Fys3110 Hjemmeeksamen

Kadnr.:

9. oktober 2017

1 Exercise 1:

1.1 1.1)

We have a qubit consisting of the two spin-1/2 basis states

$$|0\rangle \equiv |\downarrow\rangle \simeq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1}$$

$$|1\rangle \equiv |\uparrow\rangle \simeq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2}$$

we then introduce the operator $\hat{\sigma}_x$ represented by the Pauli-matrix

$$\hat{\sigma}_x \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3}$$

We can then look at how the operator acts on the basis states:

$$\hat{\sigma}_x |0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |1\rangle \tag{4}$$

$$\hat{\sigma}_x |1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |0\rangle \tag{5}$$

So the operator $\hat{\sigma}_x$ inverts the states:

$$\hat{\sigma}_x |0\rangle = |1\rangle, \qquad \hat{\sigma}_x |1\rangle = |0\rangle \tag{6}$$

1.2 1.2)

We have two states $|i\rangle$ and $|o\rangle$, and the operator G which acts on the states as follows:

$$|o\rangle = G|i\rangle, \qquad |i\rangle = G|o\rangle \tag{7}$$

We can see that

$$G(G|o\rangle) = G|i\rangle = |o\rangle \tag{8}$$

Thus we see that $GG = I \Rightarrow G = G^{-1}$. We can check if G is hermitian looking at the definition of the hermitian conjugate for some operator K:

$$\langle\psi| K |\phi\rangle = \langle\phi| K^\dagger |\psi\rangle^* \tag{9}$$

And if K is hermitian then $K = K^\dagger$ and

$$\langle\psi| K |\phi\rangle = \langle\phi| K |\psi\rangle^* \tag{10}$$

We can check if this holds for G. For this we calculate

$$\langle o| G |i\rangle = \langle o|o\rangle \tag{11}$$

$$\langle i| G |o\rangle^* = \langle i|i\rangle^* \tag{12}$$

But these qubits are normalized, so $\langle i|i\rangle^* = \langle i|i\rangle = \langle o|o\rangle = 1$, and we there for get

$$\langle o|\, G\, |i\rangle = 1 = \langle i|\, G\, |o\rangle^* \tag{13}$$

And thus we have used (1.2) to show that G is hermitian, $G = G^\dagger$. We have also shown that $G = G^{-1}$, but because G under multiplication is a operator group, we are ensured that the identity is unique, and we an therefore conclude that $G^{-1} = G^\dagger$, which means that G is unitary and hermitian.

1.3 1.3)

We want to find the NOT gate that switches the two states. But the operator G from sec. 1.2 does exactly that: it switches $|0\rangle$ to $|i\rangle$ and vice versa. Also, we showed in sec. 1.1 that the operator $\hat{\sigma}_x$ represented as the Pauli-matrix, took a spin up to down, and down to up. We therefore know that G is the operator for the NOT gate, represented by a Pauli-Matrix:

$$G = \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{14}$$

And as shown in sec. 1.2, this operator, G, is unitary hermitian.

1.4 1.4)

For the Hadamard gate (H-gate) we have the following operation:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{15}$$

We can look at its properties. First we find the hermitian transform of H:

$$H^\dagger = (H^T)^* = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H \tag{16}$$

Thus showing that H is hermitian. If we multiply H with it self we get

$$H^2 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{17}$$

This means that $H^2 = I \Rightarrow H = H^{-1}$ and that H is unitary.
We can now see what H does to qubit basis states

$$H\, |0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{18}$$

$$H\, |1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{19}$$

We can recognize these as the eigenstates for spin in x-direction, so

$$H\, |0\rangle = |{\downarrow_x}\rangle \tag{20}$$

$$H\, |1\rangle = |{\uparrow_x}\rangle \tag{21}$$

1.5 1.5)

We want to find a magnetic field that that results in the effect of H found is eq. (1.4) and (1.4). We look at H and see that

$$H = \frac{1}{\sqrt{2}} (\sigma_x + \sigma_z) \tag{22}$$

So we make an educated guess that the magnetic field has to we in $\hat{\imath} + \hat{k}$ direction. So our Hamiltonian for the magnetic field will be:

$$\hat{H} = -\mu \cdot \mathbf{B} = -g\frac{\mu_B}{\hbar}\left(\frac{h}{\sqrt{2}}\frac{\hbar}{2}\sigma_x + \frac{h}{\sqrt{2}}\frac{\hbar}{2}\sigma_z\right) = -g\frac{h\mu_B}{2\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \tag{23}$$

With

$$\mathbf{B} = \frac{h}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} \tag{24}$$

The $\sqrt{2}$ being there to ensure that the magnitude of h.

