The “quadratic family” of continued fractions and combinatorial sequences

Bishal Deb (he/him)

University College London

May 11, 2023
Topics in Special Functions and Number Theory

Based on Joint Work With

Alan D. Sokal

Structure

1. Introduction
2. Tangent, Secant, Genocchi, Genocchi medians
3. The permutations story
4. The D-permutations story
5. The cycle-alternating permutations story
6. Jacobi–Rogers matrix

Structure

1. Introduction
2. Tangent, Secant, Genocchi, Genocchi medians
3. The permutations story
4. The D-permutations story
5. The cycle-alternating permutations story
6. Jacobi–Rogers matrix

Stieltjes-type continued fraction (S-fraction):

$$\cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \ddots}}}$$

Stieltjes-type continued fraction (S-fraction):

$$\cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \ddots}}}$$

Also called regular C-fraction outside of combinatorial literature.

Given sequence $(a_n)_{n \geq 0}$

Given sequence $(a_n)_{n\geq 0}$

want to obtain

$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \ddots}}}$$

Given sequence $(a_n)_{n\geq 0}$

want to obtain

$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \ddots}}}$$

Classify sequences by growth of α

Linear α

- Catalan numbers:
  α's are $1, 1, 1, 1, \ldots$

Linear α

- Catalan numbers:
  α's are $1, 1, 1, 1, \ldots$
- $n!$:
  α's are $1, 1, 2, 2, 3, 3, \ldots$

Linear α

- Catalan numbers:
  α's are $1, 1, 1, 1, \ldots$
- $n!$:
  α's are $1, 1, 2, 2, 3, 3, \ldots$
- Bell numbers (number of set partitions):
  α's are $1, 1, 1, 2, 1, 3, 1, 4 \ldots$

Linear α

- Catalan numbers:
  α's are $1, 1, 1, 1, \ldots$
- $n!$:
  α's are $1, 1, 2, 2, 3, 3, \ldots$
- Bell numbers (number of set partitions):
  α's are $1, 1, 1, 2, 1, 3, 1, 4 \ldots$
- $(2n-1)!! = 1 \cdot 3 \cdots (2n-1)$:
  α's are $1, 2, 3, 4, 5, \ldots$

Quadratic α

- Tangent numbers A000182
- Secant numbers A000364

Quadratic α

- Tangent numbers A000182
- Secant numbers A000364
- Genocchi numbers A110501
- Median Genocchi numbers A005439

Quadratic α

- Tangent numbers A000182
- Secant numbers A000364
- Genocchi numbers A110501
- Median Genocchi numbers A005439
- Even Springer numbers A000281

Structure

Tangent and secant numbers

Maclaurin series of $\sec t + \tan t$

$$\sec t + \tan t = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$$

Tangent and secant numbers

Maclaurin series of $\sec t + \tan t$

$$\sec t + \tan t = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$$

E_{2n} - Secant numbers
α's are $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$

Maclaurin series of $\sec t + \tan t$

$$\sec t + \tan t = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$$

E_{2n} - Secant numbers
α's are $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$

E_{2n+1} - Tangent numbers
α's are $1 \cdot 2, 2 \cdot 3, 3 \cdot 4, 4 \cdot 5, 5 \cdot 6, \ldots$

Tangent and secant numbers

Maclaurin series of $\sec t + \tan t$

$$\sec t + \tan t \;=\; \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$$

E_{2n} - Secant numbers
α's are $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$

E_{2n+1} - Tangent numbers
α's are $1 \cdot 2, 2 \cdot 3, 3 \cdot 4, 4 \cdot 5, 5 \cdot 6, \ldots$
Classically expressed using Borel summation

Genocchi numbers

Genocchi numbers

$$t \tan\left(\frac{t}{2}\right) = \sum_{n=0}^{\infty} g_n \frac{t^{2n+2}}{(2n+2)!}$$

The first few numbers are $1, 1, 3, 17, 155, 2073, \ldots$

Genocchi numbers

$$t \tan\left(\frac{t}{2}\right) = \sum_{n=0}^{\infty} g_n \frac{t^{2n+2}}{(2n+2)!}$$

The first few numbers are $1, 1, 3, 17, 155, 2073, \ldots$

$$g_n = 4^{-n}(n+1)E_{2n+1}$$

Genocchi numbers

Genocchi numbers

$$t\tan\left(\frac{t}{2}\right) = \sum_{n=0}^{\infty} g_n \frac{t^{2n+2}}{(2n+2)!}$$

The first few numbers are $1, 1, 3, 17, 155, 2073, \ldots$

$$g_n = 4^{-n}(n+1)E_{2n+1}$$

$$g_n = (-1)^{n+1}\, 2(1-2^{2n+2})\, B_{2n+2}$$

Genocchi numbers

Genocchi numbers

$$t \tan\left(\frac{t}{2}\right) = \sum_{n=0}^{\infty} g_n \frac{t^{2n+2}}{(2n+2)!}$$

The first few numbers are $1, 1, 3, 17, 155, 2073, \ldots$

$$g_n \;=\; 4^{-n}(n+1)E_{2n+1}$$

$$g_n \;=\; (-1)^{n+1}\, 2(1-2^{2n+2})\, B_{2n+2}$$

α's are $1\cdot 1, 1\cdot 2,\ 2\cdot 2, 2\cdot 3,\ 3\cdot 3, 3\cdot 4\ldots$
(Viennot 1981)

Median Genocchi numbers

$$h_n = \sum_{i=0}^{n-1} (-1)^i \binom{n}{2i+1} g_{n-1-i}$$

Median Genocchi numbers

$$h_n = \sum_{i=0}^{n-1} (-1)^i \binom{n}{2i+1} g_{n-1-i}$$

The first few numbers are $1, 1, 2, 8, 56, 608, 9440, \ldots$

Median Genocchi numbers

$$h_n = \sum_{i=0}^{n-1} (-1)^i \binom{n}{2i+1} g_{n-1-i}$$

The first few numbers are $1, 1, 2, 8, 56, 608, 9440, \ldots$

No nice closed form known for exponential generating function

Median Genocchi numbers

$$h_n = \sum_{i=0}^{n-1} (-1)^i \binom{n}{2i+1} g_{n-1-i}$$

The first few numbers are $1, 1, 2, 8, 56, 608, 9440, \ldots$

No nice closed form known for exponential generating function

α's are $1^2, 1^2,\ 2^2, 2^2,\ 3^2, 3^2 \ldots$
(Viennot 1981)

