FYS1120 Oblig 2

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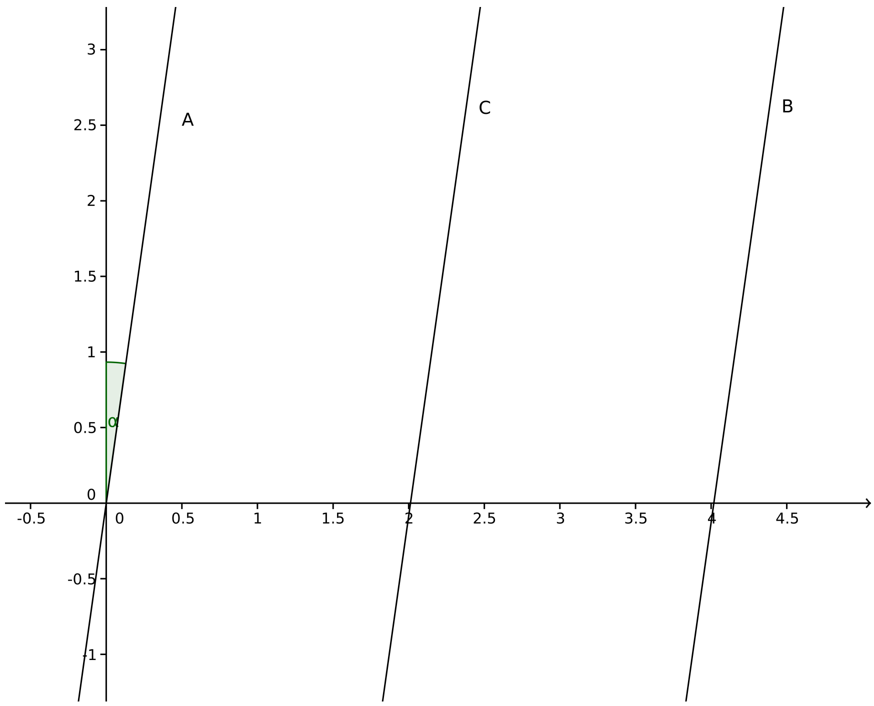
1)

Remark: The names of the axes seems to have disappeared in the pngs. But in this exercise, the y-axis is defined as ct and the x-axis as x.

a)

The Minkowski diagram:



Figur 1: The back of the cart at time 0 is defined as the origin. The angle α is draw between the world line of A and the ct-axis

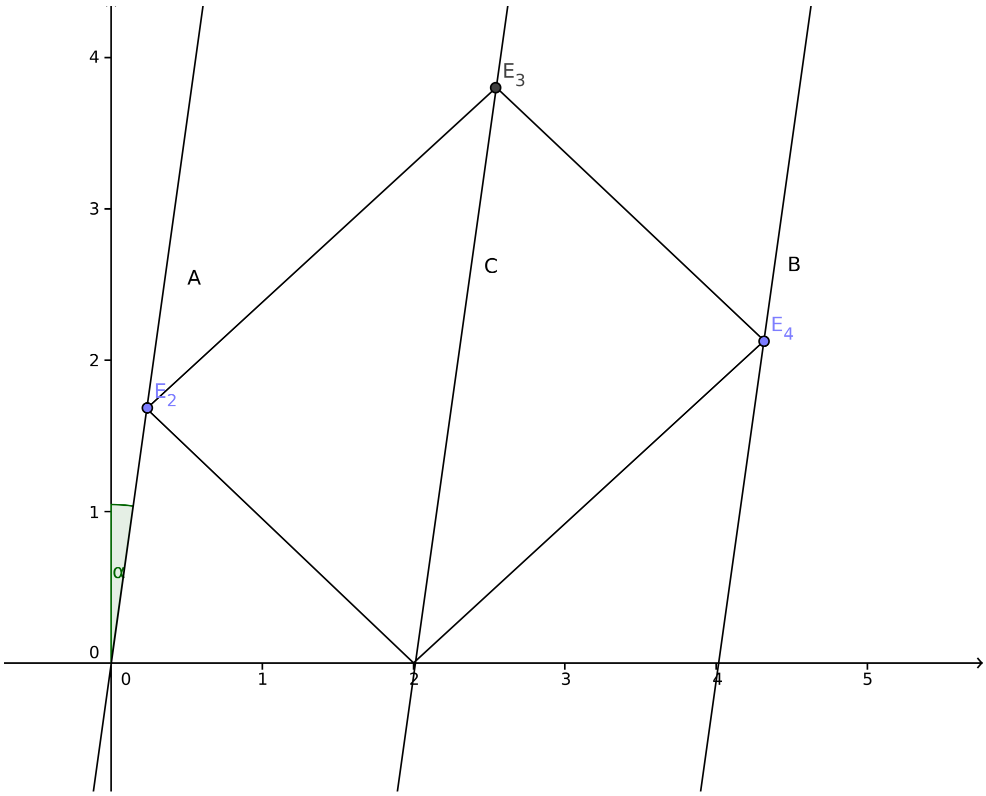
For the angle α between the ct-axis and the world lines is defined as

