Total positivity of the Eulerian triangle: A big generalisation of Brenti's conjecture

Bishal Deb (he/him)

University College London

April 24, 2021
Graduate Student Combinatorics Conference (GSCC) 2021

Based on Ongoing Joint Work With
Xi Chen, Alexander Dyachenko, Tomack Gilmore and Alan D. Sokal

Introduction

Definition (Total Positivity (TP))

A matrix of real numbers said to be totally positive (TP) if all its minors are non-negative

Introduction

Definition (Total Positivity (TP))

A matrix of real numbers said to be totally positive (TP) if all its minors are non-negative i.e., determinants of all finite square submatrices are non-negative.

Introduction

Definition (Total Positivity (TP))

A matrix of real numbers said to be totally positive (TP) if all its minors are non-negative i.e., determinants of all finite square submatrices are non-negative.

Array of numbers and not linear operator.

Definition (Total Positivity (TP))

A matrix of real numbers said to be totally positive (TP) if all its minors are non-negative i.e., determinants of all finite square submatrices are non-negative.

Array of numbers and not linear operator.

Need not be a square matrix,

Introduction

Definition (Total Positivity (TP))

A matrix of real numbers said to be totally positive (TP) if all its minors are non-negative i.e., determinants of all finite square submatrices are non-negative.

Array of numbers and not linear operator.

Need not be a square matrix, or finite!

Definition (Total Positivity (TP))

A matrix of real numbers said to be totally positive (TP) if all its minors are non-negative i.e., determinants of all finite square submatrices are non-negative.

Array of numbers and not linear operator.

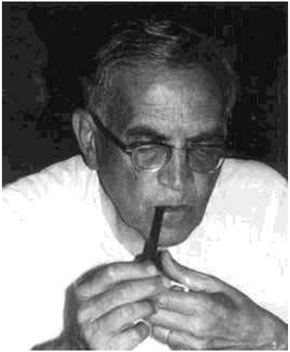
Need not be a square matrix, or finite!

Will consider matrix of polynomials soon!

Historical Note

First defined independently by two different groups in the 30s

(a) M.G. Krein (1907-1989)



(b) I.G. Schoenberg (1903-1990)

Source: MacTutor History of Mathematics Archive

We use Schoenberg's terminology.

Example and basic properties

- Example: Bidiagonal matrices with entries ≥ 0

Example and basic properties

- Example: Bidiagonal matrices with entries ≥ 0

- Matrix sum doesn't preserve TP

Example and basic properties

- Example: Bidiagonal matrices with entries ≥ 0
- Matrix sum doesn't preserve TP (Has 2×2 example)

Example and basic properties

- Example: Bidiagonal matrices with entries ≥ 0
- Matrix sum doesn't preserve TP (Has 2×2 example)
- Matrix product preserves TP

Example and basic properties

- Example: Bidiagonal matrices with entries ≥ 0
- Matrix sum doesn't preserve TP (Has 2×2 example)
- Matrix product preserves TP (Cauchy-Binet formula)

Applications to Combinatorics

Applications to Combinatorics

- Brenti in 80s started studying TP in combinatorics.

Applications to Combinatorics

- Brenti in 80s started studying TP in combinatorics.
- Many important combinatorial triangles are TP.

Applications to Combinatorics

- Brenti in 80s started studying TP in combinatorics.

- Many important combinatorial triangles are TP.

  Examples

Applications to Combinatorics

- Brenti in 80s started studying TP in combinatorics.
- Many important combinatorial triangles are TP.

  Examples

  - Binomial triangle. Entries $\binom{n}{k}$

Applications to Combinatorics

- Brenti in 80s started studying TP in combinatorics.

- Many important combinatorial triangles are TP.

  Examples

  - Binomial triangle. Entries $\binom{n}{k}$
  - Stirling cycle triangle. Entries $\left[{n+1 \atop k}\right]$

Applications to Combinatorics

- Brenti in 80s started studying TP in combinatorics.

- Many important combinatorial triangles are TP.

