Review

Definition (Improper Integral (Infinite Bound)). Let a_n be a sequence. Fix k_0 to be a real number. We define an infinite series as the following.

$$\sum_{k=k_0}^{\infty} a_k = \lim_{t \to \infty} \sum_{k=k_0}^{t} a_k$$

Furthermore, we say that such an infinite series converges or diverges depending on whether the infinite series evaluates to a fininite or infinite number.

$$\text{Converges} \implies \sum_{k=k_0}^{\infty} a_k = R \qquad \text{Diverges} \implies \sum_{k=k_0}^{\infty} a_k = \pm\infty$$

Lemma (Divergence Test.). Let a_n be a sequence. If $\lim_{n\to\infty} a_n \neq 0$, then the infinite series $\sum_{k=k_0}^{\infty} a_k$ necessarily diverges.

Theorem (Basic Comparison Test.). Let $a_n, b_n \geq 0$ be sequences such that $a_n \leq b_n$. Then the following holds.

- If $\sum_{k=k_0}^{\infty} b_k$ converges then $\sum_{k=k_0}^{\infty} a_k$ also converges.
- If $\sum_{k=k_0}^{\infty} a_k$ diverges then $\sum_{k=k_0}^{\infty} b_k$ also diverges.

Theorem (Integral Test.). Let a_n be a sequence. Suppose there exists a function f such that $f(n) = a_n$, where the holding holds.

- positive
- decreasing
- continuous
- $\int_{k_0}^{\infty} f(n)dn$ is easy to compute

Then we may claim that the improper integral $\int_{k_0}^{\infty} f(n)dn$ and infinite series $\sum_{k=k_0}^{\infty} a_k$ either both converge or both diverge.

Theorem (Ratio Test.). Let a_n be a sequence. We may compute the limit of the (absolute) ratio between a_{n+1} and a_n to conclude the following about $\sum_{k=k_0}^{\infty} a_k$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \quad \implies \quad \sum_{k=k_0}^{\infty} a_k = \begin{cases} \text{converges} & L < 1 \\ \text{diverges} & L > 1 \\ \text{we cannot tell} & L = 1 \end{cases}$$

Questions

1. True or False.

   (o) Let $f(x), g(x) \geq 0$ be functions. Then the following holds.

   – If $\lim_{k\to\infty} a_k = 0$, then the infinite series $\sum_{k=1}^{\infty} a_n$ must converge.
   (FALSE). Consider the sequence $a_k = \frac{1}{k^p}$ where $0 < p \leq 1$. We may show that the limit as k tends towards infinity is indeed 0, and yet using our integral test we know that the infinite series would not converge, as a result of our p-series test for improper integrals. To use our integral test, we fix $f(n) = a_n$. Indeed, we may use the integral test here since f is positive, decreasing, and continuous.

   $$\sum_{k=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)dx \text{ both converge or both diverge.}$$

   Using our p-series test, we know that the improper integral $\int_1^i nftyf(x)dx$ diverges, and so, the infinite series $\sum_{k=1}^{\infty} a_n$ must also. Hence, we have shown that the function sequence $a_k = \frac{1}{k^p}$ satisfies $\lim_{k\to\infty} a_k = 0$, yet the infinite series $\sum_{k=1}^{\infty} a_n$ diverges, demonstrating that the claim is false, as needed.

   – If $b_n \leq a_n \leq 5b_n$, then the infinite series $\sum_{k=1}^{\infty} a_n$ and $\sum_{k=1}^{\infty} b_n$ either both converge or both diverge.

   (TRUE). To prove the claim above, we use the basic comparison test. In particular, there are two statements we must demonstrate hold under these assumptions.

   (Convergence Statement) $\sum_{k=1}^{\infty} a_n$ converges $\Longleftrightarrow \sum_{k=1}^{\infty} b_n$ converges

   (Divergence Statement) $\sum_{k=1}^{\infty} a_n$ diverges $\Longleftrightarrow \sum_{k=1}^{\infty} b_n$ diverges

   Therefore, we have the following.

   Claim 1 $\sum_{k=1}^{\infty} a_n$ converges $\Longrightarrow \sum_{k=1}^{\infty} b_n$ converges

   Notice that we have $0 \leq b_n \leq a_n$. Further, by assumption the infinite series $\sum_{k=1}^{\infty} a_n$ converges. Consequently, by the basic comparison test, it holds that $\sum_{k=1}^{\infty} b_n$ must also converge.