$$\tan\alpha = \frac{x_i}{ct_i}$$

At some point (ct_i, x_i). But since the cart has a constant velocity, that velocity is given as $v = \frac{x_i}{t_i}$ for every point. So

$$\tan\alpha = \frac{x_i}{t_i} = \frac{1}{c} = \frac{v}{c}$$

b)



Figur 2: The lights starts at E_0 and goes backwards and forwards at 45°, hitting the world lines at E_1 and E_2. The signals then reflects back, still going at 45° as light must. Then meeting again on world line C at E_3.

c)

If we have a reference system $S^{'}$ co-moving with S, then the relative velocity is zero. Then the world lines goes straight up. This means that E_1 and E_2 happens at the same time, since neither of the ends are moving faster or slower with respect to the signal. Since the signal is moving with the same relative velocity to the ends in $S^{'}$, they will use the same time from E_0 to E_1/E_2 and back, therefor meeting at same point in space at E_0. This seems to be same as in 2.

d)

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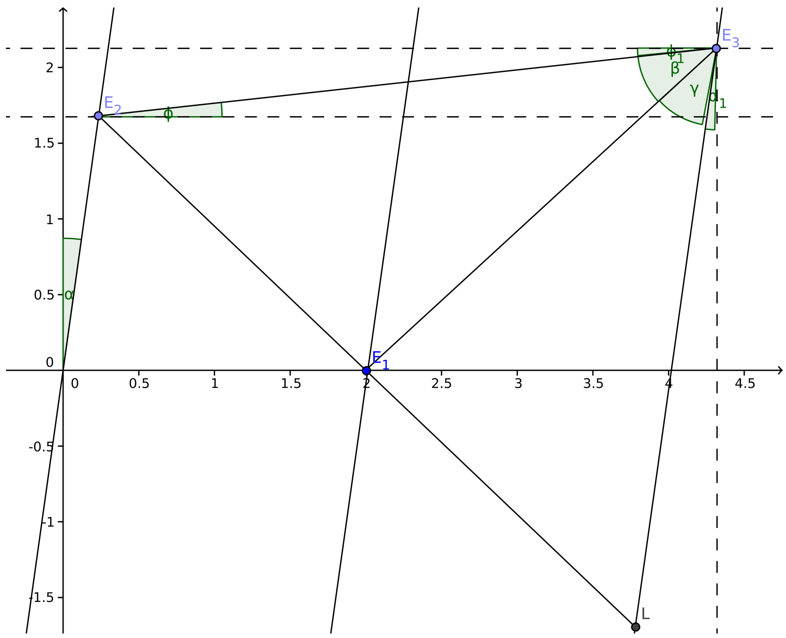

Figur 3: Helping lines and angles are drawn to make the proof easier.

