Fys3110 Hjemmeeksamen

Kadnr.:

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1 Exercise 1:

1.1 1.1)

We have a qubit consisting of the two spin-1/2 basis states

$$|0\rangle \equiv |\downarrow\rangle \simeq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1}$$

$$|1\rangle \equiv |\uparrow\rangle \simeq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2}$$

we then introduce the operator $\hat{\sigma}_x$ represented by the Pauli-matrix

$$\hat{\sigma}_x \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3}$$

We can then look at how the operator acts on the basis states:

$$\hat{\sigma}_x |0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |1\rangle \tag{4}$$

$$\hat{\sigma}_x |1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |0\rangle \tag{5}$$

So the operator $\hat{\sigma}_x$ inverts the states:

$$\hat{\sigma}_x |0\rangle = |1\rangle, \qquad \hat{\sigma}_x |1\rangle = |0\rangle \tag{6}$$

1.2 1.2)

We have two states $|i\rangle$ and $|o\rangle$, and the operator G which acts on the states as follows:

$$|o\rangle = G|i\rangle, \qquad |i\rangle = G|o\rangle \tag{7}$$

We can see that

$$G(G|o\rangle) = G|i\rangle = |o\rangle \tag{8}$$

Thus we see that $GG = I \Rightarrow G = G^{-1}$. We can check if G is hermitian looking at the definition of the hermitian conjugate for some operator K:

$$\langle\psi| K |\phi\rangle = \langle\phi| K^\dagger |\psi\rangle^* \tag{9}$$

And if K is hermitian then $K = K^\dagger$ and

$$\langle\psi| K |\phi\rangle = \langle\phi| K |\psi\rangle^* \tag{10}$$

We can check if this holds for G. For this we calculate

$$\langle o| G |i\rangle = \langle o|o\rangle \tag{11}$$

$$\langle i| G |o\rangle^* = \langle i|i\rangle^* \tag{12}$$

But these qubits are normalized, so $\langle i|i\rangle^* = \langle i|i\rangle = \langle o|o\rangle = 1$, and we there for get

$$\langle o| G |i\rangle = 1 = \langle i| G |o\rangle^* \tag{13}$$

And thus we have used (1.2) to show that G is hermitian, $G = G^\dagger$. We have also shown that $G = G^{-1}$, but because G under multiplication is a operator group, we are ensured that the identity is unique, and we an therefore conclude that $G^{-1} = G^\dagger$, which means that G is unitary and hermitian.

1.3 1.3)

We want to find the NOT gate that switches the two states. But the operator G from sec. 1.2 does exactly that: it switches $|0\rangle$ to $|i\rangle$ and vice versa. Also, we showed in sec. 1.1 that the operator $\hat{\sigma}_x$ represented as the Pauli-matrix, took a spin up to down, and down to up. We therefore know that G is the operator for the NOT gate, represented by a Pauli-Matrix:

$$G = \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{14}$$

And as shown in sec. 1.2, this operator, G, is unitary hermitian.

1.4 1.4)

For the Hadamard gate (H-gate) we have the following operation:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{15}$$

We can look at its properties. First we find the hermitian transform of H:

$$H^\dagger = (H^T)^* = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H \tag{16}$$

Thus showing that H is hermitian. If we multiply H with it self we get

$$H^2 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{17}$$

This means that $H^2 = I \Rightarrow H = H^{-1}$ and that H is unitary.
We can now see what H does to qubit basis states

$$H |0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{18}$$

$$H |1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{19}$$

We can recognize these as the eigenstates for spin in x-direction, so

$$H |0\rangle = |{\downarrow_x}\rangle \tag{20}$$

$$H |1\rangle = |{\uparrow_x}\rangle \tag{21}$$

1.5 1.5)

We want to find a magnetic field that that results in the effect of H found is eq. (1.4) and (1.4). We look at H and see that

$$H = \frac{1}{\sqrt{2}} (\sigma_x + \sigma_z) \tag{22}$$

So we make an educated guess that the magnetic field has to we in $\hat{\imath} + \hat{k}$ direction. So our Hamiltonian for the magnetic field will be:

$$\hat{H} = -\mu \cdot \mathbf{B} = -g\frac{\mu_B}{\hbar} \left(\frac{h}{\sqrt{2}} \frac{\hbar}{2} \sigma_x + \frac{h}{\sqrt{2}} \frac{\hbar}{2} \sigma_z \right) = -g\frac{h\mu_B}{2\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{23}$$

With

$$\mathbf{B} = \frac{h}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{24}$$

The $\sqrt{2}$ being there to ensure that the magnitude of h.

