Quiz 6 (Sections 2.3)

You will have 30 minutes to complete the quiz.

Name:	
Student Number:	

Q1 Consider the linear transformation given by $\mathbb{R}^2 \to \mathbb{R}^2$ that reflects vectors along the line $y = x$. (4 Points)

(a) Determine the standard matrix representation of T. (1 Point)
HINT. It may be useful to draw a graph containing $\vec{e}_1, \vec{e}_2, T(\vec{e}_1), T(\vec{e}_2)$.

(b) Any reflection along a line through the origin is an invertible transformation. Determine the standard matrix representation of T^{-1}. (2 Points)

(c) Based on your previous computation, what can we say about the standard matrix representation of a reflection and its inverse? (1 Point)

Q2 For the following linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$, justify whether they are injective, not injective, or we cannot tell. (6 Points)

(a) For some nonzero $\vec{v} \in \mathbb{R}^n$, $T(\vec{v}) = \vec{0}$.

(b) T is a surjective transformation.

(c) There exists some linear combination $T(\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n) = \vec{0}$, where $\alpha_1, ..., \alpha_n \in \mathbb{R}; \vec{v}_1, ..., \vec{v}_n \in \mathbb{R}^n$.

Q3 (Bonus) Let $S : \mathbb{R}^n \to \mathbb{R}^m, T : \mathbb{R}^m \to \mathbb{R}^n$ be linear transformations. Assume S is surjective and T is injective (6 Points).

(a) Determine whether $S \circ T$ can be injective, surjective, bijective, or none.

(b) Determine whether $T \circ S$ can be injective, surjective, bijective, or none.

Q1

(a) Graphically, we see that T swaps $\vec{e}_1$ and $\vec{e}_2$. That is, $T(\vec{e}_1) = \vec{e}_2, T(\vec{e}_2) = \vec{e}_1$. Hence, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) From part (a), we know that T is induced by the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Recall that we can find an inverse matrix using the super-augmented approach. Moreover, that A is the matrix I_2 with the two rows swapped. Therefore, we have the following.

$$\left[A \mid I\right] = \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \underset{\text{RREF}}{\Longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] = \left[I \mid A\right]$$

Hence, the standard matrix representation of T^{-1} is also A.

(c) For any linear transformation describing a reflection, we see that the standard matrix of representation is its own inverse.

Q2

(a) **NOT INJECTIVE.** Recall that for any linear transformation, it follows by definition that the zero vector of the domain must map to the zero vector of the codomain. Hence, we would have $T(\vec{v}) = T(\vec{0}_n)$, but $\vec{v} \neq \vec{0}_n$ since $\vec{v}$ is nonzero. This contradicts the definition of injective.

(b) **INJECTIVE.** Recall that for any linear transformation, we have the following triad of propositions, where if any two are true, the third must also hold.

- $T : \mathbb{R}^n \to \mathbb{R}^n$.
- T is surjective.
- T is injective.

Consequently, knowing that $T : \mathbb{R}^n \to \mathbb{R}^n$ and that T is surjective, it must also be that T is injective.

(c) **WE CANNOT TELL.** Consider the following two possibilities: the set $\{\vec{v}_1, ..., \vec{v}_n\}$ is linearly independent or linearly dependent. If the set $\{\vec{v}_1, ..., \vec{v}_n\}$ is linearly independent, and further that one of the scalars α_i is nonzero, then we could claim that the transformation is indeed not injective. Conversely, if the set $\{\vec{v}_1, ..., \vec{v}_n\}$ is linearly dependent, then it could be that $\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n$ is equal to the vector vector, which would not contradict T being injective. So in general, we do not have enough information to decide.

Q3

NOTE. It was my intention to add the requirement that $n \neq m$. Recall that for matrices A, B where AB is defined, it holds that $\mathrm{Col}(AB) \subseteq \mathrm{Col}(A)$. We can use this property to produce an analogous result for the image of linear transformations. Moreover, we know that for any two sets, if one set is a subset, then it must have less or equal dimension.

(a) Notice that $S \circ T$ is a transformation from $\mathbb{R}^m$ to $\mathbb{R}^m$. Immediately, we can infer that the only possibilities are bijective or none.

If $n = m$, then $S \circ T$ is necessarily bijective (composition of bijective transformations is itself bijective). Otherwise, if $n > m$, it depends on the outputs of the standard vectors under T.

Let $\{\vec{b}_1, ..., \vec{b}_m\} \subseteq \mathbb{R}^n$ be the outputs of the standard vectors under T. That is, we have $T(\vec{e}_i) = \vec{b}_i$. Ultimately, if we have that for any $1 \leq i, j \leq m$, where $i \neq j$, $S(\vec{b}_i) \neq S(\vec{b}_j)$, then the transformation $S \circ T$ is bijective, and otherwise, it is none.

(b) Notice that $T \circ S$ is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. Moreover, we have the following.

$$
\begin{aligned}
\mathrm{Im}(T \circ S) &\subseteq \mathrm{Im}(T) && \text{Column Space Argument.} \\
\dim(\mathrm{Im}(T \circ S)) &\leq \dim(\mathrm{Im}(T)) && \text{Dimension Argument.} \\
&= m && \text{Since } T \text{ is injective.} \\
&\leq n && \text{Since } S \text{ is surjective.}
\end{aligned}
$$

Case 1 If $m = n$, then we obtain the fact that $\dim(\mathrm{Im}(T \circ S)) = n$, which implies that the transformation must be bijective (since the associated matrix is both square and has full rank).

Case 2 If $m \neq n$, then we obtain the fact that $\dim(\mathrm{Im}(T \circ S)) < n$, which implies that the transformation cannot be surjective, or equivalently for transformations represented by square matrices, injective and bijective. In this case, it would be none.

Hence, it is possible for $T \circ S$ to be bijective or none.