Some Total Positivity Problems in Combinatorics

MPhil to PhD Upgrade Report

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Funded by the Department of Mathematics, University College London

Started working on November 25, 2020
Last updated on December 7, 2021

Abstract

A matrix of real numbers is said to be totally positive if all its minors are non-negative and a matrix of polynomials is said to be coefficientwise-totally positive if all its minors are polynomials with non-negative coefficients. This report consists of two parts: the first part is a literature review and the second part is a description of some ongoing projects. In Part I, we provide a brief introduction to the theory of total positivity and look at some of its connections to combinatorics. We then provide brief descriptions of four ongoing projects in Part II and mention our partial progress in each of the projects. The first project is an attempt at solving Brenti's conjecture from 1996 regarding the total positivity of the Eulerian triangle. The second project is an attempt to answer three total positivity questions for a one parameter infinite family of matrices whose first matrix is the well studied Stirling cycle triangle. The third project aims at obtaining necessary and sufficient conditions under which the Generalised Lah triangle studied by Pétréolle and Sokal (2021) is totally positive. Sufficient conditions were obtained by Pétréolle and Sokal but they were far from necessary. This project is an exercise in experimental mathematics aided by powerful computers. The fourth project aims at providing a bijective picture to the total positivity of the trees and forests matrices whose total positivity was recently proved independently by Sokal, who used the theory of Riordan arrays, and Gilmore, who used the Lindström-Gessel-Viennot lemma.

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Introduction

1.1 Enumerative Combinatorics

A *combinatorial class* or a *combinatorial family of objects* is a class of sets $\mathcal{S} = \{S_i\}_{i \in I}$ where I is an indexing set, and each set S_i is finite. (Most often $I = \mathbb{N}$.) *Enumerative combinatorics* is the branch of mathematics that deals with exact enumeration of combinatorial families, i.e., given a combinatorial class $\mathcal{S} = \{S_i\}_{i \in I}$, we are interested in computing $|S_i|$. Some of the textbooks for studying enumerative combinatorics are Comtet [1974], Stanley [2009], Stanley and Fomin [1999], Flajolet and Sedgewick [2009].

When $I = \mathbb{N}$, we may have several different ways to enumerate $\mathcal{S}$. Some of the common methods of enumeration are:

- We may provide an explicit formula for $|S_i|$ in terms of i.
- We may provide a recurrence relation for $|S_i|$.
- We may provide a description of the formal power series

  $$\sum_{i=0}^{\infty} |S_i| t^i$$

  as a solution to some polynomial equation in t, or as a solution to a differential equation or a system of differential equations.

1.2 Totally Positive Matrices

Positivity problems form a central theme in enumerative combinatorics (see for example Stanley [1999]). The primary idea, which will be a recurring theme in this report and my PhD, is that an integer can be proven to be non-negative if it is the cardinality of a set. In particular, we focus on total positivity problems in combinatorics.

A matrix of real numbers is said to be *totally positive* (TP in short) if all its minors are non-negative. This matrix need not be a square matrix or even finite. We can have an infinite number of rows and columns because, all we need is the determinant of all finite square submatrices to be non-negative. We say that a matrix is *totally positive of order r* (TP_r in short) if all its minors of size $\leq r$ are non-negative. We say that a matrix is *strictly totally positive* (STP in short) if all its minors are strictly positive and *strictly totally positive of order* r (STP_r) if all its minors of size $\leq r$ are strictly positive.

We would like to warn the reader that there could be some confusion in terminology while looking at the relevant literature from different sources. In the sources such as Gantmacher and Krein [2002], Fallat and Johnson [2011] the term total nonnegativity and total positivity is used to refer to what we call total positivity and strict total positivity. However, we follow the terminology in sources such as Karlin [1968], Pinkus [2009].

1.2.1 Historical Remark:

The study of total positivity was started independently in the 1930s by two different schools: Schoenberg and his school while studying the distribution of roots of polynomials in the complex plane, and Krein and his school while studying problems in mechanics. See the Foreword and the Remarks at the end of each chapter of Pinkus [2009] for more detailed historical notes on the theory of total positivity.

1.2.2 Few examples of Totally Positive Matrices

In this report, rows and columns of infinite size will be indexed from 0, and finite rows and columns will be indexed from 1.

Notice that the 1-minors of a TP matrix are non-negative and hence all its entries are non-negative. Certainly *zero matrices* are TP and so are matrices all of whose entries are equal to a *non-negative constant. Diagonal matrices* with non-negative diagonal entries constitute another family of TP matrices.

For $\boldsymbol{d} = (d_i)_{i\geq 0}$, let $D(\boldsymbol{d})$ denote the diagonal matrix with entries $d_0, d_1, d_2, \ldots$.

$$D(\boldsymbol{d}) = \begin{bmatrix} d_0 & 0 & 0 & 0 & \cdots \\ 0 & d_1 & 0 & 0 & \cdots \\ 0 & 0 & d_2 & 0 & \cdots \\ 0 & 0 & 0 & d_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Bidiagonal matrices are our next example. These are matrices with all entries 0, other than the entries in two consecutive diagonals. For example,

$$A = \begin{bmatrix} 0 & 0 & a_1 & b_1 & 0 & 0 \\ 0 & 0 & 0 & a_2 & b_2 & 0 \\ 0 & 0 & 0 & 0 & a_3 & b_3 \\ 0 & 0 & 0 & 0 & 0 & a_3 \end{bmatrix}$$

is an example of a bidiagonal matrix.

If all the entries are non-negative in a bidiagonal matrix, then it is totally positive. This is because, any square submatrix is either upper triangular or lower triangular and hence the corresponding minor is the product of its diagonal entries all of which are non-negative. Thus, in this example, A will be totally positive when $a_i \geq 0$ and $b_i \geq 0$.

Let us introduce some notation for bidiagonal matrices as they are very useful for us. By a *lower bidiagonal matrix*, we shall refer to a lower-triangular bidiagonal matrix whose non-zero entries are only allowed to be on the diagonal or the subdiagonal, and similarly, by an *upper bidiagonal matrix* we shall refer to an upper-triangular bidiagonal matrix whose non-zero entries are only allowed to be on the diagonal or the superdiagonal. For $\boldsymbol{d} = (d_i)_{i\geq 0}$ and $\boldsymbol{x} = (x_i)_{i\geq 1}$, let $L(\boldsymbol{d}, \boldsymbol{x})$ be the lower bidiagonal matrix with entries $d_0, d_1, d_2, \ldots$ on the diagonal and entries $x_1, x_2, \ldots$ on the subdiagonal and zero elsewhere, i.e.,

$$L(\boldsymbol{d}, \boldsymbol{x}) = \begin{bmatrix} d_0 & 0 & 0 & 0 & \ldots \\ x_1 & d_1 & 0 & 0 & \ldots \\ 0 & x_2 & d_2 & 0 & \ldots \\ 0 & 0 & x_3 & d_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Similarly, let $U(\boldsymbol{d}, \boldsymbol{x})$ be the upper bidiagonal matrix with entries $d_0, d_1, d_2, \ldots$ on the diagonal and entries $x_1, x_2, \ldots$ on the superdiagonal and zero elsewhere, i.e.,

$$U(\boldsymbol{d}, \boldsymbol{x}) = \begin{bmatrix} d_0 & x_1 & 0 & 0 & \ldots \\ 0 & d_1 & x_2 & 0 & \ldots \\ 0 & 0 & d_2 & x_3 & \ldots \\ 0 & 0 & 0 & d_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We abbreviate $L(\mathbf{1}, \boldsymbol{x})$ as $L(\boldsymbol{x})$ and $U(\mathbf{1}, \boldsymbol{x})$ as $U(\boldsymbol{x})$.

We shall soon look at a few examples which are more interesting. But before doing that we will look at some operations that preserve TP, thus obtaining recipes to construct new TP matrices out of existing TP matrices.

1.2.3 Operations that Preserve Total Positivity

We first state a few properties of TP matrices without proof which aren't too difficult to verify:

1. Submatrix of a TP matrix is TP.
2. Inserting rows or columns of zeros in a TP matrix preserves TP.
3. Duplicating a row or a column and adding it right next to the original row or column preserves TP.
4. Transpose of a TP matrix is TP.
5. Let A and B be both TP matrices such that A is a finite matrix. Then the block matrix

   $$C = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]$$

   is TP. This is because any square submatrix C' of C has the block matrix form

   $$C' = \left[\begin{array}{c|c} A' & 0 \\ \hline 0 & B' \end{array}\right]$$

where A' is a submatrix of A and B' is a submatrix of B. (It may happen that A' or B', or both A' and B' have no rows or columns.) It is clear that C' has a non-negative determinant as A' and B' both have non-zero determinant.

Our next example, is slightly more involved.

Theorem 1.2.3.1. *Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix. Then AB is TP.*

This theorem can be proved by using the Cauchy-Binet formula (see for e.g. [Fallat and Johnson, 2011, Section 1.1]) that expresses the determinant of a product of two matrices as a sum of products of their minors.

1.2.4 Two Operations that Do Not Preserve Total Positivity

1. The sum of two TP matrices need not be TP. For example, consider the two matrices,

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

   Clearly, A and B are both TP as they are both bidiagonal matrices with non-negative entries. However,

$$A + B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

   and $\det(A+B) = -1 < 0$. Hence $A+B$ is not TP.

2. The Hadamard product or the entry-wise product of two TP matrices need not be TP. For example, consider the two matrices,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

   Their Hadamard product is

$$A \circ B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

   which has determinant -1.

   However, Hadamard product preserves TP_2. This is not difficult to check and we omit the details here.

1.2.5 Some New Examples

1. Our first example are *inverse bidiagonal matrices* with non-negative entries. For $\boldsymbol{x} = (x_i)_{\imath \geq 1}$, let

$$T(\boldsymbol{x}) = L(-\boldsymbol{x})^{-1}. \tag{1.1}$$

   Its matrix elements are

$$T(\boldsymbol{x})_{i,j} = \begin{cases} 1 & \text{for } j = i, \\ x_{j+1}x_{j+2}\cdots x_i & \text{for } j < i, \\ 0 & \text{for } j > i. \end{cases} \tag{1.2}$$

The first few rows of $T(\boldsymbol{x})$ are

$$\begin{bmatrix} 1 & & & & \\ x_1 & 1 & & & \\ x_1x_2 & x_2 & 1 & & \\ x_1x_2x_3 & x_2x_3 & x_3 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{1.3}$$

We have the following identity:

$$\begin{aligned} &\begin{bmatrix} 1 & & & & \\ x_1 & 1 & & & \\ x_1x_2 & x_2 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_1x_2\cdots x_n & x_2\cdots x_n & \dots & x_n & 1 \end{bmatrix} \\ &= \left[\begin{array}{c|c} I & 0 \\ \hline 0 \ \dots \ 0 \ x_n & 1 \end{array}\right] \cdot \left[\begin{array}{ccccc|c} 1 & & & & & \\ x_1 & 1 & & & & \\ x_1x_2 & x_2 & 1 & & & 0 \\ \vdots & \vdots & \ddots & \ddots & & \\ x_1x_2\cdots x_{n-1} & x_2\cdots x_{n-1} & \dots & x_{n-1} & 1 & \\ \hline & & 0 & & & 1 \end{array}\right]. \end{aligned} \tag{1.4}$$

By using induction, Equation (1.4) and Theorem 1.2.3.1, we get that $T(\boldsymbol{x})$ is TP whenever $x_i \geq 0$.

2. Our second example is the *binomial triangle*. Let B_n be the $n \times n$ lower triangular matrix of binomial coefficients i.e.,

$$B_n := \left(\binom{i-1}{k-1}\right)_{1 \leq i \leq n, 1 \leq k \leq n}.$$

Using Pascal's identity, we can show that

$$B_n = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & B_{n-1} \end{array}\right] \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \end{bmatrix}. \tag{1.5}$$

Using (1.5) and induction, we get that $\forall n \in \mathbb{N}$, B_n is totally positive.

In fact, let B be the infinite lower triangular matrix $\left(\binom{n}{k}\right)_{n \geq 0, k \geq 0}$. Taking $n \to \infty$ in identity (1.5), we get the new identity

$$B = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & B \end{array}\right] L(\mathbf{1}). \tag{1.6}$$

We also get that the matrix B is TP as any square submatrix of B is contained inside B_n, for some sufficiently large n.

3. Now we look at the *binomial square*. Let $\mathcal{B}$ be the infinite matrix $\left(\binom{n+k}{k}\right)_{n\geq 0, k\geq 0}$. It has the factorisation

$$\mathcal{B} = B \cdot B^{\intercal} \tag{1.7}$$

which is a consequence of the Chu-Vandermonde identity. As B and $B^{\intercal}$ are both TP, and as the identity (1.6) still holds when restricted to the first n columns and first n rows, the matrix $\mathcal{B}$ is TP.

