FYS3121 Oblig 3

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9. februar 2017

1)

a)

To find the Lagrangian, we need to find the Cartesian coordinates expressed with the general coordinate s

$$y = h - s\sin 30^\circ = h - \frac{s}{2}, \qquad x = s\cos 30^\circ = s\frac{\sqrt{3}}{2} \tag{1}$$

We then find that

$$\dot{y} = -\frac{\dot{s}}{2}, \qquad \dot{x} = \dot{s}\frac{\sqrt{3}}{2}$$

We can then write the energies

$$V = mgy = mg(h - \frac{s}{2})$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\frac{\dot{s}^2}{4} + \frac{3}{4}\dot{s}^2) = \frac{1}{2}m\dot{s}^2$$

We then find that the Lagrangian is

$$\mathcal{L} = T - V = \frac{1}{2}m\dot{s}^2 - mg(h - \frac{s}{2})$$

b)

With an accelerated incline, we have to rewrite the coordinates

$$y = h - \frac{s}{2}, \qquad x = \frac{1}{2}at^2 + s\frac{\sqrt{3}}{2}$$

$$\dot{y} = -\frac{\dot{s}}{2}, \qquad \dot{x} = at + \dot{s}\frac{\sqrt{3}}{2}$$

For the energies we see that

$$V = mgy = mg(h - \frac{s}{2})$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\frac{\dot{s}^2}{4} + a^2t^2 + \dot{s}at\sqrt{3} + \dot{s}^2\frac{3}{4})$$

$$= \frac{1}{2}m()\dot{s}^2 + at(at + \dot{s}\sqrt{3}))$$

So we get the Lagrangian

$$\mathcal{L} = T - V = \frac{1}{2}m(\dot{s}^2 + at(at + \dot{s}\sqrt{3})) - mg(h - \frac{s}{2})$$

c)

We can now find the e.o.m. for this system by solving Lagrange's equation. We first find that

$$\frac{\partial \mathcal{L}}{\partial s} = \frac{mg}{2}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{s}} = m\dot{s} + mat\frac{\sqrt{3}}{2}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{s}}\right) = m\ddot{s} + ma\frac{\sqrt{3}}{2}$$

We can then write the e.o.m

$$m\ddot{s} + ma\frac{\sqrt{3}}{2} - \frac{mg}{2} = 0$$

$$\Rightarrow \ddot{s} = \frac{1}{2}(g - \sqrt{3}a)$$

We can now integrate this twice with respect to s.

$$s(t) = s_0 + v_0 t + \frac{1}{4}(g - \sqrt{3}a)t^2$$

Knowing that at $t = 0$: $r_0 = 0$ and $v_0 = 0$ we finally get that

$$s(t) = \frac{1}{4}(g - \sqrt{3}a)t^2$$

2)

First we write the coordinates of the ball

$$x = r\sin\theta\cos(\omega t), \qquad y = r\sin\theta\sin(\omega t), \qquad z = r\cos\theta$$

$$\dot{x} = r\dot{\theta}\cos\theta\cos(\omega t) - r\omega\sin\theta\sin(\omega t)$$

$$\dot{y} = r\dot{\theta}\cos\theta\sin(\omega t) - r\omega\sin\theta\cos(\omega t)$$

$$\dot{z} = -r\dot{\theta}\sin\theta$$

with out writing down the quite long expressions for $\dot{x}$, $\dot{y}$ and $\dot{z}$, we get that

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = r^2\dot{\theta}^2 + r^2\omega^2\sin^2\theta$$

Since there are no gravitation, we get that $V = 0$, and therefor

$$\mathcal{L} = T = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\omega^2\sin^2\theta)$$

We can now find Lagrange's equation

$$\frac{\partial \mathcal{L}}{\partial \theta} = mr^2\omega^2\sin\theta\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta}$$

An we find the e.o.m

$$mr^2\ddot{\theta} - mr^2\omega^2 \sin\theta\cos\theta = 0$$

b)

