Review

Definition (Improper Integral (Infinite Bound))**.** Let f be a function. Fix a to be a real number. We define an improper integral (from infinity occurring as one of the bounds), as the following.

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$$

Furthermore, we say that such an improper integral converges or diverges depending on whether the improper integral evaluates to a fininite or infinite number.

$$\text{Converges} \implies \int_a^\infty f(x)dx = R \qquad \text{Diverges} \implies \int_a^\infty f(x)dx = \pm\infty$$

Remark (Non-zero Limit Towards Infinity)**.** It is not the case that $\lim_{x\to\infty} f(x) = 0$ is a strong enough condition to ensure convergence (see Q1i.). However, in contrast, we may remark that if $\lim_{x\to\infty} f(x) \neq 0$, then it cannot be that an improper integral of the form $\int_a^\infty f(x)dx$ converges. Briefly, it may be useful to view this claim geometrically using the notion of area and Riemann Sums.

Lemma (Scalar Property)**.** Consider the improper integral $\int_a^\infty kf(x)dx$, where k is some constant. Then the following holds.

$$\int_a^\infty f(x)dx \text{ converges} \implies \int_a^\infty kf(x)dx \text{ converges} \qquad \int_a^\infty f(x)dx \text{ diverges} \implies \int_a^\infty kf(x)dx \text{ diverges}$$

Theorem (Basic Comparison Test.)**.** Let $f(x), g(x) \geq 0$ be functions such that $f(x) \leq g(x)$. Then the following holds.

- If $\int_a^\infty g(x)dx$ converges then $\int_a^\infty f(x)dx$ also converges.
- If $\int_a^\infty f(x)dx$ diverges then $\int_a^\infty g(x)dx$ also diverges.

Definition (Curves)**.** There are two approaches we may utilize when defining curves.

Explicit. Here, we define the curve as some function f with respect to a variable, such as x.

Parametric. Here, we define the curve by mapping how the individuals coordinates (e.g., x- and y-coordinates) change through time. In this form, we write the curve in the following form, where $x(t)$ represents the x-coordinate at t and $y(t)$ represents the y-coordinate at t; the curve is then written using the following notation $\gamma(t) = (x(t), y(t))$.

Theorem (Arc-Length Formula)**.** Consider the curve $\gamma(t) = (x(t), y(t))$ where x, y are functions of t. We may compute the arc-length of said curve along an interval $[a, b]$ using the following definite integral.

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt = \int_a^b \sqrt{\left(\frac{d}{dt}x(t)\right)^2 + \left(\frac{d}{dt}y(t)\right)^2}dt$$

Briefly, we remark that by writing the explicit form of any curve in parametric form, we obtain the following relationship, $(t, f(t)) = (x(t), y(t))$. Consequently, we may derive the explicit curve formulation of the arc-length formula

$$\int_a^b \sqrt{\left(\frac{d}{dt}x(t)\right)^2 + \left(\frac{d}{dt}y(t)\right)^2}dt = \int_a^b \sqrt{\left(\frac{d}{dt}t\right)^2 + \left(\frac{d}{dt}f(t)\right)^2}dt = \int_a^b \sqrt{1 + f'(t)^2}dt$$

Questions

1. True or False. Let $f(x), g(x) \geq 0$ be functions.

   (i) If $\lim_{x\to\infty} f(x) = 0$, then the improper integral $\int_a^\infty f(x)dx$ must converge.

   (FALSE). Consider the function $f(x) = \frac{1}{x^p}$ where $0 < p \leq 1$. We may show that the limit as x tends towards infinity is indeed 0, and yet using our p-series test we know that the improper integral would not converge.

   $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x^p} = \frac{1}{\infty} = 0 \qquad \int_a^\infty f(x)dx = \int_1^\infty \frac{1}{x^p}dx = \infty \text{ (By } p\text{-series)}$$

   Note, for simplicity, we fix $a = 1$. Hence, we have shown that the function $f(x) = \frac{1}{x}$ satisfies $\lim_{x\to\infty} f(x) = 0$, yet the improper integral $\int_a^\infty f(x)dx$ diverges, demonstrating that the claim is false, as needed.