4. Let H be the Hankel matrix of factorials, $((n+k)!)_{n\geq 0, k\geq 0}$. Then we have the identity:

$$H = D \cdot \mathcal{B} \cdot D \tag{1.8}$$

where D is the diagonal matrix $D = D((1/n!)_{n\geq 0})$. As B and D are both TP, and as the identity still holds when restricted to the first n rows and columns, the matrix H is TP.

5. Let $\left[{n \atop k}\right]$ denote the number of permutations of n into k cycles. These numbers are also called the unsigned Stirling numbers of the first kind or the Stirling cycle numbers. These numbers satisfy a simple 3-term recurrence relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = n \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix} \tag{1.9}$$

(see for e.g. [Stanley, 2009, Lemma 1.3.6. p. 26]) with the initial conditions

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1 \qquad \text{and} \qquad \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ k \end{bmatrix} = 0 \tag{1.10}$$

for $n > 0$.

Let M be the lower triangular matrix of the Stirling cycle numbers, i.e.,

$$M = \left(\begin{bmatrix} n \\ k \end{bmatrix} \right)_{n\geq 0, k\geq 0}. \tag{1.11}$$

The first few rows and columns of M are

$$\begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & 1 & 1 & & & & & & \\ & 2 & 3 & 1 & & & & & \\ & 6 & 11 & 6 & 1 & & & & \\ & 24 & 50 & 35 & 10 & 1 & & & \\ & 120 & 274 & 225 & 85 & 15 & 1 & & \\ & 720 & 1764 & 1624 & 735 & 175 & 21 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The matrix M satisfies the identity

$$M = Q \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & M \end{array} \right] \tag{1.12}$$

where $Q = T((n-1)_{n\geq 1})$. This is a consequence of the classical identity:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{i=0}^{n} \frac{(n-1)!}{(i-1)!} \begin{bmatrix} i-1 \\ k-1 \end{bmatrix} \tag{1.13}$$

(see for e.g. [Graham et al., 1989, Equation (6.16) Table 265]).

6. Let $\left\{ {n \atop k} \right\}$ denote the number of set partitions of n elements into k blocks. These numbers are also called the Stirling numbers of the second kind or Stirling subset numbers. They satisfy a simple 3-term recurrence

$$\left\{ {n+1 \atop k} \right\} = k \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\} \tag{1.14}$$

with initial conditions

$$\left\{ {0 \atop 0} \right\} = 1 \qquad \text{and} \qquad \left\{ {n \atop 0} \right\} = \left\{ {0 \atop k} \right\} = 0 \tag{1.15}$$

for $n > 0$.

Let M be the lower triangular matrix of the Stirling cycle numbers, i.e.,

$$M = \left(\left\{ {n \atop k} \right\} \right)_{n\geq 0, k\geq 0}. \tag{1.16}$$

The first few rows and columns of M are

$$\begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & 1 & 1 & & & & & & \\ & 1 & 3 & 1 & & & & & \\ & 1 & 7 & 6 & 1 & & & & \\ & 1 & 15 & 25 & 10 & 1 & & & \\ & 1 & 31 & 90 & 65 & 15 & 1 & & \\ & 1 & 63 & 301 & 350 & 140 & 21 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The matrix M satisfies the identity

$$M = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & M \end{array} \right] P \tag{1.17}$$

where $P = L((k-1)_{k\geq 1})$. This matrix identity is a consequence of the recurrence (1.14).

1.2.6 Loewner-Whitney theorem and Neville Factorisation

Let $\mathcal{G} = \mathrm{GL}_n(\mathbb{R})$ be the set of all $n \times n$ invertible matrices with real entries. Let $\mathcal{G}_{\geq 0}$ be the subset consisting of all invertible TP matrices. From subsection 1.2.3 we know that $\mathcal{G}_{\geq 0}$ is closed under matrix multiplication and we know that the identity matrix $I \in \mathcal{G}_{\geq 0}$. Thus, $\mathcal{G}_{\geq 0}$ forms a submonoid of $\mathcal{G}$. We are interested in finding a set of generators for $\mathcal{G}_{\geq 0}$. We know that invertible diagonal and bidiagonal matrices with non-negative entries are elements of $\mathcal{G}_{\geq 0}$. Note that a diagonal or a bidiagonal matrix is invertible if and only if its diagonal entries are non-zero.

Consider the following matrices:

$$D_i(x) = D(\underbrace{1, \ldots, 1}_{i-1}, x, \underbrace{1, \ldots, 1}_{n-i}), \tag{1.18}$$

$$L_i(x) = L(\underbrace{0, \ldots, 0}_{i-1}, x, \underbrace{0, \ldots, 0}_{n-i}), \tag{1.19}$$

$$U_i(x) = (L_i(x))^{\intercal}. \tag{1.20}$$

Let $\boldsymbol{d} = (d_i)_{1 \leq i \leq n}$ and $\boldsymbol{x} = (x)_{1 \leq i \leq n-1}$. We state a few observations without proof:

1.
$$D(x_1, \ldots, x_n) = D_1(x_1) \cdots D_n(x_n),$$

2.
$$L(\boldsymbol{d}, \boldsymbol{x}) = L\left(\frac{x_1}{d_1}, \ldots, \frac{x_{n-1}}{d_{n-1}}\right) \cdot D(\boldsymbol{d}) = D(\boldsymbol{d}) \cdot L\left(\frac{x_1}{d_2}, \ldots, \frac{x_{n-1}}{d_n}\right),$$

3.
$$U(\boldsymbol{d}, \boldsymbol{x}) = D(\boldsymbol{d}) \cdot U\left(\frac{x_1}{d_1}, \ldots, \frac{x_{n-1}}{d_{n-1}}\right) = U\left(\frac{x_1}{d_2}, \ldots, \frac{x_{n-1}}{d_n}\right) \cdot D(\boldsymbol{d}),$$

4.
$$L(\boldsymbol{x}) = L_1(x_1) \cdots L_{n-1}(x_{n-1}),$$

5.
$$U(\boldsymbol{x}) = U_{n-1}(x_{n-1}) \cdots U_1(x_1).$$

The above equations show that for $x > 0$, the matrices $D_i(x)$, $L_i(x)$ and $U_i(x)$ generate all diagonal and bidiagonal matrices in $\mathcal{G}_{\geq 0}$. In fact, Whitney [1952] first noticed and Loewner [1955] first stated that these matrices generate the entire monoid $\mathcal{G}_{\geq 0}$.

Theorem 1.2.6.1 (Whitney [1952], Loewner [1955])**.** *For $x > 0$, the matrices $D_i(x)$, $L_i(x)$ and $U_i(x)$ generate $\mathcal{G}_{\geq 0}$.*

We shall refer to this theorem as the *Loewner-Whitney theorem*. In fact, their proof is constructive and given $A \in \mathcal{G}_{\geq 0}$, they provide a factorisation of A into $D_i(x)$, $L_i(x)$ and $U_i(x)$. Gasca and Peña [1996] studied the factorisations of TP matrices in depth and gave an efficient polynomial time algorithm. In fact, their algorithm is more general and works for singular matrices as well. See Section 6.4 in Pinkus [2009] for historical remarks on this entire story.

1.3 Refined Enumeration by Counting Statistics

Enumerative Combinatorics, as mentioned in Section 1.1, deals with exact enumeration of combinatorial families. It is a common practice in enumerative combinatorics to refine the counting by keeping track of one or several *statistics* along with computing the cardinality of the sets of each size. Given a combinatorial family $\mathcal{S} = \{S_n\}_{n\in\mathbb{N}}$, a statistic is a function $f : \cup S_n \to \mathbb{N}$. Given a statistic f, consider the polynomials

$$p_n(x) = \sum_{i\in S_n} x^{f(i)} = \sum_{i\in\mathbb{N}} \left|f^{-1}(i)\cap S_n\right| x^i.$$

Thus,

$$p_n(1) = |S_n|\,.$$

The polynomials $p_n(x)$ provide refined information about the cardinalities of S_n: the coefficients of $p_n(x)$ keep track of $\left|f^{-1}(i)\cap S_n\right|$, the number of elements of S_n for which the statistic f has value i. Thus, we upgrade from counting using natural numbers to counting using polynomials with non-negative integer coefficients to obtain a more refined enumeration. We may simultaneously count multiple statistics by taking multivariate polynomials.

Let us now look at a few examples of various statistics.

1. Subsets of $[n]$ with their sizes.

   Let S_n be the collection of all subsets of $[n]$. We know that $\left|f^{-1}(i)\cap S_n\right| = 2^n$. Let $f : S_n \to \mathbb{N}$ be the statistic that counts the cardinality of each subset. The polynomials $p_n(x)$ are given by

   $$p_n(x) = \sum_{A\subset[n]} x^{|A|} = (1+x)^n.$$

   Let g be the statistic that counts the cardinality of the complement of each subset. We can generalise $p_n(x)$ with respect to both statistics as

   $$p_n(x,y) = \sum_{A\subset[n]} x^{|A|}y^{|[n]\setminus A|} = (x+y)^n.$$

2. Let S_n be the collection of all permutations on the letters $[n]$. Thus, $|S_n| = n!$. We have already defined the Stirling cycle numbers $\left[{n \atop k}\right]$, the number of permutations on the letters $[n]$ with k cycles. With this statistic we get that

   $$p_n(x) = \sum_{k=0}^{n} \left[{n \atop k}\right] x^k = x(x+1)\cdots(x+n-1).$$

   (see for e.g. [Stanley, 2009, Proposition 1.3.7 p. 27]).

   The study of statistics on permutations is a very rich area of study and we list some of the statistics:

   - Number of inversions,
   - Number of ascents and descents,
   - Major index,

- Left-to-right minima, left-to-right maxima, right-to-left minima, right-to-left maxima,
- Peaks, valleys, double ascents and double descents.

1.3.1 Coefficientwise Total Positive

A large number of matrices occuring in enumerative combinatorics have entries which are the cardinalities of some set. In the spirit of introducing statistics, one would want to replace these entries with polynomials in one or several variables with non-negative integer coefficients. As huge number of matrices occuring in enumerative combinatorics turn out to be totally positive, we want to extend these results to matrices with polynomial entries. To be able to do so, we need to define total positivity for matrices with polynomial entries.

We say that a polynomial (in one or several variables) is *coefficientwise-positive*, if all its coefficients are non-negative. A matrix whose entries are polynomials with real coefficients is said to be *coefficientwise-totally positive* (*coefficientwise-TP* in short) if all its minors, which are polynomials themselves, are coefficientwise-positive. We say that a matrix is *coefficientwise-totally positive of order* r if all its minors of size $\leq r$ are coefficientwise-positive.

We say that a polynomial (in one or several variables) is *pointwise-positive*, if it is non-negative when substituted with non-negative real numbers. A matrix whose entries are polynomials with real coefficients is said to be *pointwise-totally positive* (*pointwise-TP* in short) if all its minors are pointwise-positive. We say that a matrix is *pointwise-totally positive of order* r if all its minors of size $\leq r$ are pointwise-positive.

Note that coefficientwise-positivity implies pointwise-positivity. However, the converse is not true. For example, $x^2 - 2x + 1$ is not coefficientwise-positive even though it is pointwise-positive. Thus, coefficientwise total positivity implies pointwise total positivity but the converse is not true.

1.3.2 Few Examples of Coefficientwise-TP Matrices

Let $\boldsymbol{d} = (d_i)_{i \geq 0}$, $\boldsymbol{x} = (x_i)_{i \geq 1}$ be two families of indeterminates. It is not difficult to see that the following matrices are coefficientwise-TP:

- $D(\boldsymbol{D})$,
- $L(\boldsymbol{x}, \boldsymbol{d})$,
- $U(\boldsymbol{x}, \boldsymbol{d})$.

A square submatrix of any of these matrices is either upper triangular or lower-triangular, and hence its determinant is the product of the diagonal entries, which is either 0 or a monomial.

1.3.3 Operations that Preserve Coefficientwise-TP

We state several properties of TP matrices without proof:

1. Submatrix of a coefficientwise-TP matrix is coefficientwise-TP.
2. Inserting rows or columns of zeros in a coefficientwise-TP matrix preserves coefficientwise-TP.

3. Duplicating a row or a column and adding it right next to the original row or column preserves coefficientwise-TP.
4. Transpose of a coefficientwise-TP matrix is coefficientwise-TP.
5. Let A and B be both coefficientwise-TP matrices where A is a finite matrix. Then the block matrix
$$C = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]$$
is also coefficientwise-TP.
6. Matrix product preserves coefficientwise-TP. This is again a consequence of the Cauchy-Binet formula.

