Show that $i\frac{d}{dx}$ is a Hermitian operator.

$$\int_{-\infty}^{+\infty} \psi_m^* \left(\mathrm{i}\frac{d}{dx}\right)\psi_n\, dx = i[\psi_m^*\psi_n]_{-\infty}^{\pm\infty} - i\int_{-\infty}^{+\infty} \frac{d\psi_m^*}{dx}\psi_n\, dx = \int_{-\infty}^{+\infty}\left(-\mathrm{i}\frac{d\psi_m^*}{dx}\right)\psi_n\, dx = \int_{-\infty}^{+\infty}\left(\mathrm{i}\frac{d}{dx}\right)^*\psi_m^*\psi_n\, dx$$

Show that $\frac{d^2}{dx^2}$ is a Hermitian operator.

$$\int_{-\infty}^{+\infty} \psi_m^*\left(\frac{d^2}{dx^2}\right)\psi_n\, dx = \left[\psi_m^*\frac{d\psi_n}{dx}\right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\frac{d\psi_m^*}{dx}\frac{d\psi_n}{dx}dx = -\int_{-\infty}^{+\infty}\frac{d\psi_m^*}{dx}\frac{d\psi_n}{dx}dx$$
$$= -\left[\frac{d\psi_m^*}{dx}\psi_n\right]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty}\frac{d^2\psi_m^*}{dx^2}\psi_n\, dx = \int_{-\infty}^{+\infty}\left(\frac{d^2}{dx^2}\right)^*\psi_m^*\psi_n\, dx$$

Show that $-i\hbar x\frac{d}{dx}$ is a Hermitian operator.

$$\int_{-\infty}^{+\infty}\psi_m^*\left(-\mathrm{i}\hbar x\frac{d}{dx}\right)\psi_n\, dx = -\mathrm{i}\hbar\int_{-\infty}^{+\infty} x\psi_m^*\frac{d\psi_n}{dx}dx = -\mathrm{i}\hbar[x\psi_m^*\psi_n]_{-\infty}^{\pm\infty} + \mathrm{i}\hbar\int_{-\infty}^{+\infty}\frac{d(x\psi_m^*)}{dx}\psi_n\, dx$$
$$= \mathrm{i}\hbar\int_{-\infty}^{+\infty}\frac{d(x\psi_m^*)}{dx}\psi_n\, dx = \mathrm{i}\hbar\int_{-\infty}^{+\infty}\left(\psi_m^* + x\frac{d\psi_m^*}{dx}\right)\psi_n\, dx$$
$$= \mathrm{i}\hbar\int_{-\infty}^{+\infty} x\frac{d\psi_m^*}{dx}\psi_n\, dx = \int_{-\infty}^{+\infty}\left(-\mathrm{i}\hbar x\frac{d}{dx}\right)^*\psi_m^*\psi_n\, dx$$

If $\hat{A}$ and $\hat{B}$ are two Hermitian operators, show that $(\hat{A}\hat{B}+\hat{B}\hat{A})$ is a Hermitian operator.

$$\int_{-\infty}^{+\infty}\psi_m^*(\hat{A}\hat{B}+\hat{B}\hat{A})\psi_n\, dx = \int_{-\infty}^{+\infty}\psi_m^*\hat{A}\hat{B}\psi_n\, dx + \int_{-\infty}^{+\infty}\psi_m^*\hat{B}\hat{A}\psi_n\, dx = \int_{-\infty}^{+\infty}\hat{A}^*\psi_m^*\hat{B}\psi_n\, dx + \int_{-\infty}^{+\infty}\hat{B}^*\psi_m^*\hat{A}\psi_n\, dx$$
$$= \int_{-\infty}^{+\infty}\hat{B}^*\hat{A}^*\psi_m^*\psi_n\, dx + \int_{-\infty}^{+\infty}\hat{A}^*\hat{B}^*\psi_m^*\psi_n\, dx = \int_{-\infty}^{+\infty}(\hat{B}^*\hat{A}^*+\hat{A}^*\hat{B}^*)\psi_m^*\psi_n\, dx$$
$$= \int_{-\infty}^{+\infty}(\hat{A}^*\hat{B}^*+\hat{B}^*\hat{A}^*)\psi_m^*\psi_n\, dx = \int_{-\infty}^{+\infty}(\hat{A}\hat{B}+\hat{B}\hat{A})^*\psi_m^*\psi_n\, dx$$
$$\because \int_{-\infty}^{+\infty}\psi_m^*(\hat{A}\hat{B}+\hat{B}\hat{A})\psi_n\, dx = \int_{-\infty}^{+\infty}(\hat{A}\hat{B}+\hat{B}\hat{A})^*\psi_m^*\psi_n\, dx \quad \therefore (\hat{A}\hat{B}+\hat{B}\hat{A}) \text{ is a Hermitian operator.}$$

If $\hat{A}$ and $\hat{B}$ are two Hermitian operators, show that $(\hat{A}\hat{B}-\hat{B}\hat{A})$ is not a Hermitian operator.

$$\int_{-\infty}^{+\infty}\psi_m^*(\hat{A}\hat{B}-\hat{B}\hat{A})\psi_n\, dx = \int_{-\infty}^{+\infty}\psi_m^*\hat{A}\hat{B}\psi_n\, dx - \int_{-\infty}^{+\infty}\psi_m^*\hat{B}\hat{A}\psi_n\, dx = \int_{-\infty}^{+\infty}\hat{A}^*\psi_m^*\hat{B}\psi_n\, dx - \int_{-\infty}^{+\infty}\hat{B}^*\psi_m^*\hat{A}\psi_n\, dx$$
$$= \int_{-\infty}^{+\infty}\hat{B}^*\hat{A}^*\psi_m^*\psi_n\, dx - \int_{-\infty}^{+\infty}\hat{A}^*\hat{B}^*\psi_m^*\psi_n\, dx = \int_{-\infty}^{+\infty}(\hat{B}^*\hat{A}^*-\hat{A}^*\hat{B}^*)\psi_m^*\psi_n\, dx$$
$$= -\int_{-\infty}^{+\infty}(\hat{A}^*\hat{B}^*-\hat{B}^*\hat{A}^*)\psi_m^*\psi_n\, dx = -\int_{-\infty}^{+\infty}(\hat{A}\hat{B}-\hat{B}\hat{A})^*\psi_m^*\psi_n\, dx$$
$$\because \int_{-\infty}^{+\infty}\psi_m^*(\hat{A}\hat{B}-\hat{B}\hat{A})\psi_n\, dx \neq \int_{-\infty}^{+\infty}(\hat{A}\hat{B}-\hat{B}\hat{A})^*\psi_m^*\psi_n\, dx \quad \therefore (\hat{A}\hat{B}-\hat{B}\hat{A}) \text{ is not a Hermitian operator.}$$

Eigenvalue of $-i\hbar\vec{\nabla}$

$$-i\hbar\vec{\nabla}\left(e^{-i\vec{k}\cdot\vec{r}}\right) = -i\hbar\vec{\nabla}\left(e^{i\vec{k}\cdot\vec{r}}\right) = -i\hbar e^{i\vec{k}\cdot\vec{r}}\vec{\nabla}(i\vec{k}\cdot\vec{r}) = \hbar e^{i\vec{k}\cdot\vec{r}}\vec{\nabla}(\vec{k}\cdot\vec{r}) = e^{i\vec{k}\cdot\vec{r}}\vec{\nabla}(\vec{p}\cdot\vec{r}) \quad [\because \vec{p} = \hbar\vec{k}]$$
$$= me^{i\vec{k}\cdot\vec{r}}\vec{\nabla}\left(\frac{d\vec{r}}{dt}\cdot\vec{r}\right) \quad \left[\because \vec{p} = m\vec{v} = m\frac{d\vec{r}}{dt}\right] = \frac{m}{2}e^{i\vec{k}\cdot\vec{r}}\vec{\nabla}\left(\frac{d}{dt}(\vec{r}\cdot\vec{r})\right) = \frac{m}{2}e^{i\vec{k}\cdot\vec{r}}\vec{\nabla}\left(\frac{d}{dt}(r^2)\right) = \frac{m}{2}e^{i\vec{k}\cdot\vec{r}}\frac{d}{dt}\left(\vec{\nabla}(r^2)\right)$$
$$= \frac{m}{2}e^{i\vec{k}\cdot\vec{r}}\frac{d}{dt}\left(\vec{\nabla}(x^2+y^2+z^2)\right) = \frac{m}{2}e^{i\vec{k}\cdot\vec{r}}\frac{d}{dt}\left(2(x\hat{\imath}+y\hat{\jmath}+z\hat{k})\right) = me^{i\vec{k}\cdot\vec{r}}\frac{d\vec{r}}{dt} = \vec{p}e^{i\vec{k}\cdot\vec{r}} \quad \left[\because \vec{p} = m\vec{v} = m\frac{d\vec{r}}{dt}\right]$$

$\therefore -i\hbar\vec{\nabla} = \vec{p} = \hbar\vec{k} \quad [\because \vec{p} = \hbar\vec{k}]$

Eigenvalue of $-\hbar^2\nabla^2$

$$-\hbar^2\nabla^2\left(e^{-i\vec{k}\cdot\vec{r}}\right) = -\hbar^2\vec{\nabla}\cdot\vec{\nabla}\left(e^{-i\vec{k}\cdot\vec{r}}\right) = -i\hbar\vec{\nabla}\cdot\left(-i\hbar\vec{\nabla}\left(e^{-i\vec{k}\cdot\vec{r}}\right)\right) = -i\hbar\vec{\nabla}\cdot\left(\hbar\vec{k}e^{-i\vec{k}\cdot\vec{r}}\right) = -i\hbar\vec{\nabla}\cdot\left(\vec{p}e^{-i\vec{k}\cdot\vec{r}}\right)$$

$$= -i\hbar\left(e^{-i\vec{k}\cdot\vec{r}}\vec{\nabla}\cdot\vec{p} + \vec{p}\cdot\vec{\nabla}\left(e^{-i\vec{k}\cdot\vec{r}}\right)\right) = -i\hbar\left(me^{-i\vec{k}\cdot\vec{r}}\vec{\nabla}\cdot\frac{d\vec{r}}{dt} + i\vec{k}\cdot\vec{\nabla}\left(-i\hbar e^{-i\vec{k}\cdot\vec{r}}\right)\right)$$

$$= -i\hbar\left(me^{-i\vec{k}\cdot\vec{r}}\frac{d}{dt}\left(\vec{\nabla}\cdot\vec{r}\right) + i\vec{k}\cdot\left(\hbar\vec{k}e^{i\vec{k}\cdot\vec{r}}\right)\right) = -i\hbar\left(me^{-i\vec{k}\cdot\vec{r}}\frac{d}{dt}(3) + i\hbar k^2\right) = -i\hbar(i\hbar k^2) = \hbar^2k^2$$

Commutator

$$[\hat{x},\hat{p}_x]\psi = (\hat{x}\hat{p}_x - \hat{p}_x\hat{x})\psi = x\left(-\mathrm{i}\hbar\frac{\partial}{\partial x}\right)\psi - \left(-\mathrm{i}\hbar\frac{\partial}{\partial x}\right)x\psi = -\mathrm{i}\hbar x\frac{\partial\psi}{\partial x} + \mathrm{i}\hbar\frac{\partial}{\partial x}(x\psi)$$

$$= -\mathrm{i}\hbar x\frac{\partial\psi}{\partial x} + \mathrm{i}\hbar x\frac{\partial\psi}{\partial x} + \mathrm{i}\hbar\psi = \mathrm{i}\hbar\psi \Longrightarrow [\hat{x},\hat{p}_x] = \mathrm{i}\hbar$$

$$[\hat{p}_x,\hat{x}] = -[\hat{x},\hat{p}_x] = -\mathrm{i}\hbar$$

$$[\hat{x},\hat{H}]\psi = \left[\hat{x},\frac{\hat{p}_x{}^2}{2m} + \hat{V}\right]\psi = \left[\hat{x},\frac{\hat{p}_x{}^2}{2m}\right]\psi + [\hat{x},\hat{V}]\psi = \left(\frac{i\hbar}{m}\hat{p}_x\right)\psi - (xV - Vx)\psi = \frac{\hbar^2}{m}\frac{\partial\psi}{\partial x} \Longrightarrow [\hat{x},\hat{H}] = \frac{\hbar^2}{m}\frac{\partial}{\partial x}$$

$$[\hat{H},\hat{x}] = -[\hat{x},\hat{H}] = -\frac{i\hbar}{m}\hat{p}_x = -\frac{\hbar^2}{m}\frac{\partial}{\partial x}$$

$$[\hat{H},\hat{p}_x]\psi = \left[\frac{\hat{p}_x{}^2}{2m} + \hat{V},\hat{p}_x\right]\psi = \left[\frac{\hat{p}_x{}^2}{2m},\hat{p}_x\right]\psi + [\hat{V},\hat{p}_x]\psi = [\hat{V},\hat{p}_x]\psi = \left[\hat{V},-\mathrm{i}\hbar\frac{\partial}{\partial x}\right]\psi = -\mathrm{i}\hbar\left(V\frac{\partial}{\partial x} - \frac{\partial}{\partial x}V\right)\psi$$

$$= -\mathrm{i}\hbar\left(V\frac{\partial\psi}{\partial x} - \frac{\partial}{\partial x}(V\psi)\right) = -\mathrm{i}\hbar\left(V\frac{\partial\psi}{\partial x} - V\frac{\partial\psi}{\partial x} - \psi\frac{\partial V}{\partial x}\right) = -\mathrm{i}\hbar\left(-\psi\frac{\partial V}{\partial x}\right) = \left(\mathrm{i}\hbar\frac{\partial V}{\partial x}\right)\psi \Longrightarrow [\hat{H},\hat{p}_x] = \mathrm{i}\hbar\frac{\partial V}{\partial x}$$

$$[\hat{p}_x,\hat{H}] = -[\hat{H},\hat{p}_x] = -\mathrm{i}\hbar\frac{\partial V}{\partial x}$$

$$[\hat{x},\hat{p}_x{}^{n+1}] = [\hat{x},\hat{p}_x{}^n\hat{p}_x] = [\hat{x},\hat{p}_x{}^n]\hat{p}_x + \hat{p}_x{}^n[\hat{x},\hat{p}_x] = (n\mathrm{i}\hbar\hat{p}_x{}^{n-1})\hat{p}_x + \hat{p}_x{}^n(\mathrm{i}\hbar) = n\mathrm{i}\hbar\hat{p}_x{}^n + \mathrm{i}\hbar\hat{p}_x{}^n = (n+1)\mathrm{i}\hbar\hat{p}_x{}^n$$

$$[\hat{x}^{n+1},\hat{p}_x] = [\hat{x}^n\hat{x},\hat{p}_x] = [\hat{x}^n,\hat{p}_x]\hat{x} + \hat{x}^n[\hat{x},\hat{p}_x] = (n\mathrm{i}\hbar\hat{x}^{n-1})\hat{x} + \hat{x}^n(\mathrm{i}\hbar) = n\mathrm{i}\hbar\hat{x}^n + \mathrm{i}\hbar\hat{x}^n = (n+1)\mathrm{i}\hbar\hat{x}^n$$

$$\left[\hat{x},\frac{d}{dx}\right]\psi = \left(x\frac{d}{dx} - \frac{d}{dx}x\right)\psi = x\frac{d\psi}{dx} - \frac{d}{dx}(x\psi) = x\frac{d\psi}{dx} - x\frac{d\psi}{dx} - \psi = -\psi \Longrightarrow \left[\hat{x},\frac{d}{dx}\right] = -1$$

$$\left[\hat{x},\frac{d^2}{dx^2}\right]\psi = \left(x\frac{d^2}{dx^2} - \frac{d^2}{dx^2}x\right)\psi = x\frac{d^2\psi}{dx^2} - \frac{d^2}{dx^2}(x\psi) = x\frac{d^2\psi}{dx^2} - \frac{d}{dx}\left(x\frac{d\psi}{dx} + \psi\right)$$

$$= x\frac{d^2\psi}{dx^2} - \left(x\frac{d^2\psi}{dx^2} + \frac{d\psi}{dx} + \frac{d\psi}{dx}\right) = -2\frac{d\psi}{dx} \Longrightarrow \left[\hat{x},\frac{d^2}{dx^2}\right] = -2\frac{d}{dx}$$

$$\left[\hat{x},\frac{d^2}{dx^2}\right] = \left[\hat{x},\frac{d}{dx}\frac{d}{dx}\right] = \frac{d}{dx}\left[\hat{x},\frac{d}{dx}\right] + \left[\hat{x},\frac{d}{dx}\right]\frac{d}{dx} = \frac{d}{dx}(-1) + (-1)\frac{d}{dx} = -2\frac{d}{dx}$$

$$\left[\hat{x}^2,\frac{d}{dx}\right]\psi = \left(x^2\frac{d}{dx} - \frac{d}{dx}x^2\right)\psi = x^2\frac{d\psi}{dx} - \frac{d}{dx}(x^2\psi) = x^2\frac{d\psi}{dx} - \left(x^2\frac{d\psi}{dx} + 2x\psi\right) = -2x\psi \Longrightarrow \left[\hat{x}^2,\frac{d}{dx}\right] = -2x$$

$$\left[\hat{x}^2,\frac{d}{dx}\right] = \left[\hat{x}\hat{x},\frac{d}{dx}\right] = \hat{x}\left[\hat{x},\frac{d}{dx}\right] + \left[\hat{x},\frac{d}{dx}\right]\hat{x} = x(-1) + (-1)x = -2ix$$

$$[\vec{\nabla},\nabla^2]\psi = (\vec{\nabla}\nabla^2 - \nabla^2\vec{\nabla})\psi = \vec{\nabla}(\nabla^2\psi) - \nabla^2(\vec{\nabla}\psi) = \vec{\nabla}(\nabla^2\psi) - \left\{\vec{\nabla}\left(\vec{\nabla}\cdot(\vec{\nabla}\psi)\right) - \vec{\nabla}\times\vec{\nabla}\times(\vec{\nabla}\psi)\right\}$$

$$= \vec{\nabla}(\nabla^2\psi) - \vec{\nabla}\left(\vec{\nabla}\cdot\left(\vec{\nabla}\psi\right)\right) = \vec{\nabla}(\nabla^2\psi) - \vec{\nabla}(\nabla^2\psi) = 0 \Longrightarrow \left[\vec{\boldsymbol{\nabla}}, \boldsymbol{\nabla}^{\mathbf{2}}\right] = \mathbf{0}$$

$\left[\hat{L}^2, \hat{L}_z\right] = \left[\hat{L}_x{}^2 + \hat{L}_y{}^2 + \hat{L}_z{}^2, \hat{L}_z\right] = \left[\hat{L}_x{}^2, \hat{L}_z\right] + \left[\hat{L}_y{}^2, \hat{L}_z\right] + \left[\hat{L}_z{}^2, \hat{L}_z\right]$
$= \left(\hat{L}_x[\hat{L}_x, \hat{L}_z] + [\hat{L}_x, \hat{L}_z]\hat{L}_x\right) + \left(\hat{L}_y[\hat{L}_y, \hat{L}_z] + [\hat{L}_y, \hat{L}_z]\hat{L}_y\right) + \left(\hat{L}_z[\hat{L}_z, \hat{L}_z] + [\hat{L}_z, \hat{L}_z]\hat{L}_z\right)$
$= \left(\hat{L}_x(-i\hbar\hat{L}_y) + (-i\hbar\hat{L}_y)\hat{L}_x\right) + \left(\hat{L}_y(i\hbar\hat{L}_x) + (-i\hbar\hat{L}_x)\hat{L}_y\right) = 0$
Similarly, $[\hat{L}^2, \hat{L}_x] = 0$ and $[\hat{L}^2, \hat{L}_y] = 0$

$[\hat{L}^2, \hat{L}_+] = [\hat{L}^2, \hat{L}_x + i\hat{L}_y] = [\hat{L}^2, \hat{L}_x] + i[\hat{L}^2, \hat{L}_y] = 0 \quad \because [\hat{L}^2, \hat{L}_x] = 0$ and $[\hat{L}^2, \hat{L}_y] = 0$
$[\hat{L}^2, \hat{L}_-] = [\hat{L}^2, \hat{L}_x - i\hat{L}_y] = [\hat{L}^2, \hat{L}_x] - i[\hat{L}^2, \hat{L}_y] = 0 \quad \because [\hat{L}^2, \hat{L}_x] = 0$ and $[\hat{L}^2, \hat{L}_y] = 0$

$[\hat{L}_z, \hat{L}_+] = [\hat{L}_z, \hat{L}_x + i\hat{L}_y] = [\hat{L}_z, \hat{L}_x] + i[\hat{L}_z, \hat{L}_y] = i\hbar\hat{L}_y + i(-i\hbar\hat{L}_x) = i\hbar\hat{L}_y + \hbar\hat{L}_x = \hbar(\hat{L}_x + i\hat{L}_y) = \hbar\hat{L}_+$
$[\hat{L}_z, \hat{L}_-] = [\hat{L}_z, \hat{L}_x - i\hat{L}_y] = [\hat{L}_z, \hat{L}_x] - i[\hat{L}_z, \hat{L}_y] = i\hbar\hat{L}_y - i(-i\hbar\hat{L}_x) = i\hbar\hat{L}_y - \hbar\hat{L}_x = -\hbar(\hat{L}_x - i\hat{L}_y) = \hbar\hat{L}_-$

$[\hat{L}_+, \hat{L}_-] = [\hat{L}_x + i\hat{L}_y, \hat{L}_x - i\hat{L}_y] = [\hat{L}_x, \hat{L}_x] - [\hat{L}_x, i\hat{L}_y] + [i\hat{L}_y, \hat{L}_x] - [i\hat{L}_y, i\hat{L}_y]$
$= -i[\hat{L}_x, \hat{L}_y] + i[\hat{L}_y, \hat{L}_x] = -i(i\hbar\hat{L}_z) + i(-i\hbar\hat{L}_z) = 2\hbar\hat{L}_z$

$\hat{L}_+\hat{L}_- = (\hat{L}_x + i\hat{L}_y)(\hat{L}_x - i\hat{L}_y) = \hat{L}_x{}^2 + i\hat{L}_y\hat{L}_x - i\hat{L}_x\hat{L}_y + \hat{L}_y{}^2 = \left(\hat{L}_x{}^2 + \hat{L}_y{}^2\right) - i(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x)$
$= \left(\hat{L}^2 - \hat{L}_z{}^2\right) - i[\hat{L}_x, \hat{L}_y] = \hat{L}^2 - \hat{L}_z{}^2 - i(i\hbar\hat{L}_z) = \hat{L}^2 - \hat{L}_z{}^2 + \hbar\hat{L}_z$

$\hat{L}_-\hat{L}_+ = (\hat{L}_x - i\hat{L}_y)(\hat{L}_x + i\hat{L}_y) = \hat{L}_x{}^2 - i\hat{L}_y\hat{L}_x + i\hat{L}_x\hat{L}_y + \hat{L}_y{}^2 = \left(\hat{L}_x{}^2 + \hat{L}_y{}^2\right) + i(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x)$
$= \left(\hat{L}^2 - \hat{L}_z{}^2\right) + i[\hat{L}_x, \hat{L}_y] = \hat{L}^2 - \hat{L}_z{}^2 + i(i\hbar\hat{L}_z) = \hat{L}^2 - \hat{L}_z{}^2 - \hbar\hat{L}_z$

$[\hat{L}_x, \hat{L}_+] = [\hat{L}_x, \hat{L}_x + i\hat{L}_y] = [\hat{L}_x, \hat{L}_x] + i[\hat{L}_x, \hat{L}_y] = i(i\hbar\hat{L}_z) = -\hbar\hat{L}_z$
$[\hat{L}_x, \hat{L}_-] = [\hat{L}_x, \hat{L}_x - i\hat{L}_y] = [\hat{L}_x, \hat{L}_x] - i[\hat{L}_x, \hat{L}_y] = -i(i\hbar\hat{L}_z) = \hbar\hat{L}_z$
$[\hat{L}_y, \hat{L}_+] = [\hat{L}_y, \hat{L}_x + i\hat{L}_y] = [\hat{L}_y, \hat{L}_x] + i[\hat{L}_y, \hat{L}_y] = -i\hbar\hat{L}_z$
$[\hat{L}_y, \hat{L}_-] = [\hat{L}_y, \hat{L}_x - i\hat{L}_y] = [\hat{L}_y, \hat{L}_x] - i[\hat{L}_y, \hat{L}_y] = -i\hbar\hat{L}_z$

General Uncertainty Relation

The uncertainty ΔA in a dynamical variable A is defined as the root mean square deviation from the mean $\langle A\rangle$.
$(\Delta\boldsymbol{A})^{\mathbf{2}} = \langle(\boldsymbol{A} - \langle\boldsymbol{A}\rangle)^2\rangle = \langle A^2 + \langle A\rangle^2 - 2A\langle A\rangle\rangle = \langle A^2\rangle + \langle A\rangle^2 - 2\langle A\rangle\langle A\rangle = \langle A^2\rangle + \langle A\rangle^2 - 2\langle A\rangle^2 = \langle A^2\rangle - \langle A\rangle^2$
Consider two Hermitian operators, A and B. Let their commutator be $[A, B] = iC$. Let $R = A + imB$ where m is an arbitrary real number. The inner product of $R\psi$ with itself must be greater than or equal to zero.

$(R\psi, R\psi) = \langle R\psi|R\psi\rangle = \int (R\psi)^*(R\psi)\, d\tau \geq 0 \Longrightarrow \int (A + imB)^*\psi^*(A + imB)\psi\, d\tau \geq 0$

$\Longrightarrow \int (A^* - imB^*)\psi^*(A + imB)\psi\, d\tau \geq 0 \Longrightarrow \int A^*\psi^*(A + imB)\psi\, d\tau - im\int B^*\psi^*(A + imB)\psi\, d\tau \geq 0$

$\Longrightarrow \int \psi^*A(A + imB)\psi\, d\tau - im\int \psi^*B(A + imB)\psi\, d\tau \geq 0$ [$\because$ A and B are Hermitian.]

$\Longrightarrow \int \psi^*(A - imB)(A + imB)\psi\, d\tau - im\int \psi^*(A + imB)\psi\, d\tau \geq 0$

$\Longrightarrow \int \psi^*(A^2 + m^2B^2 - imBA + imAB)\psi\, d\tau \geq 0 \Longrightarrow \int \psi^*\left(A^2 - m^2B^2 + im(AB - BA)\right)\psi\, d\tau \geq 0$

$\Longrightarrow \int \psi^*\left(A^2 + m^2B^2 + im(iC)\right)\psi\, d\tau \geq 0$ [$\because (AB - BA) = [A, B] = iC$]

$\Longrightarrow \int \psi^*\left(A^2 + m^2B^2 + im(iC)\right)\psi\, d\tau \geq 0 \Longrightarrow \int \psi^*(A^2 - m^2B^2 - mC)\psi\, d\tau \geq 0$

$$\Rightarrow \int \psi^* A^2 \psi\, d\tau + m^2 \int \psi^* B^2 \psi\, d\tau - m \int \psi^* C \psi\, d\tau \geq 0 \Rightarrow \langle A^2 \rangle + m^2 \langle B^2 \rangle - m\langle C \rangle \geq 0$$

This inequality must hold regardless of the size of m. To find the value of m for which LHS is minimum,

$$\frac{d}{dm}[\langle A^2 \rangle + m^2\langle B^2 \rangle - m\langle C \rangle] = 0 \Rightarrow 2m\langle B^2 \rangle - \langle C \rangle = 0 \Rightarrow m = \frac{\langle C \rangle}{2\langle B^2 \rangle}$$

$$\therefore \langle A^2 \rangle + m^2\langle B^2 \rangle - m\langle C \rangle \geq 0 \Rightarrow \langle A^2 \rangle + \left(\frac{\langle C \rangle}{2\langle B^2 \rangle}\right)^2 \langle B^2 \rangle - \left(\frac{\langle C \rangle}{2\langle B^2 \rangle}\right)\langle C \rangle \geq 0 \Rightarrow \langle A^2 \rangle + \frac{\langle C \rangle^2}{4\langle B^2 \rangle} - \frac{\langle C \rangle^2}{2\langle B^2 \rangle} \geq 0$$

$$\Rightarrow \langle A^2 \rangle - \frac{\langle C \rangle^2}{4\langle B^2 \rangle} \geq 0 \Rightarrow \langle A^2 \rangle\langle B^2 \rangle \geq \frac{\langle C \rangle^2}{4}$$

This inequality is valid for any two Hermitian operators obeying $[A, B] = iC$. So it must also hold for $A - \langle A \rangle$ and $B - \langle B \rangle$ since $[A - \langle A \rangle, B - \langle B \rangle] = (A - \langle A \rangle)(B - \langle B \rangle) - (B - \langle B \rangle)(A - \langle A \rangle) =$

$= (AB + \langle A \rangle\langle B \rangle - A\langle B \rangle - \langle A \rangle B) - (BA + \langle B \rangle\langle A \rangle - \langle B \rangle A - B\langle A \rangle)$

$= (AB + \langle A \rangle\langle B \rangle - A\langle B \rangle - B\langle A \rangle) - (BA + \langle A \rangle\langle B \rangle - A\langle B \rangle - B\langle A \rangle) = AB - BA = [A, B] = iC$

$$\therefore \langle (A - \langle A \rangle)^2 \rangle \langle (B - \langle B \rangle)^2 \rangle \geq \frac{\langle C \rangle^2}{4} \Rightarrow (\Delta A)^2(\Delta B)^2 \geq \frac{\langle C \rangle^2}{4} \Rightarrow (\Delta \boldsymbol{A})(\Delta \boldsymbol{B}) \geq \frac{\boldsymbol{\langle C \rangle}}{\boldsymbol{2}}$$

If $A = x$ and $B = p_x$ then $C = \hbar$ $[\because [x, p_x] = iC]$ $\therefore (\Delta \boldsymbol{x})(\Delta \boldsymbol{p_x}) \geq \frac{\hbar}{\boldsymbol{2}}$ $[\because \langle \hbar \rangle = \hbar]$

$\because J^2$ and J_z commute, they can have simultaneous eigenkets, say, $|\lambda m\rangle$ **[$|\boldsymbol{\lambda m}\rangle$ is another way of writing $|\boldsymbol{\lambda}, \boldsymbol{m}\rangle$]**

$$\therefore \begin{cases} J^2|\lambda m\rangle = \lambda|\lambda m\rangle \\ J_z|\lambda m\rangle = m|\lambda m\rangle \end{cases} \Rightarrow ({J_x}^2 + {J_y}^2)|\lambda m\rangle + m^2|\lambda m\rangle = \lambda|\lambda m\rangle \Rightarrow ({J_x}^2 + {J_y}^2)|\lambda m\rangle = (\lambda - m^2)|\lambda m\rangle$$

$\Rightarrow \langle \lambda m|({J_x}^2 + {J_y}^2)|\lambda m\rangle = \langle \lambda m|(\lambda - m^2)|\lambda m\rangle \Rightarrow \langle \lambda m|{J_x}^2|\lambda m\rangle + \langle \lambda m|{J_y}^2|\lambda m\rangle = (\lambda - m^2)$ $[\because \langle \lambda m|\lambda m\rangle = 1]$

$\Rightarrow \lambda - m^2 \geq 0$ [$\because$ eigenvalues of J_x and J_x are real as they are Hermitian] $\Rightarrow \lambda \geq m^2$

$J^2|\lambda m\rangle = \lambda|\lambda m\rangle \Rightarrow J_+ J^2|\lambda m\rangle = \lambda J_+|\lambda m\rangle \Rightarrow J^2 J_+ |\lambda m\rangle = \lambda J_+|\lambda m\rangle$

$[\because [J^2, J_+]=0 \Rightarrow J^2 J_+ - J_+ J^2 = 0 \Rightarrow J^2 J_+ = J_+ J^2]$

$\therefore$ $|\lambda m\rangle$ and $J_+ |\lambda m\rangle$ are eigenkets of J^2 with the same eigenvalue λ

$J_z|\lambda m\rangle = m|\lambda m\rangle \Rightarrow J_+ J_z|\lambda m\rangle = mJ_+|\lambda m\rangle \Rightarrow (J_z J_+ - \hbar J_+)|\lambda m\rangle = mJ_+|\lambda m\rangle \Rightarrow J_z J_+|\lambda m\rangle = (m + \hbar)J_+|\lambda m\rangle$

$[\because [J_z, J_+] = \hbar J_+ \Rightarrow J_z J_+ - J_+ J_z = \hbar J_+ \Rightarrow J_+ J_z = J_z J_+ - \hbar J_+]$

$\therefore$ $J_+ |\lambda m\rangle$ is an eigenket of J_z with the eigenvalue $(m + \hbar)$

$J^2|\lambda m\rangle = \lambda|\lambda m\rangle \Rightarrow J_- J^2|\lambda m\rangle = \lambda J_-|\lambda m\rangle \Rightarrow J^2 J_- |\lambda m\rangle = \lambda J_-|\lambda m\rangle$

$[\because [J^2, J_-]=0 \Rightarrow J^2 J_- - J_+ J^2 = 0 \Rightarrow J^2 J_- = J_- J^2]$

$\therefore$ $|\lambda m\rangle$ and $J_- |\lambda m\rangle$ are eigenkets of J^2 with the same eigenvalue λ

$J_z|\lambda m\rangle = m|\lambda m\rangle \Rightarrow J_- J_z|\lambda m\rangle = mJ_+|\lambda m\rangle \Rightarrow (J_z J_- + \hbar J_-)|\lambda m\rangle = mJ_-|\lambda m\rangle \Rightarrow J_z J_-|\lambda m\rangle = (m - \hbar)J_-|\lambda m\rangle$

$[\because [J_z, J_-] = -\hbar J_- \Rightarrow J_z J_- - J_- J_z = -\hbar J_- \Rightarrow J_- J_z = J_z J_- + \hbar J_-]$

$\therefore$ $J_- |\lambda m\rangle$ is an eigenket of J_z with the eigenvalue $(m - \hbar)$

$\because$ operation by J_+ generates a state with the same angular momentum but with a z-component higher by $\hbar$, it is called the **raising operator.** Repeated operation by J_+ goes on raising J_z in steps of $\hbar$ so that at some point, $\lambda \geq m^2$ is violated. This requires a maximum allowed value of m, say μ, for a given λ to prevent the violation.

$\therefore J_z|\lambda\mu\rangle = \mu|\lambda\mu\rangle \Rightarrow J_+ J_z|\lambda\mu\rangle = \mu J_+|\lambda\mu\rangle \Rightarrow J_z J_+|\lambda\mu\rangle = (\mu + \hbar)J_+|\lambda\mu\rangle \Rightarrow J_+|\lambda\mu\rangle = 0$ $[\because (\mu + \hbar)$ is not possible$]$

$\Rightarrow J_- J_+|\lambda\mu\rangle = 0 \Rightarrow (J^2 - {J_z}^2 - \hbar J_z)|\lambda\mu\rangle = 0 \Rightarrow (\lambda - \mu^2 - \mu\hbar)|\lambda\mu\rangle = 0$ $[\because J_- J_+ = J^2 - {J_z}^2 - \hbar J_z]$

$\Rightarrow (\lambda - \mu^2 - \mu\hbar) = 0 \Rightarrow \lambda = \mu(\mu + \hbar)$

$J_z|\lambda\mu\rangle = \mu|\lambda\mu\rangle \Rightarrow J_- J_z|\lambda\mu\rangle = \mu J_-|\lambda\mu\rangle \Rightarrow J_z J_-|\lambda\mu\rangle = (\mu - \hbar)J_-|\lambda\mu\rangle \Rightarrow J_z {J_-}^n|\lambda\mu\rangle = (\mu - n\hbar)J_-|\lambda\mu\rangle$

$J_z|jm\rangle = m\hbar|jm\rangle \Rightarrow \langle j'm'|J_z|jm\rangle = m\hbar\langle j'm'|jm\rangle \Rightarrow \langle j'm'|J_z|jm\rangle = m\hbar\delta_{jj'}\delta_{mm'}$

$J^2|jm\rangle = j(j+1)\hbar^2|jm\rangle \Rightarrow \langle j'm'|J^2|jm\rangle = j(j+1)\hbar^2\langle j'm'|jm\rangle \Rightarrow \langle j'm'|J^2|jm\rangle = j(j+1)\hbar^2\delta_{jj'}\delta_{mm'}$

$$
\begin{array}{ccc|c|cc|ccc|cccc|c}
 & & j & 0 & \multicolumn{2}{c|}{\overbrace{\qquad\qquad}^{1/2}} & \multicolumn{3}{c|}{\overbrace{\qquad\qquad}^{1}} & \multicolumn{4}{c|}{\overbrace{\qquad\qquad\qquad}^{3/2}} & \cdots \\
j' & m' & m & 0 & 1/2 & -1/2 & 1 & 0 & -1 & 3/2 & 1/2 & -1/2 & 3/2 & \cdots \\
\hline
0 & 0 & & 0 & \multicolumn{2}{c|}{(0)} & \multicolumn{3}{c|}{(0)} & \multicolumn{4}{c|}{(0)} & \cdots \\
1/2 & \begin{cases} 1/2 \\ -1/2 \end{cases} & & (0) & \multicolumn{2}{c|}{\begin{bmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{bmatrix}} & \multicolumn{3}{c|}{(0)} & \multicolumn{4}{c|}{(0)} & \cdots \\
1 & \begin{cases} 1 \\ 0 \\ -1 \end{cases} & & (0) & \multicolumn{2}{c|}{(0)} & \multicolumn{3}{c|}{\begin{bmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{bmatrix}} & \multicolumn{4}{c|}{(0)} & \cdots \\
3/2 & \begin{cases} 3/2 \\ 1/2 \\ -1/2 \\ -3/2 \end{cases} & & (0) & \multicolumn{2}{c|}{(0)} & \multicolumn{3}{c|}{(0)} & \multicolumn{4}{c|}{\begin{bmatrix} 3\hbar/2 & 0 & 0 & 0 \\ 0 & \hbar/2 & 0 & 0 \\ 0 & 0 & -\hbar/2 & 0 \\ 0 & 0 & 0 & -3\hbar/2 \end{bmatrix}} & \cdots \\
 & \vdots & & \vdots & \multicolumn{2}{c|}{\vdots} & \multicolumn{3}{c|}{\vdots} & \multicolumn{4}{c|}{\vdots} & \ddots
\end{array}
$$

$\boldsymbol{J_z}$:

$$
\begin{array}{ccc|c|cc|ccc|cccc|c}
 & & j & 0 & \multicolumn{2}{c|}{\overbrace{\qquad\qquad}^{1/2}} & \multicolumn{3}{c|}{\overbrace{\qquad\qquad}^{1}} & \multicolumn{4}{c|}{\overbrace{\qquad\qquad\qquad}^{3/2}} & \cdots \\
j' & m' & m & 0 & 1/2 & -1/2 & 1 & 0 & -1 & 3/2 & 1/2 & -1/2 & 3/2 & \cdots \\
\hline
0 & 0 & & 0 & \multicolumn{2}{c|}{(0)} & \multicolumn{3}{c|}{(0)} & \multicolumn{4}{c|}{(0)} & \cdots \\
1/2 & \begin{cases} 1/2 \\ -1/2 \end{cases} & & (0) & \multicolumn{2}{c|}{\begin{bmatrix} 3\hbar^2/4 & 0 \\ 0 & 3\hbar^2/4 \end{bmatrix}} & \multicolumn{3}{c|}{(0)} & \multicolumn{4}{c|}{(0)} & \cdots \\
1 & \begin{cases} 1 \\ 0 \\ -1 \end{cases} & & (0) & \multicolumn{2}{c|}{(0)} & \multicolumn{3}{c|}{\begin{bmatrix} 2\hbar^2 & 0 & 0 \\ 0 & 2\hbar^2 & 0 \\ 0 & 0 & 2\hbar^2 \end{bmatrix}} & \multicolumn{4}{c|}{(0)} & \cdots \\
3/2 & \begin{cases} 3/2 \\ 1/2 \\ -1/2 \\ -3/2 \end{cases} & & (0) & \multicolumn{2}{c|}{(0)} & \multicolumn{3}{c|}{(0)} & \multicolumn{4}{c|}{\begin{bmatrix} 15\hbar^2/4 & 0 & 0 & 0 \\ 0 & 15\hbar^2/4 & 0 & 0 \\ 0 & 0 & 15\hbar^2/4 & 0 \\ 0 & 0 & 0 & 15\hbar^2/4 \end{bmatrix}} & \cdots \\
 & \vdots & & \vdots & \multicolumn{2}{c|}{\vdots} & \multicolumn{3}{c|}{\vdots} & \multicolumn{4}{c|}{\vdots} & \ddots
\end{array}
$$

$\boldsymbol{J^2}$:

$$J_z|jm\rangle = m\hbar|jm\rangle \Longrightarrow \begin{cases} J_+J_z|jm\rangle = m\hbar J_+|jm\rangle \Longrightarrow (J_zJ_+ - \hbar J_+)|jm\rangle = m\hbar J_+|jm\rangle \Longrightarrow J_zJ_+|jm\rangle = (m+1)\hbar J_+|jm\rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad J_z|j,m+1\rangle = (m+1)\hbar|j,m+1\rangle \end{cases}$$

$$\Longrightarrow J_+|jm\rangle = a_m|j,m+1\rangle \Longrightarrow \langle j,m+1|J_+|jm\rangle = a_m\langle j,m+1||j,m+1|\rangle \Longrightarrow a_m = \langle j,m+1|J_+|jm\rangle$$

$$J_z|jm\rangle = m\hbar|jm\rangle \Longrightarrow \begin{cases} J_-J_z|jm\rangle = m\hbar J_-|jm\rangle \Longrightarrow (J_zJ_- + \hbar J_-)|jm\rangle = m\hbar J_-|jm\rangle \Longrightarrow J_zJ_+|jm\rangle = (m-1)\hbar J_+|jm\rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad J_z|j,m-1\rangle = (m-1)\hbar|j,m-1\rangle \end{cases}$$

$$\Longrightarrow J_-|jm\rangle = b_m|j,m-1\rangle \Longrightarrow \langle j,m-1|J_-|jm\rangle = a_m\langle j,m-1||j,m-1|\rangle \Longrightarrow b_m = \langle j,m-1|J_-|jm\rangle$$

$$\begin{cases} a_m = \langle j,m+1|J_+|jm\rangle \Longrightarrow {a_m}^* = \langle jm|J_-|j,m+1\rangle \\ b_m = \langle j,m-1|J_-|jm\rangle \Longrightarrow b_{m+1} = \langle jm|J_-|j,m+1\rangle \end{cases} \Longrightarrow {a_m}^* = b_{m+1}$$

$$\begin{cases} a_m = \langle j,m+1|J_+|jm\rangle \Longrightarrow a_{m-1} = \langle jm|J_+|j,m-1\rangle \\ b_m = \langle j,m-1|J_-|jm\rangle \Longrightarrow {b_m}^* = \langle jm|J_+|j,m-1\rangle \end{cases} \Longrightarrow {b_m}^* = a_{m-1}$$

$$J_+|jm\rangle = a_m|j,m+1\rangle \Longrightarrow J_-J_+|jm\rangle = a_mJ_-|j,m+1\rangle$$

$$\Longrightarrow \left(J^2 - {J_z}^2 - \hbar J_z\right)|jm\rangle = a_mb_{m+1}|jm\rangle \quad \begin{bmatrix} \because J_-J_+ = J^2 - {J_z}^2 - \hbar J_z \\ \text{and } J_-|jm\rangle = b_m|j,m-1\rangle \end{bmatrix}$$

$$\Longrightarrow \{j(j+1) - m^2 - m\}\hbar^2|jm\rangle = {a_m}^2|jm\rangle \quad \begin{bmatrix} \because J^2|jm\rangle = j(j+1)\hbar^2|jm\rangle\,, \\ J_z|jm\rangle = m\hbar|jm\rangle \text{ and } {a_m}^* = b_{m+1} \end{bmatrix}$$

$$\Longrightarrow a_m = \sqrt{j(j+1) - m(m+1)}\;\hbar = \sqrt{(j^2 - m^2) + (j-m)}\;\hbar = \sqrt{(j-m)(j+m+1)}\;\hbar$$

$$\therefore J_+|jm\rangle = a_m|j,m+1\rangle \Longrightarrow J_+|jm\rangle = \sqrt{j(j+1) - m(m+1)}\;\hbar\;|j,m+1\rangle$$

$$\Longrightarrow \langle j'm'|J_+|jm\rangle = \sqrt{j(j+1) - m(m+1)}\;\hbar\;\langle j'm'|j,m+1\rangle$$

$$\Longrightarrow \langle j'm'|J_+|jm\rangle = \sqrt{j(j+1) - m(m+1)}\;\hbar\;\delta_{jj'}\delta_{m+1,m'} = \sqrt{(j-m)(j+m+1)}\;\hbar\;\delta_{jj'}\delta_{m+1,m'}$$

$$J_-|jm\rangle = b_m|j,m-1\rangle \Longrightarrow J_+J_-|jm\rangle = b_mJ_+|j,m-1\rangle$$

$$\Longrightarrow \left(J^2 - {J_z}^2 + \hbar J_z\right)|jm\rangle = b_m a_{m-1}|jm\rangle \quad \begin{bmatrix} \because J_+J_- = J^2 - {J_z}^2 + \hbar J_z \\ \text{and } J_+|jm\rangle = a_m|j, m+1\rangle \end{bmatrix}$$

$$\Longrightarrow \{j(j+1) - m^2 + m\}\hbar^2|jm\rangle = {b_m}^2|jm\rangle \quad \begin{bmatrix} \because J^2|jm\rangle = j(j+1)\hbar^2|jm\rangle\,, \\ J_z|jm\rangle = m\hbar|jm\rangle \text{ and } {b_m}^* = a_{m-1} \end{bmatrix}$$

$$\Longrightarrow b_m = \sqrt{j(j+1) - m(m-1)}\,\hbar = \sqrt{(j^2 - m^2) + (j+m)}\,\hbar = \sqrt{(j+m)(j-m+1)}\,\hbar$$

$$\therefore J_-|jm\rangle = b_m|j, m-1\rangle \Longrightarrow J_-|jm\rangle = \sqrt{j(j+1) - m(m-1)}\,\hbar\,|j, m-1\rangle$$

$$\Longrightarrow \langle j'm'|J_-|jm\rangle = \sqrt{j(j+1) - m(m-1)}\,\hbar\,\langle j'm'|j, m-1\rangle$$

$$\Longrightarrow \langle j'm'|J_-|jm\rangle = \sqrt{j(j+1) - m(m-1)}\,\hbar\,\delta_{jj'}\delta_{m-1,m'} = \sqrt{(j+m)(j-m+1)}\,\hbar\,\delta_{jj'}\delta_{m-1,m'}$$

$$\boldsymbol{J}_+ : \quad \begin{array}{ccc|ccccc}
 & & j & 0 & 1/2 & 1 & 3/2 & \cdots \\
j' & m' & m & 0 & \overbrace{1/2 \quad -1/2} & \overbrace{1 \quad 0 \quad -1} & \overbrace{3/2 \quad 1/2 \quad -1/2 \quad 3/2} & \cdots \\
\hline
0 & 0 & & 0 & (0) & (0) & (0) & \cdots \\
1/2 & \begin{cases} 1/2 \\ -1/2 \end{cases} & & (0) & \begin{bmatrix} 0 & \hbar \\ 0 & 0 \end{bmatrix} & (0) & (0) & \cdots \\
1 & \begin{cases} 1 \\ 0 \\ -1 \end{cases} & & (0) & (0) & \begin{bmatrix} 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & \sqrt{2}\hbar \\ 0 & 0 & 0 \end{bmatrix} & (0) & \cdots \\
3/2 & \begin{cases} 3/2 \\ 1/2 \\ -1/2 \\ -3/2 \end{cases} & & (0) & (0) & (0) & \begin{bmatrix} 0 & \sqrt{3}\hbar & 0 & 0 \\ 0 & 0 & \sqrt{3}\hbar & 0 \\ 0 & 0 & 0 & \sqrt{3}\hbar \\ 0 & 0 & 0 & 0 \end{bmatrix} & \cdots \\
 & \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

$$\boldsymbol{J}_- : \quad \begin{array}{ccc|ccccc}
 & & j & 0 & 1/2 & 1 & 3/2 & \cdots \\
j' & m' & m & 0 & \overbrace{1/2 \quad -1/2} & \overbrace{1 \quad 0 \quad -1} & \overbrace{3/2 \quad 1/2 \quad -1/2 \quad 3/2} & \cdots \\
\hline
0 & 0 & & 0 & (0) & (0) & (0) & \cdots \\
1/2 & \begin{cases} 1/2 \\ -1/2 \end{cases} & & (0) & \begin{bmatrix} 0 & 0 \\ \hbar & 0 \end{bmatrix} & (0) & (0) & \cdots \\
1 & \begin{cases} 1 \\ 0 \\ -1 \end{cases} & & (0) & (0) & \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2}\hbar & 0 & 0 \\ 0 & \sqrt{2}\hbar & 0 \end{bmatrix} & (0) & \cdots \\
3/2 & \begin{cases} 3/2 \\ 1/2 \\ -1/2 \\ -3/2 \end{cases} & & (0) & (0) & (0) & \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3}\hbar & 0 & 0 & 0 \\ 0 & \sqrt{3}\hbar & 0 & 0 \\ 0 & 0 & \sqrt{3}\hbar & 0 \end{bmatrix} & \cdots \\
 & \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

$$\begin{cases} J_+ = J_x + iJ_y \\ J_- = J_x - iJ_y \end{cases} \Longrightarrow \begin{cases} J_x = \dfrac{1}{2}(J_+ + J_-) \\ J_y = \dfrac{1}{2i}(J_+ - J_-) \end{cases}$$

$$\boldsymbol{J_x}: \begin{array}{ccc|ccccc}
 & & j & 0 & 1/2 & 1 & 3/2 & \cdots \\
j' & m' & m & 0 & 1/2 \quad -1/2 & 1 \quad 0 \quad -1 & 3/2 \quad 1/2 \quad -1/2 \quad 3/2 & \cdots \\ \hline
0 & 0 & & (0) & (0) & (0) & (0) & \cdots \\
1/2 & \begin{cases} 1/2 \\ -1/2 \end{cases} & & (0) & \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & (0) & (0) & \cdots \\
1 & \begin{cases} 1 \\ 0 \\ -1 \end{cases} & & (0) & (0) & \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & (0) & \cdots \\
3/2 & \begin{cases} 3/2 \\ 1/2 \\ -1/2 \\ -3/2 \end{cases} & & (0) & (0) & (0) & \frac{\sqrt{3}\hbar}{2}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} & \cdots \\
 & \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

$$\boldsymbol{J_y}: \begin{array}{ccc|ccccc}
 & & j & 0 & 1/2 & 1 & 3/2 & \cdots \\
j' & m' & m & 0 & 1/2 \quad -1/2 & 1 \quad 0 \quad -1 & 3/2 \quad 1/2 \quad -1/2 \quad 3/2 & \cdots \\ \hline
0 & 0 & & (0) & (0) & (0) & (0) & \cdots \\
1/2 & \begin{cases} 1/2 \\ -1/2 \end{cases} & & (0) & \frac{i\hbar}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & (0) & (0) & \cdots \\
1 & \begin{cases} 1 \\ 0 \\ -1 \end{cases} & & (0) & (0) & \frac{i\hbar}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} & (0) & \cdots \\
3/2 & \begin{cases} 3/2 \\ 1/2 \\ -1/2 \\ -3/2 \end{cases} & & (0) & (0) & (0) & \frac{\sqrt{3}i\hbar}{2}\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} & \cdots \\
 & \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Consider two non-interacting systems having angular momenta $\vec{J}_1$ & $\vec{J}_2$ and eigenkets $|j_1m_1\rangle$ & $|j_2m_2\rangle$ respectively.

$$\begin{cases} J_1{}^2|j_1m_1\rangle = j_1(j_1+1)\hbar^2|j_1m_1\rangle \\ J_{1z}|j_1m_1\rangle = m_1\hbar|j_1m_1\rangle \end{cases} \quad \text{and} \quad \begin{cases} J_2{}^2|j_2m_2\rangle = j_2(j_2+1)\hbar^2|j_2m_2\rangle \\ J_{2z}|j_2m_2\rangle = m_2\hbar|j_2m_2\rangle \end{cases} \quad \begin{bmatrix} m_1 = j_1, j_1-1, j_1-2, \dots, -j_1 \\ m_2 = j_2, j_2-1, j_2-2, \dots, -j_2 \end{bmatrix}$$

$\because$ The two systems are non-interacting, $\therefore [J_1, J_2] = 0 \Longrightarrow \begin{cases} [J_1{}^2, J_2{}^2] = 0 \\ [J_{1z}, J_{2z}] = 0 \end{cases}$

Also, $[J^2, J_z] = 0$ [previously shown] $\Longrightarrow \begin{cases} [J_1{}^2, J_{1z}] = 0 \\ [J_2{}^2, J_{2z}] = 0 \end{cases}$

Hence, the operators $J_1{}^2, J_{1z}, J_2{}^2, J_{2z}$ form a complete set of simultaneous ($known$) eigenkets $|j_1m_1, j_2m_2\rangle$

$|j_1m_1, j_2m_2\rangle = |j_1m_1\rangle|j_2m_2\rangle = |j_1j_2m_1m_2\rangle = |m_1m_2\rangle$ [shorthand form]

$\because m_1$ and m_2 can have respectively $(2j_1+1)$ and $(2j_2+1)$ values, $\therefore$ for a specified j_1 and j_2, the subspace $|m_1m_2\rangle$ has $(2j_1+1)(2j_2+1)$ dimensions.

$\vec{J} = \vec{J}_1 + \vec{J}_2 \Longrightarrow \begin{cases} J^2 = J_1{}^2 + J_2{}^2 + 2J_1J_2 \\ J_z = J_{1z} + J_{2z} \end{cases}$

$[J^2, J_1{}^2] = [(J_1{}^2 + J_2{}^2 + 2J_1J_2), J_1{}^2] = [J_1{}^2, J_1{}^2] + [J_2{}^2, J_1{}^2] + 2[J_1J_2, J_1{}^2] = 2[J_1J_2, J_1{}^2] = 2\{J_1[J_2, J_1{}^2] - [J_1, J_1{}^2]J_2\}$
$= 2J_1[J_2, J_1{}^2] = 2J_1\{J_1[J_2, J_1] - [J_2, J_1]J_1\} = 0 \quad \because [J_2, J_1] = 0$ Likewise, $[J^2, J_2{}^2] = 0$

$\because [J^2, J_1{}^2] = 0$, $[J^2, J_2{}^2] = 0$ and $[J^2, J_z] = 0$

Hence, the operators $J_1{}^2, J_2{}^2, J^2, J_z$ form a complete set of simultaneous (unknown) eigenkets $|j_1j_2jm\rangle$

$|j_1j_2jm\rangle = |jm\rangle$ [shorthand form]

The unknown eigenkets $|jm\rangle$ may be expressed as a linear combination of known eigenkets $|m_1m_2\rangle$

$$|jm\rangle = \sum_{m_1,m_2} C_{jmm_1m_2}|m_1m_2\rangle \quad \begin{bmatrix} \text{where } m_1 \text{ is from } -j_1 \text{ } to + j_1 \\ \text{and } m_1 \text{ is from } -j_2 \text{ } to + j_2 \end{bmatrix}$$

The coefficients of this linear combination $C_{jmm_1m_2}$ are called **Clebsch-Gordan (CG) coefficients, Wigner**

coefficients or **vector coupling coefficients.**

$$\langle m_1{}'m_2{}'|jm\rangle = \sum_{m_1,m_2} C_{jmm_1m_2}\langle m_1{}'m_2{}'|m_1m_2\rangle \Rightarrow \langle m_1{}'m_2{}'|jm\rangle = \sum_{m_1,m_2} C_{jmm_1m_2}\delta_{m_1m_1{}'}\delta_{m_2m_2{}'}$$

$$\Rightarrow \langle m_1{}'m_2{}'|jm\rangle = C_{jmm_1{}'m_2{}'} \Rightarrow C_{jmm_1m_2} = \langle m_1m_2|jm\rangle$$

$$\therefore |jm\rangle = \sum_{m_1,m_2}\langle m_1m_2|jm\rangle|m_1m_2\rangle = \sum_{m_1,m_2}|m_1m_2\rangle\langle m_1m_2|jm\rangle$$

[As the coefficients $\langle m_1m_2|jm\rangle$ relate two orthonormal bases, they form a unitary matrix whose rows are labelled m_1, m_2 and columns j, m]

$$\Rightarrow J_z|jm\rangle = \sum_{m_1,m_2} J_z|m_1m_2\rangle\langle m_1m_2|jm\rangle = \sum_{m_1,m_2}(J_{1z}+J_{2z})|m_1m_2\rangle\langle m_1m_2|jm\rangle \quad \text{[operating by } J_z\text{]}$$

$$\Rightarrow m\hbar|jm\rangle = \sum_{m_1,m_2}(m_1+m_2)\hbar|m_1m_2\rangle\langle m_1m_2|jm\rangle \Rightarrow m|jm\rangle = \sum_{m_1,m_2}(m_1+m_2)|m_1m_2\rangle\langle m_1m_2|jm\rangle$$

$$\Rightarrow m\sum_{m_1,m_2}|m_1m_2\rangle\langle m_1m_2|jm\rangle|jm\rangle = \sum_{m_1,m_2}(m_1+m_2)|m_1m_2\rangle\langle m_1m_2|jm\rangle$$

$$\Rightarrow \sum_{m_1,m_2}(m-m_1-m_2)|m_1m_2\rangle\langle m_1m_2|jm\rangle \Rightarrow m-m_1-m_2 = 0 \Rightarrow m = m_1+m_2$$

m_1	m_2	j	m	$\lvert j_1j_2m_1m_2\rangle$ (uncoupled representation)	$\lvert j_1j_2jm\rangle$ (coupled representation)
j_1	j_2	j_1+j_2	j_1+j_2	$\lvert j_1j_2, j_1j_2\rangle$	$\lvert j_1j_2, j_1+j_2, j_1+j_2\rangle$
j_1 j_1-1	j_2-1 j_2	j_1+j_2 j_1+j_2-1	j_1+j_2-1	$\lvert j_1j_2, j_1, j_2-1\rangle$ $\lvert j_1j_2, j_1-1, j_2\rangle$	$\lvert j_1j_2, j_1+j_2, j_1+j_2-1\rangle$ $\lvert j_1j_2, j_1+j_2-1, j_1+j_2-1\rangle$
j_1 j_1-1 j_1-2	j_2-2 j_2-1 j_2	j_1+j_2 j_1+j_2-1 j_1+j_2-2	j_1+j_2-2	$\lvert j_1j_2, j_1, j_2-2\rangle$ $\lvert j_1j_2, j_1-1, j_2-1\rangle$ $\lvert j_1j_2, j_1-2, j_2\rangle$	$\lvert j_1j_2, j_1+j_2, j_1+j_2-2\rangle$ $\lvert j_1j_2, j_1+j_2-1, j_1+j_2-2\rangle$ $\lvert j_1j_2, j_1+j_2-2, j_1+j_2-2\rangle$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j_1 j_1-1 j_1-2 $\cdots$ j_1-k	j_2-k j_2-k+1 j_2-k+2 $\cdots$ j_2	j_1+j_2 j_1+j_2-1 j_1+j_2-2 $\cdots$ j_1+j_2-k	j_1+j_2-k	$\lvert j_1j_2, j_1, j_2-k\rangle$ $\lvert j_1j_2, j_1-1, j_2-k+1\rangle$ $\lvert j_1j_2, j_1-2, j_2-k+2\rangle$ $\cdots$ $\lvert j_1j_2, j_1-k, j_2\rangle$	$\lvert j_1j_2, j_1+j_2, j_1+j_2-k\rangle$ $\lvert j_1j_2, j_1+j_2-1, j_1+j_2-k\rangle$ $\lvert j_1j_2, j_1+j_2-2, j_1+j_2-k\rangle$ $\cdots$ $\lvert j_1j_2, j_1+j_2-k, j_1+j_2-k\rangle$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Maximum value of m (for a given $j_1$and j_2), $m_{max} = (m_1+m_2)_{max} = m_{1_{max}} + m_{2_{max}} = j_1+j_2 = j_{max}$
Minimum value of m (for a given $j_1$and j_2), $m_{min} = (m_1+m_2)_{min} = (-j_1)+(-j_2) = -(j_1+j_2)$
Thus, the permitted values of m are $(j_1+j_2), (j_1+j_2-1), (j_1+j_2-2), \ldots, -(j_1+j_2)$
Minimum value of j occurs when $j_1-k = -j_1$ or $j_2-k = -j_2$ that is when $k = -2j_1$ or $-2j_2$
Minimum value of j is then $j_1+j_2-2j_1 = j_2-j_1$ or $j_1+j_2-2j_2 = j_1-j_2$ (whichever is positive)
Thus, the permitted values of j are $(j_1+j_2), (j_1+j_2-1), \ldots, |j_1-j_2|$ i.e. $|j_1-j_2| \le j \le (j_1+j_2)$ [triangular inequality]

Total number of states (assume $j_1 > j_2$) $= \sum_{j_1-j_2}^{j_1+j_2}(2j+1)$ [for every j value, there are $(2j+1)$ m values]

$= \{2(j_1+j_2)+1\} + 2(j_1+j_2-1)+1 + \{2(j_1+j_2-2)+1\} + \cdots + \{2(j_1-j_2)+1\}$ [$2j_2+1$ terms]
$= \{(2j_1+2j_2)+(2j_1+2j_2-2)+(2j_1+2j_2-4)+\cdots+(2j_1-2j_2)\} + (2j_2+1)$

$$= (2j_1)(2j_2+1) + \sum_{-j_2}^{+j_2} 2j + (2j_2+1) = (2j_1)(2j_2+1) + (2j_2+1) \quad \left[\because \sum_{-j_2}^{+j_2} 2j = 0\right] = (2j_1+1)(2j_2+1)$$

Recursion relations to calculate CG coefficients

$$|jm\rangle = \sum_{m_1,m_2} \langle m_1m_2|jm\rangle |m_1m_2\rangle \Longrightarrow J_-|jm\rangle = \sum_{m_1,m_2} (J_{1-} + J_{2-})|m_1m_2\rangle\langle m_1m_2|jm\rangle \quad \text{[operating by } J_-\text{]}$$

$$\Longrightarrow \sqrt{j(j+1) - m(m-1)}\,\hbar\,|j, m-1\rangle = \sum_{m_1,m_2} \sqrt{j_1(j_1+1) - m_1(m_1-1)}\,\hbar\,|m_1-1, m_2\rangle\langle m_1m_2|jm\rangle$$
$$+ \sum_{m_1,m_2} \sqrt{j_2(j_2+1) - m_2(m_2-1)}\,\hbar\,|m_1, m_2-1\rangle\langle m_1m_2|jm\rangle \quad \left[\because J_-|jm\rangle = \sqrt{j(j+1) - m(m-1)}\,\hbar\,|j, m-1\rangle\right]$$

$$\Longrightarrow \sqrt{j(j+1) - m(m-1)}\,\langle m_1{}'m_2{}'|j, m-1\rangle = \sum_{m_1,m_2} \sqrt{j_1(j_1+1) - m_1(m_1-1)}\,\langle m_1{}'m_2{}'|m_1-1, m_2\rangle\langle m_1m_2|jm\rangle$$
$$+ \sum_{m_1,m_2} \sqrt{j_2(j_2+1) - m_2(m_2-1)}\,\langle m_1{}'m_2{}'|m_1, m_2-1\rangle\langle m_1m_2|jm\rangle \quad \text{[multiplying by } \langle m_1{}'m_2{}'|\text{]}$$

$$\Longrightarrow \sqrt{j(j+1) - m(m-1)}\,\langle m_1{}'m_2{}'|j, m-1\rangle = \sum_{m_1,m_2} \sqrt{j_1(j_1+1) - m_1(m_1-1)}\,\delta_{m_1{}'m_1-1}\delta_{m_2{}'m_2}\langle m_1m_2|jm\rangle$$
$$+ \sum_{m_1,m_2} \sqrt{j_2(j_2+1) - m_2(m_2-1)}\,\delta_{m_1{}'m_1}\delta_{m_2{}'m_2-1}\langle m_1m_2|jm\rangle$$

$$\Longrightarrow \sqrt{j(j+1) - m(m-1)}\,\langle m_1{}'m_2{}'|j, m-1\rangle = \sqrt{j_1(j_1+1) - m_1{}'(m_1{}'+1)}\,\langle m_1{}'+1, m_2{}'|jm\rangle$$
$$+ \sqrt{j_2(j_2+1) - m_2{}'(m_2{}'+1)}\,\langle m_1{}', m_2{}'+1|jm\rangle$$

$$\therefore \boldsymbol{\sqrt{j(j+1) - m(m-1)}\,\langle m_1m_2|j, m-1\rangle = \sqrt{j_1(j_1+1) - m_1(m_1+1)}\,\langle m_1+1, m_2|jm\rangle}$$
$$\boldsymbol{+ \sqrt{j_2(j_2+1) - m_2(m_2+1)}\,\langle m_1, m_2+1|jm\rangle}$$

In a similar fashion, $J_+|jm\rangle = \sum_{m_1,m_2} (J_{1+} + J_{2+})|m_1m_2\rangle\langle m_1m_2|jm\rangle$ leads to

$$\boldsymbol{\sqrt{j(j+1) - m(m-1)}\,\langle m_1m_2|j, m-1\rangle = \sqrt{j_1(j_1+1) - m_1(m_1+1)}\,\langle m_1+1, m_2|jm\rangle}$$
$$\boldsymbol{+ \sqrt{j_2(j_2+1) - m_2(m_2+1)}\,\langle m_1, m_2+1|jm\rangle}$$

When $m = m_{max} = j_1 + j_2$ and $j = j_{max} = j_1 + j_2$, then $m_1 = j_1$ and $m_2 = j_2$

$$|jm\rangle = \sum_{m_1,m_2} C_{jmm_1m_2}|m_1m_2\rangle = \sum_{m_1,m_2} \langle m_1m_2|jm\rangle|m_1m_2\rangle \Longrightarrow |j_1+j_2, j_1+j_2\rangle = \langle j_1j_2|j_1+j_2, j_1+j_2\rangle|j_1, j_2\rangle$$

taking $|j_1+j_2, j_1+j_2\rangle$ to be normalized, the normalization condition would be $\sum_{m_1,m_2} C_{jmm_1m_2}{}^2 = 1$

$$\Longrightarrow \langle j_1j_2|j_1+j_2, j_1+j_2\rangle^2 = 1 \Longrightarrow \boldsymbol{\langle j_1j_2|j_1+j_2, j_1+j_2\rangle = 1}$$

Clebsch-Gordan (CG) coefficient matrix $\langle m_1m_2|jm\rangle$ has $(2j_1+1)(2j_2+1)$ rows and $(2j_1+1)(2j_2+1)$ columns. $\left[\because \sum_{j_1-j_2}^{j_1+j_2} (2j+1) = (2j_1+1)(2j_2+1)\right]$. If $(2j_1+1)(2j_2+1)$ is even, then there are $\dfrac{(2j_1+1)(2j_2+1)+2}{2}$ submatrices. If $(2j_1+1)(2j_2+1)$ is odd, then there are $\dfrac{(2j_1+1)(2j_2+1)+1}{2}$ submatrices. Thus, CG matrix has 1 submatrix only [viz. 1×1] corresponding to $j_1 = 0, j_2 = 0$; 2 submatrices [viz. $1\times1, 1\times1$] corresponding to $j_1 = 0, j_2 = \frac{1}{2}$ or $j_1 = \frac{1}{2}, j_2 = 0$ or $j_1 = 0, j_2 = 1$ or $j_1 = 1, j_2 = 0$; 3 submatrices [viz. $1\times1, 2\times2$ and 1×1] corresponding to $j_1 = 0, j_2 = \frac{3}{2}$ or $j_1 = \frac{3}{2}$, $j_2 = 0$ or $j_1 = \frac{1}{2}, j_2 = \frac{1}{2}$; 4 submatrices [viz. $1\times1, 2\times2, 2\times2$ and 1×1] corresponding to $j_1 = \frac{1}{2}, j_2 = 1$ or $j_1 = 1, j_2 = \frac{1}{2}$ $j_1 = 0, j_2 = 2$ or $j_1 = 2, j_2 = 0$; 5 submatrices [viz. $1\times1, 2\times2, 3\times3, 2\times2$ and 1×1] corresponding to $j_1 = 1, j_2 = 1$, etc.

$\therefore\ |j_1+j_2, j_1+j_2\rangle = |j_1, j_2\rangle \Longrightarrow J_-|j_1+j_2, j_1+j_2\rangle = (J_{1-}+J_{2-})|j_1, j_2\rangle$

$\Longrightarrow \sqrt{2(j_1+j_2)}\ |j_1+j_2, j_1+j_2-1\rangle = \sqrt{2j_1}\ |j_1-1, j_2\rangle + \sqrt{2j_2}\ |j_1, j_2-1\rangle \quad \left[\because J_-|jm\rangle = \sqrt{(j+m)(j-m+1)}\right]$

as $|m_1, m_2\rangle = |j_1, m_1\rangle|j_2, m_2\rangle$,

so, $J_{1-}|m_1, m_2\rangle = |j_2, m_2\rangle\, J_{1-}|j_1, m_1\rangle = |j_2, j_2\rangle\sqrt{(j_1+j_1)(j_1-j_1+1)}|j_1-1, j_2\rangle = \sqrt{2j_1}|j_1-1, j_2\rangle|j_2, j_2\rangle = \sqrt{2j_1}|j_1-1, j_2\rangle$

and $J_{2-}|m_1, m_2\rangle = |j_1, m_1\rangle\, J_{2-}|j_2, m_2\rangle = |j_1, j_1\rangle\sqrt{(j_2+j_2)(j_2-j_2+1)}|j_1, j_2-1\rangle = \sqrt{2j_2}|j_1, j_1\rangle|j_1, j_2-1\rangle = \sqrt{2j_2}|j_1, j_2-1\rangle$

$\therefore\ |j_1+j_2, j_1+j_2-1\rangle = \sqrt{\dfrac{j_1}{j_1+j_2}}\ |j_1-1, j_2\rangle + \sqrt{\dfrac{j_2}{j_1+j_2}}\ |j_1, j_2-1\rangle$

$$\text{Thus, } \begin{cases} C_{jmm_1m_2} = C_{j_1+j_2, j_1+j_2-1, j_1-1, j_2} = \langle \boldsymbol{m_1 m_2} | \boldsymbol{jm}\rangle = \langle \boldsymbol{j_1-1, j_2} | \boldsymbol{j_1+j_2, j_1+j_2-1}\rangle = \sqrt{\dfrac{\boldsymbol{j_1}}{\boldsymbol{j_1+j_2}}} \\ C_{jmm_1m_2} = C_{j_1+j_2, j_1+j_2-1, j_1, j_2-1} = \langle \boldsymbol{m_1 m_2} | \boldsymbol{jm}\rangle = \langle \boldsymbol{j_1, j_2-1} | \boldsymbol{j_1+j_2, j_1+j_2-1}\rangle = \sqrt{\dfrac{\boldsymbol{j_2}}{\boldsymbol{j_1+j_2}}} \end{cases}$$

		$\lvert jm\rangle$	
m_1	m_2	$\lvert j_1+j_2, j_1+j_2-1\rangle$	$\lvert j_1+j_2-1, j_1+j_2-1\rangle$
j_1	j_2-1	$\sqrt{\frac{j_2}{j_1+j_2}}$	$\sqrt{\frac{j_1}{j_1+j_2}}$
j_1-1	j_2	$\sqrt{\frac{j_1}{j_1+j_2}}$	$-\sqrt{\frac{j_2}{j_1+j_2}}$

Inverse of the equation, $|jm\rangle = \sum_{m_1, m_2} \langle m_1 m_2 | jm\rangle |m_1 m_2\rangle = \sum_{m_1, m_2} |m_1 m_2\rangle\langle m_1 m_2 | jm\rangle$ may be written as

$|m_1 m_2\rangle = \sum_{j,m} \langle jm | m_1 m_2\rangle\, |jm\rangle = \sum_{m_1, m_2} |jm\rangle\langle jm | m_1 m_2\rangle \quad \left[\begin{array}{c}\text{where } j \text{ is from } |j_1-j_2| \text{ to } (j_1+j_2) \\ \text{and } m \text{ is from } -j \text{ to } +j\end{array}\right]$

$\sum_{j,m} \langle jm | m_1 m_2\rangle\, |jm\rangle = |m_1 m_2\rangle \Longrightarrow \sum_{j,m} \langle jm | m_1 m_2\rangle \langle m_1' m_2' | jm\rangle = \langle m_1' m_2' | m_1 m_2\rangle = \delta_{m_1 m_1'}\delta_{m_2 m_2'}$

and $\sum_{m_1, m_2} \langle m_1 m_2 | jm\rangle |m_1 m_2\rangle = |jm\rangle \Longrightarrow \sum_{j,m} \langle m_1 m_2 | jm\rangle \langle j'm' | m_1 m_2\rangle = \langle j'm' | jm\rangle = \delta_{jj'}\delta_{mm'}$

so $J_{1-}|m_1, m_2\rangle = |j_2, m_2\rangle\, J_{1-}|j_1, m_1\rangle = |j_2, j_2\rangle\sqrt{(j_1+j_1)(j_1-j_1+1)}\ |j_1-1, j_2\rangle = \sqrt{2j_1}|j_1-1, j_2\rangle|j_2, j_2\rangle$
$= \sqrt{2j_1}\ |j_1-1, j_2\rangle$

so $J_{1-}|m_1, m_2\rangle = |j_2, m_2\rangle\, J_{1-}|j_1, m_1\rangle = |j_2, j_2\rangle\sqrt{(j_1+j_1)(j_1-j_1+1)}\ |j_1-1, j_2\rangle = \sqrt{2j_1}|j_1-1, j_2\rangle|j_2, j_2\rangle$
$= \sqrt{2j_1}\ |j_1-1, j_2\rangle$

Obtain the CG coefficients for a system having *j*₁ = 1/2 and *j*₂ = 1/2.

There are $(2j_1+1)(2j_2+1) = 4$ eigenstates which are $|jm\rangle = |1,1\rangle, |1,0\rangle, |0,0\rangle, |1,-1\rangle$

The possible $|m_1 m_2\rangle$ are $\left|\frac{1}{2}, \frac{1}{2}\right\rangle, \left|\frac{1}{2}, -\frac{1}{2}\right\rangle, \left|-\frac{1}{2}, \frac{1}{2}\right\rangle, \left|-\frac{1}{2}, -\frac{1}{2}\right\rangle$

$$\begin{cases} \left\langle \frac{1}{2}, \frac{1}{2} \middle| 1,1 \right\rangle = 1 \\ \left\langle -\frac{1}{2}, -\frac{1}{2} \middle| 1,-1 \right\rangle = 1 \end{cases} \quad [\because \langle j_1 j_2 | j_1+j_2, j_1+j_2\rangle = 1]$$

$$\begin{cases} \left\langle \frac{1}{2}, -\frac{1}{2} \middle| 1,0 \right\rangle = \dfrac{1}{\sqrt{2}} \quad \left[\because \langle j_1, j_2-1 | j_1+j_2, j_1+j_2-1\rangle = \sqrt{\dfrac{j_2}{j_1+j_2}}\right] \\ \left\langle -\frac{1}{2}, \frac{1}{2} \middle| 1,0 \right\rangle = \dfrac{1}{\sqrt{2}} \quad \left[\because \langle j_1-1, j_2 | j_1+j_2, j_1+j_2-1\rangle = \sqrt{\dfrac{j_1}{j_1+j_2}}\right] \end{cases}$$

$$\begin{cases} \left\langle \frac{1}{2}, -\frac{1}{2} \middle| 0,0 \right\rangle = \frac{1}{\sqrt{2}} & \left[\because \langle j_1, j_2 - 1 | j_1 + j_2 - 1, j_1 + j_2 - 1 \rangle = \sqrt{\frac{j_1}{j_1 + j_2}} \right] \\ \left\langle -\frac{1}{2}, \frac{1}{2} \middle| 0,0 \right\rangle = -\frac{1}{\sqrt{2}} & \left[\because \langle j_1 - 1, j_2 | j_1 + j_2 - 1, j_1 + j_2 - 1 \rangle = -\sqrt{\frac{j_2}{j_1 + j_2}} \right] \end{cases}$$

		$\|jm\rangle$			
m_1	m_2	$\|1,1\rangle$	$\|1,0\rangle$	$\|0,0\rangle$	$\|1,-1\rangle$
$\frac{1}{2}$	$\frac{1}{2}$	1			
$\frac{1}{2}$	$-\frac{1}{2}$		$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	
$-\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	
$-\frac{1}{2}$	$-\frac{1}{2}$				1

Obtain the CG coefficients for a system having *j*$_1$ = 1 and *j*$_2$ = 1/2.

There are $(2j_1+1)(2j_2+1) = 6$ eigenstates which are $|jm\rangle = \left|\frac{3}{2},\frac{3}{2}\right\rangle, \left|\frac{3}{2},\frac{1}{2}\right\rangle, \left|\frac{1}{2},\frac{1}{2}\right\rangle, \left|\frac{3}{2},-\frac{1}{2}\right\rangle, \left|\frac{1}{2},-\frac{1}{2}\right\rangle, \left|\frac{3}{2},-\frac{3}{2}\right\rangle$

The possible $|m_1 m_2\rangle$ are $\left|1,\frac{1}{2}\right\rangle, \left|1,-\frac{1}{2}\right\rangle, \left|0,\frac{1}{2}\right\rangle, \left|0,-\frac{1}{2}\right\rangle, \left|-1,\frac{1}{2}\right\rangle, \left|-1,-\frac{1}{2}\right\rangle$

		$\|jm\rangle$					
m_1	m_2	$\left\|\frac{3}{2},\frac{3}{2}\right\rangle$	$\left\|\frac{3}{2},\frac{1}{2}\right\rangle$	$\left\|\frac{1}{2},\frac{1}{2}\right\rangle$	$\left\|\frac{3}{2},-\frac{1}{2}\right\rangle$	$\left\|\frac{1}{2},-\frac{1}{2}\right\rangle$	$\left\|\frac{3}{2},-\frac{3}{2}\right\rangle$
1	$\frac{1}{2}$	1					
1	$-\frac{1}{2}$		$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{2}}{\sqrt{3}}$			
0	$\frac{1}{2}$		$\frac{\sqrt{2}}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$			
0	$-\frac{1}{2}$				$\frac{\sqrt{2}}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	
-1	$\frac{1}{2}$				$\frac{1}{\sqrt{3}}$	$-\frac{\sqrt{2}}{\sqrt{3}}$	
-1	$-\frac{1}{2}$						1

Obtain the CG coefficients for a system having *j*$_1$ = 1 and *j*$_2$ = 1.***

There are $(2j_1+1)(2j_2+1) = 9$ eigenstates which are $|jm\rangle = |2,2\rangle, |2,1\rangle, |2,0\rangle, |1,1\rangle, |1,0\rangle, |2,-1\rangle, |0,0\rangle, |1,-1\rangle, |2,-2\rangle$. The possible $|m_1 m_2\rangle$ are $|1,1\rangle, |1,0\rangle, |1,-1\rangle, |0,1\rangle, |0,0\rangle, |0,-1\rangle, |-1,1\rangle, |-1,0\rangle, |-1,-1\rangle$

$$\begin{cases} \langle 1, 1 | 2, 2 \rangle = 1 \\ \langle -1, -1 | 2, -2 \rangle = 1 \end{cases} \quad [\because \langle j_1 j_2 | j_1 + j_2, j_1 + j_2 \rangle = 1]$$

$$\begin{cases} \langle 1, 0 | 2, 1 \rangle = \frac{1}{\sqrt{2}} & \left[\because \langle j_1, j_2 - 1 | j_1 + j_2, j_1 + j_2 - 1 \rangle = \sqrt{\frac{j_2}{j_1 + j_2}} \right] \\ \langle 0, 1 | 2, 1 \rangle = \frac{1}{\sqrt{2}} & \left[\because \langle j_1 - 1, j_2 | j_1 + j_2, j_1 + j_2 - 1 \rangle = \sqrt{\frac{j_1}{j_1 + j_2}} \right] \end{cases}$$

$$\begin{cases} \left\langle \frac{1}{2}, -\frac{1}{2} \middle| 0,0 \right\rangle = \frac{1}{\sqrt{2}} & \left[\because \langle j_1, j_2 - 1 | j_1 + j_2 - 1, j_1 + j_2 - 1 \rangle = \sqrt{\frac{j_1}{j_1 + j_2}} \right] \\ \left\langle -\frac{1}{2}, \frac{1}{2} \middle| 0,0 \right\rangle = -\frac{1}{\sqrt{2}} & \left[\because \langle j_1 - 1, j_2 | j_1 + j_2 - 1, j_1 + j_2 - 1 \rangle = -\sqrt{\frac{j_2}{j_1 + j_2}} \right] \end{cases}$$

		$\|jm\rangle$								
m_1	m_2	$\|2,2\rangle$	$\|2,1\rangle$	$\|2,0\rangle$	$\|1,1\rangle$	$\|1,0\rangle$	$\|2,-1\rangle$	$\|0,0\rangle$	$\|1,-1\rangle$	$\|2,-2\rangle$
1	1		1							
1	0			$\frac{1}{\sqrt{3}}$		$\frac{\sqrt{2}}{\sqrt{3}}$				
1	-1									
0	1									
0	0			$\frac{\sqrt{2}}{\sqrt{3}}$		$-\frac{1}{\sqrt{3}}$				
0	-1									
-1	1						$\frac{\sqrt{2}}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$		
-1	0						$\frac{1}{\sqrt{3}}$	$-\frac{\sqrt{2}}{\sqrt{3}}$		
-1	-1								1	

From an aeroplane flying at an altitude *H* at a speed of *u*, a stone is thrown at an angle *ϑ* below the line of flight with a speed *v* relative to the plane. Find the trajectory of the stone for an observer on the ground.

$H - y = (v \sin\theta)t + \frac{1}{2}gt^2 \Longrightarrow y = H - (v \sin\theta)t - \frac{1}{2}gt^2$

$x = (u + v\cos\theta)t \Longrightarrow t = \frac{x}{(u + v\cos\theta)}$

$\therefore y = H - (v\sin\theta)\left(\frac{x}{u + v\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{u + v\cos\theta}\right)^2 \Longrightarrow y = H - \left(\frac{v\sin\theta}{u + v\cos\theta}\right)x - \frac{1}{2}\frac{g}{(u + v\cos\theta)^2}x^2$

When $\theta = \frac{\pi}{2},\ y = H - \left(\frac{v}{u}\right)x - \frac{1}{2}\frac{g}{u^2}x^2$

When $u = 0,\ y = H - (\tan\theta)x - \frac{1}{2}\frac{g\sec^2\theta}{v^2}x^2$

Partition Function

$$\bar{E} = \frac{\sum_r E_r e^{-\beta E_r}}{\sum_r e^{-\beta E_r}} = \frac{-\sum_r \frac{\partial}{\partial\beta}\left(e^{-\beta E_r}\right)}{\sum_r e^{-\beta E_r}} = \frac{-\frac{\partial}{\partial\beta}\left(\sum_r e^{-\beta E_r}\right)}{\sum_r e^{-\beta E_r}} = -\frac{1}{Z}\frac{\partial Z}{\partial\beta} = -\frac{\partial(\ln Z)}{\partial\beta} \quad \left[\begin{array}{c} Z = \sum_r e^{-\beta E_r} \text{ is called} \\ \text{"sum over states" or} \\ \text{"partition function"} \end{array}\right]$$

Variance, $\overline{(\Delta E)^2} = \overline{(E - \bar{E})^2} = \overline{E^2 - 2E\bar{E} + \bar{E}^2} = \overline{E^2} - 2\bar{E}\bar{E} + \overline{\bar{E}^2} = \overline{E^2} - 2\bar{E}^2 + \bar{E}^2 = \overline{E^2} - \bar{E}^2$

$$\overline{E^2} = \frac{\sum_r {E_r}^2 e^{-\beta E_r}}{\sum_r e^{-\beta E_r}} = \frac{-\sum_r \frac{\partial}{\partial\beta}\left(E_r e^{-\beta E_r}\right)}{\sum_r e^{-\beta E_r}} = \frac{-\frac{\partial}{\partial\beta}\left(\sum_r E_r e^{-\beta E_r}\right)}{\sum_r e^{-\beta E_r}} = \frac{\frac{\partial}{\partial\beta}\sum_r \frac{\partial}{\partial\beta}\left(e^{-\beta E_r}\right)}{\sum_r e^{-\beta E_r}} = \frac{\frac{\partial^2}{\partial\beta^2}\sum_r e^{-\beta E_r}}{\sum_r e^{-\beta E_r}} = \frac{1}{Z}\frac{\partial^2 Z}{\partial\beta^2}$$

$\therefore$ Mean-square energy, $\overline{E^2} = \frac{1}{Z}\frac{\partial^2 Z}{\partial\beta^2} = \frac{\partial}{\partial\beta}\left(\frac{1}{Z}\frac{\partial Z}{\partial\beta}\right) + \frac{1}{Z^2}\left(\frac{\partial Z}{\partial\beta}\right)^2 = -\frac{\partial\bar{E}}{\partial\beta} + \bar{E}^2$

Hence, $\overline{(\Delta E)^2} = \overline{E^2} - \bar{E}^2 = \left(-\frac{\partial\bar{E}}{\partial\beta} + \bar{E}^2\right) - \bar{E}^2 = -\frac{\partial\bar{E}}{\partial\beta} = -\frac{\partial}{\partial\beta}\left(-\frac{\partial(\ln Z)}{\partial\beta}\right) = \frac{\partial^2(\ln Z)}{\partial\beta^2}$

$E_r \equiv E_r(x),\ dE_r = \dfrac{\partial E_r}{\partial x}dx$

$$\therefore đW = \frac{\sum_r \left(-\frac{\partial E_r}{\partial x}dx\right)e^{-\beta E_r}}{\sum_r e^{-\beta E_r}} = \frac{\frac{1}{\beta}\sum_r \frac{\partial}{\partial x}\left(e^{-\beta E_r}\right)dx}{\sum_r e^{-\beta E_r}} = \frac{\frac{1}{\beta}\frac{\partial}{\partial x}\left(\sum_r e^{-\beta E_r}\right)dx}{\sum_r e^{-\beta E_r}} = \frac{1}{\beta Z}\frac{\partial Z}{\partial x}dx = \frac{1}{\beta}\frac{\partial \ln Z}{\partial x}dx = \bar{X}\,dx$$

generalized force, $X_r = -\dfrac{\partial E_r}{\partial x} \Rightarrow \bar{X} = -\overline{\dfrac{\partial E_r}{\partial x}} = \dfrac{-\sum_r \frac{\partial E_r}{\partial x}e^{-\beta E_r}}{\sum_r e^{-\beta E_r}} = \dfrac{1}{\beta}\dfrac{\partial \ln Z}{\partial x}$

If $x =$ volume, V then $đW = \bar{X}\,dx = \bar{P}\,dV$ and $\bar{X} =$ mean pressure, $\bar{P} = \dfrac{1}{\beta}\dfrac{\partial \ln Z}{\partial V} = \dfrac{1}{k_B T}\dfrac{\partial \ln Z}{\partial V}$ $\left[\begin{array}{l} Z \equiv Z(\beta, E_r) \\ \quad \equiv Z(T, V) \end{array}\right]$

This relates $\bar{P}$ to T and V, thus yielding the equation of state of the system.

$\because Z \equiv Z(\beta, E_r)$ and $E_r \equiv E_r(x)$, $\therefore Z \equiv Z(\beta, x) \Rightarrow \ln Z \equiv \ln Z\,(\beta, x) \Rightarrow d(\ln Z) = \dfrac{\partial \ln Z}{\partial x}dx + \dfrac{\partial \ln Z}{\partial \beta}d\beta$

$\Rightarrow d\ln Z = \beta\, đW - \bar{E}\,d\beta$ $\left[\because đW = \dfrac{1}{\beta}\dfrac{\partial \ln Z}{\partial x}dx \text{ and } \bar{E} = -\dfrac{\partial \ln Z}{\partial \beta}\right]$

$\Rightarrow d\ln Z = \beta\, đW - d(\bar{E}\beta) + \beta\, đE \Rightarrow d(\ln Z + \beta\bar{E}) = \beta(đW + d\bar{E}) = \beta\, đQ$ $[\because đQ = đW + d\bar{E}]$

$\Rightarrow d(\ln Z + \beta\bar{E}) = \dfrac{1}{k_B T}đQ$ $\left[\because \beta = \dfrac{1}{k_B T}\right] \Rightarrow d(\ln Z + \beta\bar{E}) = \dfrac{1}{k_B}dS$ $\left[\because S = \dfrac{đQ}{T}\right] \Rightarrow S = k_B(\ln Z + \beta\bar{E})$

$\Rightarrow S = k_B \ln Z + \dfrac{\bar{E}}{T} \Rightarrow TS = k_B T \ln Z + \bar{E} \Rightarrow -k_B T \ln Z = \bar{E} - TS \Rightarrow F = \bar{E} - TS$ $\left[\begin{array}{c}\text{where, } F = -k_B T \ln Z \text{ is} \\ \text{Helmholtz Free Energy}\end{array}\right]$

As $T \to 0$ $(i.e.\ \beta \to \infty)$, $Z = \sum_r e^{-\beta E_r} \to \Omega_0 e^{-\beta E_0} \Rightarrow \bar{E} = -\dfrac{\partial(\ln Z)}{\partial \beta} \to -\dfrac{\partial(\ln \Omega_0 - \beta E_0)}{\partial \beta} = E_0$

$\Rightarrow S = k_B(\ln Z + \beta\bar{E}) \to k_B\big((\ln \Omega_0 - \beta E_0) + \beta E_0\big) = k_B \ln \Omega_0$ $\left[\begin{array}{c}\text{which is the Third law of Thermodynamics} \\ \text{i.e. entropy is minimum at absolute zero.}\end{array}\right]$

$$Z = \int \cdots \int e^{-\beta E(q_1, \dots, p_f)} \frac{dq_1 \cdots dp_f}{{h_0}^f}$$

$$E = \sum_{i=1}^{N} \frac{{p_i}^2}{2m} + U(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N)$$

$$Z' = \int \cdots \int e^{-\beta\left\{\frac{1}{2m}({p_1}^2, \dots, {p_N}^2)\right\}} \frac{d^3\vec{r}_1 \cdots d^3\vec{p}_N}{{h_0}^{3N}}$$

$$= \frac{1}{{h_0}^{3N}} \int e^{-\frac{\beta}{2m}{p_1}^2} d^3\vec{p}_1 \cdots \int e^{-\frac{\beta}{2m}{p_N}^2} d^3\vec{p}_N \int e^{-\beta U(\vec{r}_1, \dots, \vec{r}_N)} d^3\vec{r}_1 \cdots d^3\vec{r}_N$$

$$= \frac{1}{{h_0}^{3N}} \left(\int_{-\infty}^{\infty} e^{-\frac{\beta}{2m}p^2} d^3\vec{p}\right)^N \left(\int d^3\vec{r}\right)^N [\because U = 0 \text{ for ideal gas}] = \frac{V^N}{{h_0}^{3N}} \left(\int_{-\infty}^{\infty} e^{-\frac{\beta}{2m}p^2} d^3\vec{p}\right)^N$$

$\therefore Z' = \zeta^N$ $\left[\begin{array}{c}\text{where } \zeta = \dfrac{V}{{h_0}^3}\displaystyle\int_{-\infty}^{\infty} e^{-\frac{\beta}{2m}p^2} d^3\vec{p} \text{ is the} \\ \text{partition function for a single molecule.}\end{array}\right] \Rightarrow \ln Z' = N \ln \zeta$

$$\int_{-\infty}^{\infty} e^{-\frac{\beta}{2m}p^2} d^3\vec{p} = \iiint_{-\infty}^{\infty} e^{-\frac{\beta}{2m}({p_x}^2 + {p_y}^2 + {p_z}^2)} dp_x dp_y dp_z = \int_{-\infty}^{\infty} e^{-\frac{\beta}{2m}{p_x}^2} dp_x \int_{-\infty}^{\infty} e^{-\frac{\beta}{2m}{p_y}^2} dp_y \int_{-\infty}^{\infty} e^{-\frac{\beta}{2m}{p_z}^2} dp_z = \left(\sqrt{\frac{2\pi m}{\beta}}\right)^3$$

$$\therefore \zeta = \frac{V}{{h_0}^3}\int_{-\infty}^{\infty} e^{-\frac{\beta}{2m}p^2} d^3\vec{p} = V\left(\sqrt{\frac{2\pi m}{{h_0}^2\beta}}\right)^3 = V\left(\frac{2\pi m}{{h_0}^2\beta}\right)^{\frac{3}{2}}$$

$$\therefore \ln Z' = N \ln \zeta = N\left\{\ln V - \frac{3}{2}\ln\beta + \frac{3}{2}\ln\left(\frac{2\pi m}{{h_0}^2}\right)\right\}$$

$$\bar{P} = \frac{1}{\beta}\frac{\partial \ln Z'}{\partial V} = k_B T\left(\frac{N}{V}\right) \Longrightarrow \bar{P}V = Nk_B T$$

$$\bar{E} = -\frac{\partial \ln Z'}{\partial \beta} = \frac{3}{2}\left(\frac{N}{\beta}\right) = \frac{3}{2}Nk_B T$$

Heat capacity at constant volume, $C_v = \left(\frac{\partial \bar{E}}{\partial T}\right)_V = \frac{3}{2}Nk_B = \frac{3}{2}nN_A k_B = \frac{3}{2}nR$ $[R = N_A k_B$ is the gas constant.$]$

$\therefore$ Molar specific heat at constant volume, $c_v = \frac{3}{2}R\bar{E} = -\frac{\partial \ln Z'}{\partial \beta} = \frac{3}{2}\left(\frac{N}{\beta}\right) = \frac{3}{2}Nk_B T$

$$S = k_B(\ln Z' + \beta\bar{E}) = k_B\left(N\left\{\ln V - \frac{3}{2}\ln\beta + \frac{3}{2}\ln\left(\frac{2\pi m}{{h_0}^2}\right)\right\} + \frac{3}{2}N\right) = Nk_B\left(\ln V - \frac{3}{2}\ln\beta + \frac{3}{2}\ln\left(\frac{2\pi m}{{h_0}^2}\right) + \frac{3}{2}\right)$$

This is, however, an incorrect expression for entropy.

$$\ln \zeta = \left\{\ln V + \frac{3}{2}\ln T + \sigma - \frac{3}{2}\right\}$$

Consider a system $A^{(0)}$ consisting of two systems $A^{(1)}$ and $A^{(2)}$ which are weakly interacting mutually. Let each state of $A^{(1)}$ be denoted by an index r and its corresponding energy by ${E_r}^{(1)}$. Likewise, let each state of $A^{(2)}$ be denoted by an index s and its corresponding energy by ${E_s}^{(2)}$. A state of the combined system $A^{(0)} = A^{(1)} + A^{(2)}$ can then be denoted by the pair of indices r, s. Since A_1 and A_2 interact only weakly, the corres- -ponding energy of state, ${E_{rs}}^{(0)} = {E_r}^{(1)} + {E_s}^{(2)}$. The partition function of $A^{(0)}$ is then, $Z^{(0)} = \sum_{r,s} e^{-\beta {E_{rs}}^{(0)}}$

$$= \sum_{r,s} e^{-\beta\left({E_r}^{(1)} + {E_s}^{(2)}\right)} = \sum_{r,s} e^{-\beta {E_r}^{(1)}} e^{-\beta {E_s}^{(2)}} = \left(\sum_r e^{-\beta {E_r}^{(1)}}\right)\left(\sum_s e^{-\beta {E_r}^{(2)}}\right) = Z^{(1)}Z^{(2)} \Longrightarrow \ln Z^{(1)} + \ln Z^{(2)}$$

$$\Longrightarrow \bar{E}^{(0)} = \bar{E}^{(1)} + \bar{E}^{(2)} \quad \left[\because \bar{E} = -\frac{\partial \ln Z}{\partial \beta}\right] \Longrightarrow S^{(0)} = S^{(1)} + S^{(2)} \quad [\because S = k_B(\ln Z + \beta\bar{E})] \quad \left[\begin{array}{c}\text{i.e. Entropy is an}\\ \text{extensive property.}\end{array}\right]$$

Consider a container of volume V containing N molecules of a gas at an entropy $S = Nk_B\left(\ln V + \frac{3}{2}\ln T + \sigma\right)$. Let a partition be introduced so that it divides the container in two parts and all the N gas molecules are confined to one side of the partition. The new entropy, $S' = S_1 + S_2 = S_1 = Nk_B\left(\ln\frac{V}{2} + \frac{3}{2}\ln T' + \sigma\right)$. As the volume is decreased, the entropy should decrease. But as the density has increased, entropy should increase as the temperature would increase.