Combinatorial Interpretation for Genocchi and Genocchi medians

Genocchi numbers g_n are counted by

$$\#\{\sigma \in \mathfrak{S}_{2n} | 2i > \sigma(2i) \text{ and } 2i-1 \leq \sigma(2i-1)\}$$

Combinatorial Interpretation for Genocchi and Genocchi medians

Genocchi numbers g_n are counted by

$$\#\{\sigma \in \mathfrak{S}_{2n} | 2i > \sigma(2i) \text{ and } 2i-1 \leq \sigma(2i-1)\}$$

Median Genocchi numbers h_n are counted by

$$\#\{\sigma \in \mathfrak{S}_{2n} | 2i > \sigma(2i) \text{ and } 2i-1 < \sigma(2i-1)\}$$

Combinatorial Interpretation for Genocchi and Genocchi medians

Genocchi numbers g_n are counted by

$$\#\{\sigma \in \mathfrak{S}_{2n} | 2i > \sigma(2i) \text{ and } 2i-1 \leq \sigma(2i-1)\}$$

Median Genocchi numbers h_n are counted by

$$\#\{\sigma \in \mathfrak{S}_{2n} | 2i > \sigma(2i) \text{ and } 2i-1 < \sigma(2i-1)\}$$

Also h_{n+1} counted by

$$\#\{\sigma \in \mathfrak{S}_{2n} | 2i \geq \sigma(2i) \text{ and } 2i-1 \leq \sigma(2i-1)\}$$

Combinatorial Interpretation for Genocchi and Genocchi medians

Genocchi numbers g_n are counted by

$$\#\{\sigma \in \mathfrak{S}_{2n} | 2i > \sigma(2i) \text{ and } 2i-1 \leq \sigma(2i-1)\}$$

Median Genocchi numbers h_n are counted by

$$\#\{\sigma \in \mathfrak{S}_{2n} | 2i > \sigma(2i) \text{ and } 2i-1 < \sigma(2i-1)\}$$

Also h_{n+1} counted by

$$\#\{\sigma \in \mathfrak{S}_{2n} | 2i \geq \sigma(2i) \text{ and } 2i-1 \leq \sigma(2i-1)\}$$

D-permutations or Dumont-like permutations (Lazar and Wachs 2019)

Combinatorial Interpretation for Genocchi and Genocchi medians

Genocchi numbers g_n are counted by

$$\#\{\sigma \in \mathfrak{S}_{2n} | 2i > \sigma(2i) \text{ and } 2i-1 \leq \sigma(2i-1)\}$$

D-e-semiderangements

Median Genocchi numbers h_n are counted by

$$\#\{\sigma \in \mathfrak{S}_{2n} | 2i > \sigma(2i) \text{ and } 2i-1 < \sigma(2i-1)\}$$

D-derangements

Also h_{n+1} counted by

$$\#\{\sigma \in \mathfrak{S}_{2n} | 2i \geq \sigma(2i) \text{ and } 2i-1 \leq \sigma(2i-1)\}$$

D-permutations or Dumont-like permutations (Lazar and Wachs 2019)

Example of a D-permutation

Example:-

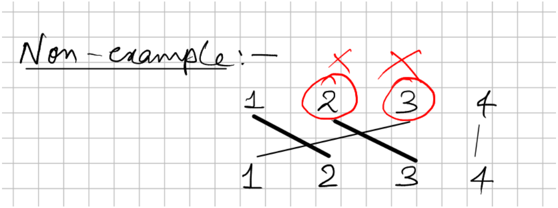
D-permutations

$2i \geqslant \sigma(2i)$ $\quad$ $2i-1 \leqslant \sigma(2i-1)$

$$\begin{pmatrix} i: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \sigma(i): & 7 & 1 & 3 & 4 & 6 & 2 & 8 & 5 \end{pmatrix}$$

$\sigma = 713\ 46285$

Non-example



Structure

Combinatorics and continued fractions: The permutations story

Combinatorics and continued fractions: The permutations story

Jacobi-type continued fraction for $n!$:

$$1 + 1!t + 2!t^2 + 3!t^3 + 4!t^4 + \ldots = \cfrac{1}{1 - 1 \cdot t - \cfrac{1 \cdot t^2}{1 - 3 \cdot t - \cfrac{4 \cdot t^2}{1 - 5 \cdot t - \cfrac{9 \cdot t^2}{1 - \ddots}}}}$$

Jacobi-type continued fraction for $n!$:

$$1 + 1!t + 2!t^2 + 3!t^3 + 4!t^4 + \ldots = \cfrac{1}{1 - 1\cdot t - \cfrac{1\cdot t^2}{1 - 3\cdot t - \cfrac{4\cdot t^2}{1 - 5\cdot t - \cfrac{9\cdot t^2}{1 - \ddots}}}}$$

Also called associated C-fraction outside of combinatorial literature.

Cycle classification

For a permutation σ, compare each i with $\sigma(i)$ and $\sigma^{-1}(i)$:

Cycle classification

For a permutation σ, compare each i with $\sigma(i)$ and $\sigma^{-1}(i)$:

- cycle valley $\sigma^{-1}(i) > i < \sigma(i)$
- cycle peaks $\sigma^{-1}(i) < i > \sigma(i)$
- cycle double rise $\sigma^{-1}(i) < i < \sigma(i)$
- cycle double fall $\sigma^{-1}(i) > i > \sigma(i)$
- fixed point $i = \sigma(i) = \sigma^{-1}(i)$

Consider 5-variable polynomials

$$P_n(x_1, x_2, y_1, y_2, w) = \sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{cpeak}(\sigma)} x_2^{\mathrm{cdfall}(\sigma)} y_1^{\mathrm{cval}(\sigma)} y_2^{\mathrm{cdrise}(\sigma)} z^{\mathrm{fix}(\sigma)}$$