1.3.4 Some More Examples of Coefficientwise-TP Matrices

1. Let $\boldsymbol{x} = (x_i)_{i\geq 0}$ be a family of indeterminates. Then $T(\boldsymbol{x})$ is coefficientwise-TP. This is true as Equation (1.4) is true even when the x_i are indeterminates.
2. Let $B_n(x, y)$ be the $n \times n$ lower triangular matrix
$$B_n(x,y) \coloneqq \left(\binom{i-1}{k-1} x^{k-1} y^{i-k}\right)_{1\leq i\leq n, 1\leq k\leq n}.$$
The equivalent of Pascal's identity in this setting is
$$\binom{n}{k} x^k y^{n-k} = x\left(\binom{n-1}{k-1} x^{k-1} y^{n-k}\right) + y\left(\binom{n-1}{k} x^k y^{n-k-1}\right).$$
Thus, we get the matrix identity
$$B_n(x,y) = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & B_{n-1}(x,y) \end{array}\right] \cdot \begin{bmatrix} x & & & \\ y & x & & \\ & \ddots & \ddots & \\ & & y & x \end{bmatrix}. \tag{1.21}$$
This also shows that the infinite lower triangular matrix $B(x, y)$ given by
$$B(x,y) \coloneqq \left(\binom{n}{k} x^k y^{n-k}\right)_{n\geq 0, k\geq 0}$$
is coefficientwise-TP.
3. Let $\mathcal{B}(x, y)$ be the infinite matrix
$$\mathcal{B}(x,y) \coloneqq \left(\binom{n+k}{k} x^n y^k\right)_{n\geq 0, k\geq 0}.$$
This matrix satisfies the identity:
$$\mathcal{B}(x,y) = B(y,x) \cdot B(x,y)^{\mathsf{T}}. \tag{1.22}$$
Thus, $\mathcal{B}$ is coefficientwise-TP.

Chapter 2

Total Positivity of Three Different Kinds of Matrices and Proof Techniques

We are primarily interested in studying the total positivity of the following three different kinds of matrices:

1. Hankel matrix of a sequence $\{a_n\}$,
2. Toeplitz matrix of a sequence $\{a_n\}$,
3. Lower triangular matrices.

We shall discuss each of the three in detail in the sections that follow.

2.1 Hankel Total Positivity

Given a sequence $\boldsymbol{a} = \{a_n\}_{n\geq 0}$, the matrix

$$H_\infty = (a_{n+k})_{n\geq 0, k\geq 0} \tag{2.1}$$

is called its *Hankel matrix.* If $\boldsymbol{a}$ is a sequence of real numbers, we say that $\boldsymbol{a}$ is *Hankel totally positive* (*Hankel-TP* in short) if $H_\infty(\boldsymbol{a})$ is TP. Similarly, if $\boldsymbol{a}$ is a sequence of polynomials with real coefficients, we say that $\boldsymbol{a}$ is *coefficientwise-Hankel-totally positive* (*coefficientwise-Hankel-TP* in short) if $H_\infty(\boldsymbol{a})$ is coefficientwise-TP.

Let $f_{\boldsymbol{a}}(t) = \sum_{n=0}^{\infty} a_n t^n$ be the ordinary generating function of $\boldsymbol{a}$. The following theorem provides equivalent conditions for $\boldsymbol{a}$ to be Hankel-TP when $\boldsymbol{a}$ is a sequence of real numbers.

Theorem 2.1.0.1 (Stieltjes [1894], Gantmakher and Krein [1937])**.** *For a sequence $\boldsymbol{a} = \{a_n\}_{n\geq 0}$ of real numbers, the following are equivalent*

1. *$\boldsymbol{a}$ is Hankel-TP.*
2. *There exists a positive measure μ on $(0, \infty)$ such that $a_n = \int_0^n x^n d\mu$ for all $n \geq 0$, i.e. $\boldsymbol{a}$ is a Stieltjes-moment sequence.*

3. There exists real numbers $\alpha_0, \alpha_1, \ldots \geq 0$ *such that*

$$f_{\boldsymbol{a}}(t) = \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{\ddots}}}. \tag{2.2}$$

Continued fractions of the form in Equation (2.2) are known as Stieltjes continued fractions or S-fractions.

When $\boldsymbol{a}$ is a sequence of polynomials, *2.* in Theorem 2.1.0.1 no longer makes sense. However, we can still talk about *1.* and *3.* by replacing Hankel-TP with coefficientwise-Hankel-TP in *1.*, and if $\alpha_0, \alpha_1, \ldots$ are a sequence of coefficientwise-positive polynomials. The following theorem still holds in the case of coefficientwise-TP:

Theorem 2.1.0.2 (Flajolet [1980], Viennot [1983], Sokal [2014])**.** *For a sequence of polynomials* $\boldsymbol{a} = \{a_n\}_{n\geq 0}$*, if there exists coefficientwise-positive polynomials* $\alpha_0, \alpha_1, \ldots$*, such that* $f_{\boldsymbol{a}}(t)$ *has the continued fraction expansion*

$$f_{\boldsymbol{a}}(t) = \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{\ddots}}} \tag{2.3}$$

then $\boldsymbol{a}$ *is Hankel totally positive.*

The above theorem was first stated in this form by Sokal in Sokal [2014]. However, it is an easy corollary of Flajolet [1980] and of [Viennot, 1983, Section 3 Chapter 4]. Flajolet showed that Dyck paths whose edges are given by height-indexed weights are combinatorial interpretations of Stieltjes-type continued fractions. Viennot showed that certain non-intersecting set of Dyck paths are the combinatorial interpretations for the Hankel determinants. The two put together along with the Lindström-Gessel-Viennot lemma prove Theorem 2.1.0.2. We will see more on this in subsection 2.1.1.

The converse of Theorem 2.1.0.2 is however not true. This has been discussed in [Pétréolle et al., 2018, Section 6.1].

2.1.1 Combinatorial Interpretation of S-Fractions

In a seminal paper, Flajolet [1980] gave combinatorial interpretations of S-fractions in terms of Dyck paths. We give a brief overview of his main result.

Let $\boldsymbol{\alpha} = \{\alpha\}_{n\geq 0}$ be a family of variables and let $R_n(\boldsymbol{\alpha})$ be the Taylor-series coefficients of its S-fraction i.e.,

$$\cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{\ddots}}} = \sum_{n=0}^{\infty} R_n(\boldsymbol{\alpha}) t^n. \tag{2.4}$$

Notice that $R_n(\boldsymbol{\alpha})$ are polynomials and these are called the Stieltjes-Rogers polynomials.

A Dyck path of semilength n, is defined to be a path in the non-negative quadrant that begins at $(0,0)$ and ends at $(2n,0)$ and only uses the steps $(1,1)$ and $(1,-1)$. We use $\mathcal{D}_n$ to denote the set of all Dyck paths of semilength n.

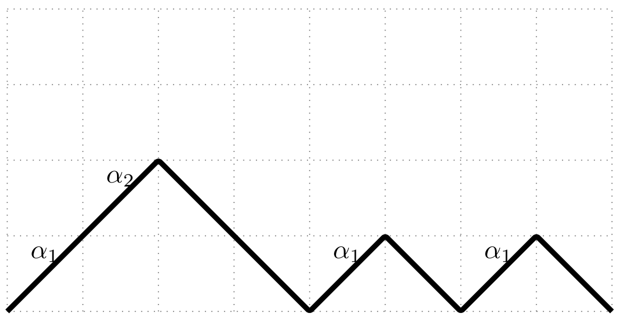

Figure 2.1: Dyck path of semilength 4 with height-dependent weights. The total weight of this path is $\alpha_1^3\alpha_2$

Given a Dyck path $D \in \mathcal{D}_n$, we assign height-dependent weights to each of the steps as follows: a step $(1, 1)$ beginning at height i is assigned the weight α_{i+1} and a step $(1, -1)$ is assigned the weight 1. The weight of D is the product of the weights of all its steps and we denote it by $\mathrm{wt}(D)$. Figure 2.1 is an example of a Dyck path of semilength 4 with weights of each individual $(1, 1)$-edges mentioned beside them.

Theorem 2.1.1.1 (Flajolet [1980])**.** *The following identity is true with $\alpha_0 = 1$,*

$$R_n(\boldsymbol{\alpha}) = \sum_{D \in \mathcal{D}_n} \mathrm{wt}(D). \tag{2.5}$$

Thus,

$$\sum_{n=0}^{\infty} \sum_{D \in \mathcal{D}_n} \mathrm{wt}(D) \cdot t^n = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{\ddots}}} \tag{2.6}$$

where $\mathcal{D}_0$ is the singleton set containing the path of semilength 0*, which is assigned weight* 1*.*

In order to show that a sequence $\boldsymbol{a} = \{a_n\}_{n \geq 0}$ is Hankel-TP, it suffices to show that there is an assignment of weights to Dyck paths, such that

$$R_n(\boldsymbol{\alpha}) = a_n.$$

2.1.2 Other Types of Continued Fractions and Path Models

The existence of S-fractions or equivalently of weighted-Dyck paths is only a sufficient condition to prove Hankel-total positivity. Other similar sufficient conditions have also been developed and used for proving coefficientwise-Hankel-total positivity. We mention some of these in Table 2.1.2 along with some references.

2.2 Toeplitz Total Positivity

Let $\boldsymbol{a} = \{a_n\}$ be a sequence of real numbers. We assume that $\boldsymbol{a}$ is an infinite sequence (if it is finite, we append an infinite sequence of 0s.) The *Toeplitz matrix* of $\boldsymbol{a}$ is defined as

$$T_\infty(\boldsymbol{a}) = (a_{n-k})_{n \geq 0, k \geq 0}$$

Continued Fractions	Path models	References
S-fractions	Dyck Paths	Flajolet [1980]
m-Branched S-fractions	m-Dyck paths	Pétréolle et al. [2018]
T-fractions and Branched T-fractions	Schröder Paths	Pétréolle et al. [2018], Elvey Price and Sokal [2020]

Table 2.1: Various continued fractions and their associated path models

where $a_{n-k} = 0$ when $n < k$. We say that the sequence $\boldsymbol{a}$ is *Toeplitz totally positive* (*Toeplitz-TP* in short) if $T_\infty(\boldsymbol{a})$ is totally positive[1].

Let $f_{\boldsymbol{a}}(t) = \sum_{n=0}^{\infty} a_n t^n$ be the ordinary generating function of $\boldsymbol{a}$ Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two sequences. We then have

$$T_\infty(\boldsymbol{a}) \cdot T_\infty(\boldsymbol{b}) = (a_{n-k})_{n\geq 0, k\geq 0} (b_{n-k})_{n\geq 0, k\geq 0} = \left(\sum_{j=k}^{n} a_{n-j} b_{n-j}\right)_{n\geq 0, k\geq 0} = T_\infty(\boldsymbol{a} * \boldsymbol{b})$$

where the sequence $\boldsymbol{a} * \boldsymbol{b}$ is the convolution of the sequences $\boldsymbol{a}$ and $\boldsymbol{b}$, i.e., the n^{th} term in $\boldsymbol{a} * \boldsymbol{b}$ is $\sum_{i=0}^{n} a_{n-i} b_i$.

Also, note that

$$f_{\boldsymbol{a}}(t) f_{\boldsymbol{b}}(t) = f_{\boldsymbol{a}*\boldsymbol{b}}(t),$$

i.e., the product of the ordinary generating functions of the sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ is the ordinary generating function of their convolution. Thus, the product of two Toeplitz matrices and the two corresponding ordinary generating functions behave similarly. In fact, the total positivity of Toeplitz matrices is interrelated with properties of the ordinary generating functions which was shown by Aissen, Schoenberg, Whitney and Edrei in 1952 in the following theorem:

Theorem 2.2.0.1 (Aissen-Schoenberg-Whitney (Aissen et al. [1952]), Edrei (Edrei [1952]))**.** *For a sequence $\boldsymbol{a} = (a_n)_{n\geq 0}$ of real numbers with $a_0 = 1$, the following are equivalent:*

1. *The sequence $\boldsymbol{a}$ is Toeplitz totally positive.*
2. *There exists sequences $\boldsymbol{\alpha} = \{\alpha\}_{n\geq}, \boldsymbol{\beta} = \{\beta_n\}_{n\geq 0}$ with $\alpha_n \geq 0$, $\beta \geq 0$, and a number $\gamma \geq 0$ such that*

$$f_{\boldsymbol{a}}(t) = e^{\gamma t} \frac{\prod_{n=1}^{\infty}(1 + \alpha_n t)}{\prod_{n=1}^{\infty}(1 - \beta_n t)}. \tag{2.7}$$

We shall refer to Theorem 2.2.0.1 as the ASWE theorem. Note that if $\boldsymbol{a}$ is a finite sequence, $f_{\boldsymbol{a}}(t)$ is a polynomial and the ASWE theorem says that $\boldsymbol{a}$ is Toeplitz-TP if and only if all the roots of $f_{\boldsymbol{a}}(t)$ are real and non-positive. The direction *2.* $\implies$ *1.* is not difficult and we shall show it in the next few paragraphs. The direction *1.* $\implies$ *2.* requires the Nevanlinna theory of meromorphic functions.