  Examples

  - Binomial triangle. Entries $\binom{n}{k}$
  - Stirling cycle triangle. Entries $\left[{n+1 \atop k}\right]$
  - Stirling subset triangle. Entries $\left\{{n+1 \atop k}\right\}$

Applications to Combinatorics

- Brenti in 80s started studying TP in combinatorics.
- Many important combinatorial triangles are TP.

  Examples
  - Binomial triangle. Entries $\binom{n}{k}$
  - Stirling cycle triangle. Entries $\left[{n+1 \atop k}\right]$
  - Stirling subset triangle. Entries $\left\{{n+1 \atop k}\right\}$
  - Reversed Stirling cycle triangle. Entries $\left[{n+1 \atop n-k+1}\right]$

Applications to Combinatorics

- Brenti in 80s started studying TP in combinatorics.
- Many important combinatorial triangles are TP.

  Examples
  - Binomial triangle. Entries $\binom{n}{k}$
  - Stirling cycle triangle. Entries $\left[{n+1 \atop k}\right]$
  - Stirling subset triangle. Entries $\left\{{n+1 \atop k}\right\}$
  - Reversed Stirling cycle triangle. Entries $\left[{n+1 \atop n-k+1}\right]$ (reversal does not preserve TP)

Reversed Stirling Subset Triangle

Reversed Stirling subset triangle:

$$\left(\left\{ {n+1 \atop k} \right\}^{\mathrm{rev}}\right)_{n,k\geq 0} = \left(\left\{ {n+1 \atop n-k+1} \right\}\right)_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 6 & 7 & 1 & & \\ 1 & 10 & 25 & 15 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Reversed Stirling Subset Triangle

Reversed Stirling subset triangle:

$$\left(\left\{ {n+1 \atop k} \right\}^{\mathrm{rev}}\right)_{n,k\geq 0} = \left(\left\{ {n+1 \atop n-k+1} \right\}\right)_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 6 & 7 & 1 & & \\ 1 & 10 & 25 & 15 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Conjecture (Us 2019)

The infinite lower triangular matrix $\left(\left\{ {n+1 \atop k} \right\}^{\mathrm{rev}}\right)_{n,k\geq 0} = \left(\left\{ {n+1 \atop n-k+1} \right\}\right)_{n,k\geq 0}$ *is TP.*

Eulerian Numbers

Eulerian Numbers

Descent of a permutation σ:

Eulerian Numbers

Descent of a permutation σ: Position i such that $\sigma(i) > \sigma(i+1)$.

Eulerian Numbers

Descent of a permutation σ: Position i such that $\sigma(i) > \sigma(i+1)$.

$\mathrm{des}(\sigma)$: number of descents of σ.

Eulerian Numbers

Descent of a permutation σ: Position i such that $\sigma(i) > \sigma(i+1)$.

$\operatorname{des}(\sigma)$: number of descents of σ.

Example: For $\sigma = 943127685$, $\operatorname{des}(\sigma) = 5$

Eulerian Numbers

Descent of a permutation σ: Position i such that $\sigma(i) > \sigma(i+1)$.

$\mathrm{des}(\sigma)$: number of descents of σ.

Example: For $\sigma = 943127685$, $\mathrm{des}(\sigma) = 5$

Definition

The Eulerian number $\left\langle {n \atop k} \right\rangle$ is defined to be the cardinality of the set $\{\sigma \in \mathfrak{S}_n | \mathrm{des}(\sigma) = k\}$.

Eulerian Numbers

Definition

The Eulerian number $\left\langle {n \atop k} \right\rangle$ is defined to be the cardinality of the set $\{\sigma \in \mathfrak{S}_n | \mathrm{des}(\sigma) = k\}$.