Consider 5-variable polynomials

$$P_n(x_1, x_2, y_1, y_2, w) = \sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{cpeak}(\sigma)} x_2^{\mathrm{cdfall}(\sigma)} y_1^{\mathrm{cval}(\sigma)} y_2^{\mathrm{cdrise}(\sigma)} z^{\mathrm{fix}(\sigma)}$$

J-fraction:

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, y_1, y_2, w) t^n$$

$$= \cfrac{1}{1 - z \cdot t - \cfrac{x_1 y_1 \cdot t^2}{1 - (x_2 + y_2 + z) \cdot t - \cfrac{4 x_1 y_1 \cdot t^2}{1 - (2x_2 + 2y_2 + z) \cdot t - \cfrac{9 x_1 y_1 \cdot t^2}{1 - \ddots}}}}$$

Record classification

Consider σ as a word $\sigma(1)\sigma(2)\dots\sigma(n)$:

- i is record if for every $j < i$ we have $\sigma(j) < \sigma(i)$
  left-to-right-maxima
- i is antirecord if for every $i > j$ we have $\sigma(i) < \sigma(j)$
  right-to-left-minima

Record classification

Consider σ as a word $\sigma(1)\sigma(2)\dots\sigma(n)$:

- i is record if for every $j < i$ we have $\sigma(j) < \sigma(i)$
  left-to-right-maxima
- i is antirecord if for every $i > j$ we have $\sigma(i) < \sigma(j)$
  right-to-left-minima

Each i is one of the following four types:

Record classification

Consider σ as a word $\sigma(1)\sigma(2)\ldots\sigma(n)$:

- i is record if for every $j < i$ we have $\sigma(j) < \sigma(i)$
  left-to-right-maxima
- i is antirecord if for every $i > j$ we have $\sigma(i) < \sigma(j)$
  right-to-left-minima

Each i is one of the following four types:

- rar - record-antirecord
- erec - exclusive record
- earec - exclusive antirecord
- nrar - neither record-antirecord

Record-and-cycle classification

Each i is one of the following ten (not 20) types:

Record-and-cycle classification

Each i is one of the following ten (not 20) types:

- ereccval
- nrcval

Record-and-cycle classification

Each i is one of the following ten (not 20) types:

- ereccval
- nrcval
- eareccpeak
- nrcpeak

Record-and-cycle classification

Each i is one of the following ten (not 20) types:

- ereccval
- nrcval
- eareccpeak
- nrcpeak
- ereccdrise
- nrcdrise

Record-and-cycle classification

Each i is one of the following ten (not 20) types:

- ereccval
- nrcval
- eareccpeak
- nrcpeak
- ereccdrise
- nrcdrise
- eareccdfall
- nrcdfall

Record-and-cycle classification

Each i is one of the following ten (not 20) types:

- ereccval
- nrcval
- eareccpeak
- nrcpeak
- ereccdrise
- nrcdrise
- eareccdfall
- nrcdfall
- rar
- nrfix

Consider 10-variable polynomials

$$P_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,w,z) = \sum_{\sigma\in\mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)}$$

Consider 10-variable polynomials

$$P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) = \sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)}$$

Nice J-fraction:

Theorem (First J-fraction of Sokal–Zeng (2022) for permutations)

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, z, w) t^n$$

$$= \cfrac{1}{1 - z \cdot t - \cfrac{x_1 y_1 \cdot t^2}{1 - (x_2 + y_2 + w) \cdot t - \cfrac{(x_1 + u_1)(y_1 + v_1) \cdot t^2}{1 - ((x_2 + v_2) + (y_2 + v_2) + w) \cdot t - \cfrac{(x_1 + 2u_1)(y_1 + 2v_1) \cdot t^2}{1 - \ddots}}}}$$

Consider 10-variable polynomials

$$P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) = \sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)}$$

Nice J-fraction:

Theorem (First J-fraction of Sokal–Zeng (2022) for permutations)

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, z, w) t^n = \cfrac{1}{1 - z \cdot t - \cfrac{x_1 y_1 \cdot t^2}{1 - (x_2 + y_2 + w) \cdot t - \cfrac{(x_1 + u_1)(y_1 + v_1) \cdot t^2}{1 - ((x_2 + v_2) + (y_2 + v_2) + w) \cdot t - \cfrac{(x_1 + 2u_1)(y_1 + 2v_1) \cdot t^2}{1 - \ddots}}}}$$

They then generalise this to 18 variables

They then generalise this to 18 variables and five families of infinitely many variables!

They then generalise this to 18 variables and five families of infinitely many variables!

Similar results were also found by Blitvić-Steingrímsson (2021) at around the same time

They then generalise this to 18 variables and five families of infinitely many variables!

Similar results were also found by Blitvić-Steingrímsson (2021) at around the same time

Randrianarivony in a little-known paper had actually interpreted almost all of the variables for different statistics in 1998!!!

Counting cycles

Consider 11-variable polynomials

$$P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) = \sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times$$
$$u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)}$$

Counting cycles

Consider 11-variable polynomials

$$P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) = \sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)}$$

No nice J-fraction!

Counting cycles

Consider 11-variable polynomials

$$P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) = \sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)}$$

No nice J-fraction!
But can obtain J-fraction by specialising $y_1 = v_1$:

Counting cycles

Consider 11-variable polynomials

$$P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) = \sum_{\sigma\in\mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times$$
$$u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)}$$

No nice J-fraction!
But can obtain J-fraction by specialising $y_1 = v_1$:

Theorem (D. (2023), Conjectured by Sokal–Zeng (2022))

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, y_1, y_2, u_1, u_2, y_1, v_2, w, z, \lambda) t^n$$
$$= \cfrac{1}{1 - z\cdot t - \cfrac{\lambda x_1\, y_1 \cdot t^2}{1 - (x_2 + y_2 + w)\cdot t - \cfrac{(\lambda+1)(x_1+u_1)y_1 \cdot t^2}{1 - ((x_2+v_2) + (y_2+v_2) + w)\cdot t - \cfrac{(\lambda+2)(x_1+2u_1)y_1\cdot t^2}{1 - \ddots}}}}$$

