FYS3120 Oblig 5

Daniel Heinesen, daniehei

22. februar 2017

1)

a)

The given coordinates for a rotating frame are

$$x = \xi \cos \omega t - \eta \sin \omega t$$
$$y = \xi \sin \omega t + \eta \cos \omega t$$

To find the Lagrangian we are going to start with the

$$\dot{x} = -\xi\omega \sin \omega t - \eta\omega \cos \omega t + \dot{\xi} \cos \omega t - \dot{\eta} \sin \omega t$$
$$\dot{y} = \xi\omega \cos \omega t - \eta\omega \sin \omega t + \dot{\xi} \sin \omega t + \dot{\eta} \cos \omega t$$

The energies of the system is as follows

$$V = mgy = mg(\xi \sin \omega t + \eta \cos \omega t)$$
$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\xi^2\omega^2 + \eta^2\omega^2 + 2\xi\dot{\eta}\omega - 2\eta\dot{\xi}\omega + \dot{\xi}^2 + \dot{\eta}^2)$$

Giving us the Lagrangian

$$L = \frac{1}{2}m(\xi^2\omega^2 + \eta^2\omega^2 + 2\xi\dot{\eta}\omega - 2\eta\dot{\xi}\omega + \dot{\xi}^2 + \dot{\eta}^2) - mg(\xi \sin \omega t + \eta \cos \omega t)$$

b)

We can now find the corresponding e.o.m's

$$\frac{\partial L}{\partial \xi} = m(\xi\omega^2 + \dot{\eta}\omega) - mg\xi \sin \omega t$$
$$\frac{\partial L}{\partial \dot{\xi}} = m(\dot{\xi} - \eta\omega)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}}\right) = m(\ddot{\xi} - \dot{\eta}\omega)$$

Giving us the first e.o.m

$$m\ddot{\xi} - 2m\dot{\eta}\omega - m\omega^2\xi + mg\xi \sin \omega t = 0 \tag{1}$$

For the second e.o.m

$$\frac{\partial L}{\partial \eta} = m(\eta\omega^2 - \dot{\xi}\omega) - mg\eta \cos \omega t$$
$$\frac{\partial L}{\partial \dot{\eta}} = m(\dot{\eta} + \xi\omega)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}}\right) = m(\ddot{\eta} + \dot{\xi}\omega)$$

Giving us the second e.o.m

$$m\ddot{\eta} + 2m\dot{\xi}\omega - m\omega^2\eta + mg\eta \cos \omega t = 0 \tag{2}$$

We can combine these and get that

$$m(\ddot{\eta}+\ddot{\xi}) - 2m\omega(\dot{\eta}-\dot{\xi}) - m\omega^2(\eta+\xi) + mg(\eta\cos\omega t + \xi\sin\omega t) = 0 \tag{3}$$

From both the e.o.m we can see that there are terms corresponding to the Coriolis- and the centrifugal force. In a general physics book the Coriolis force is given as $-2m\vec{\Omega}\times\vec{v}$, which we can recognize from our e.o.m as $-2m\omega(\dot{\xi}-\dot{\eta})$.

Centripetal force is generally given as $-m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$ which we recognize as $-m\omega^2(-\eta-\xi)$

For a rotating frame, Newton's 2nd law is given as

$$ma = F_{imp} + F_{centrifugal} + F_{Coriolis}$$

We see that this has the exact same form as our e.o.m 3

$$\underbrace{m(\ddot{\eta}+\ddot{\xi})}_{ma} = \underbrace{2m\omega(\dot{\xi}-\dot{\eta})}_{F_{Coriolis}} + \underbrace{m\omega^2(-\eta-\xi)}_{F_{centrifugal}} + \underbrace{mg(\eta\cos\omega t + \xi\sin\omega t)}_{F_{imp}}$$

2)

a)

We want to find the time it takes to get between A and B. We are going to start with our old friend

$$s = vt$$

We are the going to do this for infinitesimals

$$ds = vdt$$

We know that

$$ds = \sqrt{dx^2+dy^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \sqrt{1+y'^2}dx$$

We then need to find v. We know that energy is conserved, so

$$\frac{1}{2}mv^2 + mgy = 0 \Rightarrow v = \sqrt{-2gy}$$

Given that $y > 0$
We can now combine these 3 facts, and we get that

$$dt = \frac{ds}{v} = \sqrt{\frac{1+y'^2}{-2gy}}dx$$

We can now find the total time

$$T[y(x)] = \int dt = \int_{x_A}^{x_B} \sqrt{\frac{1+y'^2}{-2gy}}dx = \int_{x_A}^{x_B} L(y,y')dx$$

b)

The Hamiltonian H is given by

$$H = py' - L$$

Since L has no explicit time-dependence, the Hamiltonian is a constant of motion, E.

