FYS3120 Oblig 4

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1)

a)

We are going to start by finding the coordinates, we only need to find y, since we need it to find the potential energy

$$y = -\frac{b}{2}\cos\theta$$

The potential energy is then given by

$$V = mgy = -mg\frac{b}{2}\cos\theta$$

There is only rotational energy for the rod, it has a inertia of $I = \frac{1}{3}mb^2$. So its kinetic energy is given as

$$T = \frac{1}{2}I\omega^2 = \frac{1}{6}mb^2\dot{\theta}^2$$

Giving us the Lagrangian

$$L = T - V = \frac{1}{6}mb^2\dot{\theta}^2 + mg\frac{b}{2}\cos\theta$$

We can now find the e.o.m

$$\frac{\partial L}{\partial \theta} = -\frac{mgb}{2}\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{3}mb^2\dot{\theta}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{1}{3}mb^2\ddot{\theta}$$

Giving us that

$$\frac{1}{3}mb^2\ddot{\theta} + \frac{mgb}{2}\sin\theta = 0$$

or

$$\ddot{\theta} + \frac{3g}{2b}\sin\theta = 0$$

b)

We can see that for $\theta = 0$

$$\ddot{\theta} = 0$$

So this is a equilibrium. This is a minimum or maximum of the potential. We need to differentiate the potential twice to see if this is a minimum

$$\frac{d^2}{d\theta^2} - \frac{mgb}{2}\cos\theta = \frac{mgb}{2}\cos\theta$$

at $\theta = 0$

$$\Rightarrow \frac{mgb}{2}\cos(0) = \frac{mgb}{2} > 0$$

This shows that we have a minimum, and the equilibrium is thus stable.

We can now look at the e.o.m for small angles. Then $\sin\theta \approx \theta$, so

$$\ddot{\theta} = -\frac{3g}{2b}\theta$$

This is harmonic oscillator with

$$\omega^2 = \frac{3g}{2b}$$

Using that $T = \frac{2\pi}{\omega}$ we find that

$$T_0 = 2\pi\sqrt{\frac{2b}{3g}}$$

c)

Since

$$\frac{\partial L}{\partial t} = 0$$

and

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0$$

We can see that the Hamiltonian is a constant of motion. Since the constraints are holonomic and time-independent, we know that the Hamiltonian is the same as the total energy of the system. Thus the total energy is conserved.

The Hamiltonian is related to the e.o.m by Hamilton's equations

$$\dot{p} = -\frac{\partial H}{\partial \theta}, \qquad \dot{\theta} = \frac{\partial H}{\partial p}$$

d)

This derivation is heavily inspired by a similar derivation for a normal pendulum, found on wikipedia.

We are going to start by looking at the energy of the pendulum. The pendulum has no kinetic energy at θ_0 and potential $mg\cos\theta_0$. At some θ the kinetic energy has to be the difference in potential energy between these angles. So

$$\frac{1}{6}mb^2\dot{\theta}^2 = \frac{mgb}{2}(\cos\theta - \cos\theta_0)$$

giving us that

$$\dot{\theta} = \frac{d\theta}{dt} = \sqrt{\frac{3g}{b}}\sqrt{(\cos\theta - \cos\theta_0)}$$

Doing some physicist math we get that

$$dt = \sqrt{\frac{b}{3g}}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

The pendulum needs to fall from 0 to θ_0, then to $-\theta_0$, and then all the way back to θ_0, meaning that the period is 4 times the time it take to fall from 0 to θ_0. Meaning that

$$T = 4\int dt = 4\sqrt{\frac{b}{3g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

And we know that $T_0 = 2\pi\sqrt{\frac{2b}{3g}}$, giving us that

$$T = T_0\frac{\sqrt{2}}{\pi}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

We can now solve this for $\theta_0 = \pi/2$

$$T = T_0\frac{\sqrt{2}}{\pi}\int_0^{\pi/2}\frac{d\theta}{\sqrt{\cos\theta}}$$

From good old Rottmann page 161 eq. 116, we get that this is

$$T = T_0\frac{\sqrt{2}}{\pi}\frac{\pi\Gamma(\frac{1}{2})}{\sqrt{2}\Gamma(\frac{3}{4})^2} = T_0\frac{\sqrt{\pi}}{\Gamma(\frac{3}{4})^2} \approx 1.18\cdot T_0$$

2)

a)

First we write the coordinates of the ball

$$x = r\sin\theta\cos(\omega t), \qquad y = r\sin\theta\sin(\omega t), \qquad z = -r\cos\theta$$

$$\dot{x} = r\dot{\theta}\cos\theta\cos(\omega t) - r\omega\sin\theta\sin(\omega t)$$

$$\dot{y} = r\dot{\theta}\cos\theta\sin(\omega t) - r\omega\sin\theta\cos(\omega t)$$

$$\dot{z} = r\dot{\theta}\sin\theta$$

with out writing down the quite long expressions for $\dot{x}$, $\dot{y}$ and $\dot{z}$, we get that

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = r^2\dot{\theta}^2 + r^2\omega^2\sin^2\theta$$

We have the energies

$$V = mgy = -mgr\cos\theta$$
$$T = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\omega^2\sin^2\theta)$$