Structure

The D-permutations story

The D-permutations story

Consider 10-variable polynomial

$$P_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,w,z) = \sum_{\sigma\in\mathfrak{D}_{2n}} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)}$$

The D-permutations story

Consider 10-variable polynomial

$$P_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,w,z) = \sum_{\sigma\in\mathfrak{D}_{2n}} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)}$$

Thron-type continued fraction

$$\sum_{n=0}^{\infty} P_n t^n = \cfrac{1}{1-\delta_0 t - \cfrac{\alpha_1 t}{1-\cfrac{\alpha_2 t}{1-\cfrac{\alpha_3 t}{\ddots}}}}$$

The D-permutations story

Consider 10-variable polynomial

$$
\begin{aligned}
P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) &= \\
\sum_{\sigma \in \mathfrak{D}_{2n}} & x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times \\
& u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)}
\end{aligned}
$$

Thron-type continued fraction

$$
\sum_{n=0}^{\infty} P_n t^n = \cfrac{1}{1 - \delta_0 t - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{\ddots}}}}
$$

where

$$
\delta_1 = z^2
$$

$$
\alpha_{2k-1} = [x_1 + (k-1)u_1] \cdot [y_1 + (k-1)v_1]
$$

$$
\alpha_{2k} = [x_2 + (k-1)u_2 + w] \cdot [y_2 + (k-1)v_2 + w].
$$

The D-permutations story

Consider 10-variable polynomial

$$P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) = \sum_{\sigma \in \mathfrak{D}_{2n}} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times$$

$$u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)}$$

Thron-type continued fraction

$$\sum_{n=0}^{\infty} P_n t^n = \cfrac{1}{1 - \delta_0 t - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{\ddots}}}}$$

where

$$\delta_1 = z^2$$

$$\alpha_{2k-1} = [x_1 + (k-1)u_1] \cdot [y_1 + (k-1)v_1]$$

$$\alpha_{2k} = [x_2 + (k-1)u_2 + w] \cdot [y_2 + (k-1)v_2 + w].$$

Can do better!!

The D-permutations story

Separate fixed points by parity

Theorem (D.-Sokal '22 (arxiv))

$$\sum_{n=0}^{\infty} P_n t^n = \cfrac{1}{1 - \delta_0 t - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{\ddots}}}}$$

The D-permutations story

Separate fixed points by parity

Theorem (D.-Sokal '22 (arxiv))

$$\sum_{n=0}^{\infty} P_n t^n = \cfrac{1}{1 - \delta_0 t - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{\ddots}}}}$$

where

$$\delta_1 = z_{\mathrm{e}} z_{\mathrm{o}}$$

$$\alpha_{2k-1} = [x_1 + (k-1)u_1] \cdot [y_1 + (k-1)v_1]$$

$$\alpha_{2k} = [x_2 + (k-1)u_2 + w_{\mathrm{e}}] \cdot [y_2 + (k-1) + v_2 + w_{\mathrm{o}}].$$

$$\begin{tikzcd}[column sep=small]
 & \text{D-e-semiderangements} \arrow[dr, phantom, "\supseteq" sloped] & \\
\underbrace{\text{D-derangements}}_{h_n} \arrow[ur, phantom, "\subseteq" sloped] \arrow[dr, phantom, "\subseteq" sloped] & & \underbrace{\text{D-permutations}}_{h_{n+1}} \\
 & \underbrace{\text{D-o-semiderangements}}_{g_n} \arrow[ur, phantom, "\subseteq" sloped] &
\end{tikzcd}$$

$$\begin{array}{ccccc}
 & & \text{D-e-semiderangements} & & \\
 & \subseteq & & \subseteq & \\
\underbrace{\text{D-derangements}}_{h_n} & & & & \underbrace{\text{D-permutations}}_{h_{n+1}} \\
 & \subseteq & & \subseteq & \\
 & & \underbrace{\text{D-o-semiderangements}}_{g_n} & &
\end{array}$$

We can also count cycles [D.–Sokal '22, D. '23]

$$
\begin{tikzcd}
& \text{D-e-semiderangements} \arrow[dr, phantom, "\subseteq" sloped] & \\
\underbrace{\text{D-cycles}}_{g_{n-1}} \subseteq \underbrace{\text{D-derangements}}_{h_n} \arrow[ur, phantom, "\subseteq" sloped] \arrow[dr, phantom, "\subseteq" sloped] & & \underbrace{\text{D-permutations}}_{h_{n+1}} \\
& \underbrace{\text{D-o-semiderangements}}_{g_n} \arrow[ur, phantom, "\subseteq" sloped] &
\end{tikzcd}
$$

We can also count cycles [D.–Sokal '22, D. '23]

Structure

Combinatorial Interpretation for Secant numbers

Secant numbers E_{2n} are counted by

Secant numbers E_{2n} are counted by cycle-alternating permutations

Combinatorial Interpretation for Secant numbers

Secant numbers E_{2n} are counted by cycle-alternating permutations $\sigma \in \mathfrak{S}_{2n}$ where each $i \in [2n]$

- either cycle valley $(\sigma^{-1}(i) > i < \sigma(i))$
- or cycle peak $(\sigma^{-1}(i) < i > \sigma(i))$

The cycle-alternating permutations story

$$P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) = \sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)}$$

Theorem (First J-fraction of Sokal–Zeng (2022) for permutations)

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, z, w) t^n = \cfrac{1}{1 - z \cdot t - \cfrac{x_1 y_1 \cdot t^2}{1 - (x_2 + y_2 + w) \cdot t - \cfrac{(x_1 + u_1)(y_1 + v_1) \cdot t^2}{1 - ((x_2 + v_2) + (y_2 + v_2) + w) \cdot t - \cfrac{(x_1 + 2u_1)(y_1 + 2v_1) \cdot t^2}{1 - \ddots}}}}$$

The cycle-alternating permutations story

$$P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) = \sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)}$$

Theorem (First J-fraction of Sokal–Zeng (2022) for permutations)

