Continued fractions using a Laguerre digraph interpretation of the Foata–Zeilberger bijection and its variants

Bishal Deb (he/him)

Sorbonne Université and Université Paris-Cité, CNRS, LPSM

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Institut Camille Jordan

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Structure

1. Continued fractions and enumerative combinatorics
    1. Classical continued fractions
    2. Sokal–Zeng's results for factorials
    3. D.–Sokal's results for Genocchi and median Genocchi numbers
    4. Conjectures
2. Proof overview of existing results
    1. Flajolet's combinatorial interpretation
    2. Foata–Zeilberger bijection
3. What's new
    1. Laguerre digraphs
    2. New interpretation of the FZ bijection
4. The story continues . . .

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Stieltjes-type continued fractions (S-fractions)

An example

$$\sum_{n=0} n!t^n = \cfrac{1}{1 - \cfrac{1 \cdot t}{1 - \cfrac{1 \cdot t}{1 - \cfrac{2 \cdot t}{1 - \cfrac{2 \cdot t}{1 - \cfrac{3 \cdot t}{1 - \cfrac{3 \cdot t}{1 - \dots}}}}}}}$$

This line of thought goes back to Euler

DIVERGENTIBVS. 225

$$A = \cfrac{1}{1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{2x}{1+\cfrac{2x}{1+\cfrac{3x}{1+\cfrac{3x}{1+\cfrac{4x}{1+\cfrac{4x}{1+\cfrac{5x}{1+\cfrac{5x}{1+\cfrac{6x}{1+\cfrac{6x}{1+\cfrac{7x}{\text{etc.}}}}}}}}}}}}}}$$

§. 22. Quemadmodum autem huiusmodi fractionum continuarum valor ſit inveſtigandus, alibi oſtendi: Scilicet cum ſingulorum denominatorum partes integrae ſint vnitates, ſoli numeratores in computum veniunt; ſit ergo $x = 1$, atque inveſtigatio ſummae A ſequenti modo inſtituetur:

$$A = \tfrac{0}{1},\ \tfrac{1}{1},\ \tfrac{1}{2},\ \tfrac{2}{3},\ \tfrac{4}{7},\ \tfrac{8}{13},\ \tfrac{20}{34},\ \tfrac{44}{73},\ \tfrac{124}{209},\ \tfrac{300}{501},\ \text{etc.}$$

num. 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, etc:

Fractiones nimirum hic exhibitae continuo propius ad verum valorem ipſius A accedunt, et quidem alternatim eo ſunt maiores et minores; ita vt ſit:

Tom. V. Nou. Com. Ff A >

Euler (1760)

Jacobi-type continued fractions

$$\cfrac{1}{1-\gamma_0 t-\cfrac{\beta_1 t^2}{1-\gamma_1 t-\cfrac{\beta_2 t^2}{1-\ddots}}}$$

J-fractions in short

J-fraction for $n!$

$$\sum_{n=0}^{\infty} n!t^n = \cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \ddots}}}$$

where

$$\begin{aligned} \gamma_n &= 2n + 1 \\ \beta_n &= n^2 \end{aligned}$$

Sokal–Zeng's (2022) reverse program

Start with

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Instead consider

$$\begin{aligned}\gamma_0 &= w_0\\ \gamma_n &= [x_2+(n-1)u_2]+[y_2+(n-1)v_2]+w_n\\ \beta_n &= [x_1+(n-1)u_1][y_1+(n-1)v_1]\end{aligned}$$

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Question

Find the 10 permutation statistics $x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w_0, (w_n)_{n\geq 1}$

Continued fractions by Sokal–Zeng (2022) for permutations

"First" (Cycles not counted)	"Second" (Cycles counted)
J-fraction with 10-statistics	**Conjecture**: J-fraction with 10-statistics

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Proof: Foata–Zeilberger bijection (1990)	**Proof:** Biane bijection (1993)

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They later discovered that Randrianarivony (1998) had a 17-variable continued fraction.