Consider the following observations:

1. The finite sequence $(1, \alpha)$ has generating function $1 + \alpha t$. Also,

$$T_\infty((1, \alpha)) = L((\alpha, \alpha, \alpha, \ldots)),$$

[1] These sequences have also been called Pólya-frequency sequences in the literature.

which is TP if and only if $\alpha \geq 0$ as bidiagonal matrices with non-negative entries are totally positive.

2. The sequence $(1, \beta, \beta^2, \ldots)$ has the generating function $1/(1-\beta t)$. Also,

$$T_\infty\left((1, \beta, \beta^2, \ldots)\right) = T((1, \beta, \beta^2, \ldots)),$$

which is TP if and only if $\beta \geq 0$ as inverse-bidiagonal matrices with non-negative entries are TP.

3. Let $\boldsymbol{a}^1, \boldsymbol{a}^2, \boldsymbol{a}^3, \ldots$ be an infinite sequence of sequences which have the pointwise limit $\boldsymbol{a}$. If $\boldsymbol{a}^1, \boldsymbol{a}^2, \ldots$ are all Toeplitz-TP, then $\boldsymbol{a}$ is also Toeplitz-TP.

4. From *1.* and the fact that total positivity is preserved under matrix multiplication, $\left(1 + \frac{\gamma}{n}t\right)^n$ is the generating function of a Topelitz-TP sequence whenever $\gamma \geq 0$. Thus, from *3.* $e^{\gamma t}$ is the generating function of a Toeplitz-TP sequence.

From observations 1., 2. 3. and 4. above, we can conclude that the direction *2.* $\Longrightarrow$ *1.* in the ASWE theorem is true.

2.2.1 Coefficientwise Toeplitz Total Positivity

Given a sequence of polynomials $\boldsymbol{a} = \{a_n\}_{n \geq 0}$, we say that $\boldsymbol{a}$ is *coefficientwise-Toeplitz-totally positive* if $T_\infty(\boldsymbol{a})$ is coefficientwise-TP.

Consider the following observations:

1. For a coefficientwise-positive polynomial α, the finite sequence $(1, \alpha)$ has generating function $1 + \alpha t$. Also,

$$T_\infty((1, \alpha)) = L((\alpha, \alpha, \alpha, \ldots))$$

which is coefficientwise-TP.

2. For a coefficientwise-positive polynomial β, the sequence $(1, \beta, \beta^2, \ldots)$ has the generating function $1/(1-\beta t)$. Also,

$$T_\infty\left((1, \beta, \beta^2, \ldots)\right) = T((1, \beta, \beta^2, \ldots))$$

which is coefficientwise-TP.

3. Let $\boldsymbol{a}^1, \boldsymbol{a}^2, \boldsymbol{a}^3, \ldots$ be an infinite sequence of sequences which converge termwise to the sequence $\boldsymbol{a}$. If $\boldsymbol{a}^1, \boldsymbol{a}^2, \ldots$ are all coefficientwise-Toeplitz-TP, then $\boldsymbol{a}$ is also coefficientwise-Toeplitz-TP.

4. Let γ be a coefficientwise-positive polynomial. From 1. and the fact that coefficientwise total positivity is preserved under matrix multiplication, $\left(1 + \frac{\gamma}{n}t\right)^n$ is the generating function of a coefficientwise Topelitz-TP sequence. Thus, from 3., $e^{\gamma t}$ is the generating function of a coefficientwise-Toeplitz-TP sequence.

Thus, we get that *2.* $\Longrightarrow$ *1.* of the ASWE theorem still holds for the coefficientwise-Toeplitz-TP case. However, *1.* $\Longrightarrow$ *2.* need not be true.

We shall end this section by providing two well known matrices which are coefficientwise-Toeplitz-totally positive. Let $e_n(x_1, \ldots, x_m)$ be the n^{th}-elementary symmetric polynomial in m variables and let $h_n(x_1, \ldots, x_m)$ be the n^{th}-complete homogeneous symmetric polynomial in m variables. The well known Jacobi-Trudi identities and dual Jacobi-Trudi identities (see for e.g.

[Stanley and Fomin, 1999, Section 7.16]) express the skew-Schur polynomials as determinants of Toeplitz matrices of e_n and h_n. Thus, the Toeplitz matrices of e_n and h_n are coefficientwise-TP as the skew-Schur polynomials are coefficientwise-positive.

2.3 Total Positivity of Lower Triangular Matrices

Let A be the lower triangular matrix

$$A = \begin{bmatrix} a_{00} & & & \\ a_{10} & a_{11} & & \\ a_{20} & a_{21} & a_{22} & \\ \vdots & & & \ddots \end{bmatrix}$$

with non-negative entries $a_{ij} \geq 0$. Let $A_n(x)$ be the row-generating polynomials of A, i.e.,

$$A_n(x) = \sum_{k=0}^{n} a_{nk} x^k.$$

We shall refer to a lower triangular matrix as a triangle.

Then we can ask the following three different questions:

1. Is A TP?
2. Are $A_n(x)$ negative real rooted, i.e., is n^{th} row Toeplitz-TP?
3. Is the sequence $\{A_n(x)\}_{n\geq 0}$ Hankel-TP?

This investigation of these questions for several different triangles will be the central theme of my work.

Chapter 3

Lindström-Gessel-Viennot Lemma

An often used tool to prove total positivity is the well known Lindström-Gessel-Viennot lemma (LGV lemma in short), which is a lemma that has been discovered and rediscovered several times most notably in Gessel and Viennot [1989]. In this chapter, we shall provide a brief description of the LGV lemma in a setting that is useful for proving total positivity. For a proof of the LGV lemma see Aigner and Ziegler [2010] for example.

Let $\mathcal{D}$ be an acyclic digraph and let R be a commutative ring with identity. Let each edge e of $\mathcal{D}$ be assigned a weight $w_e \in R$. Let $\mathcal{U} = \{u_0, u_1, \ldots\}$ and $\mathcal{V} = \{v_0, v_1, \ldots\}$ be two distinguished sets of vertices which we call the sources and sinks, respectively. For a path $\mathcal{P}$ in $\mathcal{D}$, we write $\mathcal{P} : u_n \to v_k$ to mean that $\mathcal{P}$ starts at the source vertex u_n and ends at the sink vertex v_k. Let

$$P(u_n \to v_k) \coloneqq \sum_{\mathcal{P}: u_n \to v_k} \mathrm{wt}(\mathcal{P})$$

where $\mathrm{wt}(\mathcal{P})$ is the weight of $\mathcal{P}$. We define the path matrix of $\mathcal{D}$ to be the matrix

$$\boldsymbol{P}(\mathcal{D}) = \left(P(u_n \to v_k)\right)_{n \geq 0, k \geq 0} .$$

We call this set-up an *LGV network*.

Let $I = \{n_1, \ldots, n_r\}$ and $J = \{k_1, \ldots, k_r\}$ be two sets of indices. From the usual definition of determinant of a matrix we have that

$$\begin{aligned}
\det \boldsymbol{P}(\mathcal{D})_{I,J} &= \det \left(P(u_{n_i} \to v_{k_i})\right)_{1 \leq i \leq r} \\
&= \sum_{\sigma \in \mathfrak{S}_r} \mathrm{sign}(\sigma) \prod_{i=1}^{r} P(u_{n_i} \to v_{k_{\sigma(i)}}) \\
&= \sum_{\sigma \in \mathfrak{S}_r} \mathrm{sign}(\sigma) \sum_{(\mathcal{P}_1, \ldots, \mathcal{P}_r)} \prod_{i=1}^{r} \mathrm{wt}(\mathcal{P}_i)
\end{aligned} \tag{3.1}$$

where $\mathcal{P}_1 : u_{n_1} \to v_{n_{\sigma(1)}}, \ldots, \mathcal{P}_r : u_{n_r} \to v_{n_{\sigma(r)}}$.

A sequence of paths $(\mathcal{P}_1, \ldots, \mathcal{P}_r)$ is called *non-intersecting* if they have no vertices in common. The LGV lemma states that

$$\det \boldsymbol{P}(\mathcal{D})_{I,J} = \sum_{\sigma \in \mathfrak{S}_r} \operatorname{sign}(\sigma) \sum_{\substack{(\mathcal{P}_1,\ldots,\mathcal{P}_r) \\ \text{non-intersecting}}} \prod_{i=1}^{r} \operatorname{wt}(\mathcal{P}_i) \tag{3.2}$$

where $\mathcal{P}_1 : u_{n_1} \to v_{n_{\sigma(1)}}, \ldots, \mathcal{P}_r : u_{n_r} \to v_{n_{\sigma(r)}}$. Note that the only change in the expressions for the determinant in Equations (3.1) and (3.2) is the phrase "non-intersecting" which may remove a lot of extra terms. We shall look at some examples soon.

We say that $\mathcal{D}$ is *fully compatible* if for any subset of sources $\{u_{n_1}, \ldots, u_{n_r}\}$, and any subset of sinks $\{v_{n_1}, \ldots, v_{n_r}\}$, the only permutation $\sigma \in \mathfrak{S}_r$ which gives rise to a family of non-intersecting paths is the identity permutation. If $\mathcal{D}$ is fully compatible, Equation (3.2) reduces to

$$\det \boldsymbol{P}(\mathcal{D})_{I,J} = \sum_{\substack{(\mathcal{P}_1,\ldots,\mathcal{P}_r) \\ \text{non-intersecting}}} \prod_{i=1}^{r} \operatorname{wt}(\mathcal{P}_i) \tag{3.3}$$

where $\mathcal{P}_1 : u_{n_1} \to v_{n_1}, \ldots, \mathcal{P}_r : u_{n_r} \to v_{n_r}$.

Let $\mathcal{D}$ be planar digraph with a planar embedding in which $\mathcal{U}$ and $\mathcal{V}$ lie on the boundary of a circle and all the other vertices lie within the circle, such that the sources $\mathcal{U}$ occur in clockwise order and the sinks $\mathcal{V}$ occur in anti-clockwise order such that the sources and sinks are not interspersed. We call $\mathcal{D}$ with such an embedding a *planar network*.

From the descriptions of planar network and full compatibility, we can see that a planar network is fully compatible. This was first described by Brenti in Brenti [1995].

If the ring R is a ring of polynomials in one or several variables over $\mathbb{C}$ and if the edge weights w_e are coefficientwise-positive, then from the LGV lemma, we get that $\boldsymbol{P}(\mathcal{D})$ is coefficientwise-TP. Thus, to show that a matrix M is TP (or coefficientwise-TP), it suffices to construct a planar network $\mathcal{D}$ whose path matrix is M. This is the set-up that we often use to prove total positivity.

If a planar network $\mathcal{D}$ is provided, it is not difficult to describe $\boldsymbol{P}(\mathcal{D})$. However, the other direction is difficult, i.e., if a matrix M is provided, it is difficult to construct a planar network $\mathcal{D}$ with $\boldsymbol{P}(\mathcal{D}) = M$. However, providing such a network for M, ensures total positivity of M.

A few examples of planar digraphs are given:

1. Planar network for the lower bidiagonal matrix $L(\boldsymbol{d}, \boldsymbol{x})$ is given in Figure 3.1.
2. Planar network for the inverse lower bidiagonal matrix $T(\boldsymbol{x})$ is given in Figure 3.2.
3. Planar network for the binomial triangle is given in Figure 3.3.

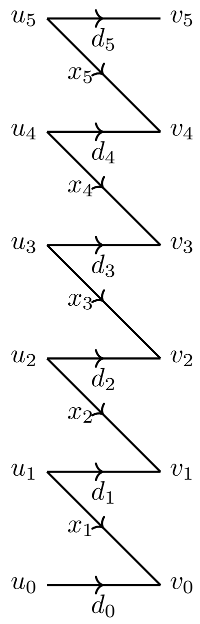

Figure 3.1: Planar network for the lower bidiagonal matrix $L(\boldsymbol{d}, \boldsymbol{x})$.

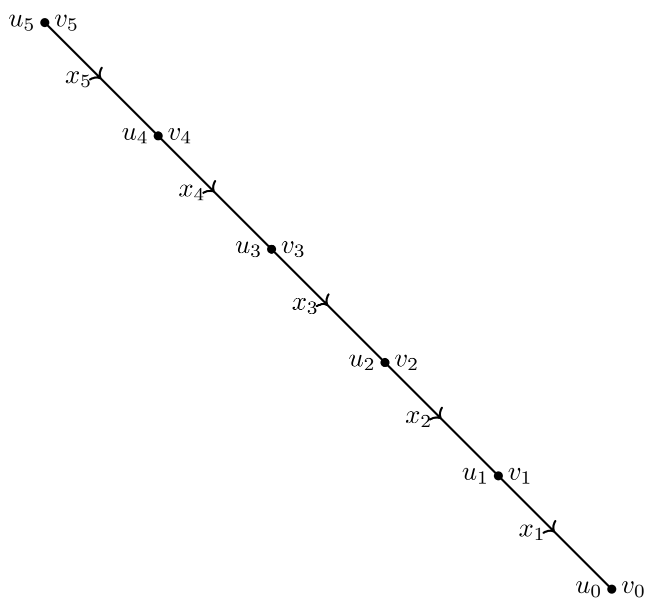

Figure 3.2: Planar network for the inverse lower bidiagonal matrix $T(\boldsymbol{x})$.