A particle in a periodic potential $V(\theta)$ would have a Lagrangian

$$\mathcal{L} = \frac{1}{2}mr^2\dot{\theta}^2 - V(\theta)$$

We then look at out Lagrangian

$$\mathcal{L} = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\omega^2\sin^2\theta)$$

we then get

$$\mathcal{L} = \frac{1}{2}mr^2(\dot{\theta}^2 + \omega^2\sin^2\theta) = \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\omega^2\sin^2\theta$$

We can see that we now have something that looks like a potential. We can say that the potential of this Lagrangian is

$$V(\theta) = -\frac{1}{2}mr^2\omega^2\sin^2\theta$$

This is a periodic potential with 2 stable equilibriums (at 0 and π) and 2 unstable equilibriums(at $\pi/2$ and $3\pi/2$). So we can say that the Lagrangian of this system is similar to that of a particle in a periodic potential.

c)

We can now start with out Lagrangian

$$\mathcal{L} = \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\omega^2\sin^2\theta$$

and find the e.o.m

$$\frac{\partial \mathcal{L}}{\partial \theta} = mr^2\omega^2\sin\theta\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta}$$

So the e.o.m is

$$mr^2\ddot{\theta} - mr^2\omega^2 \sin\theta\cos\theta = 0$$

We are now going to define $\phi = \theta - \theta_0$, where $\theta_0 = \frac{\pi}{2} + n\pi$. This gives us that $\theta = \theta_0 + \phi$, which we can use in the e.o.m

$$mr^2\ddot{\phi} - mr^2\omega^2\sin(\phi + \theta_0)\cos(\phi + \theta_0) = 0$$

We can now rewrite

$$sin(\phi+\theta_0)\cos(\phi+\theta_0) = (\cos(\phi)\sin(\theta_0)+\cos(\theta_0)\sin(\phi))(\cos(\theta_0)\cos(\phi)-\sin(\phi)\sin(\theta_0))$$

An using that $\theta = \theta_0 + \phi$

$$sin(\phi+\theta_0)\cos(\phi+\theta_0) = -\sin(\phi)\cos(\phi) = -\frac{1}{2}\sin(2\phi)$$

So

$$mr^2\ddot{\phi} + \frac{1}{2}mr^2\omega^2\sin(2\phi) = 0$$

From small angles, we know that $\sin\phi \approx \phi$. Using this we get

$$mr^2\ddot{\phi} + mr^2\omega^2\phi = 0$$

Which gives us

$$\ddot{\phi} = \omega^2\phi$$

This is a harmonic oscillator, with a angular frequency given simply (and tautologically) as

$$\omega = \omega$$

3)

a)

Here we start by finding the energies. If we define the plate as being where $V = 0$, we can see that only the lower mass has a potential energy. If the rope has length L, we see that

$$V = mg(-(L-r)) = mg(r-L)$$

The lower mass have a kinetic energy gives as

$$\frac{1}{2}m\dot{r}^2$$

While the upper mass has kinetic energy from being dragged towards the hole, and from the rotation around the hole. Its kinetic energy is then

$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$$

Giving us a total kinetic energy of

$$T = 2\cdot\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$$

And a Lagrangian

$$\mathcal{L} = \frac{1}{2}m(2\dot{r}^2 + r^2\dot{\theta}^2) - mg(r-L)$$

We can now find the e.o.m

$$\frac{\partial\mathcal{L}}{\partial\theta} = 0 \tag{2}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mr^2\dot{\theta} \tag{3}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

So the first e.o.m is:

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = 0$$

For the second

$$\frac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 - mg$$
$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = 2m\dot{r}$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = 2m\ddot{r}$$

So the second e.o.m becomes

$$2m\ddot{r} - mr\dot{\theta}^2 + mg = 0$$

b)

As we saw above from 2

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0$$

This means that θ is a cyclic coordinate, and therefor

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{constant} = \ell$$

We can there for rewrite $\dot{\theta}$ as

$$\dot{\theta} = \frac{\ell}{mr^2}$$

If we set this into the second e.o.m we get a description of the system, only dependent on r

$$2m\ddot{r} - \frac{\ell^2}{mr^3} + mg = 0$$

4)

a)