We now what to find the eigenstates of the Hamiltonian so we can express the time evolution of $|0\rangle$ and $|1\rangle$:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \to \qquad |h_1\rangle = \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}, |h_2\rangle = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix} \tag{25}$$

with eigenvalues

$$h_1 = \sqrt{2}, \qquad h_2 = -\sqrt{2} \tag{26}$$

This gives us the energies for the Hamiltonian

$$E_1 = -g\frac{h\mu_B}{2}, \qquad E_2 = g\frac{h\mu_B}{2} \tag{27}$$

We can now express our qubit states as linear combinations of the eigenvalues of the Hamiltonian:

$$|1\rangle = a\,|h_1\rangle + b\,|h_2\rangle\,, \qquad |0\rangle = c\,|h_1\rangle + d\,|h_2\rangle \tag{28}$$

This turns out to be

$$|1\rangle = \frac{\sqrt{2}}{4} \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} - \frac{\sqrt{2}}{4} \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix} \tag{29}$$

$$|0\rangle = \frac{2-\sqrt{2}}{4} \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} + \frac{2+\sqrt{2}}{4} \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix} \tag{30}$$

2 Exercise 2:

2.1 2.1)

2.2 2.2)

We have the operators which works like

$$F\,|i\rangle = \begin{cases} -\,|i^*\rangle & ,i = i^* \\ |i\rangle & ,i \neq i^* \end{cases} \tag{31}$$

We want to show that F can be written as

$$F = I - 2\left|i^*\right\rangle\left\langle i^*\right| \tag{32}$$

Let's use the operator on a ket

$$F\left|i\right\rangle = (I - 2\left|i^*\right\rangle\left\langle i^*\right|)\left|i\right\rangle = I\left|i\right\rangle - 2\left|i^*\right\rangle\left\langle i^*|i\right\rangle \tag{33}$$

$$= \left|i\right\rangle - 2\delta_{i,i^*}\left|i^*\right\rangle = \begin{cases} -\left|i^*\right\rangle & ,i = i^* \\ \left|i\right\rangle & ,i \neq i^* \end{cases} \tag{34}$$

We can see that this gives the same as (2.2), and we can therefore write $F = I - 2\left|i^*\right\rangle\left\langle i^*\right|$.

We can see from eq. 2.2 that F is a Householder transformation, which ensures that F is unitary hermitian, but we can also show this:

$$F(F\left|i\right\rangle) = (I - 2\left|i^*\right\rangle\left\langle i^*\right|)(I - 2\left|i^*\right\rangle\left\langle i^*\right|)\left|i\right\rangle \tag{35}$$

$$= I\left|i\right\rangle - 2\left|i^*\right\rangle\left\langle i^*\right| I\left|i\right\rangle - 2I\left|i^*\right\rangle\left\langle i^*|i\right\rangle + 4\left|i^*\right\rangle\left\langle i^*|i^*\right\rangle\left\langle i^*|i\right\rangle \tag{36}$$

$$= \left|i\right\rangle - 4\left|i^*\right\rangle\left\langle i^*|i\right\rangle + 4\left|i^*\right\rangle\left\langle i^*|i\right\rangle = \left|i\right\rangle \tag{37}$$