Type equation here.

Correct partition function, $Z = \frac{Z'}{N!} = \frac{\zeta^N}{N!} \Longrightarrow \ln Z = N\ln\zeta - \ln N! = N\ln\zeta - N\ln N + N$

$$S = k_B(\ln Z + \beta\bar{E}) = Nk_B\left(\ln V + \frac{3}{2}\ln T + \sigma\right) + k_B(-N\ln N + N) = Nk_B\left(\ln\frac{V}{N} + \frac{3}{2}\ln T + \sigma + 1\right)$$

$$F = -k_B T\ln Z = -Nk_B T(\ln\zeta - \ln N + 1) = -Nk_B T\left(\left\{\ln V + \frac{3}{2}\ln T + \sigma - \frac{3}{2}\right\} - \ln N + 1\right)$$

$$= -Nk_B T\left(\ln\frac{V}{N} + \frac{3}{2}\ln T + \sigma - \frac{1}{2}\right)$$

Equipartition theorem

If the Energy of a system is a function of f generalized coordinates q_k and corresponding f generalized momenta p_k, i.e. $E \equiv E(q_1, \dots\dots, q_f, p_1, \dots\dots, p_f)$ and the situation is such that $E = E_i(p_i) + E'(q_1, \dots\dots, q_f, p_1, \dots\dots p_f)$ where E_i involves only the coordinate p_i and E' involves the other coordinates, and E_i is quadratic in p_i, *i.e.* $E_i(p_i) = b_i {p_i}^2$, then when the system is in equilibrium at absolute temperature T and distributed as per the canonical distribution, the mean value,

$$\bar{E}_i = \frac{\int_{-\infty}^{\infty} e^{-\beta E(q_1,\dots\dots,p_f)} E_i dq_1 \cdots dp_f}{\int_{-\infty}^{\infty} e^{-\beta E(q_1,\dots\dots,p_f)} dq_1 \cdots dp_f} = \frac{\int_{-\infty}^{\infty} e^{-\beta (E_i+E')} E_i dq_1 \cdots dp_f}{\int_{-\infty}^{\infty} e^{-\beta (E_i+E')} dq_1 \cdots dp_f} = \frac{\int_{-\infty}^{\infty} e^{-\beta E_i} E_i dp_i \int_{-\infty}^{\infty} e^{-\beta E'} dq_1 \cdots dp_f}{\int_{-\infty}^{\infty} e^{-\beta E_i} dp_i \int_{-\infty}^{\infty} e^{-\beta E'} dq_1 \cdots dp_f} = \frac{\int_{-\infty}^{\infty} e^{-\beta E_i} E_i dp_i}{\int_{-\infty}^{\infty} e^{-\beta E_i} dp_i}$$

$$= \frac{-\frac{\partial}{\partial \beta}\int_{-\infty}^{\infty} e^{-\beta E_i} dp_i}{\int_{-\infty}^{\infty} e^{-\beta E_i} dp_i} = -\frac{\partial}{\partial \beta}\ln\left(\int_{-\infty}^{\infty} e^{-\beta E_i} dp_i\right) = -\frac{\partial}{\partial \beta}\ln\left(\int_{-\infty}^{\infty} e^{-\beta b_i {p_i}^2} dp_i\right) \; [\because E_i(p_i) = b_i {p_i}^2] = -\frac{\partial}{\partial \beta}\ln\left(\frac{\int_{-\infty}^{\infty} e^{-b_i y^2} dp_i}{\sqrt{\beta}}\right)$$

$$\left[\text{putting } p_i = \frac{y}{\sqrt{\beta}}\right] = -\frac{\partial}{\partial \beta}\left(\ln \int_{-\infty}^{\infty} e^{-b_i y^2} dp_i - \frac{1}{2}\ln\beta\right) = \frac{\partial}{\partial \beta}\left(\frac{1}{2}\ln\beta\right) \; \left[\begin{array}{l}\because \text{derivative of 1st term w.r.t. } \beta \\ \text{is zero as it is independent of } \beta\end{array}\right] = \frac{1}{2\beta} = \frac{1}{2}k_B T$$

$\bar{E}_i = \frac{1}{2}k_B T$ also when $E = E_i(q_i) + E'(q_1, \dots\dots, q_f, p_1, \dots\dots p_f)$ and $E_i(q_i) = a_i {q_i}^2$

This is the equipartition theorem which states the mean value of each independent quadratic term in the energy is $\frac{1}{2}k_B T$.

Consider a molecule of mass m in a gas (need not be ideal) at temperature T.

Translational Kinetic Energy, $K = \frac{1}{2m}({p_x}^2 + {p_y}^2 + {p_z}^2) \Longrightarrow \bar{K} = \frac{3}{2}k_B T$ [by Equipartition theorem]

For an ideal monoatomic gas, entire energy is translational kinetic energy.

So, mean energy per molecule, $\bar{E} = \bar{K} = \frac{3}{2}k_B T \Longrightarrow$ mean energy per mole, $\bar{E}_{\text{mol}} = \frac{3}{2}k_B T = N_A\left(\frac{3}{2}k_B T\right) = \frac{3}{2}RT$

$\therefore$ Molar specific heat at constant volume, $c_v = \left(\frac{\partial \bar{E}_{\text{mol}}}{\partial T}\right)_V = \frac{3}{2}R$ [$R = N_A k_B$ is the gas constant.]

Charge moving through a crossed Electric and Magnetic field

The positive and negative plates are respectively placed at $y = +\frac{d}{2}$ & $y = -\frac{d}{2}$ extending from $x = 0$ to $x = l$.

$$\left.\begin{array}{l} E_y = -\frac{V}{d} \\ F_y = (-e)E_y \end{array}\right\} \Longrightarrow F_y = \frac{eV}{d} \Longrightarrow a_y = \frac{eV}{md} \Longrightarrow \frac{dv_y}{dt} = \frac{eV}{md} \Longrightarrow v_y = \frac{eV}{md}t + C_1 = \frac{eV}{md}t \quad [\because v_y = 0 \text{ at } t = 0]$$

$$\Longrightarrow \frac{dy}{dt} = \frac{eV}{md}t \Longrightarrow y = \frac{eV}{2md}t^2 + C_2 = \frac{eV}{2md}t^2 \quad [\because y = 0 \text{ at } t = 0]$$

$$\left.\begin{array}{r} x = v_x t \Longrightarrow t = \frac{x}{v_x} \\ y = \frac{eV}{2md}t^2 \\ v_y = \frac{eV}{md}t \end{array}\right\} \Longrightarrow \left.\begin{array}{r} y = \frac{eV}{2md}\frac{x^2}{{v_x}^2} \\ v_y = \frac{eV}{md}\frac{x}{v_x} \end{array}\right\} \Longrightarrow \left.\begin{array}{r} y = \frac{eV}{2md}\frac{x^2}{\frac{2eV_a}{m}} = \frac{V}{4d}\frac{x^2}{V_a} \\ v_y = \frac{eV}{md}\frac{x}{\sqrt{\frac{2eV_a}{m}}} = \frac{V}{d}\frac{x}{\sqrt{\frac{2mV_a}{e}}} \end{array}\right\} \left[\begin{array}{l} \because \frac{1}{2}m{v_x}^2 = eV_a \Longrightarrow v_x = \sqrt{\frac{2eV_a}{m}} \\ V_a \text{ is accelerating potential.} \end{array}\right]$$

$$\text{At } x = l, \begin{cases} t = t_1 = \frac{l}{v_x} \\ y = \frac{eV}{2md}\frac{l^2}{{v_x}^2} = \frac{V}{4d}\frac{l^2}{V_a} \\ v_y = \frac{eV}{md}\frac{l}{v_x} \end{cases} \qquad \text{Beyond } x = l, \begin{cases} x = v_x t \\ y = v_y t \\ v_y = \frac{eV}{md}\frac{l}{v_x} \end{cases}$$

For the electron to emerge, $\frac{V}{4d}\frac{l^2}{V_a} < \frac{d}{2} \Rightarrow \frac{V}{V_a} < \frac{2d^2}{l^2}$

Time taken to hit the screen placed at $x = L + \frac{l}{2}$, $t_f = t_1 + t_2 = \frac{l}{v_x} + \frac{L - \frac{l}{2}}{v_x} = \frac{L + \frac{l}{2}}{v_x}$

$$\text{At } x = L + \frac{l}{2},\ \begin{cases} t = t_f = \dfrac{L + \frac{l}{2}}{v_x} \\ y = \dfrac{V}{4d}\dfrac{l^2}{V_a} + v_y t_2 = \dfrac{V}{4d}\dfrac{l^2}{V_a} + \dfrac{eV}{md}\dfrac{l}{v_x}\left(\dfrac{L - \frac{l}{2}}{v_x}\right) = \dfrac{V}{4d}\dfrac{l^2}{V_a} + \dfrac{V}{2d}\dfrac{l\left(L - \frac{l}{2}\right)}{V_a} = \dfrac{V}{2d}\dfrac{lL}{V_a} \\ v_y = \dfrac{eV}{md}\dfrac{l}{v_x} \end{cases}$$

If there also exists a magnetic field, $B_z\odot$(out of the xy-plane) in the region $x = 0$ to $x = l$, moving electrons will be deflected in the xy-plane.

$F_y = (-e)E_y + (-e)(-v_x B_z)$ $[\because v_x\hat{\imath} \times B_z\hat{k} = -v_x B_z\hat{\jmath}] = \frac{eV}{d} + ev_x B_z \Rightarrow a_y = \frac{dv_y}{dt} = \frac{e}{m}\left(\frac{V}{d} + v_x B_z\right)$

$F_x = (-e)(v_y B_z)$ $[\because v_y\hat{\jmath} \times B_z\hat{k} = v_y B_z\hat{\imath}] = -ev_y B_z \Rightarrow a_x = \frac{dv_x}{dt} = -\frac{e}{m}v_y B_z$

$\frac{d^2 v_y}{dt^2} = \frac{e}{m}B_z\frac{dv_x}{dt} = -\left(\frac{e}{m}\right)^2 B_z{}^2 v_y \Rightarrow v_y = C_1 \sin\omega_c t$ $\begin{bmatrix} C_2 = 0 \text{ [in } C_2\cos\omega_c t \text{ term]} \because v_y = 0 \text{ at } t = 0 \\ \text{Here, } \omega_c = \frac{eB_z}{m} \text{ is the cyclotron frequency.} \end{bmatrix}$

$\therefore \frac{dv_x}{dt} = -\frac{e}{m}v_y B_z = -\frac{e}{m}(C_1\sin\omega_c t)B_z \Rightarrow$ At $t = 0$, $\frac{dv_x}{dt} = 0$

$\frac{d^2 v_x}{dt^2} = -\frac{e}{m}B_z\frac{dv_y}{dt} = -\left(\frac{e}{m}\right)^2 B_z\left(\frac{V}{d} + v_x B_z\right) \Rightarrow \frac{d^2 v_x}{dt^2} = -\left(\frac{e}{m}\right)^2 B_z{}^2 v_x - \left(\frac{e}{m}\right)^2 B_z\frac{V}{d}$

This is an inhomogeneous 2nd order linear differential equation whose complete solution, $v_x = (v_x)_{\mathrm{C}} + (v_x)_{\mathrm{P}}$ where compementary solution, $(v_x)_{\mathrm{C}} = (C_3\sin\omega_c t + C_4\cos\omega_c t)$ and particular solution, $(v_x)_{\mathrm{P}} = C_5 t + C_6$

Substituting $(v_x)_{\mathrm{P}}$ into the equation gives $0 = -\left(\frac{e}{m}\right)^2 B_z{}^2(C_5 t + C_6) - \left(\frac{e}{m}\right)^2 B_z\frac{V}{d} \Rightarrow C_5 = 0$ and $C_6 = -\frac{V}{B_z d}$

$\therefore (v_x)_{\mathrm{P}} = -\frac{V}{B_z d}$ and $v_x = (C_3\sin\omega_c t + C_4\cos\omega_c t) - \frac{V}{B_z d} \Rightarrow \frac{dv_x}{dt} = C_3\omega_c\cos\omega_c t - C_4\omega_c\sin\omega_c t$

$$\because \text{At } t = 0,\ \begin{cases} v_x = v_0 \ \therefore C_4 = v_0 + \dfrac{V}{B_z d} \\ \dfrac{dv_x}{dt} = 0 \ \therefore C_3 = 0 \end{cases} \quad \therefore v_x = \left(v_0 + \frac{V}{B_z d}\right)\cos\omega_c t - \frac{V}{B_z d} = v_0\cos\omega_c t - \frac{V}{B_z d}(1 - \cos\omega_c t)$$

If $V = 0$ i.e. $E_y = 0$, $v_x = v_0\cos\omega_c t$

If $B_z = 0$, $v_x = \lim_{B_z\to 0}\left(v_0\cos\omega_c t - \frac{V}{B_z d}(1 - \cos\omega_c t)\right) = \lim_{B_z\to 0}\left(v_0\cos\frac{eB_z}{m}t - \frac{V}{B_z d}\left(1 - \cos\frac{eB_z}{m}t\right)\right) = v_0$

$$\begin{cases} \dfrac{dv_y}{dt} = C_1\omega_c\cos\omega_c t \\ \dfrac{dv_y}{dt} = \dfrac{e}{m}\left(\dfrac{V}{d} + v_x B_z\right) \end{cases} \Rightarrow C_1\omega_c\cos\omega_c t = \frac{e}{m}\left(\frac{V}{d} + v_x B_z\right) = \frac{e}{m}\left[\frac{V}{d} + \left\{\left(v_0 + \frac{V}{B_z d}\right)\cos\omega_c t - \frac{V}{B_z d}\right\}B_z\right]$$

$\Rightarrow C_1\omega_c\cos\omega_c t = \frac{e}{m}\left(v_0 B_z + \frac{V}{d}\right)\cos\omega_c t \Rightarrow C_1 = \frac{e}{m}\frac{\left(v_0 B_z + \frac{V}{d}\right)}{\omega_c} = \frac{\left(v_0 B_z + \frac{V}{d}\right)}{B_z} = v_0 + \frac{V}{B_z d} = C_4$

$\therefore v_y = C_1\sin\omega_c t = \left(v_0 + \frac{V}{B_z d}\right)\sin\omega_c t$

If $V = 0$ i.e. $E_y = 0$, $v_y = v_0\sin\omega_c t$

If $B_z = 0, v_y = \lim_{B_z\to 0}\left(v_0 + \frac{V}{B_z d}\right)\sin\frac{eB_z}{m}t = \lim_{B_z\to 0}\left(v_0 \sin\frac{eB_z}{m}t + \frac{V}{B_z d}\sin\frac{eB_z}{m}t\right) = \lim_{B_z\to 0}\frac{V}{d}\frac{et}{m}\left(\frac{\sin\frac{et}{m}B_z}{\frac{et}{m}B_z}\right) = \frac{eV}{md}t$

Now, $v_x = \frac{dx}{dt} = \left(v_0 + \frac{V}{B_z d}\right)\cos\omega_c t - \frac{V}{B_z d} \Longrightarrow x = \left(v_0 + \frac{V}{B_z d}\right)\frac{\sin\omega_c t}{\omega_c} - \frac{V}{B_z d}t + C_7$

$\because$ At $t = 0, x = 0 \quad \therefore C_7 = 0 \qquad \therefore x = \left(v_0 + \frac{V}{B_z d}\right)\frac{\sin\omega_c t}{\omega_c} - \frac{V}{B_z d}t$

If $V = 0$ i.e. $E_y = 0, x = \frac{v_0}{\omega_c}\sin\omega_c t$

If $B_z = 0, x = \lim_{B_z\to 0}\left(\left(v_0 + \frac{V}{B_z d}\right)\frac{\sin\omega_c t}{\omega_c} - \frac{V}{B_z d}t\right) = \lim_{B_z\to 0}\left(v_0 + \frac{V}{B_z d}\right)t\lim_{B_z\to 0}\frac{\sin\frac{eB_z}{m}t}{\frac{eB_z}{m}t} - \lim_{B_z\to 0}\frac{V}{B_z d}t = v_0 t$

$v_y = \frac{dy}{dt} = \left(v_0 + \frac{V}{B_z d}\right)\sin\omega_c t \Longrightarrow y = -\left(v_0 + \frac{V}{B_z d}\right)\frac{\cos\omega_c t}{\omega_c} + C_8$

$\because$ At $t = 0, y = 0 \quad \therefore C_8 = \left(v_0 + \frac{V}{B_z d}\right)\frac{1}{\omega_c} \qquad \therefore y = \left(v_0 + \frac{V}{B_z d}\right)\frac{1}{\omega_c}(1 - \cos\omega_c t)$

If $V = 0$ i.e. $E_y = 0, y = \frac{v_0}{\omega_c}(1 - \cos\omega_c t)$

If $B_z = 0, y = \lim_{B_z\to 0}\left(v_0 + \frac{V}{B_z d}\right)\frac{1}{\omega_c}(1 - \cos\omega_c t) = \lim_{B_z\to 0}\frac{V}{B_z d}\frac{1}{\omega_c}\left(2\sin^2\frac{\omega_c t}{2}\right) = \lim_{B_z\to 0}\frac{V}{d}\left(\frac{et^2}{2m}\right)\frac{\sin^2\frac{et}{2m}B_z}{\left(\frac{et}{2m}B_z\right)^2} = \frac{eV}{2md}t^2$

At $t = \frac{2\pi}{\omega_c}$, (once again) $y = 0$. Then, $x = -\frac{V}{B_z d}t = -\frac{V}{B_z d}\frac{2\pi}{\omega_c}$

$\because\ x \geq 0,\ B_z$ (or V) must be negative. So, electron emerges when $-\frac{V}{B_z d}\frac{2\pi}{\omega_c} = l \Longrightarrow B_z = -\sqrt{\frac{2\pi V}{(e/m)d}}$

y is also 0 (for any t) when $v_0 + \frac{V}{B_z d} = 0 \Longrightarrow B_z = -\frac{V}{v_0 d} = -\frac{V/d}{\sqrt{2V_a(e/m)}}$ [electron beam goes undeflected]

When $x = l$, $\left(v_0 + \frac{V}{B_z d}\right)\frac{\sin\omega_c t}{\omega_c} - \frac{V}{B_z d}t = l \Longrightarrow t = t_l$ [to be determined graphically]

When $x = 0$, $\left(v_0 + \frac{V}{B_z d}\right)\frac{\sin\omega_c t}{\omega_c} - \frac{V}{B_z d}t = 0 \Longrightarrow t = t_0$ [to be determined graphically]

When $t = t_l, y = \left(v_0 + \frac{V}{B_z d}\right)\frac{1}{\omega_c}(1 - \cos\omega_c t_l)$

For the electron to emerge from $x = l$, $t_l < t_0$ and $-\frac{d}{2} < \left(v_0 + \frac{V}{B_z d}\right)\frac{1}{\omega_c}(1 - \cos\omega_c t_l) < +\frac{d}{2}$

$$\Longrightarrow -\frac{\omega_c d}{2\left(v_0 + \frac{V}{B_z d}\right)} < (1 - \cos\omega_c t_l) < +\frac{\omega_c d}{2\left(v_0 + \frac{V}{B_z d}\right)} \Longrightarrow -\frac{\omega_c d}{2\left(v_0 + \frac{V}{B_z d}\right)} - 1 < -\cos\omega_c t_l < \frac{\omega_c d}{2\left(v_0 + \frac{V}{B_z d}\right)} - 1$$

$$\Longrightarrow \frac{\omega_c d}{2\left(v_0 + \frac{V}{B_z d}\right)} + 1 > \cos\omega_c t_l > \frac{\omega_c d}{2\left(v_0 + \frac{V}{B_z d}\right)} - 1 \Longrightarrow \frac{\omega_c d}{2\left(v_0 + \frac{V}{B_z d}\right)} - 1 < \cos\omega_c t_l < \frac{\omega_c d}{2\left(v_0 + \frac{V}{B_z d}\right)} + 1$$

For the electron to emerge from $x = 0$, $t_0 < t_l$ and $-\frac{d}{2} < \left(v_0 + \frac{V}{B_z d}\right)\frac{1}{\omega_c}(1 - \cos\omega_c t_0) < +\frac{d}{2}$

$$\Longrightarrow \frac{\omega_c d}{2\left(v_0 + \frac{V}{B_z d}\right)} - 1 < \cos\omega_c t_0 < \frac{\omega_c d}{2\left(v_0 + \frac{V}{B_z d}\right)} + 1$$

Statistical Ensembles

Microcanonical Ensemble (N, V, E) — System is isolated and cannot exchange energy or particles with a reservoir.
Canonical Ensemble (N, V, T) — System can exchange energy with a reservoir.
Grand Canonical Ensemble (μ, V, T) — System can exchange energy and particles with a reservoir.

Pressure from Energy formula

For a particle of m in a cubical box of side-length l, $E = \dfrac{n^2h^2}{8ml^2} = \dfrac{n^2h^2}{8mV^{2/3}}$ $\left[\text{where } n^2 = {n_x}^2 + {n_y}^2 + {n_y}^2\right]$

Momentum, $p = \sqrt{2mE} = \sqrt{\dfrac{n^2h^2}{4l^2}} = \dfrac{nh}{2l} \Longrightarrow lp = \dfrac{nh}{2} \Longrightarrow (lp)_{\min} = \left(\dfrac{nh}{2}\right)_{\min} = \dfrac{3h}{2} \Longrightarrow lp \geq \dfrac{3h}{2}$

Pressure, $P = -\left(\dfrac{\partial E}{\partial V}\right)_{T,N} = \dfrac{2}{3}\dfrac{n^2h^2}{8mV^{5/3}} = \dfrac{2E}{3V} \Longrightarrow E = \dfrac{3}{2}PV$

Density of States

$g(p)\,dp = \dfrac{V(4\pi p^2\,dp)}{h^3} = \dfrac{4\pi V}{h^3}p^2\,dp$

Put $p = mv \Longrightarrow dp = m\,dv \Longrightarrow p^2\,dp = m^3v^2\,dv$

$g(v)\,dv = \dfrac{4\pi V}{h^3}m^3v^2\,dv$

Put $p = \hbar\kappa \Longrightarrow dp = \hbar\,d\kappa \Longrightarrow p^2\,dp = \hbar^3\kappa^2\,d\kappa$

$g(\kappa)\,d\kappa = \dfrac{V}{2\pi^2}\kappa^2\,d\kappa$ $\left[\because \hbar = \dfrac{h}{2\pi}\right]$

For non-relativistic case, put $p = \sqrt{2m\epsilon}$ $\left[\because \epsilon = \dfrac{p^2}{2m}\right] \Longrightarrow dp = \dfrac{\sqrt{2m}}{2\sqrt{\epsilon}}d\epsilon \Longrightarrow p^2\,dp = \dfrac{2m\epsilon\sqrt{2m}}{2\sqrt{\epsilon}}d\epsilon = \sqrt{2m^3}\sqrt{\epsilon}\,d\epsilon$

$\therefore$ For non-relativistic case, $g(\epsilon)\,d\epsilon = \dfrac{4\pi V}{h^3}\sqrt{2m^3}\sqrt{\epsilon}\,d\epsilon = \dfrac{2\pi V}{h^3}\sqrt{(2m)^3}\sqrt{\epsilon}\,d\epsilon$ [Caution!]

For relativistic case, put $p^2 = \dfrac{\epsilon^2}{c^2} - m^2c^2$ $[\because \epsilon^2 \approx p^2c^2 + m^2c^4] \Longrightarrow 2p\,dp = \dfrac{2\epsilon\,d\epsilon}{c^2} \Longrightarrow p^2\,dp = \dfrac{p\epsilon\,d\epsilon}{c^2}$

$\therefore$ For relativistic case, $g(\epsilon)\,d\epsilon = \dfrac{4\pi V}{h^3c^2}p\epsilon\,d\epsilon$

For ultra-relativistic case, put $p = \dfrac{\epsilon}{c}$ $[\because \epsilon \approx pc] \Longrightarrow dp = \dfrac{d\epsilon}{c} \Longrightarrow p^2\,dp = \dfrac{\epsilon^2 d\epsilon}{c^3}$

$\therefore$ For ultra-relativistic case, $g(\epsilon)\,d\epsilon = \dfrac{4\pi V}{h^3c^3}\epsilon^2 d\epsilon$ [also applicable for photons]

$$\text{Alternatively, } \sum p_i = \frac{V\left(\frac{4\pi}{3}p^3\right)}{h^3} = \frac{4\pi V}{3h^3}p^3 \Longrightarrow \begin{cases} \sum v_i = \dfrac{4\pi V}{3h^3}(mv)^3 \\ \sum \kappa_i = \dfrac{4\pi V}{3h^3}(\hbar\kappa)^3 = \dfrac{V}{6\pi^2}\kappa^3 \\ \sum \epsilon_i = \dfrac{4\pi V}{3h^3}(2m\epsilon)^{\frac{3}{2}} \text{ [non-relativistic case]} \\ \sum \epsilon_i = \dfrac{4\pi V}{3h^3c^3}\epsilon^3 \text{ [ultra-relativistic case]} \end{cases}$$

$g(p)\,dp = \dfrac{d\sum p_i}{dp}dp = \dfrac{4\pi V}{h^3}p^2\,dp$

$g(v)\,dv = \dfrac{d\sum v_i}{dv}dv = \dfrac{4\pi V}{h^3}m^3v^2\,dv$

$g(\kappa)\,d\kappa = \dfrac{d\sum \kappa_i}{d\kappa}d\kappa = \dfrac{V}{2\pi^2}\kappa^2\,d\kappa$

$g(\epsilon)\,d\epsilon = \dfrac{d\sum \epsilon_i}{d\epsilon}d\epsilon = \dfrac{2\pi V}{h^3}(\mathbf{2m})^{\frac{\mathbf{3}}{\mathbf{2}}}\epsilon^{\frac{1}{2}}\,d\epsilon = \dfrac{4\pi V}{h^3}(\mathbf{2m^3})^{\frac{\mathbf{1}}{\mathbf{2}}}\epsilon^{\frac{1}{2}}\,d\epsilon$ [Caution!] [non-relativistic case]

$g(\epsilon)\,d\epsilon = \dfrac{d\sum \epsilon_i}{d\epsilon}d\epsilon = \dfrac{4\pi V}{h^3c^3}\epsilon^2 d\epsilon$ [ultra-relativistic case]

A (spin) degeneracy factor, g_s is multiplied to obtain the density of states in the presence of degeneracy.
A circularly polarized light may be right-handed/clockwise or left-handed/anti-clockwise. A linearly polarized light may be horizontal or vertical. So, for photons, $g_s = 2$
Electrons (charged leptons) have two spin states $\left(+\frac{1}{2} \text{ and } -\frac{1}{2}\right)$. So, for electrons, $g_s = 2$
For neutrinos (charged leptons), $g_s = 1$ whereas, for quarks, $g_s = 6$
In general, for spin-degeneracy, $g_s = 2s + 1$ [s is the spin of the particle species.]

Maxwell-Boltzmann Distribution (for identical distinguishable classical particles in a microcanonical ensemble)
Consider a system (microcanonical ensemble) partitioned into k compartments (macrostates), each compartment representing a particular energy value, ϵ_i. If the total number of particles is N, and there are n_i particles in the ith compartment, then the distribution among the compartments is realized as follows.

The number of ways of choosing n_1 particles from N particles and placing them in 1st compartment is ${}^{N}C_{n_1}$. The number of ways of choosing n_2 from a total of $(N - n_1)$ particles and placing them in 2nd compartment is ${}^{(N-n_1)}C_{n_2}$. The number of ways of choosing n_3 from a total of $(N - n_1 - n_2)$ particles and placing them in 2nd compartment is ${}^{(N-n_1-n_2)}C_{n_3}$. This continues till every particle is placed in a compartment. Therefore, the number of ways to realize the distribution is $w = {}^{N}C_{n_1} \times {}^{(N-n_1)}C_{n_2} \times {}^{(N-n_1-n_2)}C_{n_3} \times \cdots \times {}^{(N-n_1-n_2-\cdots-n_{k-1})}C_{n_k}$

$$= \frac{N!}{n_1!\,(N-n_1)!} \times \frac{(N-n_1)!}{n_2!\,(N-n_1-n_2)!} \times \frac{(N-n_1-n_2)!}{n_3!\,(N-n_1-n_2-n_3)!} \times \cdots \times \frac{(N-n_1-n_2-\cdots-n_{k-1})!}{n_k!\,(N-n_1-n_2-\cdots-n_{k-1}-n_k)!}$$

$$= \frac{N!}{n_1!\,n_2!\,n_3!\cdots n_k!\,(N-n_1-n_2-\cdots-n_{k-1}-n_k)!} = \frac{N!}{\prod n_i!\,(N-\sum n_i)!} = \frac{N!}{\prod n_i!} \quad \left[\because \sum_{i=1}^{k} n_i = N\right]$$

One can also directly infer this result by realizing that N particles can be inter-arranged *in* $N!$ ways, and within the ith compartment n_i can be inter-arranged in $n_i!$ ways.
If g_i is the number of cells (microstates) in the ith compartment i.e. the degeneracy of the ith compartment, the number of ways of distributing n_i particles among g_i cells is ${g_i}^{n_i}$ as each particle is free to occupy any of the g_i cells. $\therefore W = \left(\prod_{i=1}^{k} {g_i}^{n_i}\right) w = N! \frac{\prod {g_i}^{n_i}}{\prod n_i!}$
This expression for W, however, does not yield an extensive entropy S which is $S = k_B \ln W$. This is because while deriving it, distinguishability of the particles was not entirely taken into account.

To determine n_i for the most-probable distribution, we maximize W, or rather $\ln W$.

$$\ln W = \sum_{i=1}^{k} n_i \ln g_i - \sum_{i=1}^{k} \ln\, n_i! \approx \sum_{i=1}^{k} n_i \ln g_i - \sum_{i=1}^{k} (n_i \ln\, n_i - n_i)$$

$$\therefore \frac{\partial \ln W}{\partial n_i} = \sum_{i=1}^{k} \ln g_i - \sum_{i=1}^{k} \{(\ln n_i + 1) - 1\} = \sum_{i=1}^{k} \ln g_i - \sum_{i=1}^{k} \ln n_i = \sum_{i=1}^{k} \ln \frac{g_i}{n_i}$$

For $\ln W$ to be maximum, $\frac{\partial \ln W}{\partial n_i} = 0 = \alpha \frac{\partial N}{\partial n_i} + \beta \frac{\partial E}{\partial n_i} \quad \left[\because \frac{\partial N}{\partial n_i} = 0 \text{ and } \frac{\partial E}{\partial n_i} = 0\right]$

$$\Rightarrow \sum_{i=1}^{k} \ln \frac{g_i}{n_i} = \alpha \sum_{i=1}^{k} 1 + \beta \sum_{i=1}^{k} \epsilon_i \quad \left[\because \sum_{i=1}^{k} n_i = N \text{ and } \sum_{i=1}^{k} n_i \epsilon_i = E\right] = \sum_{i=1}^{k} (\alpha + \beta \epsilon_i)$$

$$\Rightarrow \ln \frac{g_i}{n_i} = \alpha + \beta \epsilon_i \Rightarrow \frac{g_i}{n_i} = e^{\alpha + \beta \epsilon_i} \Rightarrow \frac{n_i}{g_i} = \frac{1}{e^{\alpha + \beta \epsilon_i}} \Rightarrow n_i = \frac{g_i}{e^{\alpha + \beta \epsilon_i}} = g_i f(\epsilon_i) \quad \text{where } f(\epsilon_i) = \frac{1}{e^{\alpha + \beta \epsilon_i}}$$

$$\ln W = \sum_{i=1}^{k} n_i \left(\ln \frac{g_i}{n_i} + 1\right) = \sum_{i=1}^{k} n_i (\alpha + \beta \epsilon_i + 1) \quad \left[\because \ln \frac{g_i}{n_i} = \alpha + \beta \epsilon_i\right] = \alpha N + \beta E + N \Rightarrow E = \frac{\ln W}{\beta} - \frac{N}{\beta} - \frac{\alpha N}{\beta}$$

Comparing with the Euler-integrated fundamental equation of thermodynamics, $U = TS - PV + \mu N$, it can be identified that $E = U$ (internal energy), $\beta = \dfrac{1}{kT}$ $[\because PV = NkT]$, $S = k\ln W$ $\left[\because \beta = \dfrac{1}{kT}\right]$ and $\alpha = -\dfrac{\mu}{kT} = -\beta\mu$

$$\therefore n_i = \frac{g_i}{e^{(\epsilon_i-\mu)/kT}} = \frac{g_i}{e^{\beta(\epsilon_i-\mu)}} \quad [\text{for discrete energy}] \Longrightarrow n(\epsilon)\,d\epsilon = \frac{g(\epsilon)\,d\epsilon}{e^{(\epsilon-\mu)/kT}} = \frac{g(\epsilon)\,d\epsilon}{e^{\beta(\epsilon-\mu)}} \quad [\text{for continuous energy}]$$

$$N = \sum_{i=1}^{k} n_i = \sum_{i=1}^{k} \frac{g_i}{e^{\alpha+\beta\epsilon_i}} \approx \int_0^\infty \frac{g(\epsilon)\,d\epsilon}{e^{\alpha+\beta\epsilon}} = \frac{4\pi V}{h^3}\sqrt{2m^3}\,e^{-\alpha}\int_0^\infty \sqrt{\epsilon}\,e^{-\beta\epsilon}\,d\epsilon \quad \left[\because g(\epsilon)\,d\epsilon = \frac{4\pi V}{h^3}\sqrt{2m^3}\sqrt{\epsilon}\,d\epsilon\right]$$

$$= \frac{4\pi V}{h^3}\sqrt{2m^3}\,e^{-\alpha}\int_0^\infty 2x^2\,e^{-\beta x^2}\,dx \quad \begin{bmatrix}\text{Putting } \epsilon = x^2 \\ \Longrightarrow d\epsilon = 2x\,dx\end{bmatrix} = \frac{4\pi V}{h^3}\sqrt{2m^3}\,e^{-\alpha}\left(\frac{1}{2}\sqrt{\frac{\pi}{\beta^3}}\right) \quad \left[\because \int_0^\infty x^2\,e^{-\beta x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta^3}}\right]$$

$$\Longrightarrow e^{\alpha} = \frac{2\pi V}{h^3 N}\sqrt{2m^3}\sqrt{\frac{\pi}{\beta^3}} = \frac{V}{N}\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}} = \frac{V}{N\lambda^3} \quad \left[\lambda = \left(\frac{h^2\beta}{2\pi m}\right)^{\frac{1}{2}}\right] \Longrightarrow e^{-\alpha} = \frac{h^3 N}{2\pi V}\frac{1}{\sqrt{2m^3}}\sqrt{\frac{\beta^3}{\pi}} = \frac{\lambda^3 N}{V} = n\lambda^3$$

$$E = \sum_{i=1}^{k} n_i\epsilon_i = \sum_{i=1}^{k} \frac{g_i\epsilon_i}{e^{\alpha+\beta\epsilon_i}} \approx \int_0^\infty \frac{\epsilon\,g(\epsilon)\,d\epsilon}{e^{\alpha+\beta\epsilon}} = \frac{4\pi V}{h^3}\sqrt{2m^3}\,e^{-\alpha}\int_0^\infty \epsilon\sqrt{\epsilon}\,e^{-\beta\epsilon}\,d\epsilon \quad \left[\because g(\epsilon)\,d\epsilon = \frac{4\pi V}{h^3}\sqrt{2m^3}\sqrt{\epsilon}\,d\epsilon\right]$$

$$= \frac{4\pi V}{h^3}\sqrt{2m^3}\,e^{-\alpha}\int_0^\infty 2x^4\,e^{-\beta x^2}\,dx \quad \begin{bmatrix}\text{Putting } \epsilon = x^2 \\ \Longrightarrow d\epsilon = 2x\,dx\end{bmatrix} = \frac{4\pi V}{h^3}\sqrt{2m^3}\,e^{-\alpha}\left(\frac{3}{4}\sqrt{\frac{\pi}{\beta^5}}\right) \quad \left[\because \int_0^\infty x^4\,e^{-\beta x^2}\,dx = \frac{3}{8}\sqrt{\frac{\pi}{\beta^5}}\right]$$

$$\Longrightarrow E = \frac{3N}{2\beta} \Longrightarrow \frac{3}{2}NkT = \frac{3N}{2\beta} \quad \left[\because E = \frac{3}{2}NkT\right] \Longrightarrow \beta = \frac{1}{kT}$$

$\therefore \boldsymbol{\lambda = \left(\dfrac{h^2\beta}{2\pi m}\right)^{\frac{1}{2}} = \sqrt{\dfrac{h^2}{2\pi mkT}} = \dfrac{h}{\sqrt{2\pi mkT}}}$ which is the **thermal de Broglie wavelength**.

$$e^{\alpha} = \frac{V}{N}\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}} = \frac{V}{N}\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}} = \frac{V}{N\lambda^3} = \frac{1}{n\lambda^3} \Longrightarrow \alpha = \ln\left(\frac{V}{N}\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}}\right) = \ln\left(\frac{V}{N}\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}}\right) = \ln\left(\frac{V/N}{\lambda^3}\right)$$

$$\Longrightarrow \alpha = \frac{3}{2}\ln\left(\frac{2\pi mkT}{h^2}\left(\frac{V}{N}\right)^{\frac{2}{3}}\right) = \frac{3}{2}\ln\frac{T}{T_0} \quad \left[T_0 = \frac{h^2}{2\pi mk}\left(\frac{N}{V}\right)^{\frac{2}{3}} = \boldsymbol{\frac{h^2 n^{\frac{2}{3}}}{2\pi mk}}\right] \Longrightarrow -\frac{\mu}{kT} = \frac{3}{2}\ln\frac{T}{T_0} \Longrightarrow \boldsymbol{\mu = \frac{3}{2}kT\ln\frac{T_0}{T}}$$

Maxwell Energy Distribution

$$n(\epsilon)\,d\epsilon = \frac{g(\epsilon)\,d\epsilon}{e^{(\epsilon-\mu)/kT}} = g(\epsilon)e^{(\mu-\epsilon)/kT}\,d\epsilon = e^{-\alpha}e^{-\epsilon/kT}g(\epsilon)\,d\epsilon = (n\lambda^3)e^{-\epsilon/kT}\left(\frac{2\pi V}{h^3}(\mathbf{2m})^{\frac{3}{2}}\epsilon^{\frac{1}{2}}\,d\epsilon\right) \quad [\because e^{-\alpha} = n\lambda^3]$$

$$= \left(\frac{N}{V}\left(\frac{h^2}{2\pi mkT}\right)^{\frac{3}{2}}\right)\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\epsilon^{\frac{1}{2}}e^{-\epsilon/kT}\,d\epsilon \Longrightarrow \boldsymbol{n(\epsilon)\,d\epsilon = 2\pi N\left(\frac{1}{\pi kT}\right)^{\frac{3}{2}}\epsilon^{\frac{1}{2}}e^{-\epsilon/kT}\,d\epsilon}$$

Maxwell Speed Distribution

$$n(\epsilon)\,d\epsilon = \frac{g(\epsilon)\,d\epsilon}{e^{(\epsilon-\mu)/kT}} = g(\epsilon)e^{(\mu-\epsilon)/kT}\,d\epsilon = e^{-\alpha}e^{-\epsilon/kT}g(p)\,dp \Longrightarrow n(p)\,dp = (n\lambda^3)e^{-\frac{p^2}{2mkT}}\left(\frac{4\pi V}{h^3}p^2\,dp\right)$$

$$\Longrightarrow n(p)\,dp = (n\lambda^3)\frac{4\pi V}{h^3}p^2e^{-\frac{p^2}{2mkT}}\,dp \Longrightarrow n(v)\,dv = \left(\frac{N}{V}\left(\frac{h^2}{2\pi mkT}\right)^{\frac{3}{2}}\right)\frac{4\pi V}{h^3}m^3v^2e^{-\frac{mv^2}{2kT}}\,dv$$

$$\Longrightarrow n(v)\,dv = 4\pi N\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}v^2e^{-\frac{mv^2}{2kT}}\,dv \Longrightarrow \frac{n(v)\,dv}{N} = 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}v^2e^{-\frac{mv^2}{2kT}}\,dv$$

which gives the fraction of molecules lying in the speed range v and $v + dv$.

$\therefore \boldsymbol{\dfrac{n(v)}{N} = 4\pi\left(\dfrac{m}{2\pi kT}\right)^{\frac{3}{2}}v^2e^{-\frac{mv^2}{2kT}}}$ is the probability that a molecule will possess a speed v.

Most Probable Speed, Mean Speed and Root Mean Square Speed

$$F(v)\,dv = 4\pi N\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} v^2 e^{-\frac{mv^2}{2kT}}\,dv$$

Most probable speed: $\left.\frac{dF}{dv}\right|_{v=v_{mps}} = 0 \Rightarrow \frac{d}{dv}\left[4\pi N\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} v^2 e^{-\frac{mv^2}{2kT}}\right]\Bigg|_{v=v_{mps}} = 0 \Rightarrow \frac{d}{dv}\left[v^2 e^{-\frac{mv^2}{2kT}}\right]\Bigg|_{v=v_{mps}} = 0$

$$\Rightarrow 2ve^{-\frac{mv^2}{2kT}} - 2v^3\left(\frac{m}{2kT}\right)e^{-\frac{mv^2}{2kT}}\Bigg|_{v=v_{mps}} = 0 \Rightarrow \frac{m{v_{mps}}^2}{2kT} = 1 \Rightarrow {v_{mps}}^2 = \frac{2kT}{m} \Rightarrow \boldsymbol{v_{mps}} = \sqrt{\frac{\boldsymbol{2kT}}{\boldsymbol{m}}} \approx 1.414\sqrt{\frac{kT}{m}}$$

As T decreases and/or m increases, the most probable speed decreases.

$$\frac{d^2F}{dv^2} = 4\pi N\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\frac{d^2}{dv^2}\left[v^2 e^{-\frac{mv^2}{2kT}}\right] = 4\pi N\left(\frac{m}{2\pi kT}\right)\frac{d}{dv}\left[2ve^{-\frac{mv^2}{2kT}} - 2v^3\left(\frac{m}{2kT}\right)e^{-\frac{mv^2}{2kT}}\right]$$

$$= 4\pi N\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\left[2e^{-\frac{mv^2}{2kT}} - 4v^2\left(\frac{m}{2kT}\right)e^{-\frac{mv^2}{2kT}} - 6v^2\left(\frac{m}{2kT}\right)e^{-\frac{mv^2}{2kT}} + 4v^4\left(\frac{m}{2kT}\right)^2 e^{-\frac{mv^2}{2kT}}\right]$$

$$= 4\pi N\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\left[2 - 10v^2\left(\frac{m}{2kT}\right) + 4v^4\left(\frac{m}{2kT}\right)^2\right]e^{-\frac{mv^2}{2kT}}$$

$$\left.\frac{d^2F}{dv^2}\right|_{v=v_{mps}} = 4\pi N\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}[2 - 10 + 4]e^{-1} = -16\pi N\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}e^{-1}$$

As T decreases and/or m increases, the peak of the curve $F(v)\,dv$ vs v gets narrower as 2nd derivative becomes more negative.

Mean speed: $\boldsymbol{\bar{v}} = \frac{1}{N}\int_0^\infty F(v)v\,dv = 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\int_0^\infty v^3 e^{-\frac{mv^2}{2kT}}\,dv$

$$= 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\left[\frac{1}{2}\left(\frac{m}{2kT}\right)^{-2}\right] = \sqrt{\frac{\boldsymbol{8}}{\boldsymbol{\pi}}\frac{\boldsymbol{kT}}{\boldsymbol{m}}} \approx 1.596\sqrt{\frac{kT}{m}} \quad \left[\because \int_0^\infty x^3 e^{-\alpha x^2}dx = \frac{1}{2\alpha^2}\right]$$

Mean square speed: $\overline{v^2} = \frac{1}{N}\int_0^\infty F(v)v^2\,dv = 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\int_0^\infty v^4 e^{-\frac{mv^2}{2kT}}\,dv$

$$= 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\left[\frac{3}{8}\pi^{\frac{1}{2}}\left(\frac{m}{2kT}\right)^{-\frac{5}{2}}\right] = \frac{3kT}{m} \quad \left[\because \int_0^\infty x^4 e^{-\alpha x^2}dx = \frac{3}{8}\sqrt{\frac{\pi}{\alpha^5}}\right]$$

Root mean square speed: $\boldsymbol{v_{rms}} = \sqrt{\overline{v^2}} = \sqrt{\frac{\boldsymbol{3kT}}{\boldsymbol{m}}} \approx 1.732\sqrt{\frac{kT}{m}}$

$\therefore v_{rms} > \bar{v} > v_{mps}$

The area under the Maxwell-Boltzmann velocity distribution curve between two speeds gives the fraction of gas particles in that range of speeds. Determine the fraction of Hydrogen atoms in a gas of T = 10,000 K having speeds between 2 × 10^4 m/s and 2.5 × 10^4 m/s.

$$n(v)\,dv = 4\pi N\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-\frac{mv^2}{2kT}}\,dv \Rightarrow \frac{N_{[v_1,v_2]}}{N} = 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\int_{v_1}^{v_2} v^2 e^{-\frac{mv^2}{2kT}}\,dv \approx 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\bar{v}^2 e^{-\frac{m\bar{v}^2}{2kT}}(v_2 - v_1)$$

$\Rightarrow \frac{N_{[v_1,v_2]}}{N} \approx 0.12534$ i.e. approximately 12.5% of H atoms in the gas have speeds within the range at 10,000 K

Quantum Statistics (for identical indistinguishable quantum particles)

The number of ways to realize the distribution of N indistinguishable particles among k compartments is $w = 1 \times 1 \times 1 \times \cdots \times 1$ (k times) $= 1$

Bose-Einstein Distribution (for particles with integral spin and symmetric wavefunction)

If g_i is the number of cells (microstates) in the ith compartment i.e. the degeneracy of the ith compartment, the number of ways of distributing n_i Bosons among g_i cells is $\dfrac{(n_i+g_i-1)!}{n_i!\,(g_i-1)!}$ in which $n_i!$ and $(g_i-1)!$ are the undesired inter-arrangements of n_i Bosons and (g_i-1) dividers respectively. $\therefore W=\displaystyle\prod_{i=1}^{k}\frac{(n_i+g_i-1)!}{n_i!\,(g_i-1)!}$

Maxwell-Boltzmann Distribution follows from Bose-Einstein Distribution for temperatures much above absolute zero ($g_i \gg 1$) and very low density ($g_i \gg n_i$).

$$W=\prod_{i=1}^{k}\frac{(n_i+g_i-1)!}{n_i!\,(g_i-1)!}\approx\prod_{i=1}^{k}\frac{(n_i+g_i)!}{n_i!\,g_i!}\approx\prod_{i=1}^{k}\frac{(n_i+g_i)^{(n_i+g_i)}e^{-(n_i+g_i)}}{(n_i{}^{n_i}e^{-n_i})(g_i{}^{g_i}e^{-g_i})}\begin{bmatrix}\text{using Stirling's}\\ \text{approximation}\end{bmatrix}=\prod_{i=1}^{k}\frac{(n_i+g_i)^{(n_i+g_i)}}{n_i{}^{n_i}g_i{}^{g_i}}$$

$$=\prod_{i=1}^{k}\frac{g_i{}^{(n_i+g_i)}\left(\frac{n_i}{g_i}+1\right)^{n_i\left(\frac{g_i}{n_i}+1\right)}}{n_i{}^{n_i}g_i{}^{g_i}}=\prod_{i=1}^{k}\frac{g_i{}^{n_i}\left(\left(1+\frac{1}{g_i/n_i}\right)^{\left(\frac{g_i}{n_i}+1\right)}\right)}{n_i{}^{n_i}}\approx\prod_{i=1}^{k}\frac{g_i{}^{n_i}e^{n_i}}{n_i{}^{n_i}}\begin{bmatrix}\because \lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e\\ \text{and } \frac{g_i}{n_i}\gg 1\end{bmatrix}$$

$$=\prod_{i=1}^{k}\frac{g_i{}^{n_i}}{n_i{}^{n_i}e^{-n_i}}\approx\prod_{i=1}^{k}\frac{g_i{}^{n_i}}{n_i!}\quad\begin{bmatrix}\text{using Stirling's}\\ \text{approximation}\end{bmatrix}$$

To determine n_i for the most-probable distribution, we maximize W, or rather $\ln W$.

$$\ln W=\sum_{i=1}^{k}\ln\,(n_i+g_i-1)!-\sum_{i=1}^{k}\ln\,n_i!-\sum_{i=1}^{k}\ln\,(g_i-1)!$$

$$\approx\sum_{i=1}^{k}\{(n_i+g_i-1)\ln\,(n_i+g_i-1)-(n_i+g_i-1)\}-\sum_{i=1}^{k}(n_i\ln n_i-n_i)-\sum_{i=1}^{k}\{(g_i-1)\ln\,(g_i-1)-(g_i-1)\}$$

$$=\sum_{i=1}^{k}(n_i+g_i-1)\ln\,(n_i+g_i-1)-\sum_{i=1}^{k}n_i\ln n_i-\sum_{i=1}^{k}(g_i-1)\ln\,(g_i-1)$$

$$\approx\sum_{i=1}^{k}(n_i+g_i)\ln\,(n_i+g_i)-\sum_{i=1}^{k}n_i\ln n_i-\sum_{i=1}^{k}g_i\ln\,g_i\quad[\because g_i\gg 1]$$

$$\therefore\frac{\partial\ln W}{\partial n_i}=\sum_{i=1}^{k}\{\ln\,(n_i+g_i)+1\}-\sum_{i=1}^{k}(\ln n_i+1)=\sum_{i=1}^{k}\ln\,(n_i+g_i)-\sum_{i=1}^{k}\ln n_i=\sum_{i=1}^{k}\ln\,\left(1+\frac{g_i}{n_i}\right)$$

For $\ln W$ to be maximum, $\dfrac{\partial\ln W}{\partial n_i}=0=\alpha\dfrac{\partial N}{\partial n_i}+\beta\dfrac{\partial E}{\partial n_i}\quad\left[\because\dfrac{\partial N}{\partial n_i}=0\text{ and }\dfrac{\partial E}{\partial n_i}=0\right]$

$$\Rightarrow\sum_{i=1}^{k}\ln\,\left(1+\frac{g_i}{n_i}\right)=\alpha\sum_{i=1}^{k}1+\beta\sum_{i=1}^{k}\epsilon_i\quad\left[\because\sum_{i=1}^{k}n_i=N\text{ and }\sum_{i=1}^{k}n_i\epsilon_i=E\right]=\sum_{i=1}^{k}(\alpha+\beta\epsilon_i)$$

$$\Rightarrow\ln\left(1+\frac{g_i}{n_i}\right)=\alpha+\beta\epsilon_i\Rightarrow 1+\frac{g_i}{n_i}=e^{\alpha+\beta\epsilon_i}\Rightarrow\frac{g_i}{n_i}=e^{\alpha+\beta\epsilon_i}-1\Rightarrow\frac{n_i}{g_i}=\frac{1}{e^{\alpha+\beta\epsilon_i}-1}\Rightarrow n_i=\frac{g_i}{e^{\alpha+\beta\epsilon_i}-1}$$

$$=f_{\mathrm{BE}}(\epsilon_i)g_i\quad\begin{bmatrix}f_{\mathrm{BE}}(\epsilon_i)=\dfrac{1}{e^{\alpha+\beta\epsilon_i}-1}=\dfrac{1}{e^{(\epsilon_i-\mu)/kT}-1}\\ \text{is called the BE occupancy function/}\\ \text{BE distribution function.}\end{bmatrix}\Rightarrow n_i=\frac{g_i}{e^{(\epsilon_i-\mu)/kT}-1}\quad[\text{for discrete energy}]=f_{\mathrm{BE}}(\epsilon_i)g_i$$

$$\Rightarrow n(\epsilon)\,d\epsilon=\frac{g(\epsilon)\,d\epsilon}{e^{(\epsilon-\mu)/kT}-1}\quad[\text{for continuous energy}]=f_{\mathrm{BE}}(\epsilon)g(\epsilon)\,d\epsilon\quad[g(\epsilon)\,d\epsilon\text{ is called the density of states.}]$$

$$\ln W=\sum_{i=1}^{k}n_i\left(1+\frac{g_i}{n_i}\right)\ln\,n_i\left(1+\frac{g_i}{n_i}\right)-\sum_{i=1}^{k}n_i\ln n_i-\sum_{i=1}^{k}n_i\left(\frac{g_i}{n_i}\right)\ln\,n_i\left(\frac{g_i}{n_i}\right)$$

$$= \sum_{i=1}^{k} n_i \left(1+\frac{g_i}{n_i}\right)\left\{\ln\left(1+\frac{g_i}{n_i}\right)+\ln n_i\right\} - \sum_{i=1}^{k} n_i \ln n_i - \sum_{i=1}^{k} n_i \left(\frac{g_i}{n_i}\right)\left\{\ln\left(\frac{g_i}{n_i}\right)+\ln n_i\right\}$$

$$= \sum_{i=1}^{k} n_i \left(1+\frac{g_i}{n_i}\right)\ln\left(1+\frac{g_i}{n_i}\right) - \sum_{i=1}^{k} n_i \left(\frac{g_i}{n_i}\right)\ln\left(\frac{g_i}{n_i}\right)$$

$$N = \sum_{i=1}^{k} n_i = \sum_{i=1}^{k} \frac{g_i}{e^{\alpha+\beta\epsilon_i}-1} \approx \int_0^\infty \frac{g(\epsilon)\,d\epsilon}{e^{\alpha+\beta\epsilon}-1} = g_s \frac{4\pi V}{h^3}\sqrt{2m^3}\int_0^\infty \frac{\sqrt{\epsilon}\;d\epsilon}{e^{\alpha}e^{\beta\epsilon}-1} \quad \left[\because g(\epsilon)\,d\epsilon = g_s\frac{4\pi V}{h^3}\sqrt{2m^3}\sqrt{\epsilon}\,d\epsilon\right]$$

$$= g_s \frac{4\pi V}{h^3}\frac{\sqrt{2m^3}}{\beta^{\frac{3}{2}}}\int_0^\infty \frac{\sqrt{x}\;dx}{e^{-\beta\mu}e^{x}-1} \quad \left[\begin{matrix}\text{Putting } x=\beta\epsilon \Longrightarrow d\epsilon = dx/\beta \\ \sqrt{\epsilon}=\sqrt{x}/\sqrt{\beta}\end{matrix}\right] = g_s\frac{2\pi V}{h^3}\left(\frac{2m}{\beta}\right)^{\frac{3}{2}}\int_0^\infty \frac{\sqrt{x}\;dx}{e^{-\beta\mu}e^{x}-1}$$

$$= \frac{2g_sV}{\pi^{\frac{1}{2}}h^3}(2\pi mkT)^{\frac{3}{2}}\int_0^\infty \frac{\sqrt{x}\;dx}{\frac{1}{\eta_a}e^{x}-1} \quad \left[\begin{matrix}\text{where } \eta_a = e^{\beta\mu} \\ \eta_a \le 1 \quad \because \mu \le 0\end{matrix}\right] = \frac{2g_sV}{\sqrt{\pi}\lambda^3}\int_0^\infty \frac{\sqrt{x}\;dx}{\frac{1}{\eta_a}e^{x}-1} \quad \left[\because \lambda = \frac{h}{\sqrt{2\pi mkT}}\right]$$

$$= \frac{2g_sV}{\sqrt{\pi}\lambda^3}\int_0^\infty \frac{\sqrt{x}\;dx}{\frac{e^x}{\eta_a}(1-\eta_a e^{-x})} = \frac{2g_sV}{\sqrt{\pi}\lambda^3}\int_0^\infty \eta_a\sqrt{x}e^{-x}(1-\eta_a e^{-x})^{-1}\;dx$$

$$= \frac{2g_sV}{\sqrt{\pi}\lambda^3}\int_0^\infty \eta_a\sqrt{x}e^{-x}(1+\eta_a e^{-x}+{\eta_a}^2e^{-2x}+\cdots)\;dx$$

$$= \frac{4g_sV}{\sqrt{\pi}\lambda^3}\int_0^\infty \eta_a y^2e^{-y^2}\left(1+\eta_a e^{-y^2}+{\eta_a}^2e^{-2y^2}+\cdots\right)\;dy \quad \left[\begin{matrix}\text{Putting } x=y^2 \\ \Longrightarrow dx = 2y\,dy\end{matrix}\right]$$

$$= \frac{4g_sV}{\sqrt{\pi}\lambda^3}\int_0^\infty \left(\eta_a y^2e^{-y^2}+{\eta_a}^2y^2e^{-2y^2}+{\eta_a}^3y^2e^{-3y^2}+\cdots\right)\;dy$$

$$= \frac{4g_sV}{\sqrt{\pi}\lambda^3}\left(\frac{\eta_a}{4}\sqrt{\pi}+\frac{{\eta_a}^2}{4}\sqrt{\frac{\pi}{2^3}}+\frac{{\eta_a}^3}{4}\sqrt{\frac{\pi}{3^3}}+\cdots\right) \quad \left[\because \int_0^\infty y^2e^{-ay^2}\,dy = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}\right] = \frac{g_sV}{\lambda^3}\left(\eta_a+\frac{{\eta_a}^2}{2^{3/2}}+\frac{{\eta_a}^3}{3^{3/2}}+\cdots\right)$$

$$\Longrightarrow N = \frac{g_sV}{\lambda^3}F_{3/2}(\eta_a) \quad \left[\text{where } F_{3/2}(\eta_a) = \sum_{j=1}^{\infty}\frac{{\eta_a}^j}{j^{3/2}} \text{ is a polylogarithm function.}\right]$$

When $\eta_a = 1$ i.e. $\mu = 0$, $F_{3/2}(\eta_a) = \zeta\left(\frac{3}{2}\right) \approx 2.6124$ $\left[\text{where } \zeta\left(\frac{3}{2}\right) = \sum_{j=1}^{\infty}\frac{1}{j^{3/2}} \text{ is a Riemann zeta function.}\right]$

When $\eta_a = 1$ i.e. $\mu = 0$, $\lambda = \lambda_C$ and $T = T_C$, then $N \approx \frac{V}{{\lambda_C}^3}(2.6124) \Longrightarrow N = \frac{V}{h^3}(2\pi mkT_C)^{\frac{3}{2}}(2.6124)$

$$\Longrightarrow T_C = \frac{h^2}{2\pi mk}\left(\frac{n/g_s}{2.6124}\right)^{\frac{2}{3}} \quad \left[\text{where } n = \frac{N}{V}\right] \approx \frac{114.6\text{ K}}{M{V_M}^{\frac{2}{3}}{g_s}^{\frac{2}{3}}} \quad \left[\begin{matrix}\text{where } M = mN_A \text{ is the gram-molecular weight} \\ \text{and } V_M \text{ is the molar volume } (\text{cm}^3\text{mol}^{-1})\end{matrix}\right]$$

$$\left[\text{Calculation: } \frac{h^2N_A\cdot 10^3}{2\pi k}\left(\frac{N_A 10^6}{2.6124}\right)^{\frac{2}{3}} \approx 114.6\right]$$

Degeneracy for molecular Hydrogen

$$\eta_a \approx \frac{\lambda^3N}{g_sV} = \frac{h^3}{(2\pi mkT)^{\frac{3}{2}}}\cdot\frac{N}{g_sV}$$

For one mole of molecular Hydrogen, $\begin{cases} N = N_A \approx 6.022 \times 10^{22} \\ m = 2u \approx 2 \times 1.66 \times 10^{-27}\ \text{kg} = 3.32 \times 10^{-27}\ \text{kg} \\ V = 1.493 \times 10^{-3}\ \text{m}^3 \\ T \approx 20.37\ \text{K} \quad \text{(boiling point)} \end{cases}$

Molecular Hydrogen has 2 spin isomers — para (antiparallel proton spins) and ortho (parallel proton spins).

For paraform, $g_s = 2s + 1 = 2 \cdot 0 + 1 = 1$ [singlet state] $\therefore \eta_{a_{\text{para}}} \approx 8.258 \times 10^{-3}$

For orthoform, $g_s = 2s + 1 = 2 \cdot 1 + 1 = 3$ [triplet state] $\therefore \eta_{a_{\text{ortho}}} \approx 2.753 \times 10^{-3}$

$$P_{\text{para}} \approx \frac{RT}{V}\left(1 - \frac{\eta_{a_{\text{para}}}}{2^{5/2}}\right) \approx 0.99854\frac{RT}{V} \approx 1.13269 \times 10^5\ \text{Pa} \approx 1.118\ \text{atm}$$

$$P_{\text{ortho}} \approx \frac{RT}{V}\left(1 - \frac{\eta_{a_{\text{para}}}}{2^{5/2}}\right) \approx 0.99951\frac{RT}{V} \approx 1.113378 \times 10^5\ \text{Pa} \approx 1.119\ \text{atm}$$

Degeneracy for Helium-4

For one mole of Helium-4, $\begin{cases} N = N_A \approx 6.022 \times 10^{22} \\ m = 4u \approx 4 \times 1.66 \times 10^{-27}\ \text{kg} = 6.64 \times 10^{-27}\ \text{kg} \\ V = 0.345 \times 10^{-3}\ \text{m}^3 \\ T \approx 4.22\ \text{K} \quad \text{(boiling point)} \end{cases}$

$g_s = 2s + 1 = 2 \cdot 0 + 1 = 1$ [singlet state] $\therefore \eta_a \approx 0.134$

$$P \approx \frac{RT}{V}\left(1 - \frac{\eta_a}{2^{5/2}}\right) \approx 0.97631\frac{RT}{V} \approx 0.99287 \times 10^5\ \text{Pa} \approx 0.98\ \text{atm}$$

Fermi-Dirac Distribution (for particles with half-integral spin and anti-symmetric wavefunction)

If g_i is the number of cells (microstates) in the ith compartment i.e. the degeneracy of the ith compartment, the number of ways of distributing n_i Fermions among g_i cells, keeping in mind that one cell can be occupied by only one Fermion (as mandated by Pauli's exclusion principle), is $\dfrac{g_i(g_i - 1)(g_i - 2)\cdots(g_i - n_i + 1)}{n_i!}$

in which $n_i!$ is the undesired inter-arrangements of n_i Fermions. $\therefore W = \displaystyle\prod_{i=1}^{k} \frac{g_i!}{n_i!\,(g_i - n_i)!}$

Maxwell-Boltzmann Distribution follows from Fermi-Dirac Distribution for temperatures much above absolute zero ($g_i \gg 1$) and very low density ($g_i \gg n_i$).

$$W = \prod_{i=1}^{k} \frac{g_i!}{n_i!\,(g_i - n_i)!} \approx \prod_{i=1}^{k} \frac{{g_i}^{g_i} e^{-g_i}}{({n_i}^{n_i} e^{-n_i})(g_i - n_i)^{(g_i - n_i)} e^{-(g_i - n_i)}} \begin{bmatrix}\text{using Stirling's} \\ \text{approximation}\end{bmatrix} = \prod_{i=1}^{k} \frac{{g_i}^{g_i}}{{n_i}^{n_i}(g_i - n_i)^{(g_i - n_i)}}$$

$$= \prod_{i=1}^{k} \frac{{g_i}^{g_i}}{{n_i}^{n_i}{g_i}^{(g_i - n_i)}\left(1 - \dfrac{n_i}{g_i}\right)^{n_i\left(\frac{g_i}{n_i} - 1\right)}} = \prod_{i=1}^{k} \frac{1}{{n_i}^{n_i}{g_i}^{-n_i}\left(\left(1 - \dfrac{1}{g_i/n_i}\right)^{\left(\frac{g_i}{n_i} - 1\right)}\right)^{n_i}} \approx \prod_{i=1}^{k} \frac{{g_i}^{n_i}}{{n_i}^{n_i} e^{-n_i}} \begin{bmatrix} \because \lim_{x \to \infty}\left(1 - \dfrac{1}{x}\right)^x = \dfrac{1}{e} \\ \text{and } \dfrac{g_i}{n_i} \gg 1 \end{bmatrix}$$

$$\approx \prod_{i=1}^{k} \frac{{g_i}^{n_i}}{n_i!} \begin{bmatrix}\text{using Stirling's} \\ \text{approximation}\end{bmatrix}$$

To determine n_i for the most-probable distribution, we maximize W, or rather $\ln W$.

$$\ln W = \sum_{i=1}^{k} \ln\ g_i! - \sum_{i=1}^{k} \ln\ n_i! - \sum_{i=1}^{k} \ln\,(g_i - n_i)!$$

$$\approx \sum_{i=1}^{k} \{g_i \ln g_i - g_i\} - \sum_{i=1}^{k} (n_i \ln n_i - n_i) - \sum_{i=1}^{k} \{(g_i - n_i)\ln\,(g_i - n_i) - (g_i - n_i)\}$$

$$= \sum_{i=1}^{k} g_i \ln g_i - \sum_{i=1}^{k} n_i \ln n_i - \sum_{i=1}^{k} (g_i - n_i)\ln\,(g_i - n_i)$$

$$\therefore \frac{\partial \ln W}{\partial n_i} = -\sum_{i=1}^{k}(\ln n_i + 1) - \sum_{i=1}^{k}\{-\ln(g_i - n_i) - 1\} = -\sum_{i=1}^{k}\ln n_i + \sum_{i=1}^{k}\ln(g_i - n_i) = \sum_{i=1}^{k}\ln\left(\frac{g_i}{n_i} - 1\right)$$

For $\ln W$ to be maximum, $\frac{\partial \ln W}{\partial n_i} = 0 = \alpha\frac{\partial N}{\partial n_i} + \beta\frac{\partial E}{\partial n_i}$ $\left[\because \frac{\partial N}{\partial n_i} = 0 \text{ and } \frac{\partial E}{\partial n_i} = 0\right]$

$$\Rightarrow \sum_{i=1}^{k}\ln\left(\frac{g_i}{n_i} - 1\right) = \alpha\sum_{i=1}^{k}1 + \beta\sum_{i=1}^{k}\epsilon_i \quad \left[\because \sum_{i=1}^{k} n_i = N \text{ and } \sum_{i=1}^{k} n_i\epsilon_i = E\right] = \sum_{i=1}^{k}(\alpha + \beta\epsilon_i)$$

$$\Rightarrow \ln\left(\frac{g_i}{n_i} - 1\right) = \alpha + \beta\epsilon_i \Rightarrow \frac{g_i}{n_i} - 1 = e^{\alpha+\beta\epsilon_i} \Rightarrow \frac{g_i}{n_i} = e^{\alpha+\beta\epsilon_i} + 1 \Rightarrow \frac{n_i}{g_i} = \frac{1}{e^{\alpha+\beta\epsilon_i} + 1} \Rightarrow n_i = \frac{g_i}{e^{\alpha+\beta\epsilon_i} + 1}$$

$$= f_{\text{FD}}(\epsilon_i)g_i \begin{bmatrix} f_{\text{FD}}(\epsilon_i) = \frac{1}{e^{\alpha+\beta\epsilon_i} + 1} = \frac{1}{e^{(\epsilon_i-\mu)/kT} + 1} \\ \text{is called the FD occupancy function.} \end{bmatrix} \Rightarrow n_i = \frac{g_i}{e^{(\epsilon_i-\mu)/kT} + 1} \quad [\text{for discrete energy}] = f_{\text{FD}}(\epsilon_i)g_i$$

$$\Rightarrow n(\epsilon)\,d\epsilon = \frac{g(\epsilon)\,d\epsilon}{e^{(\epsilon-\mu)/kT} + 1} \quad [\text{for continuous energy}] = f_{\text{FD}}(\epsilon)g(\epsilon)\,d\epsilon \quad [g(\epsilon)\,d\epsilon \text{ is called the density of states.}]$$

$$\ln W = \sum_{i=1}^{k} n_i\left(\frac{g_i}{n_i}\right)\ln n_i\left(\frac{g_i}{n_i}\right) - \sum_{i=1}^{k} n_i \ln n_i - \sum_{i=1}^{k} n_i\left(\frac{g_i}{n_i} - 1\right)\ln n_i\left(\frac{g_i}{n_i} - 1\right)$$

$$= \sum_{i=1}^{k} n_i\left(\frac{g_i}{n_i}\right)\left\{\ln\left(\frac{g_i}{n_i}\right) + \ln n_i\right\} - \sum_{i=1}^{k} n_i \ln n_i - \sum_{i=1}^{k} n_i\left(\frac{g_i}{n_i} - 1\right)\left\{\ln\left(\frac{g_i}{n_i} - 1\right) + \ln n_i\right\}$$

$$= \sum_{i=1}^{k} n_i\left(\frac{g_i}{n_i}\right)\ln\left(\frac{g_i}{n_i}\right) - \sum_{i=1}^{k} n_i\left(\frac{g_i}{n_i} - 1\right)\ln\left(\frac{g_i}{n_i} - 1\right)$$

Weak Degeneracy / Slight Degeneracy

$$N = \int_0^\infty f_{\text{FD}}(\epsilon)g(\epsilon)\,d\epsilon = g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\int_0^\infty \frac{\epsilon^{\frac{1}{2}}}{e^{(\epsilon-\mu)/kT} + 1}d\epsilon = g_s\frac{2\pi V}{h^3}\left(\frac{2m}{\beta}\right)^{\frac{3}{2}}\int_0^\infty \frac{x^{\frac{1}{2}}\,dx}{e^{-\beta\mu}e^x + 1} \begin{bmatrix} \text{Putting } x = \beta\epsilon \\ \Rightarrow d\epsilon = dx/\beta \\ \epsilon^{\frac{1}{2}} = x^{\frac{1}{2}}/x^{\frac{1}{2}} \end{bmatrix}$$

$$= \frac{2g_sV}{\pi^{\frac{1}{2}}h^3}(2\pi mkT)^{\frac{3}{2}}\int_0^\infty \frac{x^{\frac{1}{2}}\,dx}{\frac{1}{\eta_a}e^x + 1} \begin{bmatrix} \text{where } \eta_a = e^{\beta\mu} \\ 0 < \eta_a \le e^{\beta\mu_0} \;\because 0 < \mu \le \mu_0 \end{bmatrix} = \frac{2g_sV}{\sqrt{\pi}\lambda^3}\int_0^\infty \frac{x^{\frac{1}{2}}\,dx}{\frac{1}{\eta_a}e^x - 1} \quad \left[\because \lambda = \frac{h}{\sqrt{2\pi mkT}}\right]$$

$$= \frac{2g_sV}{\sqrt{\pi}\lambda^3}\int_0^\infty \frac{x^{\frac{1}{2}}\,dx}{\frac{e^x}{\eta_a}(1 + \eta_a e^{-x})} = \frac{2g_sV}{\sqrt{\pi}\lambda^3}\int_0^\infty \eta_a\sqrt{x}e^{-x}(1 + \eta_a e^{-x})^{-1}\,dx$$

$$= \frac{2g_sV}{\sqrt{\pi}\lambda^3}\int_0^\infty \eta_a x^{\frac{1}{2}}e^{-x}(1 - \eta_a e^{-x} + {\eta_a}^2 e^{-2x} - \cdots)\,dx$$

$$= \frac{4g_sV}{\sqrt{\pi}\lambda^3}\int_0^\infty \eta_a y^2 e^{-y^2}(1 - \eta_a e^{-y^2} + {\eta_a}^2 e^{-2y^2} - \cdots)\,dy \quad \begin{bmatrix} \text{Putting } x = y^2 \\ \Rightarrow dx = 2y\,dy \end{bmatrix}$$

$$= \frac{4g_sV}{\sqrt{\pi}\lambda^3}\int_0^\infty (\eta_a y^2 e^{-y^2} - {\eta_a}^2 y^2 e^{-2y^2} + {\eta_a}^3 y^2 e^{-3y^2} - \cdots)\,dy$$

$$= \frac{4g_sV}{\sqrt{\pi}\lambda^3}\left(\frac{\eta_a}{4}\sqrt{\pi} - \frac{{\eta_a}^2}{4}\sqrt{\frac{\pi}{2^3}} + \frac{{\eta_a}^3}{4}\sqrt{\frac{\pi}{3^3}} - \cdots\right) \quad \left[\because \int_0^\infty y^2 e^{-ay^2}\,dy = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}\right] = \frac{g_sV}{\lambda^3}\left(\eta_a - \frac{{\eta_a}^2}{2^{3/2}} + \frac{{\eta_a}^3}{3^{3/2}} - \cdots\right)$$

$$\Rightarrow N = \frac{g_sV}{\lambda^3}G_{3/2}(\eta_a) \left[\text{where } G_{3/2}(\eta_a) = \sum_{j=1}^{\infty}\frac{(-1)^{j+1}{\eta_a}^j}{j^{3/2}} \text{ is a polylogarithm function.}\right]$$

$G_{3/2}(\eta_a)$ converges for $\eta_a \le 1$ i.e. $\mu \le 0$.

When $\eta_a = 1$ i.e. $\mu = 0$, $G_{3/2}(\eta_a) = \eta\left(\frac{3}{2}\right) \approx 0.7651$ $\left[\text{where } \eta\left(\frac{3}{2}\right) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^{3/2}} \text{ is a Dirichlet eta function.}\right]$

When $\eta_a = 1$ i.e. $\mu = 0$, $\lambda = \lambda_{\mathrm{C}}$ and $T = T_{\mathrm{C}}$, then $N \approx \frac{g_s V}{{\lambda_{\mathrm{C}}}^3}(0.7651) \Rightarrow N = \frac{g_s V}{h^3}(2\pi m k T_{\mathrm{C}})^{\frac{3}{2}}(0.7651)$

$$\Rightarrow T_{\mathrm{C}} = \frac{h^2}{2\pi m k}\left(\frac{n/g_s}{0.7651}\right)^{\frac{2}{3}} \quad \left[\text{where } n = \frac{N}{V}\right] \approx \frac{259.8 \text{ K}}{M V_M^{\frac{2}{3}} g_s^{\frac{2}{3}}} \quad \left[\begin{matrix}\text{where } M = mN_{\mathrm{A}} \text{ is the gram-molecular weight} \\ \text{and } V_M \text{ is the molar volume } (\mathrm{cm^3 mol^{-1}})\end{matrix}\right]$$

$$\left[\text{Calculation: } \frac{h^2 N_{\mathrm{A}} \cdot 10^3}{2\pi k}\left(\frac{N_{\mathrm{A}} 10^6}{0.7651}\right)^{\frac{2}{3}} \approx 259.8\right]$$

Strong Degeneracy

$G_{3/2}(\eta_a)$ does not converge for $\eta_a > 1$ i.e. $\mu > 0$. So the integral $\int_0^{\infty} \frac{\epsilon^{\frac{1}{2}}}{e^{(\epsilon-\mu)/kT} + 1} d\epsilon$ has to be approximated. It utilizes the slow variation of $f_{\mathrm{FD}}(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/kT} + 1}$ with ϵ except in the region $\epsilon \approx \mu$ where it changes rapidly so that $-\frac{\partial f_{\mathrm{FD}}(\epsilon)}{\partial \epsilon}$ resembles a delta function being negligible except when $|\epsilon - \mu|$ is small and large enough to give unit area under the peak.

