Quiz 8 (Sections 5.3, 9.4)

You will have 30 minutes to complete the quiz.

Name:
Student Number:

Q1 Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ induced by the matrix $A = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$.

(a) Sketch the unit square and the unit square under the transformation T. (1 Point)

(b) Using your sketch, determine two linearly independent eigenvectors of A.
INCLUDE THESE VECTORS IN YOUR SKETCH. (2 Point)

(c) Knowing that $\lambda_1 = 2, \lambda_2 = 3$ are the eigenvalues of A, verify computationally that the eigenvectors from (b) are indeed eigenvectors for the transformation T. (2 Points)

Q2 For the following, determine whether the matrix B is diagonalizable, not diagonalizable, or we cannot tell.

(a) The eigenvalues λ_i of B are given by $\lambda_k = 2\lambda_{k-1}$, where $\lambda_1 < 0$. (2 Points)

(b) There is some set of eigenvectors $\{\vec{v}_1, ..., \vec{v}_n\}$ that forms a basis for $\mathbb{R}^n$. (2 Points)

Q3 Let the sequence $c_0, c_1, c_2, ...$ be given by $c_k = \frac{1}{2}(c_{k-1} + c_{k-2})$ for $k \geq 2$. Here, we assume c_0, c_1 are some real numbers.

(a) Determine a matrix A that can be used for the following linear system: $A \begin{bmatrix} c_{k-1} \\ c_{k-2} \end{bmatrix} = \begin{bmatrix} c_k \\ c_{k-1} \end{bmatrix}$. (1 Point)

(b) Knowing that $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, A \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, determine c_{100}. (3 Points)