Results and conjecture for factorials

Cycle classification

For a permutation σ, compare each i with $\sigma(i)$ and $\sigma^{-1}(i)$:

Cycle classification

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- cycle valley $\sigma^{-1}(i) > i < \sigma(i)$
- cycle peaks $\sigma^{-1}(i) < i > \sigma(i)$
- cycle double rise $\sigma^{-1}(i) < i < \sigma(i)$
- cycle double fall $\sigma^{-1}(i) > i > \sigma(i)$
- fixed point $i = \sigma(i) = \sigma^{-1}(i)$

Record classification

Consider σ as $\sigma(1)\sigma(2)\ldots\sigma(n)$:

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- i is record if for every $j < i$ we have $\sigma(j) < \sigma(i)$
  left-to-right-maxima
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- rar - record-antirecord
- erec - exclusive record
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Record-and-cycle classification

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	cpeak	cval	cdrise	cdfall	fix
erec		ereccval	ereccdrise		
earec	eareccpeak			eareccdfall	
rar					rar
nrar	nrcpeak	nrcval	nrcdrise	nrcdfall	nrfix

Continued fractions counting permutation statistics

Consider 10-variable polynomials

$$P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) =$$
$$\sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times$$
$$u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)}$$

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Theorem (Sokal–Zeng (2022) First J-fraction for permutations)

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) t^n$$

$$= \cfrac{1}{1 - z \cdot t - \cfrac{x_1 y_1 \cdot t^2}{1 - (x_2 + y_2 + w) \cdot t - \cfrac{(x_1 + u_1)(y_1 + v_1) \cdot t^2}{1 - ((x_2 + u_2) + (y_2 + v_2) + w) \cdot t - \cfrac{(x_1 + 2u_1)(y_1 + 2v_1) \cdot t^2}{1 - \ddots}}}}$$

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Proof uses the Foata–Zeilberger bijection (1990)

Can we count cycles as well?

Consider 11-variable polynomials

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Conjecture (Sokal–Zeng (2022))

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, y_1, y_2, u_1, u_2, y_1, v_2, w, z, \lambda) t^n$$
$$= \cfrac{1}{1 - \lambda z \cdot t - \cfrac{\lambda x_1 y_1 \cdot t^2}{1 - (x_2 + y_2 + \lambda w) \cdot t - \cfrac{(\lambda+1)(x_1+u_1) y_1 \cdot t^2}{1 - ((x_2+v_2) + (y_2+v_2) + \lambda w) \cdot t - \cfrac{(\lambda+2)(x_1+2u_1) y_1 \cdot t^2}{1 - \ddots}}}}$$

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Genocchi and median Genocchi numbers

Genocchi numbers

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The first few numbers are $1, 1, 3, 17, 155, 2073, \ldots$.

Combinatorial Interpretation

Genocchi numbers:

$$
\begin{aligned}
g_n &= \overset{\text{D-e-semiderangements}}{\#\{\sigma \in \mathfrak{S}_{2n} | 2i > \sigma(2i) \text{ and } 2i-1 \leq \sigma(2i-1)\}} \\
&= \overset{\text{D-o-semiderangements}}{\#\{\sigma \in \mathfrak{S}_{2n} | 2i \geq \sigma(2i) \text{ and } 2i-1 < \sigma(2i-1)\}}
\end{aligned}
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Median Genocchi numbers:

$$\begin{aligned} & \text{D-permutations (introduced by Lazar and Wachs (2019))} \\ h_{n+1} &= |\mathfrak{D}_{2n}| = \#\{\sigma \in \mathfrak{S}_{2n} | 2i \geq \sigma(2i) \text{ and } 2i-1 \leq \sigma(2i-1)\} \\ & \text{D-derangements} \\ h_n &= \#\{\sigma \in \mathfrak{S}_{2n} | 2i > \sigma(2i) \text{ and } 2i-1 < \sigma(2i-1)\} \end{aligned}$$

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\end{aligned}
$$

D — Dumont-like

Continued fractions for median Genocchi numbers

The once-shifted median Genocchi numbers h_{n+1} have the following Thron-type continued fraction

$$\sum_{n=0}^{\infty} h_{n+1}\, t^n \;=\; \cfrac{1}{1-t-\cfrac{1t}{1-\cfrac{4t}{1-\cfrac{4t}{1-\cfrac{9t}{1-\cdots}}}}}$$