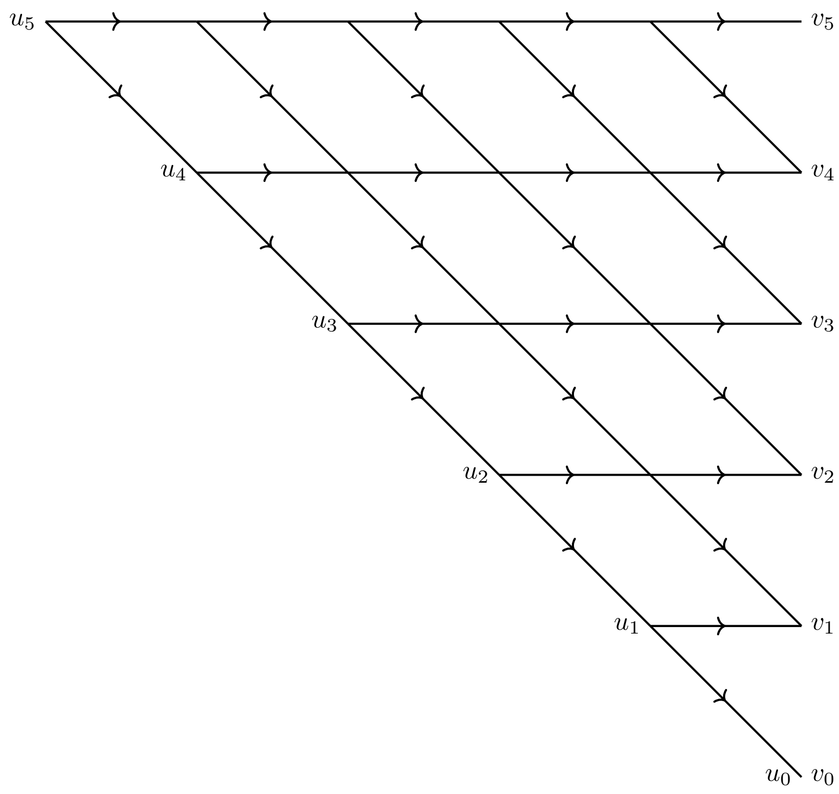

Figure 3.3: Planar network for the binomial triangle. All edges are weighted 1.

Part II

Ongoing Projects

Chapter 4

Coefficientwise Total Positivity of Some Matrices Defined by Linear Recurrences (Ongoing joint work with Xi Chen, Alexander Dyachenko, Tomack Gilmore, Alan D. Sokal)

The study of total positivity in combinatorics was started by Brenti. In Brenti [1995], he showed that several combinatorial triangles are totally positive. Some of them are:

1. Binomial triangle,
2. Stirling cycle triangle,
3. Stirling subset triangle,
4. Reversed Stirling cycle triangle.

Given a lower triangular totally positive matrix A, we can construct its reversal A^{rev} where (n, k)-entry of A^{rev} is the $(n, n-k)$-entry of A for $n \geq k$ and 0 otherwise. A^{rev} is also lower triangular.

We can ask if reversal preserves total positivity. One can construct examples to show that this is not true in general. However, for the triangles mentioned above, 1. is the reversal of itself, and 2. and 4. are reversals of each other. This motivated us to check if the reversal of 3. is totally positive. This seemed to be true empirically.

Conjecture 4.0.0.1. *The reversal of the Stirling subset triangle is totally positive.*

The first few rows of the reversed Stirling subset triangle are given in Equation 4.1.

	Triangle	Recurrence
a.	Binomial	$T(n,k) = T(n-1,k-1) + T(n-1,k)$
b.	Stirling Cycle triangle	$T(n,k) = T(n-1,k-1) + (n-1)\cdot T(n-1,k)$
c.	Stirling Subset triangle	$T(n,k) = T(n-1,k-1) + k\cdot T(n-1,k)$
d.	Reversed Stirling Cycle triangle	$T(n,k) = (n-1)\cdot T(n-1,k-1) + T(n-1,k)$
e.	Reversed Stirling Subset triangle	$T(n,k) = (n-k)\cdot T(n-1,k-1) + T(n-1,k)$
f.	Eulerian Triangle	$T(n,k) = (n-k+1)\cdot T(n-1,k-1) + (k+1)\cdot T(n-1,k)$

Table 4.1: Recurrences for various triangles

$$\begin{array}{ccccccc} 1 \\ 1 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 6 & 7 & 1 & 0 \\ 1 & 10 & 25 & 15 & 1 & 0 \\ \vdots & \vdots & \vdots & \dots & \ddots & \ddots & \ddots \end{array} \tag{4.1}$$

Another triangle which has been conjectured to be TP is the triangle of Eulerian numbers. This was first conjectured in [Brenti, 1996, Conjecture 6.10].

Conjecture 4.0.0.2 ([Brenti, 1996, Conjecture 6.10]). *The Eulerian triangle is totally positive.*

The first few rows of the Eulerian triangle are given in Equation (4.2).

$$\begin{array}{ccccccc} 1 \\ 1 & 1 \\ 1 & 4 & 1 \\ 1 & 11 & 11 & 1 \\ 1 & 26 & 66 & 26 & 1 \\ 1 & 57 & 302 & 302 & 57 & 1 \\ \vdots & \vdots & \vdots & \dots & \ddots & \ddots & \ddots \end{array} \tag{4.2}$$

A common feature of all the triangles that we have come across so far is that they are all generated by simple 3-term recurrences with the same initial condition of $T(0,0) = 1$. The recurrences are mentioned in Table 4.1.

Notice that the coefficients of the recurrences a., b. and d. in Table 4.1 are purely n-dependent and the coefficients of the recurrences a. and c. are purely k-dependent. The coefficients of e. and f. are both n and k dependent.

We state the following theorem due to Brenti:

Theorem 4.0.0.3 (Brenti [1995]). *Consider the recurrence*

$$T(0,0) = a_{0,0}, \tag{4.3a}$$

$$T(n,k) = a_{n,k}T(n-1,k-1) + b_{n,k}T(n-1,k) + c_{n,k}T(n-2,k-1) \text{ when } n \geq k, \tag{4.3b}$$

$$T(n,k) = 0 \text{ otherwise.} \tag{4.3c}$$

and let $A = (T(n,k))_{n\geq 0, k\geq 0}$.

i. If $a_{n,k}$, $b_{n,k}$, $c_{n,k}$ are purely k-dependent i.e.,

$$a_{n,k} = a_k, \tag{4.4a}$$

$$b_{n,k} = b_k, \tag{4.4b}$$

$$c_{n,k} = c_k \tag{4.4c}$$

for all $n \geq k$, then A is coefficientwise-totally positive in all indeterminates $\boldsymbol{a} = \{a_k\}, \boldsymbol{b} = \{b_k\}, \boldsymbol{c} = \{c_k\}$.

ii. If $a_{n,k}$, $b_{n,k}$, $c_{n,k}$ are purely n-dependent i.e.,

$$a_{n,k} = a_n, \tag{4.5a}$$

$$b_{n,k} = b_n, \tag{4.5b}$$

$$c_{n,k} = c_n \tag{4.5c}$$

for all $n \geq k$, then A is coefficientwise-totally positive in all indeterminates $\boldsymbol{a} = \{a_n\}, \boldsymbol{b} = \{b_n\}, \boldsymbol{c} = \{c_n\}$.

Thus, from Theorem 4.0.0.3, we see that triangles a., b., c. and d. are all totally positive. However, as the coefficients of triangles e. and f. are neither purely k-dependent nor purely n-dependent, we cannot conclude the same for these two triangles.

If we look at the recurrences of triangles e. and f. closely, we see that the recurrences look almost the same except for an extra factor k. This suggests that we should perhaps introduce some variables so that both recurrences can be obtained as specialisations of this new and more general recurrence. This followed with experimental data lead us to make the following conjectures:

Conjecture 4.0.0.4 (*acde*-Eulerian Triangle Conjecture)**.** *The triangle generated by the recurrence*

$$T(0,0) = 1, \tag{4.6a}$$

$$T(n,k) = [a(n-k)+c]T(n-1,k-1) + (dk+e)T(n-1,k) \text{ when } n \geq k, \tag{4.6b}$$

$$T(n,k) = 0 \text{ otherwise.} \tag{4.6c}$$

is coefficientwise-TP in variables a, c, d and e.

Conjecture 4.0.0.5 (*acdefg*-Eulerian Triangle Conjecture)**.** *The triangle generated by the recurrence*

$$T(0,0) = 1, \tag{4.7a}$$

$$T(n,k) = [a(n-k)+c]T(n-1,k-1)+(dk+e)T(n-1,k)+(f(n-2)+g)T(n-2,k-1) \text{ when } n \geq k, \tag{4.7b}$$

$$T(n,k) = 0 \text{ otherwise.} \tag{4.7c}$$

is coefficientwise-TP in variables a, c, d, e, f and g.

Conjecture 4.0.0.6 (*ac$\boldsymbol{d}$fg*-Eulerian Triangle Conjecture)**.** *The triangle generated by the recurrence*

$$T(0,0) = 1, \tag{4.8a}$$

$$T(n,k) = [a(n-k)+c]T(n-1,k-1)+\left(\sum_{i=0}^{k} d_i\right)T(n-1,k)+(f(n-2)+g)T(n-2,k-1) \text{ when } n \geq k, \tag{4.8b}$$

$$T(n,k) = 0 \text{ otherwise.} \tag{4.8c}$$

is coefficientwise-TP in variables a, c, $\boldsymbol{d} = \{d_i\}_{i\geq 0}$, f *and* g.

This project is an attempt at proving these conjectures. Notice that the $ac\boldsymbol{d}fg$-Eulerian Triangle Conjecture implies the $acdefg$-Eulerian Triangle Conjecture which in turn implies the $acde$-Eulerian Triangle Conjecture.

There are two possible directions one could take while attempting to solve these conjectures:

1. Algebraic - The main tool here is factorising the matrix into totally positive matrices. Standard techniques in this direction such as the production matrix method haven't been useful yet (see Pétréolle et al. [2018] for an introduction to production matrices in the context of total positivity).
2. Combinatorial - This involves the construction of a digraph and then applying the LGV lemma, as per Chapter 3. Even though we have managed to construct digraphs for the first few rows, we haven't been able to generalise this to the entire infinite matrix.

In fact, in Chen et al. [2021] we have managed to prove Conjecture 4.0.0.4 with the specialisation $(a,c,d,e) = (a,c,0,e)$ is coefficientwise-TP in a, c and e where we have taken the combinatorial route. Also, in Chen et al., which is currently under construction, we have proved the result algebraically. Notice, that this also implies that we have a proof of Conjecture 4.0.0.1. The following sections are a discussion of our result and contain sketches of our proofs.

4.1 The ace and cde triangles and their combinatorial interpretations

Consider doubly infinite lower triangular matrix $\mathbf{T}(a,c,d,e) = (T(n,k))_{n\geq 0, k\geq 0}$ where $T(n,k)$ are defined as follows:

$$T(0,0) = 1, \tag{4.9a}$$

$$T(n,k) = [a(n-k)+c]T(n-1,k-1) + (dk+e)T(n-1,k) \text{ when } n \geq k, \tag{4.9b}$$

$$T(n,k) = 0 \text{ otherwise.} \tag{4.9c}$$

We shall often refer to $\mathbf{T}(a,c,d,e)$ as the $acde$-triangle, $\mathbf{T}(0,c,d,e)$ as the cde-triangle and $\mathbf{T}(a,c,0,e)$ as the ace-triangle. Note that we have the following reversal identity

$$\mathbf{T}(a,c,d,e) = \mathbf{T}(d,e,a,c)^{\mathrm{rev}} \tag{4.10}$$

and thus

$$\mathbf{T}(a,c,0,e) = \mathbf{T}(0,e,a,c)^{\mathrm{rev}}. \tag{4.11}$$

We now state the main theorem of this chapter.

Theorem 4.1.0.1. *The matrix* $\mathbf{T}(a,c,0,e)$ *is coefficientwise totally positive in the variables* a, c, e.

In the rest of this section, we shall provide interpretations of the $\mathbf{T}(a,c,0,e)$ and $\mathbf{T}(0,c,d,e)$ triangles in terms of set partitions which play an important role in the proof of 4.1.0.1.

