

Quiz 7 (Sections 4.2, 4.3)

You will have 30 minutes to complete the quiz.

Name:
Student Number:

Q1 Let A be the following 4×4 matrix.

$$A = \begin{bmatrix} 4 & 18 & 3 & -1 \\ 2 & -5 & 0 & -3 \\ 0 & 1 & 0 & a_{34} \\ 0 & 7 & 0 & a_{44} \end{bmatrix}$$

- (a) Compute the determinant of A . (3 Points)
- (b) Assume $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is a linear transformation induced by the matrix A . For which values a_{34}, a_{44} is the transformation T invertible? (1 Point)

Q2 Let B be a $n \times n$ matrix such that $B^T B = I_n$. Prove that $|\det(B)| = 1$. (2 Points)

Q3 Assume that for some matrix C , we have $\det(C) = k$. Justify the determinant of the following.

- (a) The transpose of C . (2 Points)
- (b) We swap two rows of C . (2 Points)
- (c) We add a scalar multiple of a row of C to a different row of C . (2 Points)

Q1

- (a) Recall that when computing the determinant of a matrix, we may choose which row/column to iterate across. Noticeably, the third column contains the most zeroes of any given row/column, so we will iterate over this column.

$$\det(A) = (-1)^{1+3} 3 \begin{vmatrix} 2 & -5 & -3 \\ 0 & 1 & a_{34} \\ 0 & 7 & a_{44} \end{vmatrix} + (-1)^{2+3} 0 |A_{23}| + (-1)^{3+3} 0 |A_{33}| + (-1)^{4+3} 0 |A_{43}| = 3 \begin{vmatrix} 2 & -5 & -3 \\ 0 & 1 & a_{34} \\ 0 & 7 & a_{44} \end{vmatrix}$$

Similarly, to compute the determinant of the remaining 3×3 matrix, we will iterate across the first column as it contains the most zeroes of any row/column. For simplicity, in the computation we omit any zero terms.

$$\det(A) = 3 \begin{vmatrix} 2 & -5 & -3 \\ 0 & 1 & a_{34} \\ 0 & 7 & a_{44} \end{vmatrix} = 3 \cdot 2(a_{44} - 7a_{34}) = 6a_{44} - 42a_{34}$$

Hence, we obtain that $\det(A) = 6a_{44} - 42a_{34}$, as needed.

- (b) Recall that a linear transformation is invertible if and only if the determinant of the associated matrix is nonzero. More specifically, we must determine when $\det(A) \neq 0$.
Therefore, we have the following.

$$0 = \det(A) = 6a_{44} - 42a_{34} \implies 42a_{34} = 6a_{44} \implies 7a_{34} = a_{44}$$

Consequently, it follows that the transformation T is invertible, precisely when $7a_{34} \neq a_{44}$.

Q2

We want to show that if $B^T B = I_n$, then $|\det(B)| = 1$, or equivalently, that $\det(B) = \pm 1$. To prove this claim, we will utilize the following facts.

- (1) The identity matrix of any size has a determinant equal to 1.
- (2) For any square matrix, the determinant of the matrix and its transpose are equal.
- (3) For any two square matrices, the determinant of their product is equal to the product of their determinants.

Therefore, we have the following.

$$\begin{aligned} 1 &= \det(I_n) && \text{Fact (1).} \\ &= \det(B^T B) && \text{Assuming } B^T B = I_n. \\ &= \det(B^T) \det(B) && \text{Fact (2).} \\ &= \det(B) \det(B) && \text{Fact (3).} \\ &= \det(B)^2 && \text{Grouping Terms.} \end{aligned}$$

Hence, we obtain the fact that $\det(B)^2 = 1$, which implies that we must have $\det(B) = \pm 1$, as needed.

□

Q3

- (a) Recall that for any square matrix, the determinant of the matrix and its transpose are equal. Thus, the determinant of this matrix would also be k .
- (b) Recall that using elementary matrices, we showed that swapping two rows changes the determinant by a scalar factor of -1 . Thus, the determinant of this matrix would be $-k$.
- (c) Recall that using elementary matrices, we showed that adding a scalar multiple of a row to a different row does not change the determinant. Thus, the determinant of this matrix would also be k .