \quad (36)$$

And similarly

$$\begin{pmatrix} 1 - \sqrt{2} \\ 1 \end{pmatrix} = \sqrt{2} |\uparrow_x\rangle - |\uparrow_x\rangle - |\downarrow_x\rangle \quad (37)$$

We can now rewrite (1.5) and (1.5). We start with

$$|1(t)\rangle = \frac{\sqrt{2}}{4} \left( \left[ \sqrt{2} |\uparrow_x\rangle + |\uparrow_x\rangle + |\downarrow_x\rangle \right] e^{iEt/\hbar} - \left[ \sqrt{2} |\uparrow_x\rangle - |\uparrow_x\rangle - |\downarrow_x\rangle \right] e^{-iEt/\hbar} \right) \quad (38)$$

$$= \frac{\sqrt{2}}{4} \left( \sqrt{2} (e^{iEt/\hbar} - e^{-iEt/\hbar}) |\uparrow_x\rangle + (e^{-iEt/\hbar} + e^{iEt/\hbar}) |\uparrow_x\rangle + (e^{-iEt/\hbar} + e^{iEt/\hbar}) |\downarrow_x\rangle \right) \quad (39)$$

$$= \frac{\sqrt{2}}{4} \left( -2\sqrt{2}i \sin(Et/\hbar) |\uparrow_x\rangle + 2 \cos(Et/\hbar) |\uparrow_x\rangle + 2 \cos(Et/\hbar) |\downarrow_x\rangle \right) \quad (40)$$

But we know that we want the to evolve into  $|\uparrow_x\rangle$  as we found in (1.4). We start by finding a time where we have no  $|\downarrow_x\rangle$ , in other words  $\cos(Et/\hbar) = 0$  This gives us the time

$$t = T = \frac{(2n+1)\pi\hbar}{2E} \quad (41)$$

For this time we get that the state evolves to

$$|1(t = T)\rangle = \pm i |\uparrow_x\rangle \quad (42)$$

This is exactly what we wanted! But we see an extra  $\pm i$ , this is just an additional global phase induced by the magnetic field, and does not alter the physical state. Thus we have found the correct time for  $H|1\rangle$ . Now let us see if we get the same for  $|0\rangle$ :

$$|0(t)\rangle = \frac{1}{4} \left( (2 - \sqrt{2}) \left[ \sqrt{2} |\uparrow_x\rangle + |\uparrow_x\rangle + |\downarrow_x\rangle \right] e^{iEt/\hbar} + (2 + \sqrt{2}) \left[ \sqrt{2} |\uparrow_x\rangle - |\uparrow_x\rangle - |\downarrow_x\rangle \right] e^{-iEt/\hbar} \right) \quad (43)$$

$$= \frac{1}{4} \left( 2\sqrt{2} |\uparrow_x\rangle (e^{iEt/\hbar} + e^{-iEt/\hbar}) + 2 |\uparrow_x\rangle (e^{-iEt/\hbar} - e^{iEt/\hbar}) + \right) \quad (44)$$

$$2(|\uparrow_x\rangle + |\downarrow_x\rangle)(e^{iEt/\hbar} + e^{-iEt/\hbar}) - \sqrt{2}(|\uparrow_x\rangle + |\downarrow_x\rangle)(e^{iEt/\hbar} + e^{-iEt/\hbar}) \quad (45)$$

$$= \frac{1}{4} \left( 4\sqrt{2} \cos(Et/\hbar) + 4i \sin(Et/\hbar) |\downarrow_x\rangle - 2\sqrt{2}(|\uparrow_x\rangle + |\downarrow_x\rangle) \cos(Et/\hbar) \right) \quad (46)$$

Again we know what the state evolves to, namely  $|\downarrow_x\rangle$  (1.4). So we need to remove all  $|\uparrow_x\rangle$ . Fortunately, like last time, this can be done by letting  $\cos(Et/\hbar) = 0$ . This is true for

$$t = T = \frac{(2n+1)\pi\hbar}{2E} \quad (47)$$

Just like for  $|1(t)\rangle$ . We then get

$$|0(t=T)\rangle = \pm i |\downarrow_x\rangle \quad (48)$$

Here again we see the same global phase, and since it doesn't change the physical state, the state has evolved as desired, and we have found the correct time.

