Review

Definition **(If Then Statements).** Let S_1, S_2 be the two statements. We say that $S_1 \implies S_2$, or equivalently if S_1 then S_2, when S_1 being true implies that S_2 must also be true.

Lemma **(Differentiation Sum Rule).** Let f, g be functions. Then the following holds.

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Lemma **(Differentiation Product Rule).** Let f, g be functions. Then the following holds.

$$\frac{d}{dx}(f(x)g(x)) = \left(\frac{d}{dx}f(x)\right)g(x) + f(x)\left(\frac{d}{dx}g(x)\right)$$

Lemma **(Differentiation Scalar Rule).** Let f be a function. Fix $c \in \mathbb{R}$ to be a scalar. Then the following holds.

$$\frac{d}{dx}(cf(x)) = c\left(\frac{d}{dx}f(x)\right)$$

Lemma **(Chain Rule).** Let f, g be functions. Then the following holds.

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$

Theorem **(Fundamental Theorem of Calculus).** Let F, f be functions. Fix $a \in \mathbb{R}$. Then the following holds.

$$F(x) = \int_a^x f(t)dt \implies F'(x) = f(x)$$

We may equivalently express the above as such.

$$f(x) = \frac{d}{dx}\int_a^x f(t)dt$$

Theorem **(Fundamental Theorem of Calculus II/FTC Chain Rule Extension).** Let F, f, u be functions. Fix $a \in \mathbb{R}$. Then the following holds.

$$F(x) = \int_a^{u(x)} f(t)dt \implies F'(x) = f(u(x))\frac{d}{dx}u(x)$$

Briefly, we remark that by fixing $u(x) = x$, as is the case in the original Fundamental Theorem of Calculus, we retrieve the theorem.

$$F(x) = \int_a^{u(x)} f(t)dt = \int_a^x f(t)dt \implies F'(x) = f(u(x))\frac{d}{dx}u(x) = f(x) \cdot 1 = f(x)$$

The aim of this theorem is to generalize the Fundamental Theorem of Calculus using the Chain Rule.

Questions

1. True or False. If true, justify why, and otherwise provide a counter-example.For each part, assume $F(x)$ is the antiderivative of $f(x)$ and $G(x)$ the antiderivative of $g(x)$.

   (i) If $F(x) = G(x)$, then $f(x) = g(x)$.

   TRUE. Assume that we have $F(x) = G(x)$. Therefore, we have the following.

$$\begin{aligned} f(x) &= \frac{d}{dx}F(x) && \text{By Assumption.} \\ &= \frac{d}{dx}G(x) && \text{Since } F(x) = G(x). \\ &= g(x) && \text{By Assumption.} \end{aligned}$$

   Hence, we have shown that if $F(x) = G(x)$ then $f(x) = g(x)$, as needed.

   (ii) If $f(x) = g(x)$, then $F(x) = G(x)$.

   FALSE. Assume that we have $f(x) = g(x)$. This implies that $\frac{d}{dx}F(x) = \frac{d}{dx}G(x)$. Let $F(x) = G(x) + C$, where $C \in \mathbb{R}$ is nonzero. Therefore, we have the following.

$$\begin{aligned} f(x) &= \frac{d}{dx}F(x) && \text{By Assumption.} \\ &= \frac{d}{dx}G(x) + C && \text{Since } F(x) = G(x) + C. \\ &= \frac{d}{dx}G(x) + \frac{d}{dx}C && \text{Sum property.} \\ &= \frac{d}{dx}G(x) + 0 && \text{Derivative of Constant.} \\ &= g(x) \end{aligned}$$

   Notice that we have $f(x) = g(x)$, however, it is not the case that $F(x) = G(x)$ since $C \neq 0$. Hence, we have shown that if $f(x) = g(x)$ then we do not necessarily have $F(x) = G(x)$, as needed.

   (iii) The function $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$.

   TRUE. Using the definition of antiderivative, we must show the following.

$$\frac{d}{dx}F(x) + G(x) = f(x) + g(x)$$

   This can be done using the sum property of derivatives.

$$\frac{d}{dx}F(x) + G(x) = \frac{d}{dx}F(x) + \frac{d}{dx}G(x) = f(x) + g(x)$$

   Hence, we have shown that the function $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$, as needed.

(iv) The function $F(x)G(x)$ is an antiderivative of $f(x)g(x)$.