   Claim 2 $\sum_{k=1}^{\infty} a_n$ converges $\Longleftarrow \sum_{k=1}^{\infty} b_n$ converges

   Notice that we have $0 \leq a_n \leq 5b_n$. Further, by assumption the infinite series $\sum_{k=1}^{\infty} b_n$ converges. Noting that $\sum_{k=1}^{\infty} b_n$ converges, it holds that the infinite series $5\sum_{k=1}^{\infty} b_n = \sum_{k=1}^{\infty} 5b_n$ must also converge. Consequently, by the basic comparison test, it holds that $\sum_{k=1}^{\infty} b_n$ must also converge.

Claim 3 $\sum_{k=1}^{\infty} a_n$ diverges $\Longrightarrow \sum_{k=1}^{\infty} b_n$ diverges

Notice that we have $0 \leq a_n \leq 5b_n$. Further, by assumption the infinite series $\sum_{k=1}^{\infty} a_n$ diverges. Noting that $\sum_{k=1}^{\infty} a_n$ diverges, it holds by the basic comparison test that the infinite series $\sum_{k=1}^{\infty} 5b_n$ must also diverge. Noting that $5\sum_{k=1}^{\infty} b_n = \sum_{k=1}^{\infty} 5b_n$, we remark that the divergence of the infinite series cannot be attributed to the scalar 5, rather, it must be that $\sum_{k=1}^{\infty} b_n$ diverges.

Claim 4 $\sum_{k=1}^{\infty} a_n$ converges $\Longleftarrow \sum_{k=1}^{\infty} b_n$ converges

Notice that we have $0 \leq b_n \leq a_n)$. Further, by assumption the infinite series $\sum_{k=1}^{\infty} b_n$ diverges. Consequently, by the basic comparison test, it holds that $\sum_{k=1}^{\infty} a_n$ must also diverge.

Hence, we have shown that if $b_n \leq a_n \leq 5b_n$, then the infinite series $\sum_{k=1}^{\infty} a_n$ and $\sum_{k=1}^{\infty} b_n$ either both converge or both diverge, as needed.

– If both $\sum_{k=1}^{\infty} a_n$ and $\sum_{k=1}^{\infty} b_n$ diverge, then $\sum_{k=1}^{\infty} a_n b_n$ also diverges.

(FALSE). To disprove the claim, we will produce a counterexample utilizing the p-series infinite series. Briefly, we recall that the following holds for p-series infinite series, as a result of the integral test.

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = \begin{cases} \text{converges if} & p > 1 \\ \text{diverges if} & p \leq 1 \end{cases}$$

Fix $a_k = \frac{1}{k^{p_1}}$ and $b_k = \frac{1}{k^{p_2}}$, such that $p_1, p_2 \leq 1$ and $p_1 + p_2 > 1$. For example, we might consider $p_1 = \frac{3}{4} = p_2$. First, we remark that since $p_1, p_2 \leq 1$, the infinite series $\sum_{k=1}^{\infty} a_n$ and $\sum_{k=1}^{\infty} b_n$ diverge (p-series). However, notice that we have the following.

$$a_k b_k = \frac{1}{k^{p_1}} \cdot \frac{1}{k^{p_2}} = k^{-p_1} + k^{-p_2} = k^{-(p_1+p_2)} = \frac{1}{k^{p_1+p_2}}$$

Since $p_1 + p_2 > 1$, it holds that the infinite series $\sum_{k=1}^{\infty} a_n b_n$ converges (p-series). Under the example for p_1, p_2, we would obtain that $p_1 + p_2 = \frac{3}{2}$, and thus similarly, the infinite series converges.

(i) Changing a finite number of terms in a series does not change whether or not it converges.

(TRUE). Let a_k be a sequence. Fix b_k to be a sequence such that after the first n terms we have $a_i = b_i$. The sequence b_n represents the sequence where we have changed some finite number of terms. More specifically, we have the following.

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{n} b_k + \sum_{k=n+1}^{\infty} b_k = \sum_{k=1}^{n} b_k + \sum_{k=n+1}^{\infty} a_k$$

Notice that the summation $\sum_{k=1}^{n} b_k$ is finite, and thus must exists. Consequently, if $\sum_{k=1}^{\infty} a_k$ converges, it must be that $\sum_{k=n+1}^{\infty} a_k$ converges, and further, that $\sum_{k=1}^{\infty} b_k$ also converges. We obtain the analogous argument on divergence using the same process as convergence.

$$\sum_{k=1}^{\infty} a_k \text{ converges } \implies \sum_{k=n+1}^{\infty} a_k \text{ converges } \implies \sum_{k=1}^{\infty} b_k \text{ converges}$$

$$\sum_{k=1}^{\infty} a_k \text{ diverges } \implies \sum_{k=n+1}^{\infty} a_k \text{ diverges } \implies \sum_{k=1}^{\infty} b_k \text{ diverges}$$

(ii) If the series $\sum_{k=1}^{\infty} a_k$ converges, then the series $\sum_{k=1}^{\infty}(1-a_k)$ diverges.

