Warm-up session
Sikkim INMO training camp

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Problems covered during the session:

1. A new club flag is to be designed with 6 vertical stripes using some or all of the colours yellow, green, blue and red. In how many ways can this be done so that no two adjacent stripes have the same colour?

2. Let $n \in \mathbb{N}$ and let $p_1, p_2, \ldots, p_k$ be prime numbers. The prime factorisation of n is

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k} .$$

   What is the number of divisors of n?

   Answer: $(a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$. Prove this!

3. What is the last digit of the following numbers:

   (i) $1! + 2! + 3! + 4!$?

   (ii) $1! + 2! + 3! + 4! + \cdots + 2025!$?

4. How many zeroes are there at the end of the following numbers:

   (i) $9!$

   (ii) $10!$

   (iii) $45!$

   (iv) $81!$

   (v) $2025!$

   Now guess a general formula for the number of zeros at the end of $n!$.

   Definition 1. *Let $x \in \mathbb{R}$. The floor of x, denoted by $\lfloor x \rfloor$, and the ceiling of x, denoted by $\lceil x \rceil$ are defined to be the following integers*

$$\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\} \tag{1}$$
$$\lceil x \rceil = \min\{m \in \mathbb{Z} \mid m \geq x\} \tag{2}$$

   The number of zeroes at the end of $n!$ is the following number

$$\sum_{i=1}^{\infty} \left\lfloor \frac{n}{5^i} \right\rfloor = \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \cdots .$$

   More generally, we have the following theorem:

Theorem 1. *Let $n \in \mathbb{N}$ and let p be a prime number. Let k be such that $p^k \mid n!$ but $p^{k+1} \nmid n!$. Then*

$$k \;=\; \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots . \tag{3}$$

Homework: Prove this theorem.

More problems:

1. Prove that the number of ways of arranging p 1's and q 0's in a line such that no two 1's are adjacent is $\binom{q+1}{p}$.

2. Let n, r be given positive integers. Let $A_{n,r}$ be the number of non-negative integer solutions $(x_1, x_2, \ldots, x_n)$ of the equation

$$x_1 + x_2 + \cdots + x_n \;=\; r. \tag{4}$$

   What is the value of $A_{n,r}$?

3. Let P be a path that starts at $(0,0)$ and ends at (n,k) where $n, k \in \mathbb{N}$. The path P uses steps

   - $(i,j) \to (i+1,j)$
   - $(i,j) \to (i,j+1)$

   How many such paths P are there?

4. Prove that the number of triples (A, B, C) where A, B, C are subsets of $\{1, 2, \ldots, n\}$ such that $A \cap B \cap C = \emptyset$, $A \cap B \neq \emptyset, B \cap C \neq \emptyset$ is $7^n - 2 \cdot 6^n + 5^n$.

   - RMO 2004

5. There are n students in a class and no two of them have the same height. The students stand in a line, one behind another, in no particular order of their heights.

   (a) How many different orders are there in which the shortest student is not in the first position and the tallest student is not in the last position?

   (b) The badness of an ordering is the largest number k with the following property. There is at least one student X such that there are k students taller than X standing ahead of X. Find a formula $g_k(n)$ = number of orderings of n students with badness k.

   Example: The ordering 64 61 67 63 62 66 65 (the numbers denote heights) has badness 3 as student with height 62 has three taller students (with heights 64, 67 and 63) standing ahead in the line and nobody has more than 3 taller students standing ahead.

Possible hints for (b): It may be useful to first count orderings of badness 1 and/or to find $f_k(n)$ = the number of orderings of n students with badness less than or equal to k.

- CMI BSc Entrance 2023