Let $F(\epsilon) = \int_0^{\epsilon} g(\epsilon)\, d\epsilon$ so that $F(0) = 0$ and $\frac{dF}{d\epsilon} = g(\epsilon)$

$$\therefore N = \int_0^{\infty} f_{\mathrm{FD}}(\epsilon) g(\epsilon)\, d\epsilon = \int_0^{\infty} f_{\mathrm{FD}}(\epsilon) F'(\epsilon)\, d\epsilon = [f_{\mathrm{FD}}(\epsilon)F(\epsilon)]_0^{\infty} - \int_0^{\infty} \frac{df_{\mathrm{FD}}}{d\epsilon} F(\epsilon)\, d\epsilon \quad [\text{integrating by parts}]$$

$$\Rightarrow N = -\int_0^{\infty} \frac{df_{\mathrm{FD}}}{d\epsilon} F(\epsilon)\, d\epsilon \quad \left[\begin{matrix}[f_{\mathrm{FD}}(\epsilon)F(\epsilon)]_0^{\infty} = 0 \\ \because f_{\mathrm{FD}}(\infty) = 0 \text{ and } F(0) = 0\end{matrix}\right] \Rightarrow N = -\int_0^{\infty} f_{\mathrm{FD}}'(\epsilon) F(\epsilon)\, d\epsilon$$

The integrand is appreciable only for $\epsilon = \mu$ since $\frac{df_{\mathrm{FD}}}{d\epsilon}$ is a delta function with peak at $\epsilon = \mu$. Therefore expand $F(\epsilon)$ about μ in Taylor series i.e. $F(\epsilon) = F(\mu) + (\epsilon - \mu)F'(\mu) + \frac{(\epsilon-\mu)^2}{2!}F''(\mu) + \frac{(\epsilon-\mu)^3}{3!}F'''(\mu) + \cdots$ where $F(\mu) = F(\epsilon)|_{\epsilon=\mu}$; $F'(\mu) = F'(\epsilon)|_{\epsilon=\mu}$; $F''(\mu) = F''(\epsilon)|_{\epsilon=\mu}$; $F'''(\mu) = F'''(\epsilon)|_{\epsilon=\mu}$ and so on.

$$\therefore N = -\int_0^{\infty} f_{\mathrm{FD}}'(\epsilon) F(\epsilon)\, d\epsilon = -\int_0^{\infty} f_{\mathrm{FD}}'(\epsilon)\left[F(\mu) + (\epsilon-\mu)F'(\mu) + \frac{(\epsilon-\mu)^2}{2!}F''(\mu) + \frac{(\epsilon-\mu)^3}{3!}F'''(\mu) + \cdots\right] d\epsilon$$

$$= -F(\mu)\int_0^{\infty} f_{\mathrm{FD}}'(\epsilon)\, d\epsilon - F'(\mu)\int_0^{\infty} (\epsilon-\mu) f_{\mathrm{FD}}'(\epsilon)\, d\epsilon - \frac{F''(\mu)}{2!}\int_0^{\infty} (\epsilon-\mu)^2 f_{\mathrm{FD}}'(\epsilon)\, d\epsilon - \frac{F'''(\mu)}{3!}\int_0^{\infty} (\epsilon-\mu)^3 f_{\mathrm{FD}}'(\epsilon)\, d\epsilon + \cdots$$

$$\because f_{\mathrm{FD}}'(\epsilon) = \frac{-(1/kT)e^{(\epsilon-\mu)/kT}}{(e^{(\epsilon-\mu)/kT} + 1)^2} = \mathcal{f}(\epsilon-\mu) \Rightarrow \mathcal{f}\big(-(\epsilon-\mu)\big) = \frac{-(1/kT)e^{-(\epsilon-\mu)/kT}}{(e^{-(\epsilon-\mu)/kT} + 1)^2} = \frac{-(1/kT)e^{(\epsilon-\mu)/kT}}{(1 + e^{(\epsilon-\mu)/kT})^2} = \mathcal{f}(\epsilon-\mu)$$

$\therefore f_{\mathrm{FD}}'(\epsilon) = \mathcal{f}(\epsilon-\mu)$ is an even function of $(\epsilon - \mu)$.

$$\therefore \int_0^{\infty} f_{\mathrm{FD}}'(\epsilon)\, d\epsilon = \frac{-1}{kT}\int_0^{\infty} \frac{e^{(\epsilon-\mu)/kT}\, d\epsilon}{(e^{(\epsilon-\mu)/kT} + 1)^2} = \int_{-\mu/kT}^{\infty} \frac{-e^y\, dy}{(e^y + 1)^2} \quad \left[\begin{matrix}\text{where } y = (\epsilon-\mu)/kT \Rightarrow dy = \frac{1}{kT}d\epsilon \\ \Rightarrow d\epsilon = (kT)dy \text{ ; when } \epsilon = 0, y = \frac{-\mu}{kT}\end{matrix}\right] \approx \int_{-\infty}^{\infty} \frac{-e^y\, dy}{(e^y + 1)^2}$$

$$\left[\text{for } \mu \gg kT, \frac{-\mu}{kT} \approx -\infty\right] = 2\int_0^{\infty} \frac{-e^y\, dy}{(e^y + 1)^2} \quad \left[\begin{matrix}\because \text{ integrand is an} \\ \text{even function of } y\end{matrix}\right] = 2\left[\frac{1}{e^y + 1}\right]_0^{\infty} \approx -1$$

Next, $\int_0^\infty (\epsilon-\mu) f_{\mathrm{FD}}{}'(\epsilon)\,d\epsilon = \dfrac{-1}{kT}\int_0^\infty \dfrac{(\epsilon-\mu)\,e^{(\epsilon-\mu)/kT}}{(e^{(\epsilon-\mu)/kT}+1)^2}\,d\epsilon = kT\int_{-\mu/kT}^{\infty}\dfrac{-\,y\,e^y}{(e^y+1)^2}\,dy \approx kT\int_{-\infty}^{\infty}\dfrac{-\,y\,e^y}{(e^y+1)^2}\,dy = 0$

$\begin{bmatrix}\because \text{integrand is an}\\ \text{odd function of } y\end{bmatrix}$ Likewise, $\int_0^\infty (\epsilon-\mu)^3 f_{\mathrm{FD}}{}'(\epsilon)\,d\epsilon = 0$

Now, $\int_0^\infty (\epsilon-\mu)^2 f_{\mathrm{FD}}{}'(\epsilon)\,d\epsilon = \dfrac{-1}{kT}\int_0^\infty \dfrac{(\epsilon-\mu)^2 e^{(\epsilon-\mu)/kT}}{(e^{(\epsilon-\mu)/kT}+1)^2}\,d\epsilon = (kT)^2\int_{-\mu/kT}^{\infty}\dfrac{-\,y^2\,e^y}{(e^y+1)^2}\,dy \approx (kT)^2\int_{-\infty}^{\infty}\dfrac{-\,y^2\,e^y}{(e^y+1)^2}\,dy$

$$= -2(kT)^2\int_0^\infty \frac{y^2\,e^y}{(e^y+1)^2}\,dy = -2(kT)^2\left\{\left[\frac{-y^2}{e^y+1}\right]_0^\infty - \int_0^\infty \frac{-2y}{e^y+1}\,dy\right\} \quad \text{[integrating by parts]}$$

$$= -4(kT)^2\int_0^\infty \frac{y}{e^y+1}\,dy = -4(kT)^2\int_0^\infty ye^{-y}(1+e^{-y})^{-1}\,dy = -4(kT)^2\int_0^\infty ye^{-y}(1-e^{-y}+e^{-2y}-\cdots)\,dy$$

$$= -8(kT)^2\int_0^\infty x^3e^{-x^2}\left(1-e^{-x^2}+e^{-2x^2}-\cdots\right)\,dx \quad \begin{bmatrix}\text{Putting } y=x^2\\ \Longrightarrow dy = 2x\,dx\end{bmatrix}$$

$$= -8(kT)^2\int_0^\infty \left(x^3e^{-x^2}-x^3e^{-2x^2}+x^3e^{-3x^2}-\cdots\right)\,dx$$

$$= -8(kT)^2\left(\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2^2}+\frac{1}{2}\cdot\frac{1}{3^2}-\cdots\right)\left[\because \int_0^\infty x^3e^{-ax^2}\,dx = \frac{1}{2a^2}\right] = -4(kT)^2\left(1-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\right) = -4(kT)^2\eta(2)$$

$$= -4(kT)^2\left(1-\frac{1}{2}\right)\zeta(2) \quad [\because \eta(s) = (1-2^{1-s})\zeta(s)] = -2(kT)^2\zeta(2) = -(kT)^2\frac{\pi^2}{3} \quad \left[\because \zeta(2) = \frac{\pi^2}{6}\right]$$

$$\therefore N = -F(\mu)\cdot(-1) - F'(\mu)\cdot 0 - \frac{F''(\mu)}{2!}\left(-(kT)^2\frac{\pi^2}{3}\right) - \frac{F'''(\mu)}{3!}\cdot 0 + \cdots \approx F(\mu) + (kT)^2\frac{\pi^2}{6}F''(\mu)$$

$$\Longrightarrow N = \int_0^\mu g(\epsilon)\,d\epsilon + (kT)^2\frac{\pi^2}{6}g'(\mu)\left[\because F(\mu) = F(\epsilon)|_{\epsilon=\mu} = \int_0^\mu g(\epsilon)\,d\epsilon \text{ and } F''(\mu) = F''(\epsilon)|_{\epsilon=\mu} = g'(\epsilon)|_{\epsilon=\mu} = g'(\mu)\right]$$

$$\Longrightarrow \int_0^{\mu_0} g(\epsilon)\,d\epsilon = \int_0^\mu g(\epsilon)\,d\epsilon + (kT)^2\frac{\pi^2}{6}g'(\mu) \quad \begin{bmatrix}\because \text{At } T=0, N = \int_0^{E_{F0}} g(\epsilon)\,d\epsilon = \int_0^{\mu_0} g(\epsilon)\,d\epsilon\\ \text{and } N \text{ must be constant over all } T.\end{bmatrix}$$

$$\Longrightarrow \int_{\mu_0}^{\mu} g(\epsilon)\,d\epsilon + (kT)^2\frac{\pi^2}{6}g'(\mu) = 0 \Longrightarrow g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\left\{\int_{\mu_0}^{\mu}\epsilon^{\frac{1}{2}}\,d\epsilon + (kT)^2\frac{\pi^2}{6}\left(\frac{1}{2\mu^{\frac{1}{2}}}\right)\right\} = 0$$

$$\Longrightarrow \frac{2}{3}\left[\epsilon^{\frac{3}{2}}\right]_{\mu_0}^{\mu} + (kT)^2\frac{\pi^2}{12}\left(\frac{1}{\mu^{\frac{1}{2}}}\right) = 0 \Longrightarrow \frac{2}{3}\left(\mu^{\frac{3}{2}}-\mu_0{}^{\frac{3}{2}}\right) + (kT)^2\frac{\pi^2}{12}\left(\frac{1}{\mu^{\frac{1}{2}}}\right) = 0 \Longrightarrow \frac{2}{3}\left(\mu^2-\mu_0{}^{\frac{3}{2}}\mu^{\frac{1}{2}}\right) + (kT)^2\frac{\pi^2}{12} = 0$$

$$\Longrightarrow \mu^2 - \mu_0{}^{\frac{3}{2}}\mu^{\frac{1}{2}} + (kT)^2\frac{\pi^2}{8} = 0 \Longrightarrow w^4 - \mu_0{}^{\frac{3}{2}}w + (kT)^2\frac{\pi^2}{8} = 0 \quad \left[\text{where } w = \mu^{\frac{1}{2}}\right]$$

This is a quartic equation which can be solved by Ferrari's method which gets very complicated. So, we proceed thus:

$$\int_{\mu_0}^{\mu}\epsilon^{\frac{1}{2}}\,d\epsilon + (kT)^2\frac{\pi^2}{6}\left(\frac{1}{2\mu^{\frac{1}{2}}}\right) = 0 \Longrightarrow \mu^{\frac{1}{2}}(\mu-\mu_0) + (kT)^2\frac{\pi^2}{12}\left(\frac{1}{\mu^{\frac{1}{2}}}\right) = 0 \quad \left[\because \int_{\mu_0}^{\mu}\epsilon^{\frac{1}{2}}\,d\epsilon \approx \mu^{\frac{1}{2}}\int_{\mu_0}^{\mu}d\epsilon = \mu^{\frac{1}{2}}(\mu-\mu_0)\right]$$

$$\Longrightarrow \mu(\mu-\mu_0) + (kT)^2\frac{\pi^2}{12} = 0 \Longrightarrow \mu^2 - \mu_0\mu + (kT)^2\frac{\pi^2}{12} = 0 \Longrightarrow \mu = \frac{\mu_0 \pm \left(\mu_0{}^2 - (kT)^2\frac{\pi^2}{3}\right)^{\frac{1}{2}}}{2}$$

$$\Rightarrow \mu = \frac{\mu_0}{2}\left\{1 \pm \left(1 - \left(\frac{kT}{\mu_0}\right)^2 \frac{\pi^2}{3}\right)^{\frac{1}{2}}\right\} \approx \frac{\mu_0}{2}\left\{1 \pm \left(1 - \left(\frac{kT}{\mu_0}\right)^2 \frac{\pi^2}{6}\right)\right\} = \frac{\mu_0}{2}\left(2 - \left(\frac{kT}{\mu_0}\right)^2 \frac{\pi^2}{6}\right) = \mu_0\left(1 - \left(\frac{kT}{\mu_0}\right)^2 \frac{\pi^2}{12}\right)$$

$$\left[\because \mu = \frac{\mu_0}{2}\left\{1 - \left(1 - \left(\frac{kT}{\mu_0}\right)^2 \frac{\pi^2}{6}\right)\right\} = \mu_0\left(\frac{kT}{\mu_0}\right)^2 \frac{\pi^2}{12} \Rightarrow \mu = 0 \text{ at } T = 0 \text{ which is not applicable.}\right]$$

Alternatively:

$$\int_0^\infty \frac{\epsilon^{\frac{1}{2}}}{e^{(\epsilon-\mu)/kT}+1}\,d\epsilon = (kT)\int_{-\mu/kT}^{\infty} \frac{(\mu + (kT)y)^{\frac{1}{2}}\,dy}{e^y+1} \quad \left[\begin{array}{c}\text{where } y = (\epsilon-\mu)/kT \Rightarrow \epsilon = \mu + (kT)y \\ \Rightarrow d\epsilon = (kT)dy\text{ ; when } \epsilon = 0, y = \dfrac{-\mu}{kT}\end{array}\right]$$

$$\approx (kT)\int_{-\infty}^{\infty} \frac{(\mu + (kT)y)^{\frac{1}{2}}\,dy}{e^y+1} = (kT)\int_{-\infty}^{0} \frac{(\mu + (kT)y)^{\frac{1}{2}}\,dy}{e^y+1} + (kT)\int_0^\infty \frac{(\mu + (kT)y)^{\frac{1}{2}}\,dy}{e^y+1} \quad \left[\text{for } \mu \gg kT,\ \frac{-\mu}{kT} \approx -\infty\right]$$

$$= (kT)\int_0^\infty \frac{(\mu - (kT)y)^{\frac{1}{2}}\,dy}{e^{-y}+1} + (kT)\int_0^\infty \frac{(\mu + (kT)y)^{\frac{1}{2}}\,dy}{e^y+1} \quad \left[\begin{array}{c}\text{changing } y \text{ to } -y \\ \text{in the first integral}\end{array}\right]$$

$$= (kT)\int_0^\infty (\mu - (kT)y)^{\frac{1}{2}}\left(\frac{e^y}{1+e^y}\right)dy + (kT)\int_0^\infty \frac{(\mu + (kT)y)^{\frac{1}{2}}\,dy}{e^y+1}$$

$$= (kT)\int_0^\infty (\mu - (kT)y)^{\frac{1}{2}}\left(1 - \frac{1}{1+e^y}\right)dy + (kT)\int_0^\infty \frac{(\mu + (kT)y)^{\frac{1}{2}}\,dy}{e^y+1}$$

$$= (kT)\int_0^\infty (\mu - (kT)y)^{\frac{1}{2}}\,dy - (kT)\int_0^\infty \frac{(\mu - (kT)y)^{\frac{1}{2}}}{1+e^y}dy + (kT)\int_0^\infty \frac{(\mu + (kT)y)^{\frac{1}{2}}\,dy}{e^y+1}$$

$$= (kT)\int_{-\mu/kT}^{0} (\mu + (kT)y)^{\frac{1}{2}}\,dy + (kT)\int_0^\infty \frac{(\mu + (kT)y)^{\frac{1}{2}} - (\mu - (kT)y)^{\frac{1}{2}}}{e^y+1}dy \quad \left[\begin{array}{c}\text{changing } y \text{ to } -y \text{ in the first} \\ \text{integral and by writing } -\mu/kT \\ \text{instead of } -\infty \quad \because \mu \gg kT\end{array}\right]$$

$$= \int_0^\mu \epsilon^{\frac{1}{2}}\,d\epsilon + (kT)\int_0^\infty \left(\frac{(\mu + (kT)y)^{\frac{1}{2}} - (\mu - (kT)y)^{\frac{1}{2}}}{2(kT)y}\right)\frac{2(kT)y}{e^y+1}dy \quad \left[\begin{array}{c}\text{putting } \epsilon = \mu + (kT)y \\ \Rightarrow dy = \dfrac{d\epsilon}{kT}\text{; when } y = 0, \epsilon = \mu\end{array}\right]$$

$$= \int_0^\mu \epsilon^{\frac{1}{2}}\,d\epsilon + 2(kT)^2\int_0^\infty \frac{d}{d\epsilon}\left(\epsilon^{\frac{1}{2}}\right)\Big|_{\epsilon=\mu}\frac{y}{e^y+1}dy \quad [\because kT \text{ is small.}] = \int_{-\infty}^{\mu} \epsilon^{\frac{1}{2}}\,d\epsilon + 2(kT)^2\frac{d}{d\epsilon}\left(\epsilon^{\frac{1}{2}}\right)\Big|_{\epsilon=\mu}\int_0^\infty \frac{y}{e^y+1}dy$$

$$= \int_0^\mu \epsilon^{\frac{1}{2}}\,d\epsilon + 2(kT)^2\left(\frac{1}{2\mu^{\frac{1}{2}}}\right)\frac{\pi^2}{12} \quad \left[\because \int_0^\infty \frac{y}{e^y+1}dy = \eta(2) = \frac{\pi^2}{12}\right] = \int_0^\mu \epsilon^{\frac{1}{2}}\,d\epsilon + \frac{\pi^2}{12}\cdot\frac{(kT)^2}{\mu^{\frac{1}{2}}}$$

$$\therefore N = g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\int_0^\infty \frac{\epsilon^{\frac{1}{2}}}{e^{(\epsilon-\mu)/kT}+1}d\epsilon = g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\left(\int_{-\infty}^{\mu} \epsilon^{\frac{1}{2}}\,d\epsilon + \frac{\pi^2}{12}\cdot\frac{(kT)^2}{\mu^{\frac{1}{2}}}\right)$$

$$\Rightarrow \int_0^{\mu_0} g(\epsilon)\,d\epsilon = g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\left(\int_0^\mu \epsilon^{\frac{1}{2}}\,d\epsilon + \frac{\pi^2}{12}\cdot\frac{(kT)^2}{\mu^{\frac{1}{2}}}\right) \quad \left[\begin{array}{c}\because \text{At } T = 0, N = \displaystyle\int_0^{E_{F0}} g(\epsilon)\,d\epsilon = \int_0^{\mu_0} g(\epsilon)\,d\epsilon \\ \text{and } N \text{ must be constant over all } T.\end{array}\right]$$

$$\Rightarrow \int_0^{\mu_0} \epsilon^{\frac{1}{2}}\,d\epsilon = \int_0^\mu \epsilon^{\frac{1}{2}}\,d\epsilon + \frac{\pi^2}{12}\cdot\frac{(kT)^2}{\mu^{\frac{1}{2}}} \Rightarrow \mu^{\frac{1}{2}}(\mu - \mu_0) + \frac{\pi^2}{12}\cdot\frac{(kT)^2}{\mu^{\frac{1}{2}}} = 0 \Rightarrow \mu \approx \mu_0\left(1 - \left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right)$$

Internal energy, $U = \displaystyle\int_0^\infty \epsilon\cdot f_{\mathrm{FD}}(\epsilon)g(\epsilon)\,d\epsilon = \int_0^\infty f_{\mathrm{FD}}(\epsilon)\big(\epsilon\cdot g(\epsilon)\big)\,d\epsilon = \int_0^\mu \big(\epsilon\cdot g(\epsilon)\big)\,d\epsilon + (kT)^2\frac{\pi^2}{6}\big(\epsilon\cdot g(\epsilon)\big)'\Big|_{\epsilon=\mu}$

$$\left[\because \int_0^\infty f(\epsilon)g(\epsilon)\,d\epsilon = \int_0^\mu g(\epsilon)\,d\epsilon + (kT)^2\frac{\pi^2}{6}g'(\mu) \text{ when } f(\epsilon) \text{ is a slow-varying function of } \epsilon \text{ except at } \epsilon = \mu\right]$$

$$= g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\left(\int_0^\mu \epsilon^{\frac{3}{2}}\,d\epsilon + (kT)^2\frac{\pi^2}{6}\left(\epsilon^{\frac{3}{2}}\right)'\Big|_{\epsilon=\mu}\right) = g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\left(\frac{2}{5}\left[\epsilon^{\frac{5}{2}}\right]_0^\mu + (kT)^2\frac{\pi^2}{6}\cdot\frac{3}{2}\epsilon^{\frac{1}{2}}\Big|_{\epsilon=\mu}\right)$$

$$= g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\left(\frac{2}{5}\mu^{\frac{5}{2}} + (kT)^2\frac{\pi^2}{4}\mu^{\frac{1}{2}}\right) = g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\mu_0^{\frac{5}{2}}\left\{\frac{2}{5}\left(\frac{\mu}{\mu_0}\right)^{\frac{5}{2}} + \left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{4}\left(\frac{\mu}{\mu_0}\right)^{\frac{1}{2}}\right\}$$

$$= g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\mu_0^{\frac{5}{2}}\left\{\frac{2}{5}\left(1-\left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right)^{\frac{5}{2}} + \left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{4}\left(1-\left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right)^{\frac{1}{2}}\right\} \quad \left[\because \mu = \mu_0\left(1-\left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right)\right]$$

$$\approx g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\mu_0^{\frac{5}{2}}\left\{\frac{2}{5}\left(1-\frac{5}{2}\left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right) + \left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{4}\left(1-\frac{1}{2}\left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right)\right\}$$

$$= g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\mu_0^{\frac{5}{2}}\left\{\frac{2}{5}-\left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}+\left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{4}-\frac{3}{2}\left(\frac{kT}{\mu_0}\right)^4\left(\frac{\pi^2}{12}\right)^2\right\} \approx g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\mu_0^{\frac{5}{2}}\left\{\frac{2}{5}+\left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{6}\right\}$$

$$= g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\mu_0^{\frac{5}{2}}\cdot\frac{2}{5} + (kT)^2\frac{\pi^2}{6}\left(g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\mu_0^{\frac{1}{2}}\right) \quad [\because kT \ll \mu_0] \Longrightarrow U = U_0 + (kT)^2\frac{\pi^2}{6}g(\mu_0)$$

$$\left[\because U_0 = \int_0^{\mu_0}\epsilon\, g(\epsilon)\,d\epsilon = g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\int_0^{\mu_0}\epsilon^{\frac{3}{2}}\,d\epsilon = g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\cdot\frac{2}{5}\mu_0^{\frac{5}{2}}\right]$$

Alternatively:

$$U = \int_0^\mu \epsilon\cdot g(\epsilon)\,d\epsilon + (kT)^2\frac{\pi^2}{6}\left(\epsilon\cdot g(\epsilon)\right)'\Big|_{\epsilon=\mu} = \int_0^\mu\left(\epsilon\cdot g(\epsilon)\right)d\epsilon + g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}(kT)^2\frac{\pi^2}{6}\left(\epsilon^{\frac{3}{2}}\right)'\Big|_{\epsilon=\mu}$$

$$= \int_0^\mu \epsilon\cdot g(\epsilon)\,d\epsilon - \int_0^{\mu_0}\left(\epsilon\cdot g(\epsilon)\right)d\epsilon + \int_0^{\mu_0}\epsilon\cdot g(\epsilon)\,d\epsilon + g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}(kT)^2\frac{\pi^2}{6}\cdot\frac{3}{2}\epsilon^{\frac{1}{2}}\Big|_{\epsilon=\mu}$$

$$= \int_{\mu_0}^\mu \epsilon\cdot g(\epsilon)\,d\epsilon + U_0 + g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}(kT)^2\frac{\pi^2}{4}\mu^{\frac{1}{2}}$$

$$\approx \left(g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\mu^{\frac{1}{2}}\right)\mu_0\mu\left[\frac{\mu}{\mu_0}-1\right] + U_0 + g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}(kT)^2\frac{\pi^2}{4}\mu_0^{\frac{1}{2}}\left(1-\left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right)^{\frac{1}{2}}$$

$$\left[\because \int_{\mu_0}^\mu \epsilon\cdot g(\epsilon)\,d\epsilon \approx \mu\cdot g(\mu)\int_{\mu_0}^\mu d\epsilon = \mu\cdot g(\mu)[\mu-\mu_0] = \mu\left(g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\mu^{\frac{1}{2}}\right)\mu_0\left[\frac{\mu}{\mu_0}-1\right]\right]$$

$$\approx \left(g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\right)\mu_0\mu_0^{\frac{3}{2}}\left(1-\left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right)^{\frac{3}{2}}\left[-\left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right] + U_0 + g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}(kT)^2\frac{\pi^2}{4}\mu_0^{\frac{1}{2}}\left(1-\frac{1}{2}\left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right)$$

$$\approx U_0 + g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\left\{\mu_0^{\frac{5}{2}}\left[-\left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right]\left(1-\frac{3}{2}\left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right) + (kT)^2\frac{\pi^2}{4}\mu_0^{\frac{1}{2}}\left(1-\frac{1}{2}\left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right)\right\}$$

$$= U_0 + g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}(kT)^2\mu_0^{\frac{1}{2}}\left\{-\frac{\pi^2}{12}\left(1-\frac{3}{2}\left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right) + \frac{\pi^2}{4}\left(1-\frac{1}{2}\left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right)\right\}$$

$$= U_0 + g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}(kT)^2\mu_0^{\frac{1}{2}}\frac{\pi^2}{6} \Longrightarrow U = U_0 + (kT)^2\frac{\pi^2}{6}g(\mu_0)$$

$$U = U_0 + (kT)^2\frac{\pi^2}{6}g(\mu_0) = U_0 + (kT)^2\frac{\pi^2}{6}\left(\frac{5U_0}{2{\mu_0}^2}\right) \quad \left[\because U_0 = \frac{3}{5}N\mu_0 \Rightarrow N = \frac{5U_0}{3\mu_0} \text{ and } g(\mu_0) = \frac{3N}{2\mu_0} = \frac{5U_0}{2{\mu_0}^2}\right]$$

$$\Rightarrow U = U_0\left(1 + \left(\frac{kT}{\mu_0}\right)^2\frac{5\pi^2}{12}\right) \Rightarrow C_V = \left(\frac{\partial U}{\partial T}\right)_{V,N} = U_0\left(\frac{2k}{\mu_0}\left(\frac{kT}{\mu_0}\right)\frac{5\pi^2}{12}\right) = T\left(\frac{3}{5}N\mu_0\right)\left(\frac{k}{\mu_0}\right)^2\frac{5\pi^2}{6} = \frac{\pi^2 Nk}{2}\left(\frac{T}{T_F}\right)$$

Degeneracy Pressure, $P_D = -\left(\frac{\partial U}{\partial V}\right)_{T,N} = -\frac{\partial U}{\partial \mu_0}\frac{\partial \mu_0}{\partial V} = \frac{h^2}{2m}\left(\frac{3N}{4\pi g_s}\right)^{\frac{2}{3}}\left(\frac{2}{3}V^{-\frac{5}{3}}\right)\cdot\frac{\partial}{\partial \mu_0}\left\{\frac{3}{5}N\mu_0\left(1 + \left(\frac{kT}{\mu_0}\right)^2\frac{5\pi^2}{12}\right)\right\}$

$$\left[\because \mu_0 = \frac{h^2}{2m}\left(\frac{3n}{4\pi g_s}\right)^{\frac{2}{3}} = \frac{h^2}{2m}\left(\frac{3N}{4\pi g_s}\right)^{\frac{2}{3}} = \frac{h^2}{2m}\left(\frac{3N}{4\pi g_s}\right)^{\frac{2}{3}}V^{-\frac{2}{3}} \text{ and } \because U_0 = \frac{3}{5}N\mu_0\right]$$

$$= \frac{h^2}{2m}\left(\frac{3N}{4\pi g_s V}\right)^{\frac{2}{3}}\left(\frac{2N}{3V}\right)\cdot\frac{\partial}{\partial \mu_0}\left\{\frac{3}{5}\mu_0 + \frac{(kT)^2}{\mu_0}\frac{\pi^2}{4}\right\} = \frac{h^2}{2m}\left(\frac{3n}{4\pi g_s}\right)^{\frac{2}{3}}\left(\frac{2N}{3V}\right)\cdot\left\{\frac{3}{5} - \left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{4}\right\}$$

$$= \frac{h^2}{2m}\left(\frac{3n}{4\pi g_s}\right)^{\frac{2}{3}}\left(\frac{2N}{3V}\right)\cdot\frac{3}{5}\left\{1 - \left(\frac{kT}{\mu_0}\right)^2\frac{5\pi^2}{12}\right\} = \mu_0\left(\frac{2N}{3V}\right)\cdot\frac{3}{5}\left\{1 - \left(\frac{kT}{\mu_0}\right)^2\frac{5\pi^2}{12}\right\} = \left(\frac{2U_0}{3V}\right)\left\{1 - \left(\frac{kT}{\mu_0}\right)^2\frac{5\pi^2}{12}\right\}$$

Alternatively, $P_D = \frac{2U}{3V} = \frac{2U_0}{3V}\left(1 + \left(\frac{kT}{\mu_0}\right)^2\frac{5\pi^2}{12}\right)$ [This is **not correct** as U is a function of multiple powers of V.]

Magnetic Susceptibility of Free Electrons

Conduction electrons are found to possess a small, temperature-independent paramagnetic volume susceptibility of $\sim 10^{-6}$ which is in disagreement with the value $\sim 10^{-4}$ (at room temperature) given by the Langevin theory which varies with temperature as $1/T$. This was explained by Pauli using Fermi-Dirac statistics.
Each conduction electron has a magnetic moment whose component in the direction of magnetic field is $\pm \mu_H$.
If there are n conduction electrons per unit volume, the net magnetization, $M = (n_+ - n_-)\mu_H$ $[n = n_+ + n_-]$ where n_+ and n_- are respectively the number of electrons with parallel and anti-parallel magnetic moment components along H.

$$n = n_+ + n_- \; ; \; n_+ = \frac{1}{V}\int_0^\infty g_s f_{\text{FD}}(\epsilon)g(\epsilon + \mu_H H)\,d\epsilon \; ; \; n_- = \frac{1}{V}\int_0^\infty g_s f_{\text{FD}}(\epsilon)g(\epsilon - \mu_H H)\,d\epsilon$$

[ϵ is the total energy (kinetic + magnetic)of the electron ; $-\mu_H H$ is the magnetic energy ; $\boldsymbol{g_s = 1}$]

$$n_\pm = \frac{1}{V}\int_0^\infty g(\epsilon \pm \mu_H H)f_{\text{FD}}(\epsilon)\,d\epsilon = \frac{2\pi}{h^3}(2m)^{\frac{3}{2}}\int_0^\infty (\epsilon \pm \mu_H H)^{\frac{1}{2}}f_{\text{FD}}(\epsilon)\,d\epsilon = \frac{2\pi}{h^3}(2m)^{\frac{3}{2}}\int_0^\infty \epsilon^{\frac{1}{2}}\left(1 \pm \frac{\mu_H H}{\epsilon}\right)^{\frac{1}{2}}f_{\text{FD}}(\epsilon)\,d\epsilon$$

$$\approx \frac{2\pi}{h^3}(2m)^{\frac{3}{2}}\int_0^\infty \epsilon^{\frac{1}{2}}\left(1 \pm \frac{\mu_H H}{2\epsilon}\right)f_{\text{FD}}(\epsilon)\,d\epsilon$$

Magnetization, $M = (n_+ - n_-)\mu_H = \frac{2\pi}{h^3}(2m)^{\frac{3}{2}}\left\{\int_0^\infty \epsilon^{\frac{1}{2}}\left(1 + \frac{\mu_H H}{2\epsilon}\right)f_{\text{FD}}(\epsilon)\,d\epsilon - \int_0^\infty \epsilon^{\frac{1}{2}}\left(1 - \frac{\mu_H H}{2\epsilon}\right)f_{\text{FD}}(\epsilon)\,d\epsilon\right\}\mu_H$

$$= \frac{2\pi}{h^3}(2m)^{\frac{3}{2}}\int_0^\infty \epsilon^{\frac{1}{2}}\left(\frac{\mu_H H}{\epsilon}\right)f_{\text{FD}}(\epsilon)\,d\epsilon\,\mu_H = \frac{2\pi}{h^3}(2m)^{\frac{3}{2}}{\mu_H}^2 H\int_0^\infty \epsilon^{-\frac{1}{2}}f_{\text{FD}}(\epsilon)\,d\epsilon = \frac{4\pi}{h^3}(2m)^{\frac{3}{2}}{\mu_H}^2 H\int_0^\infty \frac{\epsilon^{-\frac{1}{2}}}{2}f_{\text{FD}}(\epsilon)\,d\epsilon$$

$$= \frac{{\mu_H}^2 H}{V}\int_0^\infty f_{\text{FD}}(\epsilon)g'(\epsilon)\,d\epsilon = \frac{{\mu_H}^2 H}{V}\left\{\int_0^\mu g'(\epsilon)\,d\epsilon + (kT)^2\frac{\pi^2}{6}g''(\mu)\right\}$$

$$\left[\because \int_0^\infty f(\epsilon)g(\epsilon)\,d\epsilon = \int_0^\mu g(\epsilon)\,d\epsilon + (kT)^2\frac{\pi^2}{6}g'(\mu) \text{ when } f(\epsilon) \text{ is a slow-varying function of } \epsilon \text{ except at } \epsilon = \mu\right]$$

$$= \frac{4\pi}{h^3}(2m)^{\frac{3}{2}}\mu_H{}^2 H\left\{\left[\epsilon^{\frac{1}{2}}\right]_0^{\mu} + (kT)^2\frac{\pi^2}{6}\left(\frac{-1}{4\mu^{\frac{3}{2}}}\right)\right\} = \frac{4\pi}{h^3}(2m)^{\frac{3}{2}}\mu_H{}^2 H\left\{\mu^{\frac{1}{2}} - (kT)^2\frac{\pi^2}{24\mu^{\frac{3}{2}}}\right\}$$

$$= \frac{4\pi}{h^3}(2m)^{\frac{3}{2}}\mu_H{}^2 H\left\{\mu_0{}^{\frac{1}{2}}\left(\frac{\mu}{\mu_0}\right)^{\frac{1}{2}} - (kT)^2\mu_0{}^{-\frac{3}{2}}\frac{\pi^2}{24}\left(\frac{\mu_0}{\mu}\right)^{\frac{3}{2}}\right\} = \frac{4\pi}{h^3}(2m)^{\frac{3}{2}}\mu_H{}^2 H\mu_0{}^{\frac{1}{2}}\left\{\left(\frac{\mu}{\mu_0}\right)^{\frac{1}{2}} - \left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{24}\left(\frac{\mu}{\mu_0}\right)^{-\frac{3}{2}}\right\}$$

$$= \frac{4\pi}{h^3}(2m)^{\frac{3}{2}}\mu_H{}^2 H\mu_0{}^{\frac{1}{2}}\left\{\left(1 - \left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right)^{\frac{1}{2}} - \left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{24}\left(1 - \left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right)^{-\frac{3}{2}}\right\} \quad \left[\because \mu = \mu_0\left(1 - \left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right)\right]$$

$$\approx \frac{4\pi}{h^3}(2m)^{\frac{3}{2}}\mu_H{}^2 H\mu_0{}^{\frac{1}{2}}\left\{\left(1 - \frac{1}{2}\left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right) - \left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{24}\left(1 + \frac{3}{2}\left(\frac{kT}{\mu_0}\right)^2\frac{\pi^2}{12}\right)\right\}$$

$$= \frac{4\pi}{h^3}(2m)^{\frac{3}{2}}\mu_H{}^2 H\mu_0{}^{\frac{1}{2}}\left\{1 - 3\left(\frac{kT}{\mu_0}\right)^2\left(\frac{\pi^2}{24}\right)^2\right\} \approx \frac{4\pi}{h^3}(2m)^{\frac{3}{2}}\frac{\mu_H{}^2 H}{\mu_0{}^{\frac{1}{2}}} \quad [\because kT \ll \mu_0] = \frac{4\pi}{h^3}(2m)^{\frac{3}{2}}\frac{\mu_H{}^2 H}{\mu_0}\mu_0{}^{\frac{3}{2}}$$

$$= \frac{4\pi}{h^3}(2m)^{\frac{3}{2}}\frac{\mu_H{}^2 H}{\mu_0}\left(\frac{h^2}{2m}\left(\frac{3n}{8\pi}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}} \quad \left[\because \mu_0 = E_{F0} = \frac{h^2}{2m_e}\left(\frac{3n_e}{8\pi}\right)^{\frac{2}{3}}\right] = \frac{4\pi}{3}\left(\frac{3n}{8\pi}\right)\frac{\mu_H{}^2 H}{\mu_0} = \frac{3}{2}\frac{n\mu_H{}^2 H}{\mu_0}$$

$$\therefore M = \frac{3n\mu_H{}^2}{2\mu_0}H \Longrightarrow \text{Magnetic susceptibility, } \chi = \frac{M}{H} = \frac{3n\mu_H{}^2}{2\mu_0} = \frac{3n\mu_H{}^2}{2kT_F} \quad \left[\begin{matrix}\because \mu_0 = E_{F0} = kT_F \\ T_F \text{ is Fermi temperature}\end{matrix}\right]$$

Electron Degeneracy Pressure of Non-Relativistic Cold Electron Gas

$g(p)\,dp = g_s\frac{4\pi V}{h^3}p^2\,dp = \frac{8\pi V}{h^3}p^2\,dp \quad [\because g_s = 2 \text{ for electrons}]$

$g(\epsilon)\,d\epsilon = g_s\frac{2\pi V}{h^3}(2m_e)^{\frac{3}{2}}\epsilon^{\frac{1}{2}}\,d\epsilon = \frac{4\pi V}{h^3}(2m_e)^{\frac{3}{2}}\epsilon^{\frac{1}{2}}\,d\epsilon$

$\because$ At $T = 0$, $f_{\text{FD}}(\epsilon) = 1$ for $\epsilon < E_{F0}$ and $f_{\text{FD}}(\epsilon) = 0$ for $\epsilon > E_{F0}$

$$\therefore N = \int_0^{\mu_0} g(\epsilon)\,d\epsilon = g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\int_0^{\mu_0}\epsilon^{\frac{1}{2}}\,d\epsilon = g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\left(\frac{2}{3}\mu_0{}^{\frac{3}{2}}\right) = \frac{2}{3}\mu_0 g(\mu_0) \Longrightarrow g(\mu_0) = \frac{3N}{2\mu_0}$$

$\because$ At $T = 0$, $f_{\text{FD}}(p) = 1$ for $p < p_{F0}$ and $f_{\text{FD}}(p) = 0$ for $p > p_{F0}$

$$N = \int_0^{p_{F0}} g(p)\,dp = g_s\frac{4\pi V}{h^3}\int_0^{p_{F0}} p^2\,dp = g_s\frac{4\pi V}{h^3}\cdot\frac{p_{F0}{}^3}{3} \Longrightarrow \boldsymbol{p_{F0}} = \left(\frac{3h^3 N}{4\pi g_s V}\right)^{\frac{1}{3}} = \boldsymbol{h\left(\frac{3n}{4\pi g_s}\right)^{\frac{1}{3}}} \quad \left[\text{where } n = \frac{N}{V}\right]$$

Fermi (angular) wavenumber at $T = 0$, $k_{F0} = \frac{p_{F0}}{\hbar} = 2\pi\left(\frac{3n}{4\pi g_s}\right)^{\frac{1}{3}} = 2\pi\left(\frac{3n_e}{8\pi}\right)^{\frac{1}{3}} \approx (3.09\text{ m})\,n_e{}^{\frac{1}{3}}\left[\begin{matrix}g_s = 2 \text{ for} \\ \text{electron gas}\end{matrix}\right]$

Fermi velocity at $T = 0$, $v_{F0} = \frac{p_{F0}}{m} = \frac{h}{m}\left(\frac{3n}{4\pi g_s}\right)^{\frac{1}{3}} = \frac{h}{m_e}\left(\frac{3n_e}{8\pi}\right)^{\frac{1}{3}} \approx (3.58\times10^{-4}\text{ m}^3/\text{s}^2)\,n_e{}^{\frac{1}{3}} \quad \left[\begin{matrix}g_s = 2\,;\,m = m_e \\ \text{for electron gas}\end{matrix}\right]$

Fermi energy at $T = 0$, $E_{F0} = \frac{p_{F0}{}^2}{2m} = \frac{h^2}{2m}\left(\frac{3n}{4\pi g_s}\right)^{\frac{2}{3}} = \frac{h^2}{2m_e}\left(\frac{3n_e}{8\pi}\right)^{\frac{2}{3}} \approx (3.65\times10^{-19}\text{ eV-m}^2)\,n_e{}^{\frac{2}{3}}$

$\left[\text{for metals, } n_e = \frac{\rho\times10^3}{a}N_{\text{A}}c \text{ where } \rho \text{ is density in kg/m}^3\text{ ; } a \text{ is } \textbf{gram}\text{-atomic mass ; } c \text{ is valency of the metal.}\right]$

For Copper at $T = 0$, $\rho = 9080\text{ kg/m}^3$; $a = 63.55$ g ; $c = 1$ $\therefore E_{F0} \approx 7.114$ eV

For Aluminium at $T = 0$, $\rho = 2735\text{ kg/m}^3$; $a = 27$ g ; $c = 3$ $\therefore E_{F0} \approx 11.765$ eV

Fermi temperature, $T_F = \frac{E_{F0}}{k} \approx (4.23\times10^{-15}\text{ K-m}^2)\,n_e{}^{\frac{2}{3}}$ [for electron gas] [Caution!]

Internal energy of fermion gas at $T = 0$, $U_0 = \int_0^{E_{F0}}\epsilon\,g(\epsilon)\,d\epsilon = g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\int_0^{E_{F0}}\epsilon^{\frac{3}{2}}\,d\epsilon = g_s\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\cdot\frac{2}{5}E_{F0}{}^{\frac{5}{2}}$

$$= g_s \frac{4\pi V}{5h^3}(2m)^{\frac{3}{2}} E_{F0}^{\frac{3}{2}} E_{F0} = g_s \frac{4\pi V}{5h^3}(2m)^{\frac{3}{2}} \left(\frac{h^2}{2m}\right)^{\frac{3}{2}} \left(\frac{3n}{4\pi g_s}\right) E_{F0} = \frac{\mathbf{3}}{\mathbf{5}} \boldsymbol{N E_{F0}} = \frac{3}{5}\frac{h^2}{2m}\left(\frac{3}{4\pi g_s}\right)^{\frac{2}{3}} \frac{N^{\frac{5}{3}}}{V^{\frac{2}{3}}} = \frac{\hbar^2 \pi^{\frac{4}{3}}}{10m}\frac{(3N)^{\frac{5}{3}}}{V^{\frac{2}{3}}}\frac{2^{\frac{2}{3}}}{g_s^{\frac{2}{3}}}$$

For electron gas at $T = 0$, $U_0 = \frac{\hbar^2 \pi^{\frac{4}{3}}}{10 m_e}\frac{(3N_e)^{\frac{5}{3}}}{V^{\frac{2}{3}}}$

Degeneracy pressure, $P_{D0} = -\left(\frac{\partial U_0}{\partial V}\right)_{T,N} = \frac{3}{5} \cdot \frac{h^2}{2m}\left(\frac{3}{4\pi g_s}\right)^{\frac{2}{3}} N^{\frac{5}{3}} \cdot \frac{2}{3} V^{-\frac{5}{3}} = \frac{2}{5} \cdot \frac{h^2}{2m}\left(\frac{3n}{4\pi g_s}\right)^{\frac{2}{3}} n = \frac{2}{5} \cdot n E_{F0}$

$P_{D0} = \frac{2}{5} \cdot n E_{F0} = \frac{2}{5V} \cdot \left(\frac{5}{3} U_0\right) \Rightarrow \boldsymbol{P_{D0} V} = \frac{\mathbf{2}}{\mathbf{3}} \boldsymbol{U_0} \Rightarrow \boldsymbol{\gamma'} = \frac{\mathbf{5}}{\mathbf{3}}$ [comparing with $PV = (\gamma' - 1)U$ for a polytrope]

Alternatively $\left(\text{if it is already assumed that } \gamma' = \frac{5}{3}\right)$, $P_D = \frac{2U_0}{3V} = \frac{2}{3V} \cdot \left(\frac{3}{5} N E_{F0}\right) = \frac{2}{5} \cdot n E_{F0}$

Atomic Nucleus as a Non-Relativistic Degenerate Fermi Gas

Radius of an atomic nucleus, $R = r_0 A^{\frac{1}{3}}$ [empirical formula ; A = mass number ; $r_0 = 1.25$ fm]

Nucleon density (concentration) $\approx \frac{A}{\frac{4\pi}{3} R^3} = \frac{A}{\frac{4\pi}{3}\left(r_0 A^{\frac{1}{3}}\right)^3} = \frac{A}{\frac{4\pi}{3} r_0{}^3 A} = \frac{1}{\frac{4\pi}{3} r_0{}^3} \approx 1.22 \times 10^{44}\ \text{m}^{-3}$

$\because$ neutrons and protons are not identical, $n_{\text{protons}} \approx n_{\text{neutrons}} \approx n = \frac{1.22 \times 10^{44}\ \text{m}^{-3}}{2} = 0.61 \times 10^{44}\ \text{m}^{-3}$

Fermi velocity, $v_F = \frac{p_F}{m} = \frac{h}{m}\left(\frac{3n}{8\pi}\right)^{\frac{1}{3}} \approx 1.5 \times 10^{22}$ m/s $\approx (5 \times 10^{13})\boldsymbol{c}$ [Surprise!]

Fermi energy, $E_F = \frac{h^2}{2m}\left(\frac{3n}{8\pi}\right)^{\frac{2}{3}} \approx 31$ MeV

Fermi temperature, $T_F = \frac{E_F}{k} \approx 3.6 \times 10^{11}$ K

Cold Non-Rotating Spherical White Dwarf Stars

Inward Gravitational Pressure balanced by Outward Electron Degeneracy Pressure

$dU_G = -\frac{Gm}{r} dm = -\frac{G}{r}\left(\frac{4\pi}{3} r^3 \rho\right)(4\pi r^2 dr \cdot \rho) = -\frac{16\pi^2}{3} G \rho^2 r^4\, dr$ [assuming uniform density]

$$U_G = \frac{16\pi^2}{3} G\rho^2 \int_0^R r^4\, dr = -\frac{16\pi^2}{15} G \rho^2 R^5 = -\frac{16\pi^2}{15} G \left(\frac{M}{\frac{4\pi}{3} R^3}\right)^2 R^5 = -\frac{3}{5}\frac{GM^2}{R}$$

$$U_D = \frac{\hbar^2 \pi^{\frac{4}{3}}}{10 m_e} \cdot \frac{(3N_e)^{\frac{5}{3}}}{V^{\frac{2}{3}}} = \frac{\hbar^2 \pi^{\frac{4}{3}}}{10 m_e} \cdot \frac{(3N_e)^{\frac{5}{3}}}{\left(\frac{4\pi}{3} R^3\right)^{\frac{2}{3}}} = \frac{\hbar^2 \pi^{\frac{2}{3}}}{10 m_e} \cdot \frac{3^{\frac{7}{3}} N_e^{\frac{5}{3}}}{2^{\frac{4}{3}} R^2} = \frac{9\hbar^2 \pi^{\frac{2}{3}}}{20 m_e} \cdot \left(\frac{3}{2}\right)^{\frac{1}{3}} \frac{N_e^{\frac{5}{3}}}{R^2}$$

Total internal energy, $U = U_D + U_G = \frac{A}{R^2} - \frac{B}{R}$ $\left[\text{where } A = \frac{9\hbar^2 \pi^{\frac{2}{3}}}{20 m_e} \cdot \left(\frac{3}{2}\right)^{\frac{1}{3}} N_e^{\frac{5}{3}} \text{ and } B = \frac{3}{5} G M^2\right]$

For minimum energy, $\frac{dU}{dR} = 0 \Rightarrow -\frac{2A}{R^3} + \frac{B}{R^2} = 0 \Rightarrow R = \frac{2A}{B} = \frac{\hbar^2 \pi^{\frac{2}{3}}}{m_e}\left(\frac{3}{2}\right)^{\frac{4}{3}} \frac{N_e^{\frac{5}{3}}}{GM^2}$

Put $m_e \approx \frac{m_N}{1823}$ and $N_e \approx \eta N_N \approx \eta \frac{M}{m_N}$ to get $R \approx \frac{\hbar^2 \pi^{\frac{2}{3}}}{\left(\frac{m_N}{1823}\right)}\left(\frac{3}{2}\right)^{\frac{4}{3}} \frac{\left(\eta \frac{M}{m_N}\right)^{\frac{5}{3}}}{GM^2} = \frac{1823 \hbar^2 \pi^{\frac{2}{3}}}{G m_N{}^{\frac{8}{3}}}\left(\frac{3}{2}\right)^{\frac{4}{3}} \frac{\eta^{\frac{5}{3}}}{M^{\frac{1}{3}}} \Rightarrow \boldsymbol{R} \propto \frac{\mathbf{1}}{\boldsymbol{M}^{\frac{\mathbf{1}}{\mathbf{3}}}}$

$$\Longrightarrow R \approx \frac{9.11574 \times 10^{16} \text{ m/kg}^3}{M^{\frac{1}{3}}} \quad [m_N = u \approx 1.66 \times 10^{-27} \text{ kg}\,; \eta = 0.5] \Longrightarrow \frac{R}{R_\odot} = 1.0419 \times 10^{-2} \left(\frac{M_\odot}{M}\right)^{\frac{1}{3}}$$

If $M = M_\odot \approx 2 \times 10^{30}$ kg, then its White Dwarf radius, $R \approx 7235$ km $\approx 0.01042\, R_\odot$

If $M = M_{\text{Ch}} = 1.44 M_\odot$, then its White Dwarf radius, $R \approx 6419$ km $\approx 0.00923\, R_\odot$

$$R = \frac{1823\hbar^2\pi^{\frac{2}{3}}}{G m_N^{\frac{8}{3}}} \cdot \left(\frac{3}{2}\right)^{\frac{4}{3}} \frac{\eta^{\frac{5}{3}}}{M^{\frac{1}{3}}} \Longrightarrow \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} = \frac{1823\hbar^2\pi^{\frac{2}{3}}}{G m_N^{\frac{8}{3}}} \cdot \left(\frac{3}{2}\right)^{\frac{4}{3}} \frac{\eta^{\frac{5}{3}}}{M^{\frac{1}{3}}} \Longrightarrow \boldsymbol{M_{\text{WD}} V_{\text{WD}} \approx 1823^3 \frac{27}{4} \frac{\hbar^6 \pi^3}{m_N{}^8 G^3}} \approx 9.4754 \times 10^{52} \text{ kg-m}^3$$

Electron Degeneracy Pressure of Relativistic Cold Electron Gas

$$g(p)\,dp = g_s \frac{4\pi V}{h^3} p^2\,dp = \frac{8\pi V}{h^3} p^2\,dp \quad [\because g_s = 2 \text{ for electrons}]$$

$$g(\epsilon)\,d\epsilon = g_s \frac{4\pi V}{h^3 c^2} p\epsilon\,d\epsilon = \frac{8\pi V}{h^3 c^2} p\epsilon\,d\epsilon$$

Number of fermions, $N = \int_0^{p_{F0}} g(p)\,dp = g_s \frac{4\pi V}{h^3} \int_0^{p_{F0}} p^2\,dp = g_s \frac{4\pi V}{h^3} \cdot \frac{p_{F0}{}^3}{3} \Longrightarrow p_{F0} = \left(\frac{3h^3 N}{4\pi g_s V}\right)^{\frac{1}{3}} = h\left(\frac{3n}{4\pi g_s}\right)^{\frac{1}{3}}$

Fermi velocity, $v_{F0} = \frac{p_{F0}}{\gamma m_e} = \frac{p_{F0}}{m_e}\sqrt{1 - \frac{v_{F0}{}^2}{c^2}} \Longrightarrow (m_e v_{F0})^2 = p_{F0}{}^2\left(1 - \frac{v_{F0}{}^2}{c^2}\right) \Longrightarrow v_{F0}{}^2 = \frac{(p_{F0}c)^2}{(m_e c)^2 + p_{F0}{}^2}$

$$\Longrightarrow v_{F0} = \left\{\frac{(hc)^2\left(\frac{3n}{4\pi g_s}\right)^{\frac{2}{3}}}{(m_e c)^2 + h^2\left(\frac{3n}{4\pi g_s}\right)^{\frac{2}{3}}}\right\}^{\frac{1}{2}} \quad \text{For electrons, } g_s = 2 \ \therefore v_{F0} = \left\{\frac{(hc)^2\left(\frac{3n}{8\pi}\right)^{\frac{2}{3}}}{(m_e c)^2 + h^2\left(\frac{3n}{8\pi}\right)^{\frac{2}{3}}}\right\}^{\frac{1}{2}} = \frac{1}{\sqrt{\frac{1}{c^2} + \left(\frac{m_e}{h}\right)^2\left(\frac{8\pi}{3n}\right)^{\frac{2}{3}}}}$$

Fermi energy, $E_F = \sqrt{p^2c^2 + m_e{}^2c^4} = \sqrt{h^2\left(\frac{3n}{8\pi}\right)^{\frac{2}{3}} c^2 + m_e{}^2 c^4}$

Atomic Nucleus as a Relativistic Degenerate Fermi Gas

Fermi velocity, $v_F = \frac{1}{\sqrt{\frac{1}{c^2} + \left(\frac{m}{h}\right)^2\left(\frac{8\pi}{3n}\right)^{\frac{2}{3}}}} \approx 0.744 \times 10^8$ m/s $\approx 0.25\boldsymbol{c}$ [Surprise!]

Fermi energy, $E_F = \sqrt{h^2\left(\frac{3n}{8\pi}\right)^{\frac{2}{3}} c^2 + m^2c^4} \approx 962$ MeV

Fermi temperature, $T_F = \frac{E_F}{k} \approx 1.1 \times 10^{13}$ K

Electron Degeneracy Pressure of Ultra-Relativistic (Extreme-Relativistic) Cold Electron Gas

$$g(p)\,dp = g_s \frac{4\pi V}{h^3} p^2\,dp = \frac{8\pi V}{h^3} p^2\,dp \quad [\because g_s = 2 \text{ for electrons}]$$

$$g(\epsilon)\,d\epsilon = g_s \frac{4\pi V}{h^3 c^3} \epsilon^2 d\epsilon = \frac{8\pi V}{h^3 c^3} \epsilon^2 d\epsilon$$

Number of fermions, $N = \int_0^{p_F} g(p)\,dp = g_s \frac{4\pi V}{h^3} \int_0^{p_F} p^2\,dp = g_s \frac{4\pi V}{h^3} \cdot \frac{p_F{}^3}{3} \Longrightarrow p_F = \left(\frac{3h^3 N}{4\pi g_s V}\right)^{\frac{1}{3}} = h\left(\frac{3n}{4\pi g_s}\right)^{\frac{1}{3}}$

Fermi energy, $E_F = p_{F0}c = hc\left(\frac{3n}{4\pi g_s}\right)^{\frac{1}{3}}$

Internal energy of fermion gas at $T = 0$, $U_0 = \int_0^{E_{F0}} \epsilon \cdot g(\epsilon)\, d\epsilon = g_s \frac{4\pi V}{h^3 c^3} \int_0^{E_{F0}} \epsilon^3\, d\epsilon = g_s \frac{4\pi V}{h^3 c^3} \cdot \frac{{E_{F0}}^4}{4} = \boldsymbol{\frac{3V}{4} n E_{F0}}$

$= \boldsymbol{\frac{3}{4} N E_{F0}} = \frac{3}{4} Nhc \left(\frac{3n}{4\pi g_s}\right)^{\frac{1}{3}} = \frac{3}{4} hc \left(\frac{3}{4\pi g_s}\right)^{\frac{1}{3}} \frac{N^{\frac{4}{3}}}{V^{\frac{1}{3}}} = \frac{\hbar c \pi^{\frac{2}{3}}}{4} \frac{(3N)^{\frac{4}{3}}}{V^{\frac{1}{3}}} \left(\frac{2}{g_s}\right)^{\frac{1}{3}}$

For electron (charged lepton), $U_0 \approx 2.32025(\hbar c) N^{\frac{4}{3}} V^{-\frac{1}{3}} = 1.43937(\hbar c) \frac{N^{\frac{4}{3}}}{R}$ $\left[\because g_s = 2 \text{ and } V = \frac{4\pi}{3} R^3\right]$

For neutrino (uncharged lepton), $U_0 \approx 2.92333(\hbar c) N^{\frac{4}{3}} V^{-\frac{1}{3}} = 1.81349(\hbar c) \frac{N^{\frac{4}{3}}}{R}$ $\left[\because g_s = 1 \text{ and } V = \frac{4\pi}{3} R^3\right]$

Degeneracy pressure, $P_{D0} = -\left(\frac{\partial U_0}{\partial V}\right)_{T,N} = \frac{3}{4} hc \left(\frac{3}{4\pi g_s}\right)^{\frac{1}{3}} N^{\frac{4}{3}} \cdot \frac{1}{3} V^{-\frac{4}{3}} = \frac{1}{4} hc \left(\frac{3}{4\pi g_s}\right)^{\frac{1}{3}} n^{\frac{4}{3}} = \frac{1}{4} \cdot n E_{F0} = \frac{1}{3} \frac{U_0}{V}$

Chandrasekhar Limit

Total internal energy, $U = U_D + U_G = \frac{A'}{R} - \frac{B}{R}$ $\left[\text{where } A' = \frac{3\hbar c \pi^{\frac{1}{3}}}{4} \left(\frac{3}{2}\right)^{\frac{2}{3}} N_e^{\frac{4}{3}} \text{ and } B = \frac{3}{5} GM^2\right]$

For minimum energy, $\frac{dU}{dR} = 0 \Longrightarrow -\frac{A'}{R^2} + \frac{B}{R^2} = 0 \Longrightarrow \left[\begin{array}{c}\text{No stable radius for a White Dwarf} \\ \text{with ultra-relativistic electrons}\end{array}\right]$

Critical point: $B = A' \Longrightarrow \frac{3}{5} GM^2 = 1.43937(\hbar c) N_e^{\frac{4}{3}} \Longrightarrow GM^2 = 2.39895(\hbar c) \left(\eta \frac{M}{m_N}\right)^{\frac{4}{3}}$ $\left[\because N_e \approx \eta N_N \approx \eta \frac{M}{m_N}\right]$

$\Longrightarrow M^{\frac{2}{3}} = 2.39895 \left(\frac{\eta}{m_N}\right)^{\frac{4}{3}} \left(\frac{\hbar c}{G}\right) \Longrightarrow \boldsymbol{M = M_{\mathrm{Ch}} \approx 3.71562\, \eta^2 \frac{{m_P}^3}{{m_N}^2}}$ $\left[\begin{array}{c} m_P = \sqrt{\frac{\hbar c}{G}} \approx 2.17647 \times 10^{-8} \text{ kg} \\ \text{is called Planck mass.}\end{array}\right]$

This gives $M_{\mathrm{Ch}} \approx 3.4138 \times 10^{30}$ kg $\approx 1.716\, M_\odot$ $\left[m_N \approx m_n\, ; \eta = 0.5\ ({}^{12}_{6}\mathrm{C}/{}^{16}_{8}\mathrm{O}/{}^{20}_{10}\mathrm{Ne})\, ; M_\odot = 1.989 \times 10^{30} \text{ kg}\right]$

$\left[\text{Exact expression for } M_{\mathrm{Ch}}: \frac{3}{5} GM^2 = \frac{3\pi^{\frac{1}{3}}}{4} \left(\frac{3}{2}\right)^{\frac{2}{3}} (\hbar c) N_e^{\frac{4}{3}} \Longrightarrow M^2 = \frac{5\pi^{\frac{1}{3}}}{4} \left(\frac{3}{2}\right)^{\frac{2}{3}} \left(\frac{\hbar c}{G}\right) \left(\eta \frac{M}{m_N}\right)^{\frac{4}{3}} \Longrightarrow \boldsymbol{M_{\mathrm{Ch}} = \frac{15\sqrt{5\pi}}{16} \eta^2 \frac{{m_P}^3}{{m_N}^2}}\right]$

Reference:
farside.ph.utexas.edu/teaching/sm1/lectures/node88.html
www.ast.cam.ac.uk/~pettini/STARS/Lecture14.pdf
www.fisica.edu.uy/~sbruzzone/FlexPaper_1.4.2_flash/prueba.pdf

Condition for degeneracy

Number of electrons per unit volume, $n_\mathrm{e} = \frac{N_\mathrm{e}}{V} = \frac{M}{A m_\mathrm{H}} Z \times \frac{1}{V} = \frac{Z}{A} \times \frac{M/V}{m_\mathrm{H}} \Longrightarrow \boldsymbol{n_\mathrm{e} = \frac{Z}{A} \frac{\rho}{m_\mathrm{H}}}$

$E_F = \frac{h^2}{2m_\mathrm{e}} \left(\frac{3n_\mathrm{e}}{8\pi}\right)^{\frac{2}{3}} = \frac{(2\pi)^2 \hbar^2}{2m_\mathrm{e}} \left(\frac{3n_\mathrm{e}}{8\pi}\right)^{\frac{2}{3}} = \frac{\hbar^2}{2m_\mathrm{e}} \left((2\pi)^3 \frac{3n_\mathrm{e}}{8\pi}\right)^{\frac{2}{3}} = \boldsymbol{\frac{\hbar^2}{2m_\mathrm{e}} (3\pi^2 n_e)^{\frac{2}{3}}} = \frac{\hbar^2}{2m_\mathrm{e}} \left\{3\pi^2 \left(\frac{Z}{A} \frac{\rho}{m_\mathrm{H}}\right)\right\}^{\frac{2}{3}} \Longrightarrow \boldsymbol{E_F \propto \rho^{\frac{2}{3}}}$

Thermal energy of an electron, $E_\mathrm{th} = \frac{3}{2} kT$

For degeneracy, $E_\mathrm{th} < E_F \Longrightarrow \frac{3}{2} kT < \frac{\hbar^2}{2m_\mathrm{e}} \left\{3\pi^2 \left(\frac{Z}{A} \frac{\rho}{m_\mathrm{H}}\right)\right\}^{\frac{2}{3}} \Longrightarrow \frac{T}{\rho^{2/3}} < \frac{\hbar^2}{3m_\mathrm{e} k} \left\{\frac{3\pi^2}{m_\mathrm{H}} \left(\frac{Z}{A}\right)\right\}^{\frac{2}{3}}$

$\Longrightarrow \boldsymbol{\frac{T}{\rho^{2/3}} < \mathcal{D}}$ $\left[\mathcal{D} = 1267.32 \text{ K-m}^2\text{kg}^{-\frac{2}{3}}\, ;\ \text{for WD, } \frac{Z}{A} \approx \frac{1}{2}\right]$

How important is electron degeneracy at the centers of Sun and Sirius B? For Sun, at centre, T_c = 1.57 × 10^7 K and ρ_c = 1.527 × 10^5 kg-m^3. For Sirius B, at centre, T_c = 7.6 × 10^7 K and ρ_c = 3.0 × 10^9 kg-m^3.

For Sun, $\frac{T}{\rho^{2/3}} = 5495.5\ \text{K-m}^2\text{kg}^{-\frac{2}{3}} > \mathcal{D} \Longrightarrow$ electron degeneracy is weak at Sun's centre

For Sirius B, $\frac{T}{\rho^{2/3}} = 36.5\ \text{K-m}^2\text{kg}^{-\frac{2}{3}} \ll \mathcal{D} \Longrightarrow$ electron degeneracy is strong at Sirius B's centre

Electron degeneracy pressure from PEP and HUP

Pressure integral, $P = \frac{1}{3}\int_0^\infty n_p pv\, dp \quad \left[n_p = \frac{\partial n_e}{\partial p}\right] \approx \frac{1}{3} n_e pv \quad \left[\begin{matrix}\text{assuming all electrons to} \\ \text{have the same momentum}\end{matrix}\right]$

In a completely degenerate electron gas, the electrons are packed astightly possible so that for a uniform number density, the separation between neighbouring electrons is $\sim {n_e}^{-\frac{1}{3}} = \Delta x$

$\Delta x \cdot \Delta p_x \approx \hbar$ [Heisenberg Uncertainty Principle] $\Longrightarrow \Delta p_x \approx \frac{\hbar}{\Delta x} = \hbar n_e^{\frac{1}{3}} \approx p_x$

In a 3D gas, by equipartion theorem, ${p_x}^2 = {p_y}^2 = {p_z}^2$

$\therefore p^2 = {p_x}^2 + {p_y}^2 + {p_z}^2 = 3{p_x}^2 \Longrightarrow p = \sqrt{3}p_x = \sqrt{3}\hbar n_e^{\frac{1}{3}}$

For non-relativistic electrons, $v = \frac{p}{m_e} = \frac{\sqrt{3}\hbar n_e^{\frac{1}{3}}}{m_e} = \frac{\sqrt{3}\hbar}{m_e}\left(\frac{Z}{A}\frac{\rho}{m_H}\right)^{\frac{1}{3}} \quad \left[\because n_e = \frac{Z}{A}\frac{\rho}{m_H}\right]$

$\therefore$ degeneracy pressure for non-relativistic electron gas, $\boldsymbol{P_{d,\mathrm{NR}}} = \frac{1}{3} n_e pv = \boldsymbol{\frac{\hbar^2}{m_e} n_e^{\frac{5}{3}}} = \boldsymbol{\frac{\hbar^2}{m_e}\left(\frac{Z}{A}\frac{\rho}{m_H}\right)^{\frac{5}{3}}} \Longrightarrow \boldsymbol{P_{d,\mathrm{NR}} \propto \rho^{\frac{5}{3}}}$

For ultra-relativistic electrons, $v \approx c$

$\therefore$ degeneracy pressure for ultra-relativistic electron gas, $\boldsymbol{P_{d,\mathrm{UR}}} = \frac{1}{3} n_e pv = \boldsymbol{\frac{\hbar c}{\sqrt{3}} n_e^{\frac{4}{3}}} = \boldsymbol{\frac{\hbar c}{\sqrt{3}}\left(\frac{Z}{A}\frac{\rho}{m_H}\right)^{\frac{4}{3}}} \Longrightarrow \boldsymbol{P_{d,\mathrm{NR}} \propto \rho^{\frac{4}{3}}}$

Mass-Volume relation

$$P_c = P_{d,\mathrm{NR}} \Longrightarrow \frac{2}{3}\pi G\rho^2 R^2 = \frac{\hbar^2}{m_e}\left(\frac{Z}{A}\frac{\rho}{m_H}\right)^{\frac{5}{3}} \Longrightarrow \frac{3\hbar^2}{2\pi G m_e}\left(\frac{Z}{A}\frac{1}{m_H}\right)^{\frac{5}{3}} = \rho^{\frac{1}{3}}R^2 = \left(\frac{M}{V}\right)^{\frac{1}{3}}\left(\frac{3V}{4\pi}\right)^{\frac{2}{3}} = (MV)^{\frac{1}{3}}\left(\frac{3}{4\pi}\right)^{\frac{2}{3}}$$

$$\Longrightarrow (MV)^{\frac{1}{3}} = \left(\frac{4\pi}{3}\right)^{\frac{2}{3}}\frac{3\hbar^2}{2\pi G m_e}\left(\frac{Z}{A}\frac{1}{m_H}\right)^{\frac{5}{3}} \Longrightarrow \boldsymbol{M_{\mathrm{WD}}V_{\mathrm{WD}}} = \left(\frac{4\pi}{3}\right)^2\left(\frac{3\hbar^2}{2\pi G m_e}\right)^3\left(\frac{Z}{A}\frac{1}{m_H}\right)^5 = \frac{6}{\pi}\left(\frac{\hbar^2}{G m_e}\right)^3\left(\frac{Z}{A}\frac{1}{m_H}\right)^5 \approx \boldsymbol{2.9 \times 10^{49}\ \text{kg-m}^3}$$

Chandrasekhar Limit (using EDP estimate)

$$P_c = P_{d,\mathrm{UR}} \Longrightarrow \frac{2}{3}\pi G{\rho_{\mathrm{Ch}}}^2 R^2 = \frac{\hbar c}{\sqrt{3}}\left(\frac{Z}{A}\frac{\rho_{\mathrm{Ch}}}{m_H}\right)^{\frac{4}{3}} \Longrightarrow \frac{\sqrt{3}}{2\pi}\frac{\hbar c}{G}\left(\frac{Z}{A}\frac{1}{m_H}\right)^{\frac{4}{3}} = \rho_{\mathrm{Ch}}^{\frac{2}{3}}R^2 = \left(\frac{3M_{\mathrm{Ch}}}{4\pi R^3}\right)^{\frac{2}{3}}R^2 = \left(\frac{3M_{\mathrm{Ch}}}{4\pi}\right)^{\frac{2}{3}}$$

$$\Longrightarrow M_{\mathrm{Ch}} = \frac{4\pi}{3}\left(\frac{\sqrt{3}}{2\pi}\right)^{\frac{3}{2}}\left(\frac{\hbar c}{G}\right)^{\frac{3}{2}}\left(\frac{Z}{A}\frac{1}{m_H}\right)^2 = \frac{1}{3^{\frac{1}{4}}}\sqrt{\frac{2}{\pi}}\left(\frac{\hbar c}{G}\right)^{\frac{3}{2}}\left(\frac{Z}{A}\frac{1}{m_H}\right)^2 \approx 5.67 \times 10^{29}\ \text{kg} \approx 0.285\ M_\odot$$

Justify the inadequacy of Newtonian mechanics to describe neutron stars. Assume M_{NS} = 1.4 $M_\odot$; R_{NS} = 10 km.

$$(v_{\mathrm{esc}})_{\mathrm{NS}} = \sqrt{\frac{2GM_{\mathrm{NS}}}{R_{\mathrm{NS}}}} = 1.93 \times 10^8\ \text{m/s} \approx 0.643c$$

Also, at the surface, $\frac{\text{gravitatonal potential of test mass}}{\text{rest mass energy of test mass}} = \frac{\left(\frac{GM_{\mathrm{NS}}m}{R_{\mathrm{NS}}}\right)}{mc^2} = \frac{GM_{\mathrm{NS}}}{R_{\mathrm{NS}}c^2} \approx 0.207$

Using virial theorem, determine the speed of an object arriving at the surface of a neutron star when it is dropped from a height of one metre. Assume M_{NS} = 1.4 $M_\odot$; R_{NS} = 10 km.

$$2K + U = 0 \Longrightarrow 2\left(\frac{1}{2}mv^2\right) - mgh = 0 \Longrightarrow v^2 = gh \Longrightarrow v = \sqrt{gh} = \sqrt{\left(\frac{GM}{R^2}\right)h} \approx 1.36 \times 10^6 \text{ m/s}$$

Using energy conservation, $v = \sqrt{2gh} \approx 1.93 \times 10^6$ m/s

White Dwarfs: Classification

DA (67%) Hydrogen-rich atmosphere (strong pressure-broadened Balmer Hydrogen spectral lines)
DB Helium-rich atmosphere (neutral Helium, He I, spectral lines)
DO Helium-rich atmosphere (ionized Helium, He II, spectral lines)
DQ (14%) Carbon-rich atmosphere (atomic or molecular Carbon spectral lines)
DZ metal-rich atmosphere (metal spectral lines)
DAB Hydrogen-rich and neutral Helium-rich atmosphere
DAO Hydrogen-rich and ionized Helium-rich atmosphere
DAZ Hydrogen-rich and metal-rich atmosphere
DBZ neutral Helium-rich and metal-rich atmosphere
DC no strong spectral lines
DX insufficiently clear spectral lines

Neutron star: Composition

Transition density (kg/m^3)	Composition	Degeneracy pressure main contributor
	Iron nuclei non-relativistic free electrons	electron
$\approx 1 \times 10^9$	electrons become relativistic	
	Iron nuclei relativistic free electrons	electron
$\approx 1 \times 10^{12}$	neutronization	
	neutron-rich nuclei relativistic free electrons	electron
$\approx 4 \times 10^{14}$	neutron drip	
	neutron-rich nuclei free neutrons relativistic free electrons	electron
$\approx 4 \times 10^{15}$	neutron degeneracy pressure dominates as electrons deplete	
subnuclear density	neutron-rich nuclei superfluid free (paired) neutrons relativistic free electrons	neutron
$\approx 2 \times 10^{17}$	nuclei dissolve	
nuclear density	superfluid free (paired) neutrons superfluid superconducting free (paired) protons relativistic free electrons	neutron
$\approx 4 \times 10^{17}$	pion production	
supernuclear density	superfluid free (paired) neutrons superfluid superconducting free (paired) protons relativistic free electrons pions	neutron

Estimate the density at which the electron capture process begins for a simple mixture of hydrogen nuclei (protons) and relativistic degenerate free electrons.

electron capture: $p^+ + e^- \rightarrow n + \nu_e$

Ignoring neutrino's rest mass energy, and in the limiting case when the neutrino carries away no kinetic energy,

$$m_e c^2\left(\frac{1}{\sqrt{1-v^2/c^2}}-1\right)=(m_n-m_p-m_e)c^2 \Rightarrow \left(\frac{m_e}{m_n-m_p}\right)^2=1-\frac{v^2}{c^2}=1-\frac{1}{1+\left(\frac{m_e c}{h}\right)^2\left(\frac{8\pi\mu m_H}{3\rho}\right)^{\frac{2}{3}}}$$

$$\left[\because v^2=\frac{1}{\frac{1}{c^2}+\left(\frac{m_e}{h}\right)^2\left(\frac{8\pi}{3n}\right)^{\frac{2}{3}}} \Rightarrow \frac{v^2}{c^2}=\frac{1}{1+\left(\frac{m_e c}{h}\right)^2\left(\frac{8\pi m_H}{3\eta\rho}\right)^{\frac{2}{3}}} \quad \begin{bmatrix}\because n=\frac{N_e}{V}=\frac{\eta N_N}{m/\rho}=\frac{\eta\rho}{m/N_N}=\frac{\eta\rho}{\overline{m}}=\frac{\eta\rho}{m_H}\\ \eta=\frac{Z}{A}\text{ assuming }\mu=1\text{ (full ionization)}\end{bmatrix}\right]$$

$$\Rightarrow 1-\left(\frac{m_e}{m_n-m_p}\right)^2=\frac{1}{1+\left(\frac{m_e c}{h}\right)^2\left(\frac{8\pi m_H}{3\eta\rho}\right)^{\frac{2}{3}}} \Rightarrow \left(\frac{m_e c}{h}\right)^2\left(\frac{8\pi m_H}{3\eta\rho}\right)^{\frac{2}{3}}=\frac{1}{1-\left(\frac{m_e}{m_n-m_p}\right)^2}-1$$

$$\Rightarrow \left(\frac{m_e c}{h}\right)^2\left(\frac{8\pi m_H}{3\eta\rho}\right)^{\frac{2}{3}}=\frac{\left(\frac{m_e}{m_n-m_p}\right)^2}{1-\left(\frac{m_e}{m_n-m_p}\right)^2} \Rightarrow \left(\frac{h}{m_e c}\right)^2\left(\frac{3\eta\rho}{8\pi m_H}\right)^{\frac{2}{3}}=\frac{1}{\left(\frac{m_e}{m_n-m_p}\right)^2}-1=\left(\frac{m_n-m_p}{m_e}\right)^2-1$$

$$\Rightarrow \left(\frac{3\eta\rho}{8\pi m_H}\right)^{\frac{2}{3}}=\frac{\left(\frac{m_n-m_p}{m_e}\right)^2-1}{\left(\frac{h}{m_e c}\right)^2} \Rightarrow \rho=\frac{8\pi c^3 m_H}{3h^3\eta}\left\{(m_n-m_p)^2-{m_e}^2\right\}^{\frac{3}{2}}=1.234\times10^{10}\ \text{kg/m}^3 \quad \begin{bmatrix}\eta=1\text{ for}\\ \text{Hydrogen}\end{bmatrix}$$

Nuclear one-decay

$A \longrightarrow B$

$N_{\mathrm{A}}+N_{\mathrm{B}}=N_{\mathrm{A0}}$

$-\frac{dN_{\mathrm{A}}}{dt}=\lambda_{\mathrm{A}}N_{\mathrm{A}} \Rightarrow \frac{dN_{\mathrm{A}}}{N_{\mathrm{A}}}=-\lambda_{\mathrm{A}}\,dt \Rightarrow \boldsymbol{N_{\mathrm{A}}=N_{\mathrm{A0}}e^{-\lambda_A t}}$ [number of non-decayed nuclei at time t]

$\Rightarrow \boldsymbol{N_B}=N_{\mathrm{A0}}-N_{\mathrm{A}}=\boldsymbol{N_{\mathrm{A0}}\left(1-e^{-\lambda_A t}\right)}$

Also, $\frac{dN_{\mathrm{B}}}{dt}=-\frac{dN_{\mathrm{A}}}{dt}=\lambda_{\mathrm{A}}N_{\mathrm{A}} \Rightarrow \frac{dN_{\mathrm{B}}}{dt}=\lambda_{\mathrm{A}}(N_{\mathrm{A0}}-N_{\mathrm{B}}) \Rightarrow \frac{dN_{\mathrm{B}}}{N_{\mathrm{A0}}-N_{\mathrm{B}}}=\lambda_{\mathrm{A}}dt \Rightarrow -\ln\left(\frac{N_{\mathrm{A0}}-N_{\mathrm{B}}}{N_{\mathrm{A0}}}\right)=\lambda_{\mathrm{A}}t$

$\Rightarrow \boldsymbol{N_{\mathrm{B}}}=N_{\mathrm{A0}}-N_{\mathrm{A0}}e^{-\lambda_A t}=\boldsymbol{N_{\mathrm{A0}}\left(1-e^{-\lambda_A t}\right)}$

$\epsilon=\boldsymbol{\epsilon_{\mathrm{A}}}=-Q_A\frac{dN_{\mathrm{A}}}{dt}=\lambda_{\mathrm{A}}Q_A N_{\mathrm{A}}=\boldsymbol{\lambda_{\mathrm{A}}Q_A N_{\mathrm{A0}}e^{-\lambda_A t}}$

Nuclear two-decay chain

$A \longrightarrow B \longrightarrow C$

$N_{\mathrm{A}}+N_{\mathrm{B}}+N_{\mathrm{C}}=N_{\mathrm{A0}}$

$$\begin{cases}\frac{dN_{\mathrm{A}}}{dt}=-\lambda_{\mathrm{A}}N_{\mathrm{A}} \Rightarrow \boldsymbol{N_{\mathrm{A}}=N_{\mathrm{A0}}e^{-\lambda_A t}}\\ \frac{dN_{\mathrm{B}}}{dt}=\lambda_{\mathrm{A}}N_{\mathrm{A}}-\lambda_{\mathrm{B}}N_{\mathrm{B}}=\lambda_{\mathrm{A}}N_{\mathrm{A0}}e^{-\lambda_A t}-\lambda_{\mathrm{B}}N_{\mathrm{B}}\\ \frac{dN_{\mathrm{C}}}{dt}=\lambda_{\mathrm{B}}N_{\mathrm{B}}\end{cases}$$

$$\frac{dN_{\mathrm{B}}}{dt}=\lambda_{\mathrm{A}}N_{\mathrm{A0}}e^{-\lambda_A t}-\lambda_{\mathrm{B}}N_{\mathrm{B}} \Rightarrow \frac{dN_{\mathrm{B}}}{dt}+\lambda_{\mathrm{B}}N_{\mathrm{B}}=\lambda_{\mathrm{A}}N_{\mathrm{A0}}e^{-\lambda_A t} \quad \begin{bmatrix}\text{first order non-homogeneous}\\ \text{linear differential equation}\end{bmatrix}$$

$$\Rightarrow e^{\lambda_{\mathrm{B}}t}\frac{dN_{\mathrm{B}}}{dt}+\lambda_{\mathrm{B}}N_{\mathrm{B}}e^{\lambda_{\mathrm{B}}t}=\lambda_{\mathrm{A}}N_{\mathrm{A0}}e^{-\lambda_{\mathrm{A}}t}e^{\lambda_{\mathrm{A}}t} \Rightarrow \frac{d\left(N_{\mathrm{B}}e^{\lambda_{\mathrm{B}}t}\right)}{dt}=\lambda_{\mathrm{A}}N_{\mathrm{A0}}e^{-(\lambda_{\mathrm{A}}-\lambda_{\mathrm{B}})t}$$

$$\Rightarrow N_{\mathrm{B}}e^{\lambda_{\mathrm{B}}t}=\frac{\lambda_{\mathrm{A}}N_{\mathrm{A0}}}{(\lambda_{\mathrm{A}}-\lambda_{\mathrm{B}})}\left(-e^{-(\lambda_{\mathrm{A}}-\lambda_{\mathrm{B}})t}+1\right) \Rightarrow \boldsymbol{N_{\mathrm{B}}}=\boldsymbol{\frac{\lambda_{\mathrm{A}}N_{\mathrm{A0}}}{(\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}})}\left(e^{-\lambda_{\mathrm{A}}t}-e^{-\lambda_{\mathrm{B}}t}\right)}$$

$$\frac{dN_{\mathrm{C}}}{dt}=\lambda_{\mathrm{B}}N_{\mathrm{B}}=\frac{\lambda_{\mathrm{B}}\lambda_{\mathrm{A}}N_{\mathrm{A0}}}{(\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}})}\left(e^{-\lambda_{\mathrm{A}}t}-e^{-\lambda_{\mathrm{B}}t}\right) \Rightarrow N_{\mathrm{C}}=\frac{\lambda_{\mathrm{B}}\lambda_{\mathrm{A}}N_{\mathrm{A0}}}{(\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}})}\left(-\frac{1}{\lambda_{\mathrm{A}}}\left(e^{-\lambda_{\mathrm{A}}t}-1\right)+\frac{1}{\lambda_{\mathrm{B}}}\left(e^{-\lambda_{\mathrm{B}}t}-1\right)\right)$$

$$\Rightarrow N_{\mathrm{C}} = \frac{\lambda_{\mathrm{B}}\lambda_{\mathrm{A}}N_{\mathrm{A0}}}{(\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}})}\left(\frac{1}{\lambda_{\mathrm{B}}}e^{-\lambda_{\mathrm{B}}t} - \frac{1}{\lambda_{\mathrm{A}}}e^{-\lambda_{\mathrm{A}}t} + \frac{1}{\lambda_{\mathrm{A}}} - \frac{1}{\lambda_{\mathrm{B}}}\right) = \boldsymbol{N_{\mathrm{A0}}\left(\frac{\lambda_{\mathrm{A}}}{\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}}}e^{-\lambda_{\mathrm{B}}t} - \frac{\lambda_{\mathrm{B}}}{\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}}}e^{-\lambda_{\mathrm{A}}t} + 1\right)}$$

Also, $N_{\mathrm{C}} = N_{\mathrm{A0}} - (N_{\mathrm{A}} + N_{\mathrm{B}}) = N_{\mathrm{A0}} - (N_{\mathrm{A}} + N_{\mathrm{B}}) = N_{\mathrm{A0}} - \left(N_{\mathrm{A0}}e^{-\lambda_A t} + \frac{\lambda_{\mathrm{A}}N_{\mathrm{A0}}}{\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}}}\left(e^{-\lambda_{\mathrm{A}}t} - e^{-\lambda_{\mathrm{B}}t}\right)\right)$

$$\Rightarrow N_{\mathrm{C}} = N_{\mathrm{A0}}\left(1 - e^{-\lambda_A t} - \frac{\lambda_{\mathrm{A}}}{\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}}}\left(e^{-\lambda_{\mathrm{A}}t} - e^{-\lambda_{\mathrm{B}}t}\right)\right) = N_{\mathrm{A0}}\left(1 - e^{-\lambda_A t}\left(1 + \frac{\lambda_{\mathrm{A}}}{\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}}}\right) + \frac{\lambda_{\mathrm{A}}}{\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}}}e^{-\lambda_{\mathrm{B}}t}\right)$$

$$\Rightarrow \boldsymbol{N_{\mathrm{C}} = N_{\mathrm{A0}}\left(1 - \frac{\lambda_{\mathrm{B}}}{\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}}}e^{-\lambda_A t} + \frac{\lambda_{\mathrm{A}}}{\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}}}e^{-\lambda_{\mathrm{B}}t}\right)}$$

$$\boldsymbol{\epsilon_{\mathrm{A}}} = -Q_{\mathrm{A}}\frac{dN_{\mathrm{A}}}{dt} = \lambda_{\mathrm{A}}Q_{\mathrm{A}}N_{\mathrm{A}} = \boldsymbol{\lambda_{\mathrm{A}}Q_{\mathrm{A}}N_{\mathrm{A0}}e^{-\lambda_A t}}$$

$$\boldsymbol{\epsilon_{\mathrm{B}}} = -Q_{\mathrm{B}}\frac{dN_{\mathrm{B}}}{dt} = -Q_{\mathrm{B}}(\lambda_{\mathrm{A}}N_{\mathrm{A}} - \lambda_{\mathrm{B}}N_{\mathrm{B}}) = -Q_{\mathrm{B}}\left(\lambda_{\mathrm{A}}N_{\mathrm{A0}}e^{-\lambda_A t} - \lambda_{\mathrm{B}}\frac{\lambda_{\mathrm{A}}N_{\mathrm{A0}}}{(\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}})}\left(e^{-\lambda_{\mathrm{A}}t} - e^{-\lambda_{\mathrm{B}}t}\right)\right)$$

$$\Rightarrow \epsilon_{\mathrm{B}} = -Q_{\mathrm{B}}N_{\mathrm{A0}}\left(\left(1 - \frac{\lambda_{\mathrm{B}}}{\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}}}\right)\lambda_{\mathrm{A}}e^{-\lambda_A t} + \frac{\lambda_{\mathrm{B}}\lambda_{\mathrm{A}}}{\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}}}e^{-\lambda_{\mathrm{B}}t}\right) = Q_{\mathrm{B}}N_{\mathrm{A0}}\frac{\lambda_{\mathrm{B}}\lambda_{\mathrm{A}}}{\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}}}\left(e^{-\lambda_A t} - e^{-\lambda_{\mathrm{B}}t}\right)$$

$$\epsilon = \epsilon_{\mathrm{A}} + \epsilon_{\mathrm{B}} = N_{\mathrm{A0}}\lambda_{\mathrm{A}}\left(Q_{\mathrm{A}}e^{-\lambda_A t} + Q_{\mathrm{B}}\frac{\lambda_{\mathrm{B}}}{\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}}}\left(e^{-\lambda_{\mathrm{A}}t} - e^{-\lambda_{\mathrm{B}}t}\right)\right) = N_{\mathrm{A0}}\frac{\lambda_{\mathrm{A}}\lambda_{\mathrm{B}}}{\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}}}\left(\left(\frac{\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}}}{\lambda_{\mathrm{B}}}Q_{\mathrm{A}} + Q_{\mathrm{B}}\right)e^{-\lambda_A t} - Q_{\mathrm{B}}e^{-\lambda_{\mathrm{B}}t}\right)$$

$$\Rightarrow \boldsymbol{\epsilon = N_{\mathrm{A0}}\frac{\lambda_{\mathrm{A}}\lambda_{\mathrm{B}}}{\lambda_{\mathrm{B}}-\lambda_{\mathrm{A}}}\left\{\left(\left(1 - \frac{\lambda_{\mathrm{A}}}{\lambda_{\mathrm{B}}}\right)Q_{\mathrm{A}} + Q_{\mathrm{B}}\right)e^{-\lambda_A t} - Q_{\mathrm{B}}e^{-\lambda_{\mathrm{B}}t}\right\}}$$

$$\Rightarrow \boldsymbol{\epsilon = N_{\mathrm{A0}}\frac{1}{\tau_{\mathrm{B}}-\tau_{\mathrm{A}}}\left\{\left(\left(\frac{\tau_{\mathrm{B}}}{\tau_{\mathrm{A}}} - 1\right)Q_{\mathrm{A}} - Q_{\mathrm{B}}\right)e^{-t/\tau_{\mathrm{A}}} + Q_{\mathrm{B}}e^{-t/\tau_{\mathrm{B}}}\right\}}$$

Type Ia Supernova from a White Dwarf

$$\begin{cases} {}^{56}_{28}\mathrm{Ni} \longrightarrow {}^{56}_{27}\mathrm{Co} + \gamma + \nu_{\mathrm{e}} \\ {}^{56}_{27}\mathrm{Co} \longrightarrow \begin{cases} {}^{56}_{26}\mathrm{Fe} + \gamma + \nu_{\mathrm{e}} & (81\%) \\ {}^{56}_{26}\mathrm{Fe} + \mathrm{e}^{+} + \gamma + \nu_{\mathrm{e}} & (19\%) \end{cases} \end{cases}$$

$$\epsilon = \frac{2.14\times10^{55}}{\tau_{\mathrm{Co}}-\tau_{\mathrm{Ni}}}\left(\frac{M_{\mathrm{Ni0}}}{M_{\odot}}\right)\left\{\left(\left(\frac{\tau_{\mathrm{Co}}}{\tau_{\mathrm{Ni}}} - 1\right)Q_{\mathrm{Ni}} - Q_{\mathrm{Co}}\right)e^{-t/\tau_{\mathrm{Ni}}} + Q_{\mathrm{Co}}e^{-t/\tau_{\mathrm{Co}}}\right\} \quad \left[N_{\mathrm{A0}} = \frac{M_{\mathrm{Ni0}}}{56\mathrm{u}} = \frac{M_{\odot}}{56\mathrm{u}}\frac{M_{\mathrm{Ni0}}}{M_{\odot}} \approx 2.14\times10^{55}\frac{M_{\mathrm{Ni0}}}{M_{\odot}}\right]$$

$$\Rightarrow \epsilon = \left(\frac{M_{\mathrm{Ni0}}}{M_{\odot}}\right)\left\{6.4443\times10^{36}e^{-t/8.8} + 1.4434\times10^{36}e^{-t/111.3}\right\}\ \mathrm{W} \quad \left[\begin{array}{c} Q_{\mathrm{Ni}} = \left(Q_{\gamma}\right)_{\mathrm{Ni}} \approx 1.75\ \mathrm{MeV}\ ;\ \tau_{\mathrm{Ni}} = 8.8\ \text{ days} \\ Q_{\mathrm{Co}} = \left(Q_{\gamma}\right)_{\mathrm{Co}} \approx 3.73\ \mathrm{MeV}\ ;\ \tau_{\mathrm{Co}} = 111.3\ \text{ days} \\ t \text{ in days ; unit is Watt.} \end{array}\right]$$

Time-integrated total energy, $\mathcal{E} = \int_0^{\infty} \epsilon\, dt = 1.878\times10^{43}\left(\frac{M_{\mathrm{Ni0}}}{M_{\odot}}\right)\ \mathrm{J}$

$$\epsilon_{\max} = \epsilon(t=0) = \left(\frac{M_{\mathrm{Ni0}}}{M_{\odot}}\right)7.8877\times10^{36}\ \mathrm{W}$$

Absolute magnitude, $\mathcal{M} = \mathcal{M}_{\odot} - 2.5\log\frac{L}{L_{\odot}} = +4.74 - 2.5\log\frac{L}{3.828\times10^{26}\ \mathrm{W}}$

If $L = \epsilon$, $\mathcal{M} = \mathcal{M}_{\odot} - 2.5\log\frac{L}{L_{\odot}} = +4.74 - 2.5\log\left\{\frac{7.8877\times10^{36}\ \mathrm{W}}{3.828\times10^{26}\ \mathrm{W}}\left(\frac{M_{\mathrm{Ni0}}}{M_{\odot}}\right)\right\}$

$$\Rightarrow \mathcal{M} \approx +4.74 - 2.5\left\{10.314 + \log\left(\frac{M_{\mathrm{Ni0}}}{M_{\odot}}\right)\right\} = -21.045 - 2.5\log\left(\frac{M_{\mathrm{Ni0}}}{M_{\odot}}\right)$$

Neutron Star (cold non-relativistic degenerate Fermi gas)

$$U_D = \frac{\hbar^2\pi^{\frac{4}{3}}}{10m_n}\frac{(3N_n)^{\frac{5}{3}}}{V^{\frac{2}{3}}} \quad [g_s = 2 \text{ for neutron}] = \left(\frac{3}{4\pi}\right)^{\frac{2}{3}}\frac{\hbar^2\pi^{\frac{4}{3}}}{10m_n}\frac{(3)^{\frac{5}{3}}}{R^2}\left(\frac{M}{m_n}\right)^{\frac{5}{3}} = \left(\frac{\pi}{4}\right)^{\frac{2}{3}}\frac{\hbar^2}{10}\frac{M^{\frac{5}{3}}}{R^2}\frac{3^{\frac{7}{3}}}{m_N^{\frac{8}{3}}} \quad [m_n \approx m_N]$$

Total internal energy, $U = U_D + U_G = \frac{C}{R^2} - \frac{B}{R}$ $\left[\text{where } C = \left(\frac{\pi}{4}\right)^{\frac{2}{3}} \frac{\hbar^2}{10} M^{\frac{5}{3}} \frac{3^{\frac{7}{3}}}{m_N^{\frac{8}{3}}} \text{ and } B = \frac{3}{5} G M^2\right]$

For minimum energy, $\frac{dU}{dR} = 0 \Longrightarrow -\frac{2C}{R^3} + \frac{B}{R^2} = 0 \Longrightarrow R = \frac{2C}{B} = \left(\frac{\pi}{4}\right)^{\frac{2}{3}} \frac{3^{\frac{4}{3}}\hbar^2}{G m_N^{\frac{8}{3}}} \frac{1}{M^{\frac{1}{3}}} = \left(\frac{3}{2}\right)^{\frac{4}{3}} \frac{\hbar^2 \pi^{\frac{2}{3}}}{G m_N^{\frac{8}{3}}} \frac{1}{M^{\frac{1}{3}}} \Longrightarrow \boldsymbol{R} \propto \frac{\mathbf{1}}{\boldsymbol{M}^{\frac{1}{3}}}$

$\Longrightarrow R \approx \frac{1.58753 \times 10^{14} \text{ m/kg}^3}{M^{\frac{1}{3}}}$ $[m_N = u \approx 1.66 \times 10^{-27} \text{ kg}] \Longrightarrow \frac{R}{R_\odot} = 1.8145 \times 10^{-5} \left(\frac{M_\odot}{M}\right)^{\frac{1}{3}}$

If $M = M_\odot \approx 2 \times 10^{30}$ kg, then its Neutron Star radius, $R \approx 12.6$ km $\approx 0.000018\, R_\odot$

If $M = M_{\text{Ch}} = 1.44 M_\odot$, then its Neutron Star radius, $R \approx 11.2$ km $\approx 0.000016\, R_\odot$

$R = \left(\frac{3}{2}\right)^{\frac{4}{3}} \frac{\hbar^2 \pi^{\frac{2}{3}}}{G m_N^{\frac{8}{3}}} \frac{1}{M^{\frac{1}{3}}} \Longrightarrow \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} = \left(\frac{3}{2}\right)^{\frac{4}{3}} \frac{\hbar^2 \pi^{\frac{2}{3}}}{G m_N^{\frac{8}{3}}} \frac{1}{M^{\frac{1}{3}}} \Longrightarrow \boldsymbol{M}_{\text{NS}} \boldsymbol{V}_{\text{NS}} = \frac{\mathbf{27}}{\mathbf{4}} \frac{\boldsymbol{\hbar}^6 \boldsymbol{\pi}^3}{\boldsymbol{m_N}^8 \boldsymbol{G}^3}$ $\left[\begin{matrix}\text{i.e. the more} \\ \text{massive it is,} \\ \text{the smaller} \\ \text{it becomes.}\end{matrix}\right] = 1.564 \times 10^{43}$ kg-m^3

Photon star

$U_R = 2.683113(\hbar c) \frac{N^{\frac{4}{3}}}{R} = 2.683113(\hbar c) \frac{1}{R} \left(\frac{\eta M}{m_N}\right)^{\frac{4}{3}}$ $\left[\begin{matrix}\because N = \frac{\eta M}{m_N} \text{ where } \eta = \text{number of leptons } (e^\pm) \text{ per nucleon} \\ \eta \leq 1\, ;\, \eta = 1 \text{ means 2 photons for every lepton pair}\end{matrix}\right]$

$\therefore U = U_R + U_G = \frac{B'}{R} - \frac{B}{R}$ $\left[\text{where } B' = 2.683113(\hbar c) \left(\frac{\eta M}{m_N}\right)^{\frac{4}{3}} \text{ and } B = \frac{3}{5} G M^2\right]$

Critical point: $B = B' \Longrightarrow \frac{3}{5} G M^2 = 2.683113(\hbar c) \left(\frac{\eta M}{m_N}\right)^{\frac{4}{3}} \Longrightarrow M^{\frac{2}{3}} = 4.471855 {m_{\text{P}}}^2 \left(\frac{\eta}{m_N}\right)^{\frac{4}{3}}$ $\left[m_{\text{P}} = \sqrt{\frac{\hbar c}{G}}\right]$

$\Longrightarrow M = 9.456525 {m_{\text{P}}}^3 \left(\frac{\eta}{m_N}\right)^2 = 3.535832 \times 10^{31} \eta^2 \leq 3.535832 \times 10^{31} = 17.777\, M_\odot$

Neutron-disintegration to prevent collapse and Proton Star (to be revised)

Critical point: $B = A' \Longrightarrow \frac{3}{5} G M^2 = \frac{3\hbar c \pi^{\frac{1}{3}}}{4} \left(\frac{3}{2}\right)^{\frac{2}{3}} {N_e}^{\frac{4}{3}} \Longrightarrow G M^2 = \frac{5\hbar c \pi^{\frac{1}{3}}}{4} \left(\frac{3}{2}\right)^{\frac{2}{3}} \left(\frac{M}{m_N}\right)^{\frac{4}{3}}$ $\left[\because N_e \approx N_N \approx \frac{M}{m_N}\right]$

$\Longrightarrow M^{\frac{2}{3}} = \frac{5\pi^{\frac{1}{3}}}{4 m_N^{\frac{4}{3}}} \left(\frac{3}{2}\right)^{\frac{2}{3}} \left(\frac{\hbar c}{G}\right) \Longrightarrow \boldsymbol{M} = \frac{3}{2} \frac{\pi^{\frac{1}{2}}}{{m_N}^2} \left(\frac{5}{4}\right)^{\frac{3}{2}} \left(\frac{\hbar c}{G}\right)^{\frac{3}{2}} \approx \mathbf{3.716} \frac{\boldsymbol{m_P}^3}{\boldsymbol{m_N}^2}$

This gives $M \approx 1.3657 \times 10^{31}$ kg $\approx 6.83\, M_\odot$

But for a proton star, $U = U_D + U_G + U_C = \frac{A'}{R} - \frac{B}{R} + \frac{D}{R}$ $\left[\text{where } D = U_C = \frac{3}{5} \frac{1}{4\pi\varepsilon_0} \frac{\left(N_p e\right)^2}{R} = \frac{3}{5} \frac{1}{4\pi\varepsilon_0} \frac{e^2}{R} \left(\frac{M}{m_p}\right)^2\right]$

$\boldsymbol{\varepsilon_0}$ is likely to have different value inside nucleonic matter.

Critical point: $B = A' + D \Longrightarrow B - D = A' \Longrightarrow \frac{3}{5} G M^2 - \frac{3}{5} \frac{1}{4\pi\varepsilon_0} e^2 \left(\frac{M}{m_p}\right)^2 = \frac{3\hbar c \pi^{\frac{1}{3}}}{4} \left(\frac{3}{2}\right)^{\frac{2}{3}} {N_e}^{\frac{4}{3}}$

$\Longrightarrow M^2 \left(1 - \frac{e^2}{4\pi\varepsilon_0} \frac{1}{{m_p}^2 G}\right) = \frac{5\pi^{\frac{1}{3}}}{4} \left(\frac{3}{2}\right)^{\frac{2}{3}} \left(\frac{M}{m_p}\right)^{\frac{4}{3}} {m_P}^2 \Longrightarrow M^{\frac{2}{3}} = \frac{\frac{5\pi^{\frac{1}{3}}}{4 m_p^{\frac{4}{3}}} \left(\frac{3}{2}\right)^{\frac{2}{3}} {m_P}^2}{1 - \frac{e^2}{4\pi\varepsilon_0} \frac{1}{{m_p}^2 G}} \Longrightarrow M = \frac{\frac{3\pi^{\frac{1}{2}}}{2{m_p}^2} \left(\frac{5}{4}\right)^{\frac{3}{2}} {m_P}^3}{\left(1 - \frac{e^2}{4\pi\varepsilon_0} \frac{1}{{m_p}^2 G}\right)^{\frac{3}{2}}}$

$\left(1 - \frac{e^2}{4\pi\varepsilon_0} \frac{1}{{m_p}^2 G}\right) = -1.3657 \times 10^{38} \ll 0$ $\therefore$ **Proton star is not feasible.**

Neutron Star: Rapid rotation

For white dwarf, $R_{\mathrm{WD}} = \dfrac{1823\hbar^2\pi^{\frac{2}{3}}}{m_N{}^{\frac{8}{3}}} \cdot \left(\dfrac{3}{2}\right)^{\frac{4}{3}} \dfrac{\eta^{\frac{5}{3}}}{GM_{\mathrm{WD}}{}^{\frac{1}{3}}} \approx \dfrac{8.05658 \times 10^{16}\ \mathrm{m/kg^3}}{M_{\mathrm{WD}}{}^{\frac{1}{3}}}$ $[\eta \approx 0.4643$ for Fe-56$]$

For neutron star, $R_{\mathrm{NS}} = \left(\dfrac{3}{2}\right)^{\frac{4}{3}} \dfrac{\hbar^2\pi^{\frac{2}{3}}}{m_N{}^{\frac{8}{3}}} \dfrac{1}{GM_{\mathrm{NS}}{}^{\frac{1}{3}}} \approx \dfrac{1.58753 \times 10^{14}\ \mathrm{m/kg^3}}{M_{\mathrm{NS}}{}^{\frac{1}{3}}}$

$\because M_{\mathrm{WD}} \approx M_{\mathrm{core}} = M_{\mathrm{NS}}$ [progenitor core] $\therefore \dfrac{\boldsymbol{R}_{\mathbf{core}}}{\boldsymbol{R}_{\mathbf{NS}}} \approx \dfrac{8.05658 \times 10^{16}\ \mathrm{m/kg^3}}{1.58753 \times 10^{14}\ \mathrm{m/kg^3}} \approx \mathbf{507.5}$

$\because I_{\mathrm{core}}\Omega_{\mathrm{core}} = I_{\mathrm{NS}}\Omega_{\mathrm{NS}}$ $\begin{bmatrix}\text{conservation of} \\ \text{angular momentum}\end{bmatrix} \Longrightarrow \boldsymbol{\Omega}_{\mathbf{NS}} = \dfrac{I_{\mathrm{core}}}{I_{\mathrm{NS}}}\Omega_{\mathrm{core}} = \left(\dfrac{\boldsymbol{R}_{\mathbf{core}}}{\boldsymbol{R}_{\mathbf{NS}}}\right)^{\mathbf{2}} \boldsymbol{\Omega}_{\mathbf{core}}$ $\begin{bmatrix}\because I = \frac{2}{5}MR^2 \text{ for a uniform} \\ \text{sphere, and } \because M_{\mathrm{core}} = M_{\mathrm{NS}}\end{bmatrix}$

$\Longrightarrow \Omega_{\mathrm{NS}} \approx 257556\ \Omega_{\mathrm{core}} \Longrightarrow \Pi_{\mathrm{NS}} \approx 3.88265 \times 10^{-6}\ \Pi_{\mathrm{core}}$

However, equatorial velocity of NS, $v_{\mathrm{NS}} = \Omega_{\mathrm{NS}}R_{\mathrm{NS}}$, must NOT exceed c (velocity of light).

In the binary system AR Scorpii consisting of a white dwarf pulsar and a red dwarf, the WD pulsar is rotating once every 1.97 minute. If it were a neutron star, what would have been its rotation period?

$\Pi_{\mathrm{NS}} \approx 3.88265 \times 10^{-6}\ \Pi_{\mathrm{WD}} \approx 4.59 \times 10^{-4}\ \mathrm{s} = 0.459\ \mathrm{ms}$

Klein-Gordon Equation

Energy expression for a relativistic free particle: $E = \dfrac{p^2}{2m}$

Put $p = -i\hbar\vec{\nabla}$ and $E = i\hbar\dfrac{\partial}{\partial t}$ to obtain $i\hbar\dfrac{\partial}{\partial t} = -\dfrac{\hbar^2}{2m}\nabla^2$ which is (time-dependent) Schrödinger operator eqn.