Thus $FF = I \Rightarrow F = F^{-1}$. We then show that F is hermitian:

$$F^\dagger = (I - 2\left|i^*\right\rangle\left\langle i^*\right|)^\dagger = I - 2(\left|i^*\right\rangle\left\langle i^*\right|)^\dagger \tag{38}$$

$$= I - 2(\left\langle i^*\right|)^\dagger(\left|i^*\right\rangle)^\dagger = I - 2\left|i^*\right\rangle\left\langle i^*\right| = F \tag{39}$$

So F is hermitian, $F = F^\dagger$. This implies that F is unitary hermitian:

$$F^{-1} = F = F^\dagger \Leftrightarrow F^\dagger = F^{-1} \tag{40}$$

2.3 2.3

We now introduce the superposition of the states $\left|i\right\rangle$

$$\left|s\right\rangle = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left|i\right\rangle \tag{41}$$

We then calculate

$$\left\langle i^*|s\right\rangle = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left\langle i^*|i\right\rangle = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\delta_{i,i^*} = \frac{1}{\sqrt{N}} \tag{42}$$

and

$$F\left|s\right\rangle = I\left|s\right\rangle - 2\left|i^*\right\rangle\left\langle i^*|s\right\rangle = \left|s\right\rangle - \frac{2}{\sqrt{N}}\left|i^*\right\rangle \tag{43}$$

2.4 2.4)

We now consider the state:

$$|g\rangle = \alpha\,|s\rangle + \beta\,|i^*\rangle \tag{44}$$

We want this to be normalized:

$$\langle g|g\rangle = 1 \tag{45}$$

Due to Riesz representation theorem, we know that such a bra $\langle g|$ exist, so we can then find the condition for α and β such that $|g\rangle$ is normalized.

$$(\alpha * \langle s| + \beta * \langle i^*|)(\alpha\,|s\rangle + \beta\,|i^*\rangle) = |\alpha|^2 + |\beta|^2 + \alpha\beta * \langle s|i^*\rangle + \alpha * \beta\,\langle i^*|s\rangle \tag{46}$$

α and β are both real. We know that $\langle i^*|s\rangle$ is real from (2.3), so $\langle s|i^*\rangle = \langle i^*|s\rangle * = \langle i^*|s\rangle$. So:

$$\alpha^2 + \beta^2 + \frac{2\alpha\beta}{\sqrt{N}} = 1 \tag{47}$$

Is the condition that makes sure that $|g\rangle$ is normalized.

2.5 2.5)

The operator for measuring the measurement i of state $|i\rangle$ is named X, so X acting on a state $|i\rangle$ is:

$$X\,|i\rangle = i\,|i\rangle \tag{48}$$

meaning that $|i\rangle$ are eigenstates of X with eigenvalue i. This means that we can use the spectral theorem to write X as a spectral representation with $|i\rangle$ and i:

$$X = \sum_{j=1}^{N} j\,|j\rangle\,\langle j| \tag{49}$$

(j and i are dummy indices and therefore interchanged).

2.6 2.6)

We want to find the probability of measuring i^* when observing $|g\rangle$. From X we find

$$X\,|i^*\rangle = i^*\,|i^*\rangle \tag{50}$$

Meaning that $|i^*\rangle$ is the state where we measure i^*, meaning that we need to find the probability of measuring $|g\rangle$ in state $|i^*\rangle$. We d this with the projection operator

$$\hat{P} = |i^*\rangle\!\langle i^*| \tag{51}$$

We then find the expectation value of P acting on $|g\rangle$:

$$P(i^*) = \langle g|\,\hat{P}\,|g\rangle = \langle g|i^*\rangle\,\langle i^*|g\rangle = |\,\langle i^*|g\rangle\,|^2 \tag{52}$$

We can calculate this

$$\langle i^*|g\rangle = \langle i^*|\,(\alpha\,|s\rangle + \beta\,|i^*\rangle) = \alpha\,\langle i^*|s\rangle + \beta\,\langle i^*|i^*\rangle = \frac{\alpha}{\sqrt{N}} + \beta \tag{53}$$