Let Π_n be the set of all set partitions of $[n]$ and let $\Pi_{n,k}$ be the set of all set partitions of $[n]$ into k nonempty block. For $i \in [n]$ and $\pi \in \Pi_n$ we write $\mathrm{smallest}((,\pi),i)$ to denote the smallest element of the block containing i. We then have:

Proposition 4.1.0.2. (i) *The matrix* $\mathbf{T}(0,c,d,e)$ *has the combinatorial interpretation*

$$T(n,k) \;=\; \sum_{\pi \in \Pi_{n+1,k+1}} \prod_{i=2}^{n+1} w_\pi(i) \tag{4.12}$$

where

$$w_\pi(i) \;=\; \begin{cases} e & \textit{if } \mathrm{smallest}(\pi,i,=)1 \\ c & \textit{if } \mathrm{smallest}(\pi,i,=)i \\ d & \textit{if } \mathrm{smallest}(\pi,i,\neq)1,i \end{cases} \tag{4.13}$$

(ii) *The matrix* $\mathbf{T}(a,c,0,e)$ *has the combinatorial*

$$T(n,k) \;=\; \sum_{\pi \in \Pi_{n+1,n+1-k}} \prod_{i=2}^{n+1} w_\pi(i) \tag{4.14}$$

where

$$w_\pi(i) \;=\; \begin{cases} c & \textit{if } \mathrm{smallest}(\pi,i,=)1 \\ e & \textit{if } \mathrm{smallest}(\pi,i,=)i \\ a & \textit{if } \mathrm{smallest}(\pi,i,\neq)1,i \end{cases} \tag{4.15}$$

Proof of Proposition 4.1.0.2. To prove (i) we will show that the quantities $T(n,k)$ defined by 4.12/4.13 satisfy the desired recurrence. Part (ii) follows immediately from (i) by way of the reversal identity (4.11).

In a partition $\pi \in \Pi_{n+1,k+1}$, consider the status of the element $n+1$ and what remains when it is deleted. If $n+1$ is a singleton, then it gets a weight c, and what remains is a partition of $[n]$ with k blocks, in which each element gets the same weight as it did in π. This gives a term $c\,T(n-1,k-1)$. If instead $n+1$ belongs to the block containing 1, then it gets a weight e, and what remains is a partition of $[n]$ with $k+1$ blocks, in which each element gets the same weight as it did in π. This gives a term $e\,T(n-1,k)$. Finally, if $n+1$ belongs to a block whose smallest element lies in $\{2,3,\ldots,n\}$, then it gets a weight d, and what remains is a partition of $[n]$ with $k+1$ blocks, in which each element gets the same weight as it did in π. There are k blocks not containing 1 to which the element $n+1$ could have been attached. This gives a term $dk\,T(n-1,k)$. Summing these terms gives the desired recurrence. □

We now state an alternate recurrence without proof that will be useful in our combinatorial proof.

Lemma 4.1.0.3 (Alternate recurrences for $\mathbf{T}(0,c,d,e)$ and $\mathbf{T}(a,c,0,e)$)**.**

(i) *The matrix* $\mathbf{T}(0,c,d,e)$ *satisfies the recurrence*

$$T(n,k) \;=\; e\,T(n-1,k) \;+\; \sum_{m=0}^{n-1} \binom{n-1}{m} d^m c\,T(n-1-m,k-1) \tag{4.16}$$

for $n \geq 1$*, where* $T(n,k) = 0$ *if* $n < 0$ *or* $k < 0$.

(ii) *The matrix* $\mathbf{T}(a,c,0,e)$ *satisfies the recurrence*

$$T(n,k) \;=\; c\,T(n-1,k-1) \;+\; \sum_{m=0}^{n-1} \binom{n-1}{m} a^m e\,T(n-1-m,k-m) \tag{4.17}$$

for $n \geq 1$*, where* $T(n,k) = 0$ *if* $n < 0$ *or* $k < 0$*.*

Lemma 4.1.0.3 can be proved by noticing the status of element $n+1$ in π and also considering the other elements in its block and the different ways to pick them.

4.2 Combinatorial route of proving Theorem 4.1.0.1: construction of a planar network

We know from Chapter 3 that constructing a planar network whose path matrix is $\mathbf{T}(a,c,0,e)$ will prove that it is totally positive. This is the purpose of this section.

4.2.1 Construction of the digraph

Figure 4.1 is a planar network which we denote by D. A special role will be played by the *triangular numbers* $\triangle(n) := \binom{n+1}{2}$ in the description of D. We also define the "triangular ceiling" $\lceil k \rceil^{\mathrm{tri}}$ to be the smallest triangular number that is $\geq k$, and the "triangular defect" $\{k\}^{\mathrm{tri}} := \lceil k \rceil^{\mathrm{tri}} - k$.

The vertices of D are the pairs (i,j) with $0 \leq i \leq j$, where i increases from right to left and j increases from bottom to top. There are two kinds of directed edges: the horizontal directed edge from (i,j) to $(i-1,j)$ [where $1 \leq i \leq j$], and the the diagonal directed edge from (i,j) to $(i-1,j-1)$. We take the source vertices to be $u_n := (\triangle(n), \triangle(n))$ and the sink vertices to be $v_k := (0, \triangle(k))$. The edges are assigned various weights which will now be described.

The diagonal edge from (i,l) to $(i-1,l-1)$ is assigned the weight $\beta_{i,l}$ where

$$\beta_{i,l} \;=\; \begin{cases} e & \text{if } i+l-1 \text{ is triangular and } i+l-1 \geq \triangle(l) \\ 1 & \text{if } i+l-1 \text{ is not triangular and } i+l-1 \geq \triangle(l) \\ 0 & \text{in all other cases} \end{cases} \tag{4.18}$$

for $i \geq 1$ and $l \geq 0$. The horizontal edge from (i,l) to $(i-1,l)$ is assigned the weight $\alpha_{i,l}$ where

$$\alpha_{i,l} \;=\; \begin{cases} a & \text{if } \triangle^{-1}(\lceil i+l-1 \rceil^{\mathrm{tri}}) - l \geq \{i+l-1\}^{\mathrm{tri}} \\ 1 & \text{if } i+l-1 \text{ is triangular and } i+l-1 < \triangle(l) \\ 0 & \text{in all other cases} \end{cases} \tag{4.19}$$

for $i,l \geq 1$. We then delete the edges with zero weight.

4.2.2 The path matrix

Let $\boldsymbol{T}_D$ be the path matrix of the digraph D. Then from the Lindström-Gessel-Viennot lemma have that:

Proposition 4.2.2.1. *The matrix* $\boldsymbol{T}_D$ *is coefficientwise totally positive in the partially ordered commutative ring* $\mathbb{Z}[a,e]$*.*

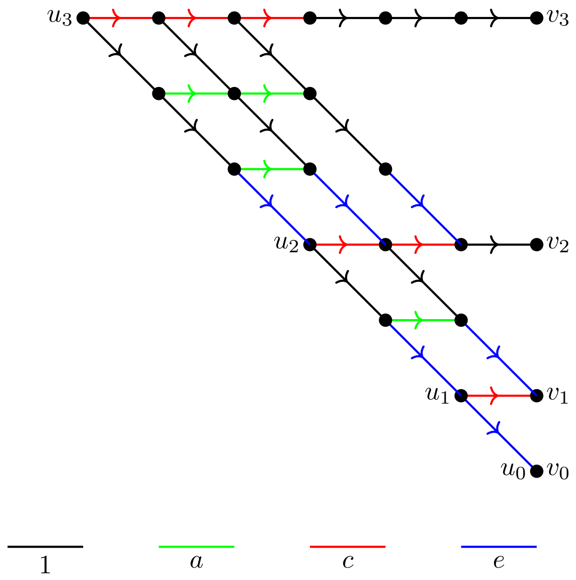

Figure 4.1: The planar network whose path matrix is $\mathbf{T}(a, c, 0, e)$ shown upto source u_4 and sink v_4

The trouble with Proposition 4.2.2.1 is that the set of paths in a digraph can be a rather complicated object; our goal is to find a simpler combinatorial interpretation. In fact, we do this by proving the following theorem:

Theorem 4.2.2.2. *The path matrix $\boldsymbol{T}_D$ coincides with the matrix $\mathbf{T}(a, c, 0, e)$.*

In order to prove Theorem 4.2.2.2, it suffices to construct a weight-preserving bijection between the set of paths from u_n to v_k and the set $\Pi_{n,k}$. We have two ways of doing this, our first proof uses Lemma 4.1.0.3 and induction on the structure of D, the second is by constructing a weight-preserving algorithm that acts as a bijection from paths between u_n to v_k and the set $\Pi_{n,k}$.

4.3 Algebraic route of proving Theorem 4.1.0.1: quasi-production matrix method

In this section, we shall introduce a powerful tool for proving the total-positivity of a lower triangular matrix which we call the method of *quasi-production matrices.*

Let $Q = (q_{i,j})_{i \geq 0, j \geq 0}$ be a lower triangular matrix with entries in a commutative ring. Consider the matrices

$$A = \left[\begin{array}{c|c} I_1 & \mathbf{0} \\ \hline \mathbf{0} & Q \end{array}\right] \left[\begin{array}{c|c} I_2 & \mathbf{0} \\ \hline \mathbf{0} & Q \end{array}\right] \left[\begin{array}{c|c} I_3 & \mathbf{0} \\ \hline \mathbf{0} & Q \end{array}\right] \cdots \tag{4.20}$$

$$\hat{A} = Q \left[\begin{array}{c|c} I_1 & \mathbf{0} \\ \hline \mathbf{0} & Q \end{array}\right] \left[\begin{array}{c|c} I_2 & \mathbf{0} \\ \hline \mathbf{0} & Q \end{array}\right] \left[\begin{array}{c|c} I_3 & \mathbf{0} \\ \hline \mathbf{0} & Q \end{array}\right] \cdots . \tag{4.21}$$

Note that Equations (4.20) and (4.21) can be rewritten as

$$A = \left[\begin{array}{c|c} I_1 & \mathbf{0} \\ \hline \mathbf{0} & QA \end{array}\right] \tag{4.22}$$

$$\hat{A} = Q \left[\begin{array}{c|c} I_1 & \mathbf{0} \\ \hline \mathbf{0} & \hat{A} \end{array}\right] \tag{4.23}$$

and we can obtain Equations (4.20) and (4.21) by iterating Equations (4.22) and (4.23) respectively. We call Q the *quasi-production matrix* and A the *quasi-output matrix* and $\hat{A}$ the *reduced quasi-output matrix*. Note that A and $\hat{A}$ are also lower triangular and they satisfy the following identities hold

$$A = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & \hat{A} \end{array}\right] \tag{4.24}$$

$$\hat{A} = QA. \tag{4.25}$$

Furthermore, we have the following theorem.

Theorem 4.3.0.1. *For a matrix $Q = (q_{i,j})_{i\geq 0, j\geq 0}$ with entries in a partially ordered commutative ring, let A and $\hat{A}$ be the quasi-output matrix and the reduced quasi-output matrix as defined in Equations* (4.20) *and* (4.21)*. If Q is totally positive of order r, then so are A, A^{rev} and $\hat{A}, \hat{A}^{\text{rev}}$.*

Thus, not only are the output matrices totally positive, their reversals are also totally positive. We omit the proof of this theorem and instead proceed to prove Theorem 4.1.0.1 using Theorem 4.3.0.1.

Proposition 4.3.0.2. *Let T_e denote the inverse bidiagonal matrix $T(e, e, e, e, e, \ldots)$ and let B_d be the binomial matrix $B(1, d)$, and let*

$$Q := T_e \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & cB_d \end{array}\right].$$

Q satisfies the identity

$$\mathbf{T}(0, c, d, e) = Q \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{T}(0, c, d, e) \end{array}\right]. \tag{4.26}$$

We can prove this proposition using the alternate recurrence in Equation (4.16) and we omit the details. Thus, Theorem 4.1.0.1 follows from Proposition 4.3.0.2 and Theorem 4.3.0.1.

Chapter 5

Total Positivity Questions for the Higher Order Stirling Cycle Triangles (Ongoing joint work with Alan D. Sokal)

Given a lower triangular array, as observed in Section 2.3, we can ask three different questions about total positivity:

1. Is the triangle totally positive?

2. Is the sequence of row-generating polynomials coefficientwise-Hankel-totally positive?

3. Are the rows Toeplitz totally positive? Are the row–generating polynomials real-rooted with non-positive roots?

One of the triangles for which the answer to all three of the above questions is true, is the Stirling cycle triangle which was introduced in Section 1.2.5. The (n, k)-entry is $\left[{n \atop k}\right]$, the number of permutations on $[n]$ with k cycles.

