# Total Positivity during distressing times

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*Supervisor*

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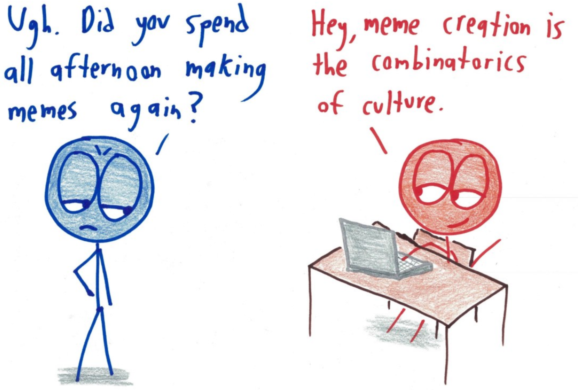


Source: Math with Bad Drawings

# Structure

1. Introduction
2. Proof techniques and some special types of matrices
    1. LGV lemma
    2. Hankel matrices
    3. Toeplitz matrices
    4. Lower triangular matrices
3. The Eulerian triangle

# Structure


Horoscope:
Something positive will happen to you this month!
COVID-19
- +

# Introduction

**Definition (TP and STP)**

We say that a matrix (not necessarily finite) is *totally positive (TP)* if all its minors (determinants of finite square submatrices) are non-negative. It is said to be *strictly totally positive (STP)* if instead of non-negativity we want positivity of the minors.
We say that a matrix is totally positive of order $r$ ($\mathrm{TP}_r$) if all its submatrices of size $\leq r$ are non-negative.

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In this talk we shall only discuss TP.

# Do such matrices exist?

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The answer to the above question is ofcourse true as we have 0 matrices.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

# Do such matrices exist?

Are there any others?

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Yes, matrices whose entries are all equal to a non-negative constant.

$$\begin{bmatrix} c & c & c \\ c & c & c \end{bmatrix}$$

where $c \geq 0$.

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Yes, diagonal matrices with non-negative entries on the diagonal.

$$\begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $d_i \geq 0$.

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Yes, bidiagonal matrices.

$$\begin{bmatrix} d_1 & 0 & 0 & 0 \\ r_1 & d_2 & 0 & 0 \\ 0 & r_2 & d_3 & 0 \\ 0 & 0 & r_3 & d_4 \\ 0 & 0 & 0 & r_4 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} d_1 & r_1 & 0 & 0 \\ 0 & d_2 & r_2 & 0 \\ 0 & 0 & d_3 & r_3 \\ 0 & 0 & 0 & d_4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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This is because any square submatrix is either upper triangular or lower triangular and hence the determinant can be computed by looking at the diagonal entries.

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Can we construct other examples?

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- Block matrices of the form

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  where both $A$ and $B$ are TP.
- Matrix Products preserve TP.
  Proof: Use Cauchy-Binet formula which we recall here:

**Theorem (Cauchy-Binet formula)**

*Let $A$ be an $m \times n$ matrix and $B$ an $n \times m$ matrix. Then we have that*

$$\det(AB) = \sum_{S \in \binom{[n]}{m}} \det(A_{[m],S}) \det(B_{S,[m]})$$

*where $[n]$ denotes the set $\{1, \ldots, n\}$, and given a set $T$, $\binom{T}{k}$ is the collection of all k-dimensional subsets of $T$.*

# Matrix Sum doesn't generally preserve TP

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

where

$$\det\left(\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\right) = -1$$

# Are there any more?

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It can be shown that all TP matrices are products of bidiagonal and diagonal matrices (Whitney 1952), (Loewner 1955), (Cryer 1972), (Gasca Peña 1990). Fallat (2001) mentions historical remarks about this.

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Infact, this result is algorithmic and we get an algorithm for bidiagonal factorisation of a matrix called Neville elimination!!

The algorithm runs in polynomial time and which also provides a polynomial time algorithm to check TP.

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We can extend the definition of TP to include matrix with polynomial (in one or several variables) entries.
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$$\text{Coefficientwise Positivity} \implies \text{Pointwise Positivity}$$

Converse is not true. Example: $x^2 - 2x + 1$.