Story for Genocchi and median Genocchi

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	Conjecture: 0-T-fraction with 12-statistics
0-T-fraction with 12-statistics $\Uparrow$ $\boldsymbol{p,q}$**-generalisation**: 0-T-fraction with 22 statistics $\Uparrow$ **Master T-fraction**: four infinite 2-parameter families two infinite 1-parameter family	**0-T-fraction** with 12-statistics $\Uparrow$ $\boldsymbol{p,q}$**-generalisation**: 0-T-fraction with 21 statistics $\Uparrow$ **Master T-fraction**: three infinite 2-parameter families, three infinite 1-parameter families, and one statistic for counting cycles
Proof: FZ-like bijection	**Proof:** Biane-like bijection
Variant forms with slightly different statistics	

Randrianarivony–Zeng conjecture (1996)

Define the polynomials

$$G_n(x, y, \bar{x}, \bar{y}) = \sum_{\sigma \in \mathfrak{D}^{o}_{2n}} x^{\mathrm{comi}(\sigma)} y^{\mathrm{lema}(\sigma)} \bar{x}^{\mathrm{cemi}(\sigma)} \bar{y}^{\mathrm{remi}(\sigma)}$$

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where

- comi – odd cycle minima,
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Conjecture

The ogf of the polynomials $G_n(x,y,\bar{x},\bar{y})$ has the following S-fraction

$$\sum_{n=0}^{\infty} G_n(x,y,\bar{x},\bar{y})t^n = \cfrac{1}{1-\cfrac{xyt}{1-\cfrac{1(\bar{x}+\bar{y})t}{1-\cfrac{(x+1)(y+1)t}{1-\cfrac{2(\bar{x}+\bar{y}+1)t}{1-\cfrac{(x+2)(y+2)t}{1-\cfrac{3(\bar{x}+\bar{y}+2)t}{1-\cdots}}}}}}} \tag{1}$$

Plot twist for Sokal–Zeng conjecture

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☑ **Proof:** Foata–Zeilberger bijection (1990)	☒ **Proof:** Biane bijection (1993)

Conjectures of D.–Sokal, Randrianarivony–Zeng

"First" (Cycles not counted)	"Second" (Cycles counted)
	~~Conjecture~~: **0-T-fraction with 12 statistics**
0-T-fraction with 12 statistics $\Uparrow$ p, q-**generalisation**: 0-T-fraction with 22 statistics $\Uparrow$ **Master T-fraction**: four infinite 2-parameter families two infinite 1-parameter families	**0-T-fraction** with 12 statistics $\Uparrow$ p, q-**generalisation** 0-T-fraction with 21 statistics $\Uparrow$ **Master T-fraction**: three infinite 2-parameter families, three infinite 1-parameter families, and one statistic for counting cycles
Proof: FZ-like bijection	**Proof:** Biane-like bijection
Variant forms	**Variant forms** $\Longrightarrow$ **Randrianarivony–Zeng (1996)** **S-fraction with 4 statistics**

Pan–Zeng ('23) came up with multivariate continued fractions for other objects enumerated by Genocchi numbers also introduced in work of Lazar's PhD thesis

Structure

Motzkin paths

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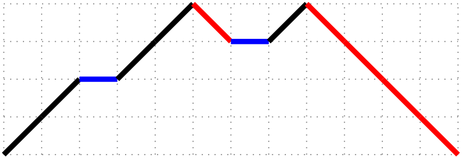

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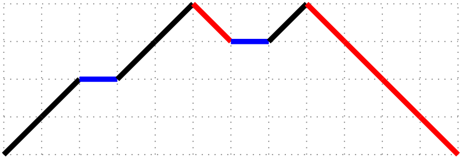

Assign weights:

- $\nearrow$: 1
- $\rightarrow$ from height $i \rightarrow i$: γ_i
- $\searrow$ from height $i \rightarrow (i-1)$: β_i

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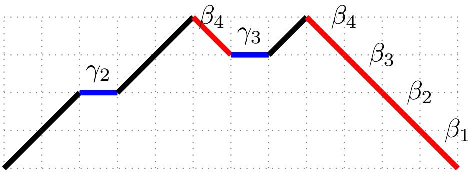