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, z, w) t^n$$

$$= \cfrac{1}{1 - z \cdot t - \cfrac{x_1 y_1 \cdot t^2}{1 - (x_2 + y_2 + w) \cdot t - \cfrac{(x_1 + u_1)(y_1 + v_1) \cdot t^2}{1 - ((x_2 + v_2) + (y_2 + v_2) + w) \cdot t - \cfrac{(x_1 + 2u_1)(y_1 + 2v_1) \cdot t^2}{1 - \ddots}}}}$$

Set $x_2 = y_2 = u_2 = v_2 = w = z = 0$

Consider 4-variable polynomials

$$P_{2n}(x,y,u,v) = \sum_{\sigma \in \mathfrak{S}_{2n}^{\mathrm{ca}}} x^{\mathrm{eareccpeak}(\sigma)} y^{\mathrm{ereccval}(\sigma)} u^{\mathrm{nrcpeak}(\sigma)} v^{\mathrm{nrcval}(\sigma)}$$

The cycle-alternating permutations story

Consider 4-variable polynomials

$$P_{2n}(x,y,u,v) = \sum_{\sigma \in \mathfrak{S}_{2n}^{\mathrm{ca}}} x^{\mathrm{eareccpeak}(\sigma)} y^{\mathrm{ereccval}(\sigma)} u^{\mathrm{nrcpeak}(\sigma)} v^{\mathrm{nrcval}(\sigma)}$$

Theorem (First J-fraction of Sokal–Zeng (2022) for cycle-alternating permutations)

$$\sum_{n=0}^{\infty} P_{2n}(x,y,u,v) t^n = \cfrac{1}{1 - \cfrac{x\,y \cdot t}{1 - \cfrac{(x+u)\,(y+v)\cdot t}{1 - \cfrac{(x+2u)\,(y+2v)\cdot t}{1 - \ddots}}}}$$

The cycle-alternating permutations story

Consider 4-variable polynomials

$$P_{2n}(x,y,u,v) = \sum_{\sigma \in \mathfrak{S}_{2n}^{\mathrm{ca}}} x^{\mathrm{eareccpeak}(\sigma)} y^{\mathrm{ereccval}(\sigma)} u^{\mathrm{nrcpeak}(\sigma)} v^{\mathrm{nrcval}(\sigma)}$$

Theorem (First J-fraction of Sokal–Zeng (2022) for cycle-alternating permutations)

$$\sum_{n=0}^{\infty} P_{2n}(x,y,u,v) t^n = \cfrac{1}{1 - \cfrac{x\,y \cdot t}{1 - \cfrac{(x+u)\,(y+v) \cdot t}{1 - \cfrac{(x+2u)\,(y+2v) \cdot t}{1 - \ddots}}}}$$

Can do better

The cycle-alternating permutations story

Separate by parity

The cycle-alternating permutations story

Separate by parity

$$
\begin{aligned}
& P_{2n}(x_{\rm e}, x_{\rm o}, y_{\rm e}, y_{\rm o}, u_{\rm e}, u_{\rm o}, v_{\rm e}, v_{\rm o}) \\
= & \sum_{\sigma \in \mathfrak{S}_{2n}^{\rm ca}} x_{\rm e}^{\rm eareccpeakeven(\sigma)} x_{\rm o}^{\rm eareccpeakodd(\sigma)} y_{\rm e}^{\rm ereccvaleven(\sigma)} y_{\rm o}^{\rm ereccvalodd(\sigma)} \quad \times \\
& \qquad u_{\rm e}^{\rm nrcpeakeven(\sigma)} u_{\rm o}^{\rm nrcpeakodd(\sigma)} v_{\rm e}^{\rm nrcvaleven(\sigma)} v_{\rm o}^{\rm nrcvalodd(\sigma)}
\end{aligned}
$$

The cycle-alternating permutations story

Separate by parity

$$
\begin{aligned}
& P_{2n}(x_{\rm e}, x_{\rm o}, y_{\rm e}, y_{\rm o}, u_{\rm e}, u_{\rm o}, v_{\rm e}, v_{\rm o}) \\
= & \sum_{\sigma \in \mathfrak{S}_{2n}^{\rm ca}} x_{\rm e}^{\rm eareccpeakeven(\sigma)} x_{\rm o}^{\rm eareccpeakodd(\sigma)} y_{\rm e}^{\rm ereccvaleven(\sigma)} y_{\rm o}^{\rm ereccvalodd(\sigma)} \quad \times \\
& \qquad u_{\rm e}^{\rm nrcpeakeven(\sigma)} u_{\rm o}^{\rm nrcpeakodd(\sigma)} v_{\rm e}^{\rm nrcvaleven(\sigma)} v_{\rm o}^{\rm nrcvalodd(\sigma)}
\end{aligned}
$$

Theorem (D.–Sokal '23 (arxiv))

$$
\sum_{n=0}^{\infty} P_{2n} t^n = \cfrac{1}{1 - \cfrac{x_{\rm e}\, y_{\rm o} \cdot t}{1 - \cfrac{(x_{\rm o} + u_{\rm o})\,(y_{\rm e} + v_{\rm e}) \cdot t}{1 - \cfrac{(x_{\rm e} + 2u_{\rm e})\,(y_{\rm o} + 2v_{\rm o}) \cdot t}{1 - \cfrac{(x_{\rm o} + 3u_{\rm o})\,(y_{\rm e} + 3v_{\rm e}) \cdot t}{\ddots}}}}}
$$

The cycle-alternating permutations story

Separate by parity

$$
\begin{aligned}
& P_{2n}(x_{\rm e}, x_{\rm o}, y_{\rm e}, y_{\rm o}, u_{\rm e}, u_{\rm o}, v_{\rm e}, v_{\rm o}) \\
= & \sum_{\sigma \in \mathfrak{S}_{2n}^{\rm ca}} x_{\rm e}^{\rm eareccpeakeven(\sigma)} x_{\rm o}^{\rm eareccpeakodd(\sigma)} y_{\rm e}^{\rm ereccvaleven(\sigma)} y_{\rm o}^{\rm ereccvalodd(\sigma)} \quad \times \\
& \quad u_{\rm e}^{\rm nrcpeakeven(\sigma)} u_{\rm o}^{\rm nrcpeakodd(\sigma)} v_{\rm e}^{\rm nrcvaleven(\sigma)} v_{\rm o}^{\rm nrcvalodd(\sigma)}
\end{aligned}
$$