We got the same time for both  $|0(t)\rangle$  and  $|1(t)\rangle$ .

$$T = \frac{(2n+1)\pi\hbar}{2E} \quad (49)$$

Since we can choose the integer  $n$  the simplest thing is to set it to 0. So in summary:

To implement the H-gate we can send the qubit through a magnetic field of strength  $h$  pointed in the  $\frac{h}{\sqrt{2}}(\hat{i} + \hat{k})$  direction. The magnetic field is turned on at  $t = 0$  and turned off at

$$t = T = \frac{\pi\hbar}{2E} = \frac{\pi\hbar}{gh\mu_B} = \frac{2m\pi}{ghe} \quad (50)$$

## 2 Exercise 2:

### 2.1 2.1)

### 2.2 2.2)

We have the operators which work like

$$F|i\rangle = \begin{cases} -|i^*\rangle & , i = i^* \\ |i\rangle & , i \neq i^* \end{cases} \quad (51)$$

We want to show that  $F$  can be written as

$$F = I - 2|i^*\rangle\langle i^*| \quad (52)$$

Let's use the operator on a ket

$$F|i\rangle = (I - 2|i^*\rangle\langle i^*|)|i\rangle = I|i\rangle - 2|i^*\rangle\langle i^*|i\rangle \quad (53)$$

$$= |i\rangle - 2\delta_{i,i^*}|i^*\rangle = \begin{cases} -|i^*\rangle & , i = i^* \\ |i\rangle & , i \neq i^* \end{cases} \quad (54)$$

We can see that this gives the same as (2.2), and we can therefore write  $F = I - 2|i^*\rangle\langle i^*|$ .

We can see from eq. 2.2 that  $F$  is a Householder transformation, which ensures that  $F$  is unitary hermitian, but we can also show this:

$$F(F|i\rangle) = (I - 2|i^*\rangle\langle i^*|)(I - 2|i^*\rangle\langle i^*|)|i\rangle \quad (55)$$

$$= I|i\rangle - 2|i^*\rangle\langle i^*|I|i\rangle - 2I|i^*\rangle\langle i^*|i\rangle + 4|i^*\rangle\langle i^*|i^*\rangle\langle i^*|i\rangle \quad (56)$$

$$= |i\rangle - 4|i^*\rangle\langle i^*|i\rangle + 4|i^*\rangle\langle i^*|i\rangle = |i\rangle \quad (57)$$

Thus  $FF = I \Rightarrow F = F^{-1}$ . We then show that  $F$  is hermitian:

$$F^\dagger = (I - 2|i^*\rangle\langle i^*|)^\dagger = I - 2(|i^*\rangle\langle i^*|)^\dagger \quad (58)$$

$$= I - 2(\langle i^*|)^\dagger(|i^*\rangle)^\dagger = I - 2|i^*\rangle\langle i^*| = F \quad (59)$$

So  $F$  is hermitian,  $F = F^\dagger$ . This implies that  $F$  is unitary hermitian:

$$F^{-1} = F = F^\dagger \Leftrightarrow F^\dagger = F^{-1} \quad (60)$$

### 2.3 2.3

We now introduce the superposition of the states  $|i\rangle$

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^N |i\rangle \quad (61)$$

We then calculate

$$\langle i^*|s\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^N \langle i^*|i\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^N \delta_{i,i^*} = \frac{1}{\sqrt{N}} \quad (62)$$

and

$$F|s\rangle = I|s\rangle - 2|i^*\rangle\langle i^*|s\rangle = |s\rangle - \frac{2}{\sqrt{N}}|i^*\rangle \quad (63)$$

### 2.4 2.4)

We now consider the state:

$$|g\rangle = \alpha|s\rangle + \beta|i^*\rangle \quad (64)$$

We want this to be normalized:

$$\langle g|g\rangle = 1 \quad (65)$$

Due to Riesz representation theorem, we know that such a bra  $\langle g|$  exist, so we can then find the condition for  $\alpha$  and  $\beta$  such that  $|g\rangle$  is normalized.

$$(\alpha * \langle s| + \beta * \langle i^*|)(\alpha|s\rangle + \beta|i^*\rangle) = |\alpha|^2 + |\beta|^2 + \alpha\beta * \langle s|i^*\rangle + \alpha * \beta \langle i^*|s\rangle \quad (66)$$

$\alpha$  and  $\beta$  are both real. We know that  $\langle i^*|s\rangle$  is real from (2.3), so  $\langle s|i^*\rangle = \langle i^*|s\rangle^* = \langle i^*|s\rangle$ . So:

$$\alpha^2 + \beta^2 + \frac{2\alpha\beta}{\sqrt{N}} = 1 \quad (67)$$

Is the condition that makes sure that  $|g\rangle$  is normalized.

