Quiz 5 (Sections 2.3)

You will have 30 minutes to complete the quiz.

Name:
Student Number:

Q1 (a) What are the two essential properties that make a transformation linear? (1 Point)

(b) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that for any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ and scalars $r, s \in \mathbb{R}$, we have $T(r\vec{u} + s\vec{v}) = rT(\vec{u}) + sT(\vec{v})$. (2 Points)

Q2 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that the folowing holds.

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad T(\vec{e}_2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad T\left(2\vec{e}_1 + 3\vec{e}_2 + \vec{e}_3\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Find the standard matrix representation of the transformation T. (3 Points)
HINT: Consider the domain and codomain to ensure you have the correct matrix dimensions.

Q3 Determine whether the following statements are true or false. You do not need to justify your work. (2 Points)

(a) If $S, T : \mathbb{R}^n \to \mathbb{R}^n$ are linear transformations, then $T \circ S(\vec{x}) = S \circ T(\vec{x})$. (0.5 Points)

(b) There exists a linear transformation T such that $\operatorname{Im}(T)$ cannot be written as a span of vectors. (0.5 Points)

(c) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is invertible if and only if $n = m$. (0.5 Points)

(d) For any subspace $S \subseteq \mathbb{R}^n$ and linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, the set $T(S) = \{T(\vec{x}) \in \mathbb{R}^m | \vec{x} \in S\}$ is a subspace of $\mathbb{R}^m$. (0.5 Points)

Q1

(a) A transformation T is linear if it satisfies the following for all vectors $\vec{u}, \vec{v}$ in the domain of T and scalar $r \in \mathbb{R}$.

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
2. $T(r\vec{u}) = rT(\vec{u})$

(b) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $r, s \in \mathbb{R}$.
To prove our claim, we will use the definition proposition above. Therefore, we have the following.

$$\begin{aligned} T(r\vec{v} + s\vec{u}) &= T(r\vec{v}) + T(s\vec{u}) && \text{Definition Prop. 1.} \\ &= rT(\vec{v}) + sT(\vec{u}) && \text{Definition Prop. 2.} \end{aligned}$$

Hence, we have obtained that $T(r\vec{v} + s\vec{u}) = rT(\vec{v}) + sT(\vec{u})$, as needed.

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Q2

Fix A to be the standard matrix representation of T. Recall that the standard matrix representation of a linear transformation is the matrix which induces the linear transformation using the vectors of the standard basis (i.e., $\vec{e}_1, ..., \vec{e}_n$). Consequently, we have the following.

$$A = \begin{bmatrix} | & | & | \\ T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \\ | & | & | \end{bmatrix}$$

By assumption, we already know $T(\vec{e}_1)$ and $T(\vec{e}_2)$ which provides us the first two columns of A. Thus, we must determine $T(\vec{e}_3)$ to complete the matrix A. Here, we will use the linearity properties of the transformation T to isolate $T(\vec{e}_3)$ in the expression for $T(2\vec{e}_1 + 3\vec{e}_2 + \vec{e}_3)$ provided by assumption. Therefore, we have the following.

$$\begin{aligned}
\begin{bmatrix} 3 \\ 2 \end{bmatrix} &= T(2\vec{e}_1 + 3\vec{e}_2 + \vec{e}_3) && \text{By Assumption.} \\
&= T(2\vec{e}_1) + T(3\vec{e}_2) + T(\vec{e}_3) && \text{Linear Transformation Prop. 1} \\
&= 2T(\vec{e}_1) + 3T(\vec{e}_2) + T(\vec{e}_3) && \text{Linear Transformation Prop. 2} \\
&= 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -1 \end{bmatrix} + T(\vec{e}_3) && \text{By Assumption.} \\
&= \begin{bmatrix} 5 \\ -1 \end{bmatrix} + T(\vec{e}_3) && \text{Linear Transformation Prop. 1} \\
\begin{bmatrix} -2 \\ 3 \end{bmatrix} &= T(\vec{e}_3) && \text{Additive Inverse.}
\end{aligned}$$

Hence, we have computed $T(\vec{e}_3)$, as needed. Putting it all together, we know that the standard matrix representation of T is given by the following matrix A.

$$A = \begin{bmatrix} | & | & | \\ T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 3 \end{bmatrix}$$

Q3

(a) **TRUE** ***FALSE***

Let $S, T : \mathbb{R}^n \to \mathbb{R}^n$ be linear transformations. Fix A, B to be the matrices that induce S, T, respectively. Notice that in general, we have $T \circ S(\vec{x}) = BA\vec{x} \neq BA = S \circ T(\vec{x})$ since most matrices do not commute (i.e., $AB \neq BA$). Hence, FALSE.

(b) **TRUE** ***FALSE***

Let T be a linear transformation. Fix A to be the matrix that induces T. We know that $\mathrm{Im}(T) = \mathrm{Col}(A)$. Moreover, we know that the column space of any matrix is necessarily a subspace. Consequently, it is always the case that $\mathrm{Im}(T)$ is a subspace. Hence, FALSE.

(c) **TRUE** ***FALSE***

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. While it is the case that if T is invertible then $n = m$, the converse is not necessarily true. For example, consider the transformation induced by the $n \times m$ zero matrix (where $n = m$). Consequently, the converse implication does not hold so the statement cannot be if and only if. Hence, FALSE.

(d) ***TRUE*** **FALSE**

Let $S \subseteq \mathbb{R}^n$ be a subspace and $T : \mathbb{R}^n \to \mathbb{R}^m$ a linear transformation. We can show that $T(S)$ is a subspace by proving it satisfies the definition of a subspace (i.e., closed under addition and scalar multiplication). Consider $\vec{x}_1, \vec{x}_2 \in S$ and $r \in \mathbb{R}$. Since S is a subspace, we know that by definition $\vec{x}_1 + \vec{x}_2 \in S$. Therefore, $T(\vec{x}_1) + T(\vec{x}_2) = T(\vec{x}_1 + \vec{x}_2) \in T(S)$ as $\vec{x}_1 + \vec{x}_2 \in S$. Similarly, since S is a subspace we know that by definition $r\vec{x}_1 \in S$. Therefore, $rT(\vec{x}_1) = T(r\vec{x}_1) \in T(S)$ as $r\vec{x}_1 \in S$. From the above, we see that $T(S)$ is both closed under addition and scalar multiplication, and thus must be a subspace of $\mathbb{R}^m$. Hence, TRUE.