We now what to find the eigenstates of the Hamiltonian so we can express the time evolution of $|0\rangle$ and $|1\rangle$:

$$\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \qquad \rightarrow \qquad |h_1\rangle = \begin{pmatrix}1+\sqrt{2}\\ 1\end{pmatrix}, |h_2\rangle = \begin{pmatrix}1-\sqrt{2}\\ 1\end{pmatrix} \tag{25}$$

with eigenvalues

$$h_1 = \sqrt{2}, \qquad h_2 = -\sqrt{2} \tag{26}$$

This gives us the energies for the Hamiltonian

$$E_1 = -g\frac{h\mu_B}{2}, \qquad E_2 = g\frac{h\mu_B}{2} \tag{27}$$

We can now express our qubit states as linear combinations of the eigenvalues of the Hamiltonian:

$$|1\rangle = a\,|h_1\rangle + b\,|h_2\rangle, \qquad |0\rangle = c\,|h_1\rangle + d\,|h_2\rangle \tag{28}$$

This turns out to be

$$|1\rangle = \frac{\sqrt{2}}{4}\begin{pmatrix}1+\sqrt{2}\\ 1\end{pmatrix} - \frac{\sqrt{2}}{4}\begin{pmatrix}1-\sqrt{2}\\ 1\end{pmatrix} \tag{29}$$

$$|0\rangle = \frac{2-\sqrt{2}}{4}\begin{pmatrix}1+\sqrt{2}\\ 1\end{pmatrix} + \frac{2+\sqrt{2}}{4}\begin{pmatrix}1-\sqrt{2}\\ 1\end{pmatrix} \tag{30}$$

2 Exercise 2:

2.1 2.1)

2.2 2.2)

We have the operators which works like

$$F\,|i\rangle = \begin{cases} -\,|i^*\rangle & , i = i^* \\ |i\rangle & , i \neq i^* \end{cases} \tag{31}$$

We want to show that F can be written as

$$F = I - 2 |i^*\rangle \langle i^*| \tag{32}$$

Let's use the operator on a ket

$$F |i\rangle = (I - 2 |i^*\rangle \langle i^*|) |i\rangle = I |i\rangle - 2 |i^*\rangle \langle i^*|i\rangle \tag{33}$$

$$= |i\rangle - 2\delta_{i,i^*} |i^*\rangle = \begin{cases} - |i^*\rangle & , i = i^* \\ |i\rangle & , i \neq i^* \end{cases} \tag{34}$$

We can see that this gives the same as (2.2), and we can therefore write $F = I - 2 |i^*\rangle \langle i^*|$.

We can see from eq. 2.2 that F is a Householder transformation, which ensures that F is unitary hermitian, but we can also show this:

$$F(F |i\rangle) = (I - 2 |i^*\rangle \langle i^*|)(I - 2 |i^*\rangle \langle i^*|) |i\rangle \tag{35}$$

$$= I |i\rangle - 2 |i^*\rangle \langle i^*| I |i\rangle - 2I |i^*\rangle \langle i^*|i\rangle + 4 |i^*\rangle \langle i^*|i^*\rangle \langle i^*|i\rangle \tag{36}$$

$$= |i\rangle - 4 |i^*\rangle \langle i^*|i\rangle + 4 |i^*\rangle \langle i^*|i\rangle = |i\rangle \tag{37}$$