   (ii) If $g(x) \leq f(x) \leq 5g(x)$, then the improper integrals $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ either both converge or both diverge.

   (TRUE). To prove the claim above, we use the basic comparison test. In particular, there are two statements we must demonstrate hold under these assumptions.

   (Convergence Statement) $\int_a^\infty f(x)dx$ converges $\iff \int_a^\infty g(x)dx$ converges

   (Divergence Statement) $\int_a^\infty f(x)dx$ diverges $\iff \int_a^\infty g(x)dx$ diverges

   Therefore, we have the following.

Claim 1 $\int_a^\infty f(x)dx$ converges $\Longrightarrow \int_a^\infty g(x)dx$ converges

Notice that we have $0 \leq g(x) \leq f(x)$. Further, by assumption the improper integral $\int_a^\infty f(x)dx$ converges. Consequently, by the basic comparison test, it holds that $\int_a^\infty g(x)dx$ must also converge.

Claim 2 $\int_a^\infty f(x)dx$ converges $\Longleftarrow \int_a^\infty g(x)dx$ converges

Notice that we have $0 \leq f(x) \leq 5g(x)$. Further, by assumption the improper integral $\int_a^\infty g(x)dx$ converges. Noting that $\int_a^\infty g(x)dx$ converges, it holds that the improper integral $5\int_a^\infty g(x)dx = \int_a^\infty 5g(x)dx$ must also converge. Consequently, by the basic comparison test, it holds that $\int_a^\infty g(x)dx$ must also converge.

Claim 3 $\int_a^\infty f(x)dx$ diverges $\Longrightarrow \int_a^\infty g(x)dx$ diverges

Notice that we have $0 \leq f(x) \leq 5g(x)$. Further, by assumption the improper integral $\int_a^\infty f(x)dx$ diverges. Noting that $\int_a^\infty f(x)dx$ diverges, it holds by the basic comparison test that the improper integral $\int_a^\infty 5g(x)dx$ must also diverge. Noting that $5\int_a^\infty g(x)dx =$

$\int_a^\infty 5g(x)dx$, we remark that the divergence of the improper integral cannot be attributed to the scalar 5, rather, it must be that $\int_a^\infty g(x)dx$ diverges.

Claim 4 $\int_a^\infty f(x)dx$ converges $\Longleftarrow \int_a^\infty g(x)dx$ converges

Notice that we have $0 \leq g(x) \leq f(x)$. Further, by assumption the improper integral $\int_a^\infty g(x)dx$ diverges. Consequently, by the basic comparison test, it holds that $\int_a^\infty f(x)dx$ must also diverge.

Hence, we have shown that if $g(x) \leq f(x) \leq 5g(x)$, then the improper integrals $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ either both converge or both diverge, as needed.

(iii) If both $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ diverge, then $\int_a^\infty f(x)g(x)dx$ also diverges.

(FALSE). To disprove the claim, we will produce a counterexample utilizing the p-series integrals. Briefly, we recall that the following holds for p-series improper integrals. Here, we assume $a = 1$.

$$\int_1^\infty \frac{1}{x^p}dx = \begin{cases} \text{converges if} & p > 1 \\ \text{diverges if} & p \leq 1 \end{cases}$$

Fix $f(x) = \frac{1}{x^{p_1}}$ and $g(x) = \frac{1}{x^{p_2}}$, such that $p_1, p_2 \leq 1$ and $p_1 + p_2 > 1$. For example, we might consider $p_1 = \frac{3}{4} = p_2$. First, we remark that since $p_1, p_2 \leq 1$, the improper integrals $\int_1^\infty f(x)dx$ and $\int_1^\infty g(x)dx$ diverge (p-series). However, notice that we have the following.

$$f(x)g(x) = \frac{1}{x^{p_1}} \cdot \frac{1}{x^{p_2}} = x^{-p_1} + x^{-p_2} = x^{-(p_1+p_2)} = \frac{1}{x^{p_1+p_2}}$$

Since $p_1 + p_2 > 1$, it holds that the improper integral $\int_1^\infty f(x)g(x)dx$ converges (p-series). Under the example for p_1, p_2, we would obtain that $p_1 + p_2 = \frac{3}{2}$, and thus similarly, the improper integral converges.