1. The total positivity of the triangle has been discussed in Section 1.2.5.

2. It is well known that the row-generating polynomial of the n^{th} row is $x(x+1)\ldots(x+n-1)$ (see for example [Comtet, 1974, Theorem A p. 213 Section 5.5] or [Stanley, 2009, Proposition 1.3.7 p. 27]) which are the Stirling cycle polynomials. They have a Stieltjes continued fraction that was found by Euler (Euler [1760]). In fact, Euler found the following

continued fraction

$$\sum_{n=0}^{\infty} x(x+y)\dots(x+(n-1)y)t^n = \cfrac{1}{1-\cfrac{xt}{1-\cfrac{yt}{1-\cfrac{(x+y)t}{1-\cfrac{2y}{1-\cfrac{(x+2y)t}{1-\cfrac{3yt}{1-\ddots}}}}}}}. \tag{5.1}$$

Putting $y = 1$, we obtain an S-fraction for the Stirling cycle polynomials. From [Pétréolle et al., 2018, Theorem 9.8], the sequence of Stirling cycle polynomials is coefficientwise-Hankel-TP in the variable x.

3. The roots of the Stirling cycle polynomials are clearly non-positive real numbers. Thus, the rows are Toeplitz totally positive.

There are several generalisation and extensions of the Stirling cycle numbers and one such generalisation is the r-associated Stirling cycle numbers $\left[{n \atop k}\right]_r$, where $\left[{n \atop k}\right]_r$ denotes the number of permutations on $[n]$ with k cycles such that each cycle has at least r elements (see [Comtet, 1974, pp. 256, 293, 295]). Notice that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_1 .$$

The first few $\left[{n \atop k}\right]_2$ are given below (A008306, OEIS Foundation Inc. [2019]):

$$\begin{matrix} 1 & & & & \\ & 1 & & & \\ & 2 & & & \\ & 6 & 3 & & \\ & 24 & 20 & & \\ & 120 & 130 & 15 & \\ & 720 & 924 & 210 & \\ & 5040 & 7308 & 2380 & 105 \end{matrix} \tag{5.2}$$

The sequence of row-generating polynomials of this triangle is not Hankel-TP. This is because the n^{th} row-generating polynomial is of degree $\lceil n/2 \rceil$ and we can show that some minors must have negative coefficients.

Instead, we consider the triangle where each of the columns are shifted up so that the diagonal entries become non-zero, i.e. consider the matrix whose (n, k)-entry is:

$$\begin{bmatrix} n \\ k \end{bmatrix}^{(r)} = \begin{bmatrix} n+(r-1)k \\ k \end{bmatrix}_r . \tag{5.3}$$

So

$$\begin{bmatrix} n \\ k \end{bmatrix}^{(1)} = \begin{bmatrix} n \\ k \end{bmatrix}_1 = \begin{bmatrix} n \\ k \end{bmatrix} \tag{5.4}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}^{(2)} = \begin{bmatrix} n+k \\ k \end{bmatrix}_2 \tag{5.5}$$

The first few entries of the triangle for $r = 2$ are:

$$\begin{array}{llll} 1 & & & \\ & 1 & & \\ & 2 & 3 & \\ & 6 & 20 & 15 \\ & 24 & 130 & 210 & 105 \end{array} \tag{5.6}$$

(A259456, A269940, A111999 in OEIS Foundation Inc. [2019] and the rows sums are A032188).

For a fixed r, we call the numbers $\left[{n \atop k}\right]^{(r)}$ as the r^{th}-order Stirling cycle numbers, the triangle will be called the r^{th}-order Stirling cycle triangle which we denote by $C^{(r)}$ and we call the row-generating polynomials the r^{th}-order Stirling cycle polynomials, which we denote by $c_{r,n}(x)$.

The numbers $\left[{n \atop k}\right]^{(r)}$ satisfy the recurrence relation:

$$\begin{bmatrix} n \\ k \end{bmatrix}^{(r)} = (r-1)!\binom{n+(r-1)k-1}{r-1}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{(r)} + (n+(r-1)k-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}^{(r)}. \tag{5.7}$$

The primary goal of this project is to answer the following conjecture:

Conjecture 5.0.0.1. *The following are true:*

1. *The triangle $C^{(r)}$ is totally positive for all $r \geq 1$.*
2. *The row-generating polynomials $c_{r,n}(x)$ are real-rooted and have non-positive roots for $r = 1$ and $r = 2$.*
3. *The sequence $\{c_{r,n}(x)\}_{n\geq 0}$ is coefficientwise-Hankel-totally positive in the variable x for all $r \geq 1$.*

Using computer experiments, we have observed that *2.* does not hold when $r = 3$ and we conjecture that for all $r \geq 3$ there is an n, such that $c_{r,n}(x)$ is not non-positive real-rooted.

So far, we have managed to prove *2.* and *3.* for $r = 2$.

Chapter 6

Experiments on the Total Positivity of the Generic Lah Triangles (Ongoing joint work with Alexander Dyachenko and Alan D. Sokal)

An *ordered tree* (also called a plane tree) is a rooted tree in which the children of each vertex are linearly ordered. An *unordered forest of increasing ordered trees* is an unordered collection of ordered trees. An *unordered forest of increasing ordered trees with n vertices* is an unordered forest of ordered trees with n vertices such that each vertex has a distinct label from the set $[n]$, and the label of each vertex is strictly greater than the label of its parent. An *increasing ordered tree* is an unordered forest of increasing ordered trees with one component.

Let F be an unordered forest of increasing ordered trees. Let $\boldsymbol{\phi} = (\phi_i)_{i\geq 0}$ be a family of indeterminates. Each vertex in F with i children is assigned the weight ϕ_i and the weight of F is the product of the weight of its vertices. Let $L_{n,k}(\boldsymbol{\phi})$ be the generating polynomial of unordered forests of increasing ordered trees with n vertices and k components. Let

$$L_n(\boldsymbol{\phi}, y) = \sum_{k=0}^{n} L_{n,k}(\boldsymbol{\phi}) y^k.$$

The first few $L_{n,k}(\boldsymbol{\phi})$ can be found in [Pétréolle and Sokal, 2021, Appendix A.1]. If we specialize the indeterminates $\boldsymbol{\phi}$ to elements from a partially ordered commutative ring R, we can talk about the various kinds of total positivity of the matrix $(L_{n,k}(\boldsymbol{\phi}))_{n\geq 0, k\geq 0}$. In fact, Pétréolle and Sokal showed the followed theorem in [Pétréolle and Sokal, 2021, Theorem 1.1]:

Theorem 6.0.0.1 ([Pétréolle and Sokal, 2021, Theorem 1.1])**.** *Fix $1 \leq r \leq \infty$. Let R be a partially ordered commutative ring, and let $\boldsymbol{\phi} = (\phi_i)_{i\geq 0}$ be a sequence in R that is Toeplitz-totally positive of order r. Then:*

(a) The lower-triangular matrix $\mathrm{L} = (L_{n,k}(\boldsymbol{\phi}))_{n\geq 0, k\geq 0}$ *is totally positive of order* r *in the ring* R.

(b) The sequence of $\boldsymbol{L}(\boldsymbol{\phi}) = (L_n(\boldsymbol{\phi}, y))_{n\geq 0}$ *is Hankel-totally positive of order* r *in the ring* $R[y]$ *equipped with the coefficientwise order.*

(c) The sequence $\boldsymbol{L}^{\Delta}(\boldsymbol{\phi}) = (L_{n+1,1}(\boldsymbol{\phi}, y))_{n\geq 0}$ *is Hankel-totally positive of order* r *in the ring* R.

The hypothesis in this theorem is that the sequence $\boldsymbol{\phi}$ should be Toeplitz totally positive, which is a sufficient condition for (a), (b) and (c) to hold but it is far from necessary. The goal of this project is to experimentally investigate and find conditions which are both necessary and sufficient for (a), (b) and (c) to hold.

Pétréolle and Sokal, in their proof, showed that $(b) \Rightarrow (c)$. Also checking (a) and (b) requires more computational resources than checking (c). Hence, we first focus our attention on finding a necessary and sufficient condition for (c) to hold.

6.1 Methodology

Our investigation takes place according to the following steps:

1. We assume that ϕ_i are all arbitrary real numbers. Then the sequence $(L_{n+1,1}(\boldsymbol{\phi}, y))_{n\geq 0}$ is a sequence of real numbers. Thus, $\boldsymbol{L}^{\Delta}(\boldsymbol{\phi}) = (L_{n+1,1}(\boldsymbol{\phi}, y))_{n\geq 0}$ is Hankel-TP if and only if its generating function has an S-fraction expansion

$$\sum_{n=0}^{\infty} L_{n+1,1}(\boldsymbol{\phi}) t^n = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \ddots}}}$$

   with $\alpha_i \geq 0$. We algorithmically compute α_i as polynomials in $(\phi_i)_{i\geq 0}$ using an algorithm due to Euler [1760] and Viscovatov [1805].

   Thus, solving the system of polynomial inequalities

$$\alpha_i \geq 0 \qquad \forall i \geq 0$$

   in terms of $(\phi_i)_{i\geq 0}$ would provide necessary and sufficient conditions on $(\phi_i)_{i\geq 0}$ for $(L_{n+1,1}(\boldsymbol{\phi}, y))_{n\geq 0}$ to be Hankel-TP.

2. For ease of computation we make a further assumption that $\phi_i = 0$ for all $i > N$ for some $N \in \mathbb{N}$. Thus, $(\alpha_i)_{i\geq 1}$ are polynomials in a finite number of indeterminates $\phi_0, \ldots, \phi_N$. We want to guess the necessary and sufficient conditions for every fixed N. We hope that taking limits as $N \to \infty$ will provide more general sufficient conditions on $(\phi_i)_{i\geq 0}$ than Toeplitz-TP.

3. With some rescaling, we may assume that $\phi_0 = \phi_N = 1$. Thus, the effective number of variables becomes $N - 1$. Let $p_N(x)$ be the polynomial

$$p_N(x) = \sum_{i=0}^{N} \phi_i x^i$$

   and we shall refer to the case where $\phi_i = 0$ for all $i > N$ as the N^{th}-degree case.

4. For the N^{th}-degree case we compute $(\phi_i)_{1\leq i\leq M}$ and let $R(N,M)$ be the region defined by the inequalities

$$\alpha_i \geq 0 \qquad \forall 1 \leq i \leq M.$$

   Note that $R(N,M)$ lies within the positive quadrant of $\mathbb{R}^{N-1}$. We plot the intersections of $R(N,M)$ with 1 and 2-dimensional hyperplanes and guess parameterisations for $R(N,M)$ in terms of conditions on $(\phi_i)_{1\leq i\leq N-1}$.

6.2 Results

6.2.1 Quadratic Case

We have $p_2(x) = 1 + \phi_1 x + x^2$. We can prove that the necessary and sufficient condition in this case is

$$\phi_1 \geq 2$$

which is also the case when $p_2(x)$ is real-rooted, and hence negative real-rooted. This coincides with the condition that $\boldsymbol{\phi}$ is Toeplitz-TP according to the ASWE Theorem (Theorem 2.2.0.1).

6.2.2 Cubic Case

We have $p_3(x) = 1 + \phi_1 x + \phi_2 x^2 + x^3$. Here $R(2,M) \subseteq \mathbb{R}^2$ and hence we make plots of $R(3,M)$. From the plots we guess that the boundary of $R(2,M)$ is given by the curve defined the parameterisation

$$\phi_1 = \frac{1}{c} + 2c^2 \tag{6.1a}$$

$$\phi_2 = 3c \tag{6.1b}$$

for a parameter $c > 0$ and the Toeplitz-TP region. We have managed to prove that the boundary lies within $R(N,M)$. However we haven't showed that this is indeed the boundary of the region. See Figure 6.1 for a plot of this region.

6.2.3 Quartic Case

We have $p_4(x) = 1 + \phi_1 x + \phi_2 x^2 + \phi_3 x^3 + x^4$. In our experiments we have focused on 2-dimensional plots rather than 3-dimensional plots. We have looked at the following 2-dimensional subregions:

- Symmetric Quartic - Here we assume that $\phi_1 = \phi_3$. See Figure 6.2 for a plot of this region.
- Quartic with Pure Imaginary Roots - Here we assume that $p_4(x) = (x^2+1)(x^2+ax+b)$. See Figure 6.3 for a plot of this region.