(TRUE). Recall that if an infinite series on the sequence s_k converges, then it must be that $\lim_{k\to\infty} s_k = 0$.

By assumption, we are told that $\sum_{k=1}^{\infty} a_k$ converges. This implies that $\lim_{k\to\infty} a_k = 0$ We will use the divergence test to show that under this assumption $\sum_{k=1}^{\infty}(1-a_k)$ must diverge. Therefore, we have the following.

$$\begin{aligned}
\lim_{k\to\infty}(1-a_k) &= \lim_{k\to\infty} 1 - \lim_{k\to\infty} a_k && \text{Sum Rule.} \\
&= 1 - \lim_{k\to\infty} a_k && \text{Evaluating First Limit.} \\
&= 1 - 0 && \text{By Convergence Assumption.} \\
&= 1 && \text{Simplifying.} \\
&\neq 0 && \text{Per Divergence Test.}
\end{aligned}$$

Hence, it must be that the series $\sum_{k=1}^{\infty}(1-a_k)$ diverges if the series $\sum_{k=1}^{\infty} a_k$ converges, as we demonstrated that $\lim_{k\to\infty}(1-a_k) \neq 0$, as needed.

2. Consider the differential equation $\frac{dy}{dx} = \frac{y^2-1}{xy}$.

   (i) Find the equilibrium solutions.

   Recall that an equilibrium solution is one where $\frac{dy}{dx} = 0$, in particular, these are constant functions. Therefore, we have the following.

$$\begin{aligned}
0 &= \frac{dy}{dx} && \text{Equilibrium Solution Definition.}\\
&= \frac{y^2-1}{xy} && \text{By Assumption.}\\
0 \cdot xy &= y^2 - 1 && \text{Multiplicative Inverse.}\\
0 &= y^2 - 1 && \text{Simplifying.}\\
1 &= y^2 && \text{Additive Inverse.}\\
\pm 1 &= y && \text{Taking Square Root.}
\end{aligned}$$

   Briefly, we remark that there are no immediate constraints imposed on x from the computations above, however, we cannot have $x = 0$ as $\frac{dy}{dx}$ is undefined for such values. Hence, we may write the set of all equilibrium solutions as the function $y(x) = \pm 1$, that is, y is the constant function of value 1 or -1, as needed.

   (ii) Find all solutions.

   In order to find all solutions to the differential equation $\frac{dy}{dx} = \frac{y^2-1}{xy}$, we must find the general solution. To do this, we will use the separation of variables method (Lecture 12). Therefore, we have the following.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y^2-1}{xy} && \text{By Assumption.}\\
\frac{y}{y^2-1}dy &= \frac{1}{x}dx && \text{Seprating Variables.}\\
\int \frac{y}{y^2-1}dy &= \int \frac{1}{x}dx && \text{Applying Integral.}\\
\int \frac{1}{2u}du &= \int \frac{1}{x}dx && \text{Substituting } u = y^2 - 1.\\
\frac{1}{2}\ln|u| + C_1 &= \ln|x| + C_2 && \text{Evaluating the Integrals.}\\
\ln|u| &= 2\ln|x| + C_3 && \text{Rearranging/Fixing } C_3 = C_2 - C_1.\\
\ln|u| &= \ln|x^2| + C_3 && \text{Rearranging.}\\
\ln|y^2-1| &= \ln|x^2| + C_3 && \text{Substituting } u = y^2 - 1.\\
y^2 - 1 &= Cx^2 && \text{Applying Exponential/Fixing } C = e^{C_3}.\\
y^2 &= Cx^2 + 1 && \text{Additive Inverse.}\\
y &= \pm\sqrt{Cx^2+1} && \text{Applying Square Root.}
\end{aligned}$$

Hence, we have determined that the set of all possible solutions is given by $y = \pm\sqrt{Cx^2+1}$, where C is a constant, as needed.

(iii) Find the solution which satisfies $y(1) = 2$.