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# LGV lemma

*The essence of mathematics is proving theorems - and so, that is what mathematicians do: They prove theorems. But to tell the truth, what they really want to prove, once in their lifetime, is a Lemma, like the one by Fatou in analysis, the Lemma of Gauss in number theory, or the Burnside-Frobenius Lemma in combinatorics.*

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- Chapter 25, Lattice Paths and Determinants
Proofs from the Book

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*... In this chapter we look at one such marvelous piece of mathematical reasoning, a counting lemma that first appeared in a paper by Bernt Lindström in 1972. Largely overlooked at the time, the result became an instant classic in 1985, when Ira Gessel and Gerard Viennot rediscovered it and demonstrated in a wonderful paper how the lemma could be successfully applied to a diversity of difficult combinatorial enumeration problems.*

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$$\det(M) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) P(u_1 \to u_{\sigma(1)}) \cdots P(u_n \to u_{\sigma(n)})$$

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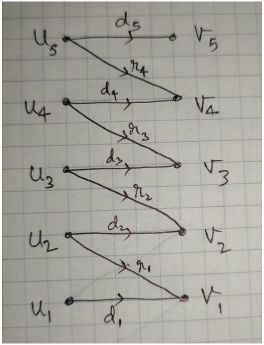
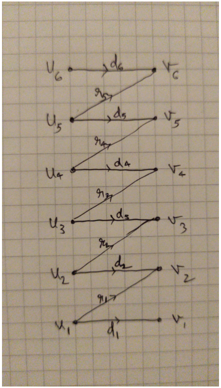
In most digraphs that we work with, the only non-intersecting paths occur when the permutation $\sigma$ is the identity permutation. Furthermore, if $R$ is the $\mathbb{R}$, $\mathbb{R}[x]$, $\mathbb{R}[x, y, z]$, etc. where each edge has non-negative weight, the determinant computed is non-negative.

# Example: Lower Bidiagonal matrix




$$\begin{bmatrix} d_1 & 0 & 0 & 0 & 0 \\ r_1 & d_2 & 0 & 0 & 0 \\ 0 & r_2 & d_3 & 0 & 0 \\ 0 & 0 & r_3 & d_4 & 0 \\ 0 & 0 & 0 & r_4 & d_5 \end{bmatrix}$$

# Example: Upper Bidiagonal matrix




$$\begin{bmatrix} d_1 & r_1 & 0 & 0 & 0 & 0 \\ 0 & d_2 & r_2 & 0 & 0 & 0 \\ 0 & 0 & d_3 & r_3 & 0 & 0 \\ 0 & 0 & 0 & d_4 & r_4 & 0 \\ 0 & 0 & 0 & 0 & d_5 & r_5 \\ 0 & 0 & 0 & 0 & 0 & d_6 \end{bmatrix}$$

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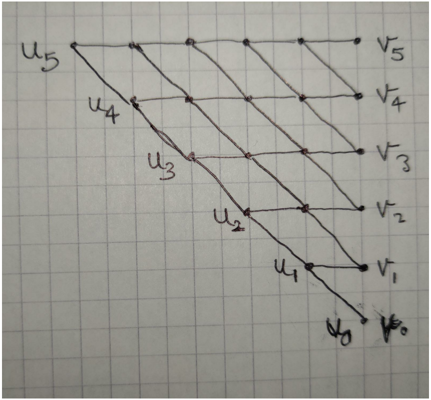
In general, Existence of an LGV digraph $\implies$ TP.

Converse not true in general. Not even true for integers as Neville factorisation involves division.

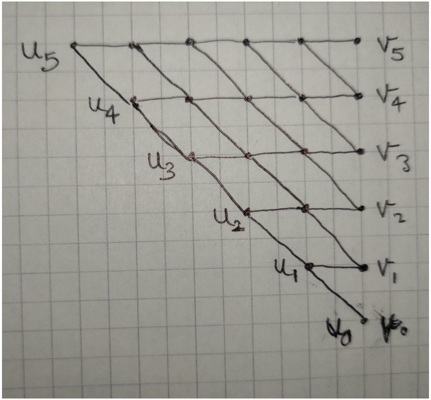
# Example:




All edges have weights 1 and are directed from left to right.

$$P(u_n \to v_k) = ??$$

# Example: Binomial Triangle

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$$P(u_n \to v_k) = \binom{n}{k}$$