I am going to do a purely geometrical proof that the angle Φ is the same as α. We can see that for the angles in the upper right corner that $\Phi_1 = \Phi$ and $\alpha_1 = \alpha$. We then draw a line E_0L, this have to have the same length the line E_2E_0. This means that the angles β and γ are equivalent. If we look at the upper right corner we have a right angle with 45° line separating it into two equal parts. This means that $\Phi_1 + \beta = \gamma + \alpha_1$. Combination this we get

$$(1) \qquad \Phi_1 + \beta = \gamma + \alpha_1$$

$$(2) \qquad \beta = \gamma$$

$$(3) \qquad \alpha_1 = \alpha$$

$$(4) \qquad \Phi_1 = \Phi$$

$$\therefore \Phi = \alpha$$

We have that the two angles are equivalent.

e)

If we were to have a signal going from E_1 to E_2 we would have

$$\tan\Phi = \frac{ct_{23}}{x_{23}} = \frac{c}{v_{23}}$$

And since $\tan\Phi = \tan\alpha = \frac{v}{c}$ we get

$$v_{23} = \frac{c}{\tan\alpha} = \frac{c^2}{v}$$

This means that the velocity of the signal has to be $> c$ which is impossible, and causality is preserved!

2)

a)

The end points B of the rod is moving with the same velocity u as the midpoint A. Since u is normal to the length of the rod, there is no effect from the Lorentz transformation on A and B. The x-position of the endpoint is $L_0/2$ away from the midpoint. So

$$B = (ct_B^{'}, L_0/2, ut_B^{'}, z^{'})$$

For $S^{'}$ $t_A^{'} = t_B^{'}$, but I have named them for later use.

b)

Now we have to use the Lorentz transformation. Since v is the velocity parallel to the x-axis

$$z = z^{'}, \qquad y = y^{'} = ut^{'}$$

But time and x-position is transformed:

$$x_A^{'} = \gamma(x_A - vt_A), \qquad t_A^{'} = \gamma(t_A - \frac{v}{c^2}x_A)$$

$$x_B^{'} = \gamma(x_B - vt_B), \qquad t_B^{'} = \gamma(t_B - \frac{v}{c^2}x_B)$$

We are after the unmarked coordinate. We can easily find them by looking from $S^{'}$ on to S by reversing the direction of v:

$$x_A = \gamma(x_A^{'} + vt_A^{'}), \qquad t_A = \gamma(t_A^{'} + \frac{v}{c^2}x_A^{'})$$

$$x_B = \gamma(x_B^{'} + vt_B^{'}), \qquad t_B = \gamma(t_B^{'} + \frac{v}{c^2}x_B^{'})$$

We now use what we know, namely that $x_A^{'} = 0$, $x_B^{'} = L_0/2$, and that $t_A = t_B = t$. We first get that

$$t_A = \gamma t_A^{'} \Rightarrow t_A^{'} = \frac{t}{\gamma}$$

$$t_B = \gamma(t_B^{'} + \frac{v}{c^2}\frac{L_0}{2}) \Rightarrow t_B^{'} = \frac{t}{\gamma} - \frac{v}{c^2}\frac{L_0}{2}$$

We then find the x-positions:

$$x_A = \gamma v t_A^{'} = vt$$

$$x_B = \gamma\left[\frac{L_0}{2} + v\left(\frac{t}{\gamma} - \frac{v}{c^2}\frac{L_0}{2}\right)\right]$$

And lastly the y-positions:

$$y_A = y_A^{'} = ut_A^{'} = \frac{u}{\gamma}t$$

$$y_B = y_B^{'} = ut_B^{'} = u\left(\frac{t}{\gamma} - \frac{v}{c^2}\frac{L_0}{2}\right)$$

We now have all the space coordinates. For A:

$$x_A = vt, \qquad y_A = \frac{u}{\gamma}t, \qquad z_A = 0$$

And for B:

$$x_B = \gamma\left[\frac{L_0}{2} + v\left(\frac{t}{\gamma} - \frac{v}{c^2}\frac{L_0}{2}\right)\right], \qquad y_B = \left(\frac{t}{\gamma} - \frac{v}{c^2}\frac{L_0}{2}\right), \qquad z_B = 0$$

c)

