FYS3140 Oblig 5

Daniel Heinesen, daniehei

24. februar 2017

6.15)

$$\frac{1}{(1-2z)(5z-4)}, \qquad z = \frac{1}{2}, \qquad z = \frac{4}{5}$$

$\underline{z = \frac{1}{2}}$:

Since we need the the nominator on the $(z - z_0)$ we can rewrite the function as

$$\frac{1}{(1-2z)(5z-4)} = \frac{-\frac{1}{2}}{(z-1/2)(5z-4)}$$

We can then find the residue

$$Res(\frac{1}{2}) = \lim_{z\to\frac{1}{2}}(z-\frac{1}{2})\frac{-\frac{1}{2}}{(z-1/2)(5z-4)} = \frac{-\frac{1}{2}}{\frac{5}{2}-4}$$
$$= \frac{-\frac{1}{2}}{-\frac{3}{2}} = \frac{1}{3}$$

$\underline{z = \frac{4}{5}}$:

We have to rewrite the function in the same fashion as above

$$\frac{1}{(1-2z)(5z-4)} = \frac{\frac{1}{5}}{(1-2z)(z-\frac{4}{5})}$$

Giving us that

$$Res(\frac{4}{5}) = \lim_{z\to\frac{4}{5}}(z-\frac{4}{5})\frac{\frac{1}{5}}{(1-2z)(z-\frac{4}{5})}$$
$$\frac{\frac{1}{5}}{1-\frac{8}{5}} = \frac{\frac{1}{5}}{\frac{5-8}{5}} = -\frac{1}{3}$$

6.15')

We now have to calculate

$$I = \oint_C \frac{1}{(1-2z)(5z-4)}dz$$

Where C is a circle at origin with radius $\frac{3}{2}$. Both of the singularities are inside this circle. From the residue theorem we get that

$$I = 2\pi i(Res(\frac{1}{2}) + Res(\frac{4}{5}) = 2\pi i(\frac{1}{3} - \frac{1}{3}) = 0$$

6.19)

$$\frac{\sin^2 z}{2z-\pi}, \qquad z=\frac{\pi}{2}$$

We have to rewrite the function

$$\frac{\sin^2 z}{2z-\pi}=\frac{1}{2}\frac{\sin^2 z}{z-\frac{\pi}{2}}$$

We get the residue from

$$Res(\frac{\pi}{2})=\lim_{z\to\frac{\pi}{2}}(z-\frac{\pi}{2})\frac{1}{2}\frac{\sin^2 z}{z-\frac{\pi}{2}}=\frac{\sin^2\frac{\pi}{2}}{2}=\frac{1}{2}$$

6.28)

$$\frac{z+2}{(z^2+9)(z^2+1)}, \qquad z=3i$$

We can see that (z^2+9) becomes zero for this point. So we can rewrite the function as

$$\frac{z+2}{(z+3i)(z-3i)(z^2+1)}$$

We can then find the residue

$$Res(3i)=\lim_{z\to 3i}(z-3i)\frac{z+2}{(z+3i)(z-3i)(z^2+1)}=\frac{3i+2}{6i(-9+1)}=\frac{3i+2}{-48i}=-\frac{1}{14}-\frac{1}{24}i$$

6.28')

$$I=\oint_C\frac{z+2}{(z^2+9)(z^2+1)}dz$$

Where C is a circle at origin with radius $\frac{3}{2}$. The only singularities if this function inside of the circle is $\pm i$. We are going to start by rewriting the integral as

$$I=\oint_C\frac{z+2}{(z^2+9)(z+i)(z-i)}dz$$

We now have to find the residues of $z=\pm i$.

$$Res(i)=\lim_{z\to i}(z-i)\frac{z+2}{(z^2+9)(z+i)(z-i)}=\frac{2+i}{(-1+9)(2i)}=\frac{1}{16}+\frac{1}{8}i$$

$$Res(-i)=\lim_{z\to -i}(z+i)\frac{z+2}{(z^2+9)(z+i)(z-i)}=\frac{2-i}{(-1+9)(-2i)}=\frac{1}{16}-\frac{1}{8}i$$

We can then solve the integral

$$I=2\pi i(Res(i)+Res(-i))=2\pi i\left(\frac{1}{16}+\frac{1}{8}i+\frac{1}{16}-\frac{1}{8}i\right)=\frac{\pi i}{4}$$

7.6)

$$I = \int_0^{\pi} \frac{d\theta}{(2+\cos\theta)^2}$$

Since this is an even function, we can write this as

$$I = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$$

We are going to integrate this around the circle $|z| = 1$. We are going to use the coordinate change $\theta \to z = e^{i\theta}$. To do this we have to use that

$$\frac{dz}{d\theta} = ie^{i\theta} = iz \Rightarrow d\theta = \frac{dz}{iz}$$

and

$$\cos\theta = \frac{1}{2}(z + \frac{1}{z})$$

So

$$I = \frac{1}{2}\oint_{|z|=1} \frac{1}{(2+\frac{1}{2})(z+\frac{1}{z}))^2}\frac{dz}{zi} = \oint_{|z|=1} \frac{2z}{i}\frac{dz}{(4z+z^2+1)^2}$$