Thus $FF = I \Rightarrow F = F^{-1}$. We then show that F is hermitian:

$$F^\dagger = (I - 2 |i^*\rangle \langle i^*|)^\dagger = I - 2(|i^*\rangle \langle i^*|)^\dagger \tag{38}$$

$$= I - 2(\langle i^*|)^\dagger (|i^*\rangle)^\dagger = I - 2 |i^*\rangle \langle i^*| = F \tag{39}$$

So F is hermitian, $F = F^\dagger$. This implies that F is unitary hermitian:

$$F^{-1} = F = F^\dagger \Leftrightarrow F^\dagger = F^{-1} \tag{40}$$

2.3 2.3

We now introduce the superposition of the states $|i\rangle$

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle \tag{41}$$

We then calculate

$$\langle i^*|s\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \langle i^*|i\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \delta_{i,i^*} = \frac{1}{\sqrt{N}} \tag{42}$$

and

$$F |s\rangle = I |s\rangle - 2 |i^*\rangle \langle i^*|s\rangle = |s\rangle - \frac{2}{\sqrt{N}} |i^*\rangle \tag{43}$$

2.4 2.4)

We now consider the state:

$$|g\rangle = \alpha |s\rangle + \beta |i^*\rangle \tag{44}$$

We want this to be normalized:

$$\langle g|g\rangle = 1 \tag{45}$$

Due to Riesz representation theorem, we know that such a bra $\langle g|$ exist, so we can then find the condition for α and β such that $|g\rangle$ is normalized.

$$(\alpha * \langle s| + \beta * \langle i^*|)(\alpha |s\rangle + \beta |i^*\rangle) = |\alpha|^2 + |\beta|^2 + \alpha\beta * \langle s|i^*\rangle + \alpha * \beta \langle i^*|s\rangle \tag{46}$$

α and β are both real. We know that $\langle i^*|s\rangle$ is real from (2.3), so $\langle s|i^*\rangle = \langle i^*|s\rangle * = \langle i^*|s\rangle$. So:

$$\alpha^2 + \beta^2 + \frac{2\alpha\beta}{\sqrt{N}} = 1 \tag{47}$$

Is the condition that makes sure that $|g\rangle$ is normalized.

2.5 2.5)

The operator for measuring the measurement i of state $|i\rangle$ is named X, so X acting on a state $|i\rangle$ is:

$$X |i\rangle = i |i\rangle \tag{48}$$

meaning that $|i\rangle$ are eigenstates of X with eigenvalue i. This means that we can use the spectral theorem to write X as a spectral representation with $|i\rangle$ and i:

$$X = \sum_{j=1}^{N} j |j\rangle \langle j| \tag{49}$$

(j and i are dummy indices and therefore interchanged).

2.6 2.6)

We want to find the probability of measuring i^* when observing $|g\rangle$. From X we find

$$X |i^*\rangle = i^* |i^*\rangle \tag{50}$$

Meaning that $|i^*\rangle$ is the state where we measure i^*, meaning that we need to find the probability of measuring $|g\rangle$ in state $|i^*\rangle$. We d this with the projection operator

$$\hat{P} = |i^*\rangle\langle i^*| \tag{51}$$

We then find the expectation value of P acting on $|g\rangle$:

$$P(i^*) = \langle g| \hat{P} |g\rangle = \langle g|i^*\rangle \langle i^*|g\rangle = | \langle i^*|g\rangle |^2 \tag{52}$$

We can calculate this

$$\langle i^*|g\rangle = \langle i^*| (\alpha |s\rangle + \beta |i^*\rangle) = \alpha \langle i^*|s\rangle + \beta \langle i^*|i^*\rangle = \frac{\alpha}{\sqrt{N}} + \beta \tag{53}$$

All the terms are real, so the norm in the probability just becomes a power of two:

$$\Rightarrow P(i^*) = |\langle i^*|g\rangle|^2 = \left(\frac{\alpha}{\sqrt{N}} + \beta\right)^2 = \frac{\alpha^2}{N} + \beta^2 + \frac{\alpha\beta}{\sqrt{N}} \tag{54}$$

From eq. (2.4) we see that

$$\beta^2 + \frac{2\alpha\beta}{\sqrt{N}} = 1 - \alpha^2 \tag{55}$$

So we get that the probability of measuring i^* becomes:

$$P(i^*) = 1 - \alpha^2 \left(1 - \frac{1}{N}\right) \tag{56}$$

2.7 2.7)

