Quiz 7 (Sections 4.2, 4.3)

You will have 30 minutes to complete the quiz.

Name:
Student Number:

Q1 Let A be the following 4×4 matrix.

$$A = \begin{bmatrix} 3 & 14 & -1 & -1 \\ 2 & -5 & 0 & -3 \\ 0 & 1 & 0 & a_{34} \\ 0 & -5 & 0 & a_{44} \end{bmatrix}$$

(a) Compute the determinant of A. (3 Points)

(b) Assume $T : \mathbb{R}^4 \to \mathbb{R}^4$ is a linear transformation induced by the matrix A. For which values a_{34}, a_{44} is the transformation T invertible? (1 Point)

Q2 Let B_1, B_2 be $n \times n$ matrices that are invertible. Show that $\det(B_1) = \det(B_2 B_1 B_2^{-1})$ (2 Points)

Q3 Assume that for some $n \times n$ matrix C, we have $\det(C) = k$. Justify the determinant of the following.

(a) The matrix $2C$. (2 Points)

(b) The matrix C^{-1}. (2 Points)

(c) We add a scalar multiple of a row of C to a different row of C. (2 Points)

Q1

(a) Recall that when computing the determinant of a matrix, we may choose which row/column to iterate across. Noticeably, the third column contains the most zeroes of any given row/column, so we will iterate over this column.

$$\det(A) = (-1)^{1+3} 3 \begin{vmatrix} 2 & -5 & -3 \\ 0 & 1 & a_{34} \\ 0 & 7 & a_{44} \end{vmatrix} + (-1)^{2+3} 0 |A_{23}| + (-1)^{3+3} 0 |A_{33}| + (-1)^{4+3} 0 |A_{43}| = 3 \begin{vmatrix} 2 & -5 & -3 \\ 0 & 1 & a_{34} \\ 0 & 7 & a_{44} \end{vmatrix}$$

Similarly, to compute the determinant of the remaining 3×3 matrix, we will iterate across the first column as it contains the most zeroes of any row/column. For simplicity, in the computation we omit any zero terms.

$$\det(A) = 3 \begin{vmatrix} 2 & -5 & -3 \\ 0 & 1 & a_{34} \\ 0 & 7 & a_{44} \end{vmatrix} = 3 \cdot 2(a_{44} - 7a_{34}) = 6a_{44} - 42a_{34}$$

Hence, we obtain that $\det(A) = 6a_{44} - 42a_{34}$, as needed.

(b) Recall that a linear transformation is invertible if and only if the determinant of the associated matrix is nonzero. More specifically, we must determine when $\det(A) \neq 0$.
Therefore, we have the following.

$$0 = \det(A) = 6a_{44} - 42a_{34} \Longrightarrow 42a_{34} = 6a_{44} \Longrightarrow 7a_{34} = a_{44}$$

Consequently, it follows that the transformation T is invertible, precisely when $7a_{34} \neq a_{44}$.

Q2

We want to show that if $B^T B = I_n$, then $|\det(B)| = 1$, or equivalently, that $\det(B) = \pm 1$. To prove this claim, we will utilize the following facts.

(1) The identity matrix of any size has a determinant equal to 1.

(2) For any square matrix, the determinant of the matrix and its transpose are equal.

(3) For any two square matrices, the determinant of their product is equal to the product of their determinants.

Therefore, we have the following.

$$\begin{aligned} 1 &= \det(I_n) && \text{Fact (1).} \\ &= \det(B^T B) && \text{Assuming } B^T B = I_n. \\ &= \det(B^T)\det(B) && \text{Fact (2).} \\ &= \det(B)\det(B) && \text{Fact (3).} \\ &= \det(B)^2 && \text{Grouping Terms.} \end{aligned}$$

Hence, we obtain the fact that $\det(B)^2 = 1$, which implies that we must have $\det(B) = \pm 1$, as needed.

□

Q3

(a) Recall that for any square matrix, the determinant of the matrix and its transpose are equal. Thus, the determinant of this matrix would also be k.

(b) Recall that using elementary matrices, we showed that swapping two rows changes the determinant by a scalar factor of -1. Thus, the determinant of this matrix would be $-k$.

(c) Recall that using elementary matrices, we showed that adding a scalar multiple of a row to a different row does not change the determinant. Thus, the determinant of this matrix would also be k.