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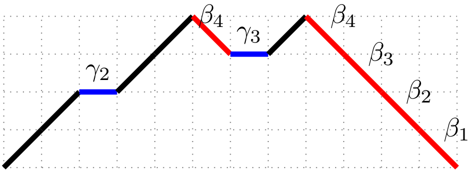

Weight $= \beta_1\beta_2\beta_3\beta_4^2\gamma_2\gamma_3$

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J-fraction

$$\cfrac{1}{1-\gamma_0 t-\cfrac{\beta_1 t^2}{1-\gamma_1 t-\cfrac{\beta_2 t^2}{1-\gamma_2 t-\cfrac{\beta_3 t^2}{\ddots}}}}$$

Combinatorial Interpretation of J-fraction

J-fraction

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Gateway for proving continued fractions using bijective combinatorics :-D

Foata–Zeilberger bijection

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  Correspond to $\sigma|_F : F \to F'$ and $\sigma|_G : G \to G'$

Description of $\sigma \to \omega$

- If i is a cycle valley, step i is $\nearrow$
- If i is a cycle peak, step i is $\searrow$
- If i is a cycle double rise, cycle double fall or fixed, step i is $\color{red}{\rightarrow}$, $\color{blue}{\rightarrow}$ or $\color{green}{\rightarrow}$ respectively.

Description of labels $\sigma \to \xi$

For $i \in [n]$

$$\xi_i = \begin{cases} \#\{j\colon j < i \text{ and } \sigma(j) > \sigma(i)\} & \text{if } \sigma(i) > i \quad \text{if } i \in \text{ Cval} \cup \text{Cdrise} \\ \#\{j\colon j > i \text{ and } \sigma(j) < \sigma(i)\} & \text{if } \sigma(i) < i \quad \text{if } i \in \text{Cpeak} \cup \text{Cdfall} \\ 0 & \text{if } \sigma(i) = i \quad \text{if } i \in \text{ Fix} \end{cases}$$

An example

Let $\sigma = 715492638 = (1762)(3598)(4) \in \mathfrak{S}_9$.

- Cval $= \{1,3\}$
- Cpeak $= \{7,9\}$
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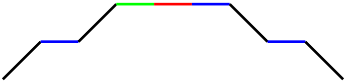

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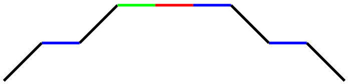

The labels ξ and the sets F, F', G, G' are:

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Structure

Laguerre digraph

Definition

A Laguerre digraph of size n is a directed graph on vertex set $\{1, \ldots, n\}$ where each vertex has indegree 0 or 1 and outdegree 0 or 1.

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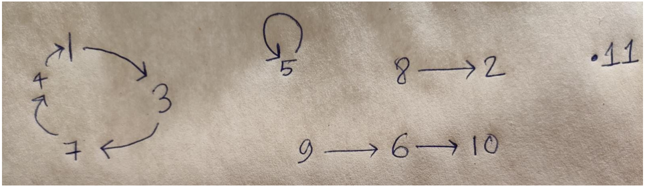

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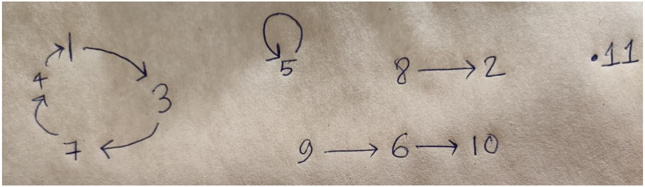

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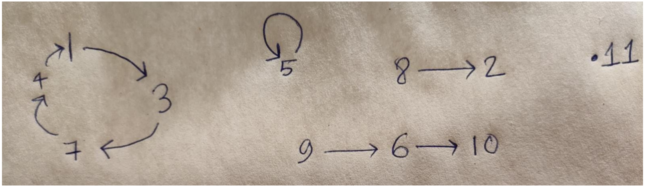

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Generalise permutations

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Laguerre digraphs after Sokal (2022)

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Stage (a): $i \in H$ (fixed points) in increasing order