Theorem (D.–Sokal '23 (arxiv))

$$
\sum_{n=0}^{\infty} P_{2n} t^n = \cfrac{1}{1 - \cfrac{x_{\rm e}\, y_{\rm o} \cdot t}{1 - \cfrac{(x_{\rm o} + u_{\rm o})\,(y_{\rm e} + v_{\rm e}) \cdot t}{1 - \cfrac{(x_{\rm e} + 2u_{\rm e})\,(y_{\rm o} + 2v_{\rm o}) \cdot t}{1 - \cfrac{(x_{\rm o} + 3u_{\rm o})\,(y_{\rm e} + 3v_{\rm e}) \cdot t}{\ddots}}}}}
$$

Special case of more general continued fraction of Sokal–Zeng involving 2 infinite families

Counting of cycles

Consider 9-variable polynomials

$$
\begin{aligned}
& P_{2n}(x_{\mathrm{e}}, x_{\mathrm{o}}, y_{\mathrm{e}}, y_{\mathrm{o}}, u_{\mathrm{e}}, u_{\mathrm{o}}, v_{\mathrm{e}}, v_{\mathrm{o}}) \\
= & \sum_{\sigma \in \mathfrak{S}_{2n}^{\mathrm{ca}}} x_{\mathrm{e}}^{\mathrm{eareccpeakeven}(\sigma)} x_{\mathrm{o}}^{\mathrm{eareccpeakodd}(\sigma)} y_{\mathrm{e}}^{\mathrm{ereccvaleven}(\sigma)} y_{\mathrm{o}}^{\mathrm{ereccvalodd}(\sigma)} \quad \times \\
& \quad u_{\mathrm{e}}^{\mathrm{nrcpeakeven}(\sigma)} u_{\mathrm{o}}^{\mathrm{nrcpeakodd}(\sigma)} v_{\mathrm{e}}^{\mathrm{nrcvaleven}(\sigma)} v_{\mathrm{o}}^{\mathrm{nrcvalodd}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)}
\end{aligned}
$$

Counting of cycles

Consider 9-variable polynomials

$$
\begin{aligned}
& P_{2n}(x_\mathrm{e}, x_\mathrm{o}, y_\mathrm{e}, y_\mathrm{o}, u_\mathrm{e}, u_\mathrm{o}, v_\mathrm{e}, v_\mathrm{o}) \\
= & \sum_{\sigma \in \mathfrak{S}_{2n}^{\mathrm{ca}}} x_\mathrm{e}^{\mathrm{eareccpeakeven}(\sigma)} x_\mathrm{o}^{\mathrm{eareccpeakodd}(\sigma)} y_\mathrm{e}^{\mathrm{ereccvaleven}(\sigma)} y_\mathrm{o}^{\mathrm{ereccvalodd}(\sigma)} \times \\
& \quad u_\mathrm{e}^{\mathrm{nrcpeakeven}(\sigma)} u_\mathrm{o}^{\mathrm{nrcpeakodd}(\sigma)} v_\mathrm{e}^{\mathrm{nrcvaleven}(\sigma)} v_\mathrm{o}^{\mathrm{nrcvalodd}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)}
\end{aligned}
$$

Need to set $y_\mathrm{e} = v_\mathrm{e}$, $y_\mathrm{o} = v_\mathrm{o}$

Counting of cycles

Consider 9-variable polynomials

$$
\begin{aligned}
& P_{2n}(x_\mathrm{e}, x_\mathrm{o}, y_\mathrm{e}, y_\mathrm{o}, u_\mathrm{e}, u_\mathrm{o}, v_\mathrm{e}, v_\mathrm{o}) \\
= & \sum_{\sigma \in \mathfrak{S}_{2n}^{\mathrm{ca}}} x_\mathrm{e}^{\mathrm{eareccpeakeven}(\sigma)} x_\mathrm{o}^{\mathrm{eareccpeakodd}(\sigma)} y_\mathrm{e}^{\mathrm{ereccvaleven}(\sigma)} y_\mathrm{o}^{\mathrm{ereccvalodd}(\sigma)} \times \\
& \quad u_\mathrm{e}^{\mathrm{nrcpeakeven}(\sigma)} u_\mathrm{o}^{\mathrm{nrcpeakodd}(\sigma)} v_\mathrm{e}^{\mathrm{nrcvaleven}(\sigma)} v_\mathrm{o}^{\mathrm{nrcvalodd}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)}
\end{aligned}
$$

Need to set $y_\mathrm{e} = v_\mathrm{e}$, $y_\mathrm{o} = v_\mathrm{o}$

Theorem (D.–Sokal '23 (arxiv))

$$
\sum_{n=0}^{\infty} P_{2n} t^n = \cfrac{1}{1 - \cfrac{\lambda x_\mathrm{e}\, y_\mathrm{o} \cdot t}{1 - \cfrac{(\lambda+1)(x_\mathrm{o}+u_\mathrm{o}) y_\mathrm{e} \cdot t}{1 - \cfrac{(\lambda+2)(x_\mathrm{e}+2u_\mathrm{e}) y_\mathrm{o} \cdot t}{1 - \cfrac{(\lambda+3)(x_\mathrm{o}+3u_\mathrm{o}) y_\mathrm{e} \cdot t}{\ddots}}}}}
$$

$$\underbrace{\text{Alternating cycles}}_{E_{2n-1}} \subseteq \underbrace{\text{Cycle-alternating permutations}}_{E_{2n}}$$

Jacobian Elliptic Function

$$u = F(\phi, k) = \int_0^{\phi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$$

Jacobian Elliptic Function

$$u = F(\phi, k) = \int_0^\phi \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$$

Jacobian amplitude

$$\mathrm{am}(u, k) = \phi = F^{-1}(u, k)$$

Jacobian Elliptic Function

$$u = F(\phi, k) = \int_0^{\phi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$$

Jacobian amplitude

$$\operatorname{am}(u, k) = \phi = F^{-1}(u, k)$$

Jacobian elliptic functions

$$\operatorname{sn}(u, k) = \sin \operatorname{am}(u, k)$$

$$\operatorname{cn}(u, k) = \cos \operatorname{am}(u, k)$$

Jacobian Elliptic Function

$$u = F(\phi, k) = \int_0^\phi \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$$

Jacobian amplitude

$$\mathrm{am}(u, k) = \phi = F^{-1}(u, k)$$

Jacobian elliptic functions

$$\mathrm{sn}(u, k) = \sin \mathrm{am}(u, k)$$

$$\mathrm{cn}(u, k) = \cos \mathrm{am}(u, k)$$

Combinatorial interpretation due to Dumont (1979,1980). He introduced Schett polynomials.

