1 Kompleks analyse:

1.1 Komplekse tall:

$$z = x + iy, \qquad \bar{z} = x - iy, \qquad z = r(\cos\theta + i\sin\theta) = re^{i\theta} \tag{1}$$

$r = |z|$ er modulus, θ er argumentet.

$$z + \bar{z} = 2Re(z) \qquad z - \bar{z} = 2Im(z), \qquad z\bar{z} = |z|^2 = r^2 = x^2 + y^2 \tag{2}$$

$$x = \frac{1}{2}(z + \bar{z}), \qquad y = \frac{1}{2i}(z - \bar{z}) \tag{3}$$

$z - z_0| < R$ er alle z innenfor en radius R

1.2 Komplekse røtter:

$$z^{1/n} = \sqrt[n]{r}e^{i\theta/n} = \sqrt[n]{r}(\cos\theta/n + i\sin\theta/n) = \omega_0 \tag{4}$$

Dette gir bare 'the principal root', resten er gitt ved

$$\omega_k = \sqrt[n]{r}e^{i\frac{\theta+2\pi k}{n}} \tag{5}$$

1.3 Analytic functions:

Def: A function is analytic in a region of the complex plane if it has a (unique) derivative at every point in that region.

All analytic functions can be written in terms of $z = x + iy$ alone.

1.3.1 Cauchy-Riemann equation:

$$f(z) = u(x, y) + iv(x, y) \tag{6}$$

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} \tag{7}$$

If this holds in a region, that f is analytic in this region, and vice versa.

- Regular points: $f(z)$ is analytic
- Singular point/singularities: A point where $f(z)$ is not analytic.
- Isolated singularity: a point where f is not analytic, but is a limit of points where f is analytic.

If f is analytic in some region, it has first order derivatives, then it also has derivatives of all orders in that region.

1.3.2 Harmonic Functions:

If $f = u + iv$ is analytic in a region, then u and v are harmonic:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{8}$$

If u is harmonic, one can find a v such that $f = u + iv$ is analytic. v is the harmonic conjugate of u.

1.4 Contour integrals:

$$\int_\Gamma f(z)dz = \lim_{z\to\infty}\sum_{k=1}^{\infty} f(c_k)\Delta z_k \tag{9}$$

For a generalized curve with parametrization $z(t)$:

$$\int_\gamma f(t)dz = \int_a^b f(z(t))z'(t)dt \tag{10}$$

1.4.1 An important integral:

$C-r = |z-z_0| = r$:

$$I = \int_{C_r} (z-z_0)^n dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases} \tag{11}$$

1.4.2 Upper bound estimate:

Generalized triangle inequality:

$$\left|\sum_k z_k\right| \leq \sum_k |z_k| \tag{12}$$

Applied to Riemann sum we get the upper bound estimate:

$$\left|\int_\gamma f(z)dz\right| \leq ML \tag{13}$$

Where $M = \max|f(z)|$ and L is the length of the curve.

1.4.3 Path:

If f is continuous everwhere in D, then contour integrals are independent of paths, and any loop integral is zero. One can also deform a contour without crossing any singularities and get that:

$$\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz \tag{14}$$

1.4.4 Cauchy Theorem:

If f is analytic in a simply connected domain D with no singularities, and Γ is any closed contour in D, then

$$\int_\Gamma f(z)dz = 0 \tag{15}$$

1.4.5 Cauchy's integral formula:

Let Γ be a simple, closed, positively oriented contour. Assume f is analytic in some simply connected domain D containing Γ, and some z_0 is inside Γ. Then:

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz \tag{16}$$

1.4.6 Generalized Cauchy integral formula:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_\Gamma \frac{f(w)}{(z-w)^{n+1}}dw \tag{17}$$

1.4.7 Cauchy inequality:

Let f be analytic on and inside a circle(C_r) of radius R, centered at z_0. If $|f(z)| \leq M$ for some z on C_r, then the derivatives satisfy:

$$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n} \tag{18}$$

This gives Liuvilles theorem: A function which is analytic and bounded in the entire complex plane, is constant.