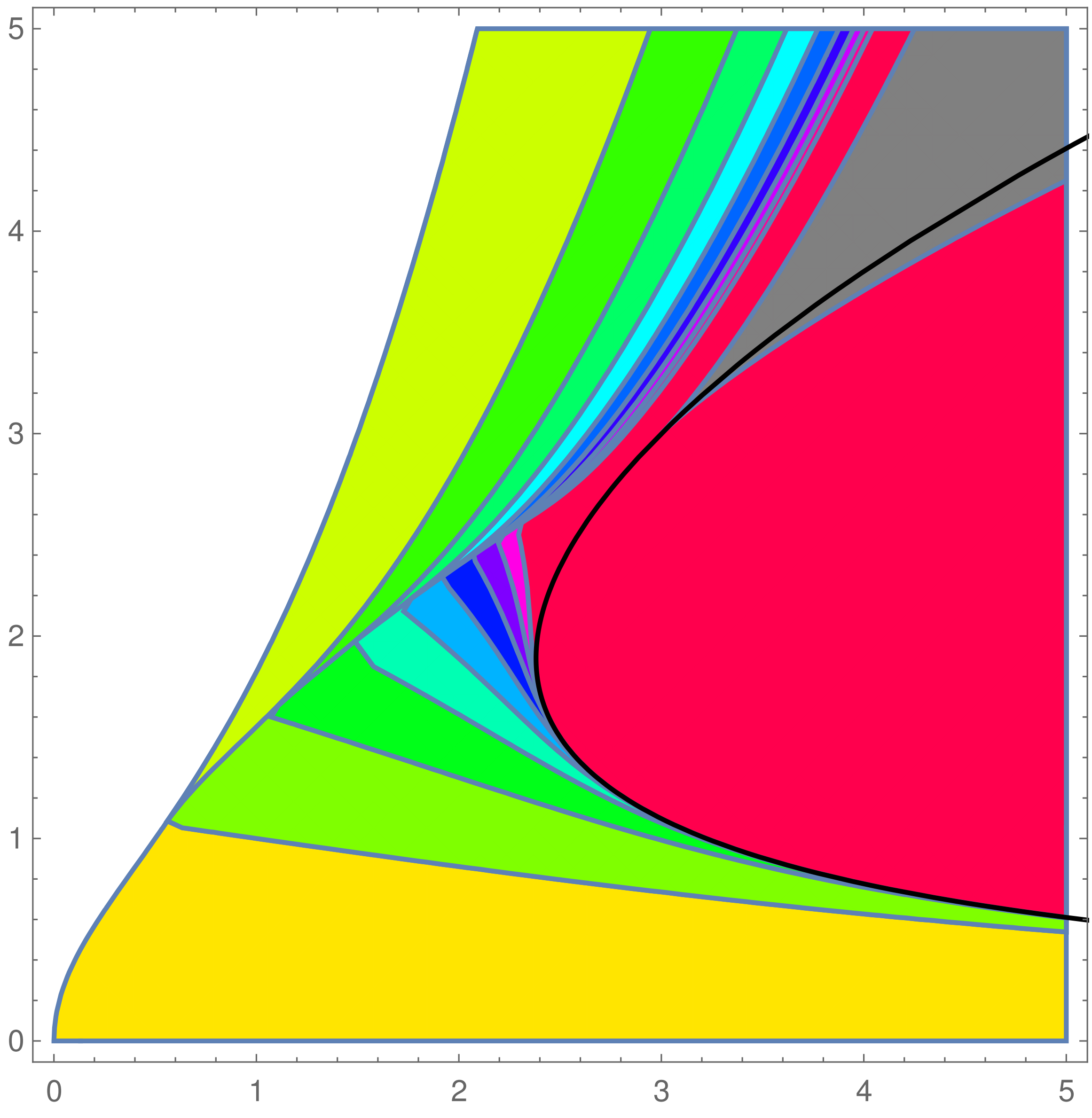

Figure 6.1: This is a plot of $R(3, M)$ for $3 \leq M \leq 19$ with ϕ_1 as the x-axis and ϕ_2 as the y-axis. Here $R(M, 19)$ is the innermost red region. The black curve is the curve parameterised by Equations (6.1). The grey region inside is the region where the sequence $\boldsymbol{\phi}$ is Toeplitz-TP. We guess that the left boundary of $R(N, M)$ is approaching the left boundary of the grey curve for $y \geq 3$ and the black line for $y \leq 3$, and the lower boundary of the region is approaching the black curve.

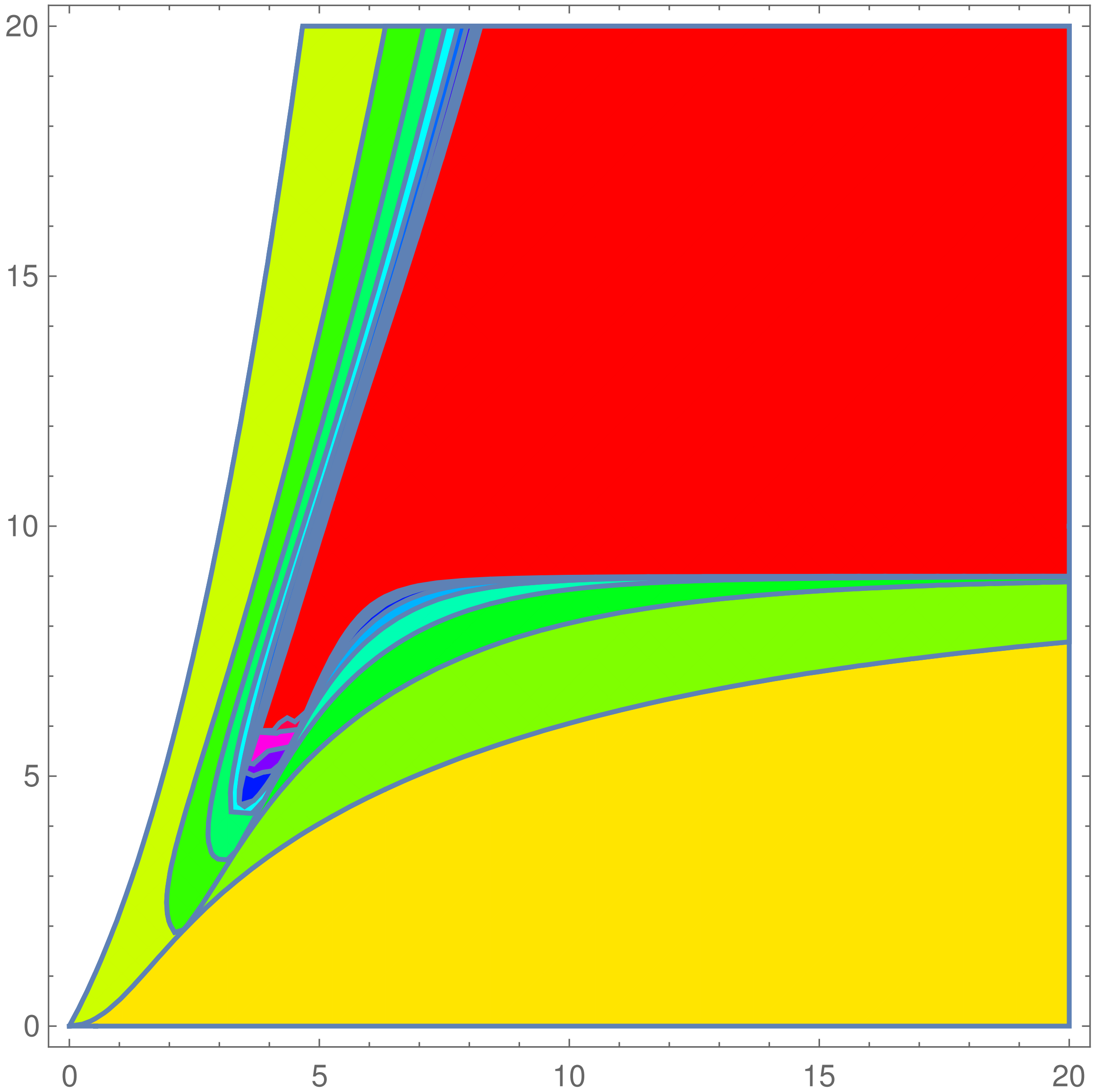

Figure 6.2: This is a plot of $R(4, M)$ where $p_4(x) = (1 + ax + bx^2 + cx^3 + x^4)$ for $3 \leq M \leq 20$ with $a/2$ as the x-axis and $b/3$ as the y-axis.

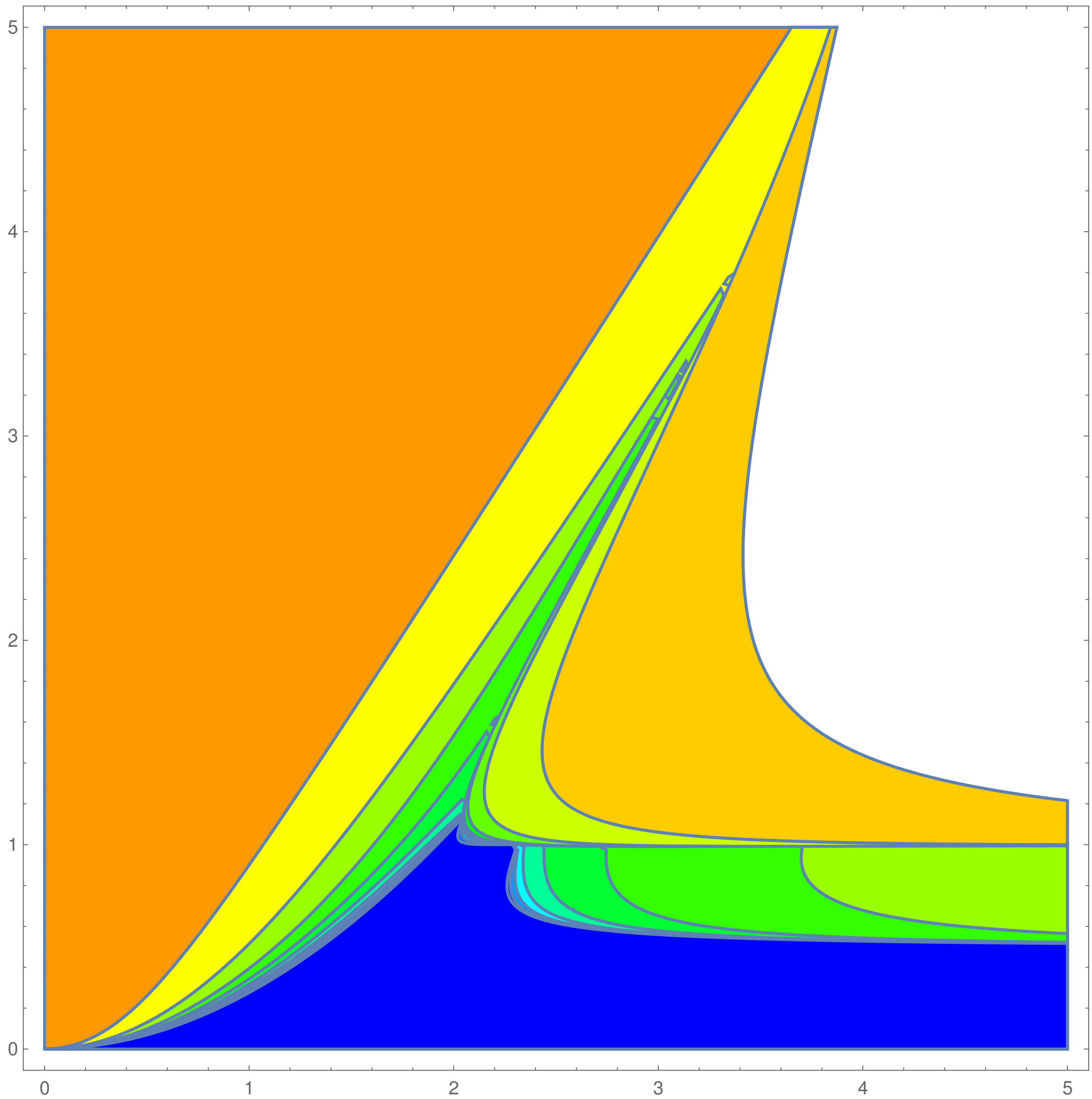

Figure 6.3: This is a plot of $R(4, M)$ where $p_4(x) = (x^2 + 1)(x^2 + ax + b)$ for $3 \leq M \leq 20$ with a as the x-axis and b as the y-axis.

Chapter 7

New Observations on the Total Positivity of the Trees and Forests Matrices

The number of labelled forests of rooted trees on n vertices is given by $f_n = (n+1)^{n-1}$ and the number of labelled forests with k rooted trees on n vertices is given by $f_{n,k} = \binom{n-1}{k-1} n^{n-k}$ (see [OEIS Foundation Inc., 2019, A061356/A137452 and A000272] and references therein). Let F be the matrix

$$F = (f_{n,k})_{n \geq 0, k \geq 0} \,. \tag{7.1}$$

In two recent papers, Sokal [2021] and Gilmore [2021], independently showed that the matrix F is totally positive by using two different methods. The former used the theory of Riordan arrays while the latter provided digraphs with non-negative edge weights whose path matrix is F and then appealed to the LGV lemma. Both authors provide a plethora of open problems and conjectures in Section 6 of their papers. The aim of this project is to try and resolve some of these conjectures and open problems, by using a bijective approach.

Even though Gilmore uses the LGV lemma, he does not have a bijection between paths in his digraph and forests. I have partially resolved this problem by constructing a bijection between paths and certain functions.

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