Energy expression for a relativistic free particle: $E^2 = p^2c^2 + m^2c^4$

Put $p = -i\hbar\vec{\nabla}$ and $E = i\hbar\dfrac{\partial}{\partial t}$ to obtain $-\hbar^2\dfrac{\partial^2}{\partial t^2} = -\hbar^2c^2\nabla^2 + m^2c^4$ which is the Klein-Gordon operator eqn.

$-\hbar^2\dfrac{\partial^2\Psi}{\partial t^2} = -\hbar^2c^2\nabla^2\Psi + m^2c^4\Psi \Longrightarrow \nabla^2\Psi - \dfrac{1}{c^2}\dfrac{\partial^2\Psi}{\partial t^2} = \dfrac{m^2c^2}{\hbar^2}\Psi \Longrightarrow \boxdot^2\Psi = \dfrac{m^2c^2}{\hbar^2}\Psi$ $\begin{bmatrix}\text{where } \boxdot^2 = \nabla^2 - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} \\ \text{is d'Alembert operator.}\end{bmatrix}$

Klein-Gordon equation is the relativistic variant of Schrödinger equation.

The plane wave $\Psi(\vec{r},t) = Ae^{i(\vec{\kappa}\cdot\vec{r}-\omega t)}$ is an eigenfunction of both energy and momentum operators with eigenvalues $\hbar\omega$ and $\hbar\vec{\kappa}$ respectively. $\left[i\hbar\dfrac{\partial\Psi}{\partial t} = \hbar\omega\Psi = E\Psi \text{ and } -i\hbar\vec{\nabla}\Psi = \hbar\vec{\kappa}\Psi\right]$

Putting $\Psi(\vec{r},t) = Ae^{i(\vec{\kappa}\cdot\vec{r}-\omega t)}$ in $i\hbar\dfrac{\partial\Psi}{\partial t} = -\dfrac{\hbar^2}{2m}\nabla^2\Psi$ we get $\hbar\omega = \dfrac{\hbar^2\kappa^2}{2m} \Longrightarrow E = \dfrac{p^2}{2m}$

Putting $\Psi(\vec{r},t) = Ae^{i(\vec{\kappa}\cdot\vec{r}-\omega t)}$ in $\boxdot^2\,\Psi = \dfrac{m^2c^2}{\hbar^2}\Psi$ we get $-\kappa^2 + \dfrac{\omega^2}{c^2} = \dfrac{m^2c^2}{\hbar^2} \Longrightarrow E^2 = p^2c^2 + m^2c^4$

$\Longrightarrow E = \pm\sqrt{p^2c^2 + m^2c^4}$ which is hard to interpret as energy cannot be negative for a free particle.

As the equation of continuity is invariant under Lorentz transformation, it should also hold for relativistic case.

Dirac's Equation for a Free Particle

$E = \pm\sqrt{p^2c^2 + m^2c^4} \Longrightarrow E\Psi = \pm\sqrt{p^2c^2 + m^2c^4}\,\Psi \Longrightarrow i\hbar\dfrac{\partial\Psi}{\partial t} = \pm\sqrt{-\hbar^2c^2\nabla^2 + m^2c^4}\,\Psi$

Put $p^2c^2 + m^2c^4 = -\hbar^2c^2\nabla^2 + m^2c^4 = (c\vec{\alpha}\cdot\vec{p} + \beta mc^2)^2$ [Dirac's idea]

$\Longrightarrow p^2c^2 + m^2c^4 = c^2\left(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta mc\right)^2 \Longrightarrow p^2 + m^2c^2 = \left(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta mc\right)^2$

$$\Longrightarrow p^2 + m^2c^2 = \left(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta mc\right)^2 \Longrightarrow \begin{cases} \alpha_x{}^2 = \alpha_y{}^2 = \alpha_z{}^2 = \beta^2 = 1 \\ \alpha_x\alpha_y + \alpha_y\alpha_x = \alpha_y\alpha_z + \alpha_z\alpha_y = \alpha_z\alpha_x + \alpha_x\alpha_z = 0 \\ \alpha_x\beta + \beta\alpha_x = \alpha_y\beta + \beta\alpha_y = \alpha_z\beta + \beta\alpha_z = 0 \end{cases}$$

i.e. α's and β anticommute in pairs and their squares are unity.

$$\therefore i\hbar\frac{\partial\Psi}{\partial t} = \pm(c\vec{\alpha}\cdot\vec{p}+\beta mc^2)\Psi = (c\vec{\alpha}\cdot\vec{p}+\beta mc^2)\Psi \begin{bmatrix}\text{replacing } \vec{\alpha} \text{ by } -\vec{\alpha} \\ \text{and } \beta \text{ by } -\beta \text{ does} \\ \text{not change the} \\ \text{relationships} \\ \text{between } \alpha\text{'s and } \beta\end{bmatrix} \Longrightarrow i\hbar\frac{\partial\Psi}{\partial t} = \left(-i\hbar c\vec{\alpha}\cdot\vec{\nabla}+\beta mc^2\right)\Psi$$

which is Dirac's wave equation for a relativistic free particle.

Dirac Matrices

$\because$ Hamiltonian, $\widehat{H} = (c\hat{\alpha}\cdot\hat{p}+\beta mc^2)$ has to be Hermitian, $\therefore \alpha_x, \alpha_y, \alpha_z$ and β must be Hermitian and square.
Also, their eigenvalues are ± 1 since their squares are unity.

$\alpha_x = \alpha_x\beta^2 \;\; [\because \beta^2 = 1] = \alpha_x\beta\beta = (\alpha_x\beta)\beta = -(\beta\alpha_x)\beta \;\; [\because \alpha_x\beta+\beta\alpha_x = 0 \Longrightarrow \alpha_x\beta = -\beta\alpha_x] = -\beta\alpha_x\beta$

$\therefore$ trace(α_x) = trace$(-\beta\alpha_x\beta)$ = −trace$(\alpha_x\beta\beta)$ $[\because$ trace(ABC) = trace$(BCA)]$ = −trace$(\alpha_x\beta^2)$ = −trace(α_x)

$\Longrightarrow$ trace$(\alpha_x) = 0$ [Trace of a square matrix is the sum of the elements of its main diagonal.]

$\Longrightarrow$ Number of $+1$ eigenvalues = Number of -1 eigenvalues $\Longrightarrow \alpha_x$ is of an even dimension.

Likewise, trace$\left(\alpha_y\right) = 0$; trace$(\alpha_z) = 0$; trace$(\beta) = 0$

Trying Pauli's three anticommuting 2×2 spin matrices $\sigma_x, \sigma_y, \sigma_z$ for $\alpha_x, \alpha_y, \alpha_z$, a fourth matrix, β, that anticommutes with these three cannot be found. So 4×4 matrices need to be formed. Now it is possible to construct four matrices fulfilling all the conditions. For that, arbitrarily, β is taken as $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix} = \begin{bmatrix}I&0\\0&-I\end{bmatrix}$ where $I = \begin{bmatrix}1&0\\0&1\end{bmatrix}$

$$\because \alpha_x\beta+\beta\alpha_x = 0 \Longrightarrow \begin{bmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\\a_{31}&a_{32}&a_{33}&a_{34}\\a_{41}&a_{42}&a_{43}&a_{44}\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}+\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\\a_{31}&a_{32}&a_{33}&a_{34}\\a_{41}&a_{42}&a_{43}&a_{44}\end{bmatrix} = 0$$

$$\Longrightarrow \begin{bmatrix}a_{11}&a_{12}&-a_{13}&-a_{14}\\a_{21}&a_{22}&-a_{23}&-a_{24}\\a_{31}&a_{32}&-a_{33}&-a_{34}\\a_{41}&a_{42}&-a_{43}&-a_{44}\end{bmatrix}+\begin{bmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\\-a_{31}&-a_{32}&-a_{33}&-a_{34}\\-a_{41}&-a_{42}&-a_{43}&-a_{44}\end{bmatrix} = 0 \Longrightarrow \begin{bmatrix}2a_{11}&2a_{12}&0&0\\2a_{21}&2a_{22}&0&0\\0&0&-2a_{33}&-2a_{34}\\0&0&-2a_{43}&-2a_{44}\end{bmatrix} \Longrightarrow 0$$

$\Longrightarrow a_{11} = a_{12} = a_{21} = a_{22} = a_{33} = a_{34} = a_{43} = a_{44} = 0$

$$\therefore \alpha_x = \begin{bmatrix}0&0&a_{13}&a_{14}\\0&0&a_{23}&a_{24}\\a_{31}&a_{32}&0&0\\a_{41}&a_{42}&0&0\end{bmatrix} \text{ ; Likewise, } \alpha_y = \begin{bmatrix}0&0&b_{13}&b_{14}\\0&0&b_{23}&b_{24}\\b_{31}&b_{32}&0&0\\b_{41}&b_{42}&0&0\end{bmatrix} \text{ ; } \alpha_z = \begin{bmatrix}0&0&c_{13}&c_{14}\\0&0&c_{23}&c_{24}\\c_{31}&c_{32}&0&0\\c_{41}&c_{42}&0&0\end{bmatrix}$$

Concisely, $\alpha_x = \begin{bmatrix}0&\alpha_{x1}\\\alpha_{x2}&0\end{bmatrix}$; $\alpha_y = \begin{bmatrix}0&\alpha_{y1}\\\alpha_{y2}&0\end{bmatrix}$; $\alpha_z = \begin{bmatrix}0&\alpha_{z1}\\\alpha_{z2}&0\end{bmatrix}$ where each element is a 2×2 matrix.

For convenience, $\alpha_{x1} = \alpha_{x2}$; $\alpha_{y1} = \alpha_{y2}$; $\alpha_{z1} = \alpha_{z2}$

$\therefore \alpha_x = \begin{bmatrix}0&\alpha_{x1}\\\alpha_{x1}&0\end{bmatrix}$; $\alpha_y = \begin{bmatrix}0&\alpha_{y1}\\\alpha_{y1}&0\end{bmatrix}$; $\alpha_z = \begin{bmatrix}0&\alpha_{z1}\\\alpha_{z1}&0\end{bmatrix}$

Now, $\alpha_x{}^2 = 1 \Longrightarrow \begin{bmatrix}0&\alpha_{x1}\\\alpha_{x1}&0\end{bmatrix}\begin{bmatrix}0&\alpha_{x1}\\\alpha_{x1}&0\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix} \Longrightarrow \begin{bmatrix}\alpha_{x1}{}^2&0\\0&\alpha_{x1}{}^2\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix} \Longrightarrow \alpha_{x1}{}^2 = 1$

Likewise, $\alpha_{y1}{}^2 = 1$; $\alpha_{z1}{}^2 = 1$ $\therefore \alpha_{x1}{}^2 = \alpha_{y1}{}^2 = \alpha_{z1}{}^2 = 1$

Now, $\alpha_x\alpha_y+\alpha_y\alpha_x = 0 \Longrightarrow \begin{bmatrix}0&\alpha_{x1}\\\alpha_{x1}&0\end{bmatrix}\begin{bmatrix}0&\alpha_{y1}\\\alpha_{y1}&0\end{bmatrix}+\begin{bmatrix}0&\alpha_{y1}\\\alpha_{y1}&0\end{bmatrix}\begin{bmatrix}0&\alpha_{x1}\\\alpha_{x1}&0\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$

$$\Longrightarrow \begin{bmatrix}\alpha_{x1}\alpha_{y1}&0\\0&\alpha_{x1}\alpha_{y1}\end{bmatrix}+\begin{bmatrix}\alpha_{y1}\alpha_{x1}&0\\0&\alpha_{y1}\alpha_{x1}\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix} \Longrightarrow \begin{bmatrix}\alpha_{x1}\alpha_{y1}+\alpha_{y1}\alpha_{x1}&0\\0&\alpha_{x1}\alpha_{y1}+\alpha_{y1}\alpha_{x1}\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$\Longrightarrow \alpha_{x1}\alpha_{y1}+\alpha_{y1}\alpha_{x1} = 0$; Likewise, $\alpha_{y1}\alpha_{z1}+\alpha_{z1}\alpha_{y1} = 0$; $\alpha_{z1}\alpha_{x1}+\alpha_{x1}\alpha_{z1} = 0$

$\therefore \alpha_{x1}\alpha_{y1}+\alpha_{y1}\alpha_{x1} = \alpha_{y1}\alpha_{z1}+\alpha_{z1}\alpha_{y1} = \alpha_{z1}\alpha_{x1}+\alpha_{x1}\alpha_{z1} = 0$

Now, $\alpha_{x1}\alpha_{y1}+\alpha_{y1}\alpha_{x1} = 0 \Longrightarrow \alpha_{x1}{}^2\alpha_{y1}+\alpha_{x1}\alpha_{y1}\alpha_{x1} = 0 \Longrightarrow \alpha_{y1}+\alpha_{x1}\alpha_{y1}\alpha_{x1} = 0$

$\Longrightarrow \alpha_{y1}{}^2+\alpha_{x1}\alpha_{y1}\alpha_{x1}\alpha_{y1} = 0 \Longrightarrow 1+\left(\alpha_{x1}\alpha_{y1}\right)^2 = 0 \Longrightarrow \left(\alpha_{x1}\alpha_{y1}\right)^2 = -1 \Longrightarrow \left(\alpha_{x1}\alpha_{y1}\right)^2 = -\alpha_{z1}{}^2$

$\Longrightarrow \alpha_{x1}\alpha_{y1} = i\alpha_{z1}$; Likewise, $\alpha_{y1}\alpha_{z1} = i\alpha_{x1}$; $\alpha_{z1}\alpha_{x1} = i\alpha_{y1}$

$\therefore\ \alpha_{x1}, \alpha_{y1}$ and α_{z1} are Pauli's spin matrices (spinors) σ_x, σ_y and σ_z (not necessarily) respectively.

$$\alpha_x = \begin{bmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix};\ \alpha_y = \begin{bmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix};\ \alpha_z = \begin{bmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

α_x, α_y and α_z are Hermitian since σ_x, σ_y and σ_z are Hermitian. They are called Dirac matrices.

Referring to Dirac's wave equation, $\because$ α's and β are 4×4 matrices $\therefore\ \Psi$ must be a 4×1 matrix.

In Dirac's scheme, $x = xI = \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{bmatrix};\ p_x = p_x I = \begin{bmatrix} p_x & 0 & 0 & 0 \\ 0 & p_x & 0 & 0 \\ 0 & 0 & p_x & 0 \\ 0 & 0 & 0 & p_x \end{bmatrix}$ Likewise for y, z, p_y and p_z.

$[x, \alpha_x] = [x, \alpha_y] = [x, \alpha_z] = [p_x, \alpha_x] = [p_x, \alpha_y] = [p_x, \alpha_z] = 0$ $\left[\begin{array}{l} \because\ \alpha_x, \alpha_y \text{ and } \alpha_z \text{ are constants.} \\ \text{Likewise for } y, z, p_y \text{ and } p_z. \end{array}\right]$

Show that the operator $c\hat{\alpha}$, where $\hat{\alpha}$ stands for Dirac matrix, can be interpreted as the velocity operator.

From Heisenberg's equation of motion, $\dfrac{d\hat{r}}{dt} = \dfrac{1}{i\hbar}[\hat{r}, \hat{H}]$ $\left[\text{where } \hat{H} = (c\hat{\alpha} \cdot \hat{p} + \beta mc^2)\right]$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{i\hbar}\left[x, c\left(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta mc\right)\right] = \frac{c}{i\hbar}\left[x, \left(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z\right)\right]$$

$$\Rightarrow \frac{dx}{dt} = \frac{c}{i\hbar}\left([x, \alpha_x p_x] + [x, \alpha_y p_y] + [x, \alpha_z p_z]\right)$$

$$= \frac{c}{i\hbar}\left(\alpha_x[x, p_x] + [x, \alpha_x]p_x + \alpha_y[x, p_y] + [x, \alpha_y]p_y + \alpha_z[x, p_z] + [x, \alpha_z]p_z\right)$$

$$= \frac{c}{i\hbar}\left(\alpha_x[x, p_x]\right)\ \ \left[\because [x, p_y] = [x, p_z] = 0\right] \Rightarrow \frac{dx}{dt} = c\alpha_x\ \ \left[\because [x, p_x] = i\hbar\right]$$

Likewise, $\dfrac{dy}{dt} = c\alpha_y$ and $\dfrac{dz}{dt} = c\alpha_z$ $\quad \therefore \dfrac{d\hat{r}}{dt} = c\hat{\alpha}$

Dielectric constant (relative permittivity) of free space or vacuum = 1

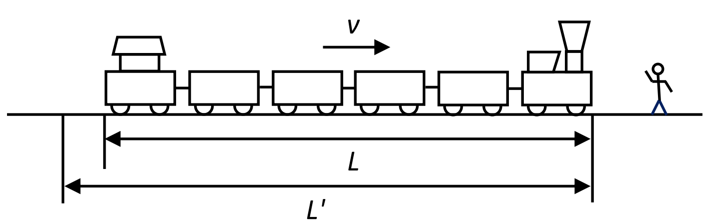
Dielectric constant of air is slightly greater than 1.

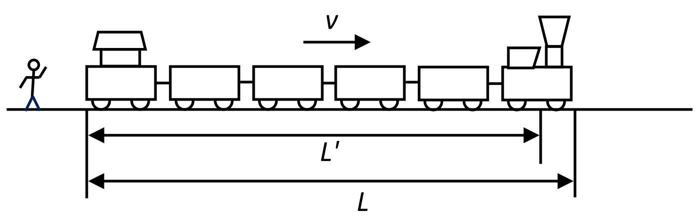
Dielectric constant of a polar liquid is greater than that of a non-polar liquid. That of a conductor is ideally infinite.

Liénard-Wiechart Potential

For an approaching train, $\begin{cases} L' = L + vt \\ L' = ct \end{cases} \Rightarrow \dfrac{L' - L}{v} = \dfrac{L'}{c} \Rightarrow L' - L = \dfrac{v}{c}L' \Rightarrow L' = \dfrac{L}{1 - v/c} \approx L(1 + v/c)$

Caboose was at $L' = L + vt$ when light started from there. The light reached the observer in time, $t = L'/c$.





Apparent length, L' would be $L + v(L'/c)$. $\therefore L' = L + v(L'/c) \Longrightarrow L' = \dfrac{L}{1 - v/c} \approx L(1 + v/c)$ [when $v \ll c$]

Apparent volume of the train is $\tau' = L'WH = \dfrac{LWH}{1 - v/c} = \dfrac{\tau}{1 - v/c}$

For a receding train, $\begin{cases} L' = L - vt \\ L' = ct \end{cases} \Longrightarrow \dfrac{L - L'}{v} = \dfrac{L'}{c} \Longrightarrow L - L' = \dfrac{v}{c}L' \Longrightarrow L' = \dfrac{L}{1 + v/c}$

Engine was at $L' = L - vt$ when light started from there. The light reached the observer in time, $t = L'/c$.

Apparent length, L' would be $L - v\,L'/c$. $\therefore L' = L - v\,L'/c \Longrightarrow L' = \dfrac{L}{1 + v/c} \approx L(1 - v/c)$ [when $v \ll c$]

Apparent volume of the train is $\tau' = L'WH = \dfrac{LWH}{1 + v/c} = \dfrac{\tau}{1 + v/c}$

When the train is approaching, its velocity is opposite to the observer's line of sight. Whereas, when the train is receding, its velocity is along the observer's line of sight.

If train's velocity makes an angle θ with the observer's line of sight, for an approaching train,

$\begin{cases} L' = L + vt \\ L'\cos\theta = ct \end{cases} \Longrightarrow \dfrac{L' - L}{v} = \dfrac{L'\cos\theta}{c} \Longrightarrow L' - L = \dfrac{v}{c}L'\cos\theta \Longrightarrow L' = \dfrac{L}{1 - v\cos\theta/c} \approx L(1 + v\cos\theta/c)$

Apparent volume of the train is $\tau' = L'WH = \dfrac{LWH}{1 - v\cos\theta/c} = \dfrac{\tau}{1 - v\cos\theta/c}$ $\left[\text{When } \theta = \dfrac{\pi}{2}, \tau' = \tau.\right]$

If train's velocity makes an angle θ with the observer's line of sight, for a receding train,

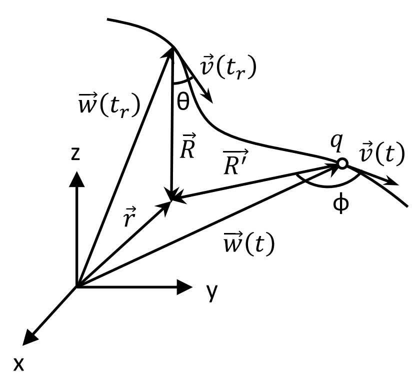
$\begin{cases} L' = L - vt \\ L'\cos\theta = ct \end{cases} \Longrightarrow \dfrac{L - L'}{v} = \dfrac{L'\cos\theta}{c} \Longrightarrow L - L' = \dfrac{v}{c}L'\cos\theta \Longrightarrow L' = \dfrac{L}{1 + v\cos\theta/c} \approx L(1 - v\cos\theta/c)$

Apparent volume of the train is $\tau' = L'WH = \dfrac{LWH}{1 + v\cos\theta/c} = \dfrac{\tau}{1 + v\cos\theta/c}$ $\left[\text{When } \theta = \dfrac{\pi}{2}, \tau' = \tau.\right]$

When the train is approaching, its velocity makes an angle $(\pi - \theta)$ with the observer's line of sight. Whereas, when the train is receding, its velocity makes an angle θ with the observer's line of sight.

This effect does **not** distort the dimensions (height and width) perpendicular to the motion of the train. Although the light from the far side is delayed in reaching the observer (when at an angle to the train) relative to the light from the near side, the two sides would look the same distance apart.

If $\hat{R}$ is the unit vector from the train to the observer (i.e. opposite to the line of sight), $\tau' = \dfrac{\tau}{1 - \hat{R}\cdot\vec{v}/c}$



$$\therefore V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r'},t_r)}{R}d\tau' = \frac{1}{4\pi\varepsilon_0}\frac{1}{R}\int\rho(\vec{r'},t_r)\,d\tau' = \frac{1}{4\pi\varepsilon_0}\frac{1}{R}\frac{q}{(1 - \hat{R}\cdot\vec{v}/c)} = \boldsymbol{\frac{1}{4\pi\varepsilon_0}\frac{qc}{(Rc - \vec{R}\cdot\vec{v})}}$$

$$\therefore \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\frac{\rho(\vec{r'},t_r)\vec{v}(t_r)}{R}d\tau' = \frac{\mu_0}{4\pi}\frac{\vec{v}}{R}\int\rho(\vec{r'},t_r)\,d\tau' = \frac{\mu_0}{4\pi}\frac{\vec{v}}{R}\frac{q}{(1 - \hat{R}\cdot\vec{v}/c)} = \boldsymbol{\frac{\mu_0}{4\pi}\frac{qc\vec{v}}{(Rc - \vec{R}\cdot\vec{v})}} = \frac{\vec{v}}{c^2}V(\vec{r},t)$$

If θ is the angle between $\vec{R}$ and $\vec{v}$, then $V(\vec{r},t) = \boldsymbol{\dfrac{1}{4\pi\varepsilon_0}\dfrac{1}{R}\dfrac{q}{(1 - v\cos\theta/c)}}$ and $\boldsymbol{\vec{A}(\vec{r},t) = \dfrac{\mu_0}{4\pi}\dfrac{\vec{v}}{R}\dfrac{q}{(1 - v\cos\theta/c)}}$

If ϕ is the angle between $\vec{R'}$ and $\vec{v}$, velocity, $\vec{v}$ is **constant**, and $\vec{R'} = \vec{R} - \vec{v}(t - t_r)$ i.e. $\vec{R} = \vec{R'} + \vec{v}(t - t_r)$ then

$$R=\sqrt{R'^2+v^2(t-t_r)^2-2R'vt\cos\,(\pi-\phi)}=\sqrt{R'^2+v^2(t-t_r)^2+2R'vt\cos\phi}\quad \text{[cosine rule]}$$

$$R^2=R'^2+v^2(t-t_r)^2-2R'vt\cos\,(\pi-\phi)=R'^2+v^2(t-t_r)^2+2R'vt\cos\phi=R'^2+\frac{v^2}{c^2}R^2+2\frac{v}{c}R'R\cos\phi$$

$$\Longrightarrow R^2\left(1-\frac{v^2}{c^2}\right)-\left(2\frac{v}{c}R'\cos\phi\right)R-R'^2=0\Longrightarrow R=\frac{\left(2\frac{v}{c}R'\cos\phi\right)\pm\sqrt{\left(2\frac{v}{c}R'\cos\phi\right)^2+4R'^2\left(1-\frac{v^2}{c^2}\right)}}{2\left(1-\frac{v^2}{c^2}\right)}$$

$$\Longrightarrow\frac{R}{R'}=\frac{\frac{v}{c}\cos\phi\pm\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}}{\left(1-\frac{v^2}{c^2}\right)}\Longrightarrow\frac{R^2}{R'^2}=\frac{\frac{v^2}{c^2}\cos^2\phi+\left(1-\frac{v^2}{c^2}\sin^2\phi\right)\pm2\frac{v}{c}\cos\phi\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}}{\left(1-\frac{v^2}{c^2}\right)^2}$$

$$\frac{R'}{\sin\theta}=\frac{R}{\sin\,(\pi-\phi)}\quad\text{[sine rule]}\Longrightarrow\sin\theta=\frac{R'}{R}\sin\,(\pi-\phi)=\frac{R'}{R}\sin\phi\Longrightarrow\cos\theta=\sqrt{1-\frac{R'^2}{R^2}\sin^2\phi}$$

$$=\sqrt{1-\frac{\left(1-\frac{v^2}{c^2}\right)^2\sin^2\phi}{\left(\frac{v}{c}\cos\phi\pm\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}\right)^2}}=\sqrt{1-\frac{\left(1-\frac{v^2}{c^2}\right)^2\sin^2\phi}{\frac{v^2}{c^2}\cos^2\phi+\left(1-\frac{v^2}{c^2}\sin^2\phi\right)\pm2\frac{v}{c}\cos\phi\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}}}$$

$$=\frac{1}{\frac{v}{c}\cos\phi\pm\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}}\sqrt{\frac{v^2}{c^2}\cos^2\phi+\left(1-\frac{v^2}{c^2}\sin^2\phi\right)\pm2\frac{v}{c}\cos\phi\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}-\left(1-\frac{v^2}{c^2}\right)^2\sin^2\phi}$$

$$=\frac{1}{\frac{v}{c}\cos\phi\pm\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}}\sqrt{\frac{v^2}{c^2}\cos^2\phi+1-\frac{v^2}{c^2}\sin^2\phi\pm2\frac{v}{c}\cos\phi\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}-\sin^2\phi-\left(\frac{v^2}{c^2}\right)^2\sin^2\phi+2\frac{v^2}{c^2}\sin^2\phi}$$

$$=\frac{1}{\frac{v}{c}\cos\phi\pm\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}}\sqrt{\frac{v^2}{c^2}\pm2\frac{v}{c}\cos\phi\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}+\cos^2\phi-\left(\frac{v^2}{c^2}\right)^2\sin^2\phi}$$

$$=\frac{1}{\frac{v}{c}\cos\phi\pm\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}}\sqrt{\frac{v^2}{c^2}\left(1-\frac{v^2}{c^2}\sin^2\phi\right)\pm2\frac{v}{c}\cos\phi\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}+\cos^2\phi}=\frac{\cos\phi\pm\frac{v}{c}\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}}{\frac{v}{c}\cos\phi\pm\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}}$$

$$\left[\text{Caution: Don't write }\frac{v}{c}\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}\pm\cos\phi\ \text{ because the belongs to }\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}\right]$$

$$\therefore R\left(1-\frac{v}{c}\cos\theta\right)=R'\left(\frac{\frac{v}{c}\cos\phi\pm\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}}{1-\frac{v^2}{c^2}}\right)\left\{1-\frac{v}{c}\left(\frac{\cos\phi\pm\frac{v}{c}\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}}{\frac{v}{c}\cos\phi\pm\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}}\right)\right\}$$

$$=\frac{R'}{\left(1-\frac{v^2}{c^2}\right)}\left\{\left(\frac{v}{c}\cos\phi\pm\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}\right)-\frac{v}{c}\left(\cos\phi\pm\frac{v}{c}\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}\right)\right\}$$

$$=\frac{R'}{\left(1-\frac{v^2}{c^2}\right)}\left\{\pm\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}\mp\frac{v^2}{c^2}\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}\right\}=\frac{R'}{\left(1-\frac{v^2}{c^2}\right)}\left(\pm\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}\right)\left(1-\frac{v^2}{c^2}\right)=\pm R'\sqrt{1-\frac{v^2}{c^2}\sin^2\phi}$$

$$\therefore \boldsymbol{V(\vec{r},t)} = \frac{\mathbf{1}}{\mathbf{4\pi\varepsilon_0}}\frac{\mathbf{1}}{\boldsymbol{R'}}\frac{\boldsymbol{q}}{\sqrt{\mathbf{1}-\frac{\boldsymbol{v^2}}{\boldsymbol{c^2}}\sin^2\boldsymbol{\phi}}} \text{ and } \boldsymbol{\vec{A}(\vec{r},t)} = \frac{\boldsymbol{\mu_0}}{\mathbf{4\pi}}\frac{\boldsymbol{\vec{v}}}{\boldsymbol{R'}}\frac{\boldsymbol{q}}{\sqrt{\mathbf{1}-\frac{\boldsymbol{v^2}}{\boldsymbol{c^2}}\sin^2\boldsymbol{\phi}}}$$

Fields due to a moving point charge

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{qc}{\left(Rc-\vec{R}\cdot\vec{v}\right)} \text{ and } \vec{A}(\vec{r},t) = \frac{\vec{v}}{c^2}V(\vec{r},t)$$

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} \text{ and } \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{R} = \vec{r}-\vec{w}(t_r) \text{ and } \vec{v} = \frac{d\vec{w}}{dt_r} \text{ and } R = |\vec{r}-\vec{w}(t_r)| = c(t-t_r) \Rightarrow \vec{\nabla}R = -c\vec{\nabla}t_r \Rightarrow -c\vec{\nabla}t_r = \vec{\nabla}\sqrt{\vec{R}\cdot\vec{R}} = \frac{\vec{\nabla}\left(\vec{R}\cdot\vec{R}\right)}{2\sqrt{\vec{R}\cdot\vec{R}}}$$

$$\Rightarrow \boldsymbol{-c\vec{\nabla}t_r} = \frac{\mathbf{1}}{\boldsymbol{R}}\left\{\left(\boldsymbol{\vec{R}}\cdot\boldsymbol{\vec{\nabla}}\right)\boldsymbol{\vec{R}}+\boldsymbol{\vec{R}}\times\left(\boldsymbol{\vec{\nabla}}\times\boldsymbol{\vec{R}}\right)\right\} \quad \left[\because \vec{\nabla}\left(\vec{F}\cdot\vec{G}\right) = \left(\vec{F}\cdot\vec{\nabla}\right)\vec{G}+\left(\vec{G}\cdot\vec{\nabla}\right)\vec{F}+\vec{F}\times\left(\vec{\nabla}\times\vec{G}\right)+\vec{G}\times\left(\vec{\nabla}\times\vec{F}\right)\right]$$

$$\left(\vec{R}\cdot\vec{\nabla}\right)\vec{R} = \left(\vec{R}\cdot\vec{\nabla}\right)(\vec{r}-\vec{w}) \ \left[\because \vec{R} = \vec{r}-\vec{w}\right] = \left(\vec{R}\cdot\vec{\nabla}\right)\vec{r}-\left(\vec{R}\cdot\vec{\nabla}\right)\vec{w} \Rightarrow \left(\boldsymbol{\vec{R}}\cdot\boldsymbol{\vec{\nabla}}\right)\boldsymbol{\vec{R}} = \boldsymbol{\vec{R}}-\boldsymbol{\vec{v}}\left(\boldsymbol{\vec{R}}\cdot\boldsymbol{\vec{\nabla}t_r}\right)$$

$$\left[\begin{aligned}&\because \left(\vec{R}\cdot\vec{\nabla}\right)\vec{r} = \left(R_x\frac{\partial}{\partial x}+R_y\frac{\partial}{\partial y}+R_z\frac{\partial}{\partial z}\right)\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right) = R_x\hat{\imath}+R_y\hat{\jmath}+R_z\hat{k} \Rightarrow \left(\boldsymbol{\vec{R}}\cdot\boldsymbol{\vec{\nabla}}\right)\boldsymbol{\vec{r}} = \boldsymbol{\vec{R}}\\ &\text{and } \left(\vec{R}\cdot\vec{\nabla}\right)\vec{w} = \left(R_x\frac{\partial}{\partial x}+R_y\frac{\partial}{\partial y}+R_z\frac{\partial}{\partial z}\right)\vec{w} = \left(R_x\frac{\partial t_r}{\partial x}\frac{d}{dt_r}+R_y\frac{\partial t_r}{\partial y}\frac{d}{dt_r}+R_z\frac{\partial t_r}{\partial z}\frac{d}{dt_r}\right)\vec{w} \Rightarrow \left(\boldsymbol{\vec{R}}\cdot\boldsymbol{\vec{\nabla}}\right)\boldsymbol{\vec{w}} = \left(\boldsymbol{\vec{R}}\cdot\boldsymbol{\vec{\nabla}t_r}\right)\boldsymbol{\vec{v}}\end{aligned}\right]$$

$$\vec{R}\times\left(\vec{\nabla}\times\vec{R}\right) = \vec{R}\times\left(\vec{\nabla}\times\vec{r}-\vec{\nabla}\times\vec{w}\right) \ \left[\because \vec{R} = \vec{r}-\vec{w}\right] = \vec{R}\times\left(-\vec{\nabla}\times\vec{w}\right) \Rightarrow \boldsymbol{\vec{R}}\times\left(\boldsymbol{\vec{\nabla}}\times\boldsymbol{\vec{R}}\right) = \boldsymbol{\vec{R}}\times\left(\boldsymbol{\vec{v}}\times\boldsymbol{\vec{\nabla}t_r}\right)$$

$$\left[\begin{aligned}&\because \vec{\nabla}\times\vec{w} = \left(\frac{\partial w_z}{\partial y}-\frac{\partial w_y}{\partial z}\right)\hat{\imath}+\left(\frac{\partial w_x}{\partial z}-\frac{\partial w_z}{\partial x}\right)\hat{\jmath}+\left(\frac{\partial w_y}{\partial x}-\frac{\partial w_x}{\partial y}\right)\hat{k}\\ &= \left(\frac{dw_z}{dt_r}\frac{\partial t_r}{\partial y}-\frac{dw_y}{dt_r}\frac{\partial t_r}{\partial z}\right)\hat{\imath}+\left(\frac{dw_x}{dt_r}\frac{\partial t_r}{\partial z}-\frac{dw_z}{dt_r}\frac{\partial t_r}{\partial x}\right)\hat{\jmath}+\left(\frac{dw_y}{dt_r}\frac{\partial t_r}{\partial x}-\frac{dw_x}{dt_r}\frac{\partial t_r}{\partial y}\right)\hat{k} \Rightarrow \boldsymbol{\vec{\nabla}}\times\boldsymbol{\vec{w}} = -\boldsymbol{\vec{v}}\times\boldsymbol{\vec{\nabla}t_r}\end{aligned}\right]$$

$$\therefore -c\vec{\nabla}t_r = \frac{1}{R}\left\{\left(\vec{R}\cdot\vec{\nabla}\right)\vec{R}+\vec{R}\times\left(\vec{\nabla}\times\vec{R}\right)\right\} = \frac{1}{R}\left\{\vec{R}-\vec{v}\left(\vec{R}\cdot\vec{\nabla}t_r\right)+\vec{R}\times\left(\vec{v}\times\vec{\nabla}t_r\right)\right\}$$

$$= \frac{1}{R}\left\{\vec{R}-\vec{v}\left(\vec{R}\cdot\vec{\nabla}t_r\right)+\vec{v}\left(\vec{R}\cdot\vec{\nabla}t_r\right)-\vec{\nabla}t_r\left(\vec{R}\cdot\vec{v}\right)\right\} \ \left[\because \vec{F}\times\left(\vec{G}\times\vec{C}\right) = \vec{G}\left(\vec{F}\cdot\vec{C}\right)-\vec{C}\left(\vec{F}\cdot\vec{G}\right)\right]$$

$$\Rightarrow -c\vec{\nabla}t_r = \frac{1}{R}\left\{\vec{R}-\vec{\nabla}t_r\left(\vec{R}\cdot\vec{v}\right)\right\} \Rightarrow -Rc\vec{\nabla}t_r = \vec{R}-\vec{\nabla}t_r\left(\vec{R}\cdot\vec{v}\right) \Rightarrow -\vec{\nabla}t_r\left(Rc-\vec{R}\cdot\vec{v}\right) = \vec{R} \Rightarrow \boldsymbol{\vec{\nabla}t_r} = \frac{\boldsymbol{-\vec{R}}}{\boldsymbol{Rc-\vec{R}\cdot\vec{v}}}$$

$$\vec{\nabla}V = \frac{qc}{4\pi\varepsilon_0}\vec{\nabla}\left(\frac{1}{Rc-\vec{R}\cdot\vec{v}}\right) = \frac{qc}{4\pi\varepsilon_0}\frac{-1}{\left(Rc-\vec{R}\cdot\vec{v}\right)^2}\vec{\nabla}\left(Rc-\vec{R}\cdot\vec{v}\right) = \frac{qc}{4\pi\varepsilon_0}\frac{c^2\vec{\nabla}t_r+\vec{\nabla}\left(\vec{R}\cdot\vec{v}\right)}{\left(Rc-\vec{R}\cdot\vec{v}\right)^2} \quad \left[\because \vec{\nabla}R = -c\vec{\nabla}t_r\right]$$

$$\vec{\nabla}\left(\vec{R}\cdot\vec{v}\right) = \left(\vec{R}\cdot\vec{\nabla}\right)\vec{v}+\left(\vec{v}\cdot\vec{\nabla}\right)\vec{R}+\vec{R}\times\left(\vec{\nabla}\times\vec{v}\right)+\vec{v}\times\left(\vec{\nabla}\times\vec{R}\right)$$

$$\left(\vec{R}\cdot\vec{\nabla}\right)\vec{v} = \left(R_x\frac{\partial}{\partial x}+R_y\frac{\partial}{\partial y}+R_z\frac{\partial}{\partial z}\right)\vec{v} = \left(R_x\frac{\partial t_r}{\partial x}\frac{d}{dt_r}+R_y\frac{\partial t_r}{\partial y}\frac{d}{dt_r}+R_z\frac{\partial t_r}{\partial z}\frac{d}{dt_r}\right)\vec{v} \Rightarrow \left(\boldsymbol{\vec{R}}\cdot\boldsymbol{\vec{\nabla}}\right)\boldsymbol{\vec{v}} = \left(\boldsymbol{\vec{R}}\cdot\boldsymbol{\vec{\nabla}t_r}\right)\boldsymbol{\vec{a}}$$

$$\left(\vec{v}\cdot\vec{\nabla}\right)\vec{R} = \left(\vec{v}\cdot\vec{\nabla}\right)\vec{r}-\left(\vec{v}\cdot\vec{\nabla}\right)\vec{w} \ \left[\because \vec{R} = \vec{r}-\vec{w}\right] \Rightarrow \left(\boldsymbol{\vec{v}}\cdot\boldsymbol{\vec{\nabla}}\right)\boldsymbol{\vec{R}} = \boldsymbol{\vec{v}}-\left(\boldsymbol{\vec{v}}\cdot\boldsymbol{\vec{\nabla}t_r}\right)\boldsymbol{\vec{v}}$$

$$\left[\begin{aligned}&\because \left(\vec{v}\cdot\vec{\nabla}\right)\vec{r} = \left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}\right)\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right) = v_x\hat{\imath}+v_y\hat{\jmath}+v_z\hat{k} \Rightarrow \left(\boldsymbol{\vec{v}}\cdot\boldsymbol{\vec{\nabla}}\right)\boldsymbol{\vec{r}} = \boldsymbol{\vec{v}}\\ &\text{and } \left(\vec{v}\cdot\vec{\nabla}\right)\vec{w} = \left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}\right)\vec{w} = \left(v_x\frac{\partial t_r}{\partial x}\frac{d}{dt_r}+v_y\frac{\partial t_r}{\partial y}\frac{d}{dt_r}+v_z\frac{\partial t_r}{\partial z}\frac{d}{dt_r}\right)\vec{w} \Rightarrow \left(\boldsymbol{\vec{v}}\cdot\boldsymbol{\vec{\nabla}}\right)\boldsymbol{\vec{w}} = \left(\boldsymbol{\vec{v}}\cdot\boldsymbol{\vec{\nabla}t_r}\right)\boldsymbol{\vec{v}}\end{aligned}\right]$$

$$\vec{R}\times\left(\vec{\nabla}\times\vec{v}\right) = \vec{R}\times\left(-\vec{a}\times\vec{\nabla}t_r\right) \Rightarrow \boldsymbol{\vec{R}}\times\left(\boldsymbol{\vec{\nabla}}\times\boldsymbol{\vec{v}}\right) = -\boldsymbol{\vec{R}}\times\left(\boldsymbol{\vec{a}}\times\boldsymbol{\vec{\nabla}t_r}\right)$$

$$\left[\begin{aligned}&\because \vec{\nabla}\times\vec{v}=\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\hat{\imath}+\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\hat{\jmath}+\left(\frac{\partial v_y}{\partial x}-\frac{\partial w_x}{\partial y}\right)\hat{k}\\&=\left(\frac{dv_z}{dt_r}\frac{\partial t_r}{\partial y}-\frac{dv_y}{dt_r}\frac{\partial t_r}{\partial z}\right)\hat{\imath}+\left(\frac{dv_x}{dt_r}\frac{\partial t_r}{\partial z}-\frac{dv_z}{dt_r}\frac{\partial t_r}{\partial x}\right)\hat{\jmath}+\left(\frac{dw_y}{dt_r}\frac{\partial t_r}{\partial x}-\frac{dw_x}{dt_r}\frac{\partial t_r}{\partial y}\right)\hat{k}\Rightarrow \boldsymbol{\vec{\nabla}\times\vec{v}=-\vec{a}\times\vec{\nabla}t_r}\end{aligned}\right]$$

$$\vec{v}\times(\vec{\nabla}\times\vec{R})=\vec{v}\times(\vec{\nabla}\times\vec{r}-\vec{\nabla}\times\vec{w})\quad[\because \vec{R}=\vec{r}-\vec{w}]=-\vec{v}\times(\vec{\nabla}\times\vec{w})\quad[\because \vec{\nabla}\times\vec{r}=0]\Rightarrow \boldsymbol{\vec{v}\times(\vec{\nabla}\times\vec{R})=\vec{v}\times(\vec{v}\times\vec{\nabla}t_r)}$$

$$\therefore \vec{\nabla}(\vec{R}\cdot\vec{v})=(\vec{R}\cdot\vec{\nabla}t_r)\vec{a}+\{\vec{v}-(\vec{v}\cdot\vec{\nabla}t_r)\vec{v}\}+\{-\vec{R}\times(\vec{a}\times\vec{\nabla}t_r)\}+\{\vec{v}\times(\vec{v}\times\vec{\nabla}t_r)\}$$
$$=(\vec{R}\cdot\vec{\nabla}t_r)\vec{a}+\{\vec{v}-(\vec{v}\cdot\vec{\nabla}t_r)\vec{v}\}-\{\vec{a}(\vec{R}\cdot\vec{\nabla}t_r)-\vec{\nabla}t_r(\vec{R}\cdot\vec{a})\}+\{\vec{v}(\vec{v}\cdot\vec{\nabla}t_r)-\vec{\nabla}t_r(\vec{v}\cdot\vec{v})\}$$
$$[\because \vec{F}\times(\vec{G}\times\vec{C})=\vec{G}(\vec{F}\cdot\vec{C})-\vec{C}(\vec{F}\cdot\vec{G})]=\vec{v}+\vec{\nabla}t_r(\vec{R}\cdot\vec{a})-v^2\vec{\nabla}t_r\Rightarrow \boldsymbol{\vec{\nabla}(\vec{R}\cdot\vec{v})=\vec{v}+(\vec{R}\cdot\vec{a}-v^2)\vec{\nabla}t_r}$$

$$\therefore \vec{\nabla}V=\frac{qc}{4\pi\varepsilon_0}\frac{c^2\vec{\nabla}t_r+\vec{\nabla}(\vec{R}\cdot\vec{v})}{(Rc-\vec{R}\cdot\vec{v})^2}=\frac{qc}{4\pi\varepsilon_0}\frac{c^2\vec{\nabla}t_r+\vec{v}+(\vec{R}\cdot\vec{a}-v^2)\vec{\nabla}t_r}{(Rc-\vec{R}\cdot\vec{v})^2}=\frac{qc}{4\pi\varepsilon_0}\frac{\vec{v}+(c^2-v^2+\vec{R}\cdot\vec{a})\vec{\nabla}t_r}{(Rc-\vec{R}\cdot\vec{v})^2}$$
$$=\frac{qc}{4\pi\varepsilon_0}\frac{\vec{v}+(c^2-v^2+\vec{R}\cdot\vec{a})\dfrac{-\vec{R}}{Rc-\vec{R}\cdot\vec{v}}}{(Rc-\vec{R}\cdot\vec{v})^2}\Rightarrow \boldsymbol{\vec{\nabla}V=\frac{qc}{4\pi\varepsilon_0}\frac{(Rc-\vec{R}\cdot\vec{v})\vec{v}-(c^2-v^2+\vec{R}\cdot\vec{a})\vec{R}}{(Rc-\vec{R}\cdot\vec{v})^3}}$$

$$\frac{\partial\vec{A}}{\partial t}=\frac{\partial}{\partial t}\left(\frac{\vec{v}}{c^2}V(\vec{r},t)\right)=\frac{qc}{4\pi\varepsilon_0c^2}\frac{\partial}{\partial t}\left(\frac{\vec{v}}{Rc-\vec{R}\cdot\vec{v}}\right)\quad\left[\because V(\vec{r},t)=\frac{1}{4\pi\varepsilon_0}\frac{qc}{(Rc-\vec{R}\cdot\vec{v})}\right]\Rightarrow \boldsymbol{\frac{\partial\vec{A}}{\partial t}=\frac{\mu_0}{4\pi}(qc)\frac{\partial}{\partial t}\left(\frac{\vec{v}}{Rc-\vec{R}\cdot\vec{v}}\right)}$$
$$R=c(t-t_r)\Rightarrow\frac{1}{c}\frac{\partial R}{\partial t}=1-\frac{\partial t_r}{\partial t}\Rightarrow\frac{\partial t_r}{\partial t}=1-\frac{1}{c}\frac{\partial R}{\partial t}=1-\frac{1}{c}\frac{\partial}{\partial t}\sqrt{\vec{R}\cdot\vec{R}}$$
$$=1-\frac{1}{2c\sqrt{\vec{R}\cdot\vec{R}}}\frac{\partial}{\partial t}(\vec{R}\cdot\vec{R})=1-\frac{1}{2Rc}\left(2\vec{R}\cdot\frac{\partial\vec{R}}{\partial t}\right)=1-\frac{1}{Rc}\left(\vec{R}\cdot\frac{\partial\vec{R}}{\partial t}\right)=1-\frac{1}{Rc}\left(\vec{R}\cdot\frac{\partial}{\partial t}(\vec{r}-\vec{w})\right)\quad[\because \vec{R}=\vec{r}-\vec{w}]$$
$$=1-\frac{1}{Rc}\left(\vec{R}\cdot\left(\frac{\partial\vec{r}}{\partial t}-\frac{\partial\vec{w}}{\partial t}\right)\right)=1-\frac{1}{Rc}\left(\vec{R}\cdot\left(-\frac{\partial t_r}{\partial t}\frac{\partial\vec{w}}{\partial t_r}\right)\right)\quad\left[\because\frac{\partial\vec{r}}{\partial t}=0\right]=1-\frac{1}{Rc}\left(\vec{R}\cdot\left(-\frac{\partial t_r}{\partial t}\frac{\partial\vec{w}}{\partial t_r}\right)\right)$$
$$\Rightarrow\frac{\partial t_r}{\partial t}=1+\frac{(\vec{R}\cdot\vec{v})}{Rc}\frac{\partial t_r}{\partial t}\quad\left[\because\frac{\partial\vec{w}}{\partial t_r}=\vec{v}\right]\Rightarrow\frac{\partial t_r}{\partial t}\left(1-\frac{\vec{R}\cdot\vec{v}}{Rc}\right)=1\Rightarrow\boldsymbol{\frac{\partial t_r}{\partial t}=\frac{Rc}{Rc-\vec{R}\cdot\vec{v}}}$$

$$\frac{\partial}{\partial t_r}\left(\frac{\vec{v}}{Rc-\vec{R}\cdot\vec{v}}\right)=\left(\frac{1}{Rc-\vec{R}\cdot\vec{v}}\right)\frac{\partial\vec{v}}{\partial t_r}+\vec{v}\frac{\partial}{\partial t_r}\left(\frac{1}{Rc-\vec{R}\cdot\vec{v}}\right)=\frac{\vec{a}}{Rc-\vec{R}\cdot\vec{v}}+\frac{-\vec{v}}{(Rc-\vec{R}\cdot\vec{v})^2}\frac{\partial}{\partial t_r}(Rc-\vec{R}\cdot\vec{v})$$
$$=\frac{\vec{a}}{Rc-\vec{R}\cdot\vec{v}}-\frac{\vec{v}}{(Rc-\vec{R}\cdot\vec{v})^2}\left(c\frac{\partial R}{\partial t_r}-\frac{\partial}{\partial t_r}(\vec{R}\cdot\vec{v})\right)=\frac{\vec{a}}{Rc-\vec{R}\cdot\vec{v}}-\frac{\vec{v}}{(Rc-\vec{R}\cdot\vec{v})^2}\left(-\frac{c}{R}(\vec{R}\cdot\vec{v})-(\vec{R}\cdot\vec{a}-v^2)\right)$$
$$\left[\begin{aligned}&\because\frac{\partial R}{\partial t_r}=\frac{\partial}{\partial t_r}\sqrt{\vec{R}\cdot\vec{R}}=\frac{1}{2\sqrt{\vec{R}\cdot\vec{R}}}\frac{\partial}{\partial t_r}(\vec{R}\cdot\vec{R})=\frac{1}{2R}\left(2\vec{R}\cdot\frac{\partial\vec{R}}{\partial t_r}\right)=\frac{1}{R}\left(\vec{R}\cdot\frac{\partial\vec{R}}{\partial t_r}\right)=\frac{1}{R}\left(\vec{R}\cdot\left(\frac{\partial\vec{r}}{\partial t_r}-\frac{\partial\vec{w}}{\partial t_r}\right)\right)\Rightarrow\boldsymbol{\frac{\partial R}{\partial t_r}=-\frac{\vec{R}\cdot\vec{v}}{R}}\\&\text{and }\frac{\partial}{\partial t_r}(\vec{R}\cdot\vec{v})=\vec{R}\cdot\frac{\partial\vec{v}}{\partial t_r}+\frac{\partial\vec{R}}{\partial t_r}\cdot\vec{v}=\vec{R}\cdot\vec{a}+\frac{\partial}{\partial t_r}(\vec{r}-\vec{w})\cdot\vec{v}=\vec{R}\cdot\vec{a}+\left(\frac{\partial\vec{r}}{\partial t_r}-\frac{\partial\vec{w}}{\partial t_r}\right)\cdot\vec{v}\Rightarrow\boldsymbol{\frac{\partial}{\partial t_r}(\vec{R}\cdot\vec{v})=\vec{R}\cdot\vec{a}-v^2}\\&\text{as }\frac{\partial\vec{r}}{\partial t_r}=0\text{ and }\frac{\partial\vec{w}}{\partial t_r}=\vec{v}\end{aligned}\right]$$
$$\therefore\frac{\partial}{\partial t_r}\left(\frac{\vec{v}}{Rc-\vec{R}\cdot\vec{v}}\right)=\frac{\vec{a}}{Rc-\vec{R}\cdot\vec{v}}+\frac{\vec{v}\left(c(\vec{R}\cdot\vec{v})+R(\vec{R}\cdot\vec{a}-v^2)\right)}{R(Rc-\vec{R}\cdot\vec{v})^2}=\frac{R\vec{a}(Rc-\vec{R}\cdot\vec{v})+c\vec{v}(\vec{R}\cdot\vec{v})+R\vec{v}(\vec{R}\cdot\vec{a}-v^2)}{R(Rc-\vec{R}\cdot\vec{v})^2}$$
$$\therefore\frac{\partial}{\partial t}\left(\frac{\vec{v}}{Rc-\vec{R}\cdot\vec{v}}\right)=\frac{\partial t_r}{\partial t}\frac{\partial}{\partial t_r}\left(\frac{\vec{v}}{Rc-\vec{R}\cdot\vec{v}}\right)=\frac{Rc}{Rc-\vec{R}\cdot\vec{v}}\left(\frac{R\vec{a}(Rc-\vec{R}\cdot\vec{v})+c\vec{v}(\vec{R}\cdot\vec{v})+R\vec{v}(\vec{R}\cdot\vec{a}-v^2)}{R(Rc-\vec{R}\cdot\vec{v})^2}\right)$$
$$=\frac{Rc\vec{a}(Rc-\vec{R}\cdot\vec{v})+c^2\vec{v}(\vec{R}\cdot\vec{v})+Rc\vec{v}(\vec{R}\cdot\vec{a}-v^2)}{(Rc-\vec{R}\cdot\vec{v})^3}=\frac{Rc\vec{a}(Rc-\vec{R}\cdot\vec{v})+c^2\vec{v}(\vec{R}\cdot\vec{v}-Rc)+Rc\vec{v}(c^2-v^2+\vec{R}\cdot\vec{a})}{(Rc-\vec{R}\cdot\vec{v})^3}$$

$$= \frac{(Rc\vec{a} - c^2\vec{v})(Rc - \vec{R}\cdot\vec{v}) + Rc\vec{v}(c^2 - v^2 + \vec{R}\cdot\vec{a})}{(Rc - \vec{R}\cdot\vec{v})^3}$$

$$\therefore \frac{\partial\vec{A}}{\partial t} = \frac{\mu_0}{4\pi}(qc)\frac{\partial}{\partial t}\left(\frac{\vec{v}}{Rc - \vec{R}\cdot\vec{v}}\right) \Longrightarrow \boldsymbol{\frac{\partial\vec{A}}{\partial t} = \frac{q}{4\pi\varepsilon_0}\frac{(R\vec{a} - c\vec{v})(Rc - \vec{R}\cdot\vec{v}) + R\vec{v}(c^2 - v^2 + \vec{R}\cdot\vec{a})}{(Rc - \vec{R}\cdot\vec{v})^3}}$$

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} = \frac{-q}{4\pi\varepsilon_0}\left\{\frac{(Rc - \vec{R}\cdot\vec{v})c\vec{v} - (c^2 - v^2 + \vec{R}\cdot\vec{a})\vec{R}c}{(Rc - \vec{R}\cdot\vec{v})^3} + \frac{(R\vec{a} - c\vec{v})(Rc - \vec{R}\cdot\vec{v}) + R\vec{v}(c^2 - v^2 + \vec{R}\cdot\vec{a})}{(Rc - \vec{R}\cdot\vec{v})^3}\right\}$$

$$= \frac{-q}{4\pi\varepsilon_0}\frac{1}{(Rc - \vec{R}\cdot\vec{v})^3}\left\{(Rc - \vec{R}\cdot\vec{v})\left(c\vec{v} + (R\vec{a} - c\vec{v})\right) + (c^2 - v^2 + \vec{R}\cdot\vec{a})(R\vec{v} - \vec{R}c)\right\}$$

$$= \frac{-q}{4\pi\varepsilon_0}\frac{1}{(Rc - \vec{R}\cdot\vec{v})^3}\left\{(Rc - \vec{R}\cdot\vec{v})R\vec{a} + (c^2 - v^2 + \vec{R}\cdot\vec{a})(R\vec{v} - \vec{R}c)\right\}$$

$$= \frac{-q}{4\pi\varepsilon_0}\frac{1}{(Rc - \vec{R}\cdot\vec{v})^3}\left\{R^2\vec{a}c - (\vec{R}\cdot\vec{v})R\vec{a} + (c^2 - v^2)(R\vec{v} - \vec{R}c) + (\vec{R}\cdot\vec{a})R\vec{v} - (\vec{R}\cdot\vec{a})\vec{R}c\right\}$$

$$= \frac{-q}{4\pi\varepsilon_0}\frac{1}{(Rc - \vec{R}\cdot\vec{v})^3}\left\{(c^2 - v^2)(R\vec{v} - \vec{R}c) + \vec{a}c(\vec{R}\cdot\vec{R}) - \vec{R}c(\vec{R}\cdot\vec{a}) + R\vec{v}(\vec{R}\cdot\vec{a}) - (\vec{R}\cdot\vec{v})R\vec{a}\right\}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{(Rc - \vec{R}\cdot\vec{v})^3}\left\{(c^2 - v^2)(R\vec{v} - \vec{R}c) + \vec{R}\times(\vec{a}\times\vec{R}c) + \vec{R}\times(R\vec{v}\times\vec{a})\right\} \quad [\because \vec{F}\times(\vec{G}\times\vec{C}) = \vec{G}(\vec{F}\cdot\vec{C}) - \vec{C}(\vec{F}\cdot\vec{G})]$$

$$\Longrightarrow \boldsymbol{\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{(Rc - \vec{R}\cdot\vec{v})^3}\left\{(c^2 - v^2)(\vec{R}c - R\vec{v}) + \vec{R}\times\left((\vec{R}c - R\vec{v})\times\vec{a}\right)\right\}}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{(\vec{R}\cdot\vec{u})^3}\left\{(c^2 - v^2)(R\vec{u}) + \vec{R}\times\left((R\vec{u})\times\vec{a}\right)\right\} \quad [\boldsymbol{\vec{u} = \hat{R}c - \vec{v}}] \Longrightarrow \boldsymbol{\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{R}{(\vec{R}\cdot\vec{u})^3}\left\{(c^2 - v^2)\vec{u} + \vec{R}\times(\vec{u}\times\vec{a})\right\}}$$

If $\vec{a} = 0, \vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{R}{(\vec{R}\cdot\vec{u})^3}(c^2 - v^2)\vec{u}$

If $\vec{v} = 0, \vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{R}{(\vec{R}\cdot\hat{R}c)^3}\left\{c^2\hat{R}c + \vec{R}\times(\hat{R}c\times\vec{a})\right\} = \frac{q}{4\pi\varepsilon_0}\frac{1}{(Rc)^2}\left\{c^2\hat{R} + \vec{R}\times(\hat{R}\times\vec{a})\right\}$

If $v = c, \vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{R}{(\vec{R}\cdot\vec{u})^3}\left\{\vec{R}\times(\vec{u}\times\vec{a})\right\} = \frac{q}{4\pi\varepsilon_0}\frac{R}{\left(\vec{R}\cdot(\hat{R}c - \hat{v}c)\right)^3}\left\{\vec{R}\times\left((\hat{R}c - \hat{v}c)\times\vec{a}\right)\right\}$

$$= \frac{q}{4\pi\varepsilon_0 c^2}\frac{R}{\left(\vec{R}\cdot(\hat{R} - \hat{v})\right)^3}\left\{\vec{R}\times\left((\hat{R} - \hat{v})\times\vec{a}\right)\right\}$$

If $\vec{a} = 0$ and $\vec{v} = 0, \vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{R}}{R^2}$

If $\vec{a} = 0$ and $v = c, \vec{E} = 0$ [Surprise!]

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{c^2}\vec{\nabla}\times(V\vec{v}) \quad \left[\because \vec{A} = \frac{\vec{v}}{c^2}V\right] = \frac{1}{c^2}\left\{V(\vec{\nabla}\times\vec{v}) + \vec{\nabla}V\times\vec{v}\right\} \quad [\because \vec{\nabla}\times(f\vec{G}) = f(\vec{\nabla}\times\vec{G}) + \vec{\nabla}f\times\vec{G}]$$

$$= \frac{qc}{4\pi\varepsilon_0 c^2}\left\{\frac{1}{(Rc - \vec{R}\cdot\vec{v})}\left(\frac{\vec{a}\times\vec{R}}{Rc - \vec{R}\cdot\vec{v}}\right) + \left(\frac{(Rc - \vec{R}\cdot\vec{v})\vec{v} - (c^2 - v^2 + \vec{R}\cdot\vec{a})\vec{R}}{(Rc - \vec{R}\cdot\vec{v})^3}\right)\times\vec{v}\right\}$$

$$\left[\because \vec{\nabla}\times\vec{v} = -\vec{a}\times\vec{\nabla}t_r = \vec{a}\times\left(\frac{\vec{R}}{Rc - \vec{R}\cdot\vec{v}}\right) ;\ V = \frac{1}{4\pi\varepsilon_0}\frac{qc}{(Rc - \vec{R}\cdot\vec{v})} ;\ \vec{\nabla}V = \frac{qc}{4\pi\varepsilon_0}\frac{(Rc - \vec{R}\cdot\vec{v})\vec{v} - (c^2 - v^2 + \vec{R}\cdot\vec{a})\vec{R}}{(Rc - \vec{R}\cdot\vec{v})^3}\right]$$

$$= \frac{qc}{4\pi\varepsilon_0 c^2}\left\{\frac{\vec{a}\times\vec{R}}{(Rc - \vec{R}\cdot\vec{v})^2} - \frac{(c^2 - v^2 + \vec{R}\cdot\vec{a})}{(Rc - \vec{R}\cdot\vec{v})^3}(\vec{R}\times\vec{v})\right\} = \frac{-qc}{4\pi\varepsilon_0 c^2}\vec{R}\times\left\{\frac{\vec{a}}{(Rc - \vec{R}\cdot\vec{v})^2} + \frac{(c^2 - v^2 + \vec{R}\cdot\vec{a})\vec{v}}{(Rc - \vec{R}\cdot\vec{v})^3}\right\}$$

$$= \frac{-qc}{4\pi\varepsilon_0 c^2}\frac{1}{(Rc - \vec{R}\cdot\vec{v})^3}\vec{R}\times\left\{\vec{a}(Rc - \vec{R}\cdot\vec{v}) + (c^2 - v^2 + \vec{R}\cdot\vec{a})\vec{v}\right\}$$

$$= \frac{-qc}{4\pi\varepsilon_0 c^2}\frac{1}{\left(Rc - \vec{R}\cdot\vec{v}\right)^3}\vec{R}\times\left\{\vec{a}Rc - \vec{a}\left(\vec{R}\cdot\vec{v}\right) + \vec{v}\left(\vec{R}\cdot\vec{a}\right) + (c^2-v^2)\vec{v}\right\}$$

$$= \frac{-qc}{4\pi\varepsilon_0 c^2}\frac{1}{\left(Rc - \vec{R}\cdot\vec{v}\right)^3}\vec{R}\times\left\{\vec{a}Rc + \vec{R}\times(\vec{v}\times\vec{a}) + (c^2-v^2)\vec{v}\right\} \quad \left[\because \vec{F}\times\left(\vec{G}\times\vec{C}\right) = \vec{G}\left(\vec{F}\cdot\vec{C}\right) - \vec{C}\left(\vec{F}\cdot\vec{G}\right)\right]$$

$$= \frac{-qc}{4\pi\varepsilon_0 c^2}\frac{1}{\left(Rc - \vec{R}\cdot\vec{v}\right)^3}\hat{R}\times\left\{\vec{a}R^2c + \vec{R}\times(R\vec{v}\times\vec{a}) + (c^2-v^2)R\vec{v}\right\}$$

$$= \frac{-qc}{4\pi\varepsilon_0 c^2}\frac{1}{\left(Rc - \vec{R}\cdot\vec{v}\right)^3}\hat{R}\times\left\{\vec{a}c\left(\vec{R}\cdot\vec{R}\right) + \vec{R}\times(R\vec{v}\times\vec{a}) + (c^2-v^2)R\vec{v} - (c^2-v^2)\vec{R}c - \left(\vec{R}\cdot\vec{a}\right)\vec{R}c\right\} \quad \left[\because \hat{R}\times\vec{R}c = 0\right]$$

$$= \frac{-qc}{4\pi\varepsilon_0 c^2}\frac{1}{\left(Rc - \vec{R}\cdot\vec{v}\right)^3}\hat{R}\times\left\{\vec{R}\times\left(\vec{a}\times\vec{R}c\right) + \vec{R}\times(R\vec{v}\times\vec{a}) + (c^2-v^2)\left(R\vec{v} - \vec{R}c\right)\right\} \quad \left[\because \vec{F}\times\left(\vec{G}\times\vec{C}\right) = \vec{G}\left(\vec{F}\cdot\vec{C}\right) - \vec{C}\left(\vec{F}\cdot\vec{G}\right)\right]$$

$$\Rightarrow \boldsymbol{\vec{B}} = \frac{\boldsymbol{qc}}{\boldsymbol{4\pi\varepsilon_0 c^2}}\frac{\boldsymbol{1}}{\left(\boldsymbol{Rc - \vec{R}\cdot\vec{v}}\right)^3}\boldsymbol{\hat{R}}\times\left\{\boldsymbol{(c^2-v^2)\left(\vec{R}c - R\vec{v}\right) + \vec{R}\times\left(\left(\vec{R}c - R\vec{v}\right)\times\vec{a}\right)}\right\} \Rightarrow \boldsymbol{\vec{B}(\vec{r},t)} = \frac{\boldsymbol{1}}{\boldsymbol{c}}\boldsymbol{\hat{R}\times\vec{E}(\vec{r},t)}$$

$$= \frac{\mu_0}{4\pi}\frac{qc}{\left(\vec{R}\cdot\vec{u}\right)^3}\hat{R}\times\left\{(c^2-v^2)(R\vec{u}) + \vec{R}\times\left((R\vec{u})\times\vec{a}\right)\right\} \quad \left[\boldsymbol{\vec{u} = \hat{R}c - \vec{v}}\right] \Rightarrow \boldsymbol{\vec{B}} = \frac{\boldsymbol{\mu_0}}{\boldsymbol{4\pi}}\frac{\boldsymbol{qc}}{\left(\boldsymbol{\vec{R}\cdot\vec{u}}\right)^3}\boldsymbol{\vec{R}\times\left\{(c^2-v^2)\vec{u} + \vec{R}\times(\vec{u}\times\vec{a})\right\}}$$

If $\vec{a} = 0, \vec{B} = \frac{\mu_0}{4\pi}\frac{qc}{\left(\vec{R}\cdot\vec{u}\right)^3}\vec{R}\times(c^2-v^2)\vec{u}$

If $\vec{v} = 0, \vec{B} = \frac{\mu_0}{4\pi}\frac{qc}{\left(\vec{R}\cdot\hat{R}c\right)^3}\vec{R}\times\left\{c^2\hat{R}c + \vec{R}\times\left(\hat{R}c\times\vec{a}\right)\right\} = \frac{\mu_0}{4\pi}\frac{qc}{(Rc)^2}\hat{R}\times\left\{c^2\hat{R} + \vec{R}\times\left(\hat{R}\times\vec{a}\right)\right\}$

If $v = c, \vec{B} = \frac{\mu_0}{4\pi}\frac{qc}{\left(\vec{R}\cdot\vec{u}\right)^3}\vec{R}\times\left\{\vec{R}\times(\vec{u}\times\vec{a})\right\} = \frac{\mu_0}{4\pi}\frac{qc}{\left(\vec{R}\cdot\left(\hat{R}c - \hat{v}c\right)\right)^3}\vec{R}\times\left\{\vec{R}\times\left(\left(\hat{R}c - \hat{v}c\right)\times\vec{a}\right)\right\}$

$$= \frac{\mu_0}{4\pi c}\frac{q}{\left(\vec{R}\cdot\left(\hat{R} - \hat{v}\right)\right)^3}\vec{R}\times\left\{\vec{R}\times\left(\left(\hat{R} - \hat{v}\right)\times\vec{a}\right)\right\}$$

If $\vec{a} = 0$ and $\vec{v} = 0, \vec{B} = \frac{\mu_0}{4\pi}\frac{qc}{(Rc)^2}\hat{R}\times\left\{c^2\hat{R}\right\} = 0$

If $\vec{a} = 0$ and $v = c, \vec{B} = 0$ [Surprise!]

Lorentz force, $\vec{F} = q_0\left(\vec{E} + \vec{v}_0\times\vec{B}\right) = \frac{qq_0}{4\pi\varepsilon_0}\frac{R}{\left(\vec{R}\cdot\vec{u}\right)^3}\left(\left\{(c^2-v^2)\vec{u} + \vec{R}\times(\vec{u}\times\vec{a})\right\} + \frac{\vec{v}_0}{c}\times\left[\hat{R}\times\left\{(c^2-v^2)\vec{u} + \vec{R}\times(\vec{u}\times\vec{a})\right\}\right]\right)$

where $\vec{v}_0$ is the velocity of q_0 at present time, t, and $\vec{R}, \vec{v}, \vec{u}, \vec{a}$ are evaluated at retarded time t_r.

Calculate the electric and magnetic potentials of a point charge moving with constant velocity.

$$V = \frac{1}{4\pi\varepsilon_0}\frac{qc}{\left(Rc - \vec{R}\cdot\vec{v}\right)} = \frac{\boldsymbol{1}}{\boldsymbol{4\pi\varepsilon_0}}\frac{\boldsymbol{qc}}{\left(\boldsymbol{\vec{R}\cdot\vec{u}}\right)} \quad \left[\vec{u} = \hat{R}c - \vec{v}\right]$$

$$Rc - \vec{R}\cdot\vec{v} = R\left(c - \hat{R}\cdot\vec{v}\right) = c(t-t_r)\left(c - \frac{(\vec{r} - \vec{v}t_r)\cdot\vec{v}}{c(t-t_r)}\right) \quad \left[\because \hat{R} = \frac{\vec{R}}{R} = \frac{(\vec{r} - \vec{v}t_r)}{c(t-t_r)}\right] = c^2(t-t_r) - (\vec{r}\cdot\vec{v} - v^2t_r)$$

$$R = |\vec{r} - \vec{w}(t_r)| = c(t-t_r) \Rightarrow |\vec{r} - \vec{v}t_r| = c(t-t_r) \quad [\because \vec{w}(t_r) = \vec{v}t_r] \Rightarrow |\vec{r} - \vec{v}t_r|^2 = c^2(t-t_r)^2$$

$$\Rightarrow r^2 + v^2{t_r}^2 - 2\vec{r}\cdot\vec{v}t_r = c^2t^2 + c^2{t_r}^2 - 2c^2tt_r \Rightarrow (c^2-v^2){t_r}^2 - 2(c^2t - \vec{r}\cdot\vec{v})t_r + (c^2t^2 - r^2) = 0$$

$$\Rightarrow t_r = \frac{(c^2t - \vec{r}\cdot\vec{v}) \pm \sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 - (c^2-v^2)(c^2t^2 - r^2)}}{(c^2-v^2)}$$

To fix the sign, consider the limit $v = 0$: $t_r|_{v=0} = \frac{(c^2t) \pm \sqrt{(c^2t)^2 - c^2(c^2t^2 - r^2)}}{c^2} = t \pm \frac{r}{c}$

$\because t_r|_{v=0} = t - \frac{r}{c} \ll t \Rightarrow$ it should be minus sign. $\therefore t_r = \frac{(c^2t - \vec{r}\cdot\vec{v}) - \sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 - (c^2-v^2)(c^2t^2 - r^2)}}{(c^2-v^2)}$

$\therefore Rc - \vec{R}\cdot\vec{v} = c^2(t-t_r) - (\vec{r}\cdot\vec{v} - v^2t_r) = (c^2t - \vec{r}\cdot\vec{v}) - (c^2-v^2)t_r = \sqrt{\boldsymbol{(c^2t - \vec{r}\cdot\vec{v})^2 - (c^2-v^2)(c^2t^2 - r^2)}}$

$$\vec{R}\cdot\vec{u} = Rc - \vec{R}\cdot\vec{v} = \sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 - (c^2 - v^2)(c^2t^2 - r^2)} = \sqrt{\left(c^2t - \left(\vec{R'} + \vec{v}t\right)\cdot\vec{v}\right)^2 + (c^2 - v^2)\left(\left(\vec{R'} + \vec{v}t\right)^2 - c^2t^2\right)}$$

$$= \sqrt{\left(c^4t^2 + \left(\vec{R'}\cdot\vec{v} + v^2t\right)^2 - 2c^2t\left(\vec{R'} + \vec{v}t\right)\cdot\vec{v}\right) + \left(c^2\left(\vec{R'} + \vec{v}t\right)^2 - v^2\left(\vec{R'} + \vec{v}t\right)^2 - c^4t^2 + c^2v^2t^2\right)}$$

$$= \sqrt{\left(\left(\vec{R'}\cdot\vec{v} + v^2t\right)^2 - 2c^2t\left(\vec{R'}\cdot\vec{v} + v^2t\right)\right) + \left(c^2\left(R'^2 + v^2t^2 + 2\vec{R'}\cdot\vec{v}t\right) - v^2\left(R'^2 + v^2t^2 + 2\vec{R'}\cdot\vec{v}t\right) + c^2v^2t^2\right)}$$

$$= \sqrt{\left(\left(\vec{R'}\cdot\vec{v}\right)^2 + v^4t^2 + 2\left(\vec{R'}\cdot\vec{v}\right)v^2t\right) + \left(c^2R'^2 - v^2\left(R'^2 + v^2t^2 + 2\vec{R'}\cdot\vec{v}t\right)\right)} = \sqrt{\left(\vec{R'}\cdot\vec{v}\right)^2 + \left(c^2R'^2 - v^2R'^2\right)}$$

$$= \sqrt{R'^2\left(\hat{R}'\cdot\vec{v}\right)^2 + R'^2(c^2 - v^2)} = R'\sqrt{\left(\hat{R}'\cdot\vec{v}\right)^2 + (c^2 - v^2)} = R'\sqrt{v^2\cos^2\phi + (c^2 - v^2)} \Rightarrow \boldsymbol{\vec{R}\cdot\vec{u} = R'c\sqrt{1 - \frac{v^2}{c^2}\sin^2\phi}}$$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{qc}{\left(\vec{R}\cdot\vec{u}\right)} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R'\sqrt{1 - \frac{v^2}{c^2}\sin^2\phi}}$$

$$\vec{A} = \frac{\vec{v}}{c^2}V = \frac{1}{4\pi\varepsilon_0c^2}\frac{q\vec{v}}{R'\sqrt{1 - \frac{v^2}{c^2}\sin^2\phi}} = \frac{\mu_0}{4\pi}\frac{q\vec{v}}{R'\sqrt{1 - \frac{v^2}{c^2}\sin^2\phi}}$$

When $v \ll c$, $V = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{R'}$; $\vec{A} = \dfrac{\mu_0}{4\pi}\dfrac{q\vec{v}}{R'}$

[Note that $\vec{R'}$ is the vector from the present location of the charge to the field point.]

Calculate the electric and magnetic fields of a point charge moving with constant velocity.

$$\because \vec{a} = 0 \quad \therefore \vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{(c^2 - v^2)R\vec{u}}{\left(\vec{R}\cdot\vec{u}\right)^3} \quad [\vec{u} = \hat{R}c - \vec{v}]$$

$$R\vec{u} = \vec{R}c - R\vec{v} = \left(\vec{r} - \vec{w}(t_r)\right)c - c(t - t_r)\vec{v} = (\vec{r} - \vec{v}t_r)c - c(t - t_r)\vec{v} = \vec{r}c - c\vec{v}t = c(\vec{r} - \vec{v}t) = c\vec{R'}$$

$$\left[\begin{array}{l}\because \vec{v} = \dfrac{d\vec{w}}{dt_r} \Rightarrow \vec{v}\,dt_r = d\vec{w} \Rightarrow \vec{w}(t) = \vec{v}t \text{ for constant velocity } \vec{v} \text{ and if the charge passes through origin at } t = 0\text{, i.e. } \vec{w}(0) = 0 \\ \qquad\qquad \vec{R'} = \vec{r} - \vec{w}(t) = \vec{r} - \vec{v}t\ ;\ \vec{R} = \vec{r} - \vec{w}(t_r) = (\vec{r} - \vec{v}t_r) \text{ and } R = |\vec{r} - \vec{w}(t_r)| = c(t - t_r)\end{array}\right]$$

$$\vec{R}\cdot\vec{u} = Rc - \vec{R}\cdot\vec{v} = \sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)} =$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{(c^2 - v^2)R\vec{u}}{\left(\vec{R}\cdot\vec{u}\right)^3} = \frac{q}{4\pi\varepsilon_0}\frac{(c^2 - v^2)c\vec{R'}}{(R'c)^3\left(1 - \frac{v^2}{c^2}\sin^2\phi\right)^{\frac{3}{2}}} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{R}'\left(1 - \frac{v^2}{c^2}\right)}{R'^2\left(1 - \frac{v^2}{c^2}\sin^2\phi\right)^{\frac{3}{2}}}$$

$$\vec{B} = \frac{1}{c}\hat{R}\times\vec{E} = \frac{1}{c^2}\vec{v}\times\vec{E}$$

$$\left[\because \vec{R} = \vec{r} - \vec{w}(t_r) = \vec{r} - \vec{v}t_r = \vec{r} - \vec{v}t + \vec{v}t - \vec{v}t_r = \vec{R'} + \vec{v}(t - t_r) \Rightarrow \hat{R} = \frac{\vec{R'}}{R} + \frac{\vec{v}(t - t_r)}{R} = \frac{\vec{R'}}{R} + \frac{\vec{v}(t - t_r)}{c(t - t_r)} = \frac{\vec{R'}}{R} + \frac{\vec{v}}{c}\right]$$

When $v \ll c$, $\boldsymbol{\vec{E} = \dfrac{q}{4\pi\varepsilon_0}\dfrac{\hat{R}'}{R'^2}}$ [Coulomb's law] ; $\boldsymbol{\vec{B}} = \dfrac{1}{c^2}\vec{v}\times\left(\dfrac{q}{4\pi\varepsilon_0}\dfrac{\hat{R}'}{R'^2}\right) = \boldsymbol{\dfrac{\mu_0}{4\pi}\dfrac{q}{R'^2}\left(\vec{v}\times\hat{R}'\right)}$ [Biot-Savart law]

[Note that $\vec{R'}$ is the vector from the present location of the charge to the field point.]

Power radiated by a point charge (Larmor formula)

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{R}{\left(\vec{R}\cdot\vec{u}\right)^3}\left\{(c^2 - v^2)\vec{u} + \vec{R}\times(\vec{u}\times\vec{a})\right\} \quad [\vec{u} = \hat{R}c - \vec{v}]$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}\hat{R}\times\vec{E}(\vec{r},t)$$

Poynting vector, $\vec{S} = \vec{E}\times\vec{H} = \dfrac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right) = \dfrac{1}{\mu_0c}\left\{\vec{E}\times\left(\hat{R}\times\vec{E}\right)\right\} = \dfrac{1}{\mu_0c}\left\{\hat{R}\left(\vec{E}\cdot\vec{E}\right) - \vec{E}\left(\vec{E}\cdot\hat{R}\right)\right\}$

$\left[\because \vec{F}\times(\vec{G}\times\vec{C}) = \vec{G}(\vec{F}\cdot\vec{C}) - \vec{C}(\vec{F}\cdot\vec{G})\right] \Longrightarrow \vec{\boldsymbol{S}} = \dfrac{\mathbf{1}}{\boldsymbol{\mu_0 c}}\{\boldsymbol{E^2\hat{R}} - \vec{\boldsymbol{E}}(\boldsymbol{\hat{R}}\cdot\vec{\boldsymbol{E}})\}$

Velocity field or Emanation field, $\vec{E}_{\text{ema}} = \dfrac{q}{4\pi\varepsilon_0}\dfrac{R}{(\vec{R}\cdot\vec{u})^3}\{(c^2 - v^2)\vec{u}\}$

This will contribute nothing since it goes like ${E_{\text{ema}}}^2 = \dfrac{1}{R^4}$ which on integration over the surface $4\pi R^2$ of a sphere of radius R will become $\dfrac{1}{R^2}$ and hence will go to zero in the limit $R \to \infty$

Acceleration field or Radiation field, $\vec{E}_{\text{rad}} = \dfrac{q}{4\pi\varepsilon_0}\dfrac{R}{(\vec{R}\cdot\vec{u})^3}\{\vec{R}\times(\vec{u}\times\vec{a})\} = \dfrac{q}{4\pi\varepsilon_0}\dfrac{R}{(\vec{R}\cdot\vec{u})^3}\{\vec{u}(\vec{R}\cdot\vec{a}) - \vec{a}(\vec{R}\cdot\vec{u})\}$

Only this will contribute since it goes like ${E_{\text{ema}}}^2 = \dfrac{1}{R^2}$ which on integration over the surface $4\pi R^2$ of a sphere of radius R will become $\dfrac{1}{R^2}$ and hence will be finite in the limit $R \to \infty$

$$\vec{S}_{\text{rad}} = \frac{1}{\mu_0 c}\{{E_{\text{rad}}}^2\hat{R} - \vec{E}_{\text{rad}}(\hat{R}\cdot\vec{E}_{\text{rad}})\} = \frac{1}{\mu_0 c}{E_{\text{rad}}}^2\hat{R} \quad \left[\begin{array}{l}\because \vec{E}_{\text{rad}} = \vec{R}\times(\vec{u}\times\vec{a}) \text{ is perpendicular to } \hat{R}\\ \text{So the second term } \vec{E}_{\text{rad}}(\hat{R}\cdot\vec{E}_{\text{rad}}) \text{ vanishes.}\end{array}\right]$$

$$\vec{E}_{\text{rad}} = \frac{q}{4\pi\varepsilon_0}\frac{R}{(\vec{R}\cdot\vec{u})^3}\{\vec{u}(\vec{R}\cdot\vec{a}) - \vec{a}(\vec{R}\cdot\vec{u})\} = \frac{q}{4\pi\varepsilon_0}\frac{R}{\left(\vec{R}\cdot(\hat{R}c - \vec{v})\right)^3}\left\{(\hat{R}c - \vec{v})(\vec{R}\cdot\vec{a}) - \vec{a}\left(\vec{R}\cdot(\hat{R}c - \vec{v})\right)\right\}$$

If $\vec{v} \ll c$, $\vec{E}_{\text{rad}} \approx \dfrac{q}{4\pi\varepsilon_0}\dfrac{R}{\left(\vec{R}\cdot(\hat{R}c)\right)^3}\{\hat{R}c(\vec{R}\cdot\vec{a}) - \vec{a}(\vec{R}\cdot\hat{R}c)\} = \dfrac{q}{4\pi\varepsilon_0}\dfrac{R}{(Rc)^3}\{\hat{R}c(\vec{R}\cdot\vec{a}) - \vec{a}Rc\}$

$$= \frac{q}{4\pi\varepsilon_0}\frac{R^2 c}{(Rc)^3}\{\hat{R}(\hat{R}\cdot\vec{a}) - \vec{a}\} = \frac{1}{4\pi\varepsilon_0 c^2}\frac{q}{R}\{\hat{R}(\hat{R}\cdot\vec{a}) - \vec{a}\} \Longrightarrow \vec{E}_{\text{rad}} = \frac{\mu_0}{4\pi}\frac{q}{R}\{\hat{R}(\hat{R}\cdot\vec{a}) - \vec{a}\}$$

$$\vec{S}_{\text{rad}} = \frac{1}{\mu_0 c}{E_{\text{rad}}}^2\hat{R} = \frac{1}{\mu_0 c}\left(\frac{\mu_0}{4\pi}\frac{q}{R}\right)^2\{\hat{R}(\hat{R}\cdot\vec{a}) - \vec{a}\}^2 = \frac{\mu_0}{16\pi^2 c}\frac{q^2}{R^2}\left\{(\hat{R}\cdot\vec{a})^2 + a^2 - 2(\hat{R}\cdot\vec{a})(\hat{R}\cdot\vec{a})\right\}\hat{R}$$

$$= \frac{\mu_0}{16\pi^2 c}\frac{q^2}{R^2}\left\{a^2 - (\hat{R}\cdot\vec{a})^2\right\}\hat{R} = \frac{\mu_0}{16\pi^2 c}\frac{q^2}{R^2}\{a^2 - (a\cos\vartheta)^2\}\hat{R} \quad \left[\begin{array}{l}\vartheta \text{ is the angle}\\ \text{between } \vec{a} \text{ and } \hat{R}\end{array}\right] \Longrightarrow \vec{\boldsymbol{S}}_{\text{rad}} = \frac{\boldsymbol{\mu_0 q^2 a^2}}{\boldsymbol{16\pi^2 c}}\left(\frac{\boldsymbol{\sin^2\vartheta}}{\boldsymbol{R^2}}\right)\boldsymbol{\hat{R}}$$

where $\hat{R}$ is the vector from location of the charge at the retarded time t_r to the field point.

Total power radiated, $P = \oint \vec{S}_{\text{rad}}\cdot d\vec{s} = \dfrac{\mu_0 q^2 a^2}{16\pi^2 c}\displaystyle\iint_{0\,0}^{2\pi\,\pi}\dfrac{\sin^2\vartheta}{R^2}R^2\sin\vartheta\, d\vartheta\, d\varphi \quad [\because \hat{R}\cdot d\vec{s} = \sin\vartheta\, d\vartheta\, d\varphi]$

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c}(2\pi)\int_0^\pi \sin^3\vartheta\, d\vartheta \quad \left[\because \int_0^{2\pi} d\varphi = 2\pi\right] = \frac{\mu_0 q^2 a^2}{32\pi c}\int_0^\pi 4\sin^3\vartheta\, d\vartheta = \frac{\mu_0 q^2 a^2}{32\pi c}\int_0^\pi (3\sin\vartheta - \sin 3\vartheta)\, d\vartheta$$

$$= \frac{\mu_0 q^2 a^2}{32\pi c}\left(6 - \frac{2}{3}\right) \quad \left[\because \int_0^\pi \sin\vartheta\, d\vartheta = [-\cos\vartheta]_0^\pi = 2 \text{ and } \int_0^\pi \sin 3\vartheta\, d\vartheta = \left[-\frac{\cos 3\vartheta}{3}\right]_0^\pi = \frac{2}{3}\right] \Longrightarrow \boldsymbol{P} = \frac{\boldsymbol{\mu_0 q^2 a^2}}{\boldsymbol{6\pi c}}$$

This is the **Larmor formula.** $\vec{S}_{\text{rad}}$ the rate at which energy passes through per unit area of the sphere at time t, is not the same as the rate at which energy left the particle at time t_r.

Power radiated by the charge at time t_r, $P_0 = \dfrac{dW}{dt_r} = \dfrac{\partial t}{\partial t_r}\dfrac{dW}{dt} = \left(\dfrac{\vec{R}\cdot\vec{u}}{Rc}\right)\dfrac{dW}{dt} = \left(\dfrac{\vec{R}\cdot\vec{u}}{Rc}\right)P \left[\because \dfrac{\partial t_r}{\partial t} = \dfrac{Rc}{Rc - \vec{R}\cdot\vec{v}} = \dfrac{Rc}{\vec{R}\cdot\vec{u}}\right]$

$$\frac{\boldsymbol{dP_0}}{\boldsymbol{d\Omega}} = \left(\frac{\vec{R}\cdot\vec{u}}{Rc}\right)\frac{1}{\mu_0 c}{E_{\text{rad}}}^2 R^2 = \left(\frac{\vec{R}\cdot\vec{u}}{Rc}\right)\frac{1}{\mu_0 c}\left(\frac{q}{4\pi\varepsilon_0}\frac{R}{(\vec{R}\cdot\vec{u})^3}\{\vec{R}\times(\vec{u}\times\vec{a})\}\right)^2 R^2 = \frac{q^2 R^3}{16\pi^2\varepsilon_0}\frac{\{\vec{R}\times(\vec{u}\times\vec{a})\}^2}{(\vec{R}\cdot\vec{u})^5}$$

$$= \frac{\boldsymbol{q^2}}{\boldsymbol{16\pi^2\varepsilon_0}}\frac{\{\boldsymbol{\hat{R}}\times(\vec{\boldsymbol{u}}\times\vec{\boldsymbol{a}})\}^2}{(\boldsymbol{\hat{R}}\cdot\vec{\boldsymbol{u}})^5} = \frac{q^2}{16\pi^2\varepsilon_0}\frac{\{\vec{u}(\hat{R}\cdot\vec{a}) - \vec{a}(\hat{R}\cdot\vec{u})\}^2}{(\hat{R}\cdot\vec{u})^5} = \frac{q^2}{16\pi^2\varepsilon_0}\frac{\left\{(\hat{R}c - \vec{v})(\hat{R}\cdot\vec{a}) - \vec{a}\left(\hat{R}\cdot(\hat{R}c - \vec{v})\right)\right\}^2}{\left(\hat{R}\cdot(\hat{R}c - \vec{v})\right)^5}$$

If $\vec{v} \ll c$, $\dfrac{dP_0}{d\Omega} \approx \dfrac{q^2}{16\pi^2\varepsilon_0}\dfrac{\left\{\hat{R}c\left(\hat{R}\cdot\vec{a}\right)-\vec{a}c\right\}^2}{c^5} = \dfrac{q^2}{16\pi^2\varepsilon_0 c^3}\left\{\hat{R}\left(\hat{R}\cdot\vec{a}\right)-\vec{a}\right\}^2 = \dfrac{\mu_0 q^2}{16\pi^2 c}\left\{\hat{R}\left(\hat{R}\cdot\vec{a}\right)-\vec{a}\right\}^2$

This is the same as before as $\vec{S}_{\text{rad}} = \dfrac{dP}{d\Omega}\hat{R} = \dfrac{dP_0}{d\Omega}\hat{R}$. This is because $\dfrac{\vec{R}\cdot\vec{u}}{Rc} = \dfrac{\vec{R}\cdot\left(\hat{R}c-\vec{v}\right)}{Rc} \approx 1$ when $\vec{v} \ll c$.