1.5 Taylor and Laurent Series:

1.5.1 Taylor:

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2!} f''(z_0)(z - z_0)^2 + ... = \sum_n \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{19}$$

If f is analytic in the disk $|z - z_0| < R$ then the above Taylor series converges in that disk.[i.e. the disk touching the nearest singularity]

If f is analytic at z_0, then the Taylor series for df/dz can be obtained by termwise differentiation.

1.5.2 Laurent:

Let f be analytic in the annulus $r < |z - z_0| < R$. Then f can be expanded there as the sum of two series:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k + \sum_{k=1}^{\infty} b_k (z - z_0)^{-k} \tag{20}$$

With

$$a_n = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} dz \qquad b_n = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{-n+1}} dz \tag{21}$$

Def: The coefficient b_1 of the $1/(z - z_0)$ term is the residue of $f(z)$ at $z = z_0$.

Laurent series are unique. So to find them we can use

$$\frac{1}{1 - \omega} = \sum_{n=0}^{\infty} \omega^2 \text{ , when } |\omega| < 1 \tag{22}$$

1.6 Zeros:

- A zeros of a function is a point where f is analytic and $f(z_0) = 0$
- A zeros of order m: $f(z_0) = f'(z_0) = ... = f^{m-1}(z_0) = 0$, $f^m(z_0) \neq 0$
- Can be factorized as: $f(z) = (z - z_0)^m \cdot g(z)$, where $g(z)$ is analytic and $g(z_0) \neq 0$

1.7 Isolated singularities:

Let f have a Laurent series, then we can have:

1.7.1 Removable Singularity/Regular point:

If all $b_n = 0$ at z_0. $f(z)$ has a limit $z \to z_0$ and we can be redefined such that f is analytic at z_0.

1.7.2 Essential Singularity:

Infinitely many b-terms at z_0

1.7.3 Pole of order m

Order m is the highest exponent of the $1/(z-z_0)$ terms.

$$f(z) = \frac{b_m}{(z-z_0)^m} + ... + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + ... \tag{23}$$

$f(z)$ can be written as $\frac{g(z)}{(z-z_0)^m}$. A pole of order 1 ($m=1$) is a Simple pole.

1.8 Residue Theory:

1.8.1 Residue Theorem:

If Γ is a simple, closed, positively oriented contour, and f is analytic on and inside Γ except at the points $z_0, z_1, ..., z_n$ inside Γ, then

$$\oint_\Gamma f(z)dz = 2\pi i \sum_{k=0}^{n} Res(f, z_k) \tag{24}$$

1.8.2 Determining the residues:

1: Read off b_1 from the Laurent series.

2: Simple poles:

$$Res(z_0) = b_1 = \lim_{z\to z_0} (z-z_0)f(z) \tag{25}$$

Finite answer only if the pole is of first order.

$$f(z) = \frac{P(z)}{Q(z)} \Rightarrow Res(z_0) = \frac{P(z_0)}{Q'(z_0)} \tag{26}$$

3: Multiple poles: If f has a pole of order m at z_0, then

$$Res(z_0) = \lim_{z\Rightarrow z_0} \left[\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} ((z-z_0)^m f(z)) \right] \tag{27}$$

Ok to overshoot with value if m.

1.9 Applications to Real Integrals:

1.9.1 Type 1:

Rational and finite functions of $\sin\theta$ and $\cos\theta$ over the interval$[0, 2\pi]$. Use:

- $z = e^{i\theta}$, $\qquad d\theta = dz/iz$
- $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + 1/z)$, $\qquad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}(z - 1/z)$
- then use residue theorem.

