Review

***Theorem* (Fundamental Theorem of Calculus).** Let F, f be functions. Fix $a \in \mathbb{R}$. If f is continuous on $[a, b]$, then the following holds.

$$F(x) = \int_a^x f(t)dt \Longrightarrow F'(x) = f(x)$$

We may equivalently express the above as such.

$$f(x) = \frac{d}{dx}\int_a^x f(t)dt$$

***Theorem* (Integration by Parts).** Let u, v be functions. Then we have the following.

$$\int udv = uv - \int vdu$$

Questions

1. True or False.

   (i) We can use the fundamental theorem of calculus to conclude that the following holds.

$$\int_{-1}^{1} \frac{1}{x^2} dx = \frac{1}{x}\Bigg|_{-1}^{1} = -2$$

FALSE. For starters, let us observe a plot of the function $\frac{1}{x^2}$. Notice that for all nonzero $x \in \mathbb{R}$, $\frac{1}{x^2} > 0$. Consequently, from a geometric perspective we can reason that the signed area should be a positive number, which is not the case here.

More formally, however, the fundamental theorem of calculus requires our function f to be continuous on the interval of integration. Notably, $\frac{1}{x^2}$ has a vertical asymptote at $x = 0$, which implies the function is not continuous on $[-1, 1]$.

   (ii) Let f be a function, such that $f(x)$ is defined for $x > 0$. Let $a, b > 0$. Assume that f is integrable on $[a, b]$.

   * If $\int_a^\infty f(x)dx$ converges, then $\int_b^\infty f(x)dx$ converges.
   * If $\int_a^\infty f(x)dx$ diverges, then $\int_b^\infty f(x)dx$ diverges.

TRUE. Recall that for real numbers a, b, c we have the following.

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

Similarly, we have the following relationship between $\int_a^\infty f(x)dx$ and $\int_b^\infty f(x)dx$.

$$\int_a^\infty f(x)dx = \int_a^b f(x)dx + \int_b^\infty f(x)dx$$

Since f is integrable on $[a, b]$, we know that $\int_a^b f(x)dx$ evaluates to some real number R.

Firstly, if $\int_a^\infty f(x)dx$ converges, then we can say that $\int_a^\infty f(x)dx$ evaluates to some real number S. This would imply that $\int_b^\infty f(x)dx = S - R$, which is a real number. Equivalently, this means that $\int_b^\infty f(x)dx$ must converge.

$$S = \int_a^\infty f(x)dx = \int_a^b f(x)dx + \int_b^\infty f(x)dx = R + \int_b^\infty f(x)dx \Longrightarrow S - R = \int_b^\infty f(x)dx$$

Secondly, if $\int_a^\infty f(x)dx$ diverges, then we can say that $\int_a^b f(x)dx + \int_b^\infty f(x)dx$ also diverges. However, we know that the integral $\int_a^b f(x)dx$ evaluates to R. Consequently, it must be that $\int_b^\infty f(x)d(x)$ diverges for the expression $\int_a^b f(x)dx + \int_b^\infty f(x)dx$ to diverge.

2. Compute the following indefinite integral, $\int e^x \cos x dx$ using recursive integration by parts.

Theorem (Integration by Parts)**.** Let u, v be functions. Then we have the following.

$$\int u dv = uv - \int v du$$

FIRST ITERATION.
Fix $u = \cos x, v = e^x$. This implies that $du = -\sin x dx, dv = e^x dx$. Using integration by parts, we have the following.

$$\int e^x \cos x dx = \int u dv = uv - \int v du = e^x \cos x - \int (-e^x \sin x) dx = e^x \cos x + \int e^x \sin x dx$$

SECOND ITERATION.
Fix $u = \sin x, v = e^x$. This implies that $du = \cos x dx, dv = e^x dx$. Using integration by parts, we have the following.

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx$$

Notice that we have re-obtained our initial integral of interest, $\int e^x \cos x dx$. Therefore, we have the following.

$$\begin{aligned}
\int e^x \cos x dx &= e^x \cos x + \int e^x \sin x dx && \text{Integration by Parts.} \\
&= e^x \cos x + \left(e^x \sin x - \int e^x \cos x dx \right) && \text{Integration by Parts.} \\
2 \int e^x \cos x dx &= e^x \cos x + e^x \sin x && \text{Cancellation Property.} \\
\int e^x \cos x dx &= \frac{1}{2} e^x (\cos x + \sin x) && \text{Multiplicative Inverse.}
\end{aligned}$$

Hence, we have determined using integration by parts that the indefinite integral is given by $\int e^x \cos x dx = \frac{1}{2} e^x (\cos x + \sin x)$, as needed.

3. Consider the following limit.

$$\lim_{n\to\infty}\left(\sum_{k=0}^{n-1}\frac{e^{-(4+k/n)}}{n}\right)$$

(i) Find a definite integral that represents the limit.

Recall the definition of definite integral given by the left Riemann sum.

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=0}^{n-1} f(x_k)\Delta x_k = \lim_{n\to\infty}\sum_{k=0}^{n-1} f\left(a+\frac{k}{n}\right)\frac{b-a}{n}$$

Notice that we may express the limit as follows.

$$\lim_{n\to\infty}\left(\sum_{k=0}^{n-1}\frac{e^{-(4+k/n)}}{n}\right) = \lim_{n\to\infty}\left(\sum_{k=0}^{n-1} e^{-4-(0+k/n)}\frac{1-0}{n}\right)$$

Therefore, by fixing $f(x) = e^{-4-x}, a = 0, b = 1$, we obtain a definite integral to present the limit.

$$\int_0^1 e^{-4-x}dx = \int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=0}^{n-1} f\left(a+\frac{k}{n}\right)\frac{b-a}{n} = \lim_{n\to\infty}\left(\sum_{k=0}^{n-1}\frac{e^{-(4+k/n)}}{n}\right)$$

Notably, we may also fix $f(x) = e^{-x}, a = 4, b = 5$.

(ii) Evaluate the integral.

$$\int_0^1 e^{-4-x}dx = -e^{-4-1} - (-e^{-4-0}) = -e^{-5} + e^{-4}$$