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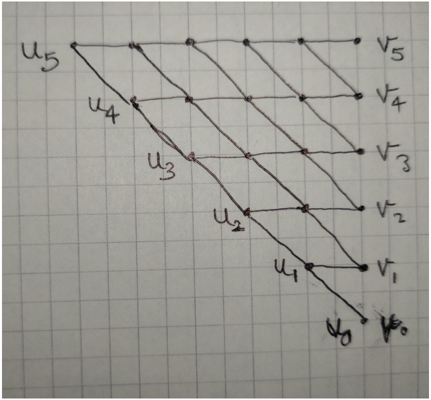


All edges have weights 1 and are directed from left to right.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

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# Hankel Matrix

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$$\begin{matrix} a_0 & a_1 & a_2 & a_3 & a_4 & \ldots \\ a_1 & a_2 & a_3 & a_4 & a_5 & \ldots \\ a_2 & a_3 & a_4 & a_5 & a_6 & \ldots \\ a_3 & a_4 & a_5 & a_6 & a_7 & \ldots \\ a_4 & a_5 & a_6 & a_7 & a_8 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

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We say that a sequence $(a_n)_{n\geq 0}$ is Hankel-totally positive (Hankel-TP in short) if its Hankel matrix is TP.

# Fundamental Fact about Hankel-TP

**Theorem (Stieltjes, Gantmacher-Krein)**

*For a sequence $(a_n)_{n\geq 0}$ of real numbers, the following are equivalent:*

1. $(a_n)_{n\geq 0}$ *is Hankel-TP.*
2. *There exists a positive measure $\mu$ on $[0,\infty)$ such that $a_n = \int_0^\infty x^n d\mu$ for all $n \geq 0$. ($(a_n)_{n\geq 0}$ is a Stieltjes moment sequence.)*
3. *There exists numbers $\alpha_0, \alpha_1, \ldots \geq 0$ such that*

$$\sum_{n=0} a_n t^n = \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t^2}{1 - \ddots}}}$$

*or in other words the generating function of $(a_n)_{n\geq 0}$ has a Stieltjes continued fraction expansion.*

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*or in other words the generating function of $(a_n)_{n\geq 0}$ has a Stieltjes continued fraction expansion.*

This doesn't hold entirely in the general case.

# Combinatorial interpretations of Stieltjes Moment Sequences

Flajolet (1980) gave a combinatorial interpretation of Stieltjes continued fractions (and also Jacobi continued fractions) by showing that Stieltjes continued fractions are generating series of Dyck paths (Motzkin paths resp.) with height dependent weights.

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Using this interpretation he gave Stieltjes continued fractions for several sequences such as Catalan numbers, Bell and Stirling numbers, tangent and secant numbers, etc. and of orthogonal polynomials such as Tchebycheff, Charlier, Hermite, Laguerre and Meixner polynomials.

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Sokal and his collaborators have used the interpretation of Flajolet and generalisations along with LGV lemma to prove several sufficient conditions for coefficientwise Hankel-TP.

# Structure

# Toeplitz Matrix

Given a sequence $a_0, a_1, \ldots$ the infinite matrix $T_\infty(\mathbf{a})$ whose $ij^{\text{th}}$ entry is $a_{i-j}$ for $i \leq j$ and 0 otherwise is called the Toeplitz matrix of $(a_n)_{n\geq 0}$.

$$\begin{matrix} a_0 & 0 & 0 & 0 & 0 & \ldots \\ a_1 & a_0 & 0 & 0 & 0 & \ldots \\ a_2 & a_1 & a_0 & 0 & 0 & \ldots \\ a_3 & a_2 & a_1 & a_0 & 0 & \ldots \\ a_4 & a_3 & a_2 & a_1 & a_0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

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\end{array}$$

We say that a sequence $(a_n)_{n\geq 0}$ is Toeplitz-totally positive (Toeplitz-TP in short) or a Polya frequency sequence (PF sequence) if its Toeplitz matrix is TP.

# Fundamental Fact about Toeplitz-TP

**Theorem (Aissen-Schoenberg-Whitney, Edrei 1952)**

*For a sequence $(a_n)_{n\geq 0}$ of real numbers with $a_0 = 1$, the following are equivalent:*

1. $(a_n)_{n\geq 0}$ *is Toeplitz-TP.*
2. *There exists $\alpha_i \geq 0$, $\beta_j \geq 0$ and $\gamma \geq 0$ such that*

$$\sum_{n=0}^{\infty} a_n t^n = e^{\gamma t} \frac{\prod_i (1 + \alpha_i t)}{\prod_j (1 - \beta_j t)}.$$

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$$\sum_{n=0}^{\infty} a_n t^n = e^{\gamma t} \frac{\prod_i (1 + \alpha_i t)}{\prod_j (1 - \beta_j t)}.$$

(2) $\Longrightarrow$ (1) is easy and even holds coefficientwise.