Suppose *v* and *a* are instantaneously collinear (at time *t*$_r$), as for example, in straight line motion. Find the angular distribution of the radiation and the total power emitted.

$$\frac{dP_0}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0}\frac{\left\{\hat{R}\times(\vec{u}\times\vec{a})\right\}^2}{\left(\hat{R}\cdot\vec{u}\right)^5} = \frac{q^2}{16\pi^2\varepsilon_0}\frac{\left\{\hat{R}\times\left(\left(\hat{R}c-\vec{v}\right)\times\vec{a}\right)\right\}^2}{\left(\hat{R}\cdot\vec{u}\right)^5} = \frac{q^2}{16\pi^2\varepsilon_0}\frac{\left\{\hat{R}\times\left(\hat{R}c\times\vec{a}\right)\right\}^2}{\left(\hat{R}\cdot\left(\hat{R}c-\vec{v}\right)\right)^5}\ \left[\begin{array}{c}\because \vec{v}\times\vec{a}=0 \text{ as}\\ \text{they are collinear}\end{array}\right]$$

$$= \frac{q^2c^2}{16\pi^2\varepsilon_0}\frac{\left\{\hat{R}\times\left(\hat{R}\times\vec{a}\right)\right\}^2}{\left(c-\hat{R}\cdot\vec{v}\right)^5} = \frac{q^2c^2}{16\pi^2\varepsilon_0}\frac{\left\{\hat{R}\left(\hat{R}\cdot\vec{a}\right)-\vec{a}\right\}^2}{\left(c-\hat{R}\cdot\vec{v}\right)^5} = \frac{q^2c^2}{16\pi^2\varepsilon_0}\frac{\left\{\left(\hat{R}\cdot\vec{a}\right)^2+a^2-2\left(\hat{R}\cdot\vec{a}\right)\left(\hat{R}\cdot\vec{a}\right)\right\}}{\left(c-\hat{R}\cdot\vec{v}\right)^5}$$

$$= \frac{q^2c^2}{16\pi^2\varepsilon_0}\frac{\left\{a^2-\left(\hat{R}\cdot\vec{a}\right)^2\right\}}{\left(c-\hat{R}\cdot\vec{v}\right)^5} = \frac{q^2c^2}{16\pi^2\varepsilon_0}\frac{\left\{a^2-(a\cos\vartheta)^2\right\}}{(c-v\cos\vartheta)^5}\ \left[\begin{array}{c}\because \vec{v} \text{ and } \vec{a} \text{ are collinear,}\\ \text{and } \vartheta \text{ is the angle between}\\ \vec{a} \text{ (or } \vec{v}\text{) and } \hat{R}\,;\ \beta=v/c\end{array}\right] = \boldsymbol{\frac{q^2a^2}{16\pi^2\varepsilon_0c^3}\frac{\sin^2\vartheta}{(1-\beta\cos\vartheta)^5}}$$

$$\therefore P_0 = \int\frac{dP_0}{d\Omega}d\Omega = \frac{q^2a^2}{16\pi^2\varepsilon_0c^3}\int_0^{2\pi}\!\!\int_0^{\pi}\frac{\sin^2\vartheta}{(1-\beta\cos\vartheta)^5}\sin\vartheta\,d\vartheta\,d\varphi\ [\because d\Omega=\sin\vartheta\,d\vartheta\,d\varphi] = \frac{q^2a^2(2\pi)}{16\pi^2\varepsilon_0c^3}\int_0^{\pi}\frac{\sin^3\vartheta}{(1-\beta\cos\vartheta)^5}d\vartheta$$

$$\int_0^{\pi}\frac{\sin^3\vartheta}{(1-\beta\cos\vartheta)^5}d\vartheta = \int_{-1}^{+1}\frac{(1-x^2)}{(1-\beta x)^5}dx\ \left[\begin{array}{c}\text{Putting }\cos\vartheta=x\\ \Longrightarrow -\sin\vartheta\,d\vartheta=dx\end{array}\right] = \left[(1-x^2)\frac{(1-\beta x)^{-4}}{4\beta}\right]_{-1}^{+1} - \int_{-1}^{+1}\frac{-2x}{4\beta(1-\beta x)^4}dx$$

$$[\text{integrating by parts}] = \int_{-1}^{+1}\frac{x}{2\beta(1-\beta x)^4}dx = \left[\frac{x}{2}\frac{(1-\beta x)^{-3}}{3\beta^2}\right]_{-1}^{+1} - \int_{-1}^{+1}\frac{1}{6\beta^2(1-\beta x)^3}dx\quad [\text{integrating by parts again}]$$

$$= \left[\frac{(1-\beta)^{-3}}{6\beta^2}+\frac{(1+\beta)^{-3}}{6\beta^2}\right] - \left[\frac{1}{6}\frac{(1-\beta)^{-2}}{2\beta^3}\right]_{-1}^{+1} = \left[\frac{1}{6\beta^2(1-\beta)^3}+\frac{1}{6\beta^2(1+\beta)^3}\right] - \left[\frac{1}{12\beta^3(1-\beta)^2}-\frac{1}{12\beta^3(1+\beta)^2}\right]$$

$$= \left[\frac{(1+\beta)^3}{6\beta^2(1-\beta^2)^3}+\frac{(1-\beta)^3}{6\beta^2(1-\beta^2)^3}\right] - \left[\frac{(1+\beta)^2}{12\beta^3(1-\beta^2)^2}-\frac{(1-\beta)^2}{12\beta^3(1-\beta^2)^2}\right] = \frac{2+6\beta^2}{6\beta^2(1-\beta^2)^3} - \frac{4\beta}{12\beta^3(1-\beta^2)^2}$$

$$= \frac{\beta(1+3\beta^2)}{3\beta^3(1-\beta^2)^3} - \frac{\beta(1-\beta^2)}{3\beta^3(1-\beta^2)^3} = \frac{(\beta+3\beta^3)}{3\beta^3(1-\beta^2)^3} - \frac{(\beta-\beta^3)}{3\beta^3(1-\beta^2)^3} = \frac{4}{3(1-\beta^2)^3} = \frac{4\gamma^6}{3}\ \left[\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right]$$

$$\therefore \boldsymbol{P_0} = \frac{q^2a^2}{16\pi^2\varepsilon_0c^3}(2\pi)\frac{4\gamma^6}{3} = \frac{q^2a^2\gamma^6}{6\pi\varepsilon_0c^3} = \boldsymbol{\frac{\mu_0q^2a^2\gamma^6}{6\pi c}}$$

Wave Guides

$$\begin{cases}\begin{cases}\vec{E}(x,y,z,t) = \vec{E}_0(x,y)e^{i(kz-\omega t)}\\ \vec{B}(x,y,z,t) = \vec{B}_0(x,y)e^{i(kz-\omega t)}\end{cases} \Longrightarrow \begin{cases}\dfrac{\partial}{\partial z} \equiv ik\\ \dfrac{\partial}{\partial t} \equiv -i\omega\end{cases}\\ \vec{E}_0 = E_x(x,y)\hat{\imath}+E_y(x,y)\hat{\jmath}+E_z(x,y)\hat{k}\\ \vec{B}_0 = B_x(x,y)\hat{\imath}+B_y(x,y)\hat{\jmath}+B_z(x,y)\hat{k}\end{cases}$$

$$\begin{cases}\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \Longrightarrow \vec{\nabla}\times\left(\vec{E}_0e^{i(kz-\omega t)}\right) = i\omega\vec{B}_0e^{i(kz-\omega t)}\\ \vec{\nabla}\times\vec{B} = \dfrac{1}{c^2}\dfrac{\partial\vec{E}}{\partial t} \Longrightarrow \vec{\nabla}\times\left(\vec{B}_0e^{i(kz-\omega t)}\right) = -\dfrac{i\omega}{c^2}\vec{E}_0e^{i(kz-\omega t)}\end{cases}$$

$$\Rightarrow \begin{cases} \left(\dfrac{\partial E_z}{\partial y} - ikE_y\right)\hat{\imath} + \left(ikE_x - \dfrac{\partial E_z}{\partial x}\right)\hat{\jmath} + \left(\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y}\right)\hat{k} = i\omega\left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right) \\ \left(\dfrac{\partial B_z}{\partial y} - ikB_y\right)\hat{\imath} + \left(ikB_x - \dfrac{\partial B_z}{\partial x}\right)\hat{\jmath} + \left(\dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y}\right)\hat{k} = -\dfrac{i\omega}{c^2}\left(E_x\hat{\imath} + E_y\hat{\jmath} + E_z\hat{k}\right) \end{cases}$$

$$\Rightarrow \begin{cases} \begin{cases} \left.\begin{aligned} \dfrac{\partial E_z}{\partial y} - ikE_y = i\omega B_x \\ ikE_x - \dfrac{\partial E_z}{\partial x} = i\omega B_y \end{aligned}\right\} \\ \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = i\omega B_z \end{cases} \\ \begin{cases} \left.\begin{aligned} \dfrac{\partial B_z}{\partial y} - ikB_y = -\dfrac{i\omega}{c^2}E_x \\ ikB_x - \dfrac{\partial B_z}{\partial x} = -\dfrac{i\omega}{c^2}E_y \end{aligned}\right\} \\ \dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y} = -\dfrac{i\omega}{c^2}E_z \end{cases} \end{cases} \Rightarrow \begin{cases} \begin{cases} \left.\begin{aligned} i\omega B_x + ikE_y = \dfrac{\partial E_z}{\partial y} \\ ikE_x - i\omega B_y = \dfrac{\partial E_z}{\partial x} \end{aligned}\right\} \\ \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = i\omega B_z \end{cases} \\ \begin{cases} \left.\begin{aligned} -\dfrac{i\omega}{c^2}E_x + ikB_y = \dfrac{\partial B_z}{\partial y} \\ ikB_x + \dfrac{i\omega}{c^2}E_y = \dfrac{\partial B_z}{\partial x} \end{aligned}\right\} \\ \dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y} = -\dfrac{i\omega}{c^2}E_z \end{cases} \end{cases} \Rightarrow \begin{cases} \begin{cases} \left.\begin{aligned} i\omega B_x + ikE_y = \dfrac{\partial E_z}{\partial y} \\ ikB_x + \dfrac{i\omega}{c^2}E_y = \dfrac{\partial B_z}{\partial x} \end{aligned}\right\} \\ \left.\begin{aligned} ikE_x - i\omega B_y = \dfrac{\partial E_z}{\partial x} \\ -\dfrac{i\omega}{c^2}E_x + ikB_y = \dfrac{\partial B_z}{\partial y} \end{aligned}\right\} \end{cases} \\ \begin{cases} \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = i\omega B_z \\ \dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y} = -\dfrac{i\omega}{c^2}E_z \end{cases} \end{cases}$$

$$\Rightarrow \begin{cases} \begin{cases} B_x = \dfrac{\begin{vmatrix} \dfrac{\partial E_z}{\partial y} & ik \\ \dfrac{\partial B_z}{\partial x} & \dfrac{i\omega}{c^2} \end{vmatrix}}{\begin{vmatrix} i\omega & ik \\ ik & \dfrac{i\omega}{c^2} \end{vmatrix}} = \dfrac{\left(\dfrac{i\omega}{c^2}\dfrac{\partial E_z}{\partial y} - ik\dfrac{\partial B_z}{\partial x}\right)}{\left(-\dfrac{\omega^2}{c^2} + k^2\right)} \Rightarrow \boldsymbol{B_x = \dfrac{i}{\left(\dfrac{\omega^2}{c^2} - k^2\right)}\left(k\dfrac{\partial B_z}{\partial x} - \dfrac{\omega}{c^2}\dfrac{\partial E_z}{\partial y}\right)} \\ E_y = \dfrac{\begin{vmatrix} i\omega & \dfrac{\partial E_z}{\partial y} \\ ik & \dfrac{\partial B_z}{\partial x} \end{vmatrix}}{\begin{vmatrix} i\omega & ik \\ ik & \dfrac{i\omega}{c^2} \end{vmatrix}} = \dfrac{\left(i\omega\dfrac{\partial B_z}{\partial x} - ik\dfrac{\partial E_z}{\partial y}\right)}{\left(-\dfrac{\omega^2}{c^2} + k^2\right)} \Rightarrow \boldsymbol{E_y = \dfrac{i}{\left(\dfrac{\omega^2}{c^2} - k^2\right)}\left(k\dfrac{\partial E_z}{\partial y} - \omega\dfrac{\partial B_z}{\partial x}\right)} \\ E_x = \dfrac{\begin{vmatrix} \dfrac{\partial E_z}{\partial x} & -i\omega \\ \dfrac{\partial B_z}{\partial y} & ik \end{vmatrix}}{\begin{vmatrix} ik & -i\omega \\ -\dfrac{i\omega}{c^2} & ik \end{vmatrix}} = \dfrac{\left(ik\dfrac{\partial E_z}{\partial x} + i\omega\dfrac{\partial B_z}{\partial y}\right)}{\left(k^2 - \dfrac{\omega^2}{c^2}\right)} \Rightarrow \boldsymbol{E_x = \dfrac{i}{\left(\dfrac{\omega^2}{c^2} - k^2\right)}\left(k\dfrac{\partial E_z}{\partial x} + \omega\dfrac{\partial B_z}{\partial y}\right)} \\ B_y = \dfrac{\begin{vmatrix} ik & \dfrac{\partial E_z}{\partial x} \\ -\dfrac{i\omega}{c^2} & \dfrac{\partial B_z}{\partial y} \end{vmatrix}}{\begin{vmatrix} ik & -i\omega \\ -\dfrac{i\omega}{c^2} & ik \end{vmatrix}} = \dfrac{\left(ik\dfrac{\partial B_z}{\partial y} + \dfrac{i\omega}{c^2}\dfrac{\partial E_z}{\partial x}\right)}{\left(k^2 - \dfrac{\omega^2}{c^2}\right)} \Rightarrow \boldsymbol{B_y = \dfrac{i}{\left(\dfrac{\omega^2}{c^2} - k^2\right)}\left(k\dfrac{\partial B_z}{\partial y} + \dfrac{\omega}{c^2}\dfrac{\partial E_z}{\partial x}\right)} \end{cases} \\ \begin{cases} \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = i\omega B_z \\ \dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y} = -\dfrac{i\omega}{c^2}E_z \end{cases} \Rightarrow \begin{cases} \dfrac{\partial}{\partial x}\left(k\dfrac{\partial E_z}{\partial y} - \omega\dfrac{\partial B_z}{\partial x}\right) - \dfrac{\partial}{\partial y}\left(k\dfrac{\partial E_z}{\partial x} + \omega\dfrac{\partial B_z}{\partial y}\right) = \omega\left(\dfrac{\omega^2}{c^2} - k^2\right)B_z \\ \dfrac{\partial}{\partial x}\left(k\dfrac{\partial B_z}{\partial y} + \dfrac{\omega}{c^2}\dfrac{\partial E_z}{\partial x}\right) - \dfrac{\partial}{\partial y}\left(k\dfrac{\partial B_z}{\partial x} - \dfrac{\omega}{c^2}\dfrac{\partial E_z}{\partial y}\right) = -\dfrac{i\omega}{c^2}\left(\dfrac{\omega^2}{c^2} - k^2\right)E_z \end{cases} \end{cases}$$

$$\Rightarrow \begin{cases} -\dfrac{\partial}{\partial x}\left(\dfrac{\partial B_z}{\partial x}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{\partial B_z}{\partial y}\right) = \left(\dfrac{\omega^2}{c^2} - k^2\right)B_z \\ \dfrac{\partial}{\partial x}\left(\dfrac{\partial E_z}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{\partial E_z}{\partial y}\right) = -\left(\dfrac{\omega^2}{c^2} - k^2\right)E_z \end{cases} \Rightarrow \begin{cases} \boldsymbol{\left[\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \left(\dfrac{\omega^2}{c^2} - k^2\right)\right]B_z = 0} \\ \boldsymbol{\left[\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \left(\dfrac{\omega^2}{c^2} - k^2\right)\right]E_z = 0} \end{cases}$$

Alternatively:

$$\begin{cases}\vec{\nabla}\cdot\vec{E}=0\Longrightarrow\vec{\nabla}\cdot\left(\vec{E}_0e^{i(kz-\omega t)}\right)=0\\ \vec{\nabla}\cdot\vec{B}=0\Longrightarrow\vec{\nabla}\cdot\left(\vec{B}_0e^{i(kz-\omega t)}\right)=0\end{cases}\Longrightarrow\begin{cases}\dfrac{\partial E_x}{\partial x}+\dfrac{\partial E_y}{\partial y}+ikE_z=0\\ \dfrac{\partial B_x}{\partial x}+\dfrac{\partial B_y}{\partial y}+ikB_z=0\end{cases}$$

$$\Longrightarrow\begin{cases}\dfrac{\partial}{\partial x}\left(k\dfrac{\partial E_z}{\partial x}+\omega\dfrac{\partial B_z}{\partial y}\right)+\dfrac{\partial}{\partial y}\left(k\dfrac{\partial E_z}{\partial y}-\omega\dfrac{\partial B_z}{\partial x}\right)+k\left(\dfrac{\omega^2}{c^2}-k^2\right)E_z=0\\ \dfrac{\partial}{\partial x}\left(k\dfrac{\partial B_z}{\partial x}-\dfrac{\omega}{c^2}\dfrac{\partial E_z}{\partial y}\right)+\dfrac{\partial}{\partial y}\left(k\dfrac{\partial B_z}{\partial y}+\dfrac{\omega}{c^2}\dfrac{\partial E_z}{\partial x}\right)+k\left(\dfrac{\omega^2}{c^2}-k^2\right)B_z=0\end{cases}\Longrightarrow\begin{cases}\boldsymbol{\left[\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\left(\dfrac{\omega^2}{c^2}-k^2\right)\right]E_z=0}\\ \boldsymbol{\left[\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\left(\dfrac{\omega^2}{c^2}-k^2\right)\right]B_z=0}\end{cases}$$

TE propagation mode: When $\boldsymbol{E_z=0}$, the waves are called TE (Transverse Electric) waves.
TM propagation mode: When $\boldsymbol{B_z=0}$, the waves are called TM (Transverse Magnetic) waves.
TEM propagation mode: When $\boldsymbol{E_z=B_z=0}$, the waves are called TEM (Transverse Electromagnetic) waves.

TE waves in Rectangular Wave Guide

$$B_z(x,y)=X(x)Y(y)$$

$$\begin{cases}B_x=\dfrac{ik}{\left(\dfrac{\omega^2}{c^2}-k^2\right)}\dfrac{\partial B_z}{\partial x}\quad\begin{bmatrix}\because E_z=0\\ \text{for TE mode}\end{bmatrix}\\ B_y=\dfrac{ik}{\left(\dfrac{\omega^2}{c^2}-k^2\right)}\dfrac{\partial B_z}{\partial y}\quad\begin{bmatrix}\because E_z=0\\ \text{for TE mode}\end{bmatrix}\end{cases}\Longrightarrow\begin{cases}B_x=\dfrac{ikY}{\left(\dfrac{\omega^2}{c^2}-k^2\right)}\dfrac{\partial X}{\partial x}\\ B_y=\dfrac{ikX}{\left(\dfrac{\omega^2}{c^2}-k^2\right)}\dfrac{\partial Y}{\partial y}\end{cases}$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega^2}{c^2}-k^2\right)\right]B_z=0\Longrightarrow Y\frac{\partial^2X}{\partial x^2}+X\frac{\partial^2Y}{\partial y^2}+\left(\frac{\omega^2}{c^2}-k^2\right)XY=0\Longrightarrow\frac{1}{X}\frac{\partial^2X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2Y}{\partial y^2}+\left(\frac{\omega^2}{c^2}-k^2\right)=0$$

$$\Longrightarrow\begin{cases}\dfrac{1}{X}\dfrac{\partial^2X}{\partial x^2}=-{k_x}^2\\ \dfrac{1}{Y}\dfrac{\partial^2Y}{\partial y^2}=-{k_y}^2\\ -{k_x}^2-{k_y}^2+\left(\dfrac{\omega^2}{c^2}-k^2\right)=0\end{cases}\Longrightarrow\begin{cases}X(x)=A\sin(k_xx)+B\cos(k_xx)\\ Y(y)=C\sin(k_yx)+D\cos(k_yx)\\ k^2=\dfrac{\omega^2}{c^2}-\left({k_x}^2+{k_y}^2\right)\end{cases}\Longrightarrow\begin{cases}\dfrac{\partial X}{\partial x}=Ak_x\cos(k_xx)-Bk_x\sin(k_xx)\\ \dfrac{\partial Y}{\partial y}=Ck_y\cos(k_yx)-Dk_y\sin(k_yx)\\ k=\sqrt{\dfrac{\omega^2}{c^2}-\left({k_x}^2+{k_y}^2\right)}\end{cases}$$

$$\Longrightarrow\begin{cases}X(x)=B\cos(k_xx)\quad\left[\because\text{At }x=0,B_x=0=\dfrac{\partial X}{\partial x}\right]\\ Y(y)=D\cos(k_yx)\quad\left[\because\text{At }y=0,B_y=0=\dfrac{\partial Y}{\partial y}\right]\end{cases}\Longrightarrow\begin{cases}X(x)=B\cos\left(\dfrac{m\pi x}{a}\right)\quad\left[\because\text{At }x=a,B_x=0=\dfrac{\partial X}{\partial x}\right]\\ Y(y)=D\cos\left(\dfrac{n\pi y}{b}\right)\quad\left[\because\text{At }y=b,B_y=0=\dfrac{\partial Y}{\partial y}\right]\end{cases}$$

$$\begin{bmatrix}\sin(k_xa)=0\Longrightarrow k_x=\dfrac{m\pi}{a}\\ \because\sin(k_yb)=0\Longrightarrow k_y=\dfrac{m\pi}{b}\end{bmatrix}\quad\therefore B_z(x,y)=X(x)Y(y)\Longrightarrow\boldsymbol{B_z=B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)}$$

$$k=\sqrt{\frac{\omega^2}{c^2}-\left({k_x}^2+{k_y}^2\right)}=\sqrt{\frac{\omega^2}{c^2}-\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}}$$

If $\dfrac{\omega^2}{c^2}<\left\{\left(\dfrac{m\pi}{a}\right)^2+\left(\dfrac{n\pi}{b}\right)^2\right\}$ k will be imaginary and the waves will be exponentially attenuated due to $e^{i(kz-\omega t)}$

$$\therefore\frac{\omega^2}{c^2}\geq\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}\Longrightarrow\omega\geq\omega_{mn}\quad\begin{bmatrix}\text{where }\boldsymbol{\omega_{mn}=\pi c\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}}\text{ is called the}\\ \textbf{cut-off frequency}\text{ for the mode in question.}\end{bmatrix}\therefore\boldsymbol{k=\frac{1}{c}\sqrt{\omega^2-{\omega_{mn}}^2}}$$

The lowest cut-off frequency occurs for the mode TE_{10} (assuming $a \geq b$): $\boldsymbol{\omega_{10}} = \dfrac{\boldsymbol{\pi c}}{\boldsymbol{a}}$

The wave-velocity, $v = \dfrac{\omega}{k} = \dfrac{\omega}{\frac{1}{c}\sqrt{\omega^2 - {\omega_{mn}}^2}} \Rightarrow \boldsymbol{v} = \dfrac{\boldsymbol{c}}{\sqrt{\boldsymbol{1} - \left(\frac{\boldsymbol{\omega_{mn}}}{\boldsymbol{\omega}}\right)^{\boldsymbol{2}}}} > c$

The group velocity, $v_g = \dfrac{d\omega}{dk} = \left(\dfrac{dk}{d\omega}\right)^{-1} = \left(\dfrac{1}{c}\dfrac{\omega}{\sqrt{\omega^2 - {\omega_{mn}}^2}}\right)^{-1} = \dfrac{c\sqrt{\omega^2 - {\omega_{mn}}^2}}{\omega} \Rightarrow \boldsymbol{v_g} = \boldsymbol{c}\sqrt{\boldsymbol{1} - \left(\frac{\boldsymbol{\omega_{mn}}}{\boldsymbol{\omega}}\right)^{\boldsymbol{2}}} < c$

TM waves in Rectangular Wave Guide

$E_z(x,y) = X(x)Y(y)$

$$\begin{cases} E_x = \dfrac{ik}{\left(\frac{\omega^2}{c^2} - k^2\right)}\dfrac{\partial E_z}{\partial y} & \begin{bmatrix} \because B_z = 0 \\ \text{for TE mode}\end{bmatrix} \\ E_y = \dfrac{ik}{\left(\frac{\omega^2}{c^2} - k^2\right)}\dfrac{\partial E_z}{\partial x} & \begin{bmatrix} \because B_z = 0 \\ \text{for TE mode}\end{bmatrix} \end{cases} \Rightarrow \begin{cases} E_x = \dfrac{ikY}{\left(\frac{\omega^2}{c^2} - k^2\right)}\dfrac{\partial X}{\partial x} \\ E_y = \dfrac{ikX}{\left(\frac{\omega^2}{c^2} - k^2\right)}\dfrac{\partial Y}{\partial y} \end{cases}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega^2}{c^2} - k^2\right)\right]E_z = 0 \Rightarrow Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + \left(\frac{\omega^2}{c^2} - k^2\right)XY = 0 \Rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \left(\frac{\omega^2}{c^2} - k^2\right) = 0$$

$$\Rightarrow \begin{cases} \dfrac{1}{X}\dfrac{\partial^2 X}{\partial x^2} = -{k_x}^2 \\ \dfrac{1}{Y}\dfrac{\partial^2 Y}{\partial y^2} = -{k_y}^2 \\ -{k_x}^2 - {k_y}^2 + \left(\dfrac{\omega^2}{c^2} - k^2\right) = 0 \end{cases} \Rightarrow \begin{cases} X(x) = A\sin(k_x x) + B\cos(k_x x) \\ Y(y) = C\sin(k_y x) + D\cos(k_y x) \\ k^2 = \dfrac{\omega^2}{c^2} - \left({k_x}^2 + {k_y}^2\right) \end{cases} \Rightarrow \begin{cases} \dfrac{\partial X}{\partial x} = Ak_x\cos(k_x x) - Bk_x\sin(k_x x) \\ \dfrac{\partial Y}{\partial y} = Ck_y\cos(k_y x) - Dk_y\sin(k_y x) \\ k = \sqrt{\dfrac{\omega^2}{c^2} - \left({k_x}^2 + {k_y}^2\right)} \end{cases}$$

$$\Rightarrow \begin{cases} X(x) = B\cos(k_x x) & \left[\because \text{At } x = 0, E_x = 0 = \dfrac{\partial X}{\partial x}\right] \\ Y(y) = D\cos(k_y x) & \left[\because \text{At } y = 0, E_y = 0 = \dfrac{\partial Y}{\partial y}\right] \end{cases} \Rightarrow \begin{cases} X(x) = B\cos\left(\dfrac{m\pi x}{a}\right) & \left[\because \text{At } x = a, B_x = 0 = \dfrac{\partial X}{\partial x}\right] \\ Y(y) = D\cos\left(\dfrac{n\pi y}{b}\right) & \left[\because \text{At } y = b, B_y = 0 = \dfrac{\partial Y}{\partial y}\right] \end{cases}$$

$$\begin{bmatrix} \sin(k_x a) = 0 \Rightarrow k_x = \dfrac{m\pi}{a} \\ \because \sin(k_y b) = 0 \Rightarrow k_y = \dfrac{m\pi}{b} \end{bmatrix} \therefore E_z(x,y) = X(x)Y(y) \Rightarrow \boldsymbol{E_z} = \boldsymbol{E_0}\cos\left(\frac{\boldsymbol{m\pi x}}{\boldsymbol{a}}\right)\cos\left(\frac{\boldsymbol{n\pi y}}{\boldsymbol{b}}\right)$$

$$k = \sqrt{\frac{\omega^2}{c^2} - \left({k_x}^2 + {k_y}^2\right)} = \sqrt{\frac{\omega^2}{c^2} - \left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\}}$$

If $\dfrac{\omega^2}{c^2} < \left\{\left(\dfrac{m\pi}{a}\right)^2 + \left(\dfrac{n\pi}{b}\right)^2\right\}$ k will be imaginary and the waves will be exponentially attenuated due to $e^{i(kz - \omega t)}$

$\therefore \dfrac{\omega^2}{c^2} \geq \left\{\left(\dfrac{m\pi}{a}\right)^2 + \left(\dfrac{n\pi}{b}\right)^2\right\} \Rightarrow \omega \geq \omega_{mn}$ [where $\boldsymbol{\omega_{mn}} = \boldsymbol{\pi c}\sqrt{\left(\frac{\boldsymbol{m}}{\boldsymbol{a}}\right)^{\boldsymbol{2}} + \left(\frac{\boldsymbol{n}}{\boldsymbol{b}}\right)^{\boldsymbol{2}}}$ is called the **cut-off frequency** for the mode in question.] $\therefore \boldsymbol{k} = \dfrac{\boldsymbol{1}}{\boldsymbol{c}}\sqrt{\boldsymbol{\omega^2} - \boldsymbol{\omega_{mn}}^{\boldsymbol{2}}}$

The lowest cut-off frequency occurs for the mode TM_{11} (assuming $a \geq b$): $\boldsymbol{\omega_{11}} = \boldsymbol{\pi c}\sqrt{\dfrac{\boldsymbol{1}}{\boldsymbol{a^2}} + \dfrac{\boldsymbol{1}}{\boldsymbol{b^2}}}$

The wave-velocity, $v = \dfrac{\omega}{k} = \dfrac{\omega}{\frac{1}{c}\sqrt{\omega^2 - {\omega_{mn}}^2}} \Rightarrow \boldsymbol{v} = \dfrac{\boldsymbol{c}}{\sqrt{\boldsymbol{1} - \left(\frac{\boldsymbol{\omega_{mn}}}{\boldsymbol{\omega}}\right)^{\boldsymbol{2}}}} > c$

The group velocity, $v_g = \dfrac{d\omega}{dk} = \left(\dfrac{dk}{d\omega}\right)^{-1} = \left(\dfrac{1}{c}\dfrac{\omega}{\sqrt{\omega^2 - {\omega_{mn}}^2}}\right)^{-1} = \dfrac{c\sqrt{\omega^2 - {\omega_{mn}}^2}}{\omega} \Rightarrow \boldsymbol{v_g} = \boldsymbol{c}\sqrt{\boldsymbol{1} - \left(\frac{\boldsymbol{\omega_{mn}}}{\boldsymbol{\omega}}\right)^{\boldsymbol{2}}} < c$

Ratio of lowest TM cut-off frequency to lowest TE cut-off frequency, $\frac{\boldsymbol{\omega_{11}}}{\boldsymbol{\omega_{10}}} = \frac{\boldsymbol{\pi c}\sqrt{\frac{\mathbf{1}}{\boldsymbol{a^2}}+\frac{\mathbf{1}}{\boldsymbol{b^2}}}}{\frac{\boldsymbol{\pi c}}{\boldsymbol{a}}} = \sqrt{\mathbf{1}+\frac{\boldsymbol{a^2}}{\boldsymbol{b^2}}} \geq \sqrt{2}$

Relativistic motion under a constant force: **A particle of mass m is subject to a constant force *F*. If it starts from rest at the origin at time *t* = 0, find its position *x*(*t*).** **[Example 12.10, Electrodynamics by Griffiths]**

$$\frac{d\vec{p}}{dt} = \vec{F} \Rightarrow \vec{p} = \vec{F}t + \vec{C} \;\; [\vec{C} \text{ is constant of integration.}] \Rightarrow \vec{p} = \vec{F}t \;\; [\because \vec{p} = 0 \text{ at } t = 0] \Rightarrow \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} = \vec{F}t$$

$$\Rightarrow m^2v^2 = F^2t^2\left(1-\frac{v^2}{c^2}\right) = F^2t^2 - \frac{v^2}{c^2}F^2t^2 \Rightarrow v^2\left(m^2+\frac{F^2t^2}{c^2}\right) = F^2t^2 \Rightarrow v^2 = \frac{F^2t^2}{m^2+\frac{F^2t^2}{c^2}} = \frac{\frac{F^2t^2}{m^2}}{1+\frac{F^2t^2}{m^2c^2}}$$

$$\Rightarrow v = \frac{\left(\frac{F}{m}\right)t}{\sqrt{1+\left(\frac{Ft}{mc}\right)^2}} \;\; \left[\begin{array}{l}\text{If } \frac{Ft}{m} \ll c \text{ then } v \approx \left(\frac{F}{m}\right)t \\ \text{which is the classical result.}\end{array}\right] \Rightarrow x = \int_0^t \frac{\left(\frac{F}{m}\right)t'\,dt'}{\sqrt{1+\left(\frac{Ft'}{mc}\right)^2}} = \frac{mc^2}{F}\int_0^{\frac{Ft}{mc}} \frac{u\,du}{\sqrt{1+u^2}}$$

$$\left[\text{putting } \left(\frac{Ft'}{mc}\right) = u \Rightarrow dt' = \frac{mc}{F}du\right] = \frac{mc^2}{F}\left[\sqrt{1+u^2}\right]_0^{\frac{Ft}{mc}} \Rightarrow x = \frac{mc^2}{F}\left\{\sqrt{1+\left(\frac{Ft}{mc}\right)^2}-1\right\}$$

which is the equation of a hyperbola. If $\frac{Ft}{m} \ll c$, $x \approx \frac{mc^2}{F}\left\{\left(1+\frac{1}{2}\left(\frac{Ft}{mc}\right)^2\right)-1\right\} = \frac{mc^2}{F}\cdot\frac{1}{2}\left(\frac{Ft}{mc}\right)^2 \Rightarrow x = \frac{Ft^2}{2m}$ which is the equation of a parabola. Thus, motion under a constant force is called hyperbolic motion.

Boltzmann Transport Equation (BTE)

Distribution function, $f(t,\vec{r},\vec{v}) \equiv$ number of particles in hypervolume element $dx\,dy\,dz\,dv_x dv_y dv_z$ at r, v at time t

By the Liouville theorem, as time lapses, the hypervolume element moves along a flowline in such a way that the distribution is conserved, i.e. $f(t+dt,\vec{r}+d\vec{r},\vec{v}+d\vec{v}) = f(t,\vec{r},\vec{v})$ [in the absence of collisions]

$\Rightarrow f(t+dt,\vec{r}+d\vec{r},\vec{v}+d\vec{v}) - f(t,\vec{r},\vec{v}) = 0 \Rightarrow dt\left(\frac{\partial f}{\partial t}\right) + d\vec{r}\cdot\vec{\nabla}_r f + d\vec{v}\cdot\vec{\nabla}_v f = 0$ [in the absence of collisions]

and $dt\left(\frac{\partial f}{\partial t}\right) + d\vec{r}\cdot\vec{\nabla}_r f + d\vec{v}\cdot\vec{\nabla}_v f = dt\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}$ [in the presence of collisions or scattering interactions]

where $\vec{\nabla}_r = \hat{\imath}\frac{\partial f}{\partial x}+\hat{\jmath}\frac{\partial f}{\partial y}+\hat{k}\frac{\partial f}{\partial z}$ and $\vec{\nabla}_v = \hat{\imath}\frac{\partial f}{\partial v_x}+\hat{\jmath}\frac{\partial f}{\partial v_y}+\hat{k}\frac{\partial f}{\partial v_z}$

In the steady state (equilibrium), $\frac{\partial f}{\partial t} = \frac{\partial f_0}{\partial t} = 0$ [where f_0 is the equilibrium value]

If f does not vary greatly from its equilibrium value f_0, then $\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \approx -\frac{(f-f_0)}{\tau}$ [where τ is the relaxation time]

In an unsteady state, $\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \approx -\frac{(f-f_0)}{\tau} \Rightarrow \frac{\partial f}{\partial t} = -\frac{(f-f_0)}{\tau} \Rightarrow \frac{\partial (f-f_0)}{\partial t} = -\frac{(f-f_0)}{\tau}$ $\left[\because \frac{\partial f_0}{\partial t} = 0\right]$

$\Rightarrow \frac{d(f-f_0)}{(f-f_0)} = -\frac{dt}{\tau} \Rightarrow \ln(f-f_0)_t = -\frac{t}{\tau} + \ln C \Rightarrow (f-f_0)_t = Ce^{-\frac{t}{\tau}} \Rightarrow \boldsymbol{(f-f_0)_t = (f-f_0)_{t=0}e^{-\frac{t}{\tau}}}$ $\left[\begin{array}{c}\text{at } t = 0, \\ f \neq f_0\end{array}\right]$

If a non-equilibrium distribution of velocities result due to external forces which are suddenly removed, the rate at which the distribution approaches the equilibrium state is proportional to the deviation of the distribution function f from the equilibrium value f_0.

$\therefore dt\left(\frac{\partial f}{\partial t}\right) + d\vec{r}\cdot\vec{\nabla}_r f + d\vec{v}\cdot\vec{\nabla}_v f = dt\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \Rightarrow \left(\frac{\partial f}{\partial t}\right) + \frac{d\vec{r}}{dt}\cdot\vec{\nabla}_r f + \frac{d\vec{v}}{dt}\cdot\vec{\nabla}_v f = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}$

$\Rightarrow \boldsymbol{\left(\frac{\partial f}{\partial t}\right) + \vec{v}\cdot\vec{\nabla}_r f + \vec{a}\cdot\vec{\nabla}_v f = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}}$ [Boltzmann transport equation] $\Rightarrow \left(\frac{\partial f}{\partial t}\right) + \vec{v}\cdot\vec{\nabla}_r f + \vec{a}\cdot\vec{\nabla}_v f = -\left(\frac{f-f_0}{\tau}\right)$

Particle Diffusion

Consider an isothermal system with a gradient in the particle concentration. The steady state Boltzmann equation in the τ-approximation is $\vec{v}\cdot\vec{\nabla}_r f=-\left(\frac{f-f_0}{\tau}\right)$ $\left[\begin{array}{l}\because \frac{\partial f}{\partial t}=0 \text{ for steady state, and}\\ \vec{a}=0 \text{ as no external field is there.}\end{array}\right]$

Solving this by iterative technique, $\vec{v}\cdot\vec{\nabla}_r f_0=-\frac{(f_1-f_0)}{\tau}\Rightarrow f_1-f_0=-\tau\vec{v}\cdot\vec{\nabla}_r f_0\Rightarrow \boldsymbol{f_1=f_0-\tau\vec{v}\cdot\vec{\nabla}_r f_0}$ $\left[\begin{array}{c}\text{1st order}\\ \text{solution}\end{array}\right]$

Similarly, $\vec{v}\cdot\vec{\nabla}_r f_1=-\frac{(f_2-f_0)}{\tau}\Rightarrow f_2-f_0=-\tau\vec{v}\cdot\vec{\nabla}_r f_1\Rightarrow f_2=f_0-\tau\vec{v}\cdot\vec{\nabla}_r f_1=f_0-\tau\vec{v}\cdot\vec{\nabla}_r\left(f_0-\tau\vec{v}\cdot\vec{\nabla}_r f_0\right)$

$\Rightarrow f_2=f_0-\tau\vec{v}\cdot\vec{\nabla}_r f_0+\tau^2\vec{v}\cdot\vec{\nabla}_r\left(\vec{v}\cdot\vec{\nabla}_r f_0\right)$ $\left[\begin{array}{c}\text{2nd order}\\ \text{solution}\end{array}\right]$

$\Rightarrow f_2=f_0-\tau\vec{v}\cdot\vec{\nabla}_r f_0+\tau^2\vec{v}\cdot\left\{\left(\vec{v}\cdot\vec{\nabla}_r\right)\vec{\nabla}_r f_0+\left(\vec{\nabla}_r f_0\cdot\vec{\nabla}_r\right)\vec{v}+\vec{v}\times\left(\vec{\nabla}_r\times\vec{\nabla}_r f_0\right)+\vec{\nabla}_r f_0\times\left(\vec{\nabla}_r\times\vec{v}\right)\right\}$

$\Rightarrow f_2=f_0-\tau\vec{v}\cdot\vec{\nabla}_r f_0+\tau^2\vec{v}\cdot\left\{\left(\vec{v}\cdot\vec{\nabla}_r\right)\vec{\nabla}_r f_0+\left(\vec{\nabla}_r f_0\cdot\vec{\nabla}_r\right)\vec{v}+\vec{\nabla}_r f_0\times\left(\vec{\nabla}_r\times\vec{v}\right)\right\}$

For classical distribution, $f_0=\frac{1}{e^{(\epsilon-\mu)/kT}}\Rightarrow\frac{df_0}{d\mu}=\frac{-1}{e^{(\epsilon-\mu)/kT}}\left(\frac{-1}{kT}\right)=\frac{f_0}{kT}$

For first order, $\boldsymbol{f=f_0-\tau\vec{v}\cdot\vec{\nabla}_r f_0}=f_0-\tau\vec{v}\cdot\left(\frac{df_0}{d\mu}\vec{\nabla}_r\mu\right)=f_0-\tau\left(\frac{f_0}{kT}\right)\vec{v}\cdot\left(\vec{\nabla}_r\mu\right)\Rightarrow f=f_0-f_0\left(\frac{\tau}{kT}\right)\vec{v}\cdot\left(\vec{\nabla}_r\mu\right)$

Diffusion flux density, $\vec{j}=\int\vec{v}f\frac{g(\epsilon)}{V}d\epsilon=\frac{1}{V}\int\vec{v}\left\{f_0-f_0\left(\frac{\tau}{kT}\right)\vec{v}\cdot\left(\vec{\nabla}_r\mu\right)\right\}g(\epsilon)\,d\epsilon$

$=\frac{1}{V}\int\vec{v}f_0 g(\epsilon)\,d\epsilon-\frac{1}{V}\int\vec{v}\left\{f_0\left(\frac{\tau}{kT}\right)\vec{v}\cdot\left(\vec{\nabla}_r\mu\right)\right\}g(\epsilon)\,d\epsilon=-\int\vec{v}\left\{f_0\left(\frac{\tau}{kT}\right)\vec{v}\cdot\left(\vec{\nabla}_r\mu\right)\right\}g(\epsilon)\,d\epsilon$

$\left[\begin{array}{l}\because\int\vec{v}f_0 g(\epsilon)\,d\epsilon=\int\vec{v}f_0\left(\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\epsilon^{\frac{1}{2}}\right)d\epsilon=0 \text{ as the integrand is an odd function of } \vec{v}\\ \qquad\text{and the integral is from } -\infty \text{ to } \infty \text{ in the velocity domain.}\end{array}\right]$

$=-\left(\frac{\tau}{kT}\right)\frac{1}{V}\int\left\{\vec{v}\left(\vec{v}\cdot\left(\vec{\nabla}_r\mu\right)\right)\right\}f_0 g(\epsilon)\,d\epsilon$ [if τ is independent of velocity]

$\Rightarrow\vec{j}=-\tilde{R}\left(\vec{\nabla}_r\mu\right)\left[\begin{array}{l}\text{where } \tilde{R}_{\alpha\beta}=\left(\frac{\tau}{kT}\right)\frac{1}{V}\int v_\alpha v_\beta f_0 g(\epsilon)\,d\epsilon=\left(\frac{\tau}{kT}\right)\frac{1}{V}\int\overline{v_\alpha v_\beta}f_0 g(\epsilon)\,d\epsilon\\ =\left(\frac{\tau}{kT}\right)\frac{1}{V}\int\delta_{\alpha\beta}\left(\frac{2\epsilon}{3m}\right)f_0 g(\epsilon)\,d\epsilon=\left(\frac{2\tau\delta_{\alpha\beta}}{3mkT}\right)\frac{1}{V}\int\epsilon f_0 g(\epsilon)\,d\epsilon=\left(\frac{2\tau\delta_{\alpha\beta}}{3mkT}\right)\frac{1}{V}\left(\frac{3}{2}NkT\right)=\frac{n\tau}{m}\delta_{\alpha\beta}\end{array}\right]$

$\Rightarrow\vec{j}=-\left(\frac{n\tau}{m}\delta_{\alpha\beta}\right)\left(\vec{\nabla}_r\mu\right)=-\left(\frac{n\tau}{m}\delta_{\alpha\beta}\right)\left(\frac{kT}{n}\vec{\nabla}_r n\right)=-\left(\frac{\tau kT}{m}\delta_{\alpha\beta}\right)\vec{\nabla}_r n\Rightarrow\boldsymbol{\vec{j}=-\tilde{D}\left(\vec{\nabla}_r n\right)}$ $\left[\begin{array}{l}\tilde{D}=\frac{\tau kT}{m}\delta_{\alpha\beta}=\frac{1}{3}\tau\overline{v^2}\delta_{\alpha\beta}\\ \text{is the diffusivity tensor}\end{array}\right]$

$\left[\because\mu=-\frac{3}{2}kT\ln\left(\frac{2\pi mkT}{h^2}n^{-\frac{2}{3}}\right)=kT\ln n-\frac{3}{2}kT\ln\left(\frac{2\pi mkT}{h^2}\right)=kT\ln n-u(T)\Rightarrow\vec{\nabla}_r\mu=\frac{kT}{n}\vec{\nabla}_r n\quad\left[\begin{array}{c}\vec{\nabla}_r u(T)=0\\ \text{(isothermal)}\end{array}\right]\right]$

This is **Fick's first law of diffusion** which states that in an isothermal system under steady state, diffusion flux density is proportional to the negative gradient of concentration.

Thermal Conductivity

Consider a metal with a uniform stationary temperature $\vec{\nabla}_r T$ gradient. The steady state Boltzmann equation in the τ-approximation is $\vec{v}\cdot\vec{\nabla}_r f=-\left(\frac{f-f_0}{\tau}\right)$ $\left[\begin{array}{l}\because\frac{\partial f}{\partial t}=0 \text{ for steady state, and}\\ \vec{a}=0 \text{ as no external field is there.}\end{array}\right]$

For FD distribution, $f_0=\frac{1}{e^{(\epsilon-\mu)/kT}+1}$

For first order, $\boldsymbol{f=f_0-\tau\vec{v}\cdot\vec{\nabla}_r f_0}=f_0-\tau\vec{v}\cdot\left(\frac{df_0}{d\left(\frac{\epsilon-\mu}{T}\right)}\vec{\nabla}_r\left(\frac{\epsilon-\mu}{T}\right)\right)=f_0-\tau\vec{v}\cdot\left(\frac{df_0}{d\epsilon}\frac{d\epsilon}{d\left(\frac{\epsilon-\mu}{T}\right)}\vec{\nabla}_r\left(\frac{\epsilon-\mu}{T}\right)\right)$

$=f_0-\tau\vec{v}\cdot\left\{\frac{df_0}{d\epsilon}\left(\frac{d\left(\frac{\epsilon-\mu}{T}\right)}{d\epsilon}\right)^{-1}\vec{\nabla}_r\left(\frac{\epsilon-\mu}{T}\right)\right\}=f_0-\tau\vec{v}\cdot\left\{\frac{df_0}{d\epsilon}\left(\frac{1}{T}\right)^{-1}\vec{\nabla}_r\left(\frac{\epsilon-\mu}{T}\right)\right\}=f_0-\frac{df_0}{d\epsilon}\tau\vec{v}\cdot\left\{T\vec{\nabla}_r\left(\frac{\epsilon-\mu}{T}\right)\right\}$

$=f_0-\frac{df_0}{d\epsilon}\tau\vec{v}\cdot\left\{T\vec{\nabla}_r\left(\frac{\epsilon}{T}\right)-T\vec{\nabla}_r\left(\frac{\mu}{T}\right)\right\}=f_0-\frac{df_0}{d\epsilon}\tau\vec{v}\cdot\left\{T\left(-\boldsymbol{\frac{\epsilon}{T^2}\vec{\nabla}_r T}\right)-T\vec{\nabla}_r\left(\frac{\mu}{T}\right)\right\}=f_0-\frac{df_0}{d\epsilon}\tau\vec{v}\cdot\left\{-\frac{\epsilon}{T}\vec{\nabla}_r T-T\vec{\nabla}_r\left(\frac{\mu}{T}\right)\right\}$

Electron flow, $\vec{j} = \int \vec{v} f \frac{g(\epsilon)}{V} d\epsilon = \frac{1}{V}\int \vec{v}\left\{f_0 - \frac{df_0}{d\epsilon}\tau\vec{v}\cdot\left\{-\frac{\epsilon}{T}\vec{\nabla}_r T - T\vec{\nabla}_r\left(\frac{\mu}{T}\right)\right\}\right\} g(\epsilon)\, d\epsilon$

$= \frac{1}{V}\int \vec{v} f_0 g(\epsilon)\, d\epsilon - \int \vec{v}\left\{\frac{df_0}{d\epsilon}\tau\vec{v}\cdot\left\{-\frac{\epsilon}{T}\vec{\nabla}_r T - T\vec{\nabla}_r\left(\frac{\mu}{T}\right)\right\}\right\} g(\epsilon)\, d\epsilon = -\frac{1}{V}\int \vec{v}\left\{\frac{df_0}{d\epsilon}\tau\vec{v}\cdot\left\{-\frac{\epsilon}{T}\vec{\nabla}_r T - T\vec{\nabla}_r\left(\frac{\mu}{T}\right)\right\}\right\} g(\epsilon)\, d\epsilon$

$\left[\because \int \vec{v} f_0 g(\epsilon)\, d\epsilon = \frac{1}{V}\int \vec{v} f_0\left(\frac{2\pi V}{h^3} g_s (2m)^{\frac{3}{2}}\epsilon^{\frac{1}{2}}\right) d\epsilon = 0\right.$ as the integrand is an odd function of $\vec{v}$ and the integral is from $-\infty$ *to* ∞ in the velocity domain.$\left.\right]$

$= -\frac{1}{V}\int \tau\frac{df_0}{d\epsilon}\vec{v}\left\{\vec{v}\cdot\left\{-\frac{\epsilon}{T}\vec{\nabla}_r T - T\vec{\nabla}_r\left(\frac{\mu}{T}\right)\right\}\right\} g(\epsilon)\, d\epsilon$

$= -\frac{1}{V}\int \tau\left(-\frac{df_0}{d\epsilon}\right)\left\{\vec{v}\left(\vec{v}\cdot\vec{\nabla}_r(\ln T)\right)\right\}\epsilon g(\epsilon)\, d\epsilon - \frac{1}{V}\int \tau\left(-\frac{df_0}{d\epsilon}\right)\left\{\vec{v}\left(\vec{v}\cdot T\vec{\nabla}_r\left(\frac{\mu}{T}\right)\right)\right\} g(\epsilon)\, d\epsilon$

$\Longrightarrow \vec{j} = -\tilde{R}_1\left(\vec{\nabla}_r(\ln T)\right) - \tilde{R}_0\left(T\vec{\nabla}_r\left(\frac{\mu}{T}\right)\right) \quad \left[\text{where } \left(\tilde{R}_q\right)_{\alpha\beta} = \frac{1}{V}\int \tau v_\alpha v_\beta\left(-\frac{df_0}{d\epsilon}\right)\epsilon^q g(\epsilon)\, d\epsilon\right]$

For $\vec{j}$ to vanish i.e. $\vec{j} = 0 \Longrightarrow -\tilde{R}_1\left(\vec{\nabla}_r(\ln T)\right) - \tilde{R}_0\left(T\vec{\nabla}_r\left(\frac{\mu}{T}\right)\right) = 0 \Longrightarrow T\vec{\nabla}_r\left(\frac{\mu}{T}\right) = -\frac{\tilde{R}_1}{\tilde{R}_0}\left(\vec{\nabla}_r(\ln T)\right) \quad \left[\frac{\tilde{R}_1}{\tilde{R}_0} = \tilde{R}_1\tilde{R}_0^{\ -1}\right]$

Heat flow, $\vec{Q} = \int \epsilon\vec{v} f\frac{g(\epsilon)}{V} d\epsilon = \frac{1}{V}\int \epsilon\vec{v}\left\{f_0 - \frac{df_0}{d\epsilon}\tau\vec{v}\cdot\left\{-\frac{\epsilon}{T}\vec{\nabla}_r T - T\vec{\nabla}_r\left(\frac{\mu}{T}\right)\right\}\right\} g(\epsilon)\, d\epsilon$

$= \frac{1}{V}\int \epsilon\vec{v} f_0 g(\epsilon)\, d\epsilon - \frac{1}{V}\int \epsilon\vec{v}\left\{\frac{df_0}{d\epsilon}\tau\vec{v}\cdot\left\{-\frac{\epsilon}{T}\vec{\nabla}_r T - T\vec{\nabla}_r\left(\frac{\mu}{T}\right)\right\}\right\} g(\epsilon)\, d\epsilon = -\frac{1}{V}\int \epsilon\vec{v}\left\{\frac{df_0}{d\epsilon}\tau\vec{v}\cdot\left\{-\frac{\epsilon}{T}\vec{\nabla}_r T - T\vec{\nabla}_r\left(\frac{\mu}{T}\right)\right\}\right\} g(\epsilon)\, d\epsilon$

$\left[\because \int \epsilon\vec{v} f_0 g(\epsilon)\, d\epsilon = \int \epsilon\vec{v} f_0\left(\frac{2\pi V}{h^3} g_s (2m)^{\frac{3}{2}}\epsilon^{\frac{1}{2}}\right) d\epsilon = 0\right.$ as the integrand is an odd function of $\vec{v}$ and the integral is from $-\infty$ *to* ∞ in the velocity domain.$\left.\right]$

$= \frac{1}{V}\int \tau\left(-\frac{df_0}{d\epsilon}\right)\vec{v}\left\{\vec{v}\cdot\left\{-\frac{\epsilon}{T}\vec{\nabla}_r T - T\vec{\nabla}_r\left(\frac{\mu}{T}\right)\right\}\right\}\epsilon g(\epsilon)\, d\epsilon$

$= -\frac{1}{V}\int \tau\left(-\frac{df_0}{d\epsilon}\right)\left\{\vec{v}\left(\vec{v}\cdot\vec{\nabla}_r(\ln T)\right)\right\}\epsilon^2 g(\epsilon)\, d\epsilon - \frac{1}{V}\int \tau\left(-\frac{df_0}{d\epsilon}\right)\left\{\vec{v}\left(\vec{v}\cdot T\vec{\nabla}_r\left(\frac{\mu}{T}\right)\right)\right\}\epsilon g(\epsilon)\, d\epsilon$

$\Longrightarrow \vec{Q} = -\tilde{R}_2\left(\vec{\nabla}_r(\ln T)\right) - \tilde{R}_1\left(T\vec{\nabla}_r\left(\frac{\mu}{T}\right)\right) \quad \left[\text{where } \left(\tilde{R}_q\right)_{\alpha\beta} = \int \tau v_\alpha v_\beta\left(-\frac{df_0}{d\epsilon}\right)\epsilon^q g(\epsilon)\, d\epsilon\right]$

When $\vec{j} = 0$, $\vec{Q} = -\tilde{R}_2\left(\vec{\nabla}_r(\ln T)\right) - \tilde{R}_1\left\{-\frac{\tilde{R}_1}{\tilde{R}_0}\left(\vec{\nabla}_r(\ln T)\right)\right\} \quad \left[\because T\vec{\nabla}_r\left(\frac{\mu}{T}\right) = -\frac{\tilde{R}_1}{\tilde{R}_0}\left(\vec{\nabla}_r(\ln T)\right) \text{ for } \vec{j} = 0\right]$

$= \left\{\tilde{R}_2 - \tilde{R}_1\frac{\tilde{R}_1}{\tilde{R}_0}\right\}\left(\vec{\nabla}_r(\ln T)\right) = -\frac{\tilde{R}_2\tilde{R}_0 - \tilde{R}_1\tilde{R}_1}{\tilde{R}_0 T}\left(\vec{\nabla}_r T\right) \Longrightarrow \boldsymbol{\vec{Q} = -\tilde{K}(\vec{\nabla}_r T)} \quad \left[\text{Conductivity tensor, } \tilde{K} = \frac{\tilde{R}_2\tilde{R}_0 - \tilde{R}_1\tilde{R}_1}{\tilde{R}_0 T}\right]$

$\boldsymbol{\left(\tilde{R}_q\right)_{\alpha\beta} = \frac{1}{V}\int \tau v_\alpha v_\beta\left(-\frac{df_0}{d\epsilon}\right)\epsilon^q g(\epsilon)\, d\epsilon}$

$= \frac{4\pi}{h^3}(2m)^{\frac{3}{2}}\int \tau v_\alpha v_\beta\left(-\frac{df_0}{d\epsilon}\right)\epsilon^{q+\frac{1}{2}} d\epsilon \quad \left[\begin{matrix}\because g_s = 2\\ \text{for electrons}\end{matrix}\right] = \frac{4\pi}{h^3}(2m)^{\frac{3}{2}}\int \tau\overline{v_\alpha v_\beta}\left(-\frac{df_0}{d\epsilon}\right)\epsilon^{q+\frac{1}{2}} d\epsilon$

$= \frac{4\pi}{h^3}(2m)^{\frac{3}{2}}\int \tau\delta_{\alpha\beta}\left(\frac{2\epsilon}{3m}\right)\left(\frac{df_0}{d\epsilon}\right)\epsilon^{q+\frac{1}{2}} d\epsilon = \frac{8\pi}{3h^3 m}(2m)^{\frac{3}{2}}\delta_{\alpha\beta}\int \tau\left(\frac{df_0}{d\epsilon}\right)\epsilon^{q+\frac{3}{2}} d\epsilon = \frac{8\pi}{3h^3 m}(2m)^{\frac{3}{2}}\int (Av^s)\left(-\frac{df_0}{d\epsilon}\right)\epsilon^{q+\frac{3}{2}} d\epsilon$

$[\because \tau = Av^s] = \frac{8\pi}{3h^3 m}(2m)^{\frac{3}{2}}\delta_{\alpha\beta}\int A\left(\frac{2\epsilon}{m}\right)^{\frac{s}{2}}\left(-\frac{df_0}{d\epsilon}\right)\epsilon^{q+\frac{3}{2}} d\epsilon = \frac{8\pi}{3h^3 m}(2m)^{\frac{3}{2}}\left(\frac{2}{m}\right)^{\frac{s}{2}} A\int\left(-\frac{df_0}{d\epsilon}\right)\epsilon^{q+\frac{3}{2}+\frac{s}{2}} d\epsilon$

$\approx \frac{8\pi}{3h^3 m}(2m)^{\frac{3}{2}}\delta_{\alpha\beta}\left(\frac{2}{m}\right)^{\frac{s}{2}} A\left\{\left(\mu^{q+\frac{3}{2}+\frac{s}{2}}\right) + (kT)^2\frac{\pi^2}{6}\left(q + \frac{3}{2} + \frac{s}{2}\right)\left(q + \frac{3}{2} + \frac{s}{2} - 1\right)\left(\mu^{q+\frac{3}{2}+\frac{s}{2}-2}\right)\right\}$

$\left[\because \int_0^\infty\left(-\frac{df}{d\epsilon}\right)F(\epsilon)\, d\epsilon \approx F(\mu) + (kT)^2\frac{\pi^2}{6}F''(\mu) \text{ when } \left(-\frac{df}{d\epsilon}\right) \text{ is a delta function with peak at } \epsilon = \mu\right]$

$\therefore \left(\tilde{R}_q\right)_{\alpha\beta} = \frac{8\pi}{3h^3 m}(2m)^{\frac{3}{2}}\delta_{\alpha\beta}\left(\frac{2\mu}{m}\right)^{\frac{s}{2}} A\mu^{\frac{3}{2}}\mu^q\left\{1 + \frac{(\pi kT)^2}{6}\left(q + \frac{3}{2} + \frac{s}{2}\right)\left(q + \frac{1}{2} + \frac{s}{2}\right)\frac{1}{\mu^2}\right\}$

$$= (\tilde{R}_q)_{\alpha\beta} = \frac{n}{m}\delta_{\alpha\beta}\left(\frac{2\mu}{m}\right)^{\frac{s}{2}} A\mu^q \left\{1 + \frac{(\pi kT)^2}{6}\left(q + \frac{3}{2} + \frac{s}{2}\right)\left(q + \frac{1}{2} + \frac{s}{2}\right)\frac{1}{\mu^2}\right\} \left[\because \mu \approx \mu_0 = \frac{h^2}{2m}\left(\frac{3n}{4\pi g_s}\right)^{\frac{2}{3}} = \frac{h^2}{2m}\left(\frac{3n}{8\pi}\right)^{\frac{2}{3}}\right]$$

$$\Rightarrow \begin{cases} (\tilde{R}_0)_{\alpha\beta} = \frac{n}{m}\delta_{\alpha\beta}\left(\frac{2\mu}{m}\right)^{\frac{s}{2}} A\left\{1 + \frac{(\pi kT)^2}{6}\left(\frac{3}{2} + \frac{s}{2}\right)\left(\frac{1}{2} + \frac{s}{2}\right)\frac{1}{\mu^2}\right\} \\ (\tilde{R}_1)_{\alpha\beta} = \frac{n}{m}\delta_{\alpha\beta}\left(\frac{2\mu}{m}\right)^{\frac{s}{2}} A\mu\left\{1 + \frac{(\pi kT)^2}{6}\left(1 + \frac{3}{2} + \frac{s}{2}\right)\left(1 + \frac{1}{2} + \frac{s}{2}\right)\frac{1}{\mu^2}\right\} \\ (\tilde{R}_2)_{\alpha\beta} = \frac{n}{m}\delta_{\alpha\beta}\left(\frac{2\mu}{m}\right)^{\frac{s}{2}} A\mu^2\left\{1 + \frac{(\pi kT)^2}{6}\left(2 + \frac{3}{2} + \frac{s}{2}\right)\left(2 + \frac{1}{2} + \frac{s}{2}\right)\frac{1}{\mu^2}\right\} \end{cases}$$

$$\Rightarrow \begin{cases} (\tilde{R}_0)_{\alpha\beta} = \frac{n}{m}\delta_{\alpha\beta}\left(\frac{2\mu}{m}\right)^{\frac{s}{2}} A\left\{1 + \frac{(\pi kT)^2}{6}\left(\frac{3}{2} + \frac{s}{2}\right)\left(\frac{1}{2} + \frac{s}{2}\right)\frac{1}{\mu^2}\right\} \\ (\tilde{R}_1)_{\alpha\beta} = \frac{n}{m}\delta_{\alpha\beta}\left(\frac{2\mu}{m}\right)^{\frac{s}{2}} A\mu\left\{1 + \frac{(\pi kT)^2}{6}\left(\frac{5}{2} + \frac{s}{2}\right)\left(\frac{3}{2} + \frac{s}{2}\right)\frac{1}{\mu^2}\right\} \\ (\tilde{R}_2)_{\alpha\beta} = \frac{n}{m}\delta_{\alpha\beta}\left(\frac{2\mu}{m}\right)^{\frac{s}{2}} A\mu^2\left\{1 + \frac{(\pi kT)^2}{6}\left(\frac{7}{2} + \frac{s}{2}\right)\left(\frac{5}{2} + \frac{s}{2}\right)\frac{1}{\mu^2}\right\} \end{cases}$$

$$\therefore \tilde{K} = \frac{\tilde{R}_2\tilde{R}_0 - \tilde{R}_1\tilde{R}_1}{\tilde{R}_0 T} = \frac{1}{T}\left\{\tilde{R}_2 - \frac{\tilde{R}_1{}^2}{\tilde{R}_0}\right\}$$

$$= \left\{\frac{n}{m}\delta_{\alpha\beta}\left(\frac{2\mu}{m}\right)^{\frac{s}{2}}\frac{A}{T}\right\}\left(\mu^2\left\{1 + \frac{(\pi kT)^2}{6}\left(\frac{7}{2} + \frac{s}{2}\right)\left(\frac{5}{2} + \frac{s}{2}\right)\frac{1}{\mu^2}\right\} - \frac{\mu^2\left\{1 + \frac{(\pi kT)^2}{6}\left(\frac{5}{2} + \frac{s}{2}\right)\left(\frac{3}{2} + \frac{s}{2}\right)\frac{1}{\mu^2}\right\}^2}{\left\{1 + \frac{(\pi kT)^2}{6}\left(\frac{3}{2} + \frac{s}{2}\right)\left(\frac{1}{2} + \frac{s}{2}\right)\frac{1}{\mu^2}\right\}}\right)$$

$$= \left\{\frac{n}{m}\delta_{\alpha\beta}\left(\frac{2\mu}{m}\right)^{\frac{s}{2}}\frac{A}{T}\right\}\mu^2\left(1 - \frac{\left\{1 + \frac{(\pi kT)^2}{6}(2)\frac{1}{\mu^2}\right\}^2}{\left\{1 + \frac{(\pi kT)^2}{6}(6)\frac{1}{\mu^2}\right\}}\right) \quad [\text{if } s = -7 \text{ i.e. } \tau = Av^{-7}]$$

$$= \left\{\frac{n}{m}\delta_{\alpha\beta}\left(\frac{2\mu}{m}\right)^{\frac{s}{2}}\frac{A\mu^2}{T}\right\}\left(1 - \frac{\left\{1 + \frac{(\pi kT)^2}{3}\frac{1}{\mu^2}\right\}^2}{\left\{1 + (\pi kT)^2\frac{1}{\mu^2}\right\}}\right) = \left\{\frac{n}{m}\delta_{\alpha\beta}\left(\frac{2\mu}{m}\right)^{\frac{s}{2}}\frac{A\mu^2}{T}\right\}\left(\frac{\frac{(\pi kT)^2}{3}\frac{1}{\mu^2} - \frac{(\pi kT)^4}{9}\frac{1}{\mu^4}}{1 + (\pi kT)^2\frac{1}{\mu^2}}\right)$$

$$= \left\{\frac{n}{m}\delta_{\alpha\beta}\left(\frac{2\mu}{m}\right)^{\frac{s}{2}}\frac{A\mu^2}{T}\right\}\frac{(\pi kT)^2}{3}\frac{1}{\mu^2}\left(\frac{1 - \frac{(\pi kT)^2}{3}\frac{1}{\mu^2}}{1 + (\pi kT)^2\frac{1}{\mu^2}}\right) \approx \left\{\frac{n}{m}\delta_{\alpha\beta}\left(\frac{2\mu}{m}\right)^{\frac{s}{2}} A\right\}\frac{(\pi k)^2}{3}T \quad [\because kT \ll \mu]$$

$$\Rightarrow \tilde{K} = \left\{\frac{n}{m}\delta_{\alpha\beta}\tau(\mu)\right\}\frac{(\pi k)^2}{3}T \quad \left[\because \tau = Av^s = A\left(\frac{2\epsilon}{m}\right)^{\frac{s}{2}} \Rightarrow \tau(\mu) = A\left(\frac{2\mu}{m}\right)^{\frac{s}{2}}\right]$$

Calculate the density of proton and neutron assuming that they have a common radius of 0.875 fm.

$$\rho_{\text{proton}} \approx \frac{1.673 \times 10^{-27}\ \text{kg}}{\frac{4\pi}{3} \times (0.875 \times 10^{-15}\ \text{m})^3} \approx 5.96 \times 10^{17}\ \text{kg/m}^3$$

$$\rho_{\text{neutron}} \approx \frac{1.675 \times 10^{-27}\ \text{kg}}{\frac{4\pi}{3} \times (0.875 \times 10^{-15}\ \text{m})^3} \approx 5.97 \times 10^{17}\ \text{kg/m}^3$$

$$\therefore \rho_{\text{nucleon}} \approx 6 \times 10^{17}\ \text{kg/m}^3$$

Show that the density of atomic nucleus is independent of mass number, A, i.e. all nuclei have the same density.

$$\rho_{\text{nucleus}} \approx \frac{Am_{\text{nucleon}}}{\frac{4\pi}{3}\times R^3} \approx \frac{Am_{\text{nucleon}}}{\frac{4\pi}{3}\times\left(R_0A^{\frac{1}{3}}\right)^3} \quad \left[\because R \approx R_0A^{\frac{1}{3}}\right] = \frac{Am_{\text{nucleon}}}{\frac{4\pi}{3}\times\left(R_0A^{\frac{1}{3}}\right)^3} = \frac{3m_{\text{nucleon}}}{4\pi R_0{}^3}$$
$$\approx \frac{3(1.66\times10^{-27}\text{ kg})}{4\pi(1.25\times10^{-15}\text{ m})^3} \approx 2.03\times10^{17}\text{ kg/m}^3 \approx \frac{1}{3}\rho_{\text{nucleon}}$$

Leptons

Particle/Antiparticle	Symbol	Q (e)	S	L_e	L_μ	L_τ	Mass (MeV/c²)	Lifetime (sec)	Decay (leptonic)
Electron / Positron	e^- / e^+	−1 / +1	½	+1 / −1	0	0	0.511	stable	stable
Muon / Antimuon	μ^- / μ^+	−1 / +1	½	0	+1 / −1	0	105.66	2.2×10^{-6}	$e^- + \bar{\nu}_e + \nu_\mu$ $e^+ + \nu_e + \bar{\nu}_\mu$
Tauon / Antitauon	τ^- / τ^+	−1 / +1	½	0	0	+1 / −1	1776.99	2.9×10^{-13}	$e^- + \bar{\nu}_e + \nu_\tau$ $\mu^- + \bar{\nu}_\mu + \nu_\tau$ $e^+ + \nu_e + \bar{\nu}_\tau$ $\mu^+ + \nu_\mu + \bar{\nu}_\tau$
Electron neutrino/antineutrino	ν_e / $\bar{\nu}_e$	0	½	+1 / −1	0	0	$< 2.2\times10^{-6}$	unknown	
Muon neutrino/antineutrino	ν_μ / $\bar{\nu}_\mu$	0	½	0	+1 / −1	0	< 0.17	unknown	
Tauon neutrino/antineutrino	ν_τ / $\bar{\nu}_\tau$	0	½	0	0	+1 / −1	< 15.5	unknown	

Koide Formulae

$$\frac{m_e+m_\mu+m_\tau}{\left(\sqrt{m_e}+\sqrt{m_\mu}+\sqrt{m_\tau}\right)^2} = \frac{2}{3} \quad \text{[for the charged leptons]}$$
$$\frac{m_U+m_D+m_S}{\left(\sqrt{m_U}+\sqrt{m_D}+\sqrt{m_S}\right)^2} = \frac{2}{3} \quad \text{[for the light quarks]}$$
$$\frac{m_C+m_B+m_T}{\left(\sqrt{m_C}+\sqrt{m_B}+\sqrt{m_T}\right)^2} = \frac{2}{3} \quad \text{[for the heavy quarks]}$$

Gell-Mann-Nishijima Formula

Electric charge, $Q = I_3 + \frac{1}{2}Y$ where Hypercharge, $Y = B + S + C + B' + T'$

Baryon number, $B = \frac{1}{3}\left(n_{\text{q}} - n_{\bar{\text{q}}}\right)$; Isospin, $I_3 = \frac{1}{2}[(n_{\text{u}} - n_{\bar{\text{u}}}) - (n_{\text{d}} - n_{\bar{\text{d}}})]$

Strangeness, $S = -(n_{\text{s}} - n_{\bar{\text{s}}})$; Charmness, $C = (n_{\text{c}} - n_{\bar{\text{c}}})$; Bottomness, $B' = -(n_{\text{b}} - n_{\bar{\text{b}}})$; Topness, $T' = (n_{\text{t}} - n_{\bar{\text{t}}})$

$\therefore Y = B + S + C + B' + T' = \frac{1}{3}\left(n_{\text{q}} - n_{\bar{\text{q}}}\right) - (n_{\text{s}} - n_{\bar{\text{s}}}) + (n_{\text{c}} - n_{\bar{\text{c}}}) - (n_{\text{b}} - n_{\bar{\text{b}}}) + (n_{\text{t}} - n_{\bar{\text{t}}})$

$= \frac{1}{3}[(n_{\text{u}} + n_{\text{d}} + n_{\text{s}} + n_{\text{c}} + n_{\text{b}} + n_{\text{t}}) - (n_{\bar{\text{u}}} + n_{\bar{\text{d}}} + n_{\bar{\text{s}}} + n_{\bar{\text{c}}} + n_{\bar{\text{b}}} + n_{\bar{\text{t}}})] - (n_{\text{s}} - n_{\bar{\text{s}}}) + (n_{\text{c}} - n_{\bar{\text{c}}}) - (n_{\text{b}} - n_{\bar{\text{b}}}) + (n_{\text{t}} - n_{\bar{\text{t}}})$

$= \frac{1}{3}\left[(n_{\text{u}} - n_{\bar{\text{u}}}) + (n_{\text{d}} - n_{\bar{\text{d}}}) + (n_{\text{s}} - n_{\bar{\text{s}}}) + (n_{\text{c}} - n_{\bar{\text{c}}}) + (n_{\text{b}} - n_{\bar{\text{b}}}) + (n_{\text{t}} - n_{\bar{\text{t}}})\right] - (n_{\text{s}} - n_{\bar{\text{s}}}) + (n_{\text{c}} - n_{\bar{\text{c}}}) - (n_{\text{b}} - n_{\bar{\text{b}}}) + (n_{\text{t}} - n_{\bar{\text{t}}})$

$= \frac{1}{3}[(n_{\text{u}} - n_{\bar{\text{u}}}) + (n_{\text{d}} - n_{\bar{\text{d}}})] - \frac{2}{3}(n_{\text{s}} - n_{\bar{\text{s}}}) + \frac{4}{3}(n_{\text{c}} - n_{\bar{\text{c}}}) - \frac{2}{3}(n_{\text{b}} - n_{\bar{\text{b}}}) + \frac{4}{3}(n_{\text{t}} - n_{\bar{\text{t}}})$

$\therefore Q = I_3 + \frac{1}{2}Y = \frac{1}{2}[(n_{\text{u}} - n_{\bar{\text{u}}}) - (n_{\text{d}} - n_{\bar{\text{d}}})] + \frac{1}{6}[(n_{\text{u}} - n_{\bar{\text{u}}}) + (n_{\text{d}} - n_{\bar{\text{d}}})]$
$\qquad - \frac{2}{6}(n_{\text{s}} - n_{\bar{\text{s}}}) + \frac{4}{6}(n_{\text{c}} - n_{\bar{\text{c}}}) - \frac{2}{6}(n_{\text{b}} - n_{\bar{\text{b}}}) + \frac{4}{6}(n_{\text{t}} - n_{\bar{\text{t}}})$

$= \frac{2}{3}(n_{\text{u}} - n_{\bar{\text{u}}}) - \frac{1}{3}(n_{\text{d}} - n_{\bar{\text{d}}}) - \frac{1}{3}(n_{\text{s}} - n_{\bar{\text{s}}}) + \frac{2}{3}(n_{\text{c}} - n_{\bar{\text{c}}}) - \frac{1}{3}(n_{\text{b}} - n_{\bar{\text{b}}}) + \frac{2}{3}(n_{\text{t}} - n_{\bar{\text{t}}})$

$\Longrightarrow Q = \frac{2}{3}[(n_{\text{u}} - n_{\bar{\text{u}}}) + (n_{\text{c}} - n_{\bar{\text{c}}}) + (n_{\text{t}} - n_{\bar{\text{t}}})] - \frac{1}{3}[(n_{\text{d}} - n_{\bar{\text{d}}}) + (n_{\text{s}} - n_{\bar{\text{s}}}) + (n_{\text{b}} - n_{\bar{\text{b}}})]$

This is the expression of charge in terms of quark content.

Particle Family

	Elementary Particles										
	Fermions								**Bosons**		
	Quarks				**Leptons**				**Gauge**		
	Upper		**Lower**		**Primons**		**Neutrinos**				
Gen 1st	Up	Up antiquark	Down	Down antiquark	Electron	Positron	Electron neutrino	Electron antineutrino	Photon		
Gen 2nd	Charm	Charm antiquark	Strange	Strange antiquark	Muon	Antimuon	Muon neutrino	Muon antineutrino	W^+	W^-	Z^0
Gen 3rd	Top	Top antiquark	Bottom	Bottom antiquark	Tauon	Antitauon	Tauon	Tauon antineutrino	Gluon		

Composite Particles										
Fermions										
Hadrons (corresponding antiparticles implicit by replacing the quarks by respective antiquarks; some mesons are self-antiparticles) (owing to its heavy mass, top quark decays before it hadronizes)										
Mesons (first letter is quark; second, antiquark)						**Baryons** (first letter denotes the repeating quark; bold ones are **Hyperons**)				
Simple		**Compound**		**Mixed**		**Uniquark**	**Biquark**		**Triquark**	
$J^P = 0^-$	$J^P = 1^-$	$J^P = 0^-$	$J^P = 1^-$	$J^P = 0^-$	$J^P = 1^-$	$J^P = {}^3/_2{}^+$	$J^P = {}^1/_2{}^+$	$J^P = {}^3/_2{}^+$	$J^P = {}^1/_2{}^+$	$J^P = {}^3/_2{}^+$
Eta-c	Phi-s	Pi-ud	Rho-ud	Pi-u-d	Rho-u-d	Delta-u	Proton	Delta-ud	**Lambda-uds**	
Eta-b	Psi-c	K-us	K*-us	Eta-ud-ss	Omega-ud	Delta-d	Neutron	Delta-du	Lambda-udc	
	Upsilon-b	K-ds	K*-ds	Eta’-uds		Omega-c	**Sigma-us**	**Sigma*-us**	Lamda-udb	
		D-cu	D*-cu	K-d·s		**Omega-s**	**Sigma-ds**	**Sigma*-ds**	**Sigma-uds**	**Sigma-uds**
		D-cd	D*-cd	K-d/s		Omega-b	Sigma-uc	Sigma*-uc	Sigma-udc	Sigma-udc
		D-cs	D*-cs				Sigma-dc	Sigma*-dc	Sigma-udb	Sigma-udb
		B-ub	B*-ub				Sigma-ub	Sigma*-ub	Omega-scb	Omega-scb
		B-db	B*-db				Sigma-db	Sigma*-db	Omega’-scb	
		B-sb	B*-sb				Omega-sc	Omega*-sc	Xi-usc	Xi*-usc
		B-cb	B*-cb				Omega-sb	Omega*-sb	Xi-dsc	Xi*-dsc
							Omega-cs	Omega*-cs	Xi-usb	Xi*-usb
							Omega-bs	Omega*-bs	Xi-dsb	Xi*-dsb
							Omega-cb	Omega*-cb	Xi-ucb	Xi*-ucb
							Omega-bc	Omega*-bc	Xi-dcb	Xi*-dcb
							Xi-su	**Xi*-su**	Xi’-usc	
							Xi-sd	**Xi*-sd**	Xi’-dsc	
							Xi-cu	Xi*-cu	Xi’-usb	
							Xi-cd	Xi*-cd	Xi’-dsb	
							Xi-bu	Xi*-bu	Xi’-ucb	
							Xi-bd	Xi*-bd	Xi’-dcb	

Magnitude scale for celestial object

Greek astronomer Hipparchus (190 BC – 120 BC) assigned an **apparent magnitude** m = 1 to the brightest star in the night sky and m = 6 to the dimmest star visible to the naked eye so that a smaller apparent magnitude implies an apparently brighter star. It was subsequently realized that the brightest star in the scale is 100 times brighter than the dimmest star in the same scale.

$$\begin{cases} m_1 - m_2 = 5 \Longrightarrow \dfrac{m_1 - m_2}{5} = 1 \\ \dfrac{\boldsymbol{F}_2}{\boldsymbol{F}_1} = 100 = 100^1 = \mathbf{100}^{\left(\frac{m_1 - m_2}{5}\right)} \left[\begin{array}{c} \text{power instead of product or power-product because human eye responds} \\ \text{to the difference in the logarithms of the brightness of two luminous objects} \end{array}\right] \end{cases}$$

$$\therefore \frac{F_2}{F_1} = 10^{2\left(\frac{m_1-m_2}{5}\right)} = 10^{\left(\frac{m_1-m_2}{2.5}\right)} = 10^{0.4(m_1-m_2)} \Rightarrow \boldsymbol{m_1 - m_2 = 2.5\log\frac{F_2}{F_1}}$$

$$\Rightarrow m_1 - m_2 = 2.5\log\frac{\frac{L_2}{4\pi {d_2}^2}}{\frac{L_1}{4\pi {d_1}^2}} = 2.5\log\frac{L_2 {d_1}^2}{L_1 {d_2}^2} \Rightarrow \boldsymbol{m_1 - m_2 = 2.5\log\frac{L_2}{L_1} - 5\log\frac{d_2}{d_1}}$$

Absolute magnitude of an object is defined as the apparent magnitude of the object if it were situated at a distance of 10 pc. It is denoted by M. Since the object is same, so $L_1 = L_2$. Note that absolute magnitude is essentially apparent magnitude albeit a standardized one, and satisfies all the relations for apparent magnitude.

$$m_1 - m_2 = 2.5\log\frac{L_2}{L_1} - 5\log\frac{d_2}{d_1} \Rightarrow m - M = -5\log\frac{10}{d} \Rightarrow \boldsymbol{m - M = 5\log d - 5} \Rightarrow \boldsymbol{d = 10^{\left(1+\frac{m-M}{5}\right)}}$$

The quantity ***m* − *M*** is called the object's **distance modulus** and serves as a measure of the distance to the object.

$$\therefore M_1 - M_2 = 2.5\log\frac{L_2}{L_1} - 5\log\frac{d_2}{d_1} \Rightarrow \boldsymbol{M_1 - M_2 = 2.5\log\frac{L_2}{L_1}} \quad [\because d_1 = d_2 = 10\text{ pc}]$$

Extinction Correction: The above expression incorporates only the inverse square fall-off of the intensity. If the light from the source reaches through absorptive media such as ISM that absorbs and scatters the photons, then the object will appear dimmer than it would be in the absence of such physical processes.

$$\therefore (m - A) - M = 5\log d - 5 \Rightarrow \boldsymbol{m - M = 5\log d - 5 + A} \quad \begin{bmatrix}\text{In general, interstellar extinction, } A \\ \text{depends strongly on the wave band.}\end{bmatrix}$$

Redshift Correction: Further, physical processes such as cosmological expansion causes redshift of radiation so that the photons received in, say, V band had been actually emitted at shorter wavelengths.

$$\therefore (m - K) - M = 5\log d - 5 + A \Rightarrow \boldsymbol{m - M = 5\log d - 5 + A + K} \quad \begin{bmatrix}K\text{ correction depends on the shape} \\ \text{of the spectrum of the object.}\end{bmatrix}$$

Triple Star Systems in the Centaurus constellation

	Alpha Centauri			Beta Centauri			HR 7151 (V766 Centauri)		
	Alpha Centauri AB (Rigil Kentaurus)		Alpha Centauri C (Proxima Centauri)	Beta Centauri A		Beta Centauri B	HR 7151A		HR 7151B
	Alpha Centauri A (HR 5459)	Alpha Centauri B (HR 5460)		Beta Centauri Aa	Beta Centauri Ab		HR 7151Aa	HR 7151Ab	
Spectral type	G2 V	K1 V	M6 Ve	B1 III		B1 V	K0 0-Ia		B0 Ibp
Variable type			UV Ceti	Beta Cephei					
U–B	+0.24	+0.68	+1.26	−0.98					
B–V	+0.71	+0.88	+1.82	−0.23					
Radial velocity	−21.4 km/s	−18.6 km/s	−22.2 km/s	+5.9 km/s					
Proper Motion, RA			−3775.75 mas/yr	−33.27 mas/yr					
Proper Motion, Dec			−765.54 mas/yr	−23.16 mas/yr					
Parallax	754.81 mas		768.13 mas	8.32 mas					
Distance	4.37 ly (1.34 pc)		4.246 ly (1.302 pc)	390 ly					
Apparent Magnitude (V)	0.01	1.33	11.13						
Absolute Magnitude (M_V)	4.38	5.71	15.60	−4.53					
Mass	1.1 $M_\odot$	0.91 $M_\odot$	0.122 $M_\odot$	10.7	10.3	4.61			

Radius	1.22 $R_\odot$	0.86 $R_\odot$	0.154 $R_\odot$						
Luminosity (bol)	1.52 $L_\odot$	0.5 $L_\odot$	0.0017 $L_\odot$	41700 $L_\odot$					
Surface Gravity log g (earth)	1.31	1.38	2.21	0.51					
Temperature	5790 K	5260 K	3042 K	25000 K					
Metallicity [Fe/H]	0.20 dex	0.23 dex	0.21 dex						
Rotation	23 days	41 days	82.6 days						
Age	4.4 Gyr	6.5 Gyr	4.85 Gyr	14.1 Myr					
Orbit	Primary	Companion	Satellite						
Period	79.91 yr		547000 yr	357 days	288.27 yr				
Semi-major axis	23.3 au		8700 au						
Eccentricity	0.52		0.50	0.825					
Inclination	79.2°		107.6°	67.4°					

If a certain region of the Sun is observed to have a synodic rotation period of 28 days, what would the sidereal period be?

A region of the Sun is akin to an inferior planet. So if P denotes the sidereal period, S, the synodic period, and $P_\odot$, the sidereal year, then $\frac{1}{P} = \frac{1}{S} + \frac{1}{P_\odot} = \frac{1}{28\text{ d}} + \frac{1}{365.2563\text{ d}} \Rightarrow P = 26\text{ d}$ $\left[\begin{matrix} \frac{\mathbf{1}}{\boldsymbol{P}} - \frac{\mathbf{1}}{\boldsymbol{S}} = \frac{\mathbf{1}}{\boldsymbol{P}_\odot} & \text{[for } \textbf{inferior}\text{]} \\ \frac{\mathbf{1}}{\boldsymbol{P}} + \frac{\mathbf{1}}{\boldsymbol{S}} = \frac{\mathbf{1}}{\boldsymbol{P}_\odot} & \text{[for } \textbf{superior}\text{]} \end{matrix}\right]$

What is the combined apparent magnitude of a binary system of (two) stars of apparent magnitudes m₁ and m₂?

$m_1 - m_2 = 2.5\log\frac{F_2}{F_1} \Rightarrow \frac{F_2}{F_1} = 10^{\frac{m_1-m_2}{2.5}}$ [where F denotes luminous flux and m denotes apparent magnitude]

$m_1 - m_{1+2} = 2.5\log\frac{F_{1+2}}{F_1} = 2.5\log\frac{F_1+F_2}{F_1} = 2.5\log\left(1+\frac{F_2}{F_1}\right) = 2.5\log\left(1+10^{\frac{m_1-m_2}{2.5}}\right) = 2.5\log\left(1+2.5^{(m_1-m_2)}\right)$

$\Rightarrow m_{1+2} = m_1 - 2.5\log\left(1+2.5^{(m_1-m_2)}\right) \Rightarrow m_{1+2} < m_1$

$m_{1+2} - m_2 = 2.5\log\frac{F_2}{F_{1+2}} = 2.5\log\frac{F_2}{F_1+F_2} = -2.5\log\left(1+\frac{F_1}{F_2}\right) = -2.5\log\left(1+10^{\frac{m_2-m_1}{2.5}}\right) = -2.5\log\left(1+2.5^{(m_2-m_1)}\right)$

$\Rightarrow m_{1+2} = m_2 - 2.5\log\left(1+2.5^{(m_2-m_1)}\right) \Rightarrow m_{1+2} < m_2$

$\therefore m_{1+2} < m_1$ and $m_{1+2} < m_2$

If $m_1 = 3$ and $m_2 = 4$, $m_{1+2} = 2.63$

If $m_1 = m_2 = m$, $m_{1+2} = m - 2.5\log 2 = m - 0.752575 \approx m - 0.75 \Rightarrow m \approx m_{1+2} + 0.75$

If $m_1 = \infty$ (invisible) and $m_2 = m_2$, $m_{1+2} = m_2$ [use the second formula]

What is the combined apparent magnitude of a system of three stars of apparent magnitudes m₁, m₂ and m₃?

$m_{1+2} = m_1 - 2.5\log\left(1+2.5^{(m_1-m_2)}\right) \Rightarrow m_{1+2+3} = m_{1+2} - 2.5\log\left(1+2.5^{(m_{1+2}-m_3)}\right)$

$\Rightarrow m_{1+2+3} = \left\{m_1 - 2.5\log\left(1+2.5^{(m_1-m_2)}\right)\right\} - 2.5\log\left(1+2.5^{\left(m_1-2.5\log\left(1+2.5^{(m_1-m_2)}\right)-m_3\right)}\right)$

Betelguese (α-Orionis) and Rigel (β-Orionis) have surface temperatures of 3500 K and 11000 K respectively. Treating it as a blackbody, find the wavelengths at which their continuous spectra peak.

Betelguese: $\lambda_{\max}T = b$ $\left[\begin{matrix}\text{Wien's displacement law} \\ \text{for a blackbody spectrum}\end{matrix}\right] \Rightarrow \lambda_{\max} = \frac{b}{T} \approx \frac{0.0029\text{ m-K}}{3500\text{ K}} \approx 829\text{ nm}$ [infrared region]

Rigel: $\lambda_{\max}T = b$ $\left[\begin{matrix}\text{Wien's displacement law} \\ \text{for a blackbody spectrum}\end{matrix}\right] \Rightarrow \lambda_{\max} = \frac{b}{T} \approx \frac{0.0029\text{ m-K}}{11000\text{ K}} \approx 264\text{ nm}$ [ultraviolet region]

Luminosity of Sun is $L_\odot$ = 3.828×10^{26} W. Radius of Sun is $R_\odot$ = 6.957×10^8 m. Find its effective surface temperature and hence the wavelength at which its continuous spectrum peaks. Also calculate the radiant flux received by Earth above its absorbing atmosphere knowing the Sun is at a distance of 1.496×10^{11} m.

$$L = 4\pi R^2\sigma T^4 \quad \begin{bmatrix}\text{Stefan's law}\\ \text{for a blackbody}\end{bmatrix} \Rightarrow T = \left(\frac{L}{4\pi R^2\sigma}\right)^{\frac{1}{4}} \approx 5772\text{ K}$$

$$\lambda_{\max}T = b \quad \begin{bmatrix}\text{Wien's displacement law}\\ \text{for a blackbody spectrum}\end{bmatrix} \Rightarrow \lambda_{\max} = \frac{b}{T} \approx \frac{0.0029\text{ m-K}}{5772\text{ K}} \approx 502.4\text{ nm} \quad [\text{visible green region}]$$

$$F = \frac{L}{4\pi d^2} = 1361\text{ Wm}^{-2} \quad [\text{This value is called } \textbf{Solar irradiance} \text{ or } \textbf{Solar constant.}]$$

The apparent magnitude of Sun is −26.83, and its distance is 1 au = 4.848×10^{-6} pc. Find its absolute magnitude and distance modulus.

$m - M = 5\log d - 5 \Rightarrow M = m - 5\log d + 5 = -26.83 - 5\log(4.848\times10^{-6}) + 5 \approx +4.74$

Distance modulus, $m - M = -26.83 - 4.74 = -31.57$

Sun's U−B and B−V color indices are +0.195 and +0.650 respectively. Its absolute visual magnitude is +4.74. Determine its U, B and V apparent magnitudes.

Distance modulus, $m - M = 5\log d - 5 = 5\log(4.848\times10^{-6}) = -31.57 \quad [\because d = 1\text{ au} \approx 4.848\times10^{-6}\text{ pc}]$

$\Rightarrow m = -31.57 + M \Rightarrow V = -31.57 + 4.74 = -26.83$

$B - V = 0.650 \Rightarrow B = 0.650 + V = 0.650 - 26.83 = -26.18$

$U - B = 0.195 \Rightarrow U = 0.195 + B = 0.195 - 26.18 = -25.99$

Assuming it to be a spherical blackbody of radius 4.66×10^9 m at a distance of 136 pc = 4.2×10^{18} m with a surface temperature of 27400 K, determine for the star Dschubba (δ-Scorpii): Luminosity, Absolute bolometric magnitude, Apparent bolometric magnitude, Distance modulus, Radiant flux at the star's surface, Radiant flux at Earth's surface and peak wavelength.

Bolometric (Total) Luminosity, $L = 4\pi R^2\sigma T^4 \quad \begin{bmatrix}\text{Stefan's law}\\ \text{for a blackbody}\end{bmatrix} \approx 8.72\times10^{30}\text{ W} \approx 12773.33\, L_\odot$

Absolute bolometric magnitude, $M = M_\odot - 2.5\log\frac{L}{L_\odot} = +4.74 - 2.5\log(12773.33) \approx -5.53$

Apparent bolometric magnitude, $m = M + 5\log d - 5 = 0.14$

alternatively, Apparent bolometric magnitude, $m = m_\odot - 2.5\log\frac{F}{F_\odot} = m_\odot - 2.5\log\frac{L{d_\odot}^2}{L_\odot d^2}$

$$= m_\odot - 2.5\log\frac{L}{L_\odot} + 5\log\frac{d}{d_\odot} = 0.14$$

Distance modulus, $m - M = 5\log d - 5 \approx 5.67$

At star's surface, $F = \frac{L}{4\pi R^2} = 3.2\times10^{10}\text{ Wm}^{-2}$

At Earth's surface (above atmosphere), $F = \frac{L}{4\pi d^2} = 3.9\times10^{-8}\text{ Wm}^{-2}$

Peak wavelength, $\lambda_{\max} = \frac{b}{T} \approx \frac{0.0029\text{ m-K}}{27400\text{ K}} \approx 106\text{ nm} \quad [\text{ultraviolet region}]$

An average person has 1.4 m^2 of skin at a skin temperature of roughly 306 K (92°F). Considering the person to be an ideal radiator in a room at a temperature of 293 K (68°F), calculate the energy radiated per second in the form of blackbody radiation, and hence, determine the peak wavelength of the radiation. Also calculate the energy absorbed per second, and hence determine the net energy lost per second.

Energy radiated per second, $L = \sigma AT^4 \approx 696\text{ W} \quad [\because T = 306\text{ K}]$

$$\lambda_{\max} = \frac{b}{T} = \frac{0.0029\text{ m-K}}{306\text{ K}} \approx 9477\text{ nm} = 9.477\ \mu\text{m} \quad [\text{long infrared region}]$$

Energy absorbed per second, $L = \sigma A T^4 \approx 585$ W $[\because T = 293 \text{ K}]$
Net energy lost per second, $\Delta L = 111$ W

At what distance from a 100 W light-bulb is the radiant flux equal to the solar irradiance?

$$\frac{L}{4\pi d^2} = 1361 \Rightarrow d = \sqrt{\frac{L}{4\pi \times 1361}} = 0.0765 \text{ m} = 7.65 \text{ cm} \quad [\because L = 100 \text{ W}]$$

Calculate the energy store of the Sun if 0.01 M⊙ is converted into radiated energy.

