Quiz 3 (Sections 1.3, 1.4)

You will have 30 minutes to complete the quiz.

Name:
Student Number:

Q1 Consider the following matrices A and B.

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} -4 & 1 \\ 0 & 1 \end{bmatrix}$$

Compute the following quantities, where defined. (3 Points)

a. AB b. B^2 c. A^2

Q2 Let $C \in M_n(\mathbb{R})$.

a. In one sentence, describe the computation of the trace. (1 Point)

b. Prove or disprove the fact that $\mathrm{tr}(C) = \mathrm{tr}(C^T)$. (2 Points)

Q3 Determine whether the following statements are true or false. You do not need to justify your work. Here, A, B, C are matrices, and O is the zero matrix.

a. If $AB = O$, then $A = O$ or $B = O$. (0.5 Points)

b. If $AB = C$ and two of the matrices are square, then so is the third. (0.5 Points)

c. If AB and BA exists, then $AB = BA$. (0.5 Points)

d. If $AC = BC$, then $A = B$. (0.5 Points)

Q1

$$AB = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (1)(-4)+(0)(0) & (1)(1)+(0)(1) \\ (-2)(-4)+(1)(0) & (-2)(1)+(1)(1) \\ (1)(-4)+(3)(0) & (1)(1)+(3)(1) \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 8 & -1 \\ -4 & 4 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} -4 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (-4)(-4)+(1)(0) & (-4)(1)+(1)(1) \\ (0)(-4)+(1)(0) & (0)(1)+(1)(1) \end{bmatrix} = \begin{bmatrix} 16 & -3 \\ 0 & 1 \end{bmatrix}$$

$$A \text{ is not a square matrix} \implies A^2 = \text{D.N.E.}$$

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Q2

a. The trace is a computation defined for square matrices and is obtained by taking the sum of the entries along the main diagonal.

b. We want to show that if C is a square matrix then $\text{tr}(C) = \text{tr}(C^T)$.
Let C be a square matrix given by the following.

$$C = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

Fix $A = C, B = C^T$. Consider that the transpose of a matrix is given by swapping the rows and columns. That is, we must have $a_{ij} = b_{ji}$ for all $1 \leq i, j \leq n$.
Therefore, we have the following.

$$\begin{aligned} \text{tr}(C) &= \text{tr}(A) && \text{Since } C = A. \\ &= a_{11} + \cdots + a_{nn} && \text{Definition of trace.} \\ &= b_{11} + \cdots + a_{nn} && \text{Since } a_{ij} = b_{ji}. \\ &= \text{tr}(B) && \text{Definition of trace.} \\ &= \text{tr}(C^T) && \text{Since } B = C^T. \end{aligned}$$

Hence, we have that $\text{tr}(C) = \text{tr}(C^T)$, as needed.

Q3

a. **FALSE.**
Fix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Notably, $AB = 0$ and yet neither $A = 0$ nor $B = 0$.

b. **TRUE.**
Let A be a $n \times k$ matrix and B a $k \times m$ matrix. Then $AB = C$ is well-defined and C is a $n \times m$ matrix.

If A, B are square matrices, then $n = k, k = m \implies n = m$. So C is a square matrix. If A, C are square matrices, then $n = k, n = m \implies k = m$. So B is a square matrix. If B, C are square matrices, then $k = m, n = m \implies k = n$. So A is a square matrix.

c. **FALSE.**
Fix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Notably, AB is the 2×2 identity matrix whereas BA is a 3×3 matrix.

d. **FALSE.**
Let C be the zero matrix, and let A and B be distinct matrices of the same dimension. Notably, $AC = 0 = BC$ and yet $A \neq B$.