FYS3140 Oblig 1

Daniel Heinesen, daniehei

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Problem 1.2

a)

$$\sum_{n=0}^{\infty} n(n+1)(z-2i)^n$$

To find the disk of convergence, we can use the ratios test: a series S_n converges if

$$\rho = \lim_{n\to\infty} \left|\frac{S_{n+1}}{S_n}\right| < 1 \tag{1}$$

$$\rho = \lim_{n\to\infty} \left|\frac{(n+1)(n+2)(z-2i)^{n+1}}{n(n+1)(z-2i)^n}\right| = \lim_{n\to\infty} \left|\frac{n+2}{n}(z-2i)\right|$$

$$|z-2i| < 1$$

This means that the disk of convergence is a disk of radius 1 and the center at $2i$.

b

$$\sum_{n=0}^{\infty} 2^n(z+i-3)^{2n}$$

Again using the ratio test:

$$\rho = \lim_{n\to\infty} \left|\frac{2^{n+1}(z+i-3)^{2(n+1)}}{2^n(z+i-3)^{2n}}\right| = \lim_{n\to\infty} |2(z+i-3)^2|$$

$$|(z+i-3)^2| < \frac{1}{2} \Rightarrow |z-(3-i)| < \frac{1}{\sqrt{2}}$$

The disk of convergence is a disk with radius $1/\sqrt{2}$ and the center at $3-i$

Problem 1.3

a)

$\sqrt{2}e^{\frac{5\pi}{4}i}$ can be written on the form $r(\cos(\theta) + i\sin(\theta))$, where:

$$r = \sqrt{2}$$

$$\cos(\frac{5}{4}) = \sin(\frac{5}{4}) = -\frac{1}{\sqrt{2}}$$

so

$$\sqrt{2}e^{\frac{5\pi}{4}i} = \sqrt{2}(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}) = -1-i$$

b

To evaluate this, we have to write the expressions in polar form

$$\frac{(1+i)^{48}}{(\sqrt{3}-i)^{25}} = \frac{(\sqrt{2}e^{i\pi/2})^{48}}{(2e^{\frac{11i}{6}\pi})^{25}}$$

$$= \frac{2^{24}e^{24\pi i}}{2^{25}e^{\frac{25}{6}\pi i}} = \frac{1}{2}e^{\frac{169}{6}\pi i} = \frac{1}{2}e^{\frac{pi}{6}i}$$

$$= \frac{1}{2}(\cos(\pi/6) + i\sin(\pi/6)) = \frac{1}{4}(\sqrt{3}+i)$$

c)

We can find $(8i\sqrt{3}-8)^{1/4}$ we rewrite the expression in polar form.

$$(8i\sqrt{3}-8) = 8(i\sqrt{3}-1) = 16e^{\frac{2\pi}{3}i}$$

so

$$(8i\sqrt{3}-8)^{1/4} = (16e^{\frac{2\pi}{3}i})^{1/4} = \sqrt[4]{16}e^{i\frac{2\pi/3+2\pi k}{4}}, k = 0,1,2,3$$

$$= 2e^{\frac{pi}{6}+\frac{pi}{2}k}$$

So

$$(8i\sqrt{3}-8)^{1/4} = \{2e^{\frac{pi}{6}i}, 2e^{\frac{2pi}{3}i}, 2e^{\frac{7pi}{6}i}, 2e^{\frac{5pi}{3}i}\}$$

d

From $Example 2, section 10$ in Boas, we know that

$$\sqrt[3]{8} = \{2, -1+i\sqrt{3}, -1-i\sqrt{3}\}$$

We can see that

$$2 + (-1+i\sqrt{3}) + (-1-i\sqrt{3}) = 0$$

So the sum of the cube roots of 8 is zero.

We can generalize this. The nth of a complex number is given by:

$$\sqrt[n]{r}e^{i(\frac{\theta}{n}+\frac{2\pi k}{n})}, k = 0, 1, ..., n-1$$

So the sum of the roots are:

$$\sum_{k=0}^{n-1}\sqrt[n]{r}e^{i(\frac{\theta}{n}+\frac{2\pi k}{n})} = \sum_{k=0}^{n-1}\sqrt[n]{r}e^{i\frac{\theta}{n}}e^{\frac{2\pi k}{n}} = \sqrt[n]{r}e^{i\frac{\theta}{n}}\sum_{k=0}^{n-1}e^{\frac{2\pi k}{n}}$$

To evaluate this sum we must look at $\sum_{k=0}^{n-1}e^{\frac{2\pi k}{n}}$. This is a geometric series, and we can solve by using:

$$S_n = \sum_{k=1}^{n} r^k = \frac{r(1-r^n)}{1-r}$$

We can rotate our expression by 2π and get:

$$\sum_{k=0}^{n-1} e^{\frac{2\pi k}{n}} = \sum_{k=1}^{n} e^{\frac{2\pi k}{n}} = \sum_{k=1}^{n} (e^{\frac{2\pi}{n}})^k = \frac{e^{\frac{2\pi}{n}}(1 - e^{\frac{2\pi n}{n}})}{1 - e^{\frac{2\pi}{n}}}$$