For convenience I am going to label the upper mass m_1 and the pendulum bob m_2. The coordinates then becomes

$$x_1 = s, \qquad y_1 = 0$$
$$\dot{x}_1 = \dot{s}, \qquad \dot{y}_1 = 0$$

and

$$x_2 = s + d\sin\theta, \qquad y_2 = -d\cos\theta$$

$$\dot{x}_2 = \dot{s} + d\cos\theta, \qquad \dot{y}_2 = d\sin\theta$$

And the energies

$$
\begin{aligned}
V &= mg(y_1 + y_2) = -mgd\cos\theta \\
T &= \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) \\
&= \frac{1}{2}m(2\dot{s}^2 + d^2\dot{\theta}^2 + 2\dot{\theta}\dot{s}d\cos\theta)
\end{aligned}
$$

This gives us

$$\mathcal{L} = T - V = \frac{1}{2}m(2\dot{s}^2 + d^2\dot{\theta}^2 + 2\dot{\theta}\dot{s}d\cos\theta) + mgd\cos\theta$$

b)

If we differentiate $\mathcal{L}$ with respect to s

$$\frac{\partial\mathcal{L}}{\partial s} = 0$$

So s is a cyclic coordinate, meaning that the Lagrangian does not depend on s. It can also be shown that since s is cyclic, there is a conserved quantity in this system, namely

$$\frac{\partial\mathcal{L}}{\partial\dot{s}} = 2m\dot{s} + md\dot{\theta}\cos\theta$$

Since this quantity is conserved, it is constant

$$2m\dot{s} + md\dot{\theta}\cos\theta = \ell$$

We can therefor rewrite this as

$$\dot{s} = \frac{1}{2}(\frac{\ell}{m} - d\dot{\theta}\cos\theta)$$

If we insert this into the Lagrangian, we get

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}m(\frac{1}{2}(\frac{\ell}{m} - d\dot{\theta}\cos\theta)^2 + d^2\dot{\theta}^2 + \dot{\theta}(\frac{\ell}{m} - d\dot{\theta}\cos\theta)d\cos\theta) + mgd\cos\theta \\
&= \frac{1}{2}m(\frac{\ell^2}{2m^2} + \frac{d^2\dot{\theta}^2}{2}\cos^2\theta + d^2\dot{\theta}) + mgd\sin\theta
\end{aligned}
$$

c)

We can now find the e.o.m

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\theta} &= -mgd\sin\theta + \frac{1}{2}md^2\dot{\theta}^2\cos\theta\sin\theta \\
\frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= -\frac{1}{2}md^2\dot{\theta}\cos^2\theta + md^2\dot{\theta} = md^2(\dot{\theta} - \frac{1}{2}\dot{\theta}\cos^2\theta) \\
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) &= md^2(\ddot{\theta} - \frac{1}{2}\ddot{\theta}\cos^2\theta + \dot{\theta}^2\sin\theta\cos\theta)
\end{aligned}
$$

This gives us that

$$md^2(\ddot{\theta} - \frac{1}{2}\ddot{\theta}\cos^2\theta + \dot{\theta}^2\sin\theta\cos\theta) + mgd\sin\theta - \frac{1}{2}md^2\dot{\theta}^2\cos\theta\sin\theta = 0$$

$$\Rightarrow \ddot{\theta}(1 - \frac{1}{2}\cos^2\theta) + \frac{\dot{\theta}^2}{2}\sin\theta\cos\theta + \frac{g}{d}\sin\theta = 0$$

d)

We can find a small-angle solution to this problem. If we Taylor expand the above expression around $\theta = 0$. We get that the first order Taylor series is

$$T_2(f(\theta)) = \frac{1}{2}\ddot{\theta} + \frac{g}{d}\theta + \frac{1}{2}\theta\dot{\theta}^2 = 0$$

We are only interested in the first order, and for small θ see get that $\dot{\theta}^2 \to 0$, so this expression becomes

$$\frac{1}{2}\ddot{\theta} + \frac{g}{d}\theta = 0$$

This is a harmonic oscillator equation

$$\ddot{\theta} = -\frac{2g}{d}\theta$$

The angular frequency of this system will be

$$\omega = \sqrt{\frac{2g}{d}}$$