To calculate the Hamiltonian we start by calculating

$$p = \frac{\partial L}{\partial y'} = \frac{y'}{-2gy}\sqrt{\frac{2gy}{1+y'^2}}$$

And then we get

$$H = \frac{y'^2}{-2gy}\sqrt{\frac{2gy}{1+y'^2}} - \sqrt{\frac{1+y'^2}{-2gy}} = E$$

Dividing the whole expression by $\sqrt{\frac{2gy}{1+y'^2}}$ we get

$$\frac{1}{-2gy}y'^2 - \frac{1+y'^2}{-2gy} = E\sqrt{\frac{1+y'^2}{-2gy}}$$

Cleaning up some, and then squaring both sides gives us

$$\left(\frac{1}{-2gy}\right)^2 = E^2\frac{1+y'^2}{-2gy}$$

$$\Rightarrow \frac{1}{-2gy} = E^2(1+y'^2)$$

This gives us the differential equation

$$(1+y'^2)y = \frac{1}{-2gE^2} = -k^2 \tag{4}$$

c)

We are given

$$x = \frac{1}{2}k^2(\theta - \sin\theta) \tag{5}$$

$$y = \frac{1}{2}k^2(\cos\theta - 1) \tag{6}$$

We need to find y', for this we need

$$\frac{dx}{d\theta} = \frac{1}{2}k^2(1 - \cos\theta) \tag{7}$$

$$\frac{dy}{d\theta} = -\frac{1}{2}k^2\sin\theta \tag{8}$$

Combining them gives us

$$\frac{dy}{dx} = \frac{\sin\theta}{\cos\theta - 1} \tag{9}$$

We now insert this into 4

$$(1 + y'^2)y = (1 + \left(\frac{\sin\theta}{\cos\theta - 1}\right)^2)\frac{1}{2}k^2(\cos\theta - 1)$$

$$= \frac{1}{2}k^2(\cos\theta - 1) + \frac{1}{2}k^2\frac{\sin^2\theta}{\cos\theta - 1} = \frac{1}{2}k^2\left[\frac{(\cos\theta - 1)^2 + \sin^2\theta}{\cos\theta - 1}\right]$$

$$= \frac{1}{2}k^2\left[\frac{2 - 2\cos\theta}{\cos\theta - 1}\right] = -k^2$$

This shows that 5 and 6 solves the differential equation 4.

If we have boundary conditions (x_A, y_A) and (x_B, y_B) we need to find the corresponding pair (r, θ) for both points, where $r = \frac{1}{2}k^2$ is constant for both points. We are going to assume for the sake of discussion that the point A is at the origin, so that $\theta_A = 0$. We can then find θ_B by solving

$$\frac{x_B}{y_B} = \frac{\theta_B - \sin\theta_B}{\cos\theta_B - 1}$$

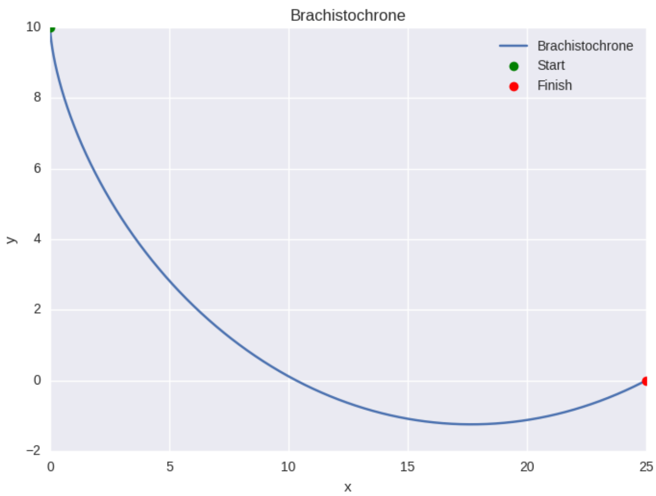
This is difficult to solve analytically for θ_B, but easy numerically.