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This order is suggested by the inverse bijection and the inversion tables

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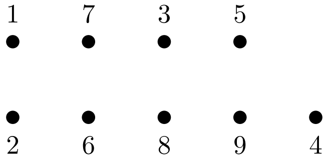

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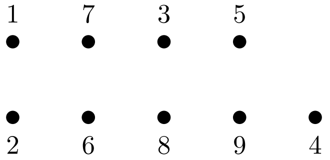
Stage (a): H in increasing order



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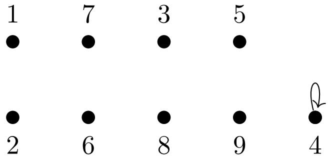

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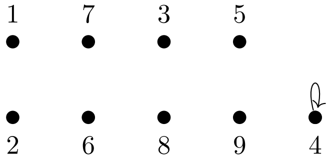

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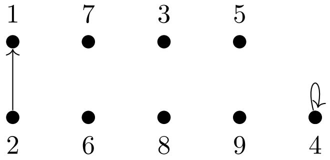

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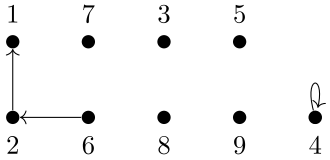

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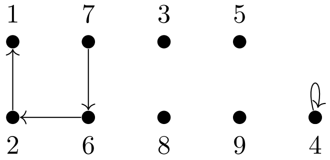

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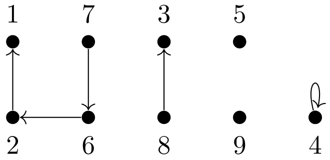

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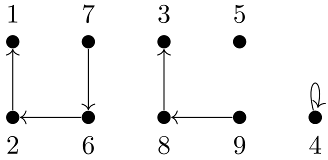

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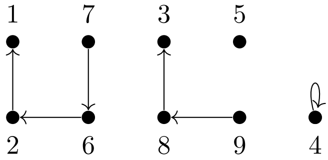
Stage (c): F in decreasing order



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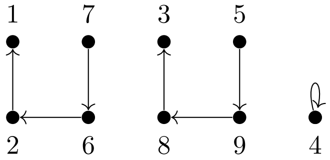


History with an example

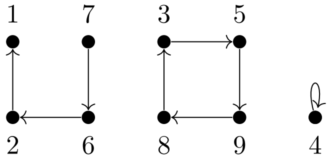
Let $\sigma = 715492638 = (1762)(3598)(4) \in \mathfrak{S}_9$.

$H = \{4\}$

$i \in F$	1	3	5
$\sigma(i) \in F'$	7	5	9
ξ_i	0	1	0

$i \in G$	2	6	7	8	9
$\sigma(i) \in G'$	1	2	6	3	8
ξ_i	0	0	1	0	0

Stage (c): F in decreasing order



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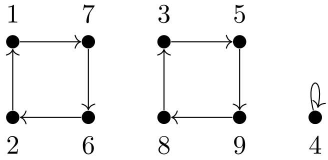

Structure

1. Continued fractions and enumerative combinatorics
   1. Classical continued fractions
   2. Sokal–Zeng's results for factorials
   3. D.–Sokal's results for Genocchi and median Genocchi numbers
   4. Conjectures
2. Proof overview of existing results
   1. Flajolet's combinatorial interpretation
   2. Foata–Zeilberger bijection
3. What's new
   1. Laguerre digraphs
   2. New interpretation of the FZ bijection
4. The story continues . . .