We can now find the angle. Fist we start with:

$$x_B - x_A = \gamma\left[\frac{L_0}{2} + v\left(\frac{t}{\gamma} - \frac{v}{c^2}\frac{L_0}{2}\right)\right] - vt = \gamma\frac{L_0}{2}\left(1 - \frac{v^2}{c^2}\right) = \frac{1}{\gamma}\frac{L_0}{2}$$

And

$$y_B - y_A = u\left(\frac{t}{\gamma} - \frac{v}{c^2}\frac{L_0}{2}\right) - \frac{u}{\gamma}t = -\frac{vu}{c^2}\frac{L_0}{2}$$

We find the angle by finding $\tan\alpha$

$$\tan\alpha = \frac{y_B - y_A}{x_B - x_A} = -\frac{\gamma vu}{c^2}$$

The observer in S will only see the rod as being parallel to the x-axis if $u = 0$, so in this case the observer will see the rod as tilted.

The length of the rod measured S is given as

$$L = 2\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} = L_0\sqrt{\gamma^{-2} + \frac{v^2u^2}{c^4}}$$

3)

From Fermat's principle we have that

$$S[y(x)] = \int_{x_1}^{x_2} n(x, y)\sqrt{1 + y'^2}dx$$

This gives us that

$$L(y, y', x) = n(x, y)\sqrt{1 + y'^2}$$

We can then use the Euler-Lagrange equation:

$$\frac{d}{dx}\frac{\partial L}{\partial y'} - \frac{\partial L}{\partial y} = 0$$

First we find the derivatives of $L(y, y', x)$:

$$\frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}n(x, y), \qquad \frac{\partial L}{\partial y} = \sqrt{1 + y'^2}\frac{\partial}{\partial y}n(x, y)$$

Giving us the equation of motion

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}n(x, y)\right) - \sqrt{1 + y'^2}\frac{\partial}{\partial y}n(x, y) = 0$$

Now, if the index of refraction is constant, $n(x, y) = n$, this means that $\partial_y n = 0$, so:

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}n\right) = 0$$

$$\frac{y'}{\sqrt{1+y'^2}}n = \alpha \tag{1}$$

Where α is a constant. Doing some algebra we get:

$$y' = \frac{\alpha}{\sqrt{n^2 - \alpha^2}}$$

Which is also a constant. This means that

$$y(x) = ax + b$$

A straight line.

b)

We now have that

$$n(x) = \begin{cases} n_1 & x < 0 \\ n_2 & x > 0 \end{cases}$$

This means we have a straight line who switches its slope at a point $(0, y_0)$. For the line before and after this point 1 holds. Meaning that the only unknown is the y-coordinate y_0 where $x = 0$. So all we have to find to find the optimal path is y_0.

For a straight line, the slope can be found as

$$y' = \frac{y - y_0}{x - x_0} = \frac{y - y_0}{x}$$

(since $x_0 = 0$). Now, inserting this into 1, we find that

$$\frac{\frac{y_1 - y_0}{x}}{\sqrt{1 + \left(\frac{y_1 - y_0}{x_1}\right)^2}} n_1 = \frac{y_1 - y_0}{\sqrt{x_1^2 + (y_1 - y_0)^2}} n_1 = \alpha$$

The same is true for $x > 0$, so:

$$\frac{y_1 - y_0}{\sqrt{x_1^2 + (y_1 - y_0)^2}} n_1 = \frac{y_2 - y_0}{\sqrt{x_2^2 + (y_2 - y_0)^2}} n_2$$

This is a function of y_0, so to find the optimal path, we have to solve it for y_0

c)

If we look at

$$\frac{y_1 - y_0}{\sqrt{x_1^2 + (y_1 - y_0)^2}}$$

We see that $\sqrt{x_1^2 + (y_1 - y_0)^2}$ is the length of the slope, while $y_1 - y_0$ is the 'opposite' side, meaning that

$$\frac{y_1 - y_0}{\sqrt{x_1^2 + (y_1 - y_0)^2}} = \sin\theta_1$$

Where θ_1 is the angle between the slope and a normal of the line where the media change the index of reflection(here the y-axis). We have now obtained Snell's law!

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$