$\Delta M c^2 = (0.01 M_\odot) c^2 \approx 1.8 \times 10^{45}$ J

Calculate the frequency shift produced by normal Zeeman effect at the centre of a sunspot where the magnetic field is 0.3 T.

$$\Delta E = \mu_\mathrm{B} B \Rightarrow h\Delta\nu = \frac{e\hbar}{2m_\mathrm{e}} B \Rightarrow \Delta\nu = \frac{e}{4\pi m_\mathrm{e}} B \approx 4.2 \text{ GHz}$$

By what fraction would the wavelength of one component of the 630.25 nm Fe I spectral line change as a consequence of magnetic field of 0.3 T?

$$E = \frac{hc}{\lambda} \Rightarrow dE = -\frac{hc}{\lambda^2} d\lambda \Rightarrow |d\lambda| = \frac{\lambda^2}{hc}|dE| = \frac{eB}{4\pi m_\mathrm{e} c}\lambda^2 \quad \left[\because |dE| = \frac{e\hbar}{2m_\mathrm{e}} B\right] \approx 5.56 \times 10^{-12} \text{ m} \approx 0.0056 \text{ nm}$$

$$\therefore \text{fractional change in wavelength} = \frac{|d\lambda|}{\lambda} \approx 8.83 \times 10^{-3} = 0.883\%$$

The parallax angle of Sirius is 0.379". Find the distance to Sirius in parsecs, in light years and in AU. Determine the distance modulus of Sirius.

$$d = \frac{1}{p''} \text{ pc} = \frac{1}{0.379''} \approx 2.64 \text{ pc} \approx 8.61 \text{ ly} \approx 544505 \text{ au}$$

Distance modulus, $m - M = 5 \log d - 5 = -2.89$

The Sun is about 480,000 times more luminous than the full Moon. What is the difference in their apparent magnitudes?

$$m_\mathrm{Sun} - m_\mathrm{Moon} = 2.5 \log \frac{F_\mathrm{Moon}}{F_\mathrm{Sun}} = 2.5 \log \frac{1}{480000} = -14.2$$

The distance to the nearest known star, Proxima Centauri, is 4.25 ly. Determine its parallax angle and its distance modulus.

$$p = \frac{1}{d}'' = \frac{1}{1.3037}'' \left[\because 4.25 \text{ ly} = \frac{4.25}{3.26} \text{ pc} = 1.3037 \text{ pc}\right] \approx 0.767''$$

Distance modulus, $m - M = 5 \log d - 5 = -4.43$

If the apparent and the absolute magnitudes of the star Aldebaran are +0.87 and −0.63 respectively, calculate its distance from Earth.

$5 \log d - 5 = m - M = 0.87 - (-0.63) = 1.5 \Rightarrow \log d = 1.3 \Rightarrow d \approx 20$ pc

A 1.2×10^4 kg spacecraft is launched from Earth and is to be accelerated radially away from Sun using a circular solar sail. The initial acceleration of the spacecraft is to be 1 *g* (9.81 m/s^2). Assuming a flat sail, determine the radius of the sail if it is (a) black, so it absorbs Sun's light, (b) shiny, so it reflects Sun's light. Compare the radii.

Centripetal force on the spacecraft, $F = ma = 1.24 \times 10^4 \times 9.81$ kg-ms^{-2}
Force due to radiation pressure on an absorbing surface in the direction of light's propagation,

$$F_{\text{rad}} = \frac{\langle S\rangle A}{c}\cos\theta = \frac{\langle S\rangle A}{c} \quad [\because \theta = 0\,;\, A = \pi r^2]$$

$$\therefore F_{\text{rad}} = F \Longrightarrow \frac{\langle S\rangle A}{c} = ma \Longrightarrow r = \sqrt{\frac{cma}{\pi\langle S\rangle}} = \sqrt{\frac{1.24\times10^4\times9.81c}{1361\pi}} \approx 92 \text{ km} \quad \left[\because \langle S\rangle = \frac{L}{4\pi d^2} = 1361 \text{ Wm}^{-2}\right]$$

Force due to radiation pressure on a reflecting surface in the direction of light's propagation,

$$F_{\text{rad}} = \frac{2\langle S\rangle A}{c}\cos^2\theta = \frac{2\langle S\rangle A}{c} \quad [\because \theta = 0\,;\, A = \pi r^2]$$

$$\therefore F_{\text{rad}} = F \Longrightarrow \frac{2\langle S\rangle A}{c} = ma \Longrightarrow r = \sqrt{\frac{cma}{2\pi\langle S\rangle}} = \sqrt{\frac{1.24\times10^4\times9.81c}{2\times1361\pi}} \approx 65 \text{ km} \quad \left[\because \langle S\rangle = \frac{L}{4\pi d^2} = 1361 \text{ Wm}^{-2}\right]$$

Thus, the radius of the reflective sail is smaller than that of the absorptive sail by a factor of $\sqrt{2}$.

The Hipparcos Space Astrometry Mission was able to measure parallax angles down to nearly 0.001″. To get a sense of the resolution, how far from a dime (diameter 1.9 cm) would you need to be for it to subtend that angle?

$$d = \frac{0.019 \text{ m}}{\frac{0.001\pi}{3600\times180}} \approx 3919 \text{ km}$$

Assume that grass grows at the rate of 5 cm per week. How much does it grow in one second? How far from it would you need to be to see it grow at an angular rate of 4 microarcsecond per second?

Growth per second $= \dfrac{0.05 \text{ m}}{7\times24\times3600} \approx 8.27\times10^{-8}$ m. Required distance$= \dfrac{8.27\times10^{-8} \text{ m}}{\frac{0.000004\pi}{3600\times180}} \approx 4.26$ km

The luminosity of a star is 1.005×10[40] erg/s (1 erg = 10[−7] J), its distance is 1 kpc. And its bolometric correction is −3.2. Find m_{pv} and m_{pg} for the star, if its colour index is −0.6.

$$M_{\text{bol}} - M_{\text{bol}\odot} = 2.5\log\frac{L_\odot}{L} \Longrightarrow M_{\text{bol}} = M_{\text{bol}\odot} + 2.5\log\frac{L_\odot}{L} = 4.74 + 2.5\log\frac{3.828\times10^{26}}{1.005\times10^{33}} = -11.3$$

$$m_{\text{bol}} - M_{\text{bol}} = 5\log d - 5 = 5\log 1000 - 5 = 10 \Longrightarrow m_{\text{bol}} = M_{\text{bol}} + 10 = -11.3 + 10 = -1.3$$

Bolometric correction, $BC = m_{\text{bol}} - m_{\text{pv}} \Longrightarrow m_{\text{pv}} = m_{\text{bol}} - BC = -1.3 + 3.2 = +1.9$

Colour Index, $CI = m_{\text{pg}} - m_{\text{pv}} \Longrightarrow m_{\text{pg}} = m_{\text{pv}} + CI = +1.9 - 0.6 = +1.3$

A visual binary star has a parallax of 0.025″, and the angular distance between the component stars is 2.5″. Calculate the linear separation between the two members of the binary.

$$d = \frac{1}{p''} \text{ pc} = \frac{1}{0.025''} \approx 40 \text{ pc} \approx 130.4 \text{ ly}$$

Separation $= 40 \text{ pc} \times \dfrac{2.5}{3600} \times \dfrac{\pi}{180} = 4.85\times10^{-4} \text{ pc} = 0.0016 \text{ ly} \approx 100$ au

In 1672, an international effort was made to measure the parallax angle of Mars at the time of opposition, when it was closest to Earth.

(a) Consider two observers who are separated by a baseline equal to Earth's diameter. If the difference in their measurements of Mars' angular position is 33.6″, what is the distance between Earth and Mars at the time of opposition?

(b) If the distance to Mars is to be measured to within 10%, how closely must the clocks be synchronized? (Ignore the rotation of Earth. The average orbital velocities of Earth and Mars are 29.79 km/s and 24.13 km/s respectively.)

Parallax angle, $p = \dfrac{33.6''}{2} = 16.8'' = 8.145\times10^{-5}$ radians

Baseline, $B = R_\oplus = 6371 \text{ km} = 6.371\times10^6$ m

Distance, $d = \dfrac{B}{\tan p} \approx \dfrac{R_\oplus}{p} = \dfrac{6.371 \times 10^6 \text{ m}}{8.145 \times 10^{-5}} \approx 7.822 \times 10^{10} \text{ km} \approx 0.523 \text{ au}$

If the two clocks are out of sync by an amount Δt (so that one observer makes the observation some moment later than the other), the baseline for their distance measurement will be $2B + v_{\text{rel}}\Delta t$ instead of 2B. Here, v_{rel} is the relative velocity of observer on Earth and Mars at the time of opposition.

$v_{\text{rel}} = 29.79 \text{ km/s} - 24.13 \text{ km/s} = 5.67 \text{ km/s}$

Measured parallax $= p_{\text{exp}}$

Measured distance, $d_{\text{exp}} = \dfrac{B}{\tan p_{\text{exp}}} = \dfrac{B}{p_{\text{exp}}}$

Actual distance, $d_{\text{act}} = \dfrac{(2B + v_{\text{rel}}\Delta t)/2}{\tan p_{\text{exp}}} \approx \dfrac{B + 0.5 v_{\text{rel}}\Delta t}{p_{\text{exp}}}$

If distance to Mars is to be measured within 10%, then $\dfrac{|d_{\text{act}} - d_{\text{exp}}|}{d_{\text{act}}} = 0.1 \Rightarrow \dfrac{0.5 v_{\text{rel}}\Delta t}{B} = 0.1 \Rightarrow \Delta t \approx 225 \text{ s}$

i.e. their clocks must not differ by more than 225 s.

Hertzsprung-Russell diagram (HR diagram)

Developed by Ejnar Hertzsprung of Denmark and Henry Norris Russell of USA during 1911-1913, HR diagram is a plot of Bolometric luminosity 10^{-5} to 10^6 (or equivalently, absolute visual magnitude +20 to −10) against Spectral type (or equivalently, B−V colour index or Effective surface temperature 600,000 K to 0 K) for the stars. Its variants are **Colour-Magnitude diagram** and **Temperature-Luminosity diagram**. Most of the stars lie on the upper-left (hot and bright) to lower-right (cooler and dimmer) diagonal, and are called **Main sequence** or **Dwarf stars** (luminosity class V) across all spectral types O to Y (blue, yellow, red, brown). **Sun is a yellow dwarf (G2 V).** Stars lying just below the Main sequence with absolute magnitudes +10.0 to +5.0 and spectral type G to M are called **Subdwarfs** (luminosity class VI). Stars lying at the lower left with absolute magnitudes +15.0 to +10.0 and spectral type B to G are called **White dwarfs** (luminosity class VII). Stars lying just above the Main sequence with absolute magnitudes +5.0 to +1.0 and spectral type A to K are called **Subgiants** (luminosity class IV). Stars lying above the Subgiants and of nearly all spectral types O to M (blue, yellow, red, brown) are respectively **Giants** (luminosity class III ; absolute magnitude +1.0 to −1.0), **Bright Giants** (luminosity class II ; absolute magnitude −1.0 to −3.0), **Supergiants** (luminosity class Ia, Iab, Ib ; absolute magnitude −3.0 to −8.0) and **Hypergiants** (luminosity class Ia+ ; absolute magnitude −8.0 to −10.0).

Low-mass star: M < 0.5 $M_\odot$
High-mass star: M > 8 $M_\odot$
Sub-stellar objects:
Brown dwarfs: M < 0.08 $M_\odot$
Planets: rocky, gas and ice
Concept of stellar population was introduced by Walter Baade in 1944.
Population I (metal-rich): metallicity: Z ~ 0.02 (solar), old and young stars, mainly in the galactic disc, open clusters
Population II (metal-poor): metallicity: Z ~ 0.1 - 0.001 $Z_\odot$, old, high-velocity stars in the galactic halo, globular clusters
Population III (ultra-metal-poor): metallicity: Z ~ 0 (e.g. HE0107-5240, HE1327-2326, Caffau's star, Cayrel's Star with 0.8 $M_\odot$ and Z ~ 10^{-7})

Evaluate C_{bol} by using $m_\odot$ = −26.83.

$$m_{\text{bol}} = -2.5 \log\left(\int_0^\infty F_\lambda \, d\lambda\right) + C_{\text{bol}} = -2.5 \log\left(\int_0^\infty \frac{L_\lambda}{4\pi r^2} d\lambda\right) + C_{\text{bol}} = -2.5 \log\left(\frac{\int_0^\infty L_\lambda \, d\lambda}{4\pi r^2}\right) + C_{\text{bol}}$$

$$\Rightarrow m_{\text{bol}} = -2.5 \log\left(\frac{L}{4\pi r^2}\right) + C_{\text{bol}}$$

For Sun, $m_{\text{bol}\odot} = -2.5 \log\left(\dfrac{L_\odot}{4\pi r^2}\right) + C_{\text{bol}\odot} = -2.5 \log(F_{\odot,1\text{ au}}) + C_{\text{bol}\odot} \quad [\because r = 1 \text{ au}]$

$\Rightarrow -26.83 = -2.5\log(1361) + C_{\text{bol}\odot}$ $\left[\because \text{Solar constant, } F_{\odot,1\text{ au}} = 1361 \text{ Wm}^{-2}\right] \Rightarrow C_{\text{bol}\odot} \approx -19.0$

Derive Kepler's first law law (law of orbits).

$$\frac{d^2r}{dt^2} - r\omega^2 = -\frac{GM}{r^2} \Rightarrow \frac{d^2r}{dt^2} = -\frac{GM}{r^2} + \left(\frac{L}{mr^2}\right)^2 r \quad [L = m\omega^2 r] \Rightarrow u = \frac{1}{r} \Rightarrow du = -\frac{dt}{r^2}$$

Derive Kepler's second law law (law of areas) assuming circular orbits.

$$dA = \frac{1}{2}r(r\,d\theta) \quad \begin{bmatrix}\text{equilateral}\\ \text{triangle}\end{bmatrix} = \frac{1}{2}r^2 d\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}rv_\theta = \frac{1}{2}|\vec{r}\times\vec{v}| = \frac{1}{2}\left|\frac{\vec{L}}{\mu}\right| \Rightarrow \frac{dA}{dt} = \frac{L}{\mu} \quad \text{(constant)}$$

Derive Kepler's third law (law of periods / harmonic law) for a two body .

$$\begin{Bmatrix}\dfrac{Gm_1m_2}{a^2} = \dfrac{m_1{v_1}^2}{a_1}\\ \dfrac{Gm_1m_2}{a^2} = \dfrac{m_2{v_2}^2}{a_2}\end{Bmatrix} \text{ and } \begin{Bmatrix}T = \dfrac{2\pi a_1}{v_1} \Rightarrow v_1 = \dfrac{2\pi a_1}{T}\\ T = \dfrac{2\pi a_2}{v_2} \Rightarrow v_2 = \dfrac{2\pi a_2}{T}\end{Bmatrix} \text{ and } \begin{Bmatrix}a = a_1 + a_2\\ m_1a_1 = m_2a_2\end{Bmatrix} \Rightarrow \begin{Bmatrix}a_1 = \dfrac{m_2}{m_1+m_2}a\\ a_1 = \dfrac{m_1}{m_1+m_2}a\end{Bmatrix}$$

$$\therefore \frac{Gm_1m_2}{a^2} = \frac{m_1{v_1}^2}{a_1} \Rightarrow \frac{Gm_2}{a^2} = \frac{1}{a_1}\left(\frac{2\pi a_1}{T}\right)^2 = \frac{4\pi^2 a_1}{T^2} = \frac{4\pi^2}{T^2}\left(\frac{m_2}{m_1+m_2}a\right) \Rightarrow \boldsymbol{T^2 = \frac{4\pi^2}{G(m_1+m_2)}a^3}$$

Find the ratio of orbital radii, orbital velocities and time periods of two masses M_1 and M_2 in a stellar binary system (pure gravitationally bound and each mass moving in a circular orbit) separated by a constant length r_0.

$$\begin{Bmatrix}M_1r_1 = M_2r_2 \Rightarrow \dfrac{\boldsymbol{r_1}}{\boldsymbol{r_2}} = \dfrac{\boldsymbol{M_2}}{\boldsymbol{M_1}}\\ r_1 + r_2 = r_0\end{Bmatrix} \Rightarrow \begin{Bmatrix}1 + \dfrac{r_1}{r_2} = 1 + \dfrac{M_2}{M_1} \Rightarrow \dfrac{r_0}{r_2} = \dfrac{M_1+M_2}{M_1} \Rightarrow r_2 = \dfrac{M_1r_0}{M_1+M_2} = \dfrac{\mu r_0}{M_2}\\ 1 + \dfrac{r_2}{r_1} = 1 + \dfrac{M_1}{M_2} \Rightarrow \dfrac{r_0}{r_1} = \dfrac{M_1+M_2}{M_2} \Rightarrow r_1 = \dfrac{M_2r_0}{M_1+M_2} = \dfrac{\mu r_0}{M_1}\end{Bmatrix} \left[\mu = \frac{M_1M_2}{M_1+M_2}\right]$$

$$\begin{Bmatrix}\dfrac{M_1{v_1}^2}{r_1} = \dfrac{GM_1M_2}{(r_1+r_2)^2} \Rightarrow v_1 = \sqrt{\dfrac{GM_2r_1}{(r_1+r_2)^2}}\\ \dfrac{M_2{v_2}^2}{r_2} = \dfrac{GM_2M_1}{(r_1+r_2)^2} \Rightarrow v_2 = \sqrt{\dfrac{GM_1r_2}{(r_1+r_2)^2}}\end{Bmatrix} \Rightarrow \frac{\boldsymbol{v_1}}{\boldsymbol{v_2}} = \sqrt{\frac{M_2r_1}{M_1r_2}} = \frac{\boldsymbol{r_1}}{\boldsymbol{r_2}} = \frac{\boldsymbol{M_2}}{\boldsymbol{M_1}} \quad \left[\because \frac{r_1}{r_2} = \frac{M_2}{M_1}\right]$$

$$\begin{Bmatrix}P_1 = \dfrac{2\pi r_1}{v_1}\\ P_2 = \dfrac{2\pi r_2}{v_2}\end{Bmatrix} \Rightarrow \frac{\boldsymbol{P_1}}{\boldsymbol{P_2}} = \frac{r_1v_2}{r_2v_1} = \frac{r_1v_2}{r_2v_1} = \frac{r_1M_1}{r_2M_2} = \boldsymbol{1} \quad \left[\because \frac{v_1}{v_2} = \frac{M_2}{M_1}\right]$$

Formation of Binaries with Neutron Stars or Black Holes

Whether or not a binary system survives the supernova explosion of one of its component stars depends on the amount of mass ejected from the system.

Total initial energy of the binary, $E_i = \frac{1}{2}M_1{v_1}^2 + \frac{1}{2}M_2{v_2}^2 - \frac{GM_1M_2}{a} = -\frac{GM_1M_2}{2a}$ [using virial theorem]

Total final energy of the binary, $E_f = \frac{1}{2}M_R{v_1}^2 + \frac{1}{2}M_2{v_2}^2 - \frac{GM_RM_2}{a}$

If the explosion is to result in an unbound system, then $E_f \geq 0 \Rightarrow \frac{1}{2}M_R{v_1}^2 + \frac{1}{2}M_2{v_2}^2 - \frac{GM_RM_2}{a} \geq 0$

$$\Rightarrow \frac{GM_RM_2}{a} \leq \frac{1}{2}M_R{v_1}^2 + \frac{1}{2}M_2{v_2}^2 \Rightarrow \frac{M_R}{M_1+M_2} \leq \frac{1}{\left(1+\frac{M_2}{M_1}\right)\left(2+\frac{M_2}{M_1}\right)} < \frac{1}{2}$$

$$\frac{GM_RM_2}{2a} = \frac{1}{2}M_R{v_1}^2 + \frac{1}{2}M_2{v_2}^2 = \frac{1}{2}\left(\frac{M_2v_2}{v_1}\right){v_1}^2 + \frac{1}{2}M_2{v_2}^2 \quad [\because M_Rv_1 = M_2v_2] = \frac{1}{2}M_2v_2v_1 + \frac{1}{2}M_2{v_2}^2$$

$$\Rightarrow M_2 v_2(v_1+v_2) = \frac{GM_R M_2}{a} \Rightarrow \left[\because T = -\frac{1}{2}U\right]$$

Lagrange points (Euler-Lagrange points)

In a **three-body** problem with two massive bodies in orbits around their common barycenter, there are **five** positions in space where a third body, of comparatively negligible mass, could be placed so as to maintain its position relative to the two massive bodies. Let $\boldsymbol{\omega}$ denote the orbital angular velocity, and $\boldsymbol{r}$ be the separation between the Lagrange point and the orbit of the less massive body.

For the two massive bodies M_1 and M_2 with $M_1 \gg M_2$, $\frac{GM_1M_2}{R^2} = M_2{\omega_2}^2\left(\frac{M_1R}{M_1+M_2}\right) \Rightarrow {\omega_2}^2 = \frac{G(M_1+M_2)}{R^3}$

L1 point (assuming circular orbits)

$$F_1 = F_2 + F_C \Rightarrow \frac{GM_1m}{(R-r)^2} = \frac{GM_2m}{r^2} + m\omega^2\left(\frac{M_1R}{M_1+M_2} - r\right)$$

$$\Rightarrow \frac{GM_1m}{(R-r)^2} = \frac{GM_2m}{r^2} + \frac{G(M_1+M_2)m}{R^3}\left(\frac{M_1R}{M_1+M_2} - r\right) \begin{bmatrix}\because \omega = \omega_2 \\ \text{at } L_1\end{bmatrix} \Rightarrow \frac{\boldsymbol{M_1}}{(\boldsymbol{R}-\boldsymbol{r})^2} = \frac{\boldsymbol{M_2}}{\boldsymbol{r^2}} + \frac{\boldsymbol{M_1}}{\boldsymbol{R^2}} - \frac{(\boldsymbol{M_1}+\boldsymbol{M_2})\boldsymbol{r}}{\boldsymbol{R^3}}$$

$$\Rightarrow 1 = \frac{M_2}{M_1}\left(\frac{R-r}{r}\right)^2 + \left(\frac{R-r}{R}\right)^2\left\{1 - \left(1+\frac{M_2}{M_1}\right)\frac{r}{R}\right\} \Rightarrow 1 = \alpha\left(\frac{1}{\epsilon}-1\right)^2 + (1-\epsilon)^2\{1-\epsilon(1+\alpha)\} \quad \begin{bmatrix}\alpha = \frac{M_2}{M_1}\,; \\ \epsilon = \frac{r}{R}\end{bmatrix}$$

$$\Rightarrow \epsilon^2 = \alpha(1-\epsilon)^2 + \epsilon^2(1-\epsilon)^2(1-\epsilon-\alpha\epsilon) \Rightarrow \frac{\epsilon^2}{(1-\epsilon)^2} = \alpha + \epsilon^2(1-\epsilon-\alpha\epsilon) = \alpha(1-\epsilon^3) + \epsilon^2(1-\epsilon)$$

$$\Rightarrow \frac{\epsilon^2}{(1-\epsilon)^3} = \alpha(1+\epsilon+\epsilon^2) + \epsilon^2 \Rightarrow \epsilon^2\left\{\frac{1}{(1-\epsilon)^3} - 1\right\} = \alpha(1+\epsilon+\epsilon^2) \Rightarrow \frac{\epsilon^2}{(1-\epsilon)^3}\left\{\frac{3\epsilon-3\epsilon^2+\epsilon^3}{1+\epsilon+\epsilon^2}\right\} = \alpha$$

$$\Rightarrow \frac{3\epsilon^3}{(1-\epsilon)^3}\left\{\frac{1-\epsilon+\frac{1}{3}\epsilon^2}{1+\epsilon+\epsilon^2}\right\} = \alpha \Rightarrow \alpha \approx 3\epsilon^3 \quad \begin{bmatrix}\text{Realizing that } \epsilon \ll 1 \\ \text{for Earth-Sun system}\end{bmatrix} \Rightarrow \epsilon \approx \sqrt[3]{\frac{\alpha}{3}} \Rightarrow \frac{\boldsymbol{r}}{\boldsymbol{R}} \approx \sqrt[3]{\frac{\boldsymbol{1}}{\boldsymbol{3}}\frac{\boldsymbol{M_2}}{\boldsymbol{M_1}}}$$

L2 point (assuming circular orbits)

$$F_1 + F_2 = F_C \Rightarrow \frac{GM_1m}{(R+r)^2} + \frac{GM_2m}{r^2} = m\omega^2\left(\frac{M_1R}{M_1+M_2} + r\right)$$

$$\Rightarrow \frac{GM_1m}{(R+r)^2} + \frac{GM_2m}{r^2} = \frac{G(M_1+M_2)m}{R^3}\left(\frac{M_1R}{M_1+M_2} + r\right) \begin{bmatrix}\because \omega = \omega_2 \\ \text{at } L_2\end{bmatrix} \Rightarrow \frac{\boldsymbol{M_1}}{(\boldsymbol{R}+\boldsymbol{r})^2} + \frac{\boldsymbol{M_2}}{\boldsymbol{r^2}} = \frac{\boldsymbol{M_1}}{\boldsymbol{R^2}} + \frac{(\boldsymbol{M_1}+\boldsymbol{M_2})\boldsymbol{r}}{\boldsymbol{R^3}}$$

$$\Rightarrow 1 + \frac{M_2}{M_1}\left(\frac{R+r}{r}\right)^2 = \left(\frac{R+r}{R}\right)^2\left\{1 + \left(1+\frac{M_2}{M_1}\right)\frac{r}{R}\right\} \Rightarrow 1 + \alpha\left(\frac{1}{\epsilon}+1\right)^2 = (1+\epsilon)^2\{1+\epsilon(1+\alpha)\} \quad \begin{bmatrix}\alpha = \frac{M_2}{M_1}\,; \\ \epsilon = \frac{r}{R}\end{bmatrix}$$

$$\Rightarrow \epsilon^2 + \alpha(1+\epsilon)^2 = \epsilon^2(1+\epsilon)^2(1+\epsilon+\alpha\epsilon) \Rightarrow \frac{\epsilon^2}{(1+\epsilon)^2} = -\alpha + \epsilon^2(1+\epsilon+\alpha\epsilon) = -\alpha(1-\epsilon^3) + \epsilon^2(1+\epsilon)$$

$$\Rightarrow \frac{\epsilon^2}{(1+\epsilon)^3} = -\alpha(1+\epsilon+\epsilon^2)\frac{1-\epsilon}{1+\epsilon} + \epsilon^2 \Rightarrow \epsilon^2\left\{\frac{1}{(1+\epsilon)^3} - 1\right\} = -\alpha(1+\epsilon+\epsilon^2)\frac{1-\epsilon}{1+\epsilon}$$

$$\Rightarrow \frac{\epsilon^2}{(1+\epsilon)^3}\left\{\frac{3\epsilon+3\epsilon^2+\epsilon^3}{1+\epsilon+\epsilon^2}\right\} = \alpha\left(\frac{1-\epsilon}{1+\epsilon}\right) \Rightarrow \frac{3\epsilon^3}{(1+\epsilon)^2(1-\epsilon)}\left\{\frac{1+\epsilon+\frac{1}{3}\epsilon^2}{1+\epsilon+\epsilon^2}\right\} = \alpha$$

$$\Rightarrow \alpha \approx 3\epsilon^3 \quad \begin{bmatrix}\text{Realizing that } \epsilon \ll 1 \\ \text{for Earth-Sun system}\end{bmatrix} \Rightarrow \epsilon \approx \sqrt[3]{\frac{\alpha}{3}} \Rightarrow \frac{\boldsymbol{r}}{\boldsymbol{R}} \approx \sqrt[3]{\frac{\boldsymbol{1}}{\boldsymbol{3}}\frac{\boldsymbol{M_2}}{\boldsymbol{M_1}}}$$

L3 point (assuming circular orbits)

$$F_1 + F_2 = F_C \Rightarrow \frac{GM_1m}{(R-r)^2} + \frac{GM_2m}{(2R-r)^2} = m\omega^2\left(R - r + \frac{M_2R}{M_1+M_2}\right)$$

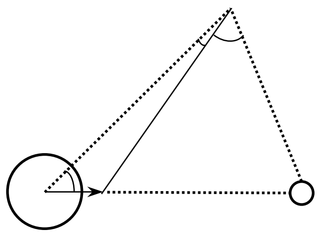
$$\Rightarrow \frac{GM_1m}{(R-r)^2} + \frac{GM_2m}{(2R-r)^2} = \frac{G(M_1+M_2)m}{R^3}\left(R - r + \frac{M_2R}{M_1+M_2}\right) \begin{bmatrix}\because \omega = \omega_2 \\ \text{at } L_3 \text{ point}\end{bmatrix}$$

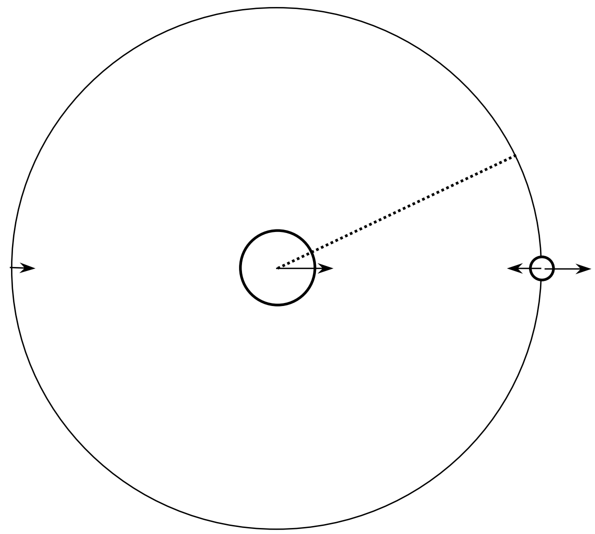
$$\Longrightarrow 1+\frac{M_2}{M_1}\left(\frac{R-r}{2R-r}\right)^2=\left(1+\frac{M_2}{M_1}\right)\left(\frac{R-r}{R}\right)^3+\frac{M_2}{M_1}\left(\frac{R-r}{R}\right)^2$$
$$\Longrightarrow 1+\alpha\left(\frac{1-\epsilon}{2-\epsilon}\right)^2=(1+\alpha)(1-\epsilon)^3+\alpha(1-\epsilon)^2\quad\left[\alpha=\frac{M_2}{M_1}\text{ and }\epsilon=\frac{r}{R}\right]=(1-\epsilon)^2(1+2\alpha-\epsilon-\alpha\epsilon)$$
$$\Longrightarrow (2-\epsilon)^2+\alpha(1-\epsilon)^2=(2-\epsilon)^2(1-\epsilon)^2(1+(2-\epsilon)\alpha-\epsilon)$$
$$\Longrightarrow -\alpha(2-\epsilon)^3(1-\epsilon)^2+\alpha(1-\epsilon)^2=(2-\epsilon)^2(1-\epsilon)^3-(2-\epsilon)^2$$
$$\Longrightarrow \alpha(1-\epsilon)^2\{1-(2-\epsilon)^3\}=(2-\epsilon)^2\{(1-\epsilon)^3-1\}$$
$$\Longrightarrow \alpha(1-\epsilon)^2\{1-(8-12\epsilon+6\epsilon^2-\epsilon^3)\}=(2-\epsilon)^2\{(1-3\epsilon+3\epsilon^2-\epsilon^3)-1\}$$
$$\Longrightarrow \alpha(1-\epsilon)^2\{-7+12\epsilon-6\epsilon^2+\epsilon^3\}=(2-\epsilon)^2\{-3\epsilon+3\epsilon^2-\epsilon^3\}\Longrightarrow \alpha=\left(\frac{2-\epsilon}{1-\epsilon}\right)^2\left\{\frac{-3\epsilon+3\epsilon^2-\epsilon^3}{-7+12\epsilon-6\epsilon^2+\epsilon^3}\right\}$$
$$\Longrightarrow \alpha=\frac{3\epsilon}{7}\left(\frac{2-\epsilon}{1-\epsilon}\right)^2\left\{\frac{1-\epsilon+\frac{1}{3}\epsilon^2}{1-\frac{12}{7}\epsilon+\frac{6}{7}\epsilon^2-\frac{1}{7}\epsilon^3}\right\}\approx\frac{3\epsilon}{7}\left(\frac{4-4\epsilon}{1-2\epsilon}\right)\left\{\frac{1-\epsilon}{1-\frac{12}{7}\epsilon}\right\}\quad\left[\begin{array}{l}\text{Realizing that }\epsilon\ll 1\\ \text{for Earth-Sun system}\end{array}\right]$$
$$\Longrightarrow \alpha\approx\frac{12\epsilon}{7\left(1-\frac{12}{7}\epsilon\right)}\approx\frac{12\epsilon}{7}\Longrightarrow \epsilon\approx\frac{7}{12}\alpha\Longrightarrow \boldsymbol{\frac{r}{R}\approx\frac{7}{12}\frac{M_2}{M_1}}$$

Alternatively:

$$F_1+F_2=F_C\Longrightarrow\frac{GM_1m}{(R+r)^2}+\frac{GM_2m}{(2R+r)^2}=m\omega^2\left(R+r+\frac{M_2R}{M_1+M_2}\right)$$
$$\Longrightarrow\frac{GM_1m}{(R+r)^2}+\frac{GM_2m}{(2R+r)^2}=\frac{G(M_1+M_2)m}{R^3}\left(R+r+\frac{M_2R}{M_1+M_2}\right)\quad\left[\begin{array}{l}\because\omega=\omega_2\\ \text{at }L_3\text{ point}\end{array}\right]$$
$$\Longrightarrow 1+\frac{M_2}{M_1}\left(\frac{R+r}{2R+r}\right)^2=\left(1+\frac{M_2}{M_1}\right)\left(\frac{R+r}{R}\right)^3+\frac{M_2}{M_1}\left(\frac{R+r}{R}\right)^2$$
$$\Longrightarrow 1+\alpha\left(\frac{1+\epsilon}{2+\epsilon}\right)^2=(1+\alpha)(1+\epsilon)^3+\alpha(1+\epsilon)^2\quad\left[\alpha=\frac{M_2}{M_1}\text{ and }\epsilon=\frac{r}{R}\right]=(1+\epsilon)^2(1+2\alpha+\epsilon+\alpha\epsilon)$$
$$\Longrightarrow (2+\epsilon)^2+\alpha(1+\epsilon)^2=(2+\epsilon)^2(1+\epsilon)^2(1+(2+\epsilon)\alpha+\epsilon)$$
$$\Longrightarrow -\alpha(2+\epsilon)^3(1+\epsilon)^2+\alpha(1+\epsilon)^2=(2+\epsilon)^2(1+\epsilon)^3-(2+\epsilon)^2$$
$$\Longrightarrow \alpha(1+\epsilon)^2\{1-(2+\epsilon)^3\}=(2+\epsilon)^2\{(1+\epsilon)^3-1\}$$
$$\Longrightarrow \alpha(1+\epsilon)^2\{1-(8+12\epsilon+6\epsilon^2+\epsilon^3)\}=(2+\epsilon)^2\{(1+3\epsilon+3\epsilon^2+\epsilon^3)-1\}$$
$$\Longrightarrow \alpha(1+\epsilon)^2\{-7+12\epsilon+6\epsilon^2+\epsilon^3\}=(2+\epsilon)^2\{3\epsilon+3\epsilon^2+\epsilon^3\}\Longrightarrow \alpha=\left(\frac{2+\epsilon}{1+\epsilon}\right)^2\left\{\frac{3\epsilon+3\epsilon^2+\epsilon^3}{-7+12\epsilon+6\epsilon^2+\epsilon^3}\right\}$$
$$\Longrightarrow \alpha=-\frac{3\epsilon}{7}\left(\frac{2+\epsilon}{1+\epsilon}\right)^2\left\{\frac{1+\epsilon+\frac{1}{3}\epsilon^2}{1-\frac{12}{7}\epsilon-\frac{6}{7}\epsilon^2-\frac{1}{7}\epsilon^3}\right\}\approx-\frac{3\epsilon}{7}\left(\frac{4+4\epsilon}{1+2\epsilon}\right)\left\{\frac{1+\epsilon}{1-\frac{12}{7}\epsilon}\right\}\quad\left[\begin{array}{l}\text{Realizing that }\epsilon\ll 1\\ \text{for Earth-Sun system}\end{array}\right]$$
$$\Longrightarrow \alpha\approx-\frac{12\epsilon}{7\left(1-\frac{12}{7}\epsilon\right)}\approx-\frac{12\epsilon}{7}\Longrightarrow \epsilon\approx-\frac{7}{12}\alpha\Longrightarrow \boldsymbol{\frac{r}{R}\approx-\frac{7}{12}\frac{M_2}{M_1}}$$

$$\boldsymbol{\frac{M_2}{M_1}}\approx\begin{cases}0.0123 & \text{for Earth-Moon system}\\ 3\times10^{-6} & \text{for Sun-Earth system}\end{cases}\quad\text{and}\quad\boldsymbol{R}\approx\begin{cases}384400\text{ km} & \text{for Earth-Moon system}\\ 1.496\times10^8\text{ km} & \text{for Sun-Earth system}\end{cases}$$
$$\therefore r(L_1)\approx r(L_2),r(L_3)=\begin{cases}61524\text{ km},2758\text{ km} & \text{for Earth-Moon system}\\ 1.496\times10^6\text{ km},261.8\text{ km} & \text{for Sun-Earth system}\end{cases}\quad[\boldsymbol{r(L_3)<r(L_1)\lesssim r(L_2)}]$$





L4 and L5 points

$R_1 = R \quad \text{and} \quad R_2 = 2R\sin\frac{\theta}{2}$

$$R_0 = \sqrt{{R_1}^2 + \left(\frac{M_2 R}{M_1 + M_2}\right)^2 - 2\frac{M_2 R R_1 \cos\theta}{M_1 + M_2}} = R\sqrt{1 + \left(\frac{\alpha}{1+\alpha}\right)^2 - 2\left(\frac{\alpha}{1+\alpha}\right)\cos\theta} \approx R\sqrt{1 + \alpha^2 - 2\alpha\cos\theta}$$

$$\sin\phi_1 = \left(\frac{M_2}{M_1 + M_2}\right)\frac{R}{R_0}\sin\theta = \left(\frac{\alpha}{1+\alpha}\right)\frac{R}{R_0}\sin\theta \approx \frac{R}{R_0}\alpha\sin\theta = \frac{\alpha\sin\theta}{\sqrt{1+\alpha^2-2\alpha\cos\theta}}$$

$$\Longrightarrow \cos\phi_1 = \sqrt{1 - \frac{\alpha^2\sin^2\theta}{1+\alpha^2-2\alpha\cos\theta}} = \sqrt{\frac{1+\alpha^2\cos^2\theta - 2\alpha\cos\theta}{1+\alpha^2-2\alpha\cos\theta}} = \frac{(1-\alpha\cos\theta)}{\sqrt{1+\alpha^2-2\alpha\cos\theta}} \approx \frac{(1-\alpha\cos\theta)}{\sqrt{1-2\alpha\cos\theta}} \approx 1$$

Alternatively, $\phi_1 \approx \frac{M_2\sin\theta}{M_1+M_2}\frac{R}{R_0} = \left(\frac{\alpha}{1+\alpha}\right)\frac{R}{R_0}\sin\theta \approx \frac{R}{R_0}\alpha\sin\theta \Longrightarrow \sin\phi_1 \approx \frac{\alpha\sin\theta}{\sqrt{1-2\alpha\cos\theta}} = \frac{\alpha\sin\theta}{1-\alpha\cos\theta}$

$$\phi_2 = \sin^{-1}\left(\frac{R}{R_2}\sin\theta\right) - \phi_1 = \sin^{-1}\left(\frac{\sin\theta}{2\sin\frac{\theta}{2}}\right) - \phi_1 = \sin^{-1}\left(\cos\frac{\theta}{2}\right) - \phi_1 = \frac{\pi}{2} - \frac{\theta}{2} - \phi_1 \Longrightarrow \begin{cases} \sin\phi_2 = \cos\left(\frac{\theta}{2} + \phi_1\right) \\ \cos\phi_2 = \sin\left(\frac{\theta}{2} + \phi_1\right) \end{cases}$$

Alternatively, $\phi_2 \approx \phi_1 + \phi_2 \;\; [\because \phi_1 \approx 0] = \frac{\pi - \theta}{2} = \frac{\pi}{2} - \frac{\theta}{2}$

$$\frac{GM_1 m}{{R_1}^2}\cos\phi_1 + \frac{GM_2 m}{{R_2}^2}\cos\phi_2 = m\omega^2 R_0 = \frac{G(M_1+M_2)m}{R^3}R_0 \Longrightarrow \frac{M_1}{{R_1}^2}\cos\phi_1 + \frac{M_2\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}{4R^2\sin^2\frac{\theta}{2}} = \frac{(M_1+M_2)R_0}{R^3}$$

$$\Longrightarrow \frac{M_1}{R^2} + \frac{M_2\sin\frac{\theta}{2}}{4R^2\sin^2\frac{\theta}{2}} \approx \frac{(M_1+M_2)R_0}{R^3} \Longrightarrow 4\sin\frac{\theta}{2} + \alpha \approx \frac{4R_0}{R}(1+\alpha)\sin\frac{\theta}{2} \approx 4\sqrt{1-2\alpha\cos\theta}\,(1+\alpha)\sin\frac{\theta}{2}$$

$$\Longrightarrow 4\sin\frac{\theta}{2} + \alpha \approx 4(1-\alpha\cos\theta)(1+\alpha)\sin\frac{\theta}{2} = 4(1-\alpha\cos\theta+\alpha)\sin\frac{\theta}{2} \Longrightarrow \alpha \approx 4\alpha(1-\cos\theta)\sin\frac{\theta}{2}$$

$$\Longrightarrow 1 \approx 4(1-\cos\theta)\sin\frac{\theta}{2} = 8\sin^3\frac{\theta}{2} \Longrightarrow 8\sin^3\frac{\theta}{2} - 1 \approx 0 \Longrightarrow \mathbf{\sin\frac{\boldsymbol{\theta}}{2} \approx \frac{1}{2}}$$

$$\frac{GM_1 m}{{R_1}^2}\sin\phi_1 = \frac{GM_2 m}{{R_2}^2}\sin\phi_2 \Longrightarrow \sin\phi_1 = \frac{\alpha}{4\sin^2\frac{\theta}{2}}\sin\phi_2 \Longrightarrow \frac{\alpha\sin\theta}{1-\alpha\cos\theta} = \frac{\alpha}{4\sin^2\frac{\theta}{2}}\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right) = \frac{\alpha}{4\sin^2\frac{\theta}{2}}\cos\frac{\theta}{2}$$

$$\Longrightarrow \frac{2\sin\frac{\theta}{2}}{1-\alpha\cos\theta} = \frac{1}{4\sin^2\frac{\theta}{2}} \Longrightarrow 8\sin^3\frac{\theta}{2} = 1 - \alpha\cos\theta = 1 - \alpha\left(1 - 2\sin^2\frac{\theta}{2}\right) \Longrightarrow 8\sin^3\frac{\theta}{2} - 1 \approx 0 \Longrightarrow \mathbf{\sin\frac{\boldsymbol{\theta}}{2} \approx \frac{1}{2}}$$

General equation:

$$\boldsymbol{R_0} = \sqrt{\boldsymbol{R_1}^2 + \left(\frac{\boldsymbol{M_2 R}}{\boldsymbol{M_1 + M_2}}\right)^2 - \boldsymbol{2}\frac{\boldsymbol{M_2 R R_1 \cos\theta}}{\boldsymbol{M_1 + M_2}}} = R\sqrt{\eta^2 + \left(\frac{\alpha}{1+\alpha}\right)^2 - 2\eta\left(\frac{\alpha}{1+\alpha}\right)\cos\theta}$$

$$\boldsymbol{R_2} = \sqrt{\boldsymbol{R_1}^2 + \boldsymbol{R}^2 - \boldsymbol{2RR_1\cos\theta}} = R\sqrt{\eta^2 + 1 - 2\eta\cos\theta} \quad \left[\text{where, } \alpha = \frac{M_2}{M_1} \text{ and } \eta = \frac{R_0}{R}\right]$$

$$\frac{\sin\phi_1}{\left(\frac{M_2 R}{M_1 + M_2}\right)} = \frac{\sin\theta}{R_0} \Rightarrow \sin\phi_1 = \left(\frac{M_2}{M_1 + M_2}\right)\frac{R}{R_0}\sin\theta \Rightarrow \boldsymbol{\phi_1} = \frac{\boldsymbol{\pi}}{\boldsymbol{2}} - \boldsymbol{\cos^{-1}}\left\{\left(\frac{\boldsymbol{\alpha}}{\boldsymbol{1+\alpha}}\right)\frac{\boldsymbol{R}}{\boldsymbol{R_0}}\boldsymbol{\sin\theta}\right\}$$

$$\Rightarrow \cos\phi_1 = \sin\left\{\cos^{-1}\left\{\left(\frac{\alpha}{1+\alpha}\right)\frac{R}{R_0}\sin\theta\right\}\right\} = \sin\left\{\cos^{-1}\left\{\frac{\left(\frac{\alpha}{1+\alpha}\right)\sin\theta}{\sqrt{\eta^2 + \left(\frac{\alpha}{1+\alpha}\right)^2 - 2\eta\left(\frac{\alpha}{1+\alpha}\right)\cos\theta}}\right\}\right\}$$

$$\frac{\sin(\phi_1 + \phi_2)}{R} = \frac{\sin\theta}{R_2} \Rightarrow \sin(\phi_1 + \phi_2) = \frac{R}{R_2}\sin\theta \Rightarrow \boldsymbol{\phi_2} = \boldsymbol{\sin^{-1}}\left(\frac{\boldsymbol{R}}{\boldsymbol{R_2}}\boldsymbol{\sin\theta}\right) - \boldsymbol{\phi_1}$$

$$\Rightarrow \cos\phi_2 = \cos\left\{\sin^{-1}\left(\frac{R}{R_2}\sin\theta\right) - \phi_1\right\} = \cos\left\{\sin^{-1}\left(\frac{\sin\theta}{\sqrt{\eta^2 + 1 - 2\eta\cos\theta}}\right) - \phi_1\right\}$$

Force equation (radial):

$$\frac{GM_1 m}{{R_1}^2}\cos\phi_1 + \frac{GM_2 m}{{R_2}^2}\cos\phi_2 = m\omega^2 R_0 = \frac{G(M_1 + M_2)m}{R^3}R_0 \Rightarrow \frac{\boldsymbol{M_1}}{\boldsymbol{R_1}^2}\boldsymbol{\cos\phi_1} + \frac{\boldsymbol{M_2}}{\boldsymbol{R_2}^2}\boldsymbol{\cos\phi_2} = \frac{\boldsymbol{(M_1 + M_2)}}{\boldsymbol{R^3}}\boldsymbol{R_0}$$

$$\Rightarrow \frac{M_1}{{R_1}^2}\cos\phi_1 + \frac{M_2}{{R_2}^2}\cos\phi_2 = \frac{(M_1 + M_2)}{R^3}R_0 \Rightarrow \frac{1}{{R_1}^2}\cos\phi_1 + \frac{\alpha}{{R_2}^2}\cos\phi_2 = \frac{(1+\alpha)}{R^3}R_0$$

$$\Rightarrow \frac{\cos\phi_1}{\eta^2} + \frac{\alpha\cos\phi_2}{(\eta^2 + 1 - 2\eta\cos\theta)} = (1+\alpha)\sqrt{\eta^2 + \left(\frac{\alpha}{1+\alpha}\right)^2 - 2\eta\left(\frac{\alpha}{1+\alpha}\right)\cos\theta}$$

Force equation (tangential):

$$\frac{GM_1 m}{{R_1}^2}\sin\phi_1 = \frac{GM_2 m}{{R_2}^2}\sin\phi_2 \Rightarrow \sin\phi_1 = \frac{{R_1}^2}{{R_2}^2}\alpha\sin\phi_2 \Rightarrow \left(\frac{\alpha}{1+\alpha}\right)\frac{R}{R_0}\sin\theta = \frac{\eta^2\alpha\sin\phi_2}{(\eta^2 + 1 - 2\eta\cos\theta)}$$

$$\Rightarrow \sin\theta = \frac{\eta^2(1+\alpha)\sin\phi_2}{(\eta^2 + 1 - 2\eta\cos\theta)}\sqrt{\eta^2 + \left(\frac{\alpha}{1+\alpha}\right)^2 - 2\eta\left(\frac{\alpha}{1+\alpha}\right)\cos\theta}$$

Life of a typical star

<table>
<tr><td rowspan="7">STELLAR NEBULA</td><td rowspan="7">PROTOSTAR</td><td rowspan="3">(high mass)</td><td rowspan="3">BLUE GIANT</td><td rowspan="3">RED SUPERGIANT</td><td rowspan="3">SUPERNOVA</td><td>SUPERNOVA REMNANT</td></tr>
<tr><td>NEUTRON STAR</td></tr>
<tr><td>BLACK HOLE</td></tr>
<tr><td rowspan="2">(medium mass)</td><td rowspan="2">YELLOW DWARF</td><td rowspan="2">RED GIANT</td><td colspan="2">PLANETARY NEBULA</td></tr>
<tr><td rowspan="2">WHITE DWARF</td><td rowspan="2">BLACK DWARF</td></tr>
<tr><td>(low mass)</td><td colspan="2">RED DWARF</td></tr>
<tr><td>(very low mass)</td><td colspan="4">BROWN DWARF</td></tr>
</table>

Big Bang cosmology (Chronology of the Universe)

Epoch (Era)	Range (approx.)	Description
Planck	10^{-43} s	Current physical therories
Grand Unification (GU)	10^{-36} s	Three forces of Standard Model are unified.
Inflationary	10^{-32} s	Cosmic inflation; Strong Nuclear force becomes distinct from Electroweak force.
Electroweak	10^{-12} s = 1 ps	Forces of Standard Model separates.

Quark	10^{-6} s = 1 μs	
Hadron	1 s	
Lepton	10 s	
Nucleosynthesis	10^3 s ≈ 20 min	
Photon (Radiation dominated)	10^{13} s ≈ 300 kyr	
Recombination (Decoupling)	300-400 kyr	
Dark age	150 Myr	
Reionization	1 Gyr	
Formation of Stars and Galaxies	10 Gyr	
Formation of Solar System	9 Gyr	
Today	13.8 Gyr	

Cosmic Distance Ladder (CDL)

Method	Range (approx.)	Description
Trigonometric parallax	1 kpc	
Secular parallax	7 kpc	
Dynamical parallax	3 Mpc	
Main-Sequence fitting (Spectroscopic parallax) / Wilson-Bappu effect	7 Mpc	
Novae	20 Mpc	
Classical Cepheids	30 Mpc	
Tip of the Red Giant Branch (TRGB)	50 Mpc	
Planetary Nebula Luminosity Function (PNLF)	50 Mpc	
Globular Cluster Luminosity Function (GCLF)	50 Mpc	
Surface Brightness Fluctuations (SBF)	100 Mpc	
Tully-Fisher relation	100 Mpc	
D-*σ* relation	100 Mpc	
Type Ia Supernovae (SNe Ia)	1 Gpc	

Hubble's law

$$r(t) = a(t)r_0 \Rightarrow \dot{r} = \dot{a}r_0 = \left(\frac{\dot{a}}{a}\right)ar_0 = H_0 r \Rightarrow \boldsymbol{v = H_0 r}$$

Classical derivation of Friedmann equation

$$U = T + V = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}m\dot{r}^2 - \frac{G(\rho V)m}{r} = \frac{1}{2}m\dot{r}^2 - \frac{4}{3}\pi G\rho m r^2 = \frac{1}{2}m(\dot{a}r_0)^2 - \frac{4}{3}\pi G\rho m(ar_0)^2$$

$$\Rightarrow \frac{2U}{m(ar_0)^2} = \left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G\rho}{3} \Rightarrow -\frac{kc^2}{a^2} = \left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G\rho}{3} \quad \left[\text{curvature, } k = -\frac{2U}{mc^2 {r_0}^2} \Rightarrow {r_0}^2 k = -\frac{2U}{mc^2}\right]$$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} \Rightarrow \boldsymbol{H^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2}} \quad \text{[Friedmann equation]}$$

$\because$ other terms are independent of r_0, $\therefore k$ must be independent of r_0, so that $U \propto {r_0}^2$

If $k > 0$, then, $U < 0 \Rightarrow V > T \Rightarrow$ expansion will halt at some time t and reverse itself.

If $k < 0$, then, $U > 0 \Rightarrow V < T \Rightarrow$ expansion will continue forever.

If $k = 0$, then, $U = 0 \Rightarrow V = T \Rightarrow$ expansion will slow down but only halt at $t = \infty$

For a flat universe, $k = 0$, so that ${H_0}^2 = \frac{8\pi G\rho_c}{3} \Rightarrow \boldsymbol{\rho_c = \frac{3{H_0}^2}{8\pi G}} \approx 9.2 \times 10^{-27}$ kg/m^3 $\left[\begin{array}{r} H_0 \approx 70 \text{ (km/s)/Mpc} \\ \approx 2.27 \times 10^{-18} \text{ s}^{-1}\end{array}\right]$

density parameter, $\boldsymbol{\Omega = \frac{\rho}{\rho_c} = \frac{8\pi G}{3{H_0}^2}\rho}$

If $k = 0, \rho = \rho_c$, so that $\Omega = 1$, and $H^2 = \frac{8\pi G\varepsilon}{3c^2} \Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\left(\varepsilon_0\left(\frac{a}{a_0}\right)^{-3(1+w)}\right) \Rightarrow \frac{\dot{a}}{a} = \sqrt{\frac{8\pi G\varepsilon_0}{3c^2}}\left(\frac{a}{a_0}\right)^{-\frac{3(1+w)}{2}}$

$\Rightarrow \frac{\dot{a}}{a^{1-\frac{3(1+w)}{2}}} = \sqrt{\frac{8\pi G\varepsilon_0}{3c^2{a_0}^{-3(1+w)}}} \Rightarrow \frac{a^{-\left\{1-\frac{3(1+w)}{2}\right\}+1}}{-\left\{1-\frac{3(1+w)}{2}\right\}+1} = \sqrt{\frac{8\pi G\varepsilon_0}{3c^2{a_0}^{-3(1+w)}}}t + C$ [when $w \neq -1$]

$\Rightarrow a = \frac{3(1+w)}{2}\sqrt{\frac{8\pi G\varepsilon_0}{3c^2{a_0}^{-3(1+w)}}}t^{\frac{2}{3(1+w)}} + a_0$

$\dot{\varepsilon} + 3H(\varepsilon + P) = 0 \Rightarrow \frac{\dot{\varepsilon}}{\varepsilon + P} = -3H \Rightarrow \frac{\dot{\varepsilon}}{\varepsilon} = -3H(1+w) = -3\left(\frac{\dot{a}}{a}\right)(1+w) \Rightarrow \ln\frac{\varepsilon}{\varepsilon_0} =- 3(1+w)\ln\frac{a}{a_0}$

$\Rightarrow \varepsilon = \varepsilon_0\left(\frac{a}{a_0}\right)^{-3(1+w)} \Rightarrow \varepsilon = \frac{\varepsilon_0}{{a_0}^{-3(1+w)}}a^{-3(1+w)}$

To describe the time evolution of the scale factor of the universe, $a(t)$, an additional equation describing the time evolution of the density ρ of material in the universe is required.

$\begin{cases} dE + p\,dV = \delta Q & \text{[1st law of thermodynamics]} \\ \delta Q = T\,dS & \text{[2nd law of thermodynamics]} \end{cases} \Rightarrow dE + p\,dV = T\,dS$

$\begin{cases} \text{energy within the expanding volume, } E = mc^2 = \frac{4}{3}\pi\rho r^3c^2 = \frac{4}{3}\pi\rho(ar_0)^3c^2 \Rightarrow \frac{dE}{dt} = 4\pi{r_0}^3\rho a^2c^2\dot{a} + \frac{4}{3}\pi{r_0}^3\dot{\rho}a^3c^2 \\ \text{volume, } V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi{r_0}^3a^3 \Rightarrow \frac{dV}{dt} = 4\pi{r_0}^3a^2\dot{a} \end{cases}$

For a reversible expansion, $dS = 0$ so that $dE + p\,dV = 0 \Rightarrow \left(4\pi\rho a^2c^2\dot{a} + \frac{4}{3}\pi\dot{\rho}a^3c^2\right) + p(4\pi a^2\dot{a}) = 0$

$\Rightarrow \left(\rho\dot{a} + \frac{1}{3}\dot{\rho}a\right) + \frac{p}{c^2}\dot{a} = 0 \Rightarrow \boldsymbol{\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0}$ [fluid equation] $\Rightarrow \dot{\rho} + 3H\left(\rho + \frac{p}{c^2}\right) = 0$

$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G\rho}{3} = -\frac{kc^2}{a^2} \Rightarrow 2\frac{\dot{a}}{a}\left(-\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right) - \frac{8\pi G}{3}\dot{\rho} = \frac{2kc^2}{a^3}\dot{a} \Rightarrow \frac{\dot{a}}{a}\left(-\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right) + 4\pi G\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = \frac{kc^2}{a^3}\dot{a}$

$\Rightarrow \left(-\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right) + 4\pi G\left(\rho + \frac{p}{c^2}\right) = \frac{kc^2}{a^2} \Rightarrow \left(-\frac{8\pi G\rho}{3} + \frac{kc^2}{a^2} + \frac{\ddot{a}}{a}\right) + 4\pi G\left(\rho + \frac{p}{c^2}\right) = \frac{kc^2}{a^2}$

$\Rightarrow \left(-\frac{8\pi G\rho}{3} + \frac{\ddot{a}}{a}\right) + 4\pi G\left(\rho + \frac{p}{c^2}\right) = 0 \Rightarrow \boldsymbol{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)}$ [acceleration equation]

Matter (Dust) dominated era

For $p = 0, \dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0$ $\begin{bmatrix}\text{fluid equation} \\ \text{with } p = 0\end{bmatrix} \Rightarrow 3\frac{\dot{a}}{a}\rho = -\dot{\rho} \Rightarrow \frac{1}{a^3}\frac{d}{dt}(\rho a^3) = 0 \Rightarrow \frac{d}{dt}(\rho a^3) = 0$

$\Rightarrow \rho a^3 = \text{constant} \Rightarrow \boldsymbol{\rho \propto \frac{1}{a^3}} \Rightarrow \rho = \frac{\rho_0}{a^3}$

$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3}$ $\begin{bmatrix}\text{Friedmann equation} \\ \text{with } k = 0\end{bmatrix} \Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\frac{\rho_0}{a^3}\right) \Rightarrow \dot{a}^2 = \frac{8\pi G}{3}\frac{\rho_0}{a} \Rightarrow \dot{a} = \sqrt{\frac{8\pi G\rho_0}{3}}a^{-\frac{1}{2}}$

$\Rightarrow a^{\frac{1}{2}}da = \sqrt{\frac{8\pi G\rho_0}{3}}dt \Rightarrow \frac{2}{3}a^{\frac{3}{2}} = \sqrt{\frac{8\pi G\rho_0}{3}}t \Rightarrow a = \left(\frac{3}{2}\sqrt{\frac{8\pi G\rho_0}{3}}\right)^{\frac{2}{3}}t^{\frac{2}{3}} \Rightarrow a = \left(\frac{t}{t_0}\right)^{\frac{2}{3}} \Rightarrow \boldsymbol{a \propto t^{\frac{2}{3}}} = \frac{t^{\frac{2}{3}}}{{t_0}^{\frac{2}{3}}}$

$$\Rightarrow \begin{cases} \boldsymbol{\rho \propto \frac{1}{t^2}} & \left[\because \rho \propto \frac{1}{a^3}\right] \\ \dot{a} = \frac{2}{3}\frac{t^{-\frac{1}{3}}}{t_0^{\frac{2}{3}}} \end{cases} \Rightarrow H = \frac{\dot{a}}{a} = \frac{2}{3}\frac{t^{-\frac{1}{3}}}{t^{\frac{2}{3}}} \Rightarrow \boldsymbol{H = \frac{2}{3t}} \quad [\textbf{Einstein-de Sitter equation}]$$

$$\Rightarrow H_0 = \frac{2}{3t_0} \Rightarrow t_0 = \frac{2}{3H_0} \approx 2.937 \times 10^{17}\ \text{s} \approx 9.3\ \text{Gyr} \quad \left[\begin{array}{l} \because H_0 \approx 70\ (\text{km/s})/\text{Mpc} \\ \quad\approx 2.27 \times 10^{-18}\ \text{s}^{-1} \end{array}\right]$$

Radiation dominated era

For $p = 0$, $\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{1}{3}\rho\right) = 0 \quad \left[\begin{array}{l}\text{fluid equation} \\ \text{with } p = \frac{1}{3}\rho c^2\end{array}\right] \Rightarrow 4\frac{\dot{a}}{a}\rho = -\dot{\rho} \Rightarrow \frac{1}{a^4}\frac{d}{dt}(\rho a^4) = 0 \Rightarrow \frac{d}{dt}(\rho a^4) = 0$

$$\Rightarrow \rho a^4 = \text{constant} \Rightarrow \boldsymbol{\rho \propto \frac{1}{a^4}} \Rightarrow \rho = \frac{\rho_0}{a^4}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} \quad \left[\begin{array}{c}\text{Friedmann equation} \\ \text{with } k = 0\end{array}\right] \Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\frac{\rho_0}{a^4}\right) \Rightarrow \dot{a}^2 = \frac{8\pi G}{3}\frac{\rho_0}{a^2} \Rightarrow \dot{a} = \sqrt{\frac{8\pi G\rho_0}{3}}\,a^{-1}$$

$$\Rightarrow a\,da = \sqrt{\frac{8\pi G\rho_0}{3}}\,dt \Rightarrow \frac{1}{2}a^2 = \sqrt{\frac{8\pi G\rho_0}{3}}\,t \Rightarrow a = \left(2\sqrt{\frac{8\pi G\rho_0}{3}}\right)^{\frac{1}{2}} t^{\frac{1}{2}} \Rightarrow a = \left(\frac{t}{t_0}\right)^{\frac{1}{2}} \Rightarrow \boldsymbol{a \propto t^{\frac{1}{2}}} = \frac{t^{\frac{1}{2}}}{t_0^{\frac{1}{2}}}$$

$$\Rightarrow \begin{cases} \boldsymbol{\rho \propto \frac{1}{t^2}} & \left[\because \rho \propto \frac{1}{a^4}\right] \\ \dot{a} = \frac{1}{2}\frac{t^{-\frac{1}{2}}}{t_0^{\frac{1}{2}}} \end{cases} \Rightarrow H = \frac{\dot{a}}{a} = \frac{1}{2}\frac{t^{-\frac{1}{2}}}{t^{\frac{1}{2}}} \Rightarrow \boldsymbol{H = \frac{1}{2t}} \quad [\textbf{Einstein-de Sitter equation}]$$

$$\Rightarrow H_0 = \frac{1}{2t_0} \Rightarrow t_0 = \frac{1}{2H_0} \approx 2.203 \times 10^{17}\ \text{s} \approx 7.0\ \text{Gyr} \quad \left[\begin{array}{l} \because H_0 \approx 70\ (\text{km/s})/\text{Mpc} \\ \quad\approx 2.27 \times 10^{-18}\ \text{s}^{-1} \end{array}\right]$$

Circle, $x_1{}^2 + x_2{}^2 = r^2$

$$\begin{cases} x_1 = r\cos\phi \\ x_2 = r\sin\phi \end{cases}$$

$dl^2 = r\,d\phi^2$

circumference of circle $= \int_0^{2\pi} r\,d\phi = 2\pi r$

area of circle $= \pi r^2$

Sphere, $x_1{}^2 + x_2{}^2 + x_3{}^2 = r^2$

$$\begin{cases} x_1 = r\sin\theta\cos\phi \\ x_2 = r\sin\theta\sin\phi \qquad dl^2 = r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) \\ x_3 = r\cos\theta \end{cases}$$

$dl^2 = r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)$

surface area of sphere, $= \int_0^{\pi} r\,d\theta \int_0^{2\pi} r\sin\theta\,d\phi = 2\pi r$

Four-sphere (hypersphere), $x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2 = r^2$

$$\begin{cases} x_1 = r\sin\psi\sin\theta\cos\phi \\ x_2 = r\sin\psi\sin\theta\sin\phi \\ x_3 = r\sin\psi\cos\theta \\ x_4 = r\cos\psi \end{cases}$$

$dl^2 = r^2\left(d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right)$

three-volume (hypervolume) of four-sphere (hypersphere) $= \int_0^{\pi} r\,d\theta \int_0^{\pi} r\sin\psi\,d\theta \int_0^{2\pi} r\sin\psi\sin\theta\,d\phi = 2\pi^2 r^3$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} \quad \text{[Friedmann equation]} \Longrightarrow \left(\frac{\dot{r}}{r}\right)^2 = \frac{8\pi G\rho}{3} - \frac{{r_0}^2 kc^2}{r^2}$$

$$\Longrightarrow \dot{r}^2 = \frac{8\pi G\rho r^2}{3} - {r_0}^2 kc^2 \Longrightarrow \dot{r}^2 - \frac{8\pi G}{3}\left(\frac{M}{2\pi^2 r}\right) = -{r_0}^2 kc^2 \quad \left[\because \rho = \frac{M}{2\pi^2 r^3} \Longrightarrow \rho r^2 = \frac{M}{2\pi^2 r}\right]$$

$$\Longrightarrow \dot{r}^2 - \frac{4GM}{3\pi r} = -{r_0}^2 kc^2 \Longrightarrow \left(\frac{d\xi}{d\tau}c\right)^2 - \frac{4GM}{3\pi}\left(\frac{3\pi c^2}{4GM\xi}\right) = -{r_0}^2 kc^2 \Longrightarrow \left(\frac{d\xi}{d\tau}\right)^2 - \frac{1}{\xi} = -{r_0}^2 k$$

$$\left[\left.\begin{matrix} \xi = \dfrac{3\pi c^2}{4GM} r \\ \tau = \dfrac{3\pi c^3}{4GM} t \end{matrix}\right\} \Longrightarrow \frac{dr}{dt} = \frac{dr}{d\xi}\frac{d\xi}{dt} = \frac{dr}{d\xi}\frac{d\xi}{d\tau}\frac{d\tau}{dt} = \left(\frac{4GM}{3\pi c^2}\right)\frac{d\xi}{d\tau}\left(\frac{3\pi c^3}{4GM}\right) = \frac{d\xi}{d\tau}c\right]$$

Thermodynamic derivation of Friedmann equation (to be clarified)

FLRW metric: $ds^2 = -c^2dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$ where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\Omega^2$ *is* the line element on 2-sphere.

Radius of the apparent horizon, $r_{\rm A}(t)$ is found from the condition $(\nabla_\mu R)(\nabla^\mu R) = 0$ where $R(t,r) = a(t)r$ is areal radius of the 2-sphere. Then, $r_{\rm A}(t) = \frac{1}{\sqrt{\frac{H^2}{c^2} + \frac{k}{a^2}}} = c\left(H^2 + \frac{kc^2}{a^2}\right)^{-\frac{1}{2}}$ $\left[H = \frac{\dot{a}}{a} \text{ is the Hubble parameter.}\right]$

$$\Longrightarrow \dot{r}_{\rm A} = -\frac{c}{2}\left(H^2 + \frac{kc^2}{a^2}\right)^{-\frac{3}{2}}\left(2H\frac{dH}{dt} - \frac{2kc^2}{a^3}\frac{da}{dt}\right) = -\frac{{r_{\rm A}}^3}{c^2}\left(H\dot{H} - \frac{kc^2}{a^2}\frac{\dot{a}}{a}\right) = -\frac{{r_{\rm A}}^3}{c^2}\left(H\dot{H} - \frac{kc^2}{a^2}H\right) \Longrightarrow \dot{r}_{\rm A} = -\frac{{r_{\rm A}}^3}{c^2}H\left(\dot{H} - \frac{kc^2}{a^2}\right)$$

Assumption: Matter in FLRW universe forms a perfect fluid with the 4-velocity u^μ. Then, energy-momentum tensor, $T_{\mu\nu} = (\varepsilon + P)u_\mu u_\nu + pg_{\mu\nu}$ [$\boldsymbol{\varepsilon}$ **is the energy density** of the perfect fluid, and P is the pressure of the perfect fluid.]

Using the conservation equation, $\nabla^\mu T_{\mu\nu} = 0$, one may then obtain $\boldsymbol{\dot{\varepsilon} + 3H(\varepsilon + P) = 0} \Longrightarrow \dot{\varepsilon} = -3H(\varepsilon + P)$

Volume enclosed by the apparent horizon, i.e. the apparent volume of the universe, $V = \frac{4}{3}\pi {r_{\rm A}}^3 \Longrightarrow dV = 4\pi {r_{\rm A}}^2 dr_{\rm A}$

Misner-Sharp energy corresponding to the total matter within the apparent horizon, $E = \varepsilon V = \frac{4}{3}\varepsilon\pi {r_{\rm A}}^3$

$$\Longrightarrow dE = 4\varepsilon\pi {r_{\rm A}}^2\, dr_{\rm A} + \frac{4}{3}\pi {r_{\rm A}}^3 d\varepsilon \Longrightarrow dE = 4\varepsilon\pi {r_{\rm A}}^2 dr_{\rm A} - 4\pi {r_{\rm A}}^3(\varepsilon + P)H dt \quad [\because d\rho = \dot{\rho}\, dt = 3H(\varepsilon + P)dt]$$

Work density, W, associated with the perfect fluid, is extracted from the energy-momentum tensor of the perfect fluid by projecting $T_{\mu\nu}$ onto the normal direction to the apparent horizon, i.e. $W = -\frac{1}{2}h^{\mu\nu}T_{\mu\nu}$, where $h^{\mu\nu}$ is the 2-metric on the normal direction. One then finds, $W = \frac{1}{2}(\varepsilon - P)$

$$\left.\begin{matrix} \textbf{1st law of Thermodynamics: } dE = \delta Q + W\, dV \\ \textbf{2nd law of Thermodynamics: } \delta Q = T\, dS \end{matrix}\right\} \Longrightarrow dE = T\, dS + W\, dV \Longrightarrow T\, dS = dE - W\, dV$$

$$\Longrightarrow T\, dS = \{4\varepsilon\pi {r_{\rm A}}^2 dr_{\rm A} - 4\pi {r_{\rm A}}^3(\varepsilon + P)H\, dt\} - \left\{\frac{1}{2}(\varepsilon - P)4\pi {r_{\rm A}}^2 dr_{\rm A}\right\} = 4\pi {r_{\rm A}}^2 dr_{\rm A}\left\{\varepsilon - \frac{1}{2}(\varepsilon - P)\right\} - 4\pi {r_{\rm A}}^3(\varepsilon + P)H\, dt$$

$$\Longrightarrow T\, dS = 4\pi {r_{\rm A}}^2 \dot{r}_{\rm A}\left\{\frac{1}{2}(\varepsilon + P)\right\}dt - 4\pi {r_{\rm A}}^3(\varepsilon + P)H\, dt \quad [\because dr_{\rm A} = \dot{r}_{\rm A}\, dt] = 4\pi {r_{\rm A}}^3(\varepsilon + P)\left(\frac{\dot{r}_{\rm A}}{2Hr_{\rm A}} - 1\right)H\, dt$$

Adopting **Hawking temperature** for the apparent horizon by associating to it the temperature, $T = \frac{\hbar\kappa}{2\pi c k_{\rm B}}$ where κ is the surface gravity evaluated on the apparent horizon. One thus finds, $\kappa = \frac{1}{2\sqrt{-h}}\partial_a\left(\sqrt{-h}h^{ab}\partial_b R\right) \Longrightarrow \kappa = \frac{c^2}{r_{\rm A}}\left(\frac{\dot{r}_{\rm A}}{2Hr_{\rm A}} - 1\right)$

[h^{ab} is the 2-metric of the (t,r)-space defined by FLRW metric.]

Adopting **Bekenstein-Hawking area law** for the apparent horizon and associating to it the entropy, $S = \frac{k_{\rm B}A}{4{L_{\rm P}}^2} = \frac{k_{\rm B}c^3 A}{4G\hbar}$ where $A = 4\pi {r_{\rm A}}^2$ is the area of the apparent horizon, $L_{\rm P}$ is the Planck length, and $k_{\rm B}$ is the Boltzmann constant.

$$\therefore dS = \frac{k_{\rm B}c^3}{4G\hbar}dA = \frac{k_{\rm B}c^3}{G\hbar}(2\pi r_{\rm A}\, dr_{\rm A}) \Longrightarrow T\, dS = \frac{c^2}{G}(\kappa r_{\rm A}\, dr_{\rm A}) \quad \left[\because T = \frac{\hbar\kappa}{2\pi c k_{\rm B}}\right] = \frac{c^4}{G}dr_{\rm A}\left(\frac{\dot{r}_{\rm A}}{2Hr_{\rm A}} - 1\right) \quad \left[\because \kappa = \frac{c^2}{r_{\rm A}}\left(\frac{\dot{r}_{\rm A}}{2Hr_{\rm A}} - 1\right)\right]$$

$$\left.\begin{aligned} T\,dS &= 4\pi r_A{}^3(\varepsilon+p)\left(\frac{\dot r_A}{2Hr_A}-1\right)H\,dt \\ T\,dS &= \frac{c^4}{G}dr_A\left(\frac{\dot r_A}{2Hr_A}-1\right)\end{aligned}\right\} \Rightarrow 4\pi r_A{}^3(\varepsilon+P)H\,dt = \frac{c^4}{G}dr_A \Rightarrow 4\pi r_A{}^3 G(\varepsilon+P)H = c^4\,\dot r_A$$

$$\Rightarrow 4\pi G(\varepsilon+P) = -c^2\left(\dot H - \frac{kc^2}{a^2}\right)\quad \left[\because \dot r_A = -\frac{r_A{}^3}{c^2}H\left(\dot H-\frac{kc^2}{a^2}\right)\right] \Rightarrow \dot H - \frac{kc^2}{a^2} = \frac{4\pi G\dot\rho}{3c^2H}\quad [\because \dot\varepsilon = -3H(\varepsilon+P)]$$

$$\Rightarrow H - \frac{kc^2}{a^2}H\,dt = \frac{4\pi G}{3c^2}d\varepsilon \Rightarrow H\,dH - \frac{kc^2\,da}{a^3}dt = \frac{4\pi G}{3c^2}d\varepsilon\quad\left[\because H=\frac{\dot a}{a}\right] \Rightarrow \int H\,dH - kc^2\int\frac{da}{a^3}dt = \frac{4\pi G\varepsilon}{3c^2}$$

$$\Rightarrow \frac{H^2}{2}+\frac{kc^2}{2a^2} = \frac{4\pi G\varepsilon}{3c^2} \Rightarrow \boldsymbol{H^2+\frac{kc^2}{a^2}=\frac{8\pi G\varepsilon}{3c^2}}$$ which is the **Friedmann equation** for an FLRW universe.

Note: If ρ is the mass density, $\boldsymbol{H^2+\frac{kc^2}{a^2}=\frac{8\pi G\rho}{3}}$

Modified Friedmann equation

The previous derivation critically depends on the use of linear area-entropy law, $S=\frac{A}{4G}$ i. e. $S\propto A$ for the apparent horizon. Any modification of this relation will modify the Friedmann equation for an FLRW universe. To examine the consequences of such a modification, consider the modified entropy-area law for the apparent horizon, $S=\frac{f(A)}{4G}$ where $f(A)$ is any arbitrary smooth function of the area A of apparent horizon. Then, $dS=\frac{f'(A)}{4G}dA=\frac{f'(A)}{G}2\pi r_A\,dr_A$

$$\Rightarrow T\,dS = \frac{f'(A)}{G}\kappa r_A\,dr_A\quad\left[\because T=\frac{\kappa}{2\pi}\right] \Rightarrow T\,dS = \frac{f'(A)\,dr_A}{G}\left(\frac{\dot r_A}{2Hr_A}-1\right)\quad\left[\because \kappa=\frac{1}{r_A}\left(\frac{\dot r_A}{2Hr_A}-1\right)\right]$$

$$\left.\begin{aligned} T\,dS &= 4\pi r_A{}^3(\rho+p)\left(\frac{\dot r_A}{2Hr_A}-1\right)H\,dt \\ T\,dS &= \frac{f'(A)\,dr_A}{G}\left(\frac{\dot r_A}{2Hr_A}-1\right)\end{aligned}\right\} \Rightarrow 4\pi r_A{}^3(\rho+p)H\,dt = \frac{f'(A)\,dr_A}{G} \Rightarrow 4\pi r_A{}^3(\rho+p)H = \frac{f'(A)\dot r_A}{G}$$

$$\Rightarrow 4\pi G(\rho+p) = \frac{\dot r_A}{r_A{}^3H}f'(A) \Rightarrow \frac{\dot r_A}{r_A{}^3}f'(A) = -\frac{4\pi G\dot\rho}{3}\quad[\because \dot\rho=-3H(\rho+p)] \Rightarrow \frac{dr_A}{4\pi r_A{}^3}f'(A) = -\frac{G\,d\rho}{3}$$

$$\Rightarrow \frac{dA}{A^2}f'(A) = -\frac{2G\,d\rho}{3}\quad[\because A=4\pi r_A{}^2] \Rightarrow \frac{2G\rho}{3} = -\int\frac{f'(A)}{A^2}dA$$ which is the **modified Friedmann equation**.

Thus, for each different area law of the entropy one associates to the apparent horizon, one gets a different modified Friedmann equation. In accordance with the holographic principle, any modification of the area law arising from the physics of black holes would necessitate a similar modification to the area law that one needs to study the apparent horizon of the universe.

Modified GUP and Apparent Horizon Entropy

Heisenberg's uncertainty principle: $\Delta x\Delta p \ge \frac{\hbar}{2}$

Generalized uncertainty principle (GUP): $\Delta x\Delta p \ge \frac{\hbar}{2}\left(1+\frac{\alpha^2 l_P{}^2}{\hbar^2}\Delta p^2\right)$ [purely quadratic ; factor α unspecified]

Modified GUP: $\Delta x\Delta p \ge \frac{\hbar}{2}\left(1-\left(\frac{4\sqrt{\mu}}{3}\right)\frac{\alpha l_P}{\hbar}\Delta p + 2(1+\mu)\frac{\alpha^2 l_P{}^2}{\hbar^2}\Delta p^2\right)$ [factor α has an upper bound to be determined experimentally, and $\mu=(2.82/\pi)^2$]

$$\Rightarrow \hbar(1+\mu)\frac{\alpha^2 l_P{}^2}{\hbar^2}\Delta p^2 - \left(\left(\frac{2\hbar\sqrt{\mu}}{3}\right)\frac{\alpha l_P}{\hbar}+\Delta x\right)\Delta p + \frac{\hbar}{2} \le 0$$ [quadratic inequality]

$$\Rightarrow \Delta p \in \frac{\left(\left(\frac{2\hbar\sqrt{\mu}}{3}\right)\frac{\alpha l_P}{\hbar}+\Delta x\right) \pm \sqrt{\left(\left(\frac{2\hbar\sqrt{\mu}}{3}\right)\frac{\alpha l_P}{\hbar}+\Delta x\right)^2 - 2\hbar^2(1+\mu)\frac{\alpha^2 l_P{}^2}{\hbar^2}}}{2\hbar(1+\mu)\frac{\alpha^2 l_P{}^2}{\hbar^2}}$$ [second order term in the quadratic is +ve, and inequality is lower (less-than), so the two solutions enclose the variable.]

$$\Rightarrow \Delta p \in \frac{\left(\left(\frac{2\sqrt{\mu}}{3}\right)\frac{\alpha l_P}{\hbar}+\Delta x\right) \pm \sqrt{\left(\left(\frac{2\sqrt{\mu}}{3}\right)\frac{\alpha l_P}{\hbar}+\Delta x\right)^2 - \gamma^2}}{\gamma^2}\quad\left[\gamma=\sqrt{2(1+\mu)}\frac{\alpha l_P}{\hbar}\right]$$

$$\Longrightarrow \Delta p \in \frac{\Delta x}{\gamma^2} + \frac{2}{3\gamma}\sqrt{\frac{\mu}{2(1+\mu)}} \pm \sqrt{\left(\frac{\Delta x}{\gamma^2} + \frac{2}{3\gamma}\sqrt{\frac{\mu}{2(1+\mu)}}\right)^2 - \frac{1}{\gamma^2}}$$

In the limit $l_P \to 0, \gamma \to 0$, so that $\Delta p \in \frac{\Delta x}{\gamma^2} + \frac{2}{3\gamma}\sqrt{\frac{\mu}{2(1+\mu)}} \pm \sqrt{\left(\frac{\Delta x}{\gamma^2} + \frac{2}{3\gamma}\sqrt{\frac{\mu}{2(1+\mu)}}\right)^2}$

$$\Longrightarrow \Delta p \in \frac{\Delta x}{\gamma^2} + \frac{2}{3\gamma}\sqrt{\frac{\mu}{2(1+\mu)}} \pm \left(\frac{\Delta x}{\gamma^2} + \frac{2}{3\gamma}\sqrt{\frac{\mu}{2(1+\mu)}}\right) \Longrightarrow \Delta p \in \left[0, \frac{2\Delta x}{\gamma^2} + \frac{4}{3\gamma}\sqrt{\frac{\mu}{2(1+\mu)}}\right] \Longrightarrow \Delta p \geq 0$$

$$\therefore \Delta p \geq \frac{\Delta x}{\gamma^2} + \frac{2}{3\gamma}\sqrt{\frac{\mu}{2(1+\mu)}} - \sqrt{\left(\frac{\Delta x}{\gamma^2} + \frac{2}{3\gamma}\sqrt{\frac{\mu}{2(1+\mu)}}\right)^2 - \frac{1}{\gamma^2}}$$

Bethe-Weizsäcker semi-empirical mass formula (SEMF)

Atomic mass, $M(A,Z) = ZM_H + Nm_n - \frac{E_B}{c^2}$ $\left[\begin{matrix} Z\text{: atomic number ; } N\text{: neutron number ; } E_B\text{: nuclear binding energy} \\ M_H\text{: mass of Hydrogen } (^1H) \text{ atom ; } m_n\text{: mass of a neutron}\end{matrix}\right]$

$\Longrightarrow M(A,Z) = ZM_H + (A-Z)m_n - \frac{E_B}{c^2}$ $[\because$ nucleon number, or mass number, $A = N + Z \Longrightarrow N = A - Z]$

Nuclear mass, $m(A,Z) = Zm_p + Nm_n - \frac{E_B}{c^2}$ $[m_p\text{: mass of proton}]$

$\Longrightarrow m(A,Z) = Zm_p + (A-Z)m_n - \frac{E_B}{c^2}$

$m(A,Z) = M(A,Z) - Zm_e - \frac{E_b}{c^2}$ $[E_b\text{: electron binding energy}] \approx M(A,Z) - Zm_e$ $\left[\because \frac{E_I}{c^2} \ll m_e\right]$

nuclear mass, and hence, atomic mass is of the order of GeV ; nuclear binding energy is of the order of MeV
electron mass is of the order of keV ; electron binding energy (ionization potential) is of the order of eV

$m(A,Z) = Zm_p + Nm_n - \frac{E_B}{c^2} \Longrightarrow (Zm_p + Nm_n) - m(A,Z) = \frac{E_B}{c^2} \Longrightarrow \frac{(Zm_p + Nm_n) - m(A,Z)}{m(A,Z)} = \frac{E_B}{m(A,Z)c^2} =$ packing fraction

Volume energy, $E_V(A) = a_V A$

Surface energy, $E_S(A) = -a_S A^{2/3}$ $\left[\begin{matrix}\because \text{Nuclear radius, } R = R_0 A^{1/3}\text{, and Surface area} = 4\pi R^2 = 4\pi {R_0}^2 A^{2/3} \\ \text{significant for low-}A\text{ values when surface-area to volume ratio is more}\end{matrix}\right]$

Coulomb energy, $E_C(A,Z) = -a_C \frac{Z^2}{A^{1/3}}$ [significant for high-Z values]

$$\left[\begin{matrix}\because \text{Coulomb potential energy of a uniform spherical charge distribution} = \frac{3}{5}\left(\frac{1}{4\pi\varepsilon_0}\right)\frac{Q^2}{R} = \frac{3}{5}\left(\frac{1}{4\pi\varepsilon_0}\right)\frac{(Ze)^2}{R_0 A^{1/3}} \\ = \frac{3}{5}\left(\frac{e^2}{4\pi\varepsilon_0 R_0}\right)\frac{Z^2}{A^{1/3}} = a_C\frac{Z^2}{A^{1/3}} \text{ where } a_C = \frac{3}{5}\left(\frac{e^2}{4\pi\varepsilon_0 R_0}\right) \approx 0.69 \text{ MeV with } R_0 = 1.25 \text{ fm}\end{matrix}\right]$$

Asymmetry energy, $E_A(A,Z) = -a_A\frac{(N-Z)^2}{A} = -a_A\frac{(A-2Z)^2}{A}$ [significant for high-A values when $N - Z$ is more]

Pairing energy, $\delta(A,Z) = \begin{cases} 0 & \text{if } A \text{ is odd} \\ +\frac{a_P}{A^k} & \text{if } A \text{ is even, and both } N \,\&\, Z \text{ are even} \\ -\frac{a_P}{A^k} & \text{if } A \text{ is even, and both } N \,\&\, Z \text{ are odd}\end{cases}$ $\left[\text{where } k = \frac{1}{2} \text{ or } \frac{3}{4}\right]$

$\therefore$ Binding energy, $E_B(A,Z) = a_V A - a_S A^{2/3} - a_C\frac{Z^2}{A^{1/3}} - a_A\frac{(A-2Z)^2}{A} + \delta$

For least-squares fit:

If $k = \frac{1}{2}$ in the asymmetry term, then $a_V = 15.8$ MeV ; $a_S = 18.3$ MeV ; $a_C = 0.714$ MeV ; $a_A = 23.2$ MeV ; $a_P = 12$ MeV

If $k = \frac{3}{4}$ in the asymmetry term, then $a_V = 15.76$ MeV ; $a_S = 17.8$ MeV ; $a_C = 0.711$ MeV ; $a_A = 23.7$ MeV ; $a_P = 34$ MeV

If $N = Z = \frac{A}{2}$, $E_B(A,Z) = a_V A - a_S A^{2/3} - a_C\frac{A^{5/3}}{4} + \delta$ [Compare this with Fermi's gas model discussed later.]

Nucleon Separation Energy

Neutron separation energy, $S_n = E_B(A,Z) - E_B(A-1,Z) \approx \frac{\partial E_B}{\partial A} dA = \frac{\partial E_B}{\partial A}$ $[\because dA = 1] \Longrightarrow S_n = A\frac{\partial(E_B/A)}{\partial A} + \frac{E_B}{A}$

$\left[\because \frac{\partial(E_B/A)}{\partial A} = \frac{1}{A}\frac{\partial E_B}{\partial A} - \frac{E_B}{A^2}\right]$ This expression is useful when the (E_B/A) vs A plot is provided.

$$\frac{\partial E_B}{\partial A} = a_V - \frac{2}{3}a_S A^{-1/3} + a_C\frac{Z^2}{3A^{4/3}} - a_A\left(-\frac{(A-2Z)^2}{A^2} + \frac{2(A-2Z)}{A}\right) = a_V - \frac{2}{3}a_S A^{-1/3} + a_C\frac{Z^2}{3A^{4/3}} - a_A\left(1 - \frac{4Z^2}{A^2}\right)$$
$$= a_V - \frac{2}{3}a_S A^{-1/3} + a_C\frac{Z^2}{3A^{4/3}} - a_A\left(1 - \frac{4Z^2}{A^2}\right) \Longrightarrow S_n = (a_V - a_A) + \frac{Z^2}{A^2}\left(\frac{1}{3}a_C A^{2/3} + a_A\right) - \frac{2}{3}a_S A^{-1/3}$$

Proton separation energy, $S_p = E_B(A,Z) - E_B(A-1,Z-1) \approx \frac{\partial E_B}{\partial A}dA + \frac{\partial E_B}{\partial Z}dZ = \frac{\partial E_B}{\partial A} + \frac{\partial E_B}{\partial Z}$ $[\because dA = dZ = 1]$

$$\frac{\partial E_B}{\partial Z} = -a_C\frac{2Z}{A^{1/3}} + a_A\frac{4(A-2Z)}{A}$$
$$\therefore S_p = (a_V - a_A) + \frac{Z^2}{A^2}\left(\frac{1}{3}a_C A^{2/3} + a_A\right) - \frac{2}{3A^{1/3}}(a_S + 3a_C Z) + 4a_A\left(1 - \frac{2Z}{A}\right)$$

The above results have been derived without considering the pairing term in the SEMF. If pairing term is considered, then following cases arise for the parent nucleus:

odd Z-odd N (daughter will be odd A-odd Z from neutron separation and odd A-even Z from proton separation):
$$\begin{cases} S_n = E_B(A,Z) - E_B(A-1,Z) = \frac{\partial E_B}{\partial A} - \frac{a_P}{A^k} \\ S_p = E_B(A,Z) - E_B(A-1,Z-1) = \frac{\partial E_B}{\partial A} + \frac{\partial E_B}{\partial Z} - \frac{a_P}{A^k} \end{cases}$$

even Z-even N (daughter will be odd A-even Z from neutron separation and odd A-odd Z from proton separation):
$$\begin{cases} S_n = E_B(A,Z) - E_B(A-1,Z) = \frac{\partial E_B}{\partial A} + \frac{a_P}{A^k} \\ S_p = E_B(A,Z) - E_B(A-1,Z-1) = \frac{\partial E_B}{\partial A} + \frac{\partial E_B}{\partial Z} + \frac{a_P}{A^k} \end{cases}$$

odd Z-even N (daughter will be even A-odd Z from neutron separation and even A-even Z from proton separation):
$$\begin{cases} S_n = E_B(A,Z) - E_B(A-1,Z) = \frac{\partial E_B}{\partial A} + \frac{a_P}{(A-1)^k} \\ S_p = E_B(A,Z) - E_B(A-1,Z-1) = \frac{\partial E_B}{\partial A} + \frac{\partial E_B}{\partial Z} - \frac{a_P}{(A-1)^k} \end{cases}$$

even Z-odd N (daughter will be even A-even Z from neutron separation and even A-odd Z from proton separation):
$$\begin{cases} S_n = E_B(A,Z) - E_B(A-1,Z) = \frac{\partial E_B}{\partial A} - \frac{a_P}{(A-1)^k} \\ S_p = E_B(A,Z) - E_B(A-1,Z-1) = \frac{\partial E_B}{\partial A} + \frac{\partial E_B}{\partial Z} + \frac{a_P}{(A-1)^k} \end{cases}$$

In the above four sets of formulae, E_B does not include the pairing term.

