FYS3140 Oblig 4

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1)

a)

$$\frac{\sin z}{3z}, \qquad z = 0$$

We are going to start by writing a Taylor expansion of $\sin z$

$$\sin z = z - \frac{z^3}{3!} + \cdots \tag{1}$$

Which gives

$$\frac{\sin z}{3z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \cdots\right) = 1 - \frac{z^2}{3!} + \cdots$$

Since we don't have any b_n-term –and therefor no singularity in the series–, $z = 0$ is a removable singularity.

b)

$$\frac{\cos z}{z^4}, \qquad z = 0$$

We can now take Taylor of $\cos z$

$$\cos z = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Which gives

$$\frac{\cos z}{z^4} = \frac{1}{z^4}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) = \frac{1}{z^4} - \frac{1}{2z^2} + \frac{1}{4!} - \frac{z^2}{6!} + \cdots$$

The highest negative power in this series is 2, this means that this function has a pole of order 2.

c)

$$\frac{z^3 - 1}{(z-1)^3}, \qquad z = 1$$

We can use partial fraction decomposition to get that

$$\frac{z^3 - 1}{(z-1)^3} = \frac{3}{z-1} + \frac{3}{(z-1)^2} + 1$$

This is the Laurent series. Since the highest negative power is 2, this has a pole of order 2

d)

$$\frac{e^z}{z-1}, \qquad z=1$$

We can do a Taylor expansion of e^z around $z=1$

$$\frac{e^z}{z-1} = e + e(z-1) + \frac{e(z-1)^2}{2!} + \cdots$$

Which gives

$$\frac{e^z}{z-1} = \frac{1}{z-1}\left(e + e(z-1) + \frac{e(z-1)^2}{2!} + \cdots\right) = \frac{e}{z-1} + e + \frac{(z-1)e}{2!} + \cdots$$

The greatest negative power here is 1, so this has a pole of order 1(simple point).

e)

$$\frac{e^z-1-z}{z^3}, \qquad z=0$$

We can use partial fraction decomposition to find

$$\frac{e^z-1-z}{z^3} = \frac{e^z}{z^3} - \frac{1}{z^3} - \frac{1}{z^2}$$

We can use Taylor on e^z which gives us

$$\begin{aligned}\frac{e^z-1-z}{z^3} &= -\frac{1}{z^3} - \frac{1}{z^2} + \frac{1}{z^3}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right)\\ &= -\frac{1}{z^3} - \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2z} + \frac{1}{3!} + \frac{z}{4!} + \cdots\\ &= \frac{1}{2z} + \frac{1}{3!} + \frac{z}{4!} + \cdots\end{aligned}$$

Which shows us that the function has a pole of order 1 (simple point).

2)

14.6.1)

$$\frac{1}{z(z+1)}, \qquad z = 0$$

$z = 0$ for $\frac{1}{z+1}$ is not a singularity, so we are going to look at the Taylor expression for this expression

$$\frac{1}{z+1} = 1 - z + z^2 - z^3 + \cdots$$

So

$$\frac{1}{z(z+1)} = \frac{1}{z}\left(1 - z + z^2 - z^3 + \cdots\right) = \frac{1}{z} - 1 + z - z^2 + \cdots$$

The residue is the coefficient of the term z^{-1}. So the residue here is $\underline{\underline{1}}$

14.6.3)

$$\frac{\sin z}{z^4}, \qquad z = 0$$

We have the Taylor expansion of $\sin z$ from 1. Thus

$$\frac{\sin z}{z^4} = \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right) = \frac{1}{z^3} - \frac{1}{6z} + \frac{z}{5!} + \cdots$$

The residue is the coefficient of the term z^{-1}. So the residue here is $\underline{\underline{-\frac{1}{6}}}$

14.6.9)

$$\frac{1}{z^2 - 5z + 6}, \qquad z = 2$$

We are going to start by doing a partial fraction decomposition

$$\frac{1}{z^2 - 5z + 6} = -\frac{1}{z-2} + \frac{1}{z-3}$$

The first term is as it should in the Laurent series. We then have to expand $\frac{1}{z-3}$ around $z = 2$. We are going to use that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

so

$$\frac{1}{z-3} = -\frac{1}{1-(z-2)} = -\sum_{n=0}^{\infty}(z-2)^n$$

So the full function

$$\frac{1}{z^2 - 5z + 6} = -\frac{1}{z-2} - \sum_{n=0}^{\infty}(z-2)^n$$

The residue is the coefficient of the term $(z-2)^{-1}$, so the residue is $\underline{\underline{-1}}$