Total positivity of the Eulerian triangle: A big generalisation of Brenti's conjecture

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Based on Ongoing Joint Work With

Xi Chen, Alexander Dyachenko, Tomack Gilmore and Alan D. Sokal

Introduction

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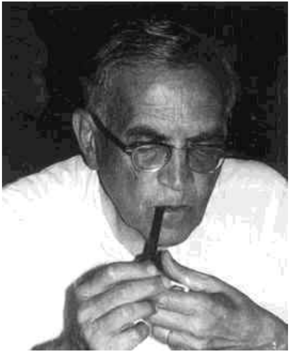
Need not be a square matrix, or finite!

Will consider matrix of polynomials soon!

First defined independently by two different groups in the 30s

(a) M.G. Krein (1907-1989)



(b) I.G. Schoenberg (1903-1990)

Source: MacTutor History of Mathematics Archive

We use Schoenberg's terminology.

Example and basic properties

- Example: Bidiagonal matrices with entries ≥ 0

$$\begin{bmatrix} 0 & a_1 & b_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & b_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_3 & b_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_4 & b_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_5 & b_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_6 \end{bmatrix}$$

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Applications to Combinatorics

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- Many important combinatorial triangles are TP.

Example 1: Binomial triangle

Entries $\binom{n}{k}$

$$\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example 2: Stirling subset triangle

Entries $\left\{ {n+1 \atop k} \right\}$: set partitions of $\{1, \ldots, n+1\}$ with k blocks

$$\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 7 & 6 & 1 & & \\ 1 & 15 & 25 & 10 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example 3: Stirling cycle triangle

Entries $\left[{n+1 \atop k}\right]$: permutations on $n+1$ letters with k cycles

$$\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 3 & 1 & & & \\ 6 & 11 & 6 & 1 & & \\ 24 & 50 & 35 & 10 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example 4: Reversed Stirling cycle triangle

Original Matrix

$$\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 3 & 1 & & & \\ 6 & 11 & 6 & 1 & & \\ 24 & 50 & 35 & 10 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Reversal

$$\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 2 & & & \\ 1 & 6 & 11 & 6 & & \\ 1 & 10 & 35 & 50 & 24 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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Arbitrary lower triangular TP matrix	Not true

Reversed Stirling Subset Triangle

Reversed Stirling subset triangle:

$$\left(\left\{ {n+1 \atop k} \right\}^{\text{rev}}\right)_{n,k\geq 0} = \left(\left\{ {n+1 \atop n-k+1} \right\}\right)_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 6 & 7 & 1 & & \\ 1 & 10 & 25 & 15 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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Conjecture (Us 2019)

The infinite lower triangular matrix $\left(\left\{ n+1 \atop k \right\}^{\text{rev}}\right)_{n,k\geq 0} = \left(\left\{ n+1 \atop n-k+1 \right\}\right)_{n,k\geq 0}$ *is TP.*

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General pattern not clear.

A comparision of the two triangles

Reversed Stirling subset triangle:

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Coefficientwise TP $\Longrightarrow$ Pointwise TP.

But much stronger.

Putting variables in the recurrence

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Introduce variables a, b, c, d, e.

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ace triangle

$$\begin{bmatrix} 1 \\ e & c \\ e^2 & ae + 2ce & c^2 \\ e^3 & 3ae^2 + 3ce^2 & a^2e + 3ace + 3c^2e & c^3 \\ e^4 & 6ae^3 + 4ce^3 & 7a^2e^2 + 12ace^2 + 6c^2e^2 & a^3e + 4a^2ce + 6ac^2e + 4c^3e & c^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

recurrence

$$A_{n,k} = [{\color{green}a}(n-k) + {\color{red}c}]A_{n-1,k-1} + {\color{blue}e}A_{n-1,k}$$

ace triangle

$$
\begin{bmatrix}
1 & & & & & \\
e & c & & & & \\
e^2 & ae + 2ce & c^2 & & & \\
e^3 & 3ae^2 + 3ce^2 & a^2e + 3ace + 3c^2e & c^3 & & \\
e^4 & 6ae^3 + 4ce^3 & 7a^2e^2 + 12ace^2 + 6c^2e^2 & a^3e + 4a^2ce + 6ac^2e + 4c^3e & c^4 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
$$