All the terms are real, so the norm in the probability just becomes a power of two:

$$\Rightarrow P(i^*) = |\langle i^*|g\rangle|^2 = \left(\frac{\alpha}{\sqrt{N}} + \beta\right)^2 = \frac{\alpha^2}{N} + \beta^2 + \frac{\alpha\beta}{\sqrt{N}} \tag{54}$$

From eq. (2.4) we see that

$$\beta^2 + \frac{2\alpha\beta}{\sqrt{N}} = 1 - \alpha^2 \tag{55}$$

So we get that the probability of measuring i^* becomes:

$$P(i^*) = 1 - \alpha^2\left(1 - \frac{1}{N}\right) \tag{56}$$

2.7 2.7)

We now introduce the operator

$$U = 2\,|s\rangle\!\langle s| - I \tag{57}$$

We want to look at $UF\,|s\rangle$. We already calculated $F\,|s\rangle$(2.3), so we get:

$$UF\,|s\rangle = U(|s\rangle - \frac{2}{\sqrt{N}}\,|i^*\rangle) \tag{58}$$

$$= 2\,|s\rangle\,\langle s|s\rangle - I\,|s\rangle - \frac{4}{\sqrt{N}}\,|s\rangle\,\langle s|i^*\rangle + \frac{2}{\sqrt{N}}I\,|i^*\rangle = |s\rangle - \frac{4}{N}\,|s\rangle + \frac{2}{\sqrt{N}}\,|i^*\rangle \tag{59}$$

So

$$UF\,|s\rangle = \left(1 - \frac{4}{N}\right)|s\rangle + \frac{2}{\sqrt{N}}\,|i^*\rangle \tag{60}$$

If we look at U(2.7) we see that it too is a Householder transformation(with a negative sign), thus we know that it is unitary hermitian. A unitary always preserves the norm. So if we first apply F(which we showed was unitary) to $|s\rangle$, its norm is still 1 ($|s\rangle$ is a normalized sum of normalized states). So when we then apply U, the norm is still 1. Thus the norm of $UF\,|s\rangle$ is 1. But we can also show this explicit:

$$\left[\left(1 - \frac{4}{N}\right)\langle s| + \frac{2}{\sqrt{N}}\,\langle i^*|\right]\left[\left(1 - \frac{4}{N}\right)|s\rangle + \frac{2}{\sqrt{N}}\,|i^*\rangle\right] \tag{61}$$

(remember that all the constants are real)

$$= \left(1 - \frac{4}{N}\right)^2\langle s|s\rangle + \frac{2}{\sqrt{N}}\left(1 - \frac{4}{N}\right)\langle s|i^*\rangle + \frac{2}{\sqrt{N}}\left(1 - \frac{4}{N}\right)\langle i^*|s\rangle + \frac{4}{N}\,\langle i^*|i^*\rangle \tag{62}$$

$$= \left(1 - \frac{4}{N}\right)^2 + \frac{4}{N}\left(1 - \frac{4}{N}\right) + \frac{4}{N} \tag{63}$$

$$= 1 - \frac{8}{N} + \frac{16}{N^2} + \frac{4}{N} - \frac{16}{N^2} + \frac{4}{N} = 1 \tag{64}$$

So we have showed that the norm of $UF\,|s\rangle$ is 1.

2.8 2.8)

We now want to calculate $UF\left|g\right\rangle$:

$$UF\left|g\right\rangle = UF(\alpha\left|s\right\rangle + \beta\left|i^*\right\rangle) = \alpha UF\left|s\right\rangle + \beta UF\left|i^*\right\rangle \tag{65}$$

We have already found $UF\left|s\right\rangle$(2.7), but we need to find

$$UF\left|i^*\right\rangle = -U\left|i^*\right\rangle = -(2\left|s\right\rangle\!\left\langle s\right| - I)\left|i^*\right\rangle = -2\left|s\right\rangle\left\langle s|i^*\right\rangle + I\left|i^*\right\rangle \tag{66}$$

$$= \left|i^*\right\rangle - \frac{2}{\sqrt{N}}\left|s\right\rangle \tag{67}$$

We can now combine these

$$\alpha UF\left|s\right\rangle + \beta UF\left|i^*\right\rangle = \alpha\left[\left(1 - \frac{4}{N}\right)\left|s\right\rangle + \frac{2}{\sqrt{N}}\left|i^*\right\rangle\right] + \beta\left[\left|i^*\right\rangle - \frac{2}{\sqrt{N}}\left|s\right\rangle\right] \tag{68}$$