FALSE. Using the definition of antiderivative, we must show the following.

$$\frac{d}{dx}F(x)G(x) \neq f(x)g(x)$$

For example, fixing $F(x) = x^4 \implies f(x) = 4x^3$ and $G(x) = x^3 \implies g(x) = 3x^2$, we obtain the following.

$$\frac{d}{dx}F(x)G(x) = \frac{d}{dx}(x^4 \cdot x^3) = \frac{d}{dx}x^7 = 7x^6 \neq 12x^5 = 4x^3 \cdot 3x^2 = f(x)g(x)$$

Hence, we have shown that the function $F(x)G(x)$ is an not necessarily an antiderivative of $f(x)g(x)$, as needed.

(v) The function $H(x) = \int_3^x tf(t)dt$ is an antiderivative of $xf(x)$.

TRUE. To show this, we may use the Fundamental Theorem of Calculus, which tells us that we must have $H'(x) = xf(x)$.

$$H(x) = \int_3^x tf(t)dt \implies \frac{d}{dx}H(x) = \frac{d}{dx}\int_3^x tf(t)dt = xf(x)$$

Hence, the function $H(x) = \int_3^x tf(t)dt$ is an antiderivative of $xf(x)$, as needed.

(vi) The function $I(x) = x\int_3^x f(t)dt$ is an antiderivative of $xf(x)$.

FALSE. To show this, we demonstrate that $I'(x) \neq xf(x)$, as this contradicts the definition of antiderivative. Therefore, we have the following.

$$\begin{aligned}
I(x) &= x\int_3^x f(t)dt && \text{By Assumption.} \\
\frac{d}{dx}I(x) &= \frac{d}{dx}x\int_3^x f(t)dt && \text{Applying Differential.} \\
&= \left(\frac{d}{dx}x\right)\int_3^x f(t)dt + x\left(\frac{d}{dx}\int_3^x f(t)dt\right) && \text{Product Rule.} \\
&= (1)\int_3^x f(t)dt + x\left(f(x)\right) && \text{Evaluating Derivative/FTC.} \\
&= \int_3^x f(t)dt + x\left(f(x)\right) && \text{Simplifying.} \\
&= F(x) + x\left(f(x)\right) && \text{Fundamental Theorem of Calculus.}
\end{aligned}$$

Notice that if we were to have $I'(x) = xf(x)$, then it must be that $F(x) = 0$, though for an arbitrary function this is not necessarily true.

Hence, we have shown that function $I(x) = x\int_3^x f(t)dt$ is not necessarily an antiderivative of $xf(x)$, as needed.

2. Consider the function $F(x) = \int_x^{x^2} \sin(\sqrt{t})dt$.

   (i) Find $F(1)$.

   Notice that when $x = 1$, we have that $x = 1 = 1^2 = x^2$. We know that if we integrate on an interval $[a, a]$, then the area is always 0 (see Tutorials 1-2). Thus, we must have that $F(1) = 0$.

   (ii) Find $F'(1)$ We first need to find the function F' before we can evaluate at the desired value. While we do not immediately have an integral that can be evaluated using the Fundamental Theorem of Calculus, we can manipulate the integral to obtain one that can.

$$
\begin{aligned}
F(x) &= \int_x^{x^2} \sin(\sqrt{t})dt && \text{By Assumption} \\
&= \int_x^{0} \sin(\sqrt{t})dt + \int_0^{x^2} \sin(\sqrt{t})dt && \text{Splitting the Integral.} \\
&= -\int_0^{x} \sin(\sqrt{t})dt + \int_0^{x^2} \sin(\sqrt{t})dt && \text{Flipping the Bounds.} \\
\frac{d}{dx}F(x) &= \frac{d}{dx}\left(-\int_0^{x} \sin(\sqrt{t})dt + \int_0^{x^2} \sin(\sqrt{t})dt\right) && \text{Applying Differential.} \\
&= -\frac{d}{dx}\int_0^{x} \sin(\sqrt{t})dt + \frac{d}{dx}\int_0^{x^2} \sin(\sqrt{t})dt && \text{Scalar Rule/Sum Rule.} \\
&= -\sin(\sqrt{x}) + \left(\frac{d}{dx}x^2\right) \cdot \sin(x) && \text{Fundamental Theorem of Calculus} \\
&= -\sin(\sqrt{x}) + 2x\sin(x) && \text{Evaluating Derivative.}
\end{aligned}
$$

   Therefore, we obtain that $F'(1) = -\sin(\sqrt{1}) + 2(1)\sin(1) = \sin(1)$.