Eulerian Numbers

Definition

The Eulerian number $\left\langle {n \atop k} \right\rangle$ is defined to be the cardinality of the set $\{\sigma \in \mathfrak{S}_n | \mathrm{des}(\sigma) = k\}$.

$$\left(\left\langle {n+1 \atop k} \right\rangle \right)_{n,k \geq 0} = \begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ 1 & 11 & 11 & 1 & & & \\ 1 & 26 & 66 & 26 & 1 & & \\ 1 & 57 & 302 & 302 & 57 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Eulerian Triangle Conjecture

$$\left(\left\langle {n+1 \atop k} \right\rangle\right)_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ 1 & 11 & 11 & 1 & & & \\ 1 & 26 & 66 & 26 & 1 & & \\ 1 & 57 & 302 & 302 & 57 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Conjecture (Brenti 1996)

The infinite lower triangular matrix $\left(\left\langle {n+1 \atop k} \right\rangle\right)_{n,k\geq 0}$ *is TP.*

Eulerian Triangle Conjecture

$$\left(\left\langle {n+1 \atop k} \right\rangle\right)_{n,k\geq 0} = \begin{bmatrix} 1 &&&&&& \\ 1 & 1 &&&&& \\ 1 & 4 & 1 &&&& \\ 1 & 11 & 11 & 1 &&& \\ 1 & 26 & 66 & 26 & 1 && \\ 1 & 57 & 302 & 302 & 57 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Conjecture (Brenti 1996)

The infinite lower triangular matrix $\left(\left\langle {n+1 \atop k} \right\rangle\right)_{n,k\geq 0}$ *is TP.*

Verified for 512×512 using an efficient bidiagonal factorisation algorithm.

Eulerian Triangle Conjecture

$$\left(\left\langle {n+1 \atop k} \right\rangle\right)_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ 1 & 11 & 11 & 1 & & & \\ 1 & 26 & 66 & 26 & 1 & & \\ 1 & 57 & 302 & 302 & 57 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Conjecture (Brenti 1996)

The infinite lower triangular matrix $\left(\left\langle {n+1 \atop k} \right\rangle\right)_{n,k\geq 0}$ *is TP.*

Verified for 512×512 using an efficient bidiagonal factorisation algorithm.

Rational numbers with large denominators appear.

Eulerian Triangle Conjecture

$$\left(\left\langle {n+1 \atop k} \right\rangle\right)_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ 1 & 11 & 11 & 1 & & & \\ 1 & 26 & 66 & 26 & 1 & & \\ 1 & 57 & 302 & 302 & 57 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Conjecture (Brenti 1996)

The infinite lower triangular matrix $\left(\left\langle {n+1 \atop k} \right\rangle\right)_{n,k\geq 0}$ *is TP.*

Verified for 512×512 using an efficient bidiagonal factorisation algorithm.

Rational numbers with large denominators appear.

General pattern not clear.

A comparision of the two triangles

Reversed Stirling subset triangle:

$$\left(\left\{ {n+1 \atop k} \right\}^{\mathrm{rev}}\right)_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 6 & 7 & 1 & & \\ 1 & 10 & 25 & 15 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Eulerian triangle:

$$\left(\left\langle {n+1 \atop k} \right\rangle\right)_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 4 & 1 & & & \\ 1 & 11 & 11 & 1 & & \\ 1 & 26 & 66 & 26 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

A comparision of the two triangles

Reversed Stirling subset triangle:

$$\left(\left\{ {n+1 \atop k} \right\}^{\mathrm{rev}}\right)_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 6 & 7 & 1 & & \\ 1 & 10 & 25 & 15 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$A_{n,k} = (n-k+1)A_{n-1,k-1} + 1 \cdot A_{n-1,k}$$

Eulerian triangle:

$$\left(\left\langle {n+1 \atop k} \right\rangle\right)_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 4 & 1 & & & \\ 1 & 11 & 11 & 1 & & \\ 1 & 26 & 66 & 26 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$A_{n,k} = (n-k+1)A_{n-1,k-1} + (k+1)A_{n-1,k}$$

A comparision of the two triangles

Reversed Stirling subset triangle:

$$\left(\left\{ {n+1 \atop k} \right\}^{\mathrm{rev}}\right)_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 6 & 7 & 1 & & \\ 1 & 10 & 25 & 15 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$A_{n,k} = (n-k+1)A_{n-1,k-1} + 1 \cdot A_{n-1,k}$$

Introduce variables

Eulerian triangle:

$$\left(\left\langle {n+1 \atop k} \right\rangle\right)_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 4 & 1 & & & \\ 1 & 11 & 11 & 1 & & \\ 1 & 26 & 66 & 26 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$A_{n,k} = (n-k+1)A_{n-1,k-1} + (k+1)A_{n-1,k}$$

Coefficientwise TP

Definition (Coefficientwise TP)

A matrix of polynomials with real coefficients is said to be coefficientwise totally positive (coefficientwise TP) if all its minors have non-negative coefficients.

