1 Kompleks analyse:

1.1 Komplekse tall:

$$z = x + iy, \qquad \bar{z} = x - iy, \qquad z = r(\cos\theta + i\sin\theta) = re^{i\theta} \tag{1}$$

$r = |z|$ er modulus, θ er argumentet.

$$z + \bar{z} = 2Re(z) \qquad z - \bar{z} = 2Im(z), \qquad z\bar{z} = |z|^2 = r^2 = x^2 + y^2 \tag{2}$$

$$x = \frac{1}{2}(z + \bar{z}), \qquad y = \frac{1}{2i}(z - \bar{z}) \tag{3}$$

$z - z_0| < R$ er alle z innenfor en radius R

1.2 Komplekse røtter:

$$z^{1/n} = \sqrt[n]{r}e^{i\theta/n} = \sqrt[n]{r}(\cos\theta/n + i\sin\theta/n) = \omega_0 \tag{4}$$

Dette gir bare 'the principal root', resten er gitt ved

$$\omega_k = \sqrt[n]{r}e^{i\frac{\theta+2\pi k}{n}} \tag{5}$$

1.3 Analytic functions:

Def: A function is analytic in a region of the complex plane if it has a (unique) derivative at every point in that region.

All analytic functions can be written in terms of $z = x + iy$ alone.

1.3.1 Cauchy-Riemann equation:

$$f(z) = u(x, y) + iv(x, y) \tag{6}$$

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} \tag{7}$$

If this holds in a region, that f is analytic in this region, and vice versa.

- Regular points: $f(z)$ is analytic
- Singular point/singularities: A point where $f(z)$ is not analytic.
- Isolated singularity: a point where f is not analytic, but is a limit of points where f is analytic.

If f is analytic in some region, it has first order derivatives, then it also has derivatives of all orders in that region.

1.3.2 Harmonic Functions:

If $f = u + iv$ is analytic in a region, then u and v are harmonic:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{8}$$

If u is harmonic, one can find a v such that $f = u + iv$ is analytic. v is the harmonic conjugate of u.

1.4 Contour integrals:

$$\int_{\Gamma} f(z)dz = \lim_{z\to\infty} \sum_{k=1}^{\infty} f(c_k)\Delta z_k \tag{9}$$

For a generalized curve with parametrization $z(t)$:

$$\int_{\gamma} f(t)dz = \int_a^b f(z(t))z'(t)dt \tag{10}$$

1.4.1 An important integral:

$C - r = |z - z_0| = r$:

$$I = \int_{C_r} (z-z_0)^n dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases} \tag{11}$$

1.4.2 Upper bound estimate:

Generalized triangle inequality:

$$\left|\sum_k z_k\right| \leq \sum_k |z_k| \tag{12}$$

Applied to Riemann sum we get the upper bound estimate:

$$\left|\int_{\gamma} f(z)dz\right| \leq ML \tag{13}$$

Where $M = \max|f(z)|$ and L is the length of the curve.

1.4.3 Path:

If f is continuous everwhere in D, then contour integrals are independent of paths, and any loop integral is zero. One can also deform a contour without crossing any singularities and get that:

$$\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz \tag{14}$$

1.4.4 Cauchy Theorem:

If f is analytic in a simply connected domain D with no singularities, and Γ is any closed contour in D, then

$$\int_{\Gamma} f(z)dz = 0 \tag{15}$$

1.4.5 Cauchy's integral formula:

Let Γ be a simple, closed, positively oriented contour. Assume f is analytic in some simply connected domain D containing Γ, and some z_0 is inside Γ. Then:

$$f(z_0) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{z-z_0}dz \tag{16}$$

1.4.6 Generalized Cauchy integral formula:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_{\Gamma} \frac{f(w)}{(z-w)^{n+1}}dw \tag{17}$$

1.4.7 Cauchy inequality:

Let f be analytic on and inside a circle(C_r) of radius R, centered at z_0. If $|f(z)| \leq M$ for some z on C_r, then the derivatives satisfy:

$$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n} \tag{18}$$

This gives Liuvilles theorem: A function which is analytic and bounded in the entire complex plane, is constant.