Recall that in the previous part, we saw that the initial differential equation had the following general solution $y = \pm\sqrt{Cx^2+1}$. Briefly, we remark that for $y_1 = -\sqrt{Cx^2+1}$, there cannot be any real solutions of the form $y_1(1) = 2$. This follows since $-\sqrt{Cx^2+1} < 0 < 2 \implies -\sqrt{Cx^2+1} \neq 2$. Consequently, we fix $y_2 = \sqrt{Cx^2+1}$ and solve for the constant C such that $y_2(1) = 2$ is satisfied.

$$
\begin{aligned}
y_2(x) &= \sqrt{C(x)^2+1} && \text{By Assumption.} \\
2 &= \sqrt{C(1)^2+1} && \text{Evaluating at } y(1) = 2. \\
&= \sqrt{C+1} && \text{Simplifying.} \\
4 &= C+1 && \text{Squaring Both Sides.} \\
3 &= C && \text{Additive Inverse.}
\end{aligned}
$$

Hence, the solution $y = \sqrt{3x^2+1}$ is the solution of the differential equation which satisfies $y(1) = 2$, as needed.

3. Determine the convergence or divergence of the following series.

(i) $\sum_{k=0}^{\infty}(-1)^k$

(DIVERGES). To show that the series above diverges, we will use the divergence test. Recall, the divergence test states that if the following limit does not equal zero then the series must diverge.

$$\lim_{n\to\infty}(-1)^n \neq 0 \Longrightarrow \text{ Series Diverges}$$

We remark that the sequence $a_n = (-1)^n$ oscillates between 1 and -1 and thus, does not approach any value as n tends towards infinity. Hence, the limit does not exist and thus the series must diverge by the divergence test.

$$\lim_{n\to\infty}(-1)^n = \text{ D.N.E.} \Longrightarrow \sum_{k=0}^{\infty}(-1)^k \text{ diverges}$$

(ii) $\sum_{k=3}^{\infty}\frac{1}{k(\ln(k))^3}$

(CONVERGES). To show that the series above converges, we will use the integral test.

$$\sum_{k=3}^{\infty}\frac{1}{k(\ln(k))^3} \text{ and } \int_3^{\infty}\frac{1}{x(\ln(x))^3}dx \text{ both converge or both diverge.}$$

In particular, we want to use the integral test since the function $f(k) = \frac{1}{k(\ln(k))^3}$ is elementary to compute (in fact, a similar function was used on A8 of the midterm). Since f is positive, decreasing, and continuous for $k \geq 3$, we may indeed use the integral test. To assess the convergence of the improper integral $\int_3^{\infty}\frac{1}{x(\ln(x))^3}dx$, we may use the substitution $u = \ln(x), du = \frac{dx}{x}$. Therefore, we have the following.

$$\begin{aligned}
\int_3^{\infty}\frac{1}{x(\ln(x))^3}dx &= \lim_{t\to\infty}\int_3^{t}\frac{1}{x(\ln(x))^3}dx && \text{Improper Integral Definition.}\\
&= \lim_{t\to\infty}\int_{\ln(3)}^{t}\frac{1}{(u)^3}du && \text{Substituting } u = \ln(x), du = \frac{dx}{x}.\\
&= \int_{\ln(3)}^{\infty}\frac{1}{u^3}du && \text{Improper Integral Definition.}\\
&\Rightarrow \text{Converges} && p\text{-series Test } (p = 3 > 1).
\end{aligned}$$

Hence, using the integral test, we demonstrated that the improper integral $\int_3^{\infty}\frac{1}{x(\ln(x))^3}dx$ converges, and thus the infinite series $\sum_{k=3}^{\infty}\frac{1}{k(\ln(k))^3}$ must also converge, as needed.

(iii) $\sum_{k=1}^{\infty} \frac{k+1}{\sqrt{k^5+1}}$ **(CONVERGES).** To show that the series above converges, we will use the basic comparison test. Consider that for all $k \geq 1$, we have that $k^5 + 1 > k^5$. Moreover, we also use the fact that $k + 1 \leq 2k$ for $k \geq 1$. Therefore, we have the following.

$$\begin{aligned}
k^5 + 1 &> k^5 && \text{Per Argument.} \\
\sqrt{k^5+1} &> \sqrt{k^5} && \text{Applying Square Root.} \\
\frac{1}{\sqrt{k^5+1}} &< \frac{1}{\sqrt{k^5}} && \text{Reciprocity Law.} \\
&= \frac{1}{k^{\frac{5}{2}}} && \text{Simplifying.} \\
\frac{k+1}{\sqrt{k^5+1}} &< \frac{k+1}{k^{\frac{5}{2}}} && \text{Multiplying by } k+1. \\
&\leq \frac{2k}{k^{\frac{5}{2}}} && \text{Since } k+1 \leq 2k. \\
&= \frac{2}{k^{\frac{3}{2}}} && \text{Simplifying.}
\end{aligned}$$