## 2.5 2.5)

The operator for measuring the measurement  $i$  of state  $|i\rangle$  is named  $X$ , so  $X$  acting on a state  $|i\rangle$  is:

$$X|i\rangle = i|i\rangle \quad (68)$$

meaning that  $|i\rangle$  are eigenstates of  $X$  with eigenvalue  $i$ . This means that we can use the spectral theorem to write  $X$  as a spectral representation with  $|i\rangle$  and  $i$ :

$$X = \sum_{j=1}^N j |j\rangle \langle j| \quad (69)$$

( $j$  and  $i$  are dummy indices and therefore interchanged).

## 2.6 2.6)

We want to find the probability of measuring  $i^*$  when observing  $|g\rangle$ . From  $X$  we find

$$X|i^*\rangle = i^*|i^*\rangle \quad (70)$$

Meaning that  $|i^*\rangle$  is the state where we measure  $i^*$ , meaning that we need to find the probability of measuring  $|g\rangle$  in state  $|i^*\rangle$ . We do this with the projection operator

$$\hat{P} = |i^*\rangle \langle i^*| \quad (71)$$

We then find the expectation value of  $P$  acting on  $|g\rangle$ :

$$P(i^*) = \langle g|\hat{P}|g\rangle = \langle g|i^*\rangle \langle i^*|g\rangle = |\langle i^*|g\rangle|^2 \quad (72)$$

We can calculate this

$$\langle i^*|g\rangle = \langle i^*|(\alpha|s\rangle + \beta|i^*\rangle) = \alpha\langle i^*|s\rangle + \beta\langle i^*|i^*\rangle = \frac{\alpha}{\sqrt{N}} + \beta \quad (73)$$

All the terms are real, so the norm in the probability just becomes a power of two:

$$\Rightarrow P(i^*) = |\langle i^*|g\rangle|^2 = \left(\frac{\alpha}{\sqrt{N}} + \beta\right)^2 = \frac{\alpha^2}{N} + \beta^2 + \frac{\alpha\beta}{\sqrt{N}} \quad (74)$$

From eq. (2.4) we see that

$$\beta^2 + \frac{2\alpha\beta}{\sqrt{N}} = 1 - \alpha^2 \quad (75)$$

So we get that the probability of measuring  $i^*$  becomes:

$$P(i^*) = 1 - \alpha^2 \left(1 - \frac{1}{N}\right) \quad (76)$$

## 2.7 2.7)

We now introduce the operator

$$U = 2|s\rangle\langle s| - I \quad (77)$$

We want to look at  $UF|s\rangle$ . We already calculated  $F|s\rangle$ (2.3), so we get:

$$UF|s\rangle = U(|s\rangle - \frac{2}{\sqrt{N}}|i^*\rangle) \quad (78)$$

$$= 2|s\rangle\langle s|s\rangle - I|s\rangle - \frac{4}{\sqrt{N}}|s\rangle\langle s|i^*\rangle + \frac{2}{\sqrt{N}}I|i^*\rangle = |s\rangle - \frac{4}{N}|s\rangle + \frac{2}{\sqrt{N}}|i^*\rangle \quad (79)$$

So

$$UF|s\rangle = \left(1 - \frac{4}{N}\right)|s\rangle + \frac{2}{\sqrt{N}}|i^*\rangle \quad (80)$$

If we look at  $U$ (2.7) we see that it too is a Householder transformation (with a negative sign), thus we know that it is unitary hermitian. A unitary always preserves the norm. So if we first apply  $F$  (which we showed was unitary) to  $|s\rangle$ , its norm is still 1 ( $|s\rangle$  is a normalized sum of normalized states). So when we then apply  $U$ , the norm is still 1. Thus the norm of  $UF|s\rangle$  is 1. But we can also show this explicit:

$$\left[\left(1 - \frac{4}{N}\right)\langle s| + \frac{2}{\sqrt{N}}\langle i^*|\right] \left[\left(1 - \frac{4}{N}\right)|s\rangle + \frac{2}{\sqrt{N}}|i^*\rangle\right] \quad (81)$$

(remember that all the constants are real)

$$= \left(1 - \frac{4}{N}\right)^2 \langle s|s\rangle + \frac{2}{\sqrt{N}}\left(1 - \frac{4}{N}\right)\langle s|i^*\rangle + \frac{2}{\sqrt{N}}\left(1 - \frac{4}{N}\right)\langle i^*|s\rangle + \frac{4}{N}\langle i^*|i^*\rangle \quad (82)$$

$$= \left(1 - \frac{4}{N}\right)^2 + \frac{4}{N}\left(1 - \frac{4}{N}\right) + \frac{4}{N} \quad (83)$$

$$= 1 - \frac{8}{N} + \frac{16}{N^2} + \frac{4}{N} - \frac{16}{N^2} + \frac{4}{N} = 1 \quad (84)$$

So we have showed that the norm of  $UF|s\rangle$  is 1.