Remarks

Flajolet's 1980 paper allowed continued fractions with non-commutative variables

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So there must be non-commutative analogues of continued fractions coming from Biane bijection or Françon–Viennot bijection

A conjecture of Baril and Kirgizov

Baril and Kirgizov (2021) conjectured the following equidistribution of statistics on $\mathfrak{S}_n$:

Conjecture

The bistatistics $(\mathrm{des}_2,\ \mathrm{cyc})$ *and* $(\mathrm{pex},\mathrm{cyc})$ *are equidistributed.*

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$$\sum_{n=0}^{\infty} y^{\mathrm{des}_2 \sigma} \lambda^{\mathrm{cyc}\,\sigma} = \cfrac{1}{1 - \lambda z - \cfrac{\lambda y z^2}{1 - (\lambda + 2)z - \cfrac{(\lambda+1)(y+1)z^2}{\cdots}}}$$

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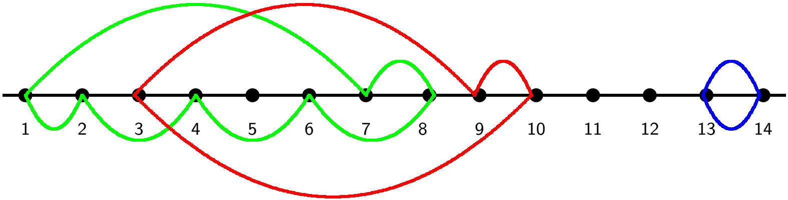
How do we resolve this???
An index i is a descent of type 2 if i is a descent and i is a record (left-to-right maxima)

Merci pour votre attention

Sokal–Zeng's master J-fraction

Pictorial representation



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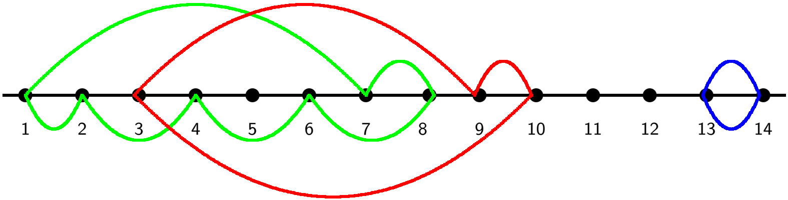
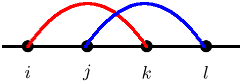


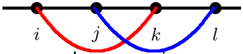
$\sigma = 7\,1\,9\,2\,5\,4\,8\,6\,10\,3\,11\,12\,14\,13 =$
$(1,7,8,6,4,2)\,(3,9,10)\,(5)\,(11)\,(12)\,(13,14) \in \mathfrak{S}_{14}$.

Due to Corteel (2007)

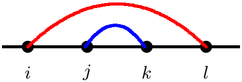
Crossings, nestings and pseudo-nestings



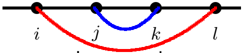
Upper crossing



Lower crossing



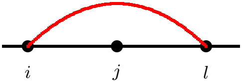
Upper nesting



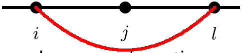
Lower nesting

$$\mathrm{ucross}(j,\sigma) \qquad \mathrm{lcross}(k,\sigma)$$

$$\mathrm{unest}(j,\sigma) \qquad \mathrm{lnest}(k,\sigma)$$



Upper pseudo-nesting



Lower pseudo-nesting

Sokal–Zeng first master J-fraction for permutations

$$Q_n(\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d},\mathsf{e}) =$$

$$\sum_{\sigma\in\mathfrak{S}_n} \prod_{i\in \mathrm{Cval}(\sigma)} \mathsf{a}_{\mathrm{ucross}(i,\sigma),\,\mathrm{unest}(i,\sigma)} \prod_{i\in \mathrm{Cpeak}(\sigma)} \mathsf{b}_{\mathrm{lcross}(i,\sigma),\,\mathrm{lnest}(i,\sigma)} \times$$

$$\prod_{i\in \mathrm{Cdfall}(\sigma)} \mathsf{c}_{\mathrm{lcross}(i,\sigma),\,\mathrm{lnest}(i,\sigma)} \prod_{i\in \mathrm{Cdrise}(\sigma)} \mathsf{d}_{\mathrm{ucross}(i,\sigma),\,\mathrm{unest}(i,\sigma)} \times$$

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Theorem

$$\sum_{n=0}^{\infty} Q_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e})t^n = \cfrac{1}{1-\gamma_0 t-\cfrac{\beta_1 t^2}{1-\gamma_1 t-\cfrac{\beta_2 t^2}{1-\gamma_2 t-\cfrac{\beta_3 t^2}{\ddots}}}}$$

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(D. arxiv '23) master J-fraction for permutations

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