1.9.2 Type 2a

Integrals of rational functions from $-\infty$ to ∞

$$I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx \tag{28}$$

- Make a contour γ_ρ from $-\rho$ to ρ
- Add a second contour from ρ via a the complex plane(half circle in the upper part of the complex plane) back to $-\rho$, C_ρ.
- use the residue theorem. Remember that the singularities have to be in the upper part
- Show that the contribution form C_ρ vanishes as $\rho \to \infty$

1.9.3 Type 2b:

$$I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(mx)dx, \qquad I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(mx)dx \tag{29}$$

Alt 1(Always safe):
use $\cos(mx) = \frac{1}{2}(e^{imx} + e^{-imx})$

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{imx} dx + \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{-imx} dx \tag{30}$$

FOr the first term, use a closed contour in the upper part of the complex plane, for the second term use one in the lower half.

Alt 2: safe as long as P/Q is real. Note that $\cos(mx) = Re(e^{mx})$(and $\sin(mx) = Im(e^{mx})$). We can therfor use $\cos(mx) \to e^{imx}$ and then take the real part at the end(or the imaginary if we have $\sin(mx)$

1.9.4 Jordan's lemma:

If $m > 0$(real) and P/Q is the quotient of two polynomial such that $deg(Q) \geq deg(P) + 1$, then

$$\lim_{\rho \to \infty} \int_{C_\rho^+} \frac{P}{Q} e^{imz} dz = 0 \tag{31}$$

Where C_ρ^+ is the contour in the upper plane. Same holds for $m < 0$ but with C_ρ^- in the lower plane.

1.9.5 Type 3:

Singularities on the real plane. We get Principal Values.

$$PV \int_a^b f(x)dx = \lim_{r \to 0} \int_a^{c-r} f(x)dx + \int_{c+r}^b f(x)dx \tag{32}$$

Where c is a singularity. If the singularities are simple poles the integral can be calculated with the residue theorem.

$$PV \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_k Res(z_k) + \pi i \sum_j Res(z_j) \tag{33}$$

Where z_k are singularities on the upper half plane, and z_j are singularities on the real line.

2 Tensor:

2.1 Cartesian Tensor

Transform properly under rotation of Cartesian coordinate system.

$$e_i' \cdot e_j = \cos\theta_{i'j} \equiv A_{ij} \tag{34}$$

2.1.1 Transformation of a position vector

$$\vec{r} = x_i e_i = x_j' e_j' \tag{35}$$

$$x_i' = \vec{r} \cdot e_i' = e_i' \cdot e_j x_j = A_{ij} x_j \tag{36}$$

Reverse:

$$x_i = e_i \cdot \vec{r} = e_i \cdot e_j' x_j' = A_{ji}' x_j = A_{ij}^T x_j' \tag{37}$$

$$A^{-1} = A^T \tag{38}$$

2.1.2 Cartesian vector/tensor

$$v' = Av, \qquad T'_{kl} = A_{ki}A_{lj}T_{ij} \tag{39}$$

$$\vec{j} = \sigma\vec{E} \text{ Ohm's law} \tag{40}$$

2.1.3 Inertia Tensor

Rigid body rotation around fixed axis:

$$\vec{L} = I\vec{\omega} \tag{41}$$

Rotation around a point. I is a rank 2 tensor

$$L_i = I_{ij}\omega_j \tag{42}$$

Determined from

$$\vec{L} = \sum_k m_k r_k \times (\omega \times r_k) \tag{43}$$

Uniform mass density: sum goes to integral.I is symmetric: can find coord. system in which I is diagonal. Eigenvetors (axes of new coord. system): Principal axes of inertia.

2.2 Levi-Civita and Kronicker Delta

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \end{cases} \tag{44}$$

$$\epsilon_{ijk} = \begin{cases} 1 & \text{even permutation} \\ -1 & \text{odd permutation} \\ 0 & \end{cases} \tag{45}$$

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{46}$$

Inner product: $\vec{u} \cdot \vec{c} = u_i v_j \delta_{ij} = u_i u_i$
Cross Product: $\vec{A} \times \vec{B} = \epsilon_{ijk} A_j B_k$
Curl: $(\nabla \times \vec{V})_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} V_k$

3 Calculus of Variation:

Minimize:

$$I = \int_{x_1}^{x_2} F(x, y, y')dx \tag{47}$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{48}$$

