FYS3140 Oblig 3

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3.1)

To prove the general Cauchy integral formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\Gamma \frac{f(w)}{(w-z)^{n+1}}dw \tag{1}$$

we are going to start with

$$f(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(w)}{w-z}dw \tag{2}$$

We are going to prove the general formula with induction. We are going to start with the case $n = 1$.

$$\frac{d}{dz_0}f(z_0) = f^{(1)}(z_0) = \frac{1}{2\pi i}\oint_\Gamma \frac{d}{dz_0}\frac{f(w)}{w-z}dw = \frac{1}{2\pi i}1\cdot\oint_\Gamma \frac{f(w)}{(w-z)^{1+1}}dw \tag{3}$$

So we can see that 1 holds for $n = 1$, in other words it holds for a $n = k$, we are now going to see that it holds for $n = k + 1$

$$\frac{d}{dz_0}f^{(k)}(z_0) = \frac{k!}{2\pi i}\oint_\Gamma \frac{d}{dz_0}\frac{f(w)}{(w-z)^{k+1}}dw = \frac{k!}{2\pi i}(k+1)\cdot\oint_\Gamma \frac{f(w)}{(w-z)^{k+2}}dw$$

$$= \frac{(k+1)!}{2\pi i}\oint_\Gamma \frac{f(w)}{(w-z)^{k+2}}dw = f^{(k+1)}(z_0)$$

We can see that it also holds for $n = k + 1$, and we have thereby proved 1 be induction.

We can use this to solve

$$\oint_\Gamma \frac{\sin 2z}{(6z-\pi)^3}dz$$

where $|z| = 2$. We get that

$$\oint_\Gamma \frac{\sin 2z}{(6z-\pi)^3}dz = \oint_\Gamma \frac{\frac{1}{6^3}\sin 2z}{(z-\frac{\pi}{6})^3}dz = \frac{2\pi i}{2!}\frac{d^2}{dz_0^2}\frac{1}{6^3}\sin 2(\frac{\pi}{6}) = -4\frac{2\pi i}{2}\frac{1}{6^3}\sin(2\frac{\pi}{6}) = -\frac{i\pi}{36\sqrt{3}}$$

3.2)

a)

$$\oint_\Gamma \frac{\cosh z}{(2\ln 2 - z)}dz$$

For where Γ is a circle $|z| = 2$.

$$\oint_\Gamma \frac{\cosh z}{(2\ln 2 - z)}dz = \oint_\Gamma \frac{-\cosh z}{(z - 2\ln 2)}dz$$

$$= 2\pi i(-\cosh(2\ln 2)) = -\frac{17\pi}{4}i$$

b)

$$\oint_{\Gamma} \frac{e^{3z}}{(z-\ln 2)^4} dz$$

where Γ is a square with vertices ± 1,$\pm i$

$$\oint_{\Gamma} \frac{e^{3z}}{(z-\ln 2)^4} dz = \frac{2\pi i}{3!} \frac{d^3}{dz_0^3} e^{3z_0}|_{z_0=\ln 2} = 9\pi i e^{3\ln 2} = 72\pi i$$

3.3)

a)

We have the function

$$f(z) = \frac{z-1}{z^2(z-2)} \tag{4}$$

And we are going to look at the Laurent series for $|z| < 2$. We can start be doing a partial fraction expansion to get that

$$f(z) = \frac{1}{2z^2} - \frac{1}{2z} + \frac{1}{4(z-2)} \tag{5}$$

The 2 first parts of the expression can stay as they are, and we only need to rewrite the last term. We can use that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

Since this expression has a singularity outside the area we are looking at, we can find its normal Taylor expansion. We can then rewrite

$$\frac{1}{4(z-2)} = -\frac{1}{8(1-\frac{z}{2})} = -\frac{1}{8}\sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+3}}$$

We then get that

$$f(z) = \frac{1}{2z^2} - \frac{1}{2z} - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+3}} \tag{6}$$

We can see that this only converges for $z < 2$.

b)

For $|z| > 2$ we have to find a series, since the last one 6 diverges for this region. We must again look at

$$\frac{1}{4(z-2)}$$

We are this time going to rewrite is slightly

$$\frac{1}{4(z-2)} = \frac{1}{4z(1-\frac{2}{z})}$$

And now we are going to use 5

$$\frac{1}{4z(1-\frac{2}{z})} = \frac{1}{4z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^{n+1} = \sum_{n=0}^{\infty}\frac{2^{n-2}}{z^{n+1}}$$

This series does not diverge at $|z| > 2$ but rather at $|z| < 2$. So we then get the result

$$f(z) = \frac{1}{2z^2} - \frac{1}{2z} + \sum_{n=0}^{\infty}\frac{2^{n-2}}{z^{n+1}} \tag{7}$$

c)

A Laurent series is given by 2 parts, the first being a series equivalent to a Taylor series, and an other given as

$$\sum_{n=1}^{\infty} b_n(z-z_0)^{-n} \tag{8}$$

The residue of a function $f(z)$ at a point z_0 is given as b_1 is the equation above 8. The part of our function(7 and 6) corresponding to this expression 8 is

$$\frac{1}{2z^2} - \frac{1}{2z}$$

b_1 is the coefficient to the term $\frac{1}{z}$, which in our case is $-\frac{1}{4}$. This means that the residue at the origin is $-\frac{1}{4}$