## 2.8 2.8)

We now want to calculate  $UF|g\rangle$ :

$$UF|g\rangle = UF(\alpha|s\rangle + \beta|i^*\rangle) = \alpha UF|s\rangle + \beta UF|i^*\rangle \quad (85)$$

We have already found  $UF|s\rangle$ (2.7), but we need to find

$$UF|i^*\rangle = -U|i^*\rangle = -(2|s\rangle\langle s| - I)|i^*\rangle = -2|s\rangle\langle s|i^*\rangle + I|i^*\rangle \quad (86)$$

$$= |i^*\rangle - \frac{2}{\sqrt{N}}|s\rangle \quad (87)$$

We can now combine these

$$\alpha UF|s\rangle + \beta UF|i^*\rangle = \alpha \left[\left(1 - \frac{4}{N}\right)|s\rangle + \frac{2}{\sqrt{N}}|i^*\rangle\right] + \beta \left[|i^*\rangle - \frac{2}{\sqrt{N}}|s\rangle\right] \quad (88)$$

We can so sort these and find



$$UF(\alpha|s\rangle + \beta|i^*\rangle) = \left[ \alpha \left( 1 - \frac{4}{N} \right) - \frac{2\beta}{\sqrt{N}} \right] |s\rangle + \left[ \frac{2\alpha}{\sqrt{N}} + \beta \right] |i^*\rangle \quad (89)$$

## 2.9 2.9)

We can numerically compute the probability of measuring  $i^*$  in the state  $(UF)^n |s\rangle$ . But as we can see from eq. (2.7) we get a contribution from  $|i^*\rangle$ , and it gets the same form as  $UF|g\rangle$ , thus this is the same as calculating  $(UF)^n |g\rangle$  with the initial conditions  $\beta = 0$ . We know from eq. 2.8 how this system evolve. We can write the coefficients in front of  $|s\rangle$  and  $|i^*\rangle$  as  $\alpha_{new}$  and  $\beta_{new}$  and get a simple algorithm for calculating evolution under  $UF$ (2.9). Each time we update the system we check the probability of measuring  $i^*$  we found in (2.6).

We want to find for what  $n$  we get 99% of measuring  $i^*$ . To do this we simply exit the loop when this probability is reached, saving the  $n$  at which this happens.

The final algorithm looks like this:

- Start with  $\alpha = 1, \beta = 0$
- loop over n:
  - Calculate  $\alpha_{new} = \left[ \alpha \left( 1 - \frac{4}{N} \right) - \frac{2\beta}{\sqrt{N}} \right]$  and  $\beta_{new} = \left[ \frac{2\alpha}{\sqrt{N}} + \beta \right]$
  - Set  $\alpha = \alpha_{new}$  and  $\beta = \beta_{new}$
  - Find the probability  $P(i^*) = 1 - \alpha^2 \left( 1 - \frac{1}{N} \right)$
  - If  $P(i^*) \geq 0.99$ :
    - Save n
    - Exit loop

The code is found at the end3.

If we don't exit the loop after  $P \neq 0.99$  but look at how the probability evolves:

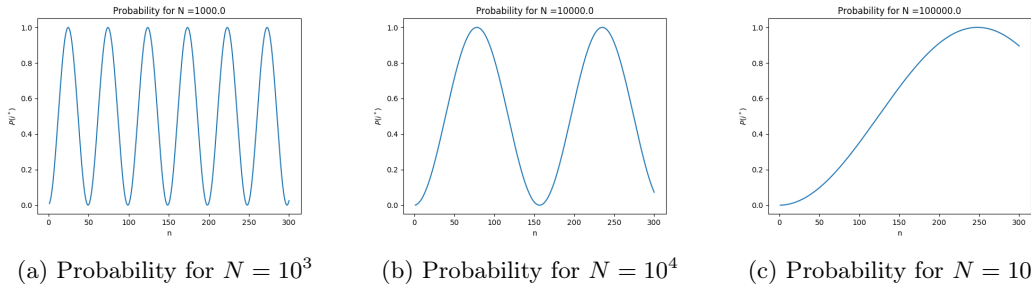
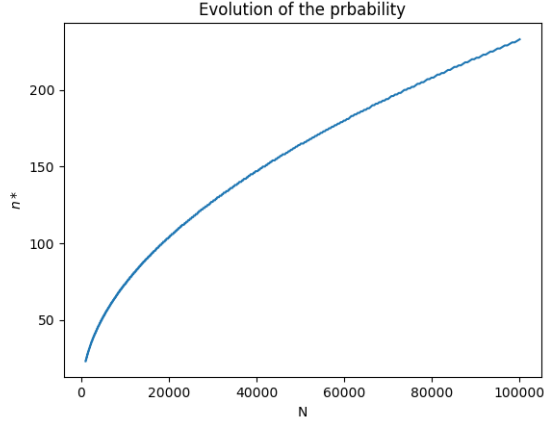