Series expansion

$$\begin{aligned}\mathrm{sn}(u,k) &= \sum_{n=0}^{\infty} (-1)^{(n-1)/2} \mathcal{E}_{2n+1}(k) \frac{u^{2n+1}}{(2n+1)!} \\ \mathrm{cn}(u,k) &= \sum_{n=0}^{\infty} (-1)^{n/2} \mathcal{E}_{2n}(k) \frac{u^{2n}}{(2n)!}\end{aligned}$$

Series expansion

$$
\begin{aligned}
\mathrm{sn}(u,k) &= \sum_{n=0}^{\infty} (-1)^{(n-1)/2} \mathcal{E}_{2n+1}(k) \frac{u^{2n+1}}{(2n+1)!} \\
\mathrm{cn}(u,k) &= \sum_{n=0}^{\infty} (-1)^{n/2} \mathcal{E}_{2n}(k) \frac{u^{2n}}{(2n)!}
\end{aligned}
$$

$$
\sum_{n=0}^{\infty} \mathcal{E}_{2n}(k) t^n = \cfrac{1}{1 - \cfrac{t}{1 - \cfrac{2^2 k^2 t}{1 - \cfrac{3^2 t}{1 - \cfrac{4^2 k^2 t}{1 - \ddots}}}}}
$$

[Stieltjes, 1889]

Series expansion

$$\begin{aligned}\mathrm{sn}(u,k) &= \sum_{n=0}^{\infty}(-1)^{(n-1)/2}\mathcal{E}_{2n+1}(k)\frac{u^{2n+1}}{(2n+1)!}\\ \mathrm{cn}(u,k) &= \sum_{n=0}^{\infty}(-1)^{n/2}\mathcal{E}_{2n}(k)\frac{u^{2n}}{(2n)!}\end{aligned}$$

$$\sum_{n=0}^{\infty}\mathcal{E}_{2n}(k)t^n = \cfrac{1}{1-\cfrac{t}{1-\cfrac{2^2k^2t}{1-\cfrac{3^2t}{1-\cfrac{4^2k^2t}{1-\ddots}}}}}$$

[Stieltjes, 1889]
Our continued fraction also generalises this

Structure

Jacobi-Rogers matrix

Consider the J-fraction

$$\cfrac{1}{1-\gamma_0 t-\cfrac{\beta_1 t^2}{1-\gamma_1 t-\cfrac{\beta_2 t^2}{1-\ddots}}}$$

Jacobi-Rogers matrix

Consider the J-fraction

$$\cfrac{1}{1-\gamma_0 t-\cfrac{\beta_1 t^2}{1-\gamma_1 t-\cfrac{\beta_2 t^2}{1-\ddots}}}$$

Define lower-triangular matrix J where

Jacobi-Rogers matrix

Consider the J-fraction

$$\cfrac{1}{1-\gamma_0 t-\cfrac{\beta_1 t^2}{1-\gamma_1 t-\cfrac{\beta_2 t^2}{1-\ddots}}}$$

Define lower-triangular matrix J where

$$\begin{aligned} \mathrm{J}_{n,n} &= 1 \\ \mathrm{J}_{n,k} &= \mathrm{J}_{n-1,k-1}+\gamma_k \mathrm{J}_{n-1,k}+\beta_{k+1}\mathrm{J}_{n-1,k+1} \end{aligned}$$

Jacobi-Rogers matrix

Consider the J-fraction

$$\cfrac{1}{1-\gamma_0 t-\cfrac{\beta_1 t^2}{1-\gamma_1 t-\cfrac{\beta_2 t^2}{1-\ddots}}}$$

Define lower-triangular matrix J where

$$\begin{aligned} \mathrm{J}_{n,n} &= 1 \\ \mathrm{J}_{n,k} &= \mathrm{J}_{n-1,k-1}+\gamma_k \mathrm{J}_{n-1,k}+\beta_{k+1}\mathrm{J}_{n-1,k+1} \end{aligned}$$

Also known as Stieltjes table/tableau

If

$$\sum_{n=0}^{\infty} a_n t_n = \cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \ddots}}}$$

then

$$\mathrm{J}_{n,0} = a_n$$

If

$$\sum_{n=0}^{\infty} a_n t_n = \cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \ddots}}}$$

then

$$\mathrm{J}_{n,0} = a_n$$

Question: If J-fraction for a_n is known, combinatorially understand matrix J

When $a_n = n!$,

$$\sum_{n=0}^{\infty} a_n t_n = \cfrac{1}{1 - t - \cfrac{1t^2}{1 - 3t - \cfrac{4t^2}{1 - \ddots}}}$$

$$\mathrm{J}_{n,k} = \binom{n}{k} \frac{n!}{k!}$$

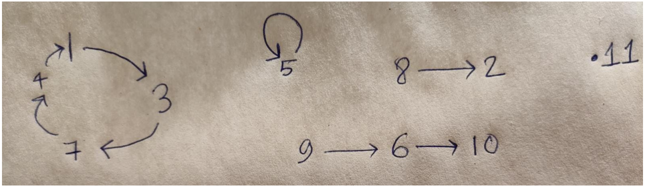
These count Laguerre digraphs with k paths

Laguerre digraph

Laguerre digraph is a labelled digraph where each vertex has in and out-degree 0 or 1

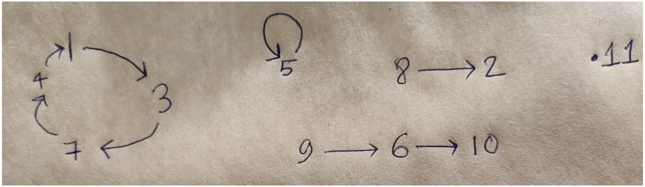
Laguerre digraph

Laguerre digraph is a labelled digraph where each vertex has in and out-degree 0 or 1