Giving us the Lagrangian

$$L = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\omega^2\sin^2\theta) + mgr\cos\theta$$

We can rewrite this as

$$L = \frac{1}{2}mr^2\dot{\theta}^2 + (\frac{1}{2}mr^2\omega^2\sin^2\theta + mgr\cos\theta)$$

We can therefor say that we have a potential

$$W(\theta) = -(\frac{1}{2}mr^2\omega^2\sin^2\theta + mgr\cos\theta)$$

The term

$$-\frac{1}{2}mr^2\omega^2\sin^2\theta$$

Can be said to be the potential energy due to the centripetal force.

b)

We can find the e.o.m

$$\frac{\partial L}{\partial\theta} = mr^2\omega^2\sin\theta\cos\theta - mgr\sin\theta = \frac{1}{2}mr^2\omega^2\sin 2\theta - mgr\sin\theta$$

$$\frac{\partial L}{\partial\dot\theta} = mr^2\dot\theta, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = mr^2\ddot\theta$$

Giving us that

$$mr^2\ddot\theta - \frac{1}{2}mr^2\omega^2\sin 2\theta + mgr\sin\theta = 0$$

We can look at this for small angles. For small angles $\sin\theta \approx \theta$, so

$$mr^2\ddot\theta - mr^2\omega^2\theta + mg\theta = 0$$

$$\Rightarrow \ddot\theta = -\left(\frac{g}{r} - \omega^2\right)\theta$$

This is a harmonic oscillation, giving us

$$\Omega = \sqrt{\frac{g - r\omega^2}{r}}$$

c)

We have the potential

$$W(\theta) = -(\frac{1}{2}mr^2\omega^2\sin^2\theta + mgr\cos\theta)$$

To find the equilibrium positions we have to differentiate

$$W'(\theta) = -mr^2\omega^2\sin\theta\cos\theta + mgr\sin\theta = -\frac{1}{2}mr^2\omega^2\sin 2\theta + mgr\sin\theta \tag{1}$$

We can see that this is 0 for $\theta = 0$, which indeed is an equilibrium. To find if it is a stable equilibrium, we have to check for which ω $W''(\theta = 0) > 0$

$$W''(\theta) = -mr^2\omega^2\cos 2\theta + mgr\cos\theta \tag{2}$$

$$W''(0) = -mr^2\omega^2 + mgr > 0$$

This gives us the condition that

$$\omega < \sqrt{\frac{g}{r}} = \omega_{cr}$$

We are now going to look at $\omega > \omega_{cr}$. We are going to presuppose that $\sin\theta \neq 0$, and can therefor rewrite $W'(\theta)$ 1 as

$$-mr^2\omega_{cr}\cos\theta = mgr$$

$$\Rightarrow \cos\theta = \frac{g}{r\omega^2}$$

Giving us the new equilibrium points

$$\theta_{\pm} = \arccos(\frac{g}{r\omega^2}) + n\pi, \qquad n = 0, 1$$

d)

When we now are going to check if these points are stable I am going to use a trick which makes it alot easier. I am going to say that

$$\omega = a\omega_{cr}$$

Where $a > 1$. This gives us that the equilibrium points can be written as

$$\theta_{\pm} = \arccos(\frac{1}{a^2}) + n\pi, \qquad n = 0, 1$$

We can now use this to find if $W''(\theta_{\pm}) > 0$. Using 2

$$W''(\arccos(\frac{1}{a^2})) = -mr^2\omega^2\cos 2(\arccos(\frac{1}{a^2})) + mgr\cos(\arccos(\frac{1}{a^2}))$$

We can obviously see that $\cos(2(\arccos(x))) = 2x^2 - 1$, so

$$-mr^2\omega^2\left(\frac{2}{a^2} - 1\right) + \frac{mgr}{a^2} > 0$$

$$\Rightarrow \frac{g}{a^2} > ra^2\omega_{cr}\left(\frac{2}{a^2} - 1\right)$$

$$\Rightarrow \frac{1}{a^2} > 2 - a^2$$

$$\Rightarrow 1 > 2a^2 - a^4 = a^2(2 - a^2)$$

Since $\cos(2(\arccos(x)) + n\pi) = 2x^2 - 1$ we get the same inequality for $W''(\arccos(\frac{1}{a^2}) + n\pi)$.

Looking at the inequality $1 > a^2(2 - a^2)$ for $a = 1 \Rightarrow a^2(2 - a^2) = 1$, we defined $a > 1$ and for $a > 1$ the function $a^2(2 - a^2)$ is monotonically decreasing function, thus the inequality holds, and $\theta_{\pm}$ are stable equilibriums.

e)

The Hamiltonian is defined here as

$$H = p\dot{\theta} - L \tag{3}$$

where

$$p = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

We then get a Hamiltonian

$$H = mr^2\dot{\theta}^2 - L = \frac{p}{mr^2} - L$$

$$= \frac{p}{mr^2} - \frac{p}{2mr^2} - \frac{1}{2}mr^2\omega^2 \sin^2\theta - mgr\cos\theta$$
$$= \frac{p}{2mr^2} - \frac{1}{2}mr^2\omega^2 \sin^2\theta - mgr\cos\theta$$