We can so sort these and find

$$UF(\alpha\left|s\right\rangle + \beta\left|i^*\right\rangle) = \left[\alpha\left(1 - \frac{4}{N}\right) - \frac{2\beta}{\sqrt{N}}\right]\left|s\right\rangle + \left[\frac{2\alpha}{\sqrt{N}} + \beta\right]\left|i^*\right\rangle \tag{69}$$

2.9 2.9)

We can numerically compute the probability of measuring i^* in the state $(UF)^n\left|s\right\rangle$. But as we can see from eq. (2.7) we get a contribution from $\left|i^*\right\rangle$, and it gets the same form as $UF\left|g\right\rangle$, thus this is the same as calculating $(UF)^n\left|g\right\rangle$ with the initial conditions $\beta = 0$. We know from eq. 2.8 how this system evolve. We can write the coefficients in front of $\left|s\right\rangle$ and $\left|i^*\right\rangle$ as α_{new} and β_{new} and get a simple algorithm for calculating evolution under UF(2.9). Each time we update the system we check the probability of measuring i^* we found in (2.6).

We want to find for what n we get 99% of measuring i^*. To do this we simply exit the loop when this probability is reached, saving the n at which this happens.

The final algorithm looks like this:

- Start with $\alpha = 1$, $\beta = 0$
- loop over n:

  Calculate $\alpha_{new} = \left[\alpha\left(1 - \frac{4}{N}\right) - \frac{2\beta}{\sqrt{N}}\right]$ and $\beta_{new} = \left[\frac{2\alpha}{\sqrt{N}} + \beta\right]$

  Set $\alpha = \alpha_{new}$ and $\beta = \beta_{new}$

  Find the probability $P(i^*) = 1 - \alpha^2\left(1 - \frac{1}{N}\right)$

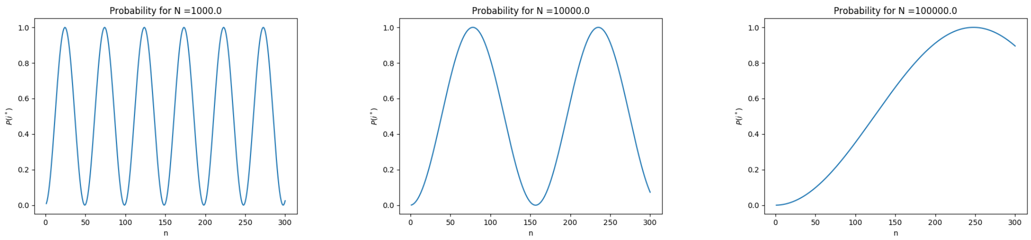
  If $P(i^*) \geq 0.99$:

  Save n

  Exit loop

The code is found at the end3.

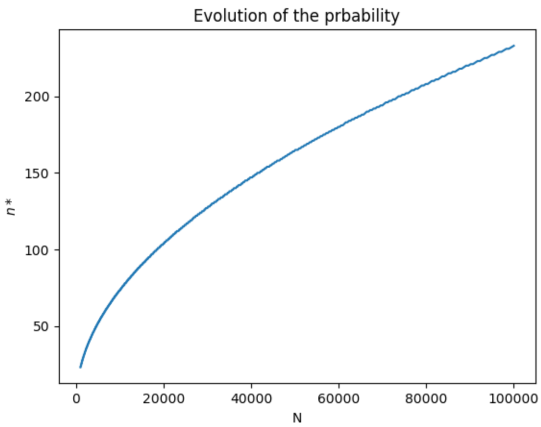
If we don't exit the loop after $P \neq 0.99$ but look at how the probability evolves:



(a) Probability for $N = 10^3$ (b) Probability for $N = 10^4$ (c) Probability for $N = 10^5$

Figur 1: We can see that the probability of measuring i^* actually goes as a sinusoidal function of n. This means that using UF doesn't always imporove the probability, and that there are regoins where applying UF actually lowers the probability. We also see that frequancy of this sinusoidal function increases with the number of unknown states N, and thus we need to apply UF more times to get a better probability.