# Fundamental Fact about Toeplitz-TP

**Theorem (Aissen-Schoenberg-Whitney, Edrei 1952)**

*For a sequence $(a_n)_{n \geq 0}$ of real numbers with $a_0 = 1$, the following are equivalent:*

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For a finite sequence we need to show that the generating polynomial is negative real rooted.

**Theorem (Katkova 2006)**

*Let* $\xi(z) = \frac{1}{2}z(z-1)\pi^{-z/2}\Gamma(z/2)\zeta(z)$ *be the Reimann-*$\xi$ *function and let* $\xi_1(z) = \xi(\sqrt{z}+1/2)$.

# Connection of Toeplitz-TP to Reimann Hypothesis

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# Log Concavity and Log Convexity

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Toeplitz-TP$_2$ implies log-concavity i.e., $a_n^2 - a_{n-1}a_{n+1} \geq 0$.

Hankel-TP$_2$ implies log-convexity i.e., $a_n^2 - a_{n-1}a_{n+1} \leq 0$.

# Lower Triangular Matrices and three types of TP

Consider an infinite lower triangular matrix $A$ with entries $a_{ij}$ where the indexing of the rows and columns begins from 0.

$$\begin{matrix} a_{00} & 0 & 0 & 0 & 0 & \dots \\ a_{10} & a_{11} & 0 & 0 & 0 & \dots \\ a_{20} & a_{21} & a_{22} & 0 & 0 & \dots \\ a_{30} & a_{31} & a_{32} & a_{33} & 0 & \dots \\ a_{40} & a_{41} & a_{42} & a_{43} & a_{44} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

# Structure

1. Introduction
2. Proof techniques and some special types of matrices
   1. LGV lemma
   2. Hankel matrices
   3. Toeplitz matrices
   4. Lower triangular matrices
3. The Eulerian triangle

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In several cases where the answer to one of these questions is true, the others also seem to be true. It is not clear which way the implications hold.

# Easy example: Binomial Triangle

For $A$ the triangle of binomial numbers,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & \dots \\ 1 & 3 & 3 & 1 & 0 & \dots \\ 1 & 4 & 6 & 4 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

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3. The sequence $(A_n(x))$ is Hankel-TP as we have the easy continued fraction expansion

$$\sum_{n=0}^{\infty} (1+x)^n t^n = \frac{1}{1-(1+x)t}.$$

# Structure

Ongoing joint work with X. Chen, A. Dyachenko, T. Gilmore, A.D. Sokal.


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# Special case of the Eulerian triangle

We can define the Eulerian numbers:

**Definition**

The *descent* of a permutation of $\sigma \in \mathfrak{S}_n$ is an $i \in [n-1]$ such that $\sigma(i) > \sigma(i+1)$. Let $\mathrm{des}(\sigma)$ denote the number of descents of a permutation $\sigma$.

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The Eulerian number $A_{n,k}$ is defined to be the cardinality of the set $\{\sigma \in \mathfrak{S}_{n+1} | \mathrm{des}(\sigma) = k\}$.

These numbers satisfy the recurrence $A_{n,k} = (n-k+1)A_{n-1,k-1} + (k+1)A_{n-1,k}$.

# Eulerian Triangle Conjecture

**Conjecture (Brenti 1996)**

*The infinite lower triangular matrix whose $(n, k)^{\mathrm{th}}$ entry is the Eulerian number $A_{n,k}$ is totally positive.*

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*The infinite lower triangular matrix whose $(n, k)^{\mathrm{th}}$ entry is the Eulerian number $A_{n,k}$ is totally positive.*

Using Neville elimination we have verified the conjecture for the upper left $500 \times 500$ submatrix. But the general pattern is not clear.

# Putting variables in the recurrence

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$$A_{n,k} = [a(n-k) + b(k-1) + c]A_{n-1,k-1} + (dk+e)A_{n-1,k}$$

and ask if it is coefficientwise TP in all five variables $a, b, c, d, e$. Setting $a = 1, b = 0, c = 1, d = 1, e = 1$ gives the Eulerian triangle and hence, coefficientwise TP of the *abcde* matrix can be used to prove that the Eulerian triangle is TP.