   (iii) Is the function increasing or decreasing at $x = 1$.

   The function is increasing at $x = 1$ since $F'(1) > 0$.

3. Solve the following integration problems.

   (i) Compute the indefinite integral $\int \frac{x+1}{x^2} dx$.

   To determine the indefinite integral, we will utilize the following two facts.

$$(1) \quad \frac{d}{dx} \ln|x| = \frac{1}{x} \qquad (2) \quad \frac{d}{dx}\left(-\frac{1}{x}\right) = \frac{1}{x^2}$$

   Here, we will manipulate the original integral such that we can use our facts above.

$$\begin{aligned}
\int \frac{x+1}{x^2} dx &= \int \left(\frac{x}{x^2} + \frac{1}{x^2}\right) dx && \text{Splitting the Fraction.} \\
&= \int \left(\frac{1}{x} + \frac{1}{x^2}\right) dx && \text{Simplifying.} \\
&= \int \frac{1}{x} dx + \int \frac{1}{x^2} dx && \text{Separating the Integral.} \\
&= \ln|x| - \frac{1}{x} + C && \text{Using (1) and (2).}
\end{aligned}$$

   Hence, we have determined that $\int \frac{x+1}{x^2} dx = \ln|x| - \frac{1}{x} + C$, where C is a constant, as needed.

   (ii) Compute the definite integral $\int_1^2 \frac{x+1}{x^2+2x} dx$.

   To solve this integral, we will utilize a concept known as u-substitution. Here, we make the following substitution.

$$\begin{aligned}
u &= x^2 + 2x \\
du &= (2x+2)dx
\end{aligned}$$

   Note that we must also change the bounds of integration. This can be done by evaluating x^2+2x at our given bounds. Thus, our upper bound becomes $(2)^2 + 2(2) = 8$ and our lower bound becomes $(1)^2 + 2(1) = 3$. Therefore, we have the following.

$$\begin{aligned}
\int_1^2 \frac{x+1}{x^2+2x} dx &= \int_1^2 \frac{1}{2(x^2+2x)}(2x+2)dx && \text{Multiplying by } \frac{2}{2}. \\
&= \int_3^8 \frac{1}{2(u)} du && \text{Substituting } u = x^2+2x. \\
&= \frac{1}{2} \int_3^8 \frac{1}{u} du && \text{Factoring } \frac{1}{2}. \\
&= \frac{1}{2}\left(\ln|u| \Big|_{x=3}^{x=8} \right) && \text{Using Antiderivative.} \\
&= \frac{1}{2}(\ln(8) - \ln(3)) && \text{Evaluating.} \\
&= \frac{1}{2} \ln\left(\frac{8}{3}\right) && \text{Logarithm Quotient Rule.}
\end{aligned}$$

ALTERNATIVE METHOD.
Later in our computations, we will need to use partial fraction decomposition. This method can be used to rewrite rationals as sums. In our case, we want to determine A, B that satisfy the following.

$$\frac{1}{x^2+2x} = \frac{1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$$

Briefly, we obtain the following, which yields $A = 1/2, B = -1/2$.

$$\frac{1}{x(x+2)} = \frac{(A+B)x+2A}{x(x+2)} \Longrightarrow A = \frac{1}{2}, B = -\frac{1}{2} \Longrightarrow \frac{1}{x^2+2x} = \frac{1}{2x} - \frac{1}{2(x+2)}$$

Next, we compute the necessary definite integral.