2. Use the comparison test to determine the convergence/divergence of the following improper integrals.

(i) $\int_1^\infty \frac{1}{x+e^x}dx$

(CONVERGES). Consider that on the interval $[1,\infty)$, we have that $x \geq 0$. Consequently, this implies that on the same interval we have $x+e^x \geq e^x$. Using the reciprocal property of inequalities, we obtain the following.

$$x+e^x \geq e^x \Longrightarrow \frac{1}{x+e^x} \leq \frac{1}{e^x}$$

Thus, we may demonstrate that $\int_1^\infty \frac{1}{e^x}dx$ converges to deduce that $\int_1^\infty \frac{1}{x+e^x}dx$ also converges by using the basic comparison test.

$$\int_1^\infty \frac{1}{e^x}dx = \lim_{t\to\infty}\int_1^t \frac{1}{e^x}dx = \left(\lim_{t\to\infty} -\frac{1}{e^t}\right) - \left(-\frac{1}{e}\right) = 0 + \frac{1}{e} = \frac{1}{e}$$

Hence, the integral converges and thus by the basic comparison test we know that $\int_1^\infty \frac{1}{x+e^x}dx$ must also converge.

$$0 \leq \frac{1}{x+e^x} \leq \frac{1}{e^x},\ \int_1^\infty \frac{1}{e^x}dx \text{ converges} \underset{\text{BCT}}{\Longrightarrow} \int_1^\infty \frac{1}{x+e^x}dx \text{ converges}$$

(ii) $\int_2^\infty \frac{1}{x+e^{-x}}dx$

$$\text{For sufficiently large } x \quad \frac{1}{x+e^{-x}} \sim \frac{1}{x} \quad \text{which diverges.}$$

(DIVERGES). Consider that on the interval $[2,\infty)$, we have that $e^{-x} \leq 1$. Consequently, this implies that on the same interval we have $x+e^{-x} \leq x+1$. Using the reciprocal property of inequalities, we obtain the following.

$$x+e^{-x} \leq x+1 \Longrightarrow \frac{1}{x+e^{-x}} \geq \frac{1}{x+1}$$

Thus, we may demonstrate that $\int_2^\infty \frac{1}{x+1}dx$ diverges to deduce that $\int_2^\infty \frac{1}{x+e^{-x}}dx$ also diverges by the basic comparison test. Briefly, we use the substitution $u+1=x, du=dx$ for simplicity.

$$\int_2^\infty \frac{1}{x+1}dx = \int_1^\infty \frac{1}{u}du \underset{p\text{-series}}{\Longrightarrow} \int_1^\infty \frac{1}{u}du \text{ diverges} \underset{\text{BCT}}{\Longrightarrow} \int_2^\infty \frac{1}{x+e^{-x}}dx \text{ diverges}$$

(iii) $\int_3^\infty \frac{x+1}{x^3-3x^2+1}dx$

$$\text{For sufficiently large } x \quad \frac{x+1}{x^3-3x^2+1} \sim \frac{x}{x^3} = \frac{1}{x^2} \quad \text{which converges.}$$

(CONVERGES). Consider that on the interval $[3,\infty)$, we have that $x^3-3x^2+1 \geq (x-2)^3 = x^3-6x^2+12x-8$. This inequality can be obtained by showing that $1 \geq -3x^2+12x-8$. In tutorial, this was done using a graphical argument recognizing that $-3x^2+12x-8$ is a downwards parabola and that evaluating at $x=3$, which occurs after the peak of the parabola, yields and output of 1. Using the reciprocal property of inequalities and after multiplying by $x+1$, we obtain the following.