Application of SEMF: alpha decay of heavy nuclei

α-decay: ${}^{A}_{Z}\mathrm{X} \longrightarrow {}^{A-4}_{Z-2}\mathrm{Y} + {}^{4}_{2}\mathrm{He}$

α-disintegration energy, $Q_\alpha = \{M(A,Z) - M(A-4,Z-2) - M({}^4\mathrm{He})\}c^2$

$$\Longrightarrow Q_\alpha = \left\{ZM_H + (A-Z)m_n - \frac{E_B(A,Z)}{c^2}\right\}c^2 - \left\{(Z-2)M_H + \big((A-4)-(Z-2)\big)m_n - \frac{E_B(A-4,Z-2)}{c^2}\right\}c^2$$
$$- \left\{2M_H + 2m_n - \frac{E_B({}^4\mathrm{He})}{c^2}\right\}c^2$$
$$\Longrightarrow Q_\alpha = \{ZM_H - (Z-2)M_H - 2M_H\}c^2 + \{(A-Z)m_n - (A-Z-2)m_n - 2m_n\}c^2$$
$$-\{E_B(A,Z) - E_B(A-4,Z-2) - E_B({}^4\mathrm{He})\}$$
$$\Longrightarrow Q_\alpha = E_B({}^4\mathrm{He}) + E_B(A-4,Z-2) - E_B(A,Z)$$
$$= E_B({}^4\mathrm{He}) + \left\{a_V(A-4) - a_S(A-4)^{2/3} - a_C\frac{(Z-2)^2}{(A-4)^{1/3}} - a_A\frac{(A-2Z)^2}{(A-4)}\right\} - \left\{a_V A - a_S A^{2/3} - a_C\frac{Z^2}{A^{1/3}} - a_A\frac{(A-2Z)^2}{A}\right\}$$
$$= E_B({}^4\mathrm{He}) - 4a_V + a_S\left\{A^{2/3} - (A-4)^{2/3}\right\} + a_C\left\{\frac{Z^2}{A^{1/3}} - \frac{(Z-2)^2}{(A-4)^{1/3}}\right\} + a_A(A-2Z)^2\left\{\frac{1}{A} - \frac{1}{(A-4)}\right\}$$
$$= E_B({}^4\mathrm{He}) - 4a_V + a_S A^{2/3}\left\{1 - \left(1 - \frac{4}{A}\right)^{2/3}\right\} + a_C\left\{\frac{Z^2}{A^{1/3}} - \frac{(Z-2)^2}{A^{1/3}\left(1-\frac{4}{A}\right)^{1/3}}\right\} - 4a_A\frac{(A-2Z)^2}{A(A-4)}$$

$$\approx E_B(^4\text{He}) - 4a_V + a_S A^{2/3}\left\{1 - \left(1 - \frac{8}{3A}\right)\right\} + \frac{a_C}{A^{1/3}}\left\{Z^2 - \frac{Z^2 - 4Z + 4}{\left(1 - \frac{4}{3A}\right)}\right\} - 4a_A\frac{(A-2Z)^2}{A(A-4)} \quad [\because A \gg 4]$$

$$= E_B(^4\text{He}) - 4a_V + \frac{8a_S}{3A^{1/3}} + \frac{a_C}{A^{1/3}}\left\{\frac{Z^2 - \frac{4Z^2}{3A}}{\left(1 - \frac{4}{3A}\right)} - \frac{Z^2 - 4Z + 4}{\left(1 - \frac{4}{3A}\right)}\right\} - 4a_A\frac{(A-2Z)^2}{A(A-4)}$$

$$= E_B(^4\text{He}) - 4a_V + \frac{8a_S}{3A^{1/3}} + \frac{4a_C}{A^{1/3}\left(1 - \frac{4}{3A}\right)}\left(-\frac{Z^2}{3A} + Z - 1\right) - 4a_A\frac{(A-2Z)^2}{A(A-4)}$$

$$\Rightarrow Q_\alpha = E_B(^4\text{He}) - 4a_V + \frac{8a_S}{3A^{1/3}} + \frac{4a_C Z}{A^{1/3}}\left(1 - \frac{Z}{3A}\right) - 4a_A\frac{(A-2Z)^2}{A(A-4)} \quad \left[\because Z \gg 1 \text{ and } A \gg \frac{4}{3}\right]$$

$$\Rightarrow Q_\alpha = E_B(^4\text{He}) - 4a_V + \frac{8a_S}{3A^{1/3}} + \frac{4a_C}{A^{1/3}}\left(\frac{2}{5}A\right)\left(1 - \frac{2}{15}\right) - 4a_A\frac{\left(A - \frac{4}{5}A\right)^2}{A^2} \quad \left[\begin{array}{c}\because A \gg 4 \text{ and } A \approx 2.5\,Z \\ \text{for heavy nuclei}\end{array}\right]$$

$$\Rightarrow Q_\alpha = 28.3\text{ MeV} - 4a_V + \frac{8a_S}{3A^{1/3}} + a_C A^{2/3}\left(\frac{104}{75}\right) - \frac{4}{25}a_A \quad [\because E_B(^4\text{He}) = 28.3\text{ MeV}]$$

$$\Rightarrow Q_\alpha = -38.6\text{ MeV} + \frac{48.8\text{ MeV}}{A^{1/3}} + A^{2/3}\text{ MeV} \quad \left[\begin{array}{l}\because a_V = 15.8\text{ MeV};\ a_S = 18.3\text{ MeV} \\ a_C = 0.714\text{ MeV};\ a_A = 23.2\text{ MeV}\end{array}\right]$$

$$Q_\alpha = 0 \Rightarrow -38.6 + \frac{48.8}{A^{1/3}} + A^{2/3} = 0 \Rightarrow A - 38.6A^{1/3} + 48.8 = 0 \Rightarrow x^3 - 38.6x + 48.8 = 0 \quad [x = A^{1/3}]$$

$$\Rightarrow x = 5.444 \Rightarrow A^{1/3} = 5.444 \Rightarrow A \approx 161.34$$

For $A < 5.444^3$ i.e. $A < 161.34$, $Q_\alpha < 0$ whereas for $A > 5.444^3$ i.e. $A > 161.34$, $Q_\alpha > 0$

However, actually α-disntegration is primarily observed for $A > 200$ because for lighter nuclei, the barrier penetration probability is very small.

Application of SEMF: stability of nuclei against beta decay

Mass approach:

$$M(A,Z) = ZM_H + (A-Z)m_n - \frac{E_B}{c^2} = ZM_H + (A-Z)m_n - \frac{1}{c^2}\left(a_V A - a_S A^{2/3} - a_C\frac{Z^2}{A^{1/3}} - a_A\frac{(A-2Z)^2}{A} + \delta\right)$$

$$\Rightarrow M(A,Z) = ZM_H + (A-Z)m_n - \frac{a_V A}{c^2} + \frac{a_S A^{2/3}}{c^2} + \frac{a_C}{c^2}\frac{Z^2}{A^{1/3}} + \frac{a_A}{c^2}\frac{(A^2 + 4Z^2 - 4AZ)}{A} - \delta$$

$$\Rightarrow M(A,Z) = \frac{1}{c^2}\left(Am_n c^2 - a_V A + a_S A^{2/3} + a_A A - \delta\right) + \left(M_H - m_n - \frac{4a_A}{c^2}\right)Z + \frac{1}{c^2}\left(\frac{a_C}{A^{1/3}} + \frac{4a_A}{A}\right)Z^2$$

$\Rightarrow M(A,Z) = f_A + pZ + q_A Z^2$ which is the equation for a parabola for a given A.

where $f_A = \frac{1}{c^2}\left(Am_n c^2 - a_V A + a_S A^{2/3} + a_A A - \delta\right)$; $p = \left(M_H - m_n - \frac{4a_A}{c^2}\right)$; $q_A = \frac{1}{c^2}\left(\frac{a_C}{A^{1/3}} + \frac{4a_A}{A}\right)$

For the most stable isobar, $\left.\left(\frac{\partial M}{\partial Z}\right)_A\right|_{Z=Z_A} = 0 \Rightarrow p + 2q_A Z_A = 0 \Rightarrow Z_A = \frac{-p}{2q_A} = \frac{-\left(M_H - m_n - \frac{4a_A}{c^2}\right)}{\frac{2}{c^2}\left(\frac{a_C}{A^{1/3}} + \frac{4a_A}{A}\right)}$

$$\Rightarrow Z_A = \frac{(-M_H c^2 + m_n c^2 + 4a_A)A}{2(a_C A^{2/3} + 4a_A)} \Rightarrow Z_A \approx \frac{A}{2 + 0.015A^{2/3}}$$

$$M(A,Z) - M(A,Z_A) = (f_A + pZ + q_A Z^2) - (f_A + pZ_A + q_A {Z_A}^2) = (pZ + q_A Z^2) - \left(-\frac{p^2}{2q_A} + \frac{p^2}{4q_A}\right) \left[\because Z_A = \frac{-p}{2q_A}\right]$$

$$\Rightarrow M(A,Z) - M(A,Z_A) = pZ + q_A Z^2 + \frac{p^2}{4q_A} = -2q_A Z_A Z + q_A Z^2 + q_A {Z_A}^2 \quad \left[\because Z_A = \frac{-p}{2q_A}\right] = q_A(Z - Z_A)^2 > 0$$

$\because M(A,Z) - M(A,Z_A) > 0$ $\therefore$ the mass parabola has its lowest point at $Z = Z_A$

$\because M(A,Z) = f_A + pZ + q_A Z^2$ and $f_A = \frac{1}{c^2}\left(Am_n c^2 - a_V A + a_S A^{2/3} + a_A A - \delta\right)$

$\therefore$ the odd A isobars form a single mass parabola, whereas the even A isobars form a double parabola with the odd-odd types' parabola slightly above the even-even types' parabola. This is for $(f_A)_{\text{odd-odd}} > (f_A)_{\text{even-even}}$ as

$$\delta = \begin{cases} +\dfrac{a_P}{A^k} & \text{for even-even} \\ -\dfrac{a_P}{A^k} & \text{for odd-odd} \end{cases}$$

Energy approach:

$$E_B(A,Z) = a_V A - a_S A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - a_A \frac{(A-2Z)^2}{A} + \delta = a_V A - a_S A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - a_A \frac{(A^2 + 4Z^2 - 4AZ)}{A} + \delta$$

$$\Rightarrow E_B(A,Z) = \left(a_V A - a_S A^{2/3} - a_A A + \delta\right) - 4a_A Z - \left(\frac{a_C}{A^{1/3}} + \frac{4a_A}{A}\right) Z^2$$

$\Rightarrow E_B(A,Z) = f_A + pZ + q_A Z^2$ which is the equation for a parabola for a given A.

where $f_A = \left(a_V A - a_S A^{2/3} - a_A A + \delta\right)$; $p = -4a_A$; $q_A = \left(\dfrac{a_C}{A^{1/3}} + \dfrac{4a_A}{A}\right)$

For the most stable isobar, $\left.\left(\dfrac{\partial E_B}{\partial Z}\right)_A\right|_{Z=Z_A} = 0 \Rightarrow p + 2q_A Z_A = 0 \Rightarrow Z_A = \dfrac{-p}{2q_A} = \dfrac{4a_A}{2\left(\dfrac{a_C}{A^{1/3}} + \dfrac{4a_A}{A}\right)}$

$$\Rightarrow Z_A = \frac{4a_A A}{2(a_C A^{2/3} + 4a_A)} \Rightarrow Z_A \approx \frac{A}{2 + 0.015A^{2/3}}$$

$$E_B(A,Z) - E_B(A,Z_A) = (f_A + pZ + q_A Z^2) - \left(f_A + pZ_A + q_A {Z_A}^2\right) = (pZ + q_A Z^2) - \left(-\frac{p^2}{2q_A} + \frac{p^2}{4q_A}\right)\left[\because Z_A = \frac{-p}{2q_A}\right]$$

$$\Rightarrow E_B(A,Z) - E_B(A,Z_A) = pZ + q_A Z^2 + \frac{p^2}{4q_A} = -2q_A Z_A Z + q_A Z^2 + q_A {Z_A}^2 \quad \left[\because Z_A = \frac{-p}{2q_A}\right] = q_A (Z - Z_A)^2 > 0$$

$\because E_B(A,Z) - E_B(A,Z_A) > 0 \;\therefore$ the binding energy parabola has its lowest point at $Z = Z_A$

Application of SEMF: beta disintegration energy of mirror nuclei

Mirror nuclei are a pair of isobaric nuclei in which the proton number and neutron number are interchanged, and differ by only one unit so that $Z - N = 1$ (i.e. odd A) for the higher Z nucleus in the pair. Then, $A = N + Z = 2Z - 1$ [$\because N = Z - 1$]. Examples are ($^3_1$H, $^3_2$He), ($^7_3$Li, $^7_4$Be), ($^{11}_5$B, $^{11}_6$C), ($^{13}_6$C, $^{13}_7$N), ($^{15}_7$N, $^{15}_8$O), etc. The higher Z member of the pair are found to be β^+ (positron) emitters such as $^{11}_6\mathrm{C} \longrightarrow {}^{11}_5\mathrm{B} + \beta^+ + \nu$

For parent atom, $M(A,Z) = ZM_H + Nm_n - \dfrac{E_B}{c^2}$

$$\Rightarrow M(A,Z) = ZM_H + (Z-1)m_n - \frac{1}{c^2}\left(a_V A - a_S A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - a_A \frac{(N-Z)^2}{A}\right) \quad \begin{bmatrix} \because \delta = 0 \\ \text{for odd } A \end{bmatrix}$$

$$\Rightarrow M(A,Z) = ZM_H + (Z-1)m_n - \frac{1}{c^2}\left(a_V A - a_S A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - \frac{a_A}{A}\right) \quad \begin{bmatrix} \because Z - N = 1 \\ \text{for parent atom} \end{bmatrix}$$

For daughter atom, $M(A,Z-1) = (Z-1)M_H + Nm_n - \dfrac{E_B}{c^2}$

$$\Rightarrow M(A,Z-1) = (Z-1)M_H + Zm_n - \frac{1}{c^2}\left(a_V A - a_S A^{2/3} - a_C \frac{(Z-1)^2}{A^{1/3}} - a_A \frac{(N-Z)^2}{A}\right) \quad \begin{bmatrix} \because \delta = 0 \\ \text{for odd } A \end{bmatrix}$$

$$\Rightarrow M(A,Z-1) = (Z-1)M_H + Zm_n - \frac{1}{c^2}\left(a_V A - a_S A^{2/3} - a_C \frac{(Z-1)^2}{A^{1/3}} - \frac{a_A}{A}\right) \quad \begin{bmatrix} \because N - Z = 1 \\ \text{for daughter atom} \end{bmatrix}$$

$$\therefore Q_{\beta^+} = \{m(A,Z) - m(A,Z-1) - m_e\}c^2 = \{(M(A,Z) - Zm_e) - (M(A,Z-1) - (Z-1)m_e) - m_e\}c^2$$

$$\Rightarrow Q_{\beta^+} = [M(A,Z) - M(A,Z-1) - 2m_e]c^2 = \left\{M_H - m_n + \frac{a_C}{c^2}\frac{Z^2 - (Z-1)^2}{A^{1/3}} - 2m_e\right\}c^2$$

$$\Rightarrow Q_{\beta^+} = \left\{M_H - m_n + \frac{a_C}{c^2}\frac{2Z-1}{A^{1/3}} - 2m_e\right\}c^2 = a_C A^{2/3} + \{M_H - m_n - 2m_e\}c^2 \quad [\because A = 2Z - 1]$$

$$\Rightarrow Q_{\beta^+} \approx a_C A^{2/3} - 1.804 \text{ MeV}$$

$$\because a_C = \frac{3}{5}\left(\frac{e^2}{4\pi\varepsilon_0 R_0}\right) \Rightarrow R_0 = \frac{3}{5}\left(\frac{e^2}{4\pi\varepsilon_0 a_C}\right) = \frac{3}{5}\left(\frac{e^2}{4\pi\varepsilon_0}\right)\frac{1}{\left(\text{slope of } Q_{\beta^+} \text{ vs } A^{2/3} \text{ graph}\right)}$$

Fermi gas model of the nucleus

Fermi (maximum) momentum at $T=0$, $p_{F0}=\left(\frac{3h^3N}{4\pi g_s\Omega}\right)^{\frac{1}{3}}=h\left(\frac{3n}{4\pi g_s}\right)^{\frac{1}{3}}$ $\left[\text{nuclear volume, } \Omega=\frac{4}{3}\pi R^3\right]$

Fermi (maximum) energy at $T=0$, $E_{F0}=\frac{{p_{F0}}^2}{2m}=\frac{h^2}{2m}\left(\frac{3n}{4\pi g_s}\right)^{\frac{2}{3}}$

For protons, $n=n_p=\frac{Z}{\frac{4}{3}\pi R^3}=\frac{3Z}{4\pi {R_0}^3 A}\approx\frac{3}{8\pi {R_0}^3}$ $\left[\text{assuming } Z=\frac{A}{2}\right]$

For neutrons, $n=n_n=\frac{N}{\frac{4}{3}\pi R^3}=\frac{3(A-Z)}{4\pi {R_0}^3 A}$ $[\because N=A-Z]\approx\frac{3}{8\pi {R_0}^3}$ $\left[\text{assuming } Z=\frac{A}{2}\right]$

$\therefore n_p=n_n\approx\frac{3}{8\pi {R_0}^3}\approx 0.061\text{ nucleon}/(\text{fm})^3$

$$E_{F0}=\frac{h^2}{2m}\left(\frac{3n}{8\pi}\right)^{\frac{2}{3}}\ \left[\begin{matrix}g_s=2\text{ for}\\\text{a nucleon}\end{matrix}\right]=\frac{h^2}{2m}\left(\frac{3}{8\pi}\left(\frac{3}{8\pi {R_0}^3}\right)\right)^{\frac{2}{3}}=\frac{h^2}{2m}\left(\frac{9}{64\pi^2 {R_0}^3}\right)^{\frac{2}{3}}=\frac{9h^2}{32m{R_0}^2(3\pi^2)^{\frac{2}{3}}}\approx 31\text{ MeV}$$

$$\text{For protons, } E_{F0,p}=\frac{h^2}{2m_p}\left(\frac{3n_p}{8\pi}\right)^{\frac{2}{3}}=\frac{h^2}{2m_p}\left(\frac{3}{8\pi}\left(\frac{3Z}{4\pi {R_0}^3A}\right)\right)^{\frac{2}{3}}=\frac{h^2}{2m_p}\left(\frac{9Z}{32\pi^2{R_0}^3A}\right)^{\frac{2}{3}}=\frac{9h^2}{32m_p{R_0}^2}\left(\frac{2Z}{3\pi^2A}\right)^{\frac{2}{3}}$$
$$=\frac{9\hbar^2}{8m_p{R_0}^2}\left(\frac{2\pi Z}{3A}\right)^{\frac{2}{3}}=\frac{\hbar^2}{2m_p{R_0}^2}\left(\frac{9\pi Z}{4A}\right)^{\frac{2}{3}}$$

$$\text{For neutrons, } E_{F0,n}=\frac{h^2}{2m_n}\left(\frac{3n_n}{8\pi}\right)^{\frac{2}{3}}=\frac{h^2}{2m_n}\left(\frac{3}{8\pi}\left(\frac{3N}{4\pi {R_0}^3A}\right)\right)^{\frac{2}{3}}=\frac{h^2}{2m_n}\left(\frac{9N}{32\pi^2{R_0}^3A}\right)^{\frac{2}{3}}=\frac{9h^2}{32m_n{R_0}^2}\left(\frac{2N}{3\pi^2A}\right)^{\frac{2}{3}}$$
$$=\frac{9\hbar^2}{8m_n{R_0}^2}\left(\frac{2\pi N}{3A}\right)^{\frac{2}{3}}=\frac{\hbar^2}{2m_n{R_0}^2}\left(\frac{9\pi N}{4A}\right)^{\frac{2}{3}}$$

Depth of potential well for protons, $V_{0,p}=E_{F0,p}+E_B/A$

Depth of potential well for neutrons, $V_{0,n}=E_{F0,n}+E_B/A$

$\because N>Z$ for same A, and $m_n\approx m_p$ $\therefore E_{F0,n}>E_{F0,p}\Longrightarrow V_{0,n}>V_{0,p}$

Internal energy (average energy) of fermi gas at $T=0$, $U_0=\frac{3}{5}N_F E_{F0}\Longrightarrow$ Average energy per fermion $=\frac{3}{5}E_{F0}$

Total internal energy of the nucleus, $E(Z,N)=Z\langle U_{0,p}\rangle+N\langle U_{0,n}\rangle=\frac{3}{5}\left(ZE_{F0,p}+NE_{F0,n}\right)$

$$=\frac{3\hbar^2}{10{R_0}^2}\left(\frac{9\pi}{4A}\right)^{\frac{2}{3}}\left(\frac{Z^{\frac{5}{3}}}{m_p}+\frac{N^{\frac{5}{3}}}{m_n}\right)\approx\boldsymbol{\frac{3\hbar^2}{10m{R_0}^2}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}}\left(\frac{Z^{\frac{5}{3}}+N^{\frac{5}{3}}}{A^{\frac{2}{3}}}\right)}=\frac{3\hbar^2}{10m{R_0}^2}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}}\left(\frac{Z^{\frac{5}{3}}+(A-Z)^{\frac{5}{3}}}{A^{\frac{2}{3}}}\right)$$

For minimum energy, $\frac{\partial E}{\partial Z}=0\Longrightarrow\frac{\partial}{\partial Z}\left(\frac{Z^{\frac{5}{3}}+(A-Z)^{\frac{5}{3}}}{A^{\frac{2}{3}}}\right)=0\Longrightarrow\frac{5}{3}Z^{\frac{2}{3}}-\frac{5}{3}(A-Z)^{\frac{2}{3}}=0\Longrightarrow Z=A-Z\Longrightarrow Z=N$

Let $N-Z=\delta$ and $\frac{\delta}{A}\ll 1$. Then, $\begin{cases}N=A-Z=A-(N-\delta)\Longrightarrow 2N=A+\delta\Longrightarrow N=\frac{A}{2}\left(1+\frac{\delta}{A}\right)\\ Z=A-N=A-(Z+\delta)\Longrightarrow 2Z=A-\delta\Longrightarrow Z=\frac{A}{2}\left(1-\frac{\delta}{A}\right)\end{cases}$

$$\therefore E(Z,N)=\frac{3\hbar^2}{10m{R_0}^2}\left(\frac{9\pi}{8}\right)^{\frac{2}{3}}\frac{A}{2}\left\{\left(1-\frac{\delta}{A}\right)^{\frac{5}{3}}+\left(1+\frac{\delta}{A}\right)^{\frac{5}{3}}\right\}$$
$$=\frac{3\hbar^2}{10m{R_0}^2}\left(\frac{9\pi}{8}\right)^{\frac{2}{3}}\frac{A}{2}\left\{\left(1-\frac{5\delta}{3A}+\frac{1}{2!}\cdot\frac{5}{3}\left(\frac{5}{3}-1\right)\left(\frac{\delta}{A}\right)^2-\cdots\right)+\left(1+\frac{5\delta}{3A}+\frac{1}{2!}\cdot\frac{5}{3}\left(\frac{5}{3}-1\right)\left(\frac{\delta}{A}\right)^2-\cdots\right)\right\}$$

$$= \frac{3\hbar^2}{10mR_0{}^2}\left(\frac{9\pi}{8}\right)^{\frac{2}{3}} A\left\{1+\frac{5}{9}\left(\frac{\delta}{A}\right)^2+\cdots\right\} \approx \frac{3\hbar^2}{10mR_0{}^2}\left(\frac{9\pi}{8}\right)^{\frac{2}{3}}\left\{A+\frac{5}{9}\frac{(N-Z)^2}{A}\right\} \approx a_F A+\frac{5}{9}a_F\frac{(N-Z)^2}{A}$$

where $a_F = \frac{3\hbar^2}{10mR_0{}^2}\left(\frac{9\pi}{8}\right)^{\frac{2}{3}} \approx 18.6$ MeV. The first term proportional to A corresponds to the volume term in SEMF, and the second term proportional to $\frac{(N-Z)^2}{A}$ corresponds to the asymmetry term in SEMF; however, the value of the coefficient of latter (≈ 10.6 MeV) is only half the measured value due to the inaccuracy of the model that underestimates the interaction between the two kinds of nucleons.

Nuclear magnetic moment

Nuclear magnetic moment, $\vec{\mu} = \vec{\mu}_L + \vec{\mu}_S$

where, $\begin{cases} \text{orbital magnetic moment, } \vec{\mu}_L = \sum_Z \vec{\mu}_l + \sum_N \vec{\mu}_l \\ \text{spin magnetic moment, } \vec{\mu}_S = \sum_Z \vec{\mu}_s + \sum_N \vec{\mu}_s \end{cases}$ and, $\begin{cases} \text{For protons, } |\vec{\mu}_s| = \mu_p = g_p\mu_N \\ \text{For neutrons, } |\vec{\mu}_s| = \mu_n = g_n\mu_N \end{cases}$

where, $\begin{cases} \text{proton } g\text{-factor, } g_p \approx +5.5857 \\ \text{neutron } g\text{-factor, } g_n \approx -3.8261 \end{cases}$ and, nuclear magneton, $\mu_N = \frac{e\hbar}{2m_p} \approx 5.051 \times 10^{-27}$ J/T

Magnetic moment of nucleus with spin (total angular momentum) I, $|\vec{\mu}| = \mu_I = g_I\sqrt{I(I+1)}\mu_N$

where g_I is the effective g-factor of the nucleus.

Measured magnetic moment, $\mu_z = \mu_I \cos\left(\vec{I}\cdot\vec{B}\right) = \frac{\mu_I m_I}{\sqrt{I(I+1)}} = g_I\mu_N m_I \approx g_I\mu_N I \quad [m_I = I, I-1, \dots -I+1, -I]$

As per extreme single particle shell model, for an even number of nucleons of one kind, $I = 0$, i.e. $(\mu_I)_{\text{e-e}} = 0$

In an odd A nucleus or an odd-odd even A nucleus, the last odd nucleon (proton or neutron) determines the magnetic moment. For such a nucleus, $I = j$ [j is the total angular momentum of the last (unpaired) nucleon.]

$\therefore$ Total angular moment, $\vec{\mu}_j = \vec{\mu}_l + \vec{\mu}_s$ where, $\begin{cases} \vec{\mu}_l \text{ is the orbital magnetic moment of the unpaired nucleon} \\ \vec{\mu}_s \text{ is the intrinsic magnetic moment of the unpaired nucleon} \end{cases}$

and, $\mu_l = g_l\sqrt{l(l+1)}\mu_N$ is the magnetic moment due to orbital motion of a nucleon of angular momentum $l\hbar$

$\begin{cases} \mu_{l,p} = \sqrt{l(l+1)}\mu_N \quad [\because g_l = 1 \text{ for proton}] \\ \mu_{l,n} = 0 \quad [\because g_l = 0 \text{ for neutron as it is neutral}] \end{cases}$ and, $\mu_s = g_s\sqrt{s(s+1)}\mu_N \quad \begin{bmatrix} g_s = g_p \text{ for proton} \\ g_s = g_n \text{ for proton} \end{bmatrix}$ where $s = \frac{1}{2}$

$\therefore$ Total magnetic moment in the direction of j, $\mu_I = \mu_j = \mu_l \cos(l,j) + \mu_s \cos(s,j)$

$$\Longrightarrow \mu_I = \mu_j = g_l\sqrt{l(l+1)}\mu_N \cos(l,j) + g_s\sqrt{s(s+1)}\mu_N \cos(s,j) \quad \begin{bmatrix} \because \mu_l = g_l\sqrt{l(l+1)}\mu_N \text{ where } l = j \pm \frac{1}{2} \\ \text{and, } \mu_s = g_s\sqrt{s(s+1)}\mu_N \text{ where } s = \frac{1}{2} \end{bmatrix}$$

From cosine law, $\cos(l,j) = \frac{j(j+1)+l(l+1)-s(s+1)}{2\sqrt{j(j+1)}\sqrt{l(l+1)}}$ and $\cos(s,j) = \frac{j(j+1)+\sqrt{s(s+1)}-l(l+1)}{2\sqrt{j(j+1)}\sqrt{s(s+1)}}$

$$\therefore \mu_I = \mu_j = g_l\mu_N\frac{j(j+1)+l(l+1)-s(s+1)}{2\sqrt{j(j+1)}} + g_s\mu_N\frac{j(j+1)+s(s+1)-l(l+1)}{2\sqrt{j(j+1)}}$$

$$\Longrightarrow g_j = g_l\frac{j(j+1)+l(l+1)-s(s+1)}{2j(j+1)} + g_s\frac{j(j+1)+s(s+1)-l(l+1)}{2j(j+1)} \quad \begin{bmatrix} \because \mu_j = g_j\sqrt{j(j+1)} \\ \text{because } I = j \end{bmatrix}$$

$$\therefore \mu_z \approx g_I I\mu_N = g_j j\mu_N = g_l\mu_N\frac{j(j+1)+l(l+1)-s(s+1)}{2(j+1)} + g_s\mu_N\frac{j(j+1)+s(s+1)-l(l+1)}{2(j+1)}$$

$$\Rightarrow \begin{cases} \mu_z = g_l \mu_N \dfrac{j(j+1) + \left(j - \frac{1}{2}\right)\left(j + \frac{1}{2}\right) - \frac{3}{4}}{2(j+1)} + g_s \mu_N \dfrac{j(j+1) + \frac{3}{4} - \left(j - \frac{1}{2}\right)\left(j + \frac{1}{2}\right)}{2(j+1)} & \text{for } l = j - \dfrac{1}{2} \\ \mu_z = g_l \mu_N \dfrac{j(j+1) + \left(j + \frac{1}{2}\right)\left(j + \frac{3}{2}\right) - \frac{3}{4}}{2(j+1)} + g_s \mu_N \dfrac{j(j+1) + \frac{3}{4} - \left(j + \frac{1}{2}\right)\left(j + \frac{3}{2}\right)}{2(j+1)} & \text{for } l = j + \dfrac{1}{2} \end{cases}$$

$$\Rightarrow \begin{cases} \mu_z = g_l \mu_N \dfrac{2j^2 + j - 1}{2(j+1)} + g_s \mu_N \dfrac{j+1}{2(j+1)} & \text{for } l = j - \dfrac{1}{2} \\ \mu_z = g_l \mu_N \dfrac{2j^2 + 3j}{2(j+1)} + g_s \mu_N \dfrac{-j}{2(j+1)} & \text{for } l = j + \dfrac{1}{2} \end{cases} \Rightarrow \begin{cases} \mu_z = g_l \mu_N \dfrac{(2j-1)(j+1)}{2(j+1)} + \dfrac{g_s \mu_N}{2} & \text{for } l = j - \dfrac{1}{2} \\ \mu_z = g_l \mu_N \dfrac{j(2j+3)}{2(j+1)} - g_s \mu_N \dfrac{j}{2(j+1)} & \text{for } l = j + \dfrac{1}{2} \end{cases}$$

$$\Rightarrow \begin{cases} \boldsymbol{\mu_z = \mu_N \left(g_l \left(j - \frac{1}{2}\right) + \frac{g_s}{2}\right)} & \textbf{for } \boldsymbol{l = j - \frac{1}{2}} \\ \boldsymbol{\mu_z = \dfrac{j}{(j+1)} \mu_N \left(g_l \left(j + \frac{3}{2}\right) - \frac{g_s}{2}\right)} & \textbf{for } \boldsymbol{l = j + \frac{1}{2}} \end{cases} \Rightarrow \begin{cases} \mu_z = \mu_N \left(g_l l + \dfrac{g_s}{2}\right) & \text{for } l = j - \dfrac{1}{2} \\ \mu_z = \dfrac{j}{(j+1)} \mu_N \left(g_l (l+1) - \dfrac{g_s}{2}\right) & \text{for } l = j + \dfrac{1}{2} \end{cases}$$

$$\Rightarrow \begin{cases} \mu_z = \mu_N \left(j - \dfrac{1}{2} + \dfrac{g_p}{2}\right) & \text{for } l = j - \dfrac{1}{2} \\ \mu_z = \dfrac{j}{(j+1)} \mu_N \left(j + \dfrac{3}{2} - \dfrac{g_p}{2}\right) & \text{for } l = j + \dfrac{1}{2} \end{cases} \begin{bmatrix} \text{for odd} \\ \text{proton,} \\ g_l = 1 \\ g_s = g_p \end{bmatrix} \text{ and } \begin{cases} \mu_z = \mu_N \dfrac{g_n}{2} & \text{for } l = j - \dfrac{1}{2} \\ \mu_z = \dfrac{-j}{(j+1)} \mu_N \dfrac{g_n}{2} & \text{for } l = j + \dfrac{1}{2} \end{cases} \begin{bmatrix} \text{for odd} \\ \text{neutron,} \\ g_l = 1 \\ g_s = g_p \end{bmatrix}$$

The values for μ_z are called **Schmidt values** and the plots of μ_z vs. j for each case are called **Schmidt diagrams**.

Compute the radii of $_{47}Ag^{107}$ and $_{92}U^{238}$, and show that the radii are related as $R_U = 1.3\ R_{Ag}$

$$R \approx R_0 A^{\frac{1}{3}} \Rightarrow \begin{cases} R_{\text{Ag}} \approx R_0 (107)^{\frac{1}{3}} \\ R_{\text{U}} \approx R_0 (238)^{\frac{1}{3}} \end{cases} \Rightarrow \frac{R_{\text{U}}}{R_{\text{Ag}}} \approx \left(\frac{238}{107}\right)^{\frac{1}{3}} \approx 1.3 \Rightarrow R_{\text{U}} \approx 1.3\ R_{\text{Ag}}$$

For the isobaric family of nuclei with A = 91, obtain (i) nuclear charge of the most stable isobar, (ii) Q_{β^+}, the energy release in β^+ decay.

(i) For the most stable isobar, $Z_A \approx \dfrac{A}{2 + 0.015 A^{2/3}} = \dfrac{91}{2 + 0.015 \times (91)^{2/3}} \approx 39.5$

$\therefore\ Z_A = 39$ or 40 most probably, and so, nuclear charge is $+39e$ or $+40e$. The observed stable isobar is $^{91}_{40}\text{Zr}$.

(ii) $Q_{\beta^+} = -1.804 \text{ MeV} + a_C A^{\frac{2}{3}} \approx 12.58 \text{ MeV}$ [$\because a_C = 0.711$ MeV]

The atomic masses of mirror nuclei ^{15}N and ^{15}O are 15.000108*u* and 15.003092*u* respectively. Determine the Coulomb coefficient a_C in the semi-empirical mass formula.

$$Q_{\beta^+} = -1.804 \text{ MeV} + a_C A^{\frac{2}{3}} \Rightarrow (15.000108 - 15.003092) \times 931.5 \text{ MeV} = -1.804 \text{ MeV} + a_C (15)^{\frac{2}{3}}$$

$$\Rightarrow a_C = \frac{4.563 \text{ MeV}}{15^{\frac{2}{3}}} \approx 0.75 \text{ MeV}$$

Using the atomic masses of mirror nuclei ^{23}Na and ^{23}Mg, calculate the Coulomb coefficient a_C and estimate R_0.

$$Q_{\beta^+} = -1.804 \text{ MeV} + a_C A^{\frac{2}{3}} \Rightarrow (22.994125 - 22.989771) \times 931.5 \text{ MeV} = -1.804 \text{ MeV} + a_C (23)^{\frac{2}{3}}$$

$$\Rightarrow a_C = \frac{4.056 \text{ MeV}}{23^{\frac{2}{3}}} \approx 0.725 \text{ MeV} \quad \therefore R_0 = \frac{3}{5}\left(\frac{e^2}{4\pi\varepsilon_0 a_C}\right) \approx 1.19 \text{ fm}$$

Calculate the binding energy of ^4He using semi-empirical mass formula. Also find binding energy per nucleon.

Binding energy for even-even nucleus, $E_B(A, Z) = a_V A - a_S A^{2/3} - a_C \dfrac{Z^2}{A^{1/3}} - a_A \dfrac{(A - 2Z)^2}{A} + \dfrac{a_P}{A^{3/4}}$

$$\Rightarrow E_B(4,2) \approx 29.08 \text{ MeV} \quad \begin{bmatrix} a_V = 15.76 \text{ MeV};\ a_S = 17.8 \text{ MeV} \\ a_C = 0.711 \text{ MeV};\ a_P = 34 \text{ MeV} \end{bmatrix} \Rightarrow \frac{E_B}{A} = 7.27 \text{ MeV}$$

Calculate the surface energy and Coulomb energy for 236U nucleus in the semi-empirical mass formula.

$$E_S(A) = -a_S A^{2/3} = -17.8\text{ MeV} \times 236^{2/3} \approx -680\text{ MeV}$$

$$E_C(A,Z) = -a_C \frac{Z^2}{A^{1/3}} = -0.711\text{ MeV} \times \frac{92^2}{236^{1/3}} \approx -974\text{ MeV}$$

Using the liquid drop model, find the most stable isobar for (i) A = 25 (ii) A = 75.

(i) For $A = 25, Z_A \approx \dfrac{A}{2+0.015A^{2/3}} = \dfrac{25}{2+0.015\times(25)^{2/3}} \approx 11.75 \approx 12$ which is ${}^{25}_{12}\text{Mg}$.

(ii) For $A = 75, Z_A \approx \dfrac{A}{2+0.015A^{2/3}} = \dfrac{75}{2+0.015\times(75)^{2/3}} \approx 33.09 \approx 33$ which is ${}^{75}_{33}\text{As}$.

Mass spectrometer values are often reported in terms of "mass defect" defined as X = (A – M) and "packing fraction" defined as P = X/A. Compute the mass defect and packing fraction for 4He, 98Mo.

(i) $X = A - M = (Zm_p + Nm_n) - M = (2\times 1.0072765 + 2\times 1.0086649)u - 4.002603u = 0.02928u \equiv 27.274\text{ MeV}$

$\because E_B({}^4_2\text{He}) = 28.3\text{ MeV} > 27.274\text{ MeV} = X,\ \therefore$ formation of ${}^4_2\text{He}$ from the 4 nucleons is endothermic, 1.026 MeV/nucleus.

$$P = \frac{X}{A} = \frac{X}{(Zm_p + Nm_n)} = \frac{0.02928u}{(42\times 1.0072765 + 56\times 1.0086649)u} \approx 0.007262$$

(ii) $X = A - M = (Zm_p + Nm_n) - M = (42\times 1.0072765 + 56\times 1.0086649)u - 97.905409u = 0.885438u \equiv 824.8\text{ MeV}$

$$P = \frac{X}{A} = \frac{X}{(Zm_p + Nm_n)} = \frac{0.885438u}{(42\times 1.0072765 + 56\times 1.0086649)u} \approx 0.008963$$

Find out the minimum energy required to remove a neutron (i.e. neutron separation energy) from 91Zr, 92Zr, 93Zr.

$$S_n = E_B(A,Z) - E_B(A-1,Z) = \begin{cases} E_B(91,40) - E_B(90,40) = 789.611\text{ MeV} - 781.904\text{ MeV} = 7.707\text{ MeV} \\ E_B(92,40) - E_B(91,40) = 799.219\text{ MeV} - 789.611\text{ MeV} = 9.608\text{ MeV} \\ E_B(93,40) - E_B(92,40) = 806.145\text{ MeV} - 799.219\text{ MeV} = 6.926\text{ MeV} \end{cases} \begin{bmatrix}\text{using SEMF directly} \\ \text{considering pairing} \\ \text{term with } k = 3/4\end{bmatrix}$$

The rotational spectrum of 79Br19F shows a series of equidistant lines spaced 0.71433 cm apart. Calculate the rotational constant *B*, and hence the moment of inertia and bond length of the molecule. Determine the wavenumber of the J = 9 → J = 10 transition, and find which transition gives rise to the most intense spectral line at room temperature (300 K). Calculate the number of revolutions per second which the BrF molecule undergoes when in (a) the J = 0 state, (b) the J = 1 state, and (c) the J = 10 state.

$2B = 0.71433\text{ cm}^{-1} \Longrightarrow B = 0.35717\text{ cm}^{-1}$

Rotational constant, $B = \dfrac{h}{8\pi^2 c I_B} \Longrightarrow I_B = \dfrac{h}{8\pi^2 c B} = 7.83748\times 10^{-46}\text{ kg-m}^2 = 7.83748\times 10^{-42}\text{ kg-cm}^2$

Reduced mass, $\mu = \dfrac{m_{\text{Br}} m_{\text{F}}}{m_{\text{Br}} + m_{\text{F}}} = \dfrac{79u\times 19u}{79u + 19u} \approx 15.31633u$ [atomic mass unit, $u = 1.66054\times 10^{-27}$]

$I_B = \mu r^2 \Longrightarrow r = \sqrt{\dfrac{I_B}{\mu}} \approx 1.75544\times 10^{-10}\text{ m} = 1.75544\text{ Å}$

$\bar{\nu}_{J\to J+1} = 2B(J+1) \Longrightarrow \bar{\nu}_{9\to 10} = 0.71433\text{ cm}^{-1}\times 10 = 7.1433\text{ cm}^{-1}$

For maximum population, $J = \sqrt{\dfrac{kT}{2hcB}} - \dfrac{1}{2} \approx 17$

$\therefore J = 17 \to J = 18$ transition gives the most intense spectral line.

$E_J = \dfrac{h^2}{8\pi^2 I_B} J(J+1) \Longrightarrow \dfrac{1}{2} I_B \omega_J{}^2 = hcBJ(J+1) \Longrightarrow \omega_J = \sqrt{\dfrac{2hcBJ(J+1)}{I_B}} \Longrightarrow \omega_J = 4\pi cB\sqrt{J(J+1)}$ $\left[\because B = \dfrac{h}{8\pi^2 c I_B}\right]$

$\Longrightarrow$ number of revolutions per second $= \dfrac{\omega_J}{2\pi} = 2cB\sqrt{J(J+1)}$

$\therefore \dfrac{\omega_{J=0}}{2\pi} \approx 0\ ;\ \dfrac{\omega_{J=1}}{2\pi} \approx 3.03\times 10^{10}\ ;\ \dfrac{\omega_{J=10}}{2\pi} \approx 22.46\times 10^{10}$

What is the change in the rotational constant *B* when H is replaced by D (Deuterium) in Hydrogen molecule?

$$\text{Rotational constant, } B = \frac{h}{8\pi^2 c I_B} = \frac{h}{8\pi^2 c(\mu r^2)} \Rightarrow \frac{B'}{B} = \frac{I_B}{I_B{}'} = \frac{\mu}{\mu'} \quad [\because I_B = \mu r^2] \Rightarrow B' = \frac{\mu}{\mu'}B = \frac{1/2}{1}B = \frac{B}{2}$$

The rotational constant for $H^{35}Cl$ is observed to be 10.5909 cm^{-1}. What are the values of B for $H^{37}Cl$ and $D^{35}Cl$?

$$\text{Rotational constant, } B = \frac{h}{8\pi^2 c I_B} = \frac{h}{8\pi^2 c(\mu r^2)} \quad [\because I_B = \mu r^2] \Rightarrow \frac{B'}{B} = \frac{\mu}{\mu'} \Rightarrow B' = \frac{\mu}{\mu'}B \quad [\because r \text{ remains constant}]$$

$$\text{For } \mathrm{H}^{35}\mathrm{Cl}, B = 10.5909 \text{ cm}^{-1} \text{ and } \mu = \frac{35 \times 1}{35 + 1}u = \frac{35}{36}u$$

$$\text{For } \mathrm{H}^{37}\mathrm{Cl}, B' = \frac{\mu}{\mu'}B = \frac{35/36}{37/38} \times 10.5909 \text{ cm}^{-1} \approx 10.5750 \text{ cm}^{-1} \quad \left[\because \mu' = \frac{37 \times 1}{37 + 1}u = \frac{37}{38}u\right]$$

$$\text{For } \mathrm{D}^{35}\mathrm{Cl}, B' = \frac{\mu}{\mu'}B = \frac{35/36}{70/37} \times 10.5909 \text{ cm}^{-1} \approx 5.4425 \text{ cm}^{-1} \quad \left[\because \mu' = \frac{35 \times 2}{35 + 2}u = \frac{70}{37}u\right]$$

The first line in the rotational spectrum of CO (Carbon monoxide) has a frequency (wavenumber) of 3.8424 cm^{-1}. Calculate the rotational constant B and C—O bond length.

$$2B = 3.8424 \text{ cm}^{-1} \Rightarrow B = 1.9212 \text{ cm}^{-1}$$

$$B = \frac{h}{8\pi^2 c I_B} = \frac{h}{8\pi^2 c(\mu r^2)} \quad [\because I_B = \mu r^2] \Rightarrow r = \sqrt{\frac{h}{8\pi^2 c(\mu B)}} \approx 1.13 \times 10^{-10} \text{ m} = 1.13 \text{ Å}$$

The separation between the J = 3 and J = 9 transitions in the far-infrared spectrum of CO molecule is 23.148 cm^{-1}. Estimate the C—O bond length.

$$\bar{\nu}_{J\to J+1} = 2B(J+1) \Rightarrow \begin{cases} \bar{\nu}_{3\to 4} = 2B(3+1) = 8B \\ \bar{\nu}_{9\to 10} = 2B(9+1) = 20B \end{cases}$$

$$\bar{\nu}_{9\to 10} - \bar{\nu}_{3\to 4} = 23.148 \text{ cm}^{-1} \Rightarrow 12B = 23.148 \text{ cm}^{-1} \Rightarrow B = 1.929 \text{ cm}^{-1}$$

$$B = \frac{h}{8\pi^2 c I_B} = \frac{h}{8\pi^2 c(\mu r^2)} \quad [\because I_B = \mu r^2] \Rightarrow r = \sqrt{\frac{h}{8\pi^2 c(\mu B)}} \approx 1.13 \times 10^{-10} \text{ m} = 1.13 \text{ Å}$$

Find the amount of energy absorbed if OH molecule makes a transition from *J* = 5 to *J* = 6. (Given: r_{OH} = 1 nm)

$$E_J = \frac{h^2}{8\pi^2 I_B}J(J+1) = \frac{h^2}{8\pi^2 \mu r^2}J(J+1) \quad [\because I_B = \mu r^2] \approx 1.13 \times 10^{-22} \text{ J} \approx 7 \times 10^{-4} \text{ eV} \quad \begin{bmatrix} \because \mu = \frac{8 \times 1}{9 + 1}u = \frac{8}{9}u \\ J = 5 \text{ and } r = 1 \text{ nm} \end{bmatrix}$$

The first rotational line of $^{12}C^{16}O$ is observed at 3.84235 cm^{-1}, and that of $^{13}C^{16}O$ at 3.67337 cm^{-1}. Calculate the atomic mass of ^{13}C assuming the mass of O to be 15.9949u.

$$B = \frac{h}{8\pi^2 c I_B} = \frac{h}{8\pi^2 c(\mu r^2)} \Rightarrow \frac{B'}{B} = \frac{I_B}{I_B{}'} = \frac{\mu}{\mu'} \quad [\because I_B = \mu r^2] \Rightarrow \frac{\mu}{\mu'} = \frac{2B'}{2B} = \frac{3.67337}{3.84235} \Rightarrow \frac{\mu}{\mu'} \approx 0.956022$$

$$\Rightarrow \frac{\frac{12 \times 15.9949}{12 + 15.9949}}{\frac{m \times 15.9949}{m + 15.9949}} \approx 0.956022 \quad \begin{bmatrix} \because \text{atomic weight of } ^{12}\mathrm{C} \\ \text{is } 12u \text{ by definition.} \end{bmatrix} \Rightarrow \frac{15.9949m}{m + 15.9949} \approx \frac{6.856206}{0.956022} \approx 7.1716$$

$$\Rightarrow 15.9949m - 7.1716m = 15.9949 \times 7.1716 \Rightarrow 8.8233m = 114.7122 \Rightarrow m = 13.0007$$

$\therefore$ The atomic mass of ^{13}C thus found is $13.0007u$.

The moment of inertia of the CO molecule is 1.46 × 10^{-46} kg-m^2. Calculate the energy in eV, and the angular velocity in the lowest rotational energy level of the CO molecule.

$$E_J = \frac{h^2}{8\pi^2 I_B}J(J+1) \Rightarrow E_1 = \frac{2h^2}{8\pi^2 I_B} \quad \begin{bmatrix} \because J = 1 \text{ for lowest} \\ \text{rotational level.} \end{bmatrix} \approx 7.62 \times 10^{-23} \text{ J} \approx 4.75 \times 10^{-4} \text{ eV}$$

$$E_J = \frac{1}{2} I_B {\omega_J}^2 \Longrightarrow \omega_J = \sqrt{\frac{2E_J}{I_B}} \Longrightarrow \omega_1 = \sqrt{\frac{2E_1}{I_B}} \approx 1.02 \times 10^{12} \text{ rad/s} \approx 1.15 \times 10^{11} \text{ rev/s}$$

What is the average period of rotation of HCl molecule when it is in J = 1 state? Internuclear distance of HCl is 0.1274 nm. Mass of H and Cl atoms are 1.674 × 10^{-27} kg and 58.12 × 10^{-27} kg respectively.

$$\mu = \frac{1.674 \times 58.12}{1.674 + 58.12} \times 10^{-27} \text{ kg} = 1.627 \times 10^{-27} \text{ kg}$$

$$E_J = \frac{h^2}{8\pi^2 I_B} J(J+1) \Longrightarrow \frac{1}{2} I_B {\omega_J}^2 = \frac{h^2}{8\pi^2 I_B} J(J+1) \Longrightarrow {\omega_J}^2 = \frac{h^2}{4\pi^2 {I_B}^2} J(J+1) \Longrightarrow \omega_J = \frac{h}{2\pi I_B}\sqrt{J(J+1)}$$

$$\Longrightarrow \omega_J = \frac{h}{2\pi(\mu r^2)}\sqrt{J(J+1)} \quad [\because I_B = \mu r^2] = \frac{h\sqrt{2}}{2\pi(\mu r^2)} \quad [\because J = 1]$$

$$\therefore \text{Time period, } T = \frac{2\pi}{\omega_J} = \frac{2\sqrt{2}\pi^2(\mu r^2)}{h} = 1.11 \times 10^{-12} \text{ s} = 1.11 \text{ ps}$$

Three consecutive lines in the rotational spectrum of $H^{79}Br$ are observed at 84.544, 101.355 and 118.112 cm^{-1}. Assign the lines to their appropriate J" → J' transitions, then deduce values for *B* and *D*, and hence evaluate the bond length and approximate frequency of the molecule.

$$2B = 101.355 - 84.544 \text{ cm}^{-1} \approx 16.8 \text{ cm}^{-1} \Longrightarrow B \approx 8.4 \text{ cm}^{-1}$$

$$\bar{\nu}_{J\to J+1} = 2B(J+1) \Longrightarrow (J+1) = \frac{\bar{\nu}_{J\to J+1}}{2B}$$

When $\bar{\nu}_{J\to J+1} = 84.544 \text{ cm}^{-1}, (J+1) = \dfrac{84.544 \text{ cm}^{-1}}{16.8 \text{ cm}^{-1}} \approx 5 \Longrightarrow J = 4$

$\therefore$ The transition corresponding to the first line is $4 \to 5$

$\because$ The lines are consecutive, and $\bar{\nu}_{J\to J+1} \propto (J+1)$, $\therefore$ The transitions corresponding to the next two lines are $5 \to 6$ and $6 \to 7$ respectively.

For a non-rigid rotator, $\bar{\nu}_{J\to J+1} = 2B(J+1) - 4D(J+1)^3$

$$\therefore \begin{cases} \bar{\nu}_{4\to5} = 2B \times 5 - 4D \times 125 \\ \bar{\nu}_{5\to6} = 2B \times 6 - 4D \times 216 \\ \bar{\nu}_{6\to7} = 2B \times 7 - 4D \times 343 \end{cases} \Longrightarrow \begin{cases} 84.544 \text{ cm}^{-1} = 10B - 500D \\ 101.355 \text{ cm}^{-1} = 12B - 852D \\ 118.112 \text{ cm}^{-1} = 14B - 1372D \end{cases}$$

$$\Longrightarrow \begin{cases} B = \dfrac{8.4738 + 8.4717 + 8.4730}{3} \text{ cm}^{-1} = 8.4728 \text{ cm}^{-1} \\ D = \dfrac{3.881 + 3.585 + 3.714}{3} \times 10^{-4} \text{ cm}^{-1} = 3.727 \times 10^{-4} \text{ cm}^{-1} \end{cases}$$

$$B = \frac{h}{8\pi^2 c I_B} = \frac{h}{8\pi^2 c(\mu r^2)} \quad [\because I_B = \mu r^2] \Longrightarrow r = \sqrt{\frac{h}{8\pi^2 c(\mu B)}} \approx 1.42 \times 10^{-10} \text{ m} = 1.42 \text{ Å}$$

$$\text{Vibrational frequency (wavenumber), } \bar{f} = \sqrt{\frac{4B^3}{D}} \approx 2555 \text{ cm}^{-1}$$

Rotational and Centrifugal distortion constants *B* and *D* for HCl are 10.593 cm^{-1} and 5.3 × 10^{-4} cm^{-1} respectively. Estimate the vibrational frequency and force constant of the molecule. The observed vibrational frequency is 2991 cm^{-1}; explain the discrepancy.

$$\text{Vibrational frequency, } \bar{f} = \sqrt{\frac{4B^3}{D}} \approx 2995 \text{ cm}^{-1}$$

The discrepancy is due to assumption of SHM for the vibration, and also due to error in the measurement of D.

$$\mu = \frac{1 \times 35}{1 + 35} u = \frac{35}{36} u$$

Force constant, $k = \left(2\pi\bar{f}c\right)^2\mu \approx 514$ N/m

The C–H and C–N bond lengths of the linear molecule H–C≡N are 0.1063 nm and 0.1155 nm respectively. Calculate *I* and *B* for HCN and DCN, using relative atomic masses of H = 1, D = 2, C = 12 and N = 14.

$$I_B = m_\mathrm{H}{r_\mathrm{CH}}^2 + m_\mathrm{N}{r_\mathrm{CN}}^2 - \frac{(m_\mathrm{H}r_\mathrm{CH} - m_\mathrm{N}r_\mathrm{CN})^2}{M} = 1.885 \times 10^{-46}\ \text{kg-m}^2$$
$$B = \frac{h}{8\pi^2 c I_B} = 148.477\ \text{cm}^{-1}$$
$$I_B{}' = m_\mathrm{D}{r_\mathrm{CD}}^2 + m_\mathrm{N}{r_\mathrm{CN}}^2 - \frac{(m_\mathrm{D}r_\mathrm{CD} - m_\mathrm{N}r_\mathrm{CN})^2}{M'} = 2.307 \times 10^{-46}\ \text{kg-m}^2$$
$$B' = \frac{h}{8\pi^2 c I_B{}'} = 121.346\ \text{cm}^{-1}$$

The OH radical has a moment of inertia of 1.48 × 10^{-40} gm-cm^2. Calculate its internuclear distance. Also calculate, for J = 5, its angular momentum and angular velocity. Determine the energy absorbed in the J = 6 ← J = 5 transition in cm^{-1} and erg/molecule.

$$\mu = \frac{16 \times 1}{16 + 1}u = \frac{16}{17}u \approx 1.563 \times 10^{-24}\ \text{gm}$$
$$I_B = \mu r^2 \Longrightarrow r = \sqrt{\frac{I_B}{\mu}} \approx 9.73 \times 10^{-9}\ \text{cm} = 0.973\ \text{Å}$$
$$L_B = \sqrt{J(J+1)}\,\hbar = \sqrt{30}\,\hbar = 5.78 \times 10^{-27}\ \text{erg-s}$$
$$L_B = I_B\omega \Longrightarrow \omega = \frac{L_B}{I_B} \approx 3.91 \times 10^{13}\ \text{rad/s} \approx 6.22 \times 10^{12}\ \text{rev/s}$$
$$E_{J\to J+1} = hcB\{(J+1)(J+2) - J(J+1)\} = 2hcB(J+1) = \frac{2h^2}{8\pi^2 I_B}(J+1) \approx 4.51 \times 10^{-14}\ \text{erg/molecule}$$

Estimate the minimum kinetic energy which a neutron, in a collision with molecule of gaseous Oxygen, can lose by exciting molecular rotation. The bond length of O_2 is 1.2 Å.

$$E_{J\to J+1} = 2hcB(J+1) \Longrightarrow (\text{KE})_\text{min} = (\Delta E)_\text{min} = 2hcB = \frac{h^2}{4\pi^2(\mu r^2)} \left[\begin{array}{l} B = \dfrac{h}{8\pi^2 c I_B} = \dfrac{h}{8\pi^2 c(\mu r^2)} \\ \text{and } \mu = \dfrac{16 \times 16}{16 + 16}u = 8u \end{array}\right] \approx 5.8 \times 10^{-23}\ \text{J}$$

The wavenumber of lines in a spectral band is given by 1000 (2n−1) cm^{-1} and −1000 (2n+1) cm^{-1} for +ve and −ve *n* respectively. Calculate the moment of inertia of the spectral emitting system.

$\bar{\nu} = 1000, 3000, 5000, \ldots$ for +ve n, and $\bar{\nu} = 1000, 3000, 5000, \ldots$ for –ve n

$\therefore\ 2B = 2000\ \text{cm}^{-1} \Longrightarrow B = 1000\ \text{cm}^{-1}$

$$I_B = \frac{h}{8\pi^2 cB} \approx 2.8 \times 10^{-42}\ \text{gm-cm}^2$$

The spacing of a series of lines in the microwave spectrum of AlH is constant at 12.604 cm^{-1}. Calculate the moment of inertia and the internuclear distance of the AlH molecule. What are the energy of rotation and the rate of rotation when J = 15? (Atomic masses of Al and H are 26.98u and 1.008u respectively.)

$2B = 12.604\ \text{cm}^{-1} \Longrightarrow B = 6.302\ \text{cm}^{-1}$

$$I_B = \frac{h}{8\pi^2 cB} \approx 4.44 \times 10^{-40}\ \text{gm-cm}^2$$
$$\mu = \frac{26.98 \times 1.008}{26.98 + 1.008}u \approx 0.9717u \approx 1.614 \times 10^{-24}\ \text{gm}$$

$$I_B = \mu r^2 \Rightarrow r = \sqrt{\frac{I_B}{\mu}} \approx 1.66 \times 10^{-8} \text{ cm} = 1.66 \text{ Å}$$

$$E_J = \frac{h^2}{8\pi^2 I_B} J(J+1) \Rightarrow E_{15} \approx 3 \times 10^{-13} \text{ erg/molecule}$$

$$E_J = \frac{1}{2} I_B {\omega_J}^2 \Rightarrow \omega_J = \sqrt{\frac{2E_J}{I_B}} \Rightarrow \omega_{15} = \sqrt{\frac{2E_{15}}{I_B}} \approx 3.68 \times 10^{13} \text{ rad/s} \approx 5.86 \times 10^{12} \text{ rev/s}$$

In the far-infrared spectrum of HCl molecule, the first line falls at 20.68 cm^{-1}. Calculate the moment of inertia, reduced mass and the bond-length of the molecule. Regarding the molecule to be rigid rotator, also calculate the wavelength of the transition J = 15 ← 14.

$$\bar{\nu}_{J \to J+1} = 2B(J+1) \Rightarrow \bar{\nu}_{0 \to 1} = 2B = 20.68 \text{ cm}^{-1} \Rightarrow B = 10.34 \text{ cm}^{-1}$$

$$\mu = \frac{1 \times 35}{1 + 35} u = \frac{35}{36} u \approx 1.614 \times 10^{-24} \text{ gm}$$

$$I_B = \frac{h}{8\pi^2 cB} \approx 2.71 \times 10^{-40} \text{ gm-cm}^2$$

$$I_B = \mu r^2 \Rightarrow r = \sqrt{\frac{I_B}{\mu}} \approx 1.29 \times 10^{-8} \text{ cm} = 1.29 \text{ Å}$$

$$\bar{\nu}_{J \to J+1} = 2B(J+1) \Rightarrow \bar{\nu}_{15 \to 14} = 30B \approx 310.2 \text{ cm}^{-1} \Rightarrow \lambda_{15 \to 14} = \frac{1}{\bar{\nu}_{15 \to 14}} = \frac{1}{310.2} \text{ cm} \approx 3.32 \times 10^{-3} \text{ cm} = 33.2 \text{ μm}$$

Calculate the ratio of the number of molecules, N_J, in a sample of HI at 300 K in the rotational states J = 5 to J = 0. (I = 4.31 × 10^{-47} kg-m^2).

$$\frac{N_J}{N_0} = (2J+1)e^{-E_J/kT} = 11 \times e^{-0.9345} \left[E_J = \frac{h^2 J(J+1)}{8\pi^2 I_B} = \frac{30h^2}{8\pi^2 I_B} \approx 3.8705 \times 10^{-21} \text{ J}\right] \approx 11 \times 0.3928 \approx 4.3208$$

PURE/ELASTIC VIBRATOR: Harmonic Oscillator

Restoring force, $\mathcal{F} = -k(r - r_{\text{eq}})$

Potential energy, $E = \frac{1}{2} k (r - r_{\text{eq}})^2$

$\omega_{\text{osc}} = \sqrt{\frac{k}{\mu}}$ where μ is the reduced mass of the system.

$\nu_{\text{osc}} = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$ and $\bar{\nu}_{\text{osc}} = \frac{1}{2\pi c}\sqrt{\frac{k}{\mu}}$

$$E_v = \left(v + \frac{1}{2}\right)\hbar\omega_{\text{osc}} = \left(v + \frac{1}{2}\right)\hbar(2\pi c \bar{\nu}_{\text{osc}}) = \left(v + \frac{1}{2}\right) hc\bar{\nu}_{\text{osc}}$$

$$\bar{\nu}_v = \frac{E_v}{hc} = \left(v + \frac{1}{2}\right)\bar{\nu}_{\text{osc}}$$

∴ Lowest vibrational energy or Zero-point energy, $E_0 = \frac{1}{2}\hbar\omega_{\text{osc}} = \frac{1}{2}h\nu_{\text{osc}}$

Selection rule for harmonic oscillator: $\Delta v = \pm 1$

For a transition from $v+1$ to v, $\bar{\nu}_{v+1 \to v} = \left(v + 1 + \frac{1}{2}\right)\bar{\nu}_{\text{osc}} - \left(v + \frac{1}{2}\right)\bar{\nu}_{\text{osc}} = \bar{\nu}_{\text{osc}}$

Vibrational spectra is observable in heteronuclear diatomic molecules, not in homonuclear diatomic molecules as they have no dipole moment to interact with the radiation.

INELASTIC VIBRATOR: Anharmonic Oscillator

Morse curve is the energy (vs internuclear distance) curve for a typical diatomic molecule undergoing anharmonic extensions and compressions together. An empirical expression that fits this curve was put forth by P M Morse, and is called the **Morse function**. $E = E_{\text{diss}}\left\{1 - e^{a(r_{\text{eq}}-r)}\right\}^2$ where E_{diss} is the dissociation energy, r_{eq} is the value of the internuclear distance r that corresponds to the minimum of Morse curve, a is a constant for a particular molecule.

With the Morse function for potential energy instead of the simple harmonic one in the Schrödinger's equation,

$$\bar{\nu}_v = \left(v+\frac{1}{2}\right)\bar{\nu}_{\text{e}} - \left(v+\frac{1}{2}\right)^2 x\bar{\nu}_{\text{e}} + \left(v+\frac{1}{2}\right)^3 y\bar{\nu}_{\text{e}} - \cdots \quad \text{where } x, y, \text{etc. are the anharmonicity constants.}$$

To first approximation, $\bar{\nu}_v = \boldsymbol{G(v)} = \left(\boldsymbol{v}+\frac{\mathbf{1}}{\mathbf{2}}\right)\bar{\boldsymbol{\nu}}_{\mathbf{e}} - \left(\boldsymbol{v}+\frac{\mathbf{1}}{\mathbf{2}}\right)^{\mathbf{2}} \boldsymbol{x}\bar{\boldsymbol{\nu}}_{\mathbf{e}} \Rightarrow \bar{\nu}_v = \left\{1 - \left(v+\frac{1}{2}\right)x\right\}\left(v+\frac{1}{2}\right)\bar{\nu}_{\text{e}}$

$$\Rightarrow \left(v+\frac{1}{2}\right)\bar{\nu}_{\text{osc}} = \left\{1 - \left(v+\frac{1}{2}\right)x\right\}\left(v+\frac{1}{2}\right)\bar{\nu}_{\text{e}} \Rightarrow \bar{\nu}_{\text{osc}} = \left\{1 - \left(v+\frac{1}{2}\right)x\right\}\bar{\nu}_{\text{e}}$$

In the ground state ($v = 0$), $\bar{\nu}_v = \frac{1}{2}\left(1 - \frac{1}{2}x\right)\bar{\nu}_{\text{e}}$ and $\bar{\nu}_{\text{osc}} = \left(1 - \frac{1}{2}x\right)\bar{\nu}_{\text{e}}$

Selection rule for anharmonic oscillator: $\Delta v = \pm 1, \pm 2, \pm 3, \ldots$

The transition with $v = 1 \to v = 0$ is called the fundamental transition/band (first harmonic). $v = 2 \to v = 0$ is called the first overtone (second harmonic). $v = 3 \to v = 0$ is called the second overtone (third harmonic).

$$(\Delta G)_1 = \bar{\nu}_{v+1\to v} = \left\{\left(v+1+\frac{1}{2}\right)\bar{\nu}_{\text{e}} - \left(v+1+\frac{1}{2}\right)^2 x\bar{\nu}_{\text{e}}\right\} - \left\{\left(v+\frac{1}{2}\right)\bar{\nu}_{\text{e}} - \left(v+\frac{1}{2}\right)^2 x\bar{\nu}_{\text{e}}\right\}$$

$$= \bar{\nu}_{\text{e}} - \left\{\left(v+1+\frac{1}{2}\right)^2 - \left(v+\frac{1}{2}\right)^2\right\}x\bar{\nu}_{\text{e}} = \bar{\nu}_{\text{e}} - 2(v+1)x\bar{\nu}_{\text{e}} \Rightarrow \bar{\boldsymbol{\nu}}_{\boldsymbol{v+1\to v}} = \{\mathbf{1} - \mathbf{2}(\boldsymbol{v}+\mathbf{1})\boldsymbol{x}\}\bar{\boldsymbol{\nu}}_{\mathbf{e}}$$

$$(\Delta G)_2 = \bar{\nu}_{v+2\to v} = \left\{\left(v+2+\frac{1}{2}\right)\bar{\nu}_{\text{e}} - \left(v+2+\frac{1}{2}\right)^2 x\bar{\nu}_{\text{e}}\right\} - \left\{\left(v+\frac{1}{2}\right)\bar{\nu}_{\text{e}} - \left(v+\frac{1}{2}\right)^2 x\bar{\nu}_{\text{e}}\right\}$$

$$= 2\bar{\nu}_{\text{e}} - \left\{\left(v+2+\frac{1}{2}\right)^2 - \left(v+\frac{1}{2}\right)^2\right\}x\bar{\nu}_{\text{e}} = 2\bar{\nu}_{\text{e}} - 2(2v+3)x\bar{\nu}_{\text{e}} \Rightarrow \bar{\boldsymbol{\nu}}_{\boldsymbol{v+2\to v}} = \{\mathbf{1} - (\mathbf{2}\boldsymbol{v}+\mathbf{3})\boldsymbol{x}\}\mathbf{2}\bar{\boldsymbol{\nu}}_{\mathbf{e}}$$

Likewise, $(\Delta G)_3 = \bar{\boldsymbol{\nu}}_{\boldsymbol{v+3\to v}} = \{1 - 3(2v+4)x\}\bar{\nu}_{\text{e}} = \{\mathbf{1} - \mathbf{2}(\boldsymbol{v}+\mathbf{2})\boldsymbol{x}\}\mathbf{3}\bar{\boldsymbol{\nu}}_{\mathbf{e}}$

Note that $\boldsymbol{v}$ denotes the vibrational quantum number of the lower level.

$$\therefore \begin{cases} \bar{\nu}_{1\to 0} = (1-2x)\bar{\nu}_{\text{e}} \\ \bar{\nu}_{2\to 1} = (1-4x)\bar{\nu}_{\text{e}} \\ \bar{\nu}_{3\to 2} = (1-6x)\bar{\nu}_{\text{e}} \end{cases}; \begin{cases} \bar{\nu}_{2\to 0} = (1-3x)2\bar{\nu}_{\text{e}} \\ \bar{\nu}_{3\to 1} = (1-5x)2\bar{\nu}_{\text{e}} \\ \bar{\nu}_{4\to 2} = (1-7x)2\bar{\nu}_{\text{e}} \end{cases}; \begin{cases} \bar{\nu}_{3\to 0} = (1-4x)3\bar{\nu}_{\text{e}} \\ \bar{\nu}_{4\to 1} = (1-6x)3\bar{\nu}_{\text{e}} \\ \bar{\nu}_{5\to 2} = (1-8x)3\bar{\nu}_{\text{e}} \end{cases}$$

Population ratio at room temperature (300 K) for $v = 1$, $\dfrac{N_{v=1}}{N_{v=0}} = e^{-\frac{\Delta E}{kT}} = e^{-\frac{hc\Delta G}{kT}} = e^{-\frac{hc(1-2x)\bar{\nu}_{\text{e}}}{k(300\text{ K})}}$

The equilibrium vibration frequency of the iodine molecule I_2 is 215 cm^{-1}, and the anharmonicity constant x is 0.003; what, at 300 K, is the intensity of the 'hot band' (v = 1 → v = 2 transition) relative to that of the fundamental (v = 0 → v = 1 transition)?

Intensity, $I_{v=0\to v=1} \propto$ Transition probability, $\dfrac{N_{v=1}}{N_{v=0}} = e^{-\frac{\Delta E(0\sim 1)}{kT}} = e^{-\frac{hc\Delta G(0\sim 1)}{kT}} = e^{-\frac{hc(1-2x)\bar{\nu}_{\text{e}}}{k(300\text{ K})}}$

Intensity, $I_{v=1\to v=2} \propto$ Transition probability, $\dfrac{N_{v=2}}{N_{v=1}} = e^{-\frac{\Delta E(1\sim 2)}{kT}} = e^{-\frac{hc\Delta G(1\sim 2)}{kT}} = e^{-\frac{hc(1-4x)\bar{\nu}_{\text{e}}}{k(300\text{ K})}}$

$$\therefore \frac{I_{v=1\to v=2}}{I_{v=0\to v=1}} = \frac{e^{-\frac{hc(1-4x)\bar{\nu}_{\text{e}}}{k(300\text{ K})}}}{e^{-\frac{hc(1-2x)\bar{\nu}_{\text{e}}}{k(300\text{ K})}}} = e^{-\left\{\frac{hc(1-4x)\bar{\nu}_{\text{e}}}{k(300\text{ K})} - \frac{hc(1-2x)\bar{\nu}_{\text{e}}}{k(300\text{ K})}\right\}} = e^{-\frac{hc(1-2x)\bar{\nu}_{\text{e}}}{k(300\text{ K})}} \approx 0.36$$

The fundamental and first overtone transitions of $^{14}N^{16}O$ are centered at 1876.06 cm^{-1} and 3724.20 cm^{-1} respectively. Evaluate the equilibrium vibration frequency, the anharmonicity, the exact zero point energy, and

the force constant of the molecule. Assuming *v* to be a continuous variable, determine the maximum value of *G*(*v*), and calculate a value for the dissociation energy of NO. Criticize this method.

$$\begin{cases} \bar{\nu}_{1\to 0} = (1-2x)\bar{\nu}_e = 1876.06\ \text{cm}^{-1} \\ \bar{\nu}_{2\to 0} = (1-3x)2\bar{\nu}_e = 3724.20\ \text{cm}^{-1} \end{cases} \Rightarrow \frac{(1-2x)\bar{\nu}_e}{(1-3x)2\bar{\nu}_e} = \frac{1876.06}{3724.20} \Rightarrow \frac{(1-2x)}{(1-3x)} = \frac{3752.12}{3724.20}$$

$$\Rightarrow 3724.20 - 7448.40x = 3752.12 - 11256.36x \Rightarrow 3807.96x = 27.92 \Rightarrow x \approx 0.007332$$

$\therefore$ equilibrium vibration frequency, $\bar{\nu}_e = \dfrac{1876.06\ \text{cm}^{-1}}{(1-2x)} = 1903.98\ \text{cm}^{-1}$

Exact zero-point energy (expressed in cm^{-1}), $\bar{\nu}_v = \dfrac{1}{2}\left(1 - \dfrac{1}{2}x\right)\bar{\nu}_e = 948.50\ \text{cm}^{-1}$

Force constant, $k = (2\pi\bar{\nu}_e c)^2\mu \approx 1595\ \text{N/m}$ $\left[\because \mu = \dfrac{14\times 16}{14+16}u\right]$

$$G(v) = \left(v + \frac{1}{2}\right)\bar{\nu}_e - \left(v + \frac{1}{2}\right)^2 x\bar{\nu}_e \Rightarrow \frac{dG}{dv} = \bar{\nu}_e - 2\left(v + \frac{1}{2}\right)x\bar{\nu}_e$$

For maximum G, $\dfrac{dG}{dv} = 0 \Rightarrow \bar{\nu}_e - 2\left(v + \dfrac{1}{2}\right)x\bar{\nu}_e = 0 \Rightarrow 2\left(v + \dfrac{1}{2}\right)x = 1 \Rightarrow v = \dfrac{1}{2x} - \dfrac{1}{2} = \dfrac{1}{2}\left(\dfrac{1}{x} - 1\right) \approx 67.7$

Dissociation energy (expressed in cm^{-1}), $G_{\max} = \left(v + \dfrac{1}{2}\right)\bar{\nu}_e - \left(v + \dfrac{1}{2}\right)^2 x\bar{\nu}_e \approx 64920.213\ \text{cm}^{-1}$

$\equiv 1.29 \times 10^{-18}\ \text{J} \equiv 776.62\ \text{kJ/mol}$. The method is inaccurate as the higher order terms in $\left(v + \dfrac{1}{2}\right)$ have been ignored that become important for large v.

The fundamental band for CO is centered at 2143.3 cm^{-1} and first overtone at 4259.7 cm^{-1}. Calculate the equilibrium vibrational frequency, the anharmonicity constant and the force constant of the molecule.

$$\begin{cases} \bar{\nu}_{1\to 0} = (1-2x)\bar{\nu}_e = 2143.3\ \text{cm}^{-1} \\ \bar{\nu}_{2\to 0} = (1-3x)2\bar{\nu}_e = 4259.7\ \text{cm}^{-1} \end{cases} \Rightarrow \frac{(1-2x)\bar{\nu}_e}{(1-3x)2\bar{\nu}_e} = \frac{2143.3}{4259.7} \Rightarrow \frac{(1-2x)}{(1-3x)} = \frac{4286.6}{4259.7}$$

$$\Rightarrow 4259.7 - 8519.4x = 4286.6 - 12859.8x \Rightarrow 4340.4x = 26.9 \Rightarrow x \approx 0.0062$$

$\therefore \bar{\nu}_e = \dfrac{2143.3\ \text{cm}^{-1}}{(1-2x)} = 2170.2\ \text{cm}^{-1}$

Force constant, $k = (2\pi\bar{\nu}_e c)^2\mu \approx 1903\ \text{N/m}$ $\left[\because \mu = \dfrac{12\times 16}{12+16}u\right]$

The vibrational wavenumbers of the following molecules in their *v* = 0 states are: HCl, 2885 cm^{-1}; DCl, 1990 cm^{-1}; D_2, 2990 cm^{-1} and HD, 3627 cm^{-1}. Calculate the energy change in kJ/mol of the reaction: HCl + D_2 → DCl + HD and determine whether energy is liberated or absorbed. (Consider the zero-point energies of the involved molecules.)

$$\Delta E \text{ per mole} = \frac{1}{2}(2885 + 2990)\ \text{cm}^{-1} - \frac{1}{2}(1990 + 3627)\ \text{cm}^{-1} = 129\ \text{cm}^{-1} \equiv 5.125 \times 10^{-21}\ \text{J} \equiv +1.543\ \text{kJ}$$

The 2886 cm^{-1} fundamental band of HCl can be shown to fit the empirical equation *v* (cm^{-1}) = 2885.90 + 20.577*m* –0.3034*m*^{2}. Calculate the values of B_e, B_0 and B_1. Given *α* = 0.3019 cm^{-1}.

$$\begin{cases} B_1 + B_0 = 20.577\ \text{cm}^{-1} \\ B_1 - B_0 = -0.3034\ \text{cm}^{-1} \end{cases} \Rightarrow \begin{cases} 2B_1 = 20.2736\ \text{cm}^{-1} \Rightarrow B_1 = 10.1368\ \text{cm}^{-1} \\ 2B_0 = 20.8804\ \text{cm}^{-1} \Rightarrow B_0 = 10.4402\ \text{cm}^{-1} \end{cases}$$

$B_0 = B_e - \alpha\left(v + \dfrac{1}{2}\right) = B_e - \dfrac{\alpha}{2}$ $[\because v = 0 \text{ for } B_0] \Rightarrow B_e = 10.59115\ \text{cm}^{-1}$

Also, $B_1 = B_e - \alpha\left(v + \dfrac{1}{2}\right) = B_e - \dfrac{3\alpha}{2}$ $[\because v = 1 \text{ for } B_1] \Rightarrow B_e = 10.58965\ \text{cm}^{-1}$

Vibrating Rotator (Rotating Vibrator)

Born-Oppenheimer approximation: Since a typical diatomic molecule has rotational energy separations of 1-10 cm^{-1} whereas the vibrational energy separations are about 1000 times larger (during one rotation, the molecule vibrates some 1000 times), the rotations and vibrations may be considered to be executed independently of each other.

$\bar{\nu}_J = \boldsymbol{F(J) = BJ(J+1) - DJ^2(J+1)^2}$

$\bar{\nu}_v = \boldsymbol{G(v)} = \left(\boldsymbol{v + \frac{1}{2}}\right)\boldsymbol{\bar{\nu}_e} - \left(\boldsymbol{v + \frac{1}{2}}\right)^2 \boldsymbol{x\bar{\nu}_e}$

$\boldsymbol{\bar{\nu}_{J',v+1\to J'',v} = \bar{\nu}_{J''+1,v+1\to J'',v} = \{2B(J''+1) - 4D(J''+1)^3\} + \{1 - 2(v+1)x\}\bar{\nu}_e}$

If $v = 0$, and D is ignored, **fundamental** change, $\boldsymbol{\bar{\nu}_{J',v+1\to J'',v} = \bar{\nu}_{J''+1,v+1\to J'',v} = 2B(J''+1) + \{1 - 2x\}\bar{\nu}_e}$

$\Delta \boldsymbol{J = J' - J''}$ **and the lines arising from** $\Delta\boldsymbol{J = -2, -1, 0, +1, +2}$ **are called** $\boldsymbol{O, P, Q, R, S}$ **branch respectively.**

Breakdown of Born-Oppenheimer approximation: Due to the simultaneous vibrations, the bond length changes continually during the rotation. If the vibration is simple harmonic, the mean bond length will be the same as the equilibrium bond length, and it will not vary with the vibration. However, the rotational constant B depends on $1/r^2$ whose average value (r_{av}) is not the same as $1/{r_{\text{eq}}}^2$. Further, an increase in vibrational energy is accom--panied an increase in the vibrational amplitude and hence the value of B will depend on the v quantum number. Since r_{av} increases with the vibrational energy, $\boldsymbol{B}$ **is smaller in the upper vibrational state than in the lower.**

$\boldsymbol{B_v = B_{\text{eq}} - \alpha\left(v + \frac{1}{2}\right)}$ where α is positive constant for each molecule.

$\therefore$ For $v = 0 \longrightarrow v = 1$ transition (fundamental vibration change),

$\boldsymbol{\bar{\nu}_{J',v=1\to J'',v=0} = \{B'J'(J'+1) - B''J''(J''+1)\} + \{1 - 2x\}\bar{\nu}_e}$

$$\Longrightarrow \begin{cases} \bar{\nu}_R = \{B'(J''+1)(J''+1+1) - B''J''(J''+1)\} + \{1-2x\}\bar{\nu}_e & [\text{for } \Delta J = +1 \text{ (R branch) ; expressed in } J''] \\ \bar{\nu}_P = \{B'J'(J'+1) - B''(J'+1)(J'+1+1)\} + \{1-2x\}\bar{\nu}_e & [\text{for } \Delta J = -1 \text{ (P Branch) ; expressed in } J'] \end{cases}$$

$$\Longrightarrow \begin{cases} \bar{\nu}_R = \{B'(J''+1)^2 + B'(J''+1) - B''(J''+1)^2 + B''(J''+1)\} + \{1-2x\}\bar{\nu}_e & [\text{for } \Delta J = +1 \text{ (R branch)}] \\ \bar{\nu}_P = \{B'(J'+1)^2 - B'(J'+1) - B''(J'+1)^2 - B''(J'+1)\} + \{1-2x\}\bar{\nu}_e & [\text{for } \Delta J = -1 \text{ (P Branch)}] \end{cases}$$

$$\Longrightarrow \begin{cases} \bar{\nu}_R = (B' - B'')(J''+1)^2 + (B' + B'')(J''+1) + \{1-2x\}\bar{\nu}_e & [\text{for } \Delta J = +1 \text{ (R branch)}] \\ \bar{\nu}_P = (B' - B'')(J'+1)^2 - (B' + B'')(J'+1) + \{1-2x\}\bar{\nu}_e & [\text{for } \Delta J = -1 \text{ (P Branch)}] \end{cases}$$

$\Longrightarrow \bar{\nu}_{P,R} = (B' + B'')m + (B' - B'')m^2 + \{1-2x\}\bar{\nu}_e$ where $m = \pm 1, \pm 2, \ldots$ $[\boldsymbol{B' < B''}]$

Raman Scattering: Classical theory

When a molecule is placed in an electric field (of the radiation), the medium gets polarized.

$\vec{E} = \vec{E}_0 \cos\omega_0 t$ [prefer cos over sin]

$$\alpha = \alpha_0 + \left(\frac{\partial\alpha}{\partial x}\right)_0 x + \cdots \approx \alpha_0 + \left(\frac{\partial\alpha}{\partial x}\right)_0 x = \alpha_0 + \left(\frac{\partial\alpha}{\partial x}\right)_0 x_0 \cos\omega_{\text{m}} t \quad [\because x = x_0 \cos\omega_{\text{m}} t]$$

$$\therefore \vec{P} = \alpha\vec{E} = \left\{\alpha_0 + \left(\frac{\partial\alpha}{\partial x}\right)_0 x_0 \cos\omega_{\text{m}} t\right\}\vec{E}_0 \cos\omega_0 t = \alpha_0\vec{E}_0\cos\omega_0 t + \left(\frac{\partial\alpha}{\partial x}\right)_0 x_0 \vec{E}_0 \cos\omega_{\text{m}} t \cos\omega_0 t$$

$$= \alpha_0\vec{E}_0\cos\omega_0 t + \frac{1}{2}\left(\frac{\partial\alpha}{\partial x}\right)_0 x_0\vec{E}_0\{\cos(\omega_0 - \omega_{\text{m}})t + \cos(\omega_0 + \omega_{\text{m}})t\}$$

$$\begin{cases} \omega = \omega_0 & \textbf{Rayleigh line} \\ \omega = \omega_0 - \omega_{\text{m}} & \textbf{Stokes line} \text{ (lower frequency)} \\ \omega = \omega_0 + \omega_{\text{m}} & \textbf{anti-Stokes line} \text{ (higher frequency)} \end{cases}$$

If $\left(\frac{\partial\alpha}{\partial x}\right)_0 = 0$, no Raman line will be observed. [**Raman-inactive**]

The classical theory fails to give the correct intensities of the Stokes lines.

The exciting line in an experiment is 5460 Å and the Stokes line is at 5520 Å. Find the wavelength of the anti-Stokes line.

wavenumber of exciting line, $\bar{\nu}_0 = \frac{1}{5460 \times 10^{-8}\ \text{cm}} = \frac{10^8}{5460}\ \text{cm}^{-1}$

wavenumber of Stokes line, $\bar{\nu}_0 - \bar{\nu}_m = \frac{10^8}{5520}\ \text{cm}^{-1} \Longrightarrow \bar{\nu}_m = \left(\frac{10^8}{5460} - \frac{10^8}{5520}\right)\ \text{cm}^{-1}$

wavenumber of anti-Stokes line, $\bar{\nu}_0 + \bar{\nu}_m = \frac{10^8}{5460}\ \text{cm}^{-1} + \left(\frac{10^8}{5460} - \frac{10^8}{5520}\right)\ \text{cm}^{-1} = \left(\frac{2 \times 10^8}{5460} - \frac{10^8}{5520}\right)\ \text{cm}^{-1}$

$\therefore$ wavelength of anti-Stokes line, $\lambda_{\text{AS}} = \dfrac{1}{\bar{\nu}_0 + \bar{\nu}_m} \approx 5401.3 \times 10^{-8}\ \text{cm} = 5401.3\ \text{Å}$

With exciting line 4358 Å, a sample gives Stokes line at 4458 Å. Deduce the wavelength of the anti-Stokes line.

wavenumber of exciting line, $\bar{\nu}_0 = \dfrac{1}{4358 \times 10^{-8}\ \text{cm}} = \dfrac{10^8}{4358}\ \text{cm}^{-1}$

wavenumber of Stokes line, $\bar{\nu}_0 - \bar{\nu}_m = \dfrac{10^8}{4458}\ \text{cm}^{-1} \Rightarrow \bar{\nu}_m = \left(\dfrac{10^8}{4358} - \dfrac{10^8}{4458}\right)\ \text{cm}^{-1}$

wavenumber of anti-Stokes line, $\bar{\nu}_0 + \bar{\nu}_m = \dfrac{10^8}{4358}\ \text{cm}^{-1} + \left(\dfrac{10^8}{4358} - \dfrac{10^8}{4458}\right)\ \text{cm}^{-1} = \left(\dfrac{2 \times 10^8}{4358} - \dfrac{10^8}{4458}\right)\ \text{cm}^{-1}$

$\therefore$ wavelength of anti-Stokes line, $\lambda_{\text{AS}} = \dfrac{1}{\bar{\nu}_0 + \bar{\nu}_m} \approx 4247.1 \times 10^{-8}\ \text{cm} = 4247.1\ \text{Å}$

The bond length of N_2 molecule is 1.097×10^{-10} m. What would be the positions of the first three Raman lines of N_2? ($^{14}N = 23.25 \times 10^{-27}$ kg)

Since the nuclear spin of nitrogen is non-zero, the first three (rotational) Raman lines of the linear molecule occur at $6B$, $10B$ and $14B$.

Reduced mass, $\mu = \dfrac{(23.25 \times 10^{-27}\ \text{kg})(23.25 \times 10^{-27}\ \text{kg})}{(23.25 \times 10^{-27}\ \text{kg}) + (23.25 \times 10^{-27}\ \text{kg})} \approx 11.625 \times 10^{-27}\ \text{kg}$

$B = \dfrac{h}{8\pi^2 c I_B} = \dfrac{h}{8\pi^2 c \mu r^2} \approx 2.001\ \text{cm}^{-1}$

The first three (rotational) Raman lines of a linear molecule occur at $12.0\ \text{cm}^{-1}$, $20.01\ \text{cm}^{-1}$ and $28.01\ \text{cm}^{-1}$.

The first three rotational Raman lines of a linear triatomic molecule are at 4.86 cm^{-1}, 8.14 cm^{-1} and 11.36 cm^{-1} from the exciting line. Estimate the rotational constant and the moment of inertia of the molecule.

The first three rotational Raman lines of a linear molecule occur at $6B$, $10B$ and $14B$.

$$\left.\begin{aligned} 6B &= 4.86\ \text{cm}^{-1} \\ 10B &= 8.14\ \text{cm}^{-1} \\ 14B &= 11.36\ \text{cm}^{-1} \end{aligned}\right\} \Rightarrow B = \frac{4.86\ \text{cm}^{-1} + 8.14\ \text{cm}^{-1} + 11.36\ \text{cm}^{-1}}{30} = 0.812\ \text{cm}^{-1} = 81.2\ \text{m}^{-1}$$

$I_B = \dfrac{h}{8\pi^2 c B} \approx 3.45 \times 10^{-46}\ \text{kg-m}^2$

Irradiation of Carbon tetrachloride by 4358 Å radiation gives Raman lines at 4400 Å, 4419 Å, 4447 Å. Calculate the Raman shift for each of these lines in cm^{-1}.

wavenumber of exciting line, $\bar{\nu}_0 = \dfrac{1}{4358 \times 10^{-8}\ \text{cm}} = \dfrac{10^8}{4358}\ \text{cm}^{-1}$

Raman shift of first line $= \left(\dfrac{10^8}{4358} - \dfrac{10^8}{4400}\right)\ \text{cm}^{-1} \approx 219.03\ \text{cm}^{-1}$

Raman shift of second line $= \left(\dfrac{10^8}{4358} - \dfrac{10^8}{4419}\right)\ \text{cm}^{-1} \approx 316.75\ \text{cm}^{-1}$

Raman shift of third line $= \left(\dfrac{10^8}{4358} - \dfrac{10^8}{4447}\right)\ \text{cm}^{-1} \approx 459.24\ \text{cm}^{-1}$

The first rotational Raman line of H_2 appears at 346 cm^{-1} from the exciting line. Calculate bond length of H_2 molecule. ($^1H = 1.673 \times 10^{-27}$ kg)

Since the nuclear spin of hydrogen is non-zero, the first (rotational) Raman line of the linear molecule occur at $6B$.

$\therefore B = \dfrac{346\ \text{cm}^{-1}}{6} \approx 5766.67\ \text{m}^{-1}$

Reduced mass, $\mu = \dfrac{1.0079 \times 1.0079}{1.0079 + 1.0079} u \approx 0.504u$

$B = \dfrac{h}{8\pi^2 c I_B} = \dfrac{h}{8\pi^2 c \mu r^2} \Longrightarrow$ Bond length, $r = \sqrt{\dfrac{h}{8\pi^2 c \mu B}} \approx 0.76 \times 10^{-10}$ m $= 0.76$ Å

The Raman line associated with a vibrational mode which is both Raman and infrared active is found at 4600 Å when excited by light of wavelength 4358 Å. Calculate the wavelength of the corresponding infrared band.

wavenumber of the exciting line, $\bar{\nu}_0 = \dfrac{1}{4358 \times 10^{-8} \text{ cm}} = \dfrac{10^8}{4358} \text{ cm}^{-1}$

wavenumber of the Raman (Stokes) line, $\bar{\nu}_0 - \bar{\nu}_m = \dfrac{10^8}{4600} \text{ cm}^{-1} \Longrightarrow \bar{\nu}_m = \left(\dfrac{10^8}{4358} - \dfrac{10^8}{4600}\right) \text{ cm}^{-1}$

wavenumber of the corresponding IR band, $\lambda_m = \dfrac{1}{\bar{\nu}_m} = 82838$ Å

If the bond length of H_2 is 0.07417 nm, what would be the positions of the first three rotational Raman lines in the spectrum? What is the effect of nuclear spin on the spectrum? ($^1H = 1.673 \times 10^{-27}$ kg)

Reduced mass, $\mu = \dfrac{1.0079 \times 1.0079}{1.0079 + 1.0079} u \approx 0.504u$

$B = \dfrac{h}{8\pi^2 c I_B} = \dfrac{h}{8\pi^2 c \mu r^2} \approx 60.801 \text{ cm}^{-1}$

The first three lines are at $\bar{\nu} = 6B, 10B$ and $14B \approx 364.8 \text{ cm}^{-1}, 608.01 \text{ cm}^{-1}$ and 851.2 cm^{-1}

The nuclear spin of hydrogen is $\dfrac{1}{2}$. Hence, there will be an alternation in the intensity of the spectral lines.

A Raman line was observed at 4768.5 Å when acetylene was irradiated by 4358.3 Å radiation. Calculate the equilibrium vibrational frequency that caused this shift.

Raman line for symmetric diatomic molecule occurs at $\bar{\nu} = \bar{\nu}_0 \pm \bar{\nu}_e$

$\bar{\nu}_0 - \bar{\nu}_e = \dfrac{1}{4768.5 \times 10^{-10} \text{ m}} = \dfrac{10^{10}}{4768.5} \text{ m}^{-1} \Longrightarrow \bar{\nu}_e = \dfrac{10^{10}}{4358.3} \text{ m}^{-1} - \dfrac{10^{10}}{4768.5} \text{ m}^{-1}$

$\therefore$ equilibrium vibrational frequency, $\nu_e = c\bar{\nu}_e \approx 59.17$ GHz

If Raman line is observed at 4768 Å, and exciting wavelength is 4358 Å, then $\nu_e \approx 59.15$ GHz

A Raman line is observed at 477 nm when acetylene interacts with 436 nm radiation. Calculate the vibrational frequency that causes this shift.

Raman line for symmetric diatomic molecule occurs at $\bar{\nu} = \bar{\nu}_0 \pm \bar{\nu}_e$

$\bar{\nu}_0 - \bar{\nu}_e = \dfrac{1}{477 \times 10^{-9} \text{ m}} = \dfrac{10^9}{477} \text{ m}^{-1} \Longrightarrow \bar{\nu}_e = \dfrac{10^9}{436} \text{ m}^{-1} - \dfrac{10^9}{477} \text{ m}^{-1}$

$\therefore$ equilibrium vibrational frequency, $\nu_e = c\bar{\nu}_e \approx 59$ GHz

If Raman line is observed at 475 nm, and exciting wavelength is 435 nm, then $\nu_e \approx 58$ GHz

In the rotational Raman spectrum of HCl, the shifts from the exciting line are represented by

$$\Delta\bar{\boldsymbol{\nu}} = (\mathbf{62.4} + \mathbf{41.6}\,\boldsymbol{J}) \text{ cm}^{-1}$$

Evaluate (i) the rotational constant, and (ii) the bond length of HCl molecule.

Since the nuclear spin of HCl is non-zero, the separation in rotational Raman spectrum of the linear molecule occur at $4B$.

$\therefore B = \dfrac{41.6 \text{ cm}^{-1}}{4} \approx 1040 \text{ m}^{-1}$

Reduced mass, $\mu = \dfrac{1 \times 35.5}{1 + 35.5} u \approx 0.9726u$

$$B = \frac{h}{8\pi^2 c I_B} = \frac{h}{8\pi^2 c\mu r^2} \Longrightarrow \text{Bond length, } r = \sqrt{\frac{h}{8\pi^2 c\mu B}} \approx 1.29 \times 10^{-10}\ \text{m} = 1.29\ \text{Å}$$

A substance shows Raman line at 4570 Å when the exciting line is at 4358 Å. Find the positions of Stokes and anti-Stokes lines for the same substance if the wavelength of the exciting line is 4047 Å.

wavenumber of old exciting line, $\bar{\nu}_0 = \frac{1}{4358 \times 10^{-8}\ \text{cm}} = \frac{10^8}{4358}\ \text{cm}^{-1}$

Raman shift of first line (Stokes), $\bar{\nu}_m = \left(\frac{10^8}{4358} - \frac{10^8}{4570}\right)\ \text{cm}^{-1}$

wavenumber of new exciting line, $\bar{\nu}_0{}' = \frac{1}{4047 \times 10^{-8}\ \text{cm}} = \frac{10^8}{4047}\ \text{cm}^{-1}$

wavenumber of new Stokes line , $\bar{\nu}_0{}' - \bar{\nu}_m = \frac{10^8}{4047}\ \text{cm}^{-1} - \left(\frac{10^8}{4358} - \frac{10^8}{4570}\right)\ \text{cm}^{-1}$

wavelength of new Stokes line , $\frac{1}{\bar{\nu}_0{}' - \bar{\nu}_m} = 4229.2\ \text{Å}$

wavelength of new anti-Stokes line , $\frac{1}{\bar{\nu}_0{}' + \bar{\nu}_m} = 3880.0\ \text{Å}$

If Raman line is shown at 4567 Å, then Stokes and anti-Stokes lines are at 4226.6 Å and 3880.0 Å respectively.