We can use θ_B to find r

$$x_B = r(\theta_B - \sin\theta_B) \Rightarrow r = \frac{x_B}{(\theta_B - \sin\theta_B)}$$

If we now let $\theta \in [0, \theta_B]$ and $\frac{1}{2}k^2 = r$ we get a brachistochrone from $(0,0)$ to (x_B, y_B). If we want to use another starting point we just need to calculate a new θ_A, and move the all the y-coordinates by y_A(Since 5 and 6 always has it origin at $(0,0)$)

d)



Figur 1: A brachistochrone going from $(0, 10)$ to $(25, 0)$

We can see that the path of the ball/mass goes below 0, but even though is has to move uphill for the last stretch it is still the fastest route.

e)

To find the extrema of $y(\theta)$ we are going set its derivatives equal to zero. So

$$\frac{dy}{d\theta} = -\frac{1}{2}k^2 \sin\theta = 0$$

The solutions are

$$\theta = n\pi, \qquad n = 0, 1, \cdots$$

Given that the slops starts at $\theta = 0$ and is a maximum, it is easy to see that the minimum is at $\theta = \pi$. If we insert this into the definitions of x and y we get that

$$x(\pi) = \frac{1}{2}k^2(\theta - \sin\theta) = \frac{1}{2}k^2\pi$$

$$y(\pi) = \frac{1}{2}k^2(\cos\pi - 1) = -k^2$$

This gives us that

$$y_B = -\frac{2}{\pi}x_B$$

We can now find the time for a mass to fall to this point. We wish to integrate between $\theta = 0$ and π. To do this we are going to use a small trick. But first, to make it easier to write I am again going to use that $\frac{1}{2}k^2 = r$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2$$

$$= r^2((1 - \cos\theta)^2 + sin^2\theta) = 2r^2(1 - \cos\theta)$$

$$\Rightarrow ds = \sqrt{2r^2(1 - \cos\theta)}$$

We remember from earlier that

$$v = \sqrt{-2gy} = \sqrt{-2gr(\cos\theta - 1)}$$

We are again going to use that

$$T = \int dt = \int \frac{ds}{v} = \int_0^\pi \sqrt{\frac{2r^2(1 - \cos\theta)}{-2gr(\cos\theta - 1)}}d\theta = \int_0^\pi \sqrt{\frac{r}{g}}d\theta$$

$$= \pi\sqrt{\frac{r}{g}} = k\sqrt{\frac{\pi^2}{2g}}$$

We then find the time if the path was a straight line. We can do this without using any integral. First we find the acceleration felt by the mass

$$a = -g\cos\theta$$

We know that the slope is the same all the way, we also know that

$$y_B = -\frac{2}{\pi}x_b \Rightarrow \frac{x_B}{y_B} = -\frac{\pi}{2} = \tan\theta$$

($\tan\theta$ is defined this way because we are using that the y-axis is at $\theta = 0$). If we use this expression in the acceleration we get

$$a = -\frac{g}{\sqrt{1+\frac{\pi^2}{4}}}$$

We now find the total length of the slope

$$s^2 = x^2 + y^2 = y^2(1 + \frac{\pi^2}{4})$$

We can now use our old friend

$$s = \frac{1}{2}at^2$$

$$\Rightarrow y\sqrt{1+\frac{\pi^2}{4}} = -\frac{1}{2}\frac{gt^2}{\sqrt{1+\frac{\pi^2}{4}}}$$

$$\Rightarrow t = T = \sqrt{\frac{-2y}{g}\left(1+\frac{\pi^2}{4}\right)}$$

We can so use that $y = y_B = \frac{1}{2}k^2(\cos\pi - 1) = -k^2$

$$T = k\sqrt{\frac{\pi^2}{2g} + \frac{2}{g}}$$

If we compare this with the time for the brachistochrone, we can see that the brachistochrone is faster by a time

$$\Delta t = -k\sqrt{\frac{2}{g}}$$

As we expected.