We can then look at how many $n = n^*$ is necessary to get a probability $P(i^*) \geq 0.99$:

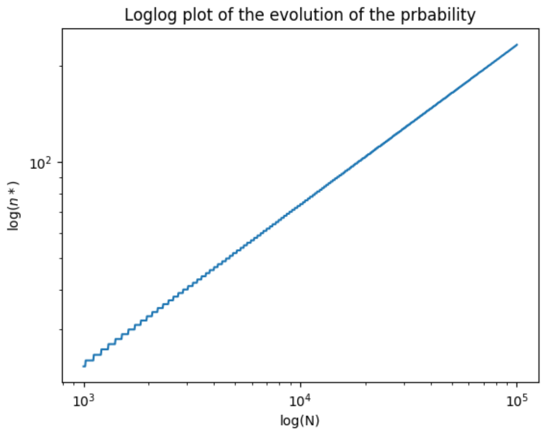


Figur 2: The number of times n^* we need to apply UF to get a probability $P(i^*) \geq 0.99$

We can see that as N increases, so does the number if times we need to apply UF this number n^*. It looks quite like a square root, but we need to make sure. For this we use a log-log plot, this is because:

$$n^* = N^i \Rightarrow \log(n^*) = i \cdot \log(N) \tag{70}$$

So we can use the log-log plot, find the slope and therefore find the order of N of which n^* increases.



Figur 3: This shows $\log(n^*)$ vs $\log(n)$ and clearly shows a straight line. The slope of this line is the order of N at which n^* increases.

If we use a normal finite difference on this curve(which is exact for a linear function), we find that slope is ~ 0.503. This is close enough to 1/2 that we can conclude that

$$n^* \sim O(\sqrt{N}) \tag{71}$$

This is an immense improvement over the classical result from sec. 2.1, but more on this below.

3 Code

```
import numpy as np
import matplotlib.pyplot as plt

alpha = 1
beta = 0

top_percentile = []

Ns = np.logspace(3,5,800) #Ns for finding n* for P > 0.99
#Ns = [1e3,1e4,1e5] #Ns for plotting the evolution of the probability
for N in Ns:
    probs = []
    ns = []
    alpha = 1
    beta = 0
    for n in range(1,301):
        new_alpha = (alpha*(1-4/N) - beta*2/np.sqrt(N))
        new_beta = alpha*2/np.sqrt(N) + beta

        alpha = new_alpha
        beta = new_beta

        p = 1-alpha**2*(1-1./N)

        probs.append(p)
        ns.append(n)
        """
        Comment the if-statement before plotting the evolution of the
```

```
        probability
        """
        if p >= 0.99:
            top_percentile.append(n)
            break


"""Uncomment for plotting the evolution of the probability"""
#    plt.plot(ns,probs)
#    plt.title("Probability for N ={}".format(N))
#    plt.xlabel("n")
#    plt.ylabel(r"$P(i^*)$")
#    plt.show()


"""Comment all below before using the plot above"""
log_top = np.log(np.array(top_percentile))
log_Ns = np.log(Ns)

print("The speed for looking up i* goes as N to the power of ", \
      np.mean((log_top[:-1] - log_top[1:])/(log_Ns[:-1] - log_Ns[1:])))

plt.loglog(Ns,top_percentile)
plt.title("Loglog plot of the evolution of the prbability")
plt.xlabel("log(N)")
plt.ylabel(r"$\log(n*)$")


#plt.plot(Ns,top_percentile)
#plt.title("Evolution of the prbability")
#plt.xlabel("N")
#plt.ylabel(r"$n*$")
#plt.show()
```