$$x^3-3x^2+1 \geq (x-2)^3 \implies \frac{1}{x^3-3x^2+1} \leq \frac{1}{(x-2)^3} \implies \frac{x+1}{x^3-3x^2+1} \leq \frac{x+1}{(x-2)^3}$$

Next, we utilize one last comparison to achieve the necessary claim of convergence. In particular, consider that on the interval $[3,\infty)$, we have that $x+1 \leq 4(x-2)$. Therefore, we have the following.

$$\frac{x+1}{x^3-3x^2+1} \leq \frac{x+1}{(x-2)^3} \leq \frac{4(x-2)}{(x-2)^3} = \frac{4}{(x-2)^2}$$

Now, we have obtained a function where we may quickly assess the convergence. Using the substitution $u=x-2, du=dx$, we see that the integral converges by our p-series test.

$$\int_3^\infty \frac{4}{(x-2)^2}dx = 4\int_1^\infty \frac{du}{u^2} \implies \text{Converges by } p\text{-series}$$

Consequently, it must be by the basic comparison test that the improper integral $\int_3^\infty \frac{x+1}{x^3-3x^2+1}dx$ converges, as needed.

$$\frac{x+1}{x^3-3x^2+1} \leq \frac{4}{(x-2)^2}, \quad \int_3^\infty \frac{4}{(x-2)^2}dx \text{ converges} \implies \int_3^\infty \frac{x+1}{x^3-3x^2+1}dx \text{ converges}$$

(iv) $\int_4^\infty \frac{\ln x}{x^2}dx$

(CONVERGES). To start, we show that on the interval $[4,\infty)$, we have that $\ln(x) \leq \sqrt{x}$. To verify this claim, we will demonstrate that the function $h(x)=\sqrt{x}-\ln(x)$ is positive and increasing. To ensure the function is positive, we evaluate $h(4)$.

$$h(4)=\sqrt{4}-\ln(4) = 2-2\ln(2) = 2(1-\ln(2)) > 0 \qquad (\ln(2) < \ln(e) = 1)$$

Next, we may compute the first derivative $h'(x)$ and verify whether $h'(x) \geq 0$ on our interval $[4,\infty)$, as this implies $h(x)$ would be increasing.

$$h'(x) = \frac{d}{dx}\left(\sqrt{x}-\ln x\right) = \frac{d}{dx}\sqrt{x} - \frac{d}{dx}\ln x = \frac{1}{\sqrt{x}} - \frac{1}{x} = \frac{\sqrt{x}-1}{x} > 0$$

Since $\sqrt{x} > 1$ on the interval $[4,\infty)$, it is indeed the case that $h(x)$ is increasing throughout across the interval. Hence, we may claim that $\ln(x) \leq \sqrt{x}$. Furthermore, by dividing both sides by $\frac{1}{x^2}$, we obtain $\frac{\ln(x)}{x^2} \leq \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}$. By the p-series test, we remark that $\frac{3}{2} > 1$ and so the integral must converge, further, by the basic comparison test we obtain that $\int_4^\infty \frac{\ln x}{x^2}dx$ converges, as needed.

3. For the following curves, compute their arc lengths.

(i) The curve $f(x) = \frac{2}{3}(x-1)^{\frac{3}{2}}$ from $x = 1$ to $x = 4$.

Notably, we are provided a definition of the curve in its explicit form. Before computing the arc-length, we must compute $f'(x)$ using the Chain Rule and Power Rule.

$$f'(x) = \frac{d}{dx}\left(\frac{2}{3}(x-1)^{\frac{3}{2}}\right) = \frac{2}{3} \cdot \frac{3}{2}(x-1)^{\frac{1}{2}} \cdot \frac{d}{dx}(x-1) = 1 \cdot (x-1)^{\frac{1}{2}} \cdot 1 = \sqrt{x-1}$$

Therefore, we have the following.