With exciting radiation of 4358 Å, the Raman spectrum of benzene showed lines corresponding to Raman shifts 608, 846, 995, 1178, 1599 and 3064 cm^{-1}. What would be the Raman wavelengths, if benzene is irradiated with light of wavelength 5461 Å.

wavenumber of new exciting line, $\bar{\nu}_0 = \frac{1}{5461 \times 10^{-8}\ \text{cm}} = \frac{10^8}{5461}\ \text{cm}^{-1}$

wavenumber of first Raman line $= \left(\frac{10^8}{5461} - 608\right)\ \text{cm}^{-1}$

$\therefore$ wavelength of first Raman line $= 5648.5\ \text{Å}$

wavenumber of second Raman line $= \left(\frac{10^8}{5461} - 846\right)\ \text{cm}^{-1}$

$\therefore$ wavelength of second Raman line $= 5725.5\ \text{Å}$

The vibrational Raman spectrum of $^{35}Cl_2$ shows a series of Stokes lines and anti-Stokes lines separated by 0.9752 cm^{-1}. Find the bond length of Cl_2 molecule.

Since the nuclear spin of chlorine is non-zero, the separation in vibrational Raman spectrum of the linear molecule occur at $4B$.

$\therefore B = \frac{0.9752\ \text{cm}^{-1}}{4} \approx 24.38\ \text{m}^{-1}$

Reduced mass, $\mu = \frac{35 \times 35}{35 + 35} u \approx 17.5u$

$$B = \frac{h}{8\pi^2 c I_B} = \frac{h}{8\pi^2 c\mu r^2} \Longrightarrow \text{Bond length, } r = \sqrt{\frac{h}{8\pi^2 c\mu B}} \approx 1.99 \times 10^{-10}\ \text{m} = 1.99\ \text{Å}$$

The first three Stokes lines in the rotational Raman spectrum of $^{16}O_2$ are separated by 14.4 cm^{-1}, 25.8 cm^{-1} and 37.4 cm^{-1} from the exciting radiation. Using the rigid rotor model, obtain the value of r_0.

Since the nuclear spin of oxygen is zero, alternate lines corresponding to even values of J will be missing. Hence, the first line will be at $10B_0$ with a separation of $8B_0$ between lines.

$10B_0 = 14.4\ \text{cm}^{-1}$

$$8B_0 = \begin{cases} 25.8\text{ cm}^{-1} - 14.4\text{ cm}^{-1} = 11.4\text{ cm}^{-1} \\ 37.4\text{ cm}^{-1} - 25.8\text{ cm}^{-1} = 11.6\text{ cm}^{-1} \end{cases} \Longrightarrow 8B_0 = \frac{11.4\text{ cm}^{-1} + 11.6\text{ cm}^{-1}}{2} = 11.5\text{ cm}^{-1}$$

$$\therefore B_0 = \frac{14.4\text{ cm}^{-1} + 11.5\text{ cm}^{-1}}{18} \approx 1.44\text{ cm}^{-1}$$

Reduced mass, $\mu = \dfrac{16 \times 16}{16 + 16}u = 8u$

$$B = \frac{h}{8\pi^2 c I_B} = \frac{h}{8\pi^2 c\mu r^2} \Longrightarrow \text{Bond length, } r = \sqrt{\frac{h}{8\pi^2 c\mu B}} \approx 1.21 \times 10^{-10}\text{ m} = 1.21\text{ Å}$$

The first Stokes line in the rotational Raman spectrum of $^{14}N^{15}N$ is observed at 11.5416 cm^{-1}. (a) What is its B value? (b) Calculate its bond length. (c) Will there be an intensity alternation in the spectrum of $^{14}N^{15}N$? (d) Will $^{14}N^{15}N$ show a rotational spectrum?

First line is formed at $6B$. So, $B = \dfrac{11.5416\text{ cm}^{-1}}{6} = 1.9236\text{ cm}^{-1} = 192.36\text{ m}^{-1}$

Reduced mass, $\mu = \dfrac{14 \times 15}{14 + 15}u \approx 7.24138u$

$$B = \frac{h}{8\pi^2 c I_B} = \frac{h}{8\pi^2 c\mu r^2} \Longrightarrow \text{Bond length, } r = \sqrt{\frac{h}{8\pi^2 c\mu B}} \approx 1.1 \times 10^{-10}\text{ m} = 1.1\text{ Å}$$

For the ground state of oxygen, the values of $\bar{\nu}_e$ and $\bar{\nu}_e x_e$ are 1580.4 and 12.07 cm^{-1} respectively. Calculate the zero point energy and the expected vibrational Raman shift.

Vibrational Raman shift, $\bar{\nu}_m = \bar{\nu}_e(1 - 2x_e) = \bar{\nu}_e - 2\bar{\nu}_e x_e = 1556.26\text{ cm}^{-1}$

Vibrational energy (in wavenumber), $\varepsilon_v = \left(v + \dfrac{1}{2}\right)\bar{\nu}_e - \left(v + \dfrac{1}{2}\right)^2 \bar{\nu}_e x_e$

For zero point energy, $\varepsilon_v = \dfrac{1}{2}\bar{\nu}_e - \left(\dfrac{1}{2}\right)^2 \bar{\nu}_e x_e = 787.18\text{ cm}^{-1}$

Nuclear Magnetic Resonance (NMR)

Atomic Number	Neutron	Nuclear spin	Examples
Even	Even	0	$_2He^4$, $_6C^{12}$, $_8O^{16}$, $_{16}S^{32}$, …
Odd	Odd	1, 2, 3, …	$_1D^2$, $_3Li^6$, $_5B^{10}$, $_7N^{14}$, …
Even	Odd	1/2, 3/2, 5/2, …	$_6C^{13}$, $_8O^{17}$, …
Odd	Even	1/2, 3/2, 5/2, …	$_1H^1$, $_3Li^7$, $_7N^{15}$, $_9F^{19}$, $_{11}Na^{23}$ …

$$\begin{cases} E = -\vec{\mu} \cdot \vec{B} \\ \vec{\mu} = g_N \mu_N \vec{I} \end{cases} \Longrightarrow E = -g_N \mu_N \vec{I} \cdot \vec{B} = -g_N \mu_N m_I B_z \quad \begin{bmatrix} \because \vec{B} = B_z \hat{k} \\ \text{and } I_z = m_I \end{bmatrix}$$

$$\therefore \Delta E = g_N \mu_N B_z \Delta m_I = g_N \mu_N B_z \quad [\because \Delta m_I = \pm 1] \Longrightarrow \begin{cases} h\nu = g_N \mu_N B_z \Longrightarrow \boldsymbol{\nu} = \dfrac{\boldsymbol{g_N \mu_N B_z}}{h} \\ \hbar\omega = g_N \mu_N B_z \Longrightarrow \boldsymbol{\omega} = \dfrac{\boldsymbol{g_N \mu_N B_z}}{\hbar} \end{cases}$$

$$\text{For a nucleus with } I = \frac{1}{2}, \begin{cases} E_{+1/2} = -\dfrac{1}{2} g_N \mu_N B_z \\ E_{-1/2} = \dfrac{1}{2} g_N \mu_N B_z \end{cases}$$

Electron Spin Resonance

$$\begin{cases} E = -\vec{\mu}_s \cdot \vec{B} \\ \vec{\mu}_s = g_e \mu_B \vec{S} = -g_s \mu_B \vec{S} \quad [\because g_s = |g_e| = -g_e] \end{cases} \Longrightarrow E = g_s \mu_B \vec{S} \cdot \vec{B} = g_s \mu_B m_s B_z \quad [\because \vec{B} = B_z \hat{k} \text{ and } S_z = m_s]$$

$$\begin{cases} E_{+1/2} = \dfrac{1}{2} g_s \mu_B B_z \\ E_{-1/2} = -\dfrac{1}{2} g_s \mu_B B_z \end{cases} \Rightarrow E_{+1/2} - E_{-1/2} = \Delta E = g_s \mu_B B_z \Rightarrow \begin{cases} h\nu = g_s \mu_B B_z \Rightarrow \boldsymbol{\nu} = \dfrac{\boldsymbol{g_s \mu_B B_z}}{h} \\ \hbar\omega = g_s \mu_B B_z \Rightarrow \boldsymbol{\omega} = \dfrac{\boldsymbol{g_s \mu_B B_z}}{\hbar} \end{cases}$$

A system of protons at a temperature of 25°C is placed in a magnetic field of 2T. What is the relative population of protons in the two spin states?

^{11}B has I = 3/2 and g_N = 1.794. Find the energy levels of ^{11}B and the number of transitions when it is placed in a magnetic field 1.5 T.

The g_N value for ^{19}F nucleus is 5.256 and $\mu_N = 5.0504 \times 10^{-27}$ J/T. Calculate the resonance frequency when it is placed in a magnetic field of strength 1.0 T.

The chemical shift of the methyl protons in acetaldehyde is δ = 2.22 ppm, and that of aldehyde proton is δ = 9.80 ppm. What is the difference in local magnetic field between the two regions of the molecule when the applied field is (a) 1.51 T (b) 7.0 T? How does the appearance of the spectrum change when recorded at 300 MHz instead of 60 MHz?

A particular NMR instrument operates at 30.256 MHz. What magnetic fields are required to bring a proton nucleus and a ^{13}C nucleus to resonance at this frequency? Magnetic moment of proton is 2.2927 μ_N and of ^{13}C is 0.7022 μ_N.

Predict the J^P for the following odd-A nuclides: $_2He_1$, $_3Li_4$, $_6C_7$, $_8O_9$, $_9F_{10}$, $_{11}Na_{12}$, $_{12}Mg_{13}$, $_{14}Si_{15}$, $_{15}P_{16}$, $_{19}K_{20}$, $_{25}Mn_{30}$.
(Conjectured configurations which yield the observed value are indicated in bold.)

Parity, $P = (-1)^l$

$_2He_1$ [neutron] — $1s_{1/2}$ [1] — $^1/_2{}^+$

$_3Li_4$ [proton] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [1] — $^3/_2{}^-$

$_6C_7$ [neutron] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [1] — $^1/_2{}^-$

$_8O_9$ [neutron] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [2] ; $1d_{5/2}$ [1] — $^5/_2{}^+$

$_9F_{10}$ [proton] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [2] ; $1d_{5/2}$ [1] — $^5/_2{}^+$

$_9F_{10}$ [proton] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [2] ; $1d_{5/2}$ [0], $2s_{1/2}$ [1] — $^1/_2{}^+$

$_{11}Na_{12}$ [proton] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [2] ; $1d_{5/2}$ [3] — $^5/_2{}^+$

$_{11}Na_{12}$ [proton] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [2] ; $1d_{5/2}$ [0], $2s_{1/2}$ [2], $1d_{3/2}$ [1] — $^3/_2{}^+$

$_{12}Mg_{13}$ [neutron] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [2] ; $1d_{5/2}$ [5] — $^5/_2{}^+$

$_{14}Si_{15}$ [neutron] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [2] ; $1d_{5/2}$ [6], $2s_{1/2}$ [1] — $^1/_2{}^+$

$_{15}P_{16}$ [proton] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [2] ; $1d_{5/2}$ [6], $2s_{1/2}$ [1] — $^1/_2{}^+$

$_{19}K_{20}$ [proton] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [2] ; $1d_{5/2}$ [6], $2s_{1/2}$ [2], $1d_{3/2}$ [3] — $^3/_2{}^+$

$_{25}Mn_{30}$ [proton] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [2] ; $1d_{5/2}$ [6], $2s_{1/2}$ [2], $1d_{3/2}$ [4] ; $1f_{7/2}$ [5] — $^7/_2{}^-$

$_{25}Mn_{30}$ [proton] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [2] ; $1d_{5/2}$ [6], $2s_{1/2}$ [2], $1d_{3/2}$ [4] ; $1f_{7/2}$ [0], $2p_{3/2}$ [4], $1f_{5/2}$ [1] — $^5/_2{}^-$

$_{28}Ni_{29}$ [neutron] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [2] ; $1d_{5/2}$ [6], $2s_{1/2}$ [2], $1d_{3/2}$ [4] ; $1f_{7/2}$ [8], $2p_{3/2}$ [1], $1f_{5/2}$ [0] — $^3/_2{}^-$

Predict the J^P for the following odd-odd nuclides: $_7N_7$, $_{19}K_{23}$, $_{27}Co_{33}$.
(Pairing effect considered configurations which yield the observed value are indicated in bold.)

$_7N_7$ [proton] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [1] — $^1/_2{}^-$

$_7N_7$ [neutron] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [1] — $^1/_2{}^-$

Nordheim number, $N^* = (j_p + j_n) - (l_p + l_n) = (^1/_2 + {}^1/_2) - (1 + 1) = 1 - 2 = -1$

$J^P = (j_p + j_n)$ with $(-1)^{(l_p + l_n)}$ as parity = 1^+

$_{15}P_{15}$ [proton] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [2] ; $1d_{5/2}$ [6], $2s_{1/2}$ [1] — $^1/_2{}^+$

$_{15}P_{15}$ [neutron] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [2] ; $1d_{5/2}$ [6], $2s_{1/2}$ [1] — $^1/_2{}^+$

Nordheim number, N* = $(j_p + j_n) - (l_p + l_n) = (^1/_2 + {}^1/_2) - (0 + 0) = 1 - 0 = 1$

$J^P = (j_p + j_n)$ with $(-1)^{(l_p + l_n)}$ as parity = 1^+

$_{19}K_{23}$ [proton] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [2] ; $1d_{5/2}$ [6], $2s_{1/2}$ [2], $1d_{3/2}$ [3] — $^3/_2{}^+$

$_{19}K_{23}$ [neutron] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [2] ; $1d_{5/2}$ [6], $2s_{1/2}$ [2], $1d_{3/2}$ [4] ; $1f_{7/2}$ [3] — $^7/_2{}^-$

Nordheim number, N* = $(j_p + j_n) - (l_p + l_n) = (^3/_2 + {}^7/_2) - (2 + 3) = 5 - 5 = 0$

$J^P = |j_p - j_n|$ with $(-1)^{(l_p + l_n)}$ as parity = 2^-

$_{27}Co_{33}$ [proton] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [2] ; $1d_{5/2}$ [6], $2s_{1/2}$ [2], $1d_{3/2}$ [4] ; $1f_{7/2}$ [7] — $^7/_2{}^-$

$_{27}Co_{33}$ [neutron] — $1s_{1/2}$ [2] ; $1p_{3/2}$ [4], $1p_{1/2}$ [2] ; $1d_{5/2}$ [6], $2s_{1/2}$ [2], $1d_{3/2}$ [4] ; $1f_{7/2}$ [8], $2p_{3/2}$ [1], $1f_{5/2}$ [4] — $^3/_2{}^-$

Nordheim number, N* = $(j_p + j_n) - (l_p + l_n) = (^7/_2 + {}^3/_2) - (3 + 1) = 5 - 4 = 1$

$J^P = (j_p + j_n)$ with $(-1)^{(l_p + l_n)}$ as parity = 5^+

One-dimensional Monoatomic Periodic Lattice

The force exerted on the n^{th} atom by the $(n+1)^{\text{th}}$ atom (spring on right) is $k(u_{n+1} - u_n)$ towards right
The force exerted on the n^{th} atom by the $(n-1)^{\text{th}}$ atom (spring on left) is $k(u_n - u_{n-1})$ towards left
u_{n-1}, u_n, u_{n+1} are the displacements of the $(n-1)^{\text{th}}, n^{\text{th}}, (n+1)^{\text{th}}$ atoms respectively, k is the force constant. of the interatomic bond, and M is the mass of an atom. Assume that the last atom of the chain is also the first.

$\therefore$ Total force on the n^{th} atom is $M\dfrac{d^2u_n}{dt^2} = k(u_{n+1} - u_n) - k(u_n - u_{n-1}) = k(u_{n-1} - 2u_n + u_{n+1})$

Trial solution: $u_n = Ae^{i(\kappa x_n - \omega t)}$ $\begin{bmatrix} x_n = na \text{ is the equilibrium} \\ \text{position of the } n^{\text{th}} \text{ atom} \end{bmatrix} \Longrightarrow u_n = Ae^{i(n\kappa a - \omega t)}$

$$\therefore M\frac{d^2u_n}{dt^2} = k(u_{n-1} - 2u_n + u_{n+1}) \Longrightarrow -M\omega^2 u_n = k(e^{-i\kappa a} - 2 + e^{i\kappa a})u_n \Longrightarrow \omega^2 = \frac{2k}{M}(1 - \cos\kappa a)$$

$$\Longrightarrow \omega^2 = \frac{4k}{M}\sin^2\frac{\kappa a}{2} \Longrightarrow \boldsymbol{\omega} = \sqrt{\frac{\boldsymbol{4k}}{\boldsymbol{M}}}\left|\mathbf{sin}\frac{\boldsymbol{\kappa a}}{\mathbf{2}}\right| \Longrightarrow \text{maximum frequency, } \boldsymbol{\omega}_{\mathbf{m}} = \sqrt{\frac{\boldsymbol{4k}}{\boldsymbol{M}}} \begin{bmatrix} \kappa a = (2n-1)\pi \\ \text{i.e. } \kappa = (2n-1)\dfrac{\pi}{a} \end{bmatrix}$$

$$\therefore \boldsymbol{\omega} = \boldsymbol{\omega}_{\mathbf{m}}\left|\mathbf{sin}\frac{\boldsymbol{\kappa a}}{\mathbf{2}}\right| \quad [\textbf{dispersion relation}] \Longrightarrow \omega = \left(\frac{\omega_{\text{m}}}{\kappa}\left|\sin\frac{\kappa a}{2}\right|\right)\kappa \Longrightarrow v_{\text{s}} = \frac{\omega_{\text{m}}}{\kappa}\left|\sin\frac{\kappa a}{2}\right| \quad \left[\because \frac{\omega}{\kappa} = v_{\text{p}} = v_{\text{s}}\right]$$

Symmetry in κ-space

First Brillouin zone/BZ (principle range): $-\dfrac{\pi}{a} \le \kappa \le \dfrac{\pi}{a}$

Translational symmetry: $\omega\left(\kappa + \dfrac{2\pi}{a}\right) = \omega(\kappa)$

Mirror symmetry about $\kappa = \dfrac{n\pi}{a}$: $\omega(-\kappa) = \omega(\kappa)$

Number of modes

$u_n = u_{n+N} \Longrightarrow Ae^{i(\kappa na - \omega t)} = Ae^{i(\kappa(n+N)a - \omega t)} \Longrightarrow 1 = e^{i\kappa Na} \Longrightarrow e^{i\kappa L} = 1$ [length of the ring, $L = Na$]

$\Longrightarrow \kappa L = 2n\pi \Longrightarrow \kappa = \dfrac{2n\pi}{L}$ [allowed values of κ]

$$\therefore \text{number of modes in the first BZ} = \frac{2\pi/a}{2\pi/L} \quad [\because n = 1] = \frac{L}{a} = N = \text{number of atoms in the lattice}$$
$$= \text{number of modes (allowed values of } \kappa\text{) in the first BZ}$$

Phase velocity and Group velocity

Phase velocity, $v_{\mathrm{p}} = \frac{\omega}{\kappa} = \frac{\omega_{\mathrm{m}}}{\kappa}\left|\sin\frac{\kappa a}{2}\right| = \left(\frac{\omega_{\mathrm{m}} a}{2}\right)\frac{\left|\sin\frac{\kappa a}{2}\right|}{\frac{\kappa a}{2}} \Rightarrow \boldsymbol{v}_{\mathbf{p}} = (\boldsymbol{v}_{\mathrm{s}})_{\mathbf{max}}\left|\mathbf{sinc}\frac{\boldsymbol{\kappa a}}{\mathbf{2}}\right|$

Group velocity, $v_{\mathrm{g}} = \frac{d\omega}{d\kappa} = \frac{\omega_{\mathrm{m}} a}{2}\left|\cos\frac{\kappa a}{2}\right| \Rightarrow \boldsymbol{v}_{\mathbf{g}} = (\boldsymbol{v}_{\mathrm{s}})_{\mathbf{max}}\left|\mathbf{cos}\frac{\boldsymbol{\kappa a}}{\mathbf{2}}\right|$

In the long wavelength limit, $v_{\mathrm{p}} = (v_{\mathrm{s}})_{\max}\lim_{\kappa\to 0}\frac{\left|\sin\frac{\kappa a}{2}\right|}{\frac{\kappa a}{2}} = (v_{\mathrm{s}})_{\max}$ and $v_{\mathrm{g}} = (v_{\mathrm{s}})_{\max}\lim_{\kappa\to 0}\left|\cos\frac{\kappa a}{2}\right| = (v_{\mathrm{s}})_{\max} = v_{\mathrm{p}}$

Long wavelength limit

$$\lim_{\lambda\to\infty}\omega = \lim_{\kappa\to 0}\omega = \lim_{\kappa\to 0}\omega_{\mathrm{m}}\left|\sin\frac{\kappa a}{2}\right| = \left(\frac{\omega_{\mathrm{m}} a}{2}\right)\kappa = \kappa a\sqrt{\frac{k}{M}} \quad \left[\because \omega_{\mathrm{m}} = \sqrt{\frac{4k}{M}}\right]$$

$$\therefore \text{ when } \lambda \gg a, \omega = \left(\frac{\omega_{\mathrm{m}} a}{2}\right)\kappa \Rightarrow 2\pi\nu = \left(\frac{\omega_{\mathrm{m}} a}{2}\right)\frac{2\pi}{\lambda} \Rightarrow \frac{\omega_{\mathrm{m}} a}{2} = \nu\lambda = v_{\mathrm{p}} \Rightarrow (\boldsymbol{v}_{\mathrm{s}})_{\mathbf{max}} = \frac{\boldsymbol{\omega}_{\mathrm{m}}\boldsymbol{a}}{\mathbf{2}} \quad \left[\begin{matrix}\because \left|\mathrm{sinc}\frac{\kappa a}{2}\right| \to 1 \\ \text{as } \kappa \to 0\end{matrix}\right]$$

$$\because (\boldsymbol{v}_{\mathrm{s}})_{\mathbf{max}} = \sqrt{\frac{\boldsymbol{Y}}{\boldsymbol{\rho}}} \quad \therefore \sqrt{\frac{Y}{\rho}} = \frac{\omega_{\mathrm{m}} a}{2} \Rightarrow a\sqrt{\frac{k}{M}} = \sqrt{\frac{Y}{\rho}} \Rightarrow k = \frac{MY}{\rho a^2} \Rightarrow \boldsymbol{k} = \boldsymbol{aY} \quad \left[\because \rho = \frac{M}{a^3}\right]$$

One-dimensional Diatomic Periodic Lattice

Total force on the $(2n)^{\mathrm{th}}$ atom is $M\frac{d^2u_{2n}}{dt^2} = k(u_{2n+1} - u_{2n}) - k(u_{2n} - u_{2n-1}) = k(u_{2n-1} - 2u_{2n} + u_{2n+1})$

Total force on the $(2n-1)^{\mathrm{th}}$ atom is $m\frac{d^2u_{2n-1}}{dt^2} = k(u_{2n} - u_{2n-1}) - k(u_{2n-1} - u_{2n-2}) = k(u_{2n-2} - 2u_{2n-1} + u_{2n})$

Trial solution: $\begin{cases} u_{2n} = Ae^{i(\kappa x_{2n} - \omega t)} \\ u_{2n-1} = Be^{i(\kappa x_{2n-1} - \omega t)}\end{cases} \left[\begin{matrix} x_n = 2na \text{ is the equilibrium} \\ \text{position of the } (2n)^{\mathrm{th}} \text{ atom}\end{matrix}\right] \Rightarrow \begin{cases} u_{2n} = Ae^{i(2n\kappa a - \omega t)} \\ u_{2n-1} = Be^{i((2n-1)\kappa a - \omega t)}\end{cases}$

$$\therefore \begin{cases} M\frac{d^2u_{2n}}{dt^2} = k(u_{2n-1} - 2u_{2n} + u_{2n+1}) \\ m\frac{d^2u_{2n-1}}{dt^2} = k(u_{2n-2} - 2u_{2n-1} + u_{2n})\end{cases} \Rightarrow \begin{cases} -M\omega^2 u_{2n} = k\left(\frac{B}{A}e^{-i\kappa a} - 2 + \frac{B}{A}e^{i\kappa a}\right)u_{2n} \\ -m\omega^2 u_{2n-1} = k\left(\frac{A}{B}e^{-i\kappa a} - 2 + \frac{A}{B}e^{i\kappa a}\right)u_{2n-1}\end{cases}$$

$$\Rightarrow \begin{cases} -M\omega^2 A = k(Be^{-i\kappa a} - 2A + Be^{i\kappa a}) \\ -m\omega^2 B = k(Ae^{-i\kappa a} - 2B + Ae^{i\kappa a})\end{cases} \Rightarrow \begin{cases}(M\omega^2 - 2k)A + (2k\cos\kappa a)B = 0 \\ (2k\cos\kappa a)A + (m\omega^2 - 2k)B = 0\end{cases} \left[\begin{matrix}\text{For a non-trivial solution} \\ \text{to exist, determinant of the} \\ \text{coeff. matrix must be zero.}\end{matrix}\right]$$

$$\Rightarrow \begin{vmatrix} M\omega^2 - 2k & 2k\cos\kappa a \\ 2k\cos\kappa a & m\omega^2 - 2k\end{vmatrix} = 0 \Rightarrow (M\omega^2 - 2k)(m\omega^2 - 2k) - 4k^2\cos^2\kappa a = 0$$

$$\Rightarrow (Mm\omega^4 - 2k(M+m)\omega^2 + 4k^2) - 4k^2\cos^2\kappa a = 0 \Rightarrow \omega^4 - \frac{2k}{\mu}\omega^2 + \frac{4k^2}{Mm}\sin^2\kappa a = 0$$

$$\Rightarrow \omega^2 = \frac{k}{\mu} \pm k\sqrt{\frac{1}{\mu^2} - \frac{4\sin^2\kappa a}{Mm}} \Rightarrow \begin{cases} {\omega^2}_- = \frac{k}{\mu} - k\sqrt{\frac{1}{\mu^2} - \frac{4\sin^2\kappa a}{Mm}} \quad \text{[Acoustic branch]} \\ {\omega^2}_+ = \frac{k}{\mu} + k\sqrt{\frac{1}{\mu^2} - \frac{4\sin^2\kappa a}{Mm}} \quad \text{[Optical branch]}\end{cases}$$

$$\begin{cases}(M\omega^2 - 2k)A + (2k\cos\kappa a)B = 0 \Rightarrow \frac{A}{B} = \frac{2k\cos\kappa a}{2k - M\omega^2} \\ (2k\cos\kappa a)A + (m\omega^2 - 2k)B = 0 \Rightarrow \frac{A}{B} = \frac{2k - m\omega^2}{2k\cos\kappa a}\end{cases} \Rightarrow \begin{cases}\text{If } \kappa a = \frac{\pi}{2} \text{ and } \omega^2 \neq \sqrt{\frac{2k}{M}} \text{ then } A = 0 \\ \text{If } \kappa a = \frac{\pi}{2} \text{ and } \omega^2 \neq \sqrt{\frac{2k}{m}} \text{ then } B = 0\end{cases}$$

Acoustic branch

$$\omega^2{}_- = \frac{k}{\mu} - k\sqrt{\frac{1}{\mu^2} - \frac{4\sin^2\kappa a}{Mm}} = \frac{k}{\mu}\left(1 - \sqrt{1 - \frac{4\mu^2\sin^2\kappa a}{Mm}}\right)$$

$$\lim_{\kappa a\to 0}\omega^2{}_- = \frac{k}{\mu}\left(1 - \lim_{\kappa a\to 0}\sqrt{1 - \frac{4\mu^2\sin^2\kappa a}{Mm}}\right) = \frac{k}{\mu}\left(1 - \left(1 - \lim_{\kappa a\to 0}\frac{2\mu^2\sin^2\kappa a}{Mm}\right)\right) = \frac{k}{\mu}\left(1 - \left(1 - \frac{2\mu^2(\kappa a)^2}{Mm}\right)\right)$$

$$= \frac{2k\mu\kappa^2 a^2}{Mm} \Longrightarrow (\omega^2{}_-)_{\min} = \frac{2k\kappa^2 a^2}{M+m} \Longrightarrow \boldsymbol{(\omega_-)_{\min} = \kappa a\sqrt{\frac{2k}{M+m}}} \Longrightarrow \begin{cases} (\omega_-)_{\min} = 0 \text{ when } \kappa = 0 \\ v_s = \omega_{m-}a \quad \left[\because \frac{\omega}{\kappa} = v_p = v_s \text{ and } \omega_{\min-} = \sqrt{\frac{2k}{M+m}}\right] \end{cases}$$

$$\begin{cases} \frac{A}{B} = \frac{2k\cos\kappa a}{2k - M\omega^2{}_-} = \frac{2k\cos\kappa a}{2k - M(\kappa a)^2\frac{2k}{M+m}} = \frac{(M+m)\cos\kappa a}{(M+m) - M(\kappa a)^2} \Longrightarrow \boldsymbol{\frac{A}{B} \to 1 \text{ as } \kappa a \to 0} \Longrightarrow A = B \\ \frac{A}{B} = \frac{2k - m\omega^2{}_-}{2k\cos\kappa a} = \frac{2k - m(\kappa a)^2\frac{2k}{M+m}}{2k\cos\kappa a} = \frac{(M+m) - m(\kappa a)^2}{(M+m)\cos\kappa a} \Longrightarrow \boldsymbol{\frac{A}{B} \to 1 \text{ as } \kappa a \to 0} \Longrightarrow A = B \end{cases}$$

$$\lim_{\kappa a\to\frac{\pi}{2}}\omega^2{}_- = \frac{k}{\mu}\left(1 - \lim_{\kappa a\to\frac{\pi}{2}}\sqrt{1 - \frac{4\mu^2\sin^2\kappa a}{Mm}}\right) = \frac{k}{\mu}\left(1 - \sqrt{1 - \frac{4\mu^2}{Mm}}\right) = \frac{k}{\mu}\left(1 - \sqrt{1 - \frac{4Mm}{(M+m)^2}}\right)$$

$$= \frac{k}{\mu}\left(1 - \frac{M-m}{M+m}\right) = \frac{k}{\mu}\left(\frac{2m}{M+m}\right) \Longrightarrow (\omega^2{}_-)_{\max} = \frac{2k}{M} \Longrightarrow \boldsymbol{(\omega_-)_{\max} = \sqrt{\frac{2k}{M}}} \left[\begin{array}{c}\text{independent of the lighter atom, and} \\ \text{identical to the monoatomic case}\end{array}\right]$$

$$\begin{cases} \frac{A}{B} = \frac{2k\cos\kappa a}{2k - M\omega^2{}_-} = \frac{2k\cos\kappa a}{2k - 2k} = \frac{\cos\kappa a}{1-1} \to \frac{1}{0}\left(\frac{\pi}{2} - \kappa a\right) \Longrightarrow \boldsymbol{\frac{A}{B} \to \infty \text{ as } \kappa a \to \frac{\pi}{2}} \Longrightarrow B = 0 \\ \frac{A}{B} = \frac{2k - m\omega^2{}_-}{2k\cos\kappa a} = \frac{2k - m\frac{2k}{M}}{2k\cos\kappa a} = \frac{1 - \frac{m}{M}}{\cos\kappa a} \to \frac{1 - \frac{m}{M}}{\frac{\pi}{2} - \kappa a} \Longrightarrow \boldsymbol{\frac{A}{B} \to \infty \text{ as } \kappa a \to \frac{\pi}{2}} \Longrightarrow B = 0 \end{cases}$$

Optical branch

$$\omega^2{}_+ = \frac{k}{\mu} + k\sqrt{\frac{1}{\mu^2} - \frac{4\sin^2\kappa a}{Mm}} = \frac{k}{\mu}\left(1 + \sqrt{1 - \frac{4\mu^2\sin^2\kappa a}{Mm}}\right)$$

$$\lim_{\kappa a\to 0}\omega^2{}_+ = \frac{k}{\mu}\left(1 + \lim_{\kappa a\to 0}\sqrt{1 - \frac{4\mu^2\sin^2\kappa a}{Mm}}\right) = \frac{k}{\mu}\left(1 + \left(1 - \lim_{\kappa a\to 0}\frac{2\mu^2\sin^2\kappa a}{Mm}\right)\right) = \frac{2k}{\mu}$$

$$\Longrightarrow (\omega^2{}_+)_{\max} = \frac{2k}{\mu} \Longrightarrow \boldsymbol{(\omega_+)_{\max} = \sqrt{\frac{2k}{\mu}}}$$

$$\begin{cases} \frac{A}{B} = \frac{2k\cos\kappa a}{2k - M\omega^2{}_+} = \frac{2k\cos\kappa a}{2k - M\frac{2k}{\mu}} = \frac{\cos\kappa a}{1 - \frac{M+m}{m}} = \frac{m\cos\kappa a}{-M} \Longrightarrow \boldsymbol{\frac{A}{B} \to -\frac{m}{M} \text{ as } \kappa a \to 0} \Longrightarrow MA + mB = 0 \\ \frac{A}{B} = \frac{2k - m\omega^2{}_+}{2k\cos\kappa a} = \frac{2k - m\frac{2k}{\mu}}{2k\cos\kappa a} = \frac{1 - \frac{M+m}{M}}{\cos\kappa a} = \frac{-m}{M\cos\kappa a} \Longrightarrow \boldsymbol{\frac{A}{B} \to -\frac{m}{M} \text{ as } \kappa a \to 0} \Longrightarrow MA + mB = 0 \end{cases}$$

$$\lim_{\kappa a\to\frac{\pi}{2}}\omega^2{}_+ = \frac{k}{\mu}\left(1 + \lim_{\kappa a\to\frac{\pi}{2}}\sqrt{1 - \frac{4\mu^2\sin^2\kappa a}{Mm}}\right) = \frac{k}{\mu}\left(1 + \sqrt{1 - \frac{4\mu^2}{Mm}}\right) = \frac{k}{\mu}\left(1 + \sqrt{1 - \frac{4Mm}{(M+m)^2}}\right)$$

$$= \frac{k}{\mu}\left(1 + \frac{M-m}{M+m}\right) = \frac{k}{\mu}\left(\frac{2M}{M+m}\right) \Longrightarrow (\omega^2{}_+)_{\min} = \frac{2k}{m} \Longrightarrow (\boldsymbol{\omega_+})_{\mathbf{min}} = \sqrt{\frac{\boldsymbol{2k}}{\boldsymbol{m}}} \begin{bmatrix}\text{independent of the heavier atom,}\\ \text{and identical to the monoatomic case}\end{bmatrix}$$

$$\begin{cases} \dfrac{A}{B} = \dfrac{2k\cos\kappa a}{2k - M\omega^2{}_+} = \dfrac{2k\cos\kappa a}{2k - M\dfrac{2k}{m}} = \dfrac{\cos\kappa a}{1 - \dfrac{M}{m}} \to \dfrac{-m\left(\frac{\pi}{2} - \kappa a\right)}{M-m} \Longrightarrow \dfrac{\boldsymbol{A}}{\boldsymbol{B}} \to \mathbf{0} \text{ as } \boldsymbol{\kappa a} \to \dfrac{\boldsymbol{\pi}}{\mathbf{2}} \Longrightarrow A = 0 \\ \dfrac{A}{B} = \dfrac{2k - m\omega^2{}_+}{2k\cos\kappa a} = \dfrac{2k-2k}{2k\cos\kappa a} = \dfrac{1-1}{\cos\kappa a} \to \dfrac{0}{\left(\frac{\pi}{2} - \kappa a\right)} \Longrightarrow \dfrac{\boldsymbol{A}}{\boldsymbol{B}} \to \mathbf{0} \text{ as } \boldsymbol{\kappa a} \to \dfrac{\boldsymbol{\pi}}{\mathbf{2}} \Longrightarrow A = 0 \end{cases}$$

The speed of sound in a certain linear monoatomic lattice is 1.08 × 10^4 m/s. If the mass of each atom is 6.81 × 10^{-26} kg and the interatomic distance at equilibrium is 4.85 Å, find the force constant and maximum normal mode (angular) frequency.

Speed of sound, $v_s \approx \dfrac{\omega_m a}{2} = a\sqrt{\dfrac{k}{m}}$ $\left[\because \omega_m = \sqrt{\dfrac{4k}{m}}\right] \Longrightarrow$ Spring constant, $k \approx \dfrac{v_s{}^2 m}{a^2} \approx 33.77$ N/m

Maximum (natural cut-off) normal mode frequency, $\omega_m = \sqrt{\dfrac{4k}{m}} \approx 4.454 \times 10^{13}$ rad/s $\Longrightarrow \nu_m \approx 7.1 \times 10^{12}$ Hz

If the velocity of sound in a solid is taken to be 3 × 10^3 m/s and the interatomic distance is 3 × 10^{-10} m, calculate the value of cut-off frequency assuming a linear monoatomic lattice.

Speed of sound, $v_s = \dfrac{\omega_m}{\kappa}\left|\sin\dfrac{\kappa a}{2}\right| \Longrightarrow$ Cut-off (angular) frequency, $\omega_m = \dfrac{\kappa v_s}{\left|\sin\dfrac{\kappa a}{2}\right|} = \dfrac{\pi v_s}{a}$ $\left[\because \kappa = \dfrac{\pi}{a} \text{ at cut-off}\right]$

Alternatively, $\because$ at cut-off, $\kappa = \dfrac{\pi}{a} \Longrightarrow \lambda = 2a$ $\therefore$ Cut-off (linear) frequency, $\nu_m = \dfrac{v_s}{2a} = 5 \times 10^{12}$ Hz

If the velocity of sound in a solid is of the order 10^3 m/s, compare the frequency of the sound wave of *λ* = 10 Å for (a) a monoatomic system, and (b) acoustic waves and optical waves in a diatomic system containing two identical atoms (*M* = *m*) per unit cell of interatomic spacing 2.5 Å.

For the monoatomic system, $\omega = v_s\kappa = \dfrac{2\pi v_s}{\lambda} = 6.283 \times 10^{12}$ rad/s

For acoustic waves in a diatomic system, $\begin{cases} (\omega_-)_{\max} = \sqrt{\dfrac{2k}{M}} \text{ for } \kappa a = \dfrac{\pi}{2} \Longrightarrow \omega_{\max} = \dfrac{\pi v_p}{2a} \\ (\omega_-)_{\min} = 0 \text{ for } \kappa a = 0 \Longrightarrow \omega_{\min} = 0 \end{cases}$ $\left[\because \kappa a = \dfrac{2\pi a}{\lambda} = \dfrac{2\pi\nu a}{v_p} = \dfrac{\omega a}{v_p}\right]$

For optical waves in a diatomic system, $\begin{cases} (\omega_+)_{\max} = \sqrt{\dfrac{4k}{M}} \text{ for } \kappa a = 0 \Longrightarrow \omega = 0 \\ (\omega_+)_{\min} = \sqrt{\dfrac{2k}{m}} \text{ for } \kappa a = \dfrac{\pi}{2} \Longrightarrow \omega = \dfrac{\pi v_s}{2a} \end{cases}$ $\left[\because \kappa a = \dfrac{2\pi a}{\lambda} = \dfrac{2\pi\nu a}{v_s} = \dfrac{\omega a}{v_s}\right]$

For the diatomic system, $\omega^2 = \dfrac{2k}{M} \pm \dfrac{2k}{M}\sqrt{1 - \sin^2\kappa a} \Longrightarrow \omega = \sqrt{\dfrac{2k}{M}(1 \pm \cos\kappa a)}$ $\left[\because \mu = \dfrac{Mm}{M+m} = \dfrac{M}{2}\right]$

At $\kappa a = \dfrac{\pi}{2}$

The unit cell parameter of NaCl crystal is 5.6 Å and the modulus of elasticity along [100] direction is 5×10^{10} N/m^2. Estimate the wavelength at which and electromagnetic radiation is strongly reflected by the crystal.

$$\mu = \left(\frac{1}{M_{\text{Na}}} + \frac{1}{M_{\text{Cl}}}\right)^{-1} = \left(\frac{1}{23\,u} + \frac{1}{35.5\,u}\right)^{-1} \approx 13.96\,u$$

$$(\omega_+)_{\max} = \sqrt{\frac{2k}{\mu}} = \sqrt{\frac{2aY}{\mu}}\ \ [\because k = 2aY] = \sqrt{\frac{2 \times 5.6 \times 10^{-10} \times 5 \times 10^{10}}{13.96\,u}}\ \text{rad/s} = 4.915 \times 10^{13}\ \text{rad/s}$$

$$(\lambda_+)_{\max} = \frac{c}{(\nu_+)_{\max}} = \frac{2\pi c}{(\omega_+)_{\max}} \approx 3.83 \times 10^{-5}\ \text{m}$$

Show for the Kronig-Penney potential with $P << 1$, the energy of the lowest energy band at $\kappa = 0$ is $E = h^2P/4\pi^2ma^2$.

$$\frac{P\sin\alpha a}{\alpha a} + \cos\alpha a = \cos\kappa a \Longrightarrow \frac{P\sin\alpha a}{\alpha a} + \cos\alpha a = 1\ \ \begin{bmatrix}\because \cos\kappa a = 1 \\ \text{as } \kappa = 0\end{bmatrix} \Longrightarrow \frac{P\sin\alpha a}{\alpha a} = 1 - \cos\alpha a$$

$$\Longrightarrow \frac{P}{\alpha a}\left(2\sin\frac{\alpha a}{2}\cos\frac{\alpha a}{2}\right) = \left(2\sin^2\frac{\alpha a}{2}\right) \Longrightarrow \frac{P}{\alpha a} = \tan\frac{\alpha a}{2} \approx \frac{\alpha a}{2} \Longrightarrow P = \frac{\alpha^2 a^2}{2} = \frac{ma^2E}{\hbar^2}\ \ \left[\because \alpha^2 = \frac{2mE}{\hbar^2}\right]$$

$$\Longrightarrow E = \frac{\hbar^2 P}{ma^2} = \frac{h^2 P}{4\pi^2 ma^2}\ \ \left[\hbar = \frac{h}{2\pi}\right]$$

Copper has an atomic weight 63.5, a density of 8.9×10^3 kg/m³, and $v_t = 2.32 \times 10^3$ m/s and $v_l = 4.76 \times 10^3$ m/s (i.e. $v_t < v_l$). Estimate the specific heat at low temperature, say 30 K.

$$\theta_D = \frac{h}{k_B}\nu_D = \frac{h}{k_B}\left\{\frac{9N}{4\pi V}\left(\frac{1}{{v_l}^3} + \frac{2}{{v_t}^3}\right)^{-1}\right\}^{\frac{1}{3}} \approx 340.4\ \text{K}\ \ \left[\because \frac{N}{V} = n = \frac{\rho}{m} \text{ and } m = 63.5\ \text{u}\right]$$

$$C_V = \frac{12}{5}\pi^4 R\left(\frac{T}{\theta_D}\right)^3 \approx 1.33\ \text{J/mol-K}\ \ [\because T = 30\ \text{K}]$$

The Debye temperature of diamond (carbon) is 2230 K. Calculate the specific heat at 10 K. Also compute the highest vibrational frequency (Debye frequency) of diamond.

$$k_B\theta_D = h\nu_D \Longrightarrow \nu_D = \frac{k_B\theta_D}{h} \approx 4.65 \times 10^{13}\ \text{Hz}$$

$$C_V = \frac{12}{5}\pi^4 R\left(\frac{T}{\theta_D}\right)^3 \approx 1.75 \times 10^{-4}\ \text{J/mol-K}\ \ [\because T = 10\ \text{K}]$$

For copper, the lattice specific heat at low temperature has the behaviour: $C_V = 4.6 \times 10^{-2}\ T^3$ J/kmol-K. Estimate the Debye temperature of copper.

$$C_V = \frac{12}{5}\pi^4 R\left(\frac{T}{\theta_D}\right)^3 \Longrightarrow (4.6 \times 10^{-2} \times 10^{-3})T^3 = 2.4\pi^4 R\left(\frac{T}{\theta_D}\right)^3 \Longrightarrow {\theta_D}^3 = \frac{2.4\pi^4 R}{4.6 \times 10^{-5}} \Longrightarrow \theta_D \approx 348.3\ \text{K}$$

In aluminium, $v_t = 2.32 \times 10^3$ m/s and $v_l = 4.76 \times 10^3$ m/s. Its density is 2.7×10^3 kg/m³ and atomic weight is 26.98.
(*a*) Calculate Debye cut-off frequency for aluminium from these data.
(*b*) From specific heat measurements, Debye temperature is found to be 375 K. Find Debye cut-off frequency from this figure and compare with the result obtained in (*a*).

$$\text{(a) } \nu_D = \left\{\frac{9N}{4\pi V}\left(\frac{1}{{v_l}^3} + \frac{2}{{v_t}^3}\right)^{-1}\right\}^{\frac{1}{3}} \approx 6.34 \times 10^{12}\ \text{Hz}$$

$$\text{(b) } k_B\theta_D = h\nu_D \Longrightarrow \nu_D = \frac{k_B\theta_D}{h} \approx 7.8 \times 10^{12}\ \text{Hz}\ \ \text{which is comparable with that obtained in (a).}$$

Estimate the Debye temperature of gold if its atomic weight is 197, the density is 1.9×10^4 kg/m³ and the velocity of sound in it is 2100 m/s.

$$\theta_{\mathrm{D}} = \frac{h}{k_{\mathrm{B}}}\nu_{\mathrm{D}} = \frac{h}{k_{\mathrm{B}}}\left\{\frac{9N}{4\pi V}\left(\frac{1}{{v_l}^3}+\frac{2}{{v_t}^3}\right)^{-1}\right\}^{\frac{1}{3}} \approx \frac{h}{k_{\mathrm{B}}}\left\{\frac{9N}{4\pi V}\left(\frac{3}{v^3}\right)^{-1}\right\}^{\frac{1}{3}} = \frac{h}{k_{\mathrm{B}}}\left\{\frac{3Nv^3}{4\pi V}\right\}^{\frac{1}{3}} \approx 242.1\ \mathrm{K}$$

Diamond (carbon) has Young's modulus of 10^{12} N/m^2 and density of 3500 kg/m^3. Compute the Debye temperature.

$$\text{velocity of sound, } v = \sqrt{\frac{Y}{\rho}} \approx 1.69 \times 10^4\ \mathrm{m/s}$$

$$\theta_{\mathrm{D}} = \frac{h}{k_{\mathrm{B}}}\nu_{\mathrm{D}} = \frac{h}{k_{\mathrm{B}}}\left\{\frac{9N}{4\pi V}\left(\frac{1}{{v_l}^3}+\frac{2}{{v_t}^3}\right)^{-1}\right\}^{\frac{1}{3}} \approx \frac{h}{k_{\mathrm{B}}}\left\{\frac{9N}{4\pi V}\left(\frac{3}{v^3}\right)^{-1}\right\}^{\frac{1}{3}} = \frac{h}{k_{\mathrm{B}}}\left\{\frac{3Nv^3}{4\pi V}\right\}^{\frac{1}{3}} \approx 2817.8\ \mathrm{K}$$

A magnetic material has a magnetization of 3300 A/m and flux density (*B*) of 0.0044 Wb/m^2. Calculate the magnetic field intensity (*H*) and the relative permeability of the material

$$B = \mu_0(H+M) \Longrightarrow H = \frac{B}{\mu_0} - M = \frac{0.0044\ \mathrm{Wb/m^2}}{\mu_0} - 3300\ \mathrm{A/m} \approx 201.4\ \mathrm{T}$$

$$B = \mu H = \mu_0\mu_r H \Longrightarrow \mu_r = \frac{B}{\mu_0 H} \approx 17.4$$

The magnetic field intensity (*H*) in copper is 10^6 A/m. If the magnetic susceptibility of copper is -0.8×10^{-5}, calculate the flux density and magnetization in copper.

$$M = \chi H = (-0.8 \times 10^{-5}) \times 10^6\ \mathrm{A/m} = -8.0\ \mathrm{A/m}$$

$$B = \mu_0(H+M) = (4\pi \times 10^{-7}\ \mathrm{H/m})(10^6\ \mathrm{A/m} - 8.0\ \mathrm{A/m}) \approx 1.257\ \mathrm{T}$$

The magnetic field intensity (*H*) in a piece of ferric oxide is 10^6 A/m. If the susceptibility of the material at room temperature is 1.5×10^{-3}, calculate the flux density and the magnetization of the material.

$$M = \chi H = (-0.8 \times 10^{-5}) \times 10^6\ \mathrm{A/m} = 1500\ \mathrm{A/m} = 1.5 \times 10^3\ \mathrm{A/m}$$

$$B = \mu_0(H+M) = (4\pi \times 10^{-7}\ \mathrm{H/m})(10^6\ \mathrm{A/m} + 1500\ \mathrm{A/m}) \approx 1.259\ \mathrm{T}$$

The magnetic susceptibility of copper is -0.5×10^{-5}. Calculate the magnetic moment per unit volume (magnetization) in copper when it is subjected to a field of intensity (*H*) of 10^4 A/m.

$$M = \chi H = (-0.5 \times 10^{-5}) \times 10^4\ \mathrm{A/m} = -0.05\ \mathrm{A/m}$$

Estimate the order of the diamagnetic susceptibility of Copper. Use a value of 0.1 nm as atomic radius, 0.3608 nm as lattice parameter, and assume that only one electron per atom contributes.

$\because$ Copper is an FCC crystal, so it has 4 atoms per unit cell. $\therefore$ electron density, $n = \frac{N}{V} = \frac{4}{a^3} \approx 8.5 \times 10^{28}\ \mathrm{m^3}$

$$\text{diamagnetic susceptibility, } \chi_{\mathrm{dia}} = -\frac{\mu_0 Z' e^2 n\overline{r^2}}{6m_e} \approx -\frac{\mu_0 Z' e^2 n\bar{r}^2}{6m_e} \approx -5 \times 10^{-6}$$

Approximately how large must the magnetic induction be for the orientation energy to be comparable to the thermal energy at room temperature? Assume magnetic moment, $\mu_{\mathrm{m}} = 5\mu_{\mathrm{B}}$

$$\text{magnetic (orientation) energy} = \text{thermal energy} \Longrightarrow \mu_{\mathrm{m}} B \approx \frac{5}{2}k_{\mathrm{B}}T \Longrightarrow B = \frac{1}{2}\frac{k_{\mathrm{B}}T}{\mu_{\mathrm{B}}} \approx 223.3\ \mathrm{T}$$

A paramagnetic salt contains 10^{28} ions/m^3 with magnetic moment of one Bohr magneton. Calculate the paramagnetic susceptibility and the magnetization produced in a uniform magnetic field of strength (*H*) 10^6 A/m at room temperature (300 K).

$$\text{paramagnetic susceptibility, } \chi_{\mathrm{para}} = \frac{\mu_0 n {\mu_{\mathrm{m}}}^2}{3k_{\mathrm{B}}T} \approx 8.7 \times 10^{-5}$$

magnetization, $M_{\text{para}} = \chi_{\text{para}} H = 87\ \text{A/m}$

A paramagnetic substance has 10^{28} atoms/m^3. The magnetic moment of each atom is 1.8×10^{-23} A-m^2. Calculate the paramagnetic susceptibility at 300 K. What would be the (magnetic) dipole moment of a bar of this material 0.1 m long and 1 cm^2 cross-section placed in a field of intensity (*H*) 8×10^4 A/m?

paramagnetic susceptibility, $\chi_{\text{para}} = \dfrac{\mu_0 n {\mu_{\text{m}}}^2}{3k_{\text{B}}T} \approx 3.28 \times 10^{-4}$
magnetization, $M_{\text{para}} = \chi_{\text{para}} H = 26.2\ \text{A/m}$
magnetic moment induced, $m = M_{\text{para}} V = 26.2\ \text{A/m} \times (0.1 \times 1 \times 10^{-4})\ \text{m}^3 = 2.62 \times 10^{-4}\ \text{A-m}^2$

Polarization, Dielectric constant, Electric displacement and Electric susceptibility

Polarization (or Polarization density), $\vec{P} = N\vec{p}$ [N is the concentration (number density) of the dielectric.]

$N = \dfrac{\rho}{m} = \dfrac{\rho}{\frac{M}{N_{\text{A}}}} = \dfrac{N_{\text{A}}\rho}{M}$ [ρ = density ; m = molecular (or atomic) mass ; M = molar mass]

Electric displacement, $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ [$\vec{E}$ is the resultant or internal electric field.]

Also, Electric displacement, $\vec{D} = \varepsilon_0 \vec{E}_0$ [$\vec{E}_0$ is the applied or external electric field.]

$\therefore \varepsilon_0 \vec{E}_0 = \varepsilon_0 \vec{E} + \vec{P} \Longrightarrow \boldsymbol{\vec{E}_0} = \boldsymbol{\vec{E}} + \dfrac{\boldsymbol{\vec{P}}}{\boldsymbol{\varepsilon_0}} \Longrightarrow \vec{E} = \vec{E}_0 - \dfrac{\vec{P}}{\varepsilon_0}$

Dipole moment, $\vec{p} = \alpha\vec{E}$ [α is the polarizability of a molecule (or atom) of the dielectric. It is a microscopic property. **This relation is valid only for gases.**]

$\therefore \vec{P} = N\alpha\vec{E} \Longrightarrow \boldsymbol{\vec{D}} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0\vec{E} + N\alpha\vec{E} = \varepsilon_0\left(1 + \dfrac{N\alpha}{\varepsilon_0}\right)\vec{E} = \boldsymbol{\varepsilon\vec{E}} = \varepsilon_0\varepsilon_r\vec{E}$ [$\varepsilon_r = 1 + \dfrac{N\alpha}{\varepsilon_0}$ is the dielectric constant or relative permittivity.]

$\therefore \vec{D} = \varepsilon_0(1+\chi)\vec{E}$ [$\chi = \dfrac{N\alpha}{\varepsilon_0}$ is the electric susceptibility.] $\Longrightarrow \vec{P} = \varepsilon_0\chi\vec{E}$ and $\varepsilon_r = 1 + \chi$

Local electric field (Lorentz field)

$\vec{E}_{\text{loc}} = \vec{E}_0 + \vec{E}_1 + \vec{E}_2 + \vec{E}_3$

$\boldsymbol{\vec{E}_0}$ **(External applied field)**

$\boldsymbol{\vec{E}_1}$ (**Depolarization field**): due to the polarization charges on the external surface; opposed to the electric field; its value depends on the geometrical shape of the external surface; by Gauss' law, for an infinite slab, $\vec{E}_1 = \dfrac{-\vec{P}}{\varepsilon_0}$

$\boldsymbol{\vec{E}_2}$: due to the polarization charges on the surface of the Lorentz cavity

$$E_2 = \int dE_2 = \int \frac{(P\cos\theta)\,dA}{4\pi\varepsilon_0 r^2}\cos\theta = \int_0^\pi \frac{(P\cos\theta)\,(2\pi r\sin\theta)r\,d\theta}{4\pi\varepsilon_0 r^2}\cos\theta = \frac{P}{2\varepsilon_0}\int_0^\pi \cos^2\theta\sin\theta\,d\theta = \frac{P}{2\varepsilon_0}\int_{-1}^{1} t^2\,dt$$
$$= \frac{P}{2\varepsilon_0}\left[\frac{t^3}{3}\right]_{-1}^{+1} = \frac{P}{2\varepsilon_0}\left(\frac{2}{3}\right) = \frac{P}{3\varepsilon_0} \Longrightarrow \vec{E}_2 = \frac{\vec{P}}{3\varepsilon_0}$$

$\boldsymbol{\vec{E}_3}$: due to other dipoles in the cavity; value depends on crystal structure of the solid; for cubic lattice, $E_3 = 0$

$$\therefore \vec{E}_{\text{loc}} = \vec{E}_0 + \frac{-\vec{P}}{\varepsilon_0} + \frac{\vec{P}}{3\varepsilon_0} + 0 \Longrightarrow \boldsymbol{\vec{E}_{\text{loc}}} = \boldsymbol{\vec{E}_0} - \frac{\boldsymbol{2\vec{P}}}{\boldsymbol{3\varepsilon_0}} = \left(\vec{E} + \frac{\vec{P}}{\varepsilon_0}\right) - \frac{2\vec{P}}{3\varepsilon_0} \Longrightarrow \boldsymbol{\vec{E}_{\text{loc}}} = \boldsymbol{\vec{E}} + \frac{\boldsymbol{\vec{P}}}{\boldsymbol{3\varepsilon_0}} \quad \textbf{[Lorentz relation]}$$

[$\vec{E}$ is (average) internal electric field or Maxwell field that enters into the Maxwell equations.]

Clausius-Mosotti relation

However for solids and liquids, $\boldsymbol{\vec{p}} = \boldsymbol{\alpha\vec{E}_{\text{loc}}}$ [$\vec{E}$ is the polarizing or local electric field.] $\Longrightarrow \boldsymbol{\vec{P}} = \boldsymbol{N\alpha\vec{E}_{\text{loc}}}$

$$\therefore \vec{P} = N\alpha\vec{E}_{\text{loc}} = N\alpha\left(\vec{E} + \frac{\vec{P}}{3\varepsilon_0}\right) \Longrightarrow \vec{P} = N\alpha\vec{E} + N\alpha\frac{\vec{P}}{3\varepsilon_0} \Longrightarrow \vec{P}\left(1 - \frac{N\alpha}{3\varepsilon_0}\right) = N\alpha\vec{E} \Longrightarrow \vec{P} = \left(\frac{N\alpha}{1 - \frac{N\alpha}{3\varepsilon_0}}\right)\vec{E}$$

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0\vec{E} + \left(\frac{N\alpha}{1 - \frac{N\alpha}{3\varepsilon_0}}\right)\vec{E} = \left(\varepsilon_0 + \frac{N\alpha}{1 - \frac{N\alpha}{3\varepsilon_0}}\right)\vec{E} = \left(\frac{\varepsilon_0 + \frac{2N\alpha}{3}}{1 - \frac{N\alpha}{3\varepsilon_0}}\right)\vec{E} = \varepsilon_0\left(\frac{1 + \frac{2N\alpha}{3}}{1 - \frac{N\alpha}{3\varepsilon_0}}\right)\vec{E} \Longrightarrow \vec{D} = \varepsilon_0\varepsilon_r\vec{E}$$

For solids and liquids, $\varepsilon_r = \dfrac{1 + \frac{2N\alpha}{3\varepsilon_0}}{1 - \frac{N\alpha}{3\varepsilon_0}} \Longrightarrow \boldsymbol{\dfrac{\varepsilon_r - 1}{\varepsilon_r + 2} = \dfrac{N\alpha}{3\varepsilon_0}}$ $\begin{bmatrix}\text{Clausius-Mosotti} \\ \text{relation}\end{bmatrix} \Longrightarrow \varepsilon_r - 1 = \dfrac{N\alpha}{\varepsilon_0}$ $\begin{bmatrix}\text{for gases,} \\ \varepsilon_r \approx 1\end{bmatrix}$

Dipolar polarizability (Orientational polarizability) *applicable for liquids and gases*

Let an external electric field $\mathcal{E}$ be applied to a gas/liquid dipolar dielectric along the x-direction causing the dipoles to orient along that direction. This orientation is opposed by the thermal agitation which tends to randomize the dipoles. This randomization resulted in a vanishing average polarization in the absence of the electric field. But the presence of the field tends to align the dipoles, resulting in a net polarization.

Potential energy of the dipole, $U = -\vec{p}\cdot\vec{\mathcal{E}} = -(\vec{p}\cdot\hat{\imath})\mathcal{E} = -p_x\mathcal{E} = -p\mathcal{E}\cos\theta$

Probability of the dipole been oriented along $\theta, f(\theta) = e^{-\frac{U}{kT}} = e^{\frac{p\mathcal{E}\cos\theta}{kT}}$ [Boltzmann factor]

Average value of p_x, $\bar{p}_x = \dfrac{\int_\Omega p_x f(\theta)d\Omega}{\int_\Omega f(\theta)d\Omega} = \dfrac{\iint_{0\,0}^{\pi\,2\pi} p\cos\theta\, e^{\frac{p\mathcal{E}\cos\theta}{kT}}\, d\phi\sin\theta\, d\theta}{\iint_{0\,0}^{\pi\,2\pi} e^{\frac{p\mathcal{E}\cos\theta}{kT}}\, d\phi\sin\theta\, d\theta} = \dfrac{2\pi\int_0^\pi p\cos\theta\, e^{\frac{p\mathcal{E}\cos\theta}{kT}}\sin\theta\, d\theta}{2\pi\int_0^\pi e^{\frac{p\mathcal{E}\cos\theta}{kT}}\sin\theta\, d\theta}$

$$= \frac{p\int_1^{-1} te^{at}\,dt}{\int_1^{-1} e^{at}\,dt} \begin{bmatrix} t = \cos\theta \Longrightarrow \\ dt = -\sin\theta\, d\theta \\ a = \frac{p\mathcal{E}}{kT}\end{bmatrix} = \frac{p\int_{-1}^{1}\frac{\partial}{\partial a}(e^{at})\,dt}{\int_{-1}^{1} e^{at}\,dt} = p\frac{\partial}{\partial a}\left(\ln\int_{-1}^{1} e^{at}\,dt\right) = p\frac{\partial}{\partial a}\left(\ln\left[\frac{e^{at}}{a}\right]_{-1}^{+1}\right)$$

$$= p\frac{\partial}{\partial a}(\ln[e^a - e^{-a}] - \ln a) = p\left(\frac{e^a + e^{-a}}{e^a - e^{-a}} - \frac{1}{a}\right) = p\left(\coth a - \frac{1}{a}\right) = p\,L(a) \approx \frac{pa}{3} \Longrightarrow \boldsymbol{\bar{p}_x \approx \frac{p^2\mathcal{E}}{3kT}}$$

Langevin function, $L(a) = \coth a - \dfrac{1}{a} = \left(\dfrac{1}{a} + \dfrac{a}{3} - \dfrac{a^3}{45} + \dfrac{2a^5}{945} - \cdots\right) - \dfrac{1}{a} = \dfrac{a}{3} - \dfrac{a^3}{45} + \dfrac{2a^5}{945} - \cdots \approx \dfrac{a}{3}$ $\begin{bmatrix}\text{for } a \ll 1, \\ \text{i.e. } p\mathcal{E} \ll kT\end{bmatrix}$

$\lim_{a\to\infty} L(a) = \lim_{a\to\infty}\left(\coth a - \dfrac{1}{a}\right) = 1 - 0 = 1$ $\begin{bmatrix}\text{Langevin function} \\ \text{saturates for } a \gg 1\end{bmatrix} \Longrightarrow \bar{p}_x = p$ [for $p\mathcal{E} \gg kT$]

Argumental derivation: $\bar{p}_x = p\left(\dfrac{\text{orientational energy}}{\text{thermal energy}}\right) = p\left(\dfrac{p\mathcal{E}/3}{kT}\right) \approx \dfrac{p^2\mathcal{E}}{3kT}$

Dipolar (Orientational) polarizability, $\alpha_{\text{d}} = \dfrac{\bar{p}_x}{\mathcal{E}} = \dfrac{p^2}{3kT}$

$\therefore \alpha = \alpha_{\text{i}} + \alpha_{\text{e}} + \alpha_{\text{d}} \Longrightarrow \boldsymbol{\alpha = \alpha_{\text{ie}} + \dfrac{p^2}{3kT}}$ $\begin{bmatrix}\text{non-dipolar polarizability,} \\ \alpha_{\text{ie}} = \alpha_{\text{i}} + \alpha_{\text{e}}\end{bmatrix}$ This is the **Langevin-Debye equation.**

$\dfrac{\varepsilon_r - 1}{\varepsilon_r + 2} = \dfrac{N\alpha}{3\varepsilon_0}$ [Clausius-Mosotti relation] $\Longrightarrow \dfrac{\varepsilon_r - 1}{\varepsilon_r + 2} = \dfrac{N}{3\varepsilon_0}\left(\alpha_{\text{ie}} + \dfrac{p^2}{3kT}\right)$

$\therefore \left(\dfrac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)$ vs $\dfrac{1}{T}$ plot is a straight line with slope proportional to p^2 and y-intercept proportional to α_{ie}.

$\varepsilon_r - 1 = \dfrac{N\alpha}{\varepsilon_0}$ [for gas] $\Longrightarrow \varepsilon_r - 1 = \dfrac{N}{3\varepsilon_0}\left(\alpha_{\text{ie}} + \dfrac{p^2}{3kT}\right) \Longrightarrow$ Electric susceptibility, $\chi_{\text{e}} = \dfrac{N}{3\varepsilon_0}\left(\alpha_{\text{ie}} + \dfrac{p^2}{3kT}\right)$

Electronic Polarizability: Static (DC)

At equilibrium, $(Ze)\mathcal{E} = (Ze)\dfrac{(Ze)d}{4\pi\varepsilon_0 r^3}$ $\left[\because \dfrac{(Ze)d}{4\pi\varepsilon_0 r^3} \text{ is the electric field inside the uniform negatively charged electron sphere that encloses the atomic nucleus.}\right]$

$$\Rightarrow (Ze)d = \mathcal{E}(4\pi\varepsilon_0 r^3) \Rightarrow p = \mathcal{E}(4\pi\varepsilon_0 r^3) \Rightarrow p = \mathcal{E}(4\pi\varepsilon_0 r^3) \Rightarrow \boldsymbol{\alpha}_{\mathbf{e}} = \frac{p}{\mathcal{E}} = \mathbf{4\pi\varepsilon_0 r^3} \quad [r = \text{atomic radius}]$$

Electronic Polarizability: Dynamic (AC)

$$F = -(Ze)\frac{(Ze)x}{4\pi\varepsilon_0 r^3} \Rightarrow F \propto -x \quad \therefore \mu\frac{d^2x}{dt^2} + \mu{\omega_0}^2 x = -Ze\mathcal{E}_0 \sin\omega t \quad \left[\because \mathcal{E} = \mathcal{E}_0 \sin\omega t\ ;\ \mu = \text{reduced mass of the nucleus-electron cloud system}\right]$$

Trial solution: $x = x_0 \sin\omega t \therefore \mu\dfrac{d^2x}{dt^2} + \mu{\omega_0}^2 x = -Ze\mathcal{E}_0 \sin\omega t \Rightarrow -\mu x + \mu{\omega_0}^2 x = -Ze\mathcal{E} \Rightarrow x = \dfrac{Ze\mathcal{E}}{\mu({\omega_0}^2 - \omega^2)}$

$$\therefore p = (Ze)x = \frac{(Ze)^2\mathcal{E}}{\mu({\omega_0}^2 - \omega^2)} \Rightarrow \boldsymbol{\alpha}_{\mathbf{e}} = \frac{p}{\mathcal{E}} = \frac{(Ze)^2}{\mu({\omega_0}^2 - \omega^2)} \approx \frac{(Ze)^2}{(Zm_e)({\omega_0}^2 - \omega^2)} \approx \frac{\boldsymbol{Ze^2}}{\boldsymbol{m_e}(\boldsymbol{\omega_0}^2 - \boldsymbol{\omega}^2)}$$

$$\left[\because \mu = \frac{(Zm_p)(Zm_e)}{Zm_p + Zm_e} \approx \frac{(Zm_p)(Zm_e)}{Zm_p} = (Zm_e) \quad \because m_e \ll m_p\right] \text{and} \left[{\omega_0}^2 = \frac{k}{\mu} = \frac{(Ze)^2}{4\pi\varepsilon_0 r^3(Zm_e)} = \frac{Ze^2}{4\pi\varepsilon_0 r^3 m_e}\right]$$

When $\omega = 0$, $\alpha_e = \dfrac{Ze^2}{m_e{\omega_0}^2} = \dfrac{Ze^2}{m_e}\left(\dfrac{4\pi\varepsilon_0 r^3 m_e}{Ze^2}\right) = 4\pi\varepsilon_0 r^3$ and, when $\omega = 0$, $\alpha_e = 0$

Electric susceptibility, $\boldsymbol{\chi}_{\mathbf{e}} = \dfrac{P}{\varepsilon_0\mathcal{E}} = \dfrac{Np}{\varepsilon_0\mathcal{E}} = \dfrac{N}{\varepsilon_0}\alpha_e = \dfrac{\boldsymbol{NZe^2}}{\boldsymbol{\varepsilon_0 m_e}(\boldsymbol{\omega_0}^2 - \boldsymbol{\omega}^2)}$

Refractive index, $\boldsymbol{\eta} = \sqrt{\varepsilon_r} = \sqrt{1 + \chi_e} = \sqrt{\mathbf{1} + \dfrac{\boldsymbol{NZe^2}}{\boldsymbol{\varepsilon_0 m_e}(\boldsymbol{\omega_0}^2 - \boldsymbol{\omega}^2)}}$

Ionic Polarizability

Total force on the $(2n)^{\text{th}}$ atom is $M\dfrac{d^2u_{2n}}{dt^2} = k(u_{2n+1} - u_{2n}) - k(u_{2n} - u_{2n-1}) + e^*\mathcal{E}$ [effective charge, $e^* < e$]

$$\Rightarrow M\frac{d^2u_{2n}}{dt^2} = k(u_{2n-1} - 2u_{2n} + u_{2n+1}) + e^*\mathcal{E}_0 \sin\omega t \quad [\because \mathcal{E} = \mathcal{E}_0 \sin\omega t]$$

Total force on the $(2n-1)^{\text{th}}$ atom is $m\dfrac{d^2u_{2n-1}}{dt^2} = k(u_{2n} - u_{2n-1}) - k(u_{2n-1} - u_{2n-2}) - e^*\mathcal{E}$ [k is force constant.]

$$\Rightarrow m\frac{d^2u_{2n-1}}{dt^2} = k(u_{2n-2} - 2u_{2n-1} + u_{2n}) - e^*\mathcal{E}_0 \sin\omega t \quad [\because \mathcal{E} = \mathcal{E}_0 \sin\omega t]$$

Trial solution: $\begin{cases} u_{2n} = A\sin\omega t \\ u_{2n-1} = B\sin\omega t \end{cases} \begin{bmatrix} A = u_{02n} \\ B = u_{02n-1} \end{bmatrix} \quad \therefore \begin{cases} M\dfrac{d^2u_{2n}}{dt^2} = k(u_{2n-1} - 2u_{2n} + u_{2n+1}) + e^*\mathcal{E}_0 \sin\omega t \\ m\dfrac{d^2u_{2n-1}}{dt^2} = k(u_{2n-2} - 2u_{2n-1} + u_{2n}) - e^*\mathcal{E}_0 \sin\omega t \end{cases}$

$$\Rightarrow \begin{cases} -M\omega^2 u_{2n} = k\left(\dfrac{B}{A} - 2 + \dfrac{B}{A}\right)u_{2n} + e^*\mathcal{E} \\ -m\omega^2 u_{2n-1} = k\left(\dfrac{A}{B} - 2 + \dfrac{A}{B}\right)u_{2n-1} - e^*\mathcal{E} \end{cases} \Rightarrow \begin{cases} -M\omega^2 A = 2k(B - A) + e^*\mathcal{E}_0 \\ -m\omega^2 u_{2n-1}B = 2k(A - B) - e^*\mathcal{E}_0 \end{cases}$$

$$\Rightarrow \begin{cases} (M\omega^2 - 2k)A + 2kB = -e^*\mathcal{E}_0 \\ 2kA + (m\omega^2 - 2k)B = e^*\mathcal{E}_0 \end{cases} \Rightarrow \begin{cases} A = \dfrac{\begin{vmatrix} -e^*\mathcal{E}_0 & 2k \\ e^*\mathcal{E}_0 & (m\omega^2 - 2k) \end{vmatrix}}{\begin{vmatrix} (M\omega^2 - 2k) & 2k \\ 2k & (m\omega^2 - 2k) \end{vmatrix}} = \dfrac{-e^*\mathcal{E}_0(m\omega^2 - 2k + 2k)}{(M\omega^2 - 2k)(m\omega^2 - 2k) - 4k^2} \\ B = \dfrac{\begin{vmatrix} (M\omega^2 - 2k) & -e^*\mathcal{E}_0 \\ 2k & e^*\mathcal{E}_0 \end{vmatrix}}{\begin{vmatrix} (M\omega^2 - 2k) & 2k \\ 2k & (m\omega^2 - 2k) \end{vmatrix}} = \dfrac{e^*\mathcal{E}_0(M\omega^2 - 2k + 2k)}{(M\omega^2 - 2k)(m\omega^2 - 2k) - 4k^2} \end{cases}$$

$$\Rightarrow \begin{cases} A = \dfrac{-e^*\mathcal{E}_0(m\omega^2)}{Mm\omega^4 - 2k(M+m)\omega^2} = \dfrac{e^*\mathcal{E}_0 m}{2k(M+m) - Mm\omega^2} \\ B = \dfrac{e^*\mathcal{E}_0(M\omega^2)}{Mm\omega^4 - 2k(M+m)\omega^2} = \dfrac{-e^*\mathcal{E}_0 M}{2k(M+m) - Mm\omega^2} \end{cases} \Rightarrow \begin{cases} \boldsymbol{u_{02n}} = \dfrac{\boldsymbol{e^*\mathcal{E}_0}}{\boldsymbol{M(\omega_0{}^2 - \omega^2)}} \\ \boldsymbol{u_{02n-1}} = \dfrac{\boldsymbol{-e^*\mathcal{E}_0}}{\boldsymbol{m(\omega_0{}^2 - \omega^2)}} \end{cases} \left[\omega_0 = \sqrt{\frac{2k}{\mu}}\right]$$

$$\therefore p = e^*(u_{2n} - u_{2n-1}) = \frac{e^{*2}\mathcal{E}_0 \sin\omega t}{M(\omega_0{}^2 - \omega^2)} + \frac{e^{*2}\mathcal{E}_0 \sin\omega t}{m(\omega_0{}^2 - \omega^2)} \Rightarrow \boldsymbol{p} = \frac{\boldsymbol{e^{*2}\mathcal{E}(m+M)}}{\boldsymbol{Mm(\omega_0{}^2 - \omega^2)}} \Rightarrow \boldsymbol{\alpha_{\mathrm{i}}} = \frac{p}{\mathcal{E}} = \frac{\boldsymbol{e^{*2}}}{\boldsymbol{\mu(\omega_0{}^2 - \omega^2)}}$$

When $\omega = 0$, $\alpha_{\mathrm{e}} = \dfrac{e^*}{\mu\omega_0{}^2} = \dfrac{e^*}{2k}$ $\left[\omega_0{}^2 = \dfrac{2k}{\mu}\right]$ and, when $\omega = \infty$ $\alpha_{\mathrm{e}} = 0$

Electric susceptibility, $\boldsymbol{\chi_{\mathrm{e}}} = \dfrac{P}{\varepsilon_0\mathcal{E}} = \dfrac{Np}{\varepsilon_0\mathcal{E}} = \dfrac{N}{\varepsilon_0}\alpha_{\mathrm{i}} = \dfrac{\boldsymbol{Ne^*(m-M)}}{\boldsymbol{\varepsilon_0\mu(\omega_0{}^2 - \omega^2)}}$

Refractive index, $\boldsymbol{\eta} = \sqrt{\varepsilon_r} = \sqrt{1+\chi_{\mathrm{e}}} = \sqrt{\boldsymbol{1} + \dfrac{\boldsymbol{Ne^*(m-M)}}{\boldsymbol{\varepsilon_0\mu(\omega_0{}^2 - \omega^2)}}}$

If the molecular dipoles in a 1 mm radius water drop are pointed in the same direction, calculate the polarization. Dipole moment of the water molecule is 6×10^{-30} C-m.

Density of water, $\rho = 10^3\ \mathrm{kg/m^3}$

Concentration of H_2O molecules, $N = \dfrac{10^3\ \mathrm{m^{-3}}}{18 \times 1.66 \times 10^{-27}} = 3.35 \times 10^{28}\ \mathrm{m^{-3}}$

Polarization, $P = Np \approx 0.2\ \mathrm{C/m^2}$

Assuming that the polarizability of Kr atom is 2.18×10^{-40} F-m², calculate its dielectric constant at 0 °C and 1 atm.

Concentration of Krypton atoms at NTP, $N = \dfrac{N_{\mathrm{A}}}{22.4 \times 10^{-3}\ \mathrm{m^3}} \approx 2.688 \times 10^{25}\ \mathrm{m^{-3}}$

Dielectric constant, $\varepsilon_r = 1 + \dfrac{N\alpha}{\varepsilon_0} \approx 1.000662$ $[\because \alpha = 2.18 \times 10^{-40}\ \mathrm{F\text{-}m^2}]$

Find the total polarizability of CO_2 if its susceptibility is 0.985×10^{-3}. Density of CO_2 is 1.977 kg/m³.

Concentration of CO_2 molecules, $N = \dfrac{1.977\ \mathrm{m^{-3}}}{44 \times 1.66 \times 10^{-27}} \approx 2.707 \times 10^{25}\ \mathrm{m^{-3}}$

Polarizability, $\alpha = \dfrac{\varepsilon_0\chi}{N} \approx 3.22 \times 10^{-40}\ \mathrm{F\text{-}m^2}$ $[\because \chi = 0.985 \times 10^{-3}]$

On being polarized, an Oxygen atom produces a dipole moment of 0.5×10^{-22} C-m. If the distance of the centre of negative charge cloud from the nucleus is 4×10^{-17} m, calculate the polarizability of Oxygen atom.

At equilibrium, Coulomb force $=$ Lorentz force $\Rightarrow \dfrac{q^2}{4\pi\varepsilon_0 d^2} = qE \Rightarrow E = \dfrac{8e}{4\pi\varepsilon_0(4 \times 10^{-17}\ \mathrm{m})^2} \approx 7.2 \times 10^{24}\ \mathrm{V/m}$

Polarizability, $\alpha = \dfrac{p}{E} = 6.94 \times 10^{-48}\ \mathrm{F\text{-}m^2}$ $[\because p = 0.5 \times 10^{-22}\ \mathrm{C\text{-}m}]$

The relative permittivity of Argon at 0 °C and 1 atm is 1.000435. Calculate the polarizability of the Argon atom.

Concentration of Argon atoms at NTP, $N = \dfrac{N_{\mathrm{A}}}{22.4 \times 10^{-3}\ \mathrm{m^3}} \approx 2.707 \times 10^{25}\ \mathrm{m^{-3}}$

Relative permittivity, $\varepsilon_r = 1 + \dfrac{N\alpha}{\varepsilon_0} \Rightarrow$ Polarizability, $\alpha = \dfrac{\varepsilon_0}{N}(\varepsilon_r - 1) \approx 1.43 \times 10^{-40}\ \mathrm{F\text{-}m^2}$

The dielectric constant of Helium at 0 °C and 1 atm is 1.000074. Calculate the dipole moment induced in each He atom when the gas is subjected to an electric field of 3×10^4 V/m.

Concentration of Helium atoms at NTP, $N = \dfrac{N_{\mathrm{A}}}{22.4 \times 10^{-3}\ \mathrm{m^3}} \approx 2.707 \times 10^{25}\ \mathrm{m^{-3}}$

Relative permittivity, $\varepsilon_r = 1 + \dfrac{N\alpha}{\varepsilon_0} \Longrightarrow$ Polarizability, $\alpha = \dfrac{\varepsilon_0}{N}(\varepsilon_r - 1) \approx 1.43 \times 10^{-40}$ F-m^2

Dipole moment, $p = \alpha E = \dfrac{\alpha E_0}{\varepsilon_r} \approx 7.26 \times 10^{-37}$ C-m

Calculate the induced dipole moment per unit volume (polarization density) of Helium gas when it is placed in a field of 6 × 10^5 V/m. The atomic polarizability of Helium is 0.18 × 10^{-40} F-m^2 and the concentration of Helium atom is 2.6 × 10^{25} m^{-3}. Also calculate the separation of positive and negative charges in each atom.

$\varepsilon_r = 1 + \dfrac{N\alpha}{\varepsilon_0} \approx 1.000041$

Dipole moment, $p = \alpha E = \dfrac{\alpha E_0}{\varepsilon_r} \approx 1.08 \times 10^{-35}$ C-m

Polarization density, $P = Np \approx 2.81 \times 10^{-10}$ C/m^2

Separation, $d = \dfrac{p}{2e} \approx 3.37 \times 10^{-17}$ m

If the density of polarization of diamond is 1.32 × 10^{-10} C/m^2, calculate the shift of the centre of negative cloud of 6e$^-$ in each atom from the nucleus. The density of diamond is 3500 kg/m^3.

Concentration of Carbon atoms, $N = \dfrac{3500 \text{ m}^{-3}}{12 \times 1.66 \times 10^{-27}} \approx 1.756 \times 10^{29}$ m^{-3}

Dipole moment, $p = \dfrac{P}{N} \approx 7.52 \times 10^{-40}$ C-m

Separation, $d = \dfrac{p}{6e} \approx 7.82 \times 10^{-22}$ m

There are 1.6 × 10^{20} molecules/m^3 in NaCl vapour. Determine the orientational polarization at room temperature if the vapour is subjected to a field of 5 × 10^4 V/m. Assume that the NaCl molecule consists of Na$^+$ and Cl$^-$ ions separated by 2.5 Å.

Dipole moment, $p = qd = e \times 2.5 \times 10^{-10}$ m $\approx 4 \times 10^{-29}$ C-m

Orientational (or dipolar) polarizability, $\alpha_\text{d} = \dfrac{p^2}{3k_\text{B}T} \approx 1.29 \times 10^{-37}$ F-m^2

Orientational (or dipolar) polarization, $P_\text{d} = N\bar{p} = N\alpha_\text{d}E = 1.032 \times 10^{-12}$ C/m^2 $[\because \bar{p} = \alpha_\text{d}E]$

Assuming that there are 10^{27} molecules/m^3 in HCl vapour, calculate the orientational polarization at room temperature if the vapour is subjected to an electric field 10^6 V/m. The dipole moment of HCl molecule is 3.46×10^{-30} C-m. Show that at this temperature and for such a high field, the value of *a* (where *a* = *pE*/*kT*) is very much small than unity.

Orientational (or dipolar) polarizability, $\alpha_\text{d} = \dfrac{p^2}{3k_\text{B}T} \approx 9.63 \times 10^{-40}$ F-m^2

Orientational (or dipolar) polarization, $P_\text{d} = N\bar{p} = N\alpha_\text{d}E = 9.63 \times 10^{-7}$ C/m^2 $[\because \bar{p} = \alpha_\text{d}E]$

Electric energy $\sim pE = 3.46 \times 10^{-24}$ J

Thermal energy $\sim kT = 4.14 \times 10^{-21}$ J

$\therefore a = \dfrac{pE}{kT} \approx 8 \times 10^{-3} \ll 1$

One gram-molecule (1 mole) of a certain polar substance is dissolved into 1000 cm^3 (1 litre) of a non-polar liquid. The liquid itself has a dielectric constant of 3 at 27°C, whereas the solution has a dielectric constant of 3.2 at the same temperature. Calculate the dipole moment of the polar molecules.

Assuming the volume to be constant after dissolving the polar substance in the liquid,

For liquid, $N_1\alpha_\text{e} = 3\varepsilon_0\left(\dfrac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) = 3\varepsilon_0\left(\dfrac{3-1}{3+2}\right) = 3\varepsilon_0(0.4)$

For solution, $N_1\alpha_e + N_2\alpha_o = 3\varepsilon_0\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) = 3\varepsilon_0\left(\frac{3.2 - 1}{3.2 + 2}\right) = 3\varepsilon_0(0.423)$

$\Longrightarrow N_2\alpha_o = 3\varepsilon_0(0.423) - N_1\alpha_e = 3\varepsilon_0(0.423) - 3\varepsilon_0(0.4) = 0.069\varepsilon_0 \Longrightarrow \alpha_o = \frac{0.069\varepsilon_0}{N_2}$

$\because \alpha_o = \frac{p^2}{3k_BT} \Longrightarrow p = \sqrt{3k_BT\alpha_o} = \sqrt{3k_BT \times \frac{0.069\varepsilon_0}{N_2}} = \sqrt{3k_B(300\text{ K}) \times \frac{0.069\varepsilon_0}{N_A \times 10^3\text{ m}^{-3}}} = 3.55 \times 10^{-30}$ C-m

For a dielectric material with ε_r = 4.94 and n^2 = 2.69 (n is the index of refraction), calculate the ratio between electronic and ionic polarizabilities.

Clausius-Mosotti relation: $\frac{N\alpha}{3\varepsilon_0} = \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \Longrightarrow \frac{N(\alpha_e + \alpha_i)}{3\varepsilon_0} = \frac{\varepsilon_r - 1}{\varepsilon_r + 2}$

At optical frequencies, $\varepsilon_r = n^2$ and $\alpha_i = 0$ $\left[\because n = \frac{c}{v} = \frac{\frac{1}{\sqrt{\mu_0\varepsilon_0}}}{\frac{1}{\sqrt{\mu\varepsilon}}} = \sqrt{\mu_r\varepsilon_r} \approx \sqrt{\varepsilon_r}\right]$

$\therefore \frac{N(\alpha_e + \alpha_i)}{3\varepsilon_0} = \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \Longrightarrow \frac{N\alpha_e}{3\varepsilon_0} = \frac{n^2 - 1}{n^2 + 2}$

$$\begin{cases} \frac{N(\alpha_e + \alpha_i)}{3\varepsilon_0} = \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \\ \frac{N\alpha_e}{3\varepsilon_0} = \frac{n^2 - 1}{n^2 + 2} \end{cases} \Longrightarrow \frac{\alpha_e + \alpha_i}{\alpha_e} = \frac{\frac{\varepsilon_r - 1}{\varepsilon_r + 2}}{\frac{n^2 - 1}{n^2 + 2}} = \frac{\frac{4.94 - 1}{4.94 + 2}}{\frac{2.69 - 1}{2.69 + 2}} = \frac{0.56772}{0.36034} = 1.5755 \Longrightarrow \frac{\alpha_i}{\alpha_e} = 0.5755 \Longrightarrow \frac{\alpha_i}{\alpha_e} = 1.7376$$

The atomic weight and density of sulphur are 32 and 2.08 g-cm^{-3}. The electronic polarizability of the atom is 3.28 × 10^{-40} F-m^2. If solid Sulphur has cubic symmetry, what will be its dielectric constant?

Clausius-Mosotti relation: $\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{N\alpha}{3\varepsilon_0} \Longrightarrow \frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \left(\frac{N_A\rho}{M}\right)\frac{\alpha}{3\varepsilon_0} = \left(\frac{N_A \times 2.08 \times 10^6}{32}\right)\frac{3.28 \times 10^{-40}}{3\varepsilon_0} = 0.48336$

$\Longrightarrow \varepsilon_r - 1 = 0.48336(\varepsilon_r + 2) \Longrightarrow 0.51664\varepsilon_r = 1.96672 \Longrightarrow \varepsilon_r = 3.80675$

Determine the percentage of ionic polarizability in the NaCl crystal has optical index of refraction and static dielectric constant 1.5 and 5.6 respectively.

Clausius-Mosotti relation: $\frac{N\alpha}{3\varepsilon_0} = \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \Longrightarrow \frac{N(\alpha_e + \alpha_i)}{3\varepsilon_0} = \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \Longrightarrow \alpha_e + \alpha_i = \frac{3\varepsilon_0}{N}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)$

At optical frequencies, $\varepsilon_r = n^2$ and $\alpha_i = 0$ $\left[\because n = \frac{c}{v} = \frac{\frac{1}{\sqrt{\mu_0\varepsilon_0}}}{\frac{1}{\sqrt{\mu\varepsilon}}} = \sqrt{\mu_r\varepsilon_r} \approx \sqrt{\varepsilon_r}\right]$

$\therefore \alpha_e + \alpha_i = \frac{3\varepsilon_0}{N}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) \Longrightarrow \alpha_e = \frac{3\varepsilon_0}{N}\left(\frac{n^2 - 1}{n^2 + 2}\right)$

$$\begin{cases} \alpha_e = \frac{3\varepsilon_0}{N}\left(\frac{n^2 - 1}{n^2 + 2}\right) \\ \alpha_e + \alpha_i = \frac{3\varepsilon_0}{N}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) \end{cases} \Longrightarrow \frac{\alpha_e}{\alpha_e + \alpha_i} = \frac{\frac{n^2 - 1}{n^2 + 2}}{\frac{\varepsilon_r - 1}{\varepsilon_r + 2}} = \frac{\frac{1.5^2 - 1}{1.5^2 + 2}}{\frac{5.6 - 1}{5.6 + 2}} = 0.4859 \Longrightarrow \frac{\alpha_i}{\alpha_e + \alpha_i} = 1 - 0.4859 = 51.41\%$$

The polarizability of NH_3 molecule is found experimentally by the measurement of dielectric constant as 2.42×10^{-39} F-m^2 at 309 K and 1.74×10^{-39} F-m^2 at 448 K respectively. Calculate for each temperature the polarizability due to permanent dipole moment and due to deformation of molecules.

Polarizability, $\alpha = \alpha_\mathrm{i} + \alpha_\mathrm{d} = \alpha_\mathrm{i} + \dfrac{p^2}{3k_\mathrm{B}T} \Longrightarrow \begin{cases} \alpha_1 = \alpha_\mathrm{i} + \dfrac{p^2}{3k_\mathrm{B}T_1} \\ \alpha_2 = \alpha_\mathrm{i} + \dfrac{p^2}{3k_\mathrm{B}T_2} \end{cases}$

$$\Longrightarrow \begin{cases} \alpha_1 - \alpha_2 = \dfrac{p^2}{3k_\mathrm{B}}\left(\dfrac{1}{T_1} - \dfrac{1}{T_2}\right) \Longrightarrow \dfrac{p^2}{3k_\mathrm{B}} = \dfrac{\alpha_1 - \alpha_2}{\dfrac{1}{T_1} - \dfrac{1}{T_2}} \approx 677 \times 10^{-39}\ \mathrm{Fm^2\text{-}K} \\ \dfrac{\alpha_1 - \alpha_\mathrm{i}}{\alpha_2 - \alpha_\mathrm{i}} = \dfrac{T_2}{T_1} \Longrightarrow T_1(\alpha_1 - \alpha_\mathrm{i}) = T_2(\alpha_2 - \alpha_\mathrm{i}) \Longrightarrow \alpha_\mathrm{i} = \dfrac{\alpha_1 T_1 - \alpha_2 T_2}{T_1 - T_2} \approx 0.23 \times 10^{-39}\ \mathrm{F\text{-}m^2} \end{cases}$$

$$\therefore \begin{cases} \alpha_\mathrm{d1} = \dfrac{p^2}{3k_\mathrm{B}T_1} = 2.19 \times 10^{-39}\ \mathrm{F\text{-}m^2} \\ \alpha_\mathrm{d2} = \dfrac{p^2}{3k_\mathrm{B}T_2} = 1.51 \times 10^{-39}\ \mathrm{F\text{-}m^2} \end{cases}$$

A metallic surface, when illuminated with light of wavelength λ_1 emits electrons with energies upto a maximum value E_1, and when illuminated with a light of wavelength λ_2, it emits electrons with energies upto a maximum value E_2. Find the expression for the Planck's constant *h* and for the Workfunction *W* of the metal.

$$\begin{cases} \dfrac{hc}{\lambda_1} = W + E_1 \\ \dfrac{hc}{\lambda_2} = W + E_2 \end{cases} \Longrightarrow \begin{cases} \dfrac{hc}{\lambda_1} - \dfrac{hc}{\lambda_2} = E_1 - E_2 \Longrightarrow hc\left(\dfrac{1}{\lambda_1} - \dfrac{1}{\lambda_2}\right) = E_1 - E_2 \Longrightarrow \boldsymbol{h} = \dfrac{\boldsymbol{\lambda_1\lambda_2(E_1 - E_2)}}{\boldsymbol{c(\lambda_2 - \lambda_1)}} \\ \dfrac{\dfrac{hc}{\lambda_1}}{\dfrac{hc}{\lambda_2}} = \dfrac{W + E_1}{W + E_2} \Longrightarrow \dfrac