Laguerre digraph

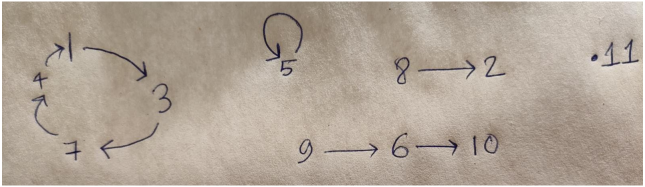
Laguerre digraph is a labelled digraph where each vertex has in and out-degree 0 or 1



Each connected component is a cycle or a path

Laguerre digraph

Laguerre digraph is a labelled digraph where each vertex has in and out-degree 0 or 1

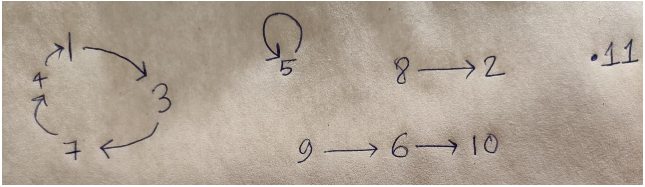


Each connected component is a cycle or a path

No paths - permutation

Laguerre digraph

Laguerre digraph is a labelled digraph where each vertex has in and out-degree 0 or 1



Each connected component is a cycle or a path

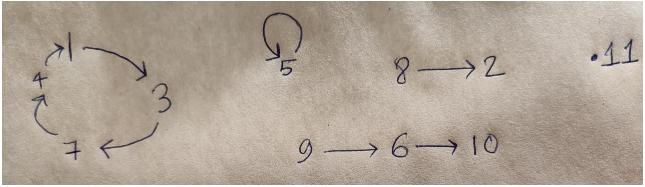
No paths - permutation

Number of Laguerre digraphs on n vertices with k elements -

$$\mathrm{J}_{n,k} = \binom{n}{k}\frac{n!}{k!}$$

Laguerre digraph

Laguerre digraph is a labelled digraph where each vertex has in and out-degree 0 or 1



Each connected component is a cycle or a path

No paths - permutation

Number of Laguerre digraphs on n vertices with k elements -

$$\mathrm{J}_{n,k} = \binom{n}{k}\frac{n!}{k!}$$

Can extend permutation statistics to Laguerre digraphs [D.–Sokal (ongoing)]

i can be classified as:

- Peak
- Valley
- Double ascent
- Double descent
- Loop

i can be classified as:

- Peak
- Valley
- Double ascent
- Double descent
- Loop

Alternating Laguerre digraph - Laguerre digraphs where each vertex is either a peak or a valley

i can be classified as:

- Peak
- Valley
- Double ascent
- Double descent
- Loop

Alternating Laguerre digraph - Laguerre digraphs where each vertex is either a peak or a valley

Interpret Jacobi-Rogers matrix for secant numbers E_{2n} [D.–Sokal '23]

Question

We have a combinatorial interpretation for

$$\cfrac{1}{1-\cfrac{1\cdot t}{1-\cfrac{1\cdot t}{1-\cfrac{2\cdot t}{1-\cfrac{2\cdot t}{1-\ddots}}}}}$$

i.e. α's given by $1,1,2,2,3,3,4,4,\ldots$. We can also read off statistics from this by putting in variables.

Question

We have a combinatorial interpretation for

$$\cfrac{1}{1-\cfrac{1\cdot t}{1-\cfrac{1\cdot t}{1-\cfrac{2\cdot t}{1-\cfrac{2\cdot t}{1-\ddots}}}}}$$

i.e. α's given by $1,1,2,2,3,3,4,4,\ldots$. We can also read off statistics from this by putting in variables.
Question: Combinatorially understand α's $1^k,1^k,2^k,2^k,3^k,3^k,\ldots$ "multivariately"

- $k=1$ quasi-linear case: $n!$

Question

We have a combinatorial interpretation for

$$\cfrac{1}{1-\cfrac{1\cdot t}{1-\cfrac{1\cdot t}{1-\cfrac{2\cdot t}{1-\cfrac{2\cdot t}{1-\ddots}}}}}$$

i.e. α's given by $1, 1, 2, 2, 3, 3, 4, 4, \ldots$. We can also read off statistics from this by putting in variables.
Question: Combinatorially understand α's $1^k, 1^k, 2^k, 2^k, 3^k, 3^k, \ldots$ "multivariately"

- $k = 1$ quasi-linear case: $n!$
- $k = 2$ quasi-quadratic case: Median Genocchi numbers

Question

We have a combinatorial interpretation for

$$\cfrac{1}{1-\cfrac{1\cdot t}{1-\cfrac{1\cdot t}{1-\cfrac{2\cdot t}{1-\cfrac{2\cdot t}{1-\ddots}}}}}$$

i.e. α's given by $1, 1, 2, 2, 3, 3, 4, 4, \ldots$. We can also read off statistics from this by putting in variables.

Question: Combinatorially understand α's $1^k, 1^k, 2^k, 2^k, 3^k, 3^k, \ldots$ "multivariately"

- $k = 1$ quasi-linear case: $n!$
- $k = 2$ quasi-quadratic case: Median Genocchi numbers
- $k = 3$ quasi-cubic case: Not on OEIS!!!

References

- Some multivariate master polynomials for permutations, set partitions, and perfect matchings, and their continued fractions, *Advances in Applied Mathematics*, A. Sokal and J. Zeng, 2022.
- A simple algorithm for expanding a power series as a continued fraction, *Expositiones Mathematicae*, A. Sokal, 2022.
- Classical continued fractions for some multivariate polynomials generalizing the Genocchi and median Genocchi numbers, B. Deb and A. Sokal, arxiv, 2022.
- Continued fractions for cycle-alternating permutations, B. Deb and A. Sokal, arxiv, 2023.
- Continued fractions using a Laguerre digraph interpretation of the Foata–Zeilberger bijection and its variants, B. Deb, arxiv, 2023.

Thank you