Figure 1: We can see that the probability of measuring  $i^*$  actually goes as a sinusoidal function of  $n$ . This means that using  $UF$  doesn't always improve the probability, and that there are regions where applying  $UF$  actually lowers the probability. We also see that frequency of this sinusoidal function increases with the number of unknown states  $N$ , and thus we need to apply  $UF$  more times to get a better probability.

We can then look at how many  $n = n^*$  is necessary to get a probability  $P(i^*) \geq 0.99$ :



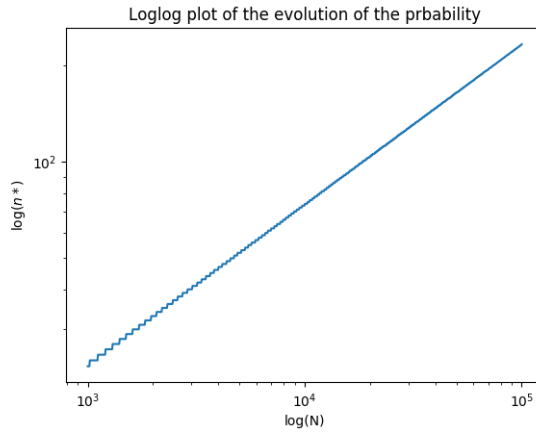
Figur 2: The number of times  $n^*$  we need to apply  $UF$  to get a probability  $P(i^*) \geq 0.99$

### TA MED TABELL SOM OPPGAVEN BER OM!!!!

We can see that as  $N$  increases, so does the number of times we need to apply  $UF$  this number  $n^*$ . It looks quite like a square root, but we need to make sure. For this we use a log-log plot, this is because:

$$n^* = N^i \Rightarrow \log(n^*) = i \cdot \log(N) \quad (90)$$

So we can use the log-log plot, find the slope and therefore find the order of  $N$  of which  $n^*$  increases.



Figur 3: This shows  $\log(n^*)$  vs  $\log(n)$  and clearly shows a straight line. The slope of this line is the order of  $N$  at which  $n^*$  increases.

If we use a normal finite difference on this curve (which is exact for a linear function), we find that slope is  $\sim 0.503$ . This is close enough to  $1/2$  that we can conclude that

$$n^* \sim O(\sqrt{N}) \quad (91)$$

This is an immense improvement over the classical result from sec. 2.1, but more on this below.

## 3 Code

```

import numpy as np
import matplotlib.pyplot as plt

alpha = 1
beta = 0

top_percentile = []

Ns = np.logspace(3,5,800) #Ns for finding n* for P > 0.99
#Ns = [1e3,1e4,1e5] #Ns for plotting the evolution of the probability
for N in Ns:
    probs = []
    ns = []
    alpha = 1
    beta = 0
    for n in range(1,301):
        new_alpha = (alpha*(1-4/N) - beta*2/np.sqrt(N))
        new_beta = alpha*2/np.sqrt(N) + beta

        alpha = new_alpha
        beta = new_beta

        p = 1-alpha**2*(1-1./N)

        probs.append(p)
        ns.append(n)
        """
        Comment the if-statement before plotting the evolution of the
        probability
        """
        if p >= 0.99:
            top_percentile.append(n)
            break

    """Uncomment for plotting the evolution of the probability"""
    # plt.plot(ns,probs)
    # plt.title("Probability for N ={}".format(N))
    # plt.xlabel("n")
    # plt.ylabel(r"$P(i^*)$")
    # plt.show()

    """Comment all below before using the plot above"""
log_top = np.log(np.array(top_percentile))
log_Ns = np.log(Ns)

print("The speed for looking up i* goes as N to the power of ", \
      np.mean((log_top[:-1] - log_top[1:]))/(log_Ns[:-1] - log_Ns[1:]))

plt.loglog(Ns,top_percentile)
plt.title("Loglog plot of the evolution of the prbability")
plt.xlabel("log(N)")
plt.ylabel(r"$\log(n^*)$")

#plt.plot(Ns,top_percentile)
#plt.title("Evolution of the prbability")

```

```
#plt.xlabel("N")  
#plt.ylabel(r"$n*$")  
#plt.show()
```