The roots of $4z + z^2 + 1$ is $2 \pm \sqrt{3}i$. Only $z = 2 - \sqrt{3}i$ is inside the circle, so we need to find the residue here

$$I = \oint \frac{1}{i}\frac{2z}{(z-(2-\sqrt{3}i))^2(z-(2+\sqrt{3}i))^2}dz$$

$$\Rightarrow Res(2-\sqrt{3}i) = \lim_{z\to 2-\sqrt{3}i} \frac{d}{dz}(z-(2-\sqrt{3}i))^2 \frac{2z}{(z-(2-\sqrt{3}i))^2(z-(2+\sqrt{3}i))^2}$$

$$= \lim_{z\to 2-\sqrt{3}i} \frac{1}{i}\frac{2(z+\sqrt{3}+2}{(2+\sqrt{3}-z)^3}$$

$$= \frac{1}{i}\frac{8}{(2\sqrt{3})^3} = \frac{1}{3i\sqrt{3}}$$

We can now solve the integral

$$I = 2\pi i Res(2-\sqrt{3}i) = \frac{2\pi}{3\sqrt{3}}$$

7.8)

$$I = \int_0^{\pi} \frac{\sin^2\theta}{13-12\cos\theta}d\theta$$

We are going to use the same tricks as above, first we can rewrite

$$I = \frac{1}{2}\int_0^{2\pi} \frac{\sin^2\theta}{13-12\cos\theta}d\theta$$

We are going to use the same coordinate change. We also have to use that

$$\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$$

We now have

$$I = \frac{1}{2}\oint_{|z|=1} \frac{-1}{4}\frac{(z-\frac{1}{z})^2}{13-6(z+\frac{1}{z})}\frac{dz}{iz} = \frac{-1}{8i}\oint \frac{(z^2-1)^2}{z^2(13z-6z^2-6)}dz$$

The roots of $z^2(13z-6z^2-6)$ are 0,2/3 and 3/2. Only the two former are inside the circle and are necessary to calculate for solving the integral. We can now rewrite the integral as

$$I = \frac{1}{48i}\oint \frac{(z^2-1)^2}{z^2(z-2/3)(z-3/2)}dz$$

And we can find the residues

$$Res(0) = \lim_{z\to 0}\frac{d}{dz}(z-0)^2\frac{(z^2-1)^2}{z^2(z-2/3)(z-3/2)}$$
$$= \lim_{z\to 0}\frac{6(z^2-1)(12z^3-39z^2+36z-13)}{(6z^2-13z+6)^2}$$
$$= \frac{13}{6}$$

and

$$Res(2/3) = \lim_{z\to 2/3}(z-2/3)\frac{(z^2-1)^2}{z^2(z-2/3)(z-3/2)}$$
$$\frac{1}{(2/3)^2(2/3-3/2)} = -\frac{5}{6}$$

We can now solve the integral

$$I = 2\pi i(Res(0)+Res(2/3) = 2\pi i(\frac{13}{6}-\frac{5}{6}) = \frac{\pi}{18}$$

7.10)

$$\int_{-\infty}^{\infty}\frac{dx}{x^2+4x+5}$$

The roots of x^+4x+5 are $-2\pm i$. So we can introduce that $x\to z$ and rewrite the integral as

$$I = \int_{-\infty}^{\infty}\frac{dz}{(z-(-2-i))(z-(-2+i))}$$

We are now going to use the technique to solve these kinds of integrals. We are going to integrate the function between $-\rho$ and ρ, then a half circle in the upper part of the complex plan back to $-\rho$. We are then going to let $|\rho|\to\infty$

The only singularity that is inside this half circle is $-2+i$. So we have to find

$$Res(-2+i) = \lim_{z\to -2+i}(z-(-2+i))\frac{1}{(z-(-2-i))(z-(-2+i))} = \frac{1}{2i}$$

So the integral is

$$I = 2\pi i(Res(-2+i)) = 2\pi i\left(\frac{1}{2i}\right) = \pi$$

7.12)

$$I = \int_0^\infty \frac{x^2}{x^4+16}dx$$

This is an even function, so we can rewrite this as

$$I = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{x^4+16}dx$$

The roots of x^4+16 is $z = \pm\sqrt{2}\pm\sqrt{2}i$. We are going to use the same technique as above. First we rewrite the integral

$$I = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{(z-(\sqrt{2}+\sqrt{2}i))(z-(\sqrt{2}-\sqrt{2}i))(z-(-\sqrt{2}+\sqrt{2}i))(z-(-\sqrt{2}-\sqrt{2}i))}dz$$

The only singularities that are inside the half circle are $z = \pm\sqrt{2}+\sqrt{2}i$. So we need to find the residues at these points. I am not going to write the full definition of the residues since I have done that in every exercise above, and more due to the fact that I won't have space to write it in a single line...

$$Res(\sqrt{2}+\sqrt{2}i) = \frac{(\sqrt{2}+\sqrt{2}i)^2}{(2\sqrt{2}+2\sqrt{2}i)(2\sqrt{2}i)(2\sqrt{2})} = \frac{1}{8\sqrt{2}} + \frac{1}{8\sqrt{2}i}$$

and

$$Res(-\sqrt{2}+\sqrt{2}i) = \frac{(-\sqrt{2}+\sqrt{2}i)^2}{(-2\sqrt{2}+2\sqrt{2}i)(2\sqrt{2}i)(2\sqrt{2})} = -\frac{1}{8\sqrt{2}} + \frac{1}{8\sqrt{2}i}$$

We can now solve the integral

$$I = \frac{1}{2}2\pi i(Res(\sqrt{2}+\sqrt{2}i) + Res(-\sqrt{2}+\sqrt{2}i)) = \pi i\left(\frac{1}{4i\sqrt{2}}\right) = \frac{\pi}{2\sqrt{2}}$$