recurrence

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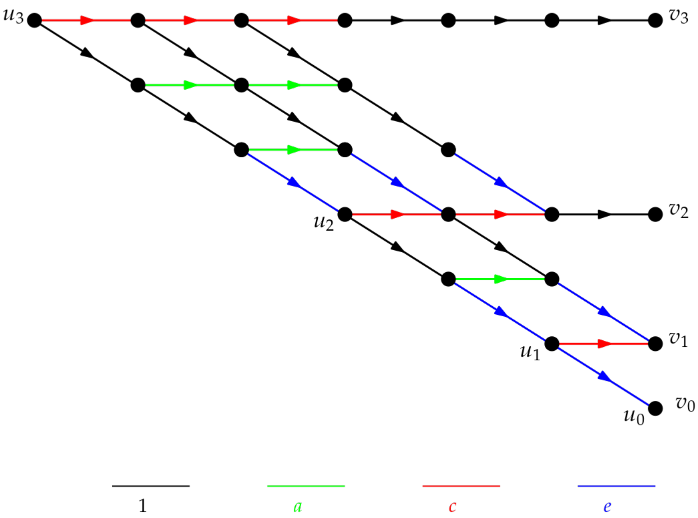
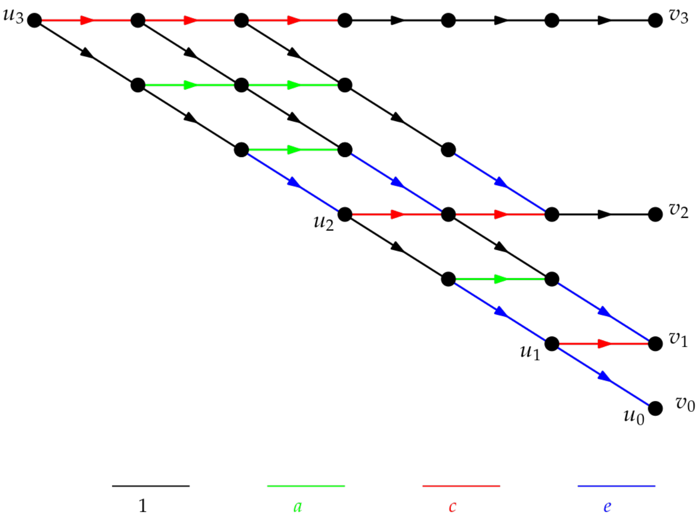
We have three proofs:

- Two Digraph proofs using the Lindström-Gessel-Viennot lemma
  - Bijection between paths in the digraph to set partitions.
  - Exploiting the structure of the digraph.
- Algebraic proof

An ace digraph



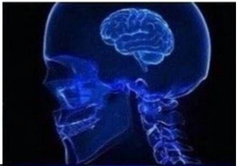
An ace digraph



We have a weight preserving bijection between paths from $u_n \to v_k$ and set partitions of $\{1, \ldots, n+1\}$ into $n - k + 1$ parts.

$$A_{n,k} = (n-k+1)A_{n-1,k-1} + (k+1)A_{n-1,k}$$

Eulerian triangle conjecture



$$A_{n,k} = [a(n-k)+c]A_{n-1,k-1} + (dk+e)A_{n-1,k}$$

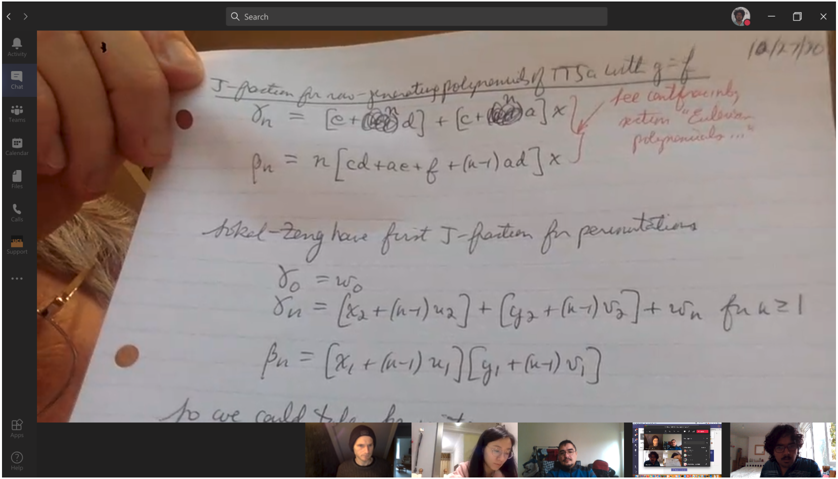
acde conjecture

$$A_{n,k} = [a(n-k)+c]A_{n-1,k-1} + (dk+e)A_{n-1,k} + [f(n-2)+g]A_{n-2,k-1}$$

acdefg conjecture

$$A_{n,k} = [a(n-k)+c]A_{n-1,k-1} + \left(\sum_{i=0}^{k} d_i\right) A_{n-1,k} + [f(n-2)+g]A_{n-2,k-1}$$

$ac\mathbf{d}fg$ conjecture



Thank you

Meme images from internet and digraph drawn by Tomack Gilmore.