$$\begin{aligned}
\int_1^2 \frac{x+1}{x^2+2x}dx &= \int_1^2 \left(\frac{x}{x^2+2x} + \frac{1}{x^2+2x}\right) dx && \text{Splitting the Fraction.} \\
&= \int_1^2 \left(\frac{1}{x+2} + \frac{1}{x^2+2x}\right) dx && \text{Cancellation Property.} \\
&= \int_1^2 \frac{1}{x+2}dx + \int_1^2 \frac{1}{x^2+2x}dx && \text{Separating the Integral.} \\
&= \int_1^2 \frac{1}{x+2}dx + \int_1^2 \frac{1}{2x} - \frac{1}{2(x+2)}dx && \text{Partial Fraction Decomposition.} \\
&= \int_1^2 \frac{1}{x+2}dx + \frac{1}{2}\int_1^2 \frac{1}{x} - \frac{1}{x+2}dx && \text{Factoring.} \\
&= \int_1^2 \frac{1}{x+2}dx + \frac{1}{2}\int_1^2 \frac{1}{x}dx - \frac{1}{2}\int_1^2 \frac{1}{x+2}dx && \text{Splitting the Integral.} \\
&= \frac{1}{2}\int_1^2 \frac{1}{x+2}dx + \frac{1}{2}\int_1^2 \frac{1}{x}dx && \text{Simplifying.} \\
&= \frac{1}{2}\ln|x+2|\,\Big|_{x=1}^{x=2} + \frac{1}{2}\ln|x|\,\Big|_{x=1}^{x=2} && \text{Taking Anti-derivatives.} \\
&= \frac{1}{2}(\ln(4) - \ln(3) + \ln(2) - \ln(1)) && \text{Factoring/Evaluating.} \\
&= \frac{1}{2}\ln\frac{8}{3} && \text{Simplifying.}
\end{aligned}$$

Hence, we have computed that $\int_1^2 \frac{x+1}{x^2+2x}dx = \frac{1}{2}\ln\frac{8}{3}$, as needed.

(iii) Solve the initial value problem on the interval $x > -2$.

$$\frac{dy}{dx} = \frac{x+1}{x+2}, \quad y(1) = 1$$

To solve this initial value problem, we must first determine the antiderivative of $\frac{x+1}{x+2}$. Therefore, we have the following.

$$y = \int \frac{x+1}{x+2}dx = \int \frac{u-1}{u}du \qquad \text{Substituting } u = x+2.$$

$$
\begin{aligned}
&= \int \left(1 - \frac{1}{u}\right) du && \text{Splitting the Fraction.} \\
&= \int 1 du - \int \frac{1}{u} du && \text{Separating the Integral.} \\
&= u - \ln|u| + C && \text{Evaluating Integrals.} \\
&= x + 2 - \ln|x+2| + C && \text{Substituting } u = x + 2.
\end{aligned}
$$

Lastly, now that we have determined that $y = x+2-\ln|x+2|+C$, we may utilize the provided value to find a particular solution.

$$1 = y(1) \implies 1 = (1) + 2 - \ln|(1)+2| + C = 3 - \ln 3 + C \implies -2 + \ln 3 = C$$

Hence, we have determined that the function $y = x - \ln|x+2| + \ln 3$ satisfies the initial value problem $\frac{dy}{dx} = \frac{x+1}{x+2}$ such that $y(1) = 1$, as needed.

BONUS. Compute the indefinite integral $\int x\sqrt{x+1}dx$.

Sometimes in integration, you will come across functions that may appear more complex and therefore more difficult to approach. Often, it may prove useful to make certain substitutions to simplify our computations. However, we must remain aware of the substitutions we make throughout are computations to i) substitute back to the original variable and ii) to ensure we maintain the appropriate differential operator. We illustrate this technique through the following problem, in particular, we will use a substitution on the square root term.

Fix $u = \sqrt{x+1}$. This implies that we have $u^2 = x+1 \implies u^2 - 1 = x$. When we make a substitution on the variable x, we must also replace the differential operator dx. Observing that $x = u^2 - 1$, this implies that $dx = 2udu$. Making the necessary substitution, we obtain the following integral.

$$\int x\sqrt{x+1}dx = \int (u^2-1)u \cdot 2udu = \int 2u^4 - 2u^2 du$$

Now, our integral only consists of polynomial terms which are more simple to integrate.

$$\begin{aligned}
\int x\sqrt{x+1}dx &= \int (u^2-1)u \cdot 2udu && \text{Substituting } u = \sqrt{x+1}. \\
&= \int 2u^4 - 2u^2 du && \text{Developing.} \\
&= \int 2u^4 du - \int 2u^2 du && \text{Separating the Integral.} \\
&= \frac{2}{5}u^5 - \frac{2}{3}u^3 + C && \text{Inverse Power Rule.} \\
&= \frac{2}{5}(\sqrt{x+1})^5 - \frac{2}{3}(\sqrt{x+1})^3 + C && \text{Substituting } u = \sqrt{x+1}. \\
&= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C && \text{Simplifying.}
\end{aligned}$$

Hence, we have computed that the indefinite integral $\int x\sqrt{x+1}dx$ equals $\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C$, where C is a constant, as needed.