

## POLYNOMIAL INEQUALITIES

# 1 **Rationals**

### Key Terms

**Polynomial** A polynomial is a specific type of function which takes the following form  $f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ , where  $a_i$  represents the  $i$ -th coefficient. What we see is that a polynomial consist exclusively of terms comprised of a coefficient and some power of  $x$ .

**Ex:**  $x - 1, x^3 + 1, 2, x^2 + x + 1$

**Factored Form** The factor form of a polynomial is obtained by writing the polynomial as a product of its roots. For quadratics, we can use quadratic formula or other factorization shortcuts. For larger degree polynomials, we need synthetic/long division.

**Remainder Theorem** If we want to divide a polynomial  $f(x)$  by  $x - a$ , then the remainder is given by  $f(a)$ . This is useful to determine if a term  $x - a$  completely divides  $f(x)$  (i.e., do we get a remainder of zero).

### Solving Polynomial Inequalities

In general, we may use the following checklist to approach solving inequalities involving polynomials.

1. Bring every term over to one side of the inequality (i.e., we want one side to be zero).
2. Factor the new polynomial entirely.
3. Determine the zeroes of the polynomial.
4. Setup table with the factored terms as the rows and the intervals around the zeroes as the columns.
5. Input values to determine the sign of the function on those intervals.
6. Select appropriate intervals and include inclusive signs if needed.

## Examples

For each of the following functions, determine all asymptotes and holes.

a)  $(x + 4)(x - 2)(x - 1) \geq 0$

1. In this case, we already have zero on one side of the inequality, so no extra work is needed.
2. In this case, the polynomial is already factored entirely, so no extra work is needed.
3. We may determine the zeroes of our polynomial by solving within each factored term individually.

$$x + 4 = 0 \implies x = -4$$

$$x - 2 = 0 \implies x = 2$$

$$x - 1 = 0 \implies x = 1$$

So, we have zeroes at  $x = -4, 1, 2$

4. Next, we setup our table with the factored terms as the rows and the intervals around the zeroes as the columns.

	$x < -4$	$-4 < x < 1$	$1 < x < 2$	$x > 2$
$x + 4$				
$x - 1$				
$x - 2$				
$f(x)$				

5. Now that we have our table, we may begin to input values to determine the sign of our function on these intervals.

- On the interval  $x < -4$ , we may select  $x = -5$ .
- On the interval  $-4 < x < 1$ , we may select  $x = 0$ .
- On the interval  $1 < x < 2$ , we may select  $x = 1.5$
- On the interval  $x > 2$ , we may select  $x = 3$ .

Therefore, we plug in these values into each of the factored terms and note their signs.

	$x < -4$	$-4 < x < 1$	$1 < x < 2$	$x > 2$
$x + 4$	–	+	+	+
$x - 1$	–	–	+	+
$x - 2$	–	–	–	+
$f(x)$	–	+	–	+

The last row is obtained by taking the product of each entry in a column and noting its sign. For example, in the first column, there are three negative signs. So, a negative times a negative times a negative, would itself be negative.

6. Here, we want to determine when the given function is greater than or equal to zero (this means we want inclusive intervals). So, we look at where our function is positive. This happens on the intervals  $-4 < x < 1$  and  $x > 2$  in our table. As we want to have inclusive intervals, this means our inequality is satisfied when  $-4 \leq x \leq 1$  and  $x \geq 2$ , as needed.

b)  $x^2 - 8x + 16 < 4$

- In this case, we do not already have zero on one side of the inequality. We remark that by subtracting 4 from both sides, this is made possible. This gives us the new following new inequality.

$$x^2 - 8x + 12 < 0$$

- The polynomial  $x^2 - 8x + 12$  is a quadratic, so we can see if any of our factorization shortcuts work. In fact, if we look for two numbers that sum to  $-8$  and multiply to  $12$ , these are given by  $-2, -6$ . So, we have the following.

$$x^2 - 8x + 12 = (x - 2)(x - 6)$$

- We may determine the zeroes of our polynomial by solving within each factored term individually.

$$x - 2 = 0 \implies x = 2$$

$$x - 6 = 0 \implies x = 6$$

So, we have zeroes at  $x = 2, 6$

- Next, we setup our table with the factored terms as the rows and the intervals around the zeroes as the columns.

	$x < 2$	$2 < x < 6$	$x > 6$
$x - 2$			
$x - 6$			
$f(x)$			

- Now that we have our table, we may begin to input values to determine the sign of our function on these intervals.

- On the interval  $x < 2$ , we may select  $x = 0$ .
- On the interval  $2 < x < 6$ , we may select  $x = 3$ .
- On the interval  $x > 6$ , we may select  $x = 7$ .

Therefore, we plug in these values into each of the factored terms and note their signs.

	$x < 2$	$2 < x < 6$	$x > 6$
$x - 2$	—	+	+
$x - 6$	—	—	+
$f(x)$	+	—	+

The last row is obtained by taking the product of each entry in a column and noting its sign. For example, in the first column, there are three negative signs. So, a negative times a negative, would itself be positive.

- Here, we want to determine when the given function is strictly less than zero (this means we do not want inclusive intervals). So, we look at where our function is negative and can take those intervals as is. This means our inequality is satisfied when  $2 < x < 6$ , as needed.