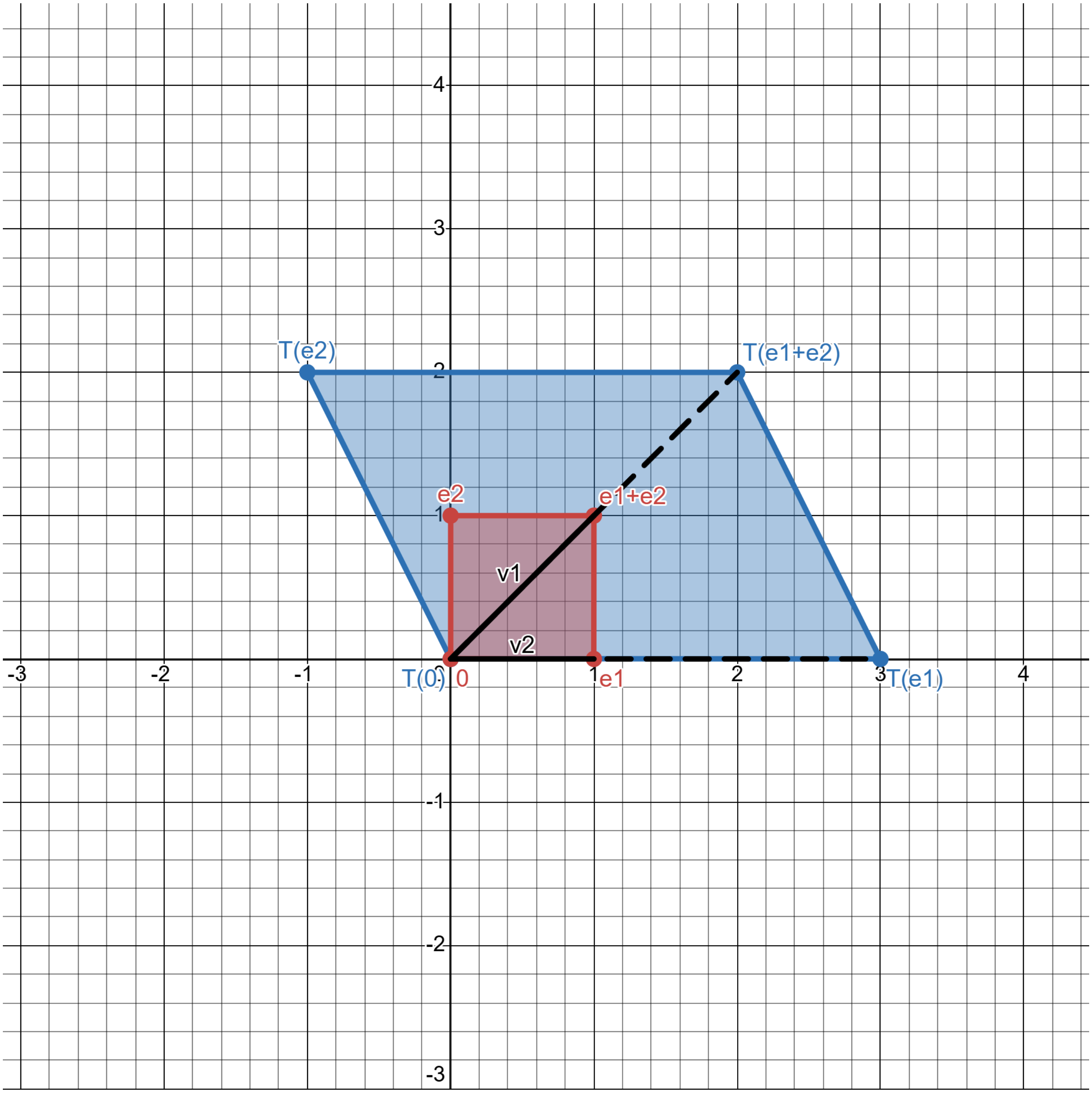
Q1

Visualizations available here (and on the next page). Graphically, we see that the eigenvectors are $\vec{v}_1 = [1, 1]$ and $\vec{v}_2 = [1, 0]$. To verify that these vectors are indeed eigenvectors, we may either verify that $A\vec{v}_i = \lambda_i \vec{v}_i$, or equivalently, that $\vec{v}_i \in \text{null}(A - \lambda_i I)$.

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \lambda_1 \vec{v}_1 \quad A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \lambda_2 \vec{v}_2$$

$$A - 2I = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \implies \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \text{null}(A - 2I) \quad A - 3I = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \implies \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \text{null}(A - 2I)$$

Hence, we have computationally verified that $\vec{v}_1$ and $\vec{v}_2$ are indeed eigenvectors.



T(e2)
T(e1+e2)
e2
e1+e2
v1
v2
T(0)
0
e1
T(e1)
-3
-2
-1
1
2
3
4
-1
-2
-3

Q2

(a) Recall that for any $n \times n$ matrix B, if there are n distinct eigenvalues, then the matrix B is necessarily diagonalizable. Notice that if $\lambda_1 < 0$ and $\lambda_k = 2\lambda_{k-1}$, then it must be that $\lambda_k = 2\lambda_{k-1} < \lambda_{k-1}$. Since we have that $\lambda_1 > \lambda_2 > \ldots > \lambda_n$, it is necessarily the case that each eigenvalue λ_i is distinct. Hence, the matrix B is diagonalizable.

(b) Recall that we say a matrix B is diagonalizable if it can be written in the form PDP^{-1}, where P is a matrix with eigenvectors as its columns, and D is a diagonal matrix with the corresponding eigenvalues on its diagonal. Fix the following matrices P and D, where λ_i is the eigenvalue associated with $\vec{v}_i$.

$$P = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \quad D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

Hence, it is sufficient to show that P is invertible to claim that B is diagonalizable. In fact, since we know that $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$, we know that the set is linearly independent, which implies that P must be invertible. Thus, we obtain that B is indeed diagonalizable.

Q3

(a) Notice that we have can write the following two equations for c_k, c_{k-1} in terms of c_{k-1} and c_{k-2}.

$$c_k = \frac{1}{2}c_{k-1} + \frac{1}{2}c_{k-2} \quad c_{k-1} = c_{k-1}$$

Putting this into the form of the desired linear system, $A\begin{bmatrix} c_{k-1} \\ c_{k-2} \end{bmatrix} = \begin{bmatrix} c_k \\ c_{k-1} \end{bmatrix}$, we obtain the following.

$$A\begin{bmatrix} c_{k-1} \\ c_{k-2} \end{bmatrix} = \begin{bmatrix} c_k \\ c_{k-1} \end{bmatrix} \Longrightarrow A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$$

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(b) We use the following remark in order to determine c_{100}.

$$A^k \begin{bmatrix} c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} c_{k+1} \\ c_k \end{bmatrix}$$

Moreover, we may use the provided assumption to diagonalize A, as it provides us with the two eigenvectors and their associated eigenvalues.

$$A = PDP^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Using properties of diagonalization, we observe that we may generalize the following.

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & -\frac{1}{2}^k \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 + \left(-\frac{1}{2}\right)^k & 1 - \left(-\frac{1}{2}\right)^k \\ 2 - 2\left(-\frac{1}{2}\right)^k & 1 + 2\left(-\frac{1}{2}\right)^k \end{bmatrix}$$

Now, we can use the above to determine c_{100}.

$$\begin{bmatrix} c_{100} \\ c_{99} \end{bmatrix} = A^{99}\begin{bmatrix} c_1 \\ c_0 \end{bmatrix} = PD^{99}P^{-1}\begin{bmatrix} c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2^{-99} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 + \left(-\frac{1}{2}\right)^{99} & 1 - \left(-\frac{1}{2}\right)^{99} \\ 2 - 2\left(-\frac{1}{2}\right)^{99} & 1 + 2\left(-\frac{1}{2}\right)^{99} \end{bmatrix} \begin{bmatrix} c_1 \\ c_0 \end{bmatrix}$$

Therefore, it must be that $c_{100} = \frac{1}{3}\left(2 + \left(-\frac{1}{2}\right)^{99}\right)c_1 + \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^{99}\right)c_0$.