LGV lemma

The essence of mathematics is proving theorems - and so, that is what mathematicians do: They prove theorems. But to tell the truth, what they really want to prove, once in their lifetime, is a Lemma, like the one by Fatou in analysis, the Lemma of Gauss in number theory, or the Burnside-Frobenius Lemma in combinatorics.

...

- Chapter 25, Lattice Paths and Determinants
Proofs from the Book

LGV lemma on planar digraphs

- Let G be a directed acyclic graph embedded in the plane such that

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  - Sources – clockwise direction

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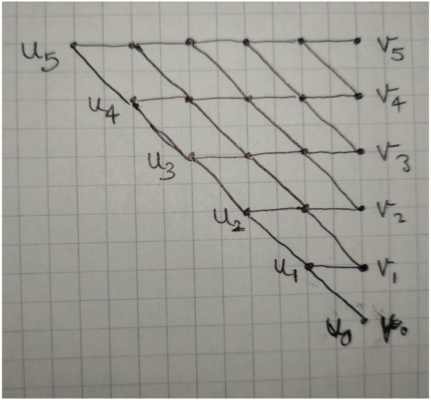

Example. Edges weighted 1.

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- Weight of a path – product of edges

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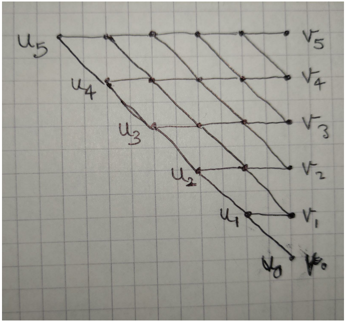
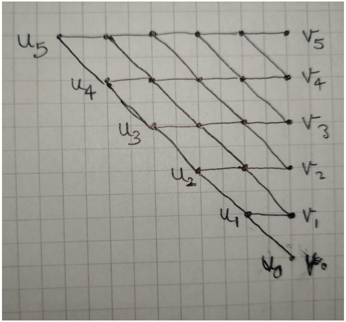
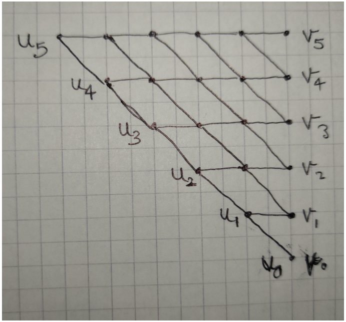

u_5
v_5
u_4
v_4
u_3
v_3
u_2
v_2
u_1
v_1
u_0
v_0



Path matrix

$$\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}$$



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Binomial triangle!

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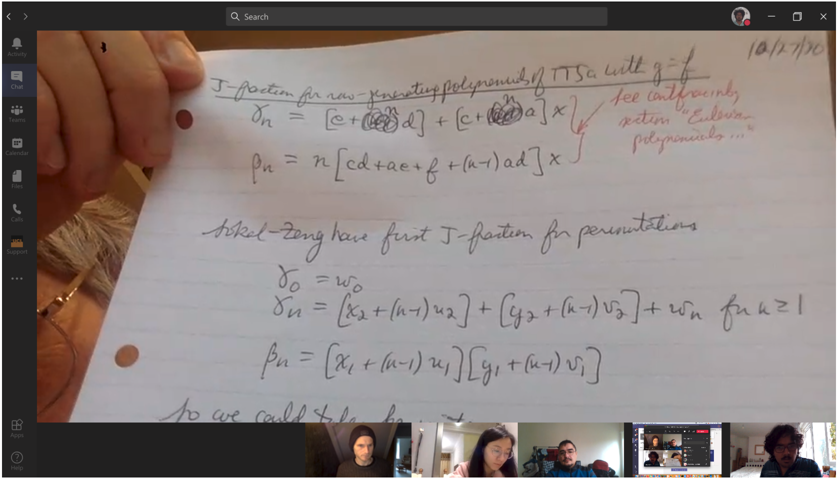
- Path matrix
  - Entry (n, k) – sum of paths from u_n to v_k
- LGV lemma says that the path matrix is totally positive



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