Can simplify Euler-Lagrange by change of variables:

$$x' = \frac{1}{y'}, \qquad dx = x'dy \text{ or } dy = y'dx \tag{49}$$

3.1 Optics – Fermat's Principle:

$$P = \int nds, \qquad ds = \sqrt{dx^2 + dy^2} \tag{50}$$

4 ODE:

4.1 Linear First order DE:

$$y' + P(x)y = Q(x) \qquad dy + [Py - Q]dx = 0 \tag{51}$$

4.1.1 Exact DE's

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy, \qquad M(x,y)dx + N(x,y)dy = 0 \tag{52}$$

The last DE is called exact if the LHS is a total differential,

$$M = \frac{\partial u}{\partial x}; \qquad N = \frac{\partial u}{\partial y} \tag{53}$$

This gives:

$$du = 0 \Leftrightarrow u = \text{ const} \tag{54}$$

4.1.2 Integrating factor:

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)dx + C\right], \qquad \mu(x) = e^{\int P(x)dx} \tag{55}$$

If $Q(x) = 0$, the homogeneous case:

$$y(x) = \frac{C}{\mu(x)} = Ce^{-\int P(x)dx} \tag{56}$$

4.2 Ordinary 2nd order DE:

$$y'' + P(x)y' + Q(x)y = R(x), \qquad y(x) = y_h + y_p = c_1y_1 + c_2y_2 + y_p \tag{57}$$

If y_1 and y_2 are linearly independent, it must hold that the Wronskian determinant

$$W(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0 \qquad \forall x_0 \tag{58}$$

4.3 Homogeneous equations ($R(x) = 0$)

4.3.1 Variations of Constants:

if y_1 is a solution, an other linearly independent solution can be found as

$$y_2(x) = c(x)y_1(x) \tag{59}$$

Where $c(x)$ can be found from the DE.

4.3.2 Homogeneous DE's with constant coefficients

$$y'' + ay' + by = 0, \qquad \lambda^2 + a\lambda + b = 0 \qquad \lambda_\pm = \frac{1}{2}[-a \pm \sqrt{a^2 - 4b}] \tag{60}$$

Case 1: $\lambda_- \neq \lambda_+$, both real ($a^2 - 4b > 0$), gives two linearly independent solutions:

$$y(x) = C_1e^{\lambda_+ x} + C_2e^{\lambda_- x} \tag{61}$$

Case 2: Double root $\lambda_+ = \lambda_- \equiv \lambda = \frac{-a}{2}$

$$y(x) = (Ax + B)e^{\lambda x} \tag{62}$$

Case 3: Complex root ($a^2 - 4b < 0$)

$$\lambda_{\pm} = -\frac{a}{2} \pm i\sqrt{4b - a^2} = -\frac{a}{2} \pm i\omega \tag{63}$$

$$y(x) = Ae^{-a/2+i\omega} + Be^{-a/2-i\omega} = e^{-ax/2}\left(A\cos(\omega x) + B\sin(\omega x)\right) \tag{64}$$

$$= e^{-ax/2}\left(\tilde{A}e^{i\omega x} + \tilde{B}e^{-i\omega x}\right) = ke^{-ax/2}\sin(\omega x + \phi) \tag{65}$$

4.3.3 Euler-Cauchy:

$$x^2y'' + a_1xy' + a_0y = 0, \text{ or } y'' + \frac{a_1}{x}y' + \frac{a_0}{x^2}y = 0 \tag{66}$$

Use:

$$x = e^z, \qquad z = \ln x, \qquad dx = e^z dz = xdz \tag{67}$$

For $x > 0$

$$x = -|x| = -e^z, \qquad z = \ln|x|, \qquad dx = -e^z dz = xdz \tag{68}$$

For $x < 0$. We then get

$$\frac{d^2y(z)}{dz^2} + (a_1 - 1)\frac{dy(z)}{dz} + a_0y(z) = 0 \tag{69}$$

Solve then transform back to $y(x)$. No solution at $x = 0$, and different coefficients for cases $x < 0$ and $x > 0$!