We now introduce the operator

$$U = 2|s\rangle\langle s| - I \tag{57}$$

We want to look at $UF|s\rangle$. We already calculated $F|s\rangle$(2.3), so we get:

$$UF|s\rangle = U(|s\rangle - \frac{2}{\sqrt{N}}|i^*\rangle) \tag{58}$$

$$= 2|s\rangle\langle s|s\rangle - I|s\rangle - \frac{4}{\sqrt{N}}|s\rangle\langle s|i^*\rangle + \frac{2}{\sqrt{N}}I|i^*\rangle = |s\rangle - \frac{4}{N}|s\rangle + \frac{2}{\sqrt{N}}|i^*\rangle \tag{59}$$

So

$$UF|s\rangle = \left(1 - \frac{4}{N}\right)|s\rangle + \frac{2}{\sqrt{N}}|i^*\rangle \tag{60}$$

If we look at U(2.7) we see that it too is a Householder transformation(with a negative sign), thus we know that it is unitary hermitian. A unitary always preserves the norm. So if we first apply F(which we showed was unitary) to $|s\rangle$, its norm is still 1 ($|s\rangle$ is a normalized sum of normalized states). So when we then apply U, the norm is still 1. Thus the norm of $UF|s\rangle$ is 1. But we can also show this explicit:

$$\left[\left(1 - \frac{4}{N}\right)\langle s| + \frac{2}{\sqrt{N}}\langle i^*|\right]\left[\left(1 - \frac{4}{N}\right)|s\rangle + \frac{2}{\sqrt{N}}|i^*\rangle\right] \tag{61}$$

(remember that all the constants are real)

$$= \left(1 - \frac{4}{N}\right)^2\langle s|s\rangle + \frac{2}{\sqrt{N}}\left(1 - \frac{4}{N}\right)\langle s|i^*\rangle + \frac{2}{\sqrt{N}}\left(1 - \frac{4}{N}\right)\langle i^*|s\rangle + \frac{4}{N}\langle i^*|i^*\rangle \tag{62}$$

$$= \left(1 - \frac{4}{N}\right)^2 + \frac{4}{N}\left(1 - \frac{4}{N}\right) + \frac{4}{N} \tag{63}$$

$$= 1 - \frac{8}{N} + \frac{16}{N^2} + \frac{4}{N} - \frac{16}{N^2} + \frac{4}{N} = 1 \tag{64}$$

So we have showed that the norm of $UF|s\rangle$ is 1.

2.8 2.8)

We now want to calculate $UF\,|g\rangle$:

$$UF\,|g\rangle = UF(\alpha\,|s\rangle + \beta\,|i^*\rangle) = \alpha UF\,|s\rangle + \beta UF\,|i^*\rangle \tag{65}$$

We have already found $UF\,|s\rangle$(2.7), but we need to find

$$UF\,|i^*\rangle = -U\,|i^*\rangle = -(2\,|s\rangle\!\langle s| - I)\,|i^*\rangle = -2\,|s\rangle\,\langle s|i^*\rangle + I\,|i^*\rangle \tag{66}$$

$$= |i^*\rangle - \frac{2}{\sqrt{N}}\,|s\rangle \tag{67}$$

We can now combine these

$$\alpha UF\,|s\rangle + \beta UF\,|i^*\rangle = \alpha\left[\left(1-\frac{4}{N}\right)|s\rangle + \frac{2}{\sqrt{N}}\,|i^*\rangle\right] + \beta\left[|i^*\rangle - \frac{2}{\sqrt{N}}\,|s\rangle\right] \tag{68}$$

We can so sort these and find

$$UF(\alpha\,|s\rangle + \beta\,|i^*\rangle) = \left[\alpha\left(1-\frac{4}{N}\right) - \frac{2\beta}{\sqrt{N}}\right]|s\rangle + \left[\frac{2\alpha}{\sqrt{N}} + \beta\right]|i^*\rangle \tag{69}$$