We see that $e^{\frac{2\pi n}{n}} = e^{2\pi} = 1$, so $(1 - e^{\frac{2\pi n}{n}}) = 0$, since $n > 1$ that means that $1 - e^{\frac{2\pi}{n}} \neq 0$, and we find that

$$\sum_{k=0}^{n-1} e^{\frac{2\pi k}{n}} = 0$$

and

$$\sum_{k=0}^{n-1} \sqrt[n]{r} e^{i(\frac{\theta}{n} + \frac{2\pi k}{n})} = 0$$

Problem 1.3

a)

We know that $\sin(z) = \frac{e^{zi} - e^{-zi}}{2i}$

$$\int_0^{2\pi} \sin^2(4x) dx = \int_0^{2\pi} (\frac{e^{4xi} - e^{-4xi}}{2i})^2 dx$$

$$= \int_0^{2\pi} (\frac{e^{8xi} - 2 + e^{-8xi}}{-4}) dx$$

$$= -\frac{1}{4}[\frac{e^{8ix}}{8i} - 2x - \frac{e^{-8ix}}{8i}]_0^{2\pi} = -\frac{1}{4}[\frac{1}{8i} - 4\pi - \frac{1}{8i} - \frac{1}{8i} + \frac{1}{8i}]$$

$$= -\frac{-4\pi}{4} = \pi$$

b)

$$\sin^2 z = \frac{e^{2iz} - e^{-2iz}}{2i} = \frac{(e^{iz})^2 - (e^{-iz})^2}{2i}$$

$$= \frac{1}{2i}(e^{iz} + e^{-iz})(e^{iz} - e^{-iz})$$

$$2\frac{e^{iz} - e^{-iz}}{2i}\frac{e^{iz} + e^{-iz}}{2} = 2\cos(z)\sin(z)$$

c)

$$\cosh^2 z - sinh^2 z = (\frac{e^z + e^{-z}}{2})^2 - (\frac{e^z - e^{-z}}{2})^2$$

From the fact that $\cosh(z) = \frac{e^z + e^{-z}}{2}$ and $\sinh(z) = \frac{e^z - e^{-z}}{2}$

$$= \frac{1}{4}(e^{2z} + e^{-2z} + 2 - e^{2z} - e^{-2z} + 2) = \frac{4}{4} = 1$$

d)

$$\sin(i \ln \frac{1-i}{1+i}) = \sin(i \ln z) = i \sinh(\ln z)$$

From $\sin(iz) = i\sinh(z)$

$$i\left(\frac{e^{\ln z} - e^{-\ln z}}{2}\right) = i\left(\frac{z - 1/z}{2}\right)$$

$$= \frac{i}{2}\left(\frac{1-i}{1+i} - \frac{1+i}{1-i}\right) = \frac{i}{2}\frac{(1-i)^2 - (1+i)^2}{2}$$

$$= \frac{i}{4}(-4i) = 1$$

e)

$$(-e)^{i\pi} = (-1)^{i\pi} e^{i\pi}$$

We know that $e^{i\pi} = -1$, so

$$(e^{i\pi})^{i\pi} e^{i\pi} = e^{-\pi^2} e^{i\pi} = -e^{-\pi^2}$$

So

$$(-e)^{i\pi} = -e^{-\pi^2} + 0i$$

f)

To show that

$$\tanh^{-1}(z) = \frac{1}{2} \ln \frac{1+z}{1-z}$$

We are first going to evaluate $\tanh(\frac{1}{2} \ln \frac{1+z}{1-z})$. Lets call $\frac{1+z}{1-z}$ y to make the algebra less tedious. We know that

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{2}{2}\frac{e^x - e^{-x}}{e^x + e^{-x}}$$

So

$$\tanh(\frac{1}{2}\ln y) = \frac{e^{\frac{1}{2}\ln y} - e^{-\frac{1}{2}\ln y}}{e^{\frac{1}{2}\ln y} + e^{-\frac{1}{2}\ln y}}$$

$$= \frac{\sqrt{y} - \frac{1}{\sqrt{y}}}{\sqrt{y} + \frac{1}{\sqrt{y}}} = \frac{y-1}{y+1} = \frac{\frac{1+z}{1-z} - 1}{\frac{1+z}{1-z} + 1} = \frac{1+z-1+z}{1+z+1-z} = \frac{2z}{2} = z$$

So if

$$\tanh(\frac{1}{2} \ln \frac{1+z}{1-z}) = z$$

then

$$\tanh^{-1}(z) = \frac{1}{2} \ln \frac{1+z}{1-z}$$