RECIPROCALS AND RATIONALS

1 Rationals

Key Terms

Polynomial: A polynomial is a specific type of function which takes the following form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where a_i represents the i-th coefficient. What we see is that a polynomial consist exclusively of terms comprised of a coefficient and some power of x.
Ex: $x - 1$, $x^3 + 1$, 2, $x^2 + x + 1$

Degree: The degree (∂) of a polynomial is the number of the largest exponent in the polynomial. **Note:** polynomials may not always be sorted from largest to smallest degree, even if we often do this by convention.

Leading Coefficient: The leading coefficient (L.C.) is the coefficient of the term of largest degree in our polynomial.

Findings Vertical Asymptotes, Holes, Horizontal Asymptotes, and Oblique Asymptotes (Section 5.3)

In general, we may use the following checklist to assess what types of asymptotes our rational function may have, as well as any holes. We then demonstrate how to use this approach to find this information for rational functions.

1. State the restrictions.

   To find the restrictions, we determine for which values of x we have $q(x) = 0$. This requires factoring $q(x)$.

2. Is the rational function in the most simplified form?

   Yes. The restrictions are the **vertical asymptotes**.

   No. Simplify to the simplest form. If any of the restrictions disappear from being cancelled out, then these restrictions represent **holes**. The remaining restrictions are **vertical asymptotes**.

   Note: to determine if we can simplify any further, we must factor $p(x)$ and $q(x)$. Further, if the denominator in the simplified form does not depend on x, then there are no horizontal nor oblique asymptotes (we may roughly think of this as all the x's get cancelled out).

3. Determine the degree of $p(x)$ and $q(x)$. Here, we use ∂ to denote the degree of a polynomial.

$\partial q(x) > \partial p(x)$: When the degree of the polynomial in the denominator is larger, we have a **horizontal asymptote** at $y = 0$.

$\partial q(x) = \partial p(x)$: When the degree of the polynomials are the same, we have a **horizontal asymptote** at $y = \frac{\text{L.C.}(p(x))}{\text{L.C.}(q(x))}$.

$\partial q(x) < \partial p(x)$: When the degree of the polynomial in the denominator is larger, we have an **oblique asymptote**. Recall that we can long divide polynomials where $\partial q(x) < \partial p(x)$ into the following form $p(x) = f(x)q(x) + r(x)$, where $r(x)$ is the remainder polynomial. The oblique asymptote is then given by $f(x)$.

Examples

For each of the following functions, determine all asymptotes and holes.

a) $f(x) = \frac{3x-6}{x-2}$

(a) We need to find the restrictions. Here, $q(x) = x - 2$ is already factored, so we solve $x - 2 = 0$. This gives us the restriction $x \neq 2$.

(b) Next, we ask ourselves is our function is in the most simplified form. We notice that there is a GCF in the numerator, so we decide to factor and observe whether there any simplifications.

$$\frac{3x-6}{x-2} = \frac{3(x-2)}{(x-2)} = 3$$

Here, we note that our restriction from $x - 2 = 0$ cancelled out in the simplification above. So that means, we have a hole at $x = 2$.

Also, since the denominator in the simplified form does not depend on x, there are no other asymptotes.

(c) Not needed, no horizontal or oblique asymptotes.

So to summarize, $f(x)$ has no asymptotes but a hole at $x = 2$.

b) $f(x) = \frac{2x+5}{x-6}$

(a) We need to find the restrictions. Here, $q(x) = x - 6$ is already factored, so we solve $x - 6 = 0$. This gives us the restriction $x \neq 6$.

(b) Next, we ask ourselves is our function is in the most simplified form. We notice that there is a GCF in the numerator, so we decide to factor and observe whether there any simplifications.

$$\frac{2x-5}{x-6} = \frac{2(x-\frac{5}{2})}{(x-6)} \Longrightarrow \text{Simplified}$$

Here, we note that our restriction from $x - 6 = 0$ does not cancel out in the simplification above. So that means, we have a vertical asymptote at $x = 6$.

Also, since the denominator in the simplified form does depend on x, there can either be a horizontal or oblique asymptote.

(c) Here, we have two polynomials of degree 1 ($\partial p(x) = 1, \partial q(x) = 1$). Since they are of the same degree, there is a horizontal asymptotes and we must look at the quotient of their leading coefficients.

$$p(x) = 2x - 5 \Longrightarrow \text{L.C.}(p(x)) = 2 \quad q(x) = x - 6 \Longrightarrow \text{L.C.}(q(x)) = 1$$

So, there is a horizontal asymptote at $y = \frac{2}{1} = 2$.

So to summarize, $f(x)$ has a vertical asymptote at $x = 6$ and a horizontal asymptote at $y = 2$.

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