FYS3120 Oblig 2

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1)

a)

$\frac{\partial \mathcal{L}}{\partial t}$ is the partial differentiation of $\mathcal{L}$ and will only differentiate on explicit dependence on t.

$\frac{d\mathcal{L}}{dt}$ is the total differentiation of $\mathcal{L}$ and will also differentiate variables in $\mathcal{L}$ dependent on t(so it will differentiate on implicit dependence on t.). So

$$\frac{d\mathcal{L}}{dt} = \sum_i \frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t}$$

b)

Lets start with

$$\frac{d}{dt}\left(\mathcal{L} - \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{d\mathcal{L}}{dt} - \sum_i \left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\dot{q}_i - \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i$$

We can see from Lagrange's equation that $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}$. We then get

$$= \frac{d\mathcal{L}}{dt} - \sum_i \frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i - \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i$$

if we now insert the expression we found for $\frac{d\mathcal{L}}{dt}$ above, all but the last expression of $\frac{d\mathcal{L}}{dt}$ disappears, and we are left with the desired result

$$\frac{d}{dt}\left(\mathcal{L} - \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial t}$$

2)

a)

The distance from the center of mass of the first rod to the origin is always $\frac{l}{2}$. The second rod can only rotate around the end if the first rod, and its center of mass is always a distance l from the origin. These are the 2 constraint. Also $d = 4 - 2 = 2$. We are going to chose the angle of the second rod ϕ, and the angle of the second rod θ, both relative to the vertical axis. These are 2 generalized coordinates.

We can now look at the potential energy. The first rod has a potential energy of $-\frac{l}{2}mg\cos\theta$. The second rod has the potential energy $-lmg\cos\theta$. So we get

$$V(\theta) = -\frac{3}{2}mgl\cos\theta$$

The first rod swings around its end point, and has the kinetic energy

$$K_1 = \frac{1}{2}I_1\dot{\theta}^2 = \frac{1}{6}ml^2\dot{\theta}^2$$

The second rod has kinetic energy akin to that of a pendulum $1/2ml^2\dot{\theta}^2$ and kinetic energy from rotating around its center of mass. So

$$K_2 = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}I_2\dot{\phi}^2 = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{24}ml^2\dot{\phi}^2$$

We therefor get the Lagrangian

$$\mathcal{L} = K - V = \frac{1}{6}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{24}ml^2\dot{\phi}^2 + \frac{3}{2}mgl\cos\theta$$

$$\mathcal{L} = \frac{2}{3}ml^2\dot{\theta}^2 + \frac{1}{24}ml^2\dot{\phi}^2 + \frac{3}{2}mgl\cos\theta \tag{1}$$

b)

We can now find the e.o.m. of the system using the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q_i}}\right) - \frac{\partial\mathcal{L}}{\partial q_i} = 0 \tag{2}$$

We are first going to find all the expressions from the equation above:

$$\frac{\partial\mathcal{L}}{\partial\phi} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = -mgl\frac{3}{2}\sin\theta$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{1}{12}ml^2\dot{\phi}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{4}{3}ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) = \frac{1}{12}ml^2\ddot{\phi}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{4}{3}ml^2\ddot{\theta}$$

If we set these expressions into eq 2 we find that the e.o.m are

$$\frac{1}{12}ml^2\ddot{\phi} = 0$$

$$\frac{4}{3}ml^2\ddot{\theta} + \frac{3}{2}mgl\sin\theta = 0$$

We are now going to look at the last to these two equations for small angles. For small angles we can through Taylor expansions find that $\sin\theta \approx \theta$, so

$$\frac{4}{3}ml^2\ddot{\theta} + \frac{3}{2}mgl\theta = 0$$

This is a ODE of the form $ax'' + bx = 0$ which have the general solution $x(t) = A\cos(\omega t)$, with $\omega = \sqrt{\frac{b}{a}}$. We can see that for us $a = \frac{4}{3}ml^2$ and $b = \frac{3}{2}mgl$ so the angular frequency of the upper rod is given as

$$\omega = \sqrt{\frac{\frac{3}{2}mgl}{\frac{4}{3}ml^2}} = \frac{3}{2}\sqrt{\frac{g}{2l}}$$

3)

We are first going to find the position vector of the mass in generalized coordinates

$$x = r\cos(\omega t), \qquad y = r\sin(\omega t)$$

and then there derivatives

$$\dot{x} = \dot{r}\cos(\omega t) - r\omega\sin(\omega t), \qquad \dot{y}\dot{r}\sin(\omega t) + r\omega\cos(\omega t)$$