$$\begin{aligned}
\underset{[1,4]}{\text{Arc Length}}\, f &= \int_1^4 \sqrt{1+f'(x)^2}dx && \text{Arc-Length Formula} \\
&= \int_1^4 \sqrt{1+(\sqrt{x-1})^2}dx && \text{Since } f'(x) = \sqrt{x-1}. \\
&= \int_1^4 \sqrt{1+x-1}dx && \text{Simplifying.} \\
&= \int_1^4 \sqrt{x}dx && \text{Cancellation Property.} \\
&= \frac{2}{3}x^{\frac{3}{2}}\Bigg|_{x=1}^{x=4} && \text{Fundamental Theorem of Calculus.} \\
&= \frac{2}{3}(4^{\frac{3}{2}} - 1^{\frac{3}{2}}) && \text{Evaluating.} \\
&= \frac{14}{3} && \text{Simplifying.}
\end{aligned}$$

Hence, we have computed that the arc length of the curve $f(x) = \frac{2}{3}(x-1)^{\frac{3}{2}}$ from $x = 1$ to $x = 4$ is $\frac{14}{3}$, as needed.

(ii) The curve given by $x(t) = \cos^3(t)$ and $y(t) = \sin^3(t)$ from $t = 0$ to $t = \frac{\pi}{2}$.

To compute the arc-length of this curve, we utilize the following facts on trigonemtric functions and their derivatives.

$$\frac{d}{dx}\sin^2 x = 2\sin x \cos x \qquad \frac{d}{dx}\cos^2 x = -2\cos x \sin x$$

$$\frac{d}{dx}\sin^3 x = 3\sin^2 x \cos x \qquad \frac{d}{dx}\cos^3 x = -3\cos^2 x \sin x$$

Moreover, we fix $\gamma(t) = (x(t), y(t))$ to represent the curve. Therefore, we have the following.

$$\begin{aligned}
\underset{[0,\frac{\pi}{2}]}{\text{Arc Length}}\, \gamma &= \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{d}{dt}x(t)\right)^2 + \left(\frac{d}{dt}y(t)\right)^2}dt && \text{Arc-Length Formula} \\
&= \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{d}{dt}\cos^3(t)\right)^2 + \left(\frac{d}{dt}\sin^3(t)\right)^2}dt && \text{Substituting } x(t), y(t).
\end{aligned}$$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{2}} \sqrt{(-3\cos^2 t \sin t)^2 + \left(3\sin^2 t \cos t\right)^2} dt && \text{Derivative Observation.}\\
&= 3\int_0^{\frac{\pi}{2}} \sqrt{(\cos^2 t \sin t)^2 + \left(\sin^2 t \cos t\right)^2} dt && \text{Factoring/Simplifying.}\\
&= 3\int_0^{\frac{\pi}{2}} \sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t} dt && \text{Developing.}\\
&= 3\int_0^{\frac{\pi}{2}} \sqrt{(\cos^2 t \sin^2 t)(\cos^2 t + \sin^2 t)} dt && \text{Factoring.}\\
&= 3\int_0^{\frac{\pi}{2}} \sqrt{(\cos^2 t \sin^2 t)(1)} dt && \text{Since } \sin^2 t + \cos^2 t = 1.\\
&= 3\int_0^{\frac{\pi}{2}} \cos t \sin t dt && \text{Simplifying}\\
&= 3\left(\frac{1}{2}\sin^2(x)\Big|_{t=0}^{t=\frac{\pi}{2}}\right) && \text{Derivative Observation/FTC.}\\
&= 3\left(\frac{1}{2}\sin^2(\frac{\pi}{2}) - \frac{1}{2}\sin^2(0)\right) && \text{Evaluating.}\\
&= \frac{3}{2} && \text{Simplifying.}
\end{aligned}$$

Hence, we have computed that the arc-length along the curve $\gamma(t) = (\cos^3(t), \sin^3(t))$ from $t = 0$ to $t = \frac{\pi}{2}$ is $\frac{3}{2}$, as needed.