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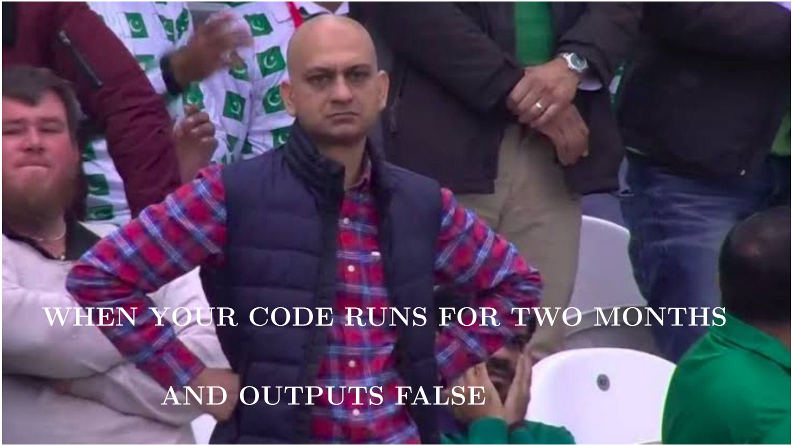
We checked it upto $11 \times 11$ which took about a week to compute.
Unfortunately the $12 \times 12$ matrix turned out false after about 2 months of real time.


WHEN YOUR CODE RUNS FOR TWO MONTHS
AND OUTPUTS FALSE

# Is all hope lost?

We can look at the matrix with $b = 0$ and which is generated by the recurrence

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Infact, we have the following special cases for different substitutions:

| Values of $a, c, d, e$ | Matrix obtained |
|---|---|
| $a = 1, c = 1, d = 1, e = 1$ | Eulerian triangle |
| $a = 1, c = 0, d = 1, e = 1$ | Down-shifted Eulerian triangle |
| $a = 1, c = 1, d = 1, e = 0$ | Diagonally down-shifted Eulerian triangle |
| $a = 0.c = 1.d = 1.e = 1$ | $k$-dependent recurrence which gives the matrix of Stirling subset numbers. Brenti 1995. |
| $a = 1, c = 1, d = 0.e = 1$ | Row reversed matrix of Stirling subset numbers |

# *ace* triangle

The coefficientwise results for cases of $\{a = 0, d = 0\}$ (weighted Binomial triangle), $a = 0$ (weighted Stirling subset triangle) were already known. We managed to resolve the $c = 0$ (weighted reverse Stirling subset) case and gave two different proofs for the result.

The first proof involves a bijecitive argument involving the LGV lemma. The second proof consists of a linear algebraic method involving matrix products which we call the quasi-production matrix method.

$$T(n,k) = (n-k+1)T(n-1,k-1)$$

$$+(k+1)T(n-1,k)$$

$$T(n,k) = [a(n-k)+c]T(n-1,k-1)$$

$$+(dk+e)T(n-1,k)$$

$$T(n,k) = (n-k+1)T(n-1,k-1) + (k+1)T(n-1,k)$$

Eulerian triangle conjecture

$$T(n,k) = [a(n-k)+c]T(n-1,k-1) + (dk+e)T(n-1,k)$$

*acde* conjecture

$$T(n,k) = [a(n-k)+c]T(n-1,k-1) + (dk+e)T(n-1,k) + [f(n-2)+g]T(n-2,k-1)$$

*acdefg* conjecture

$$T(n,k) = [a(n-k)+c]T(n-1,k-1) + \left(\sum_{i=0}^{k} d_i\right) T(n-1,k) + [f(n-2)+g]T(n-2,k-1)$$

$ac\mathbf{d}fg$ conjecture

# Summary Slide

1. Introduction - Definitions and generating examples
2. Proof techniques and some special types of matrices
    1. LGV lemma - description and illustration using some simple examples
    2. Hankel matrices - definition and fundamental theorem relating to Stieltjes moment sequences and Stieltjes Continued fractions
    3. Toeplitz matrices - definition and fundamental theorem relating to negative real zeros and positive real poles
    4. Lower triangular matrices - three different types of TP
3. The Eulerian triangle - the conjecture, partial progress and a lot of other stronger conjectures