This gives us the speed squared

$$v^2 = \dot{r}^2 + r^2\omega^2$$

And therefor

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2)$$

There are no other forces drawn in the picture, and therefor no gravitation. This means that the mass does not have any potential energy:

$$V = 0$$

So

$$\mathcal{L} = T - V = T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2)$$

We can now calculate all the expressions we need for Lagrange's equation 2.

$$\frac{\partial \mathcal{L}}{\partial r} = mr\omega^2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}, \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = m\ddot{r}$$

Setting these into 2 we find that

$$m\ddot{r} - mr\omega^2 = 0$$

This gives us the differential equation for the motion

$$\ddot{r} = r\omega^2$$

This ODE has a general solution

$$r(t) = c_1 e^{\omega t} + c_2 e^{-\omega t}$$

And

$$\dot{r}(t) = c_1\omega e^{\omega t} - c_2\omega e^{-\omega t}$$

Given the initial conditions for $t = 0$, $r = r_0$ and $\dot{r} = 0$

$$r(t = 0) = c_1 + c_2 = r_0$$

$$\dot{r}(0) = c_1\omega - c_2\omega = 0$$

We can find that

$$c_1 = c_2 = \frac{r_0}{2}$$

So are solution for these initial conditions are

$$r(t) = r_0 \frac{e^{\omega t} + e^{-\omega t}}{2} = r_0 \cosh(\omega t)$$

b)

3)

a)

As in the last exercise we are going to start with the position vector for the pendulum bob

$$x = s + l\sin\theta = l\sin\theta, \qquad y = s + l\cos\theta = l\cos\theta$$

Because $s = 0$. The speed squared of a pendulum bob is given as

$$v^2 = l^2\theta^2$$

and the kinetic energy

$$T = \frac{1}{2}ml^2\theta^2$$

The potential energy is given as

$$V = -mgl\cos\theta$$

The Lagrangian thus becomes

$$\mathcal{L} = \frac{1}{2}ml^2\theta^2 + mgl\cos\theta$$

We then find that

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mgl\sin\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

So the e.o.m is

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

Which is the e.o.m of a normal pendulum.

b)

This problem is alot worse... First the coordinates

$$x = s + l\sin\theta, \qquad y = -l\cos\theta$$

And there derivatives

$$\dot{x} = \dot{s} + l\dot{\theta}\cos\theta, \qquad \dot{y} = l\sin\theta\dot{\theta}$$

We can then find the speed squared

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{s}^2 + 2\dot{s}\dot{\theta}l\cos\theta + l^2\theta^2$$

So

$$T = \frac{1}{2}m(\dot{s}^2 + 2\dot{s}\dot{\theta}l\cos\theta + l^2\theta^2)$$
$$V = -mgl\cos\theta$$

And

$$\mathcal{L} = \frac{1}{2}m(\dot{s}^2 + 2\dot{s}\dot{\theta}l\cos\theta + l^2\theta^2) + mgl\cos\theta$$

We can then find the expressions for the Lagrangian

$$\frac{\partial\mathcal{L}}{\partial s} = 0 \tag{3}$$

I am not going to do more about this now, because s is now shown to be a cyclic coordinate. I am going to look more at s later.

$$\frac{\partial\mathcal{L}}{\partial\theta} = -ml\dot{s}\dot{\theta}\sin\theta - mgl\sin\theta$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = m\dot{s}l\cos\theta + ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = m\ddot{s}l\cos\theta - m\dot{s}l\dot{\theta}\sin\theta + ml^2\ddot{\theta}$$

Using this in 2 we can find the e.o.m

$$m\ddot{s}l\cos\theta + ml^2\ddot{\theta} + mgl\sin\theta = 0 \tag{4}$$

c)

As mentioned above we found that

$$\frac{\partial\mathcal{L}}{\partial s} = 0$$

This means that s is a cyclic coordinate. This further means that

$$\frac{\partial\mathcal{L}}{\partial\dot{s}} = m\dot{s} + mgl\dot{\theta}\cos\theta = p_s = constant$$

p_s is a conserved quantity and a constant of motion, so

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{s}}\right) = \ddot{s} = 0$$

If we use this in 4 the first part of the equation disappears and we are left with

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

Which is the equation for a normal pendulum(a mass in free fall restricted by the pendulum chain/rod).