Notably, we know that the infinite series $\sum_{k=1}^{\infty} \frac{1}{x^p}$ converges when $p > 1$ (using either Integral Test or p-series), and thus, as $\frac{k+1}{\sqrt{k^5+1}} \leq \frac{2}{k^{\frac{3}{2}}}$, by the basic comparison test it must be that $\sum_{k=1}^{\infty} \frac{k+1}{\sqrt{k^5+1}}$ converges, as needed.

$$\frac{k+1}{\sqrt{k^5+1}} \leq \frac{2}{k^{\frac{3}{2}}}, \quad 2\sum_{k=1}^{\infty} \frac{1}{x^{\frac{3}{2}}} = \sum_{k=1}^{\infty} \frac{2}{x^{\frac{3}{2}}} \text{ converges } \underset{\text{BCT}}{\Longrightarrow} \sum_{k=1}^{\infty} \frac{k+1}{\sqrt{k^5+1}} \text{ converges}$$

(iv) $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$

(CONVERGES). To show that the series above converges, we will use the ratio test. Here, we define the sequence $a_n = \frac{(n!)^2}{(2n)!}$, as is represented in the interested infinite sum.

$$\begin{aligned}
\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| &= \lim_{k\to\infty} \left| \frac{((k+1)!)^2}{(2(k+1))!} \cdot \frac{(2k)!}{(k!)^2} \right| && \text{By Assumption.} \\
&= \lim_{k\to\infty} \left| \frac{((k+1)!)^2}{(2k+2)!} \cdot \frac{(2k)!}{(k!)^2} \right| && \text{Developing.} \\
&= \lim_{k\to\infty} \left| \frac{((k+1)(k!))^2}{(2k+2)(2k+1)((2k)!)} \cdot \frac{(2k)!}{(k!)^2} \right| && \text{Expanding Factorial Terms.} \\
&= \lim_{k\to\infty} \left| \frac{(k+1)^2(k!)^2}{(2k+2)(2k+1)((2k)!)} \cdot \frac{(2k)!}{(k!)^2} \right| && \text{Developing.} \\
&= \lim_{k\to\infty} \left| \frac{(k+1)^2}{(2k+2)(2k+1)} \right| && \text{Cancelling Common Terms.} \\
&= \lim_{k\to\infty} \left| \frac{(k+1)(k+1)}{4(k+1)(k+\frac{1}{2})} \right| && \text{Developing/Factoring.} \\
&= \lim_{k\to\infty} \left| \frac{(k+1)}{4(k+\frac{1}{2})} \right| && \text{Cancelling Common Terms.}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\lim_{k\to\infty}\left|\frac{1+\frac{1}{k}}{1+\frac{1}{2k}}\right| && \text{Factoring/Dividing by } k. \\
&= \frac{1}{4}\cdot\left|\frac{1+0}{1+0}\right| && \text{Evaluating Limit.} \\
&= \frac{1}{4} && \text{Simplifying.} \\
&< 1 && \text{Converges by Ratio Test.}
\end{aligned}
$$

Hence, we have shown using the ratio test that the infinite sum $\sum_{k=1}^{\infty}\frac{(k!)^2}{(2k)!}$ converges, as needed.

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \frac{1}{4} < 1 \implies \sum_{k=1}^{\infty}\frac{(k!)^2}{(2k)!} \text{ converges}$$

4. Suppose the power series $\sum_{k=0}^{\infty} c_k(x-2)^k$ converges at $x=4$.

(i) Show that it must also converge at $x=1$.

To prove this claim, we must utilize the notion of radius of convergence for power series. Given our power series, we know that there is some number R such that the following holds.

$$|x-2| = |x-a| < R \implies \text{ converges} \quad |x-2| = |x-a| > R \implies \text{ diverges}$$

Knowing that by assumption, our power series converges at $x=4$, we may infer that $R \geq 2$. Therefore, evaluating at $x=1$, we obtain that $|x-2| = |1-2| = 1 < 2 \leq R$. Consequently, it is indeed the case that the power series would converge when $x=1$, as needed.

(ii) What about for $x=0$.

Undergoing the same process as above, we recognize that we obtain that $|x-2| = |0-2| = 2$. This leaves two possibilities.

$R > 2$ If we have that $R > 2$ then the evaluation of $x=0$ is contained in our radius of convergence, and thus, the power series would also converge.

$R = 2$ If we have that $R = 2$ then the evaluation of $x=0$ lies on the end-point of our radius of convergence, and thus, we cannot guarantee that it will converge/diverge. That is, in this scenario we cannot tell.

Taking both scenarios into consideration, we cannot guarantee that the power series would converge when $x=0$.