Coefficientwise TP

Definition (Coefficientwise TP)

A matrix of polynomials with real coefficients is said to be coefficientwise totally positive (coefficientwise TP) if all its minors have non-negative coefficients.

One or several variables.

Coefficientwise TP

Definition (Coefficientwise TP)

A matrix of polynomials with real coefficients is said to be coefficientwise totally positive (coefficientwise TP) if all its minors have non-negative coefficients.

One or several variables.

Coefficientwise TP $\implies$ Pointwise TP.

Coefficientwise TP

Definition (Coefficientwise TP)

A matrix of polynomials with real coefficients is said to be coefficientwise totally positive (coefficientwise TP) if all its minors have non-negative coefficients.

One or several variables.

Coefficientwise TP $\Longrightarrow$ Pointwise TP.

But much stronger.

Putting variables in the recurrence

Replace

$$A_{n,k} = (n-k+1)A_{n-1,k-1} + (k+1)A_{n-1,k}$$

$$A_{n,k} = (n-k+1)A_{n-1,k-1} + 1 \cdot A_{n-1,k}$$

Introduce variables a, b, c, d, e.

$$A_{n,k} = [a(n-k) + b(k-1) + c]A_{n-1,k-1} + (dk+e)A_{n-1,k}$$

Putting variables in the recurrence

Replace

$$A_{n,k} = (n-k+1)A_{n-1,k-1} + (k+1)A_{n-1,k}$$

$$A_{n,k} = (n-k+1)A_{n-1,k-1} + 1 \cdot A_{n-1,k}$$

Introduce variables a, b, c, d, e.

$$A_{n,k} = [a(n-k) + b(k-1) + c]A_{n-1,k-1} + (dk+e)A_{n-1,k}$$

Is it coefficientwise TP?
(Minors are polynomials with non-negative coefficients?)

Putting variables in the recurrence

Replace

$$A_{n,k} = (n-k+1)A_{n-1,k-1} + (k+1)A_{n-1,k}$$

$$A_{n,k} = (n-k+1)A_{n-1,k-1} + 1 \cdot A_{n-1,k}$$

Introduce variables a, b, c, d, e.

$$A_{n,k} = [a(n-k) + b(k-1) + c]A_{n-1,k-1} + (dk+e)A_{n-1,k}$$

Is it coefficientwise TP?
(Minors are polynomials with non-negative coefficients?)
More general conjecture!

Putting variables in the recurrence

Replace

$$A_{n,k} = (n-k+1)A_{n-1,k-1} + (k+1)A_{n-1,k}$$
$$A_{n,k} = (n-k+1)A_{n-1,k-1} + 1 \cdot A_{n-1,k}$$

Introduce variables a, b, c, d, e.

$$A_{n,k} = [a(n-k) + b(k-1) + c]A_{n-1,k-1} + (dk+e)A_{n-1,k}$$

Is it coefficientwise TP?
(Minors are polynomials with non-negative coefficients?)
More general conjecture!

True till 11×11 - a week.

Putting variables in the recurrence

Replace

$$A_{n,k} = (n-k+1)A_{n-1,k-1} + (k+1)A_{n-1,k}$$
$$A_{n,k} = (n-k+1)A_{n-1,k-1} + 1 \cdot A_{n-1,k}$$

Introduce variables a, b, c, d, e.

$$A_{n,k} = [a(n-k) + b(k-1) + c]A_{n-1,k-1} + (dk+e)A_{n-1,k}$$

Is it coefficientwise TP?
(Minors are polynomials with non-negative coefficients?)
More general conjecture!

True till 11×11 - a week.

False for 12×12 - 99 days CPU time.

