Fys3310 Hjemmeeksamen

Kadnr.:

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1 Exercise 1:

1.1 1.4)

For the Hadamard gate (H-gate) we have the following operation:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{1}$$

We can look at its properties. First we find the hermitian transform of H:

$$H^\dagger = (H^T)^* = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H \tag{2}$$

Thus showing that H is hermitian. If we multiply H with it self we get

$$H^2 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

This means that $H^2 = I \Rightarrow H = H^{-1}$ and that H is unitary.
We can now see what H does to qubit basis states

$$H\left|0\right\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{4}$$

$$H\left|1\right\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5}$$

We can recognize these as the eigenstates for spin in x-direction, so

$$H\left|0\right\rangle = \left|\downarrow_x\right\rangle \tag{6}$$

$$H\left|1\right\rangle = \left|\uparrow_x\right\rangle \tag{7}$$

1.2 1.5)

We want to find a magnetic field that that results in the effect of H found is eq. (1.1) and (1.1). We look at H and see that

$$H = \frac{1}{\sqrt{2}} (\sigma_x + \sigma_z) \tag{8}$$

So we make an educated guess that the magnetic field has to we in $\hat{i} + \hat{k}$ direction. So our Hamiltonian for the magnetic field will be:

$$\hat{H} = -\mu \cdot \mathbf{B} = -g\frac{\mu_B}{\hbar} \left(\frac{h}{\sqrt{2}} \frac{\hbar}{2} \sigma_x + \frac{h}{\sqrt{2}} \frac{\hbar}{2} \sigma_z \right) = -g\frac{h\mu_B}{2\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{9}$$

With

$$\mathbf{B} = \frac{h}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{10}$$

The $\sqrt{2}$ being there to ensure that the magnitude of h.

We now what to find the eigenstates of the Hamiltonian so we can express the time evolution of $|0\rangle$ and $|1\rangle$:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \rightarrow \qquad |h_1\rangle = \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}, |h_2\rangle = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix} \tag{11}$$

with eigenvalues

$$h_1 = \sqrt{2}, \qquad h_2 = -\sqrt{2} \tag{12}$$

This gives us the energies for the Hamiltonian

$$E_1 = -g\frac{h\mu_B}{2}, \qquad E_2 = g\frac{h\mu_B}{2} \tag{13}$$

We can now express our qubit states as linear combinations of the eigenvalues of the Hamiltonian:

$$|1\rangle = a\,|h_1\rangle + b\,|h_2\rangle\,, \qquad |0\rangle = c\,|h_1\rangle + d\,|h_2\rangle \tag{14}$$

This turns out to be

$$|1\rangle = \frac{\sqrt{2}}{4}\begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} - \frac{\sqrt{2}}{4}\begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix} \tag{15}$$

$$|0\rangle = \frac{2-\sqrt{2}}{4}\begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} + \frac{2+\sqrt{2}}{4}\begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix} \tag{16}$$

2 Exercise 2:

2.1 2.1)

2.2 2.2)

We have the operators which works like

$$F\,|i\rangle = \begin{cases} -\,|i^*\rangle & ,i = i^* \\ |i\rangle & ,i \neq i^* \end{cases} \tag{17}$$

We want to show that F can be written as

$$F = I - 2\,|i^*\rangle\,\langle i^*| \tag{18}$$

Let's use the operator on a ket

$$F\,|i\rangle = (I - 2\,|i^*\rangle\,\langle i^*|)\,|i\rangle = I\,|i\rangle - 2\,|i^*\rangle\,\langle i^*|i\rangle \tag{19}$$

$$= |i\rangle - 2\delta_{i,i^*}\,|i^*\rangle = \begin{cases} -\,|i^*\rangle & ,i = i^* \\ |i\rangle & ,i \neq i^* \end{cases} \tag{20}$$

We can see that this gives the same as (2.2), and we can therefore write $F = I - 2\,|i^*\rangle\,\langle i^*|$.

We can see from eq. 2.2 that F is a Householder transformation, which ensures that F is unitary hermitian, but we can also show this:

$$F(F\,|i\rangle) = (I - 2\,|i^*\rangle\,\langle i^*|)(I - 2\,|i^*\rangle\,\langle i^*|)\,|i\rangle \tag{21}$$

$$= I|i\rangle - 2|i^*\rangle\langle i^*| I|i\rangle - 2I|i^*\rangle\langle i^*|i\rangle + 4|i^*\rangle\langle i^*|i^*\rangle\langle i^*|i\rangle \tag{22}$$

$$= |i\rangle - 4|i^*\rangle\langle i^*|i\rangle + 4|i^*\rangle\langle i^*|i\rangle = |i\rangle \tag{23}$$

Thus $FF = I \Rightarrow F = F^{-1}$. We then show that F is hermitian:

$$F^\dagger = (I - 2|i^*\rangle\langle i^*|)^\dagger = I - 2(|i^*\rangle\langle i^*|)^\dagger \tag{24}$$

$$= I - 2(\langle i^*|)^\dagger(|i^*\rangle)^\dagger = I - 2|i^*\rangle\langle i^*| = F \tag{25}$$

So F is hermitian, $F = F^\dagger$. This implies that F is unitary hermitian:

$$F^{-1} = F = F^\dagger \Leftrightarrow F^\dagger = F^{-1} \tag{26}$$

2.3 2.3

We now introduce the superposition of the states $|i\rangle$

$$|s\rangle = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}|i\rangle \tag{27}$$

We then calculate

$$\langle i^*|s\rangle = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\langle i^*|i\rangle = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\delta_{i,i^*} = \frac{1}{\sqrt{N}} \tag{28}$$

and

$$F|s\rangle = I|s\rangle - 2|i^*\rangle\langle i^*|s\rangle = |s\rangle - \frac{2}{\sqrt{N}}|i^*\rangle \tag{29}$$

2.4 2.4)

We now consider the state:

$$|g\rangle = \alpha|s\rangle + \beta|i^*\rangle \tag{30}$$

We want this to be normalized:

$$\langle g|g\rangle = 1 \tag{31}$$

Due to Riesz representation theorem, we know that such a bra $\langle g|$ exist, so we can then find the condition for α and β such that $|g\rangle$ is normalized.

$$(\alpha * \langle s| + \beta * \langle i^*|)(\alpha|s\rangle + \beta|i^*\rangle) = |\alpha|^2 + |\beta|^2 + \alpha\beta * \langle s|i^*\rangle + \alpha * \beta\langle i^*|s\rangle \tag{32}$$

α and β are both real. We know that $\langle i^*|s\rangle$ is real from (2.3), so $\langle s|i^*\rangle = \langle i^*|s\rangle * = \langle i^*|s\rangle$. So:

$$\alpha^2 + \beta^2 + \frac{2\alpha\beta}{\sqrt{N}} = 1 \tag{33}$$

Is the condition that makes sure that $|g\rangle$ is normalized.

2.5 2.5)

The operator for measuring the measurement i of state $|i\rangle$ is named X, so X acting on a state $|i\rangle$ is:

$$X|i\rangle = i|i\rangle \tag{34}$$

meaning that $|i\rangle$ are eigenstates of X with eigenvalue i. This means that we can use the spectral theorem to write X as a spectral representation with $|i\rangle$ and i:

$$X = \sum_{j=1}^{N} j|j\rangle\langle j| \tag{35}$$

(j and i are dummy indices and therefore interchanged).

2.6 2.6)