1.5 Taylor and Laurent Series:

1.5.1 Taylor:

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2!}f''(z_0)(z - z_0)^2 + ... = \sum_n \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{19}$$

If f is analytic in the disk $|z - z_0| < R$ then the above Taylor series converges in that disk.[i.e. the disk touching the nearest singularity]

If f is analytic at z_0, then the Taylor series for df/dz can be obtained by termwise differentiation.

1.5.2 Laurent:

Let f be analytic in the annulus $r < |z - z_0| < R$. Then f can be expanded there as the sum of two series:

$$f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k + \sum_{k=1}^{\infty} b_k(z - z_0)^{-k} \tag{20}$$

With

$$a_n = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}dz \qquad b_n = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{(z - z_0)^{-n+1}}dz \tag{21}$$

Def: The coefficient b_1 of the $1/(z - z_0)$ term is the residue of $f(z)$ at $z = z_0$.

Laurent series are unique. So to find them we can use

$$\frac{1}{1 - \omega} = \sum_{n=0}^{\infty} \omega^2 \text{ , when } |\omega| < 1 \tag{22}$$

1.6 Zeros:

- A zeros of a function is a point where f is analytic and $f(z_0) = 0$
- A zeros of order m: $f(z_0) = f'(z_0) = ... = f^{m-1}(z_0) = 0$, $f^m(z_0) \neq 0$
- Can be factorized as: $f(z) = (z - z_0)^m \cdot g(z)$, where $g(z)$ is analytic and $g(z_0) \neq 0$

1.7 Isolated singularities:

Let f have a Laurent series, then we can have:

1.7.1 Removable Singularity/Regular point:

If all $b_n = 0$ at z_0. $f(z)$ has a limit $z \to z_0$ and we can be redefined such that f is analytic at z_0.

1.7.2 Essential Singularity:

Infinitely many b-terms at z_0

1.7.3 Pole of order m

Order m is the highest exponent of the $1/(z-z_0)$ terms.

$$f(z) = \frac{b_m}{(z-z_0)^m} + ... + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + ... \tag{23}$$

$f(z)$ can be written as $\frac{g(z)}{(z-z_0)^m}$. A pole of order 1 ($m=1$) is a Simple pole.

1.8 Residue Theory:

1.8.1 Residue Theorem:

If Γ is a simple, closed, positively oriented contour, and f is analytic on and inside Γ except at the points $z_0, z_1, ..., z_n$ inside Γ, then

$$\oint_\Gamma f(z)dz = 2\pi i \sum_{k=0}^{n} Res(f, z_k) \tag{24}$$

1.8.2 Determining the residues:

1: Read off b_1 from the Laurent series.

2: Simple poles:

$$Res(z_0) = b_1 = \lim_{z\to z_0} (z-z_0)f(z) \tag{25}$$

Finite answer only if the pole is of first order.

$$f(z) = \frac{P(z)}{Q(z)} \Rightarrow Res(z_0) = \frac{P(z_0)}{Q'(z_0)} \tag{26}$$

3: Multiple poles: If f has a pole of order m at z_0, then

$$Res(z_0) = \lim_{z\Rightarrow z_0} \left[\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z-z_0)^m f(z)\right)\right] \tag{27}$$

Ok to overshoot with value if m.

1.9 Applications to Real Integrals:

1.9.1 Type 1:

Rational and finite functions of $\sin\theta$ and $\cos\theta$ over the interval$[0, 2\pi]$. Use:

- $z = e^{i\theta}$, $\quad d\theta = dz/iz$
- $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + 1/z)$, $\quad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}(z - 1/z)$
- then use residue theorem.

1.9.2 Type 2a

Integrals of rational functions from $-\infty$ to ∞

$$I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx \tag{28}$$

- Make a contour γ_ρ from $-\rho$ to ρ
- Add a second contour from ρ via a the complex plane(half circle in the upper part of the complex plane) back to $-\rho$, C_ρ.
- use the residue theorem. Remember that the singularities have to be in the upper part
- Show that the contribution form C_ρ vanishes as $\rho \to \infty$

1.9.3 Type 2b:

$$I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(mx) dx, \qquad I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(mx) dx \tag{29}$$

Alt 1(Always safe):
use $\cos(mx) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{i\theta} dx + \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{-i\theta} dx \tag{30}$$

FOr the first term, use a closed contour in the upper part of the complex plane, for the second term use one in the lower half.

Alt 2: safe as long as P/Q is real.