Putting variables in the recurrence

Replace

$$A_{n,k} = (n-k+1)A_{n-1,k-1} + (k+1)A_{n-1,k}$$

$$A_{n,k} = (n-k+1)A_{n-1,k-1} + 1 \cdot A_{n-1,k}$$

Introduce variables a, b, c, d, e.

$$A_{n,k} = [a(n-k) + b(k-1) + c]A_{n-1,k-1} + (dk+e)A_{n-1,k}$$

Is it coefficientwise TP?
(Minors are polynomials with non-negative coefficients?)
More general conjecture!

True till 11×11 - a week.

False for 12×12 - 99 days CPU time. ☹

Is all hope lost?

Put $b = 0$ to obtain

$$A_{n,k} = [a(n-k) + c]A_{n-1,k-1} + (dk + e)A_{n-1,k}$$

Is all hope lost?

Put $b = 0$ to obtain

$$A_{n,k} = [a(n-k) + c]A_{n-1,k-1} + (dk + e)A_{n-1,k}$$

Empirically true upto 13×13.

Is all hope lost?

Put $b = 0$ to obtain

$$A_{n,k} = [a(n-k) + c]A_{n-1,k-1} + (dk + e)A_{n-1,k}$$

Empirically true upto 13×13.

Special cases:

(a, c, d, e)	Matrix obtained

Is all hope lost?

Put $b = 0$ to obtain

$$A_{n,k} = [a(n-k) + c]A_{n-1,k-1} + (dk + e)A_{n-1,k}$$

Empirically true upto 13×13.

Special cases:

(a, c, d, e)	Matrix obtained
$(1, 1, 1, 1)$	clean Eulerian triangle, conjecture
$(1, 0, 1, 1)$	shifted Eulerian triangle, conjecture
$(1, 1, 1, 0)$	shifted Eulerian triangle, conjecture

Is all hope lost?

Put $b = 0$ to obtain

$$A_{n,k} = [a(n-k)+c]A_{n-1,k-1} + (dk+e)A_{n-1,k}$$

Empirically true upto 13×13.

Special cases:

(a, c, d, e)	Matrix obtained
$(1, 1, 1, 1)$	clean Eulerian triangle, conjecture
$(1, 0, 1, 1)$	shifted Eulerian triangle, conjecture
$(1, 1, 1, 0)$	shifted Eulerian triangle, conjecture
$(0, 1, 0, 1)$	Binomial triangle, TP

Is all hope lost?

Put $b = 0$ to obtain

$$A_{n,k} = [a(n-k) + c]A_{n-1,k-1} + (dk + e)A_{n-1,k}$$

Empirically true upto 13×13.

Special cases:

(a, c, d, e)	Matrix obtained
$(1, 1, 1, 1)$	clean Eulerian triangle, conjecture
$(1, 0, 1, 1)$	shifted Eulerian triangle, conjecture
$(1, 1, 1, 0)$	shifted Eulerian triangle, conjecture
$(0, 1, 0, 1)$	Binomial triangle, TP
$(0, 1, 1, 1)$	Stirling subset numbers, TP proved by Brenti 1995

Is all hope lost?

Put $b = 0$ to obtain

$$A_{n,k} = [a(n-k) + c]A_{n-1,k-1} + (dk + e)A_{n-1,k}$$

Empirically true upto 13×13.

Special cases:

(a, c, d, e)	Matrix obtained
$(1,1,1,1)$	clean Eulerian triangle, conjecture
$(1,0,1,1)$	shifted Eulerian triangle, conjecture
$(1,1,1,0)$	shifted Eulerian triangle, conjecture
$(0,1,0,1)$	Binomial triangle, TP
$(0,1,1,1)$	Stirling subset numbers, TP proved by Brenti 1995
$(1,1,0,1)$	Row reversed matrix of Stirling subset numbers, conjecture

$$A_{n,k} = (n-k+1)A_{n-1,k-1}$$
$$+(k+1)A_{n-1,k}$$

$$A_{n,k} = (n-k+1)A_{n-1,k-1} + 1 \cdot A_{n-1,k}$$

$$A_{n,k} = [a(n-k)+c]A_{n-1,k-1}$$
$$+(dk+e)A_{n-1,k}$$

Is all hope lost?

Put $b = 0$ to obtain

$$A_{n,k} = [a(n-k) + c]A_{n-1,k-1} + (dk + e)A_{n-1,k}$$

This seems to be coefficientwise TP so far.

Infact, we have the following special cases for different substitutions:

(a, c, d, e)	Matrix obtained
$(1,1,1,1)$	clean Eulerian triangle, conjecture
$(1,0,1,1)$	Down-shifted Eulerian triangle, conjecture
$(1,1,1,0)$	Diagonally down-shifted Eulerian triangle, conjecture
$(0,1,0,1)$	Binomial triangle, TP
$(0,1,1,1)$	Stirling subset numbers, TP proved by Brenti 1995
(1,1,0,1)	**Row reversed matrix of Stirling subset numbers, conjecture**

Is all hope lost?

Put $b = 0$ to obtain

$$A_{n,k} = [a(n-k) + c]A_{n-1,k-1} + (dk + e)A_{n-1,k}$$

This seems to be coefficientwise TP so far.

Infact, we have the following special cases for different substitutions:

(a, c, d, e)	Matrix obtained
$(1, 1, 1, 1)$	clean Eulerian triangle, conjecture
$(1, 0, 1, 1)$	Down-shifted Eulerian triangle, conjecture
$(1, 1, 1, 0)$	Diagonally down-shifted Eulerian triangle, conjecture
$(0, 1, 0, 1)$	Binomial triangle, TP
$(0, 1, 1, 1)$	Stirling subset numbers, TP proved by Brenti 1995
(1,1,0,1)	**Row reversed matrix of Stirling subset numbers, ~~conjecture~~**

ace triangle

Reversed Stirling subset triangle

$$\left(\left\{{n+1 \atop k}\right\}^{\mathrm{rev}}\right)_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 6 & 7 & 1 & & \\ 1 & 10 & 25 & 15 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

ace triangle

$$
\begin{bmatrix}
1 \\
e & c \\
e^2 & ae + 2ce & c^2 \\
e^3 & 3ae^2 + 3ce^2 & a^2e + 3ace + 3c^2e & c^3 \\
e^4 & 6ae^3 + 4ce^3 & 7a^2e^2 + 12ace^2 + 6c^2e^2 & a^3e + 4a^2ce + 6ac^2e + 4c^3e & c^4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
$$

recurrence

$$A_{n,k} = [{\color{green}a}(n-k) + {\color{red}c}]A_{n-1,k-1} + {\color{blue}e}A_{n-1,k}$$

ace triangle

$$\begin{bmatrix} 1 & & & & & \\ e & c & & & & \\ e^2 & ae+2ce & c^2 & & & \\ e^3 & 3ae^2+3ce^2 & a^2e+3ace+3c^2e & c^3 & & \\ e^4 & 6ae^3+4ce^3 & 7a^2e^2+12ace^2+6c^2e^2 & a^3e+4a^2ce+6ac^2e+4c^3e & c^4 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

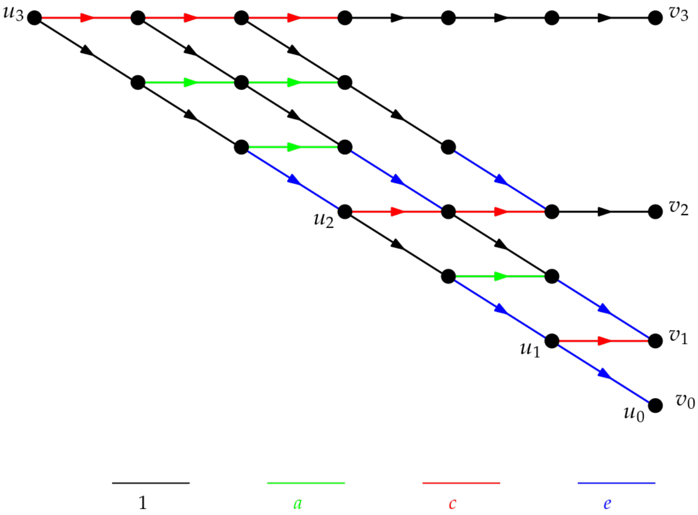
recurrence

$$A_{n,k} = [a(n-k)+c]A_{n-1,k-1} + eA_{n-1,k}$$

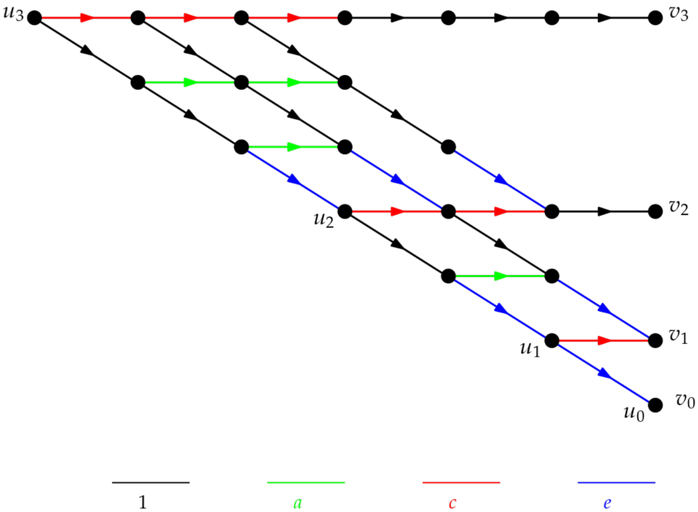
We have three proofs:

- Two Digraph proofs using the Lindström-Gessel-Viennot lemma
    - Bijection between paths in the digraph to set partitions.
    - Exploiting the structure of the digraph.
- Algebraic proof

An *ace* digraph



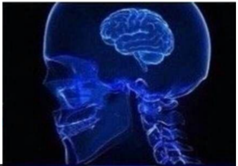
An ace digraph



We have a weight preserving bijection between paths from $u_n \to v_k$ and set partitions of $\{1, \ldots, n+1\}$ into $n-k+1$ parts.

$$A_{n,k} = (n-k+1)A_{n-1,k-1}$$
$$+(k+1)A_{n-1,k}$$

Eulerian triangle conjecture



$$A_{n,k} = [a(n-k)+c]A_{n-1,k-1}$$
$$+(dk+e)A_{n-1,k}$$

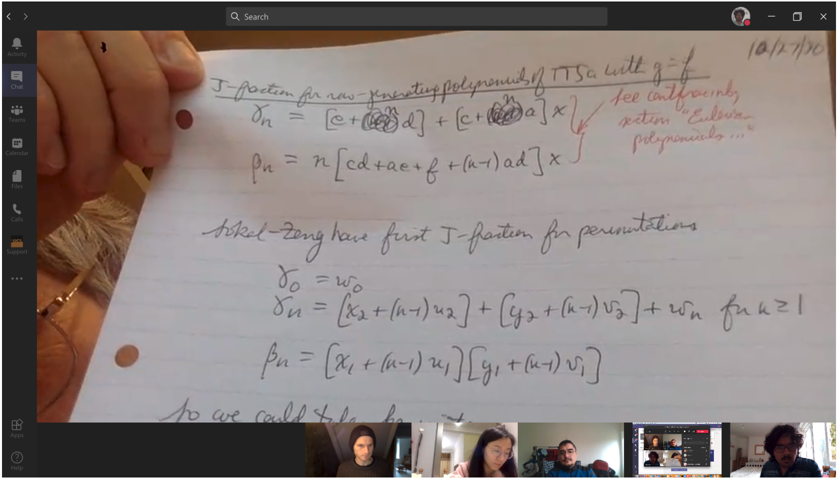
acde conjecture

$$A_{n,k} = [a(n-k)+c]A_{n-1,k-1}$$
$$+(dk+e)A_{n-1,k}$$
$$+[f(n-2)+g]A_{n-2,k-1}$$

acdefg conjecture

$$A_{n,k} = [a(n-k)+c]A_{n-1,k-1}$$
$$+\left(\sum_{i=0}^{k} d_i\right)A_{n-1,k}$$
$$+[f(n-2)+g]A_{n-2,k-1}$$

$ac\mathbf{d}fg$ conjecture



Thank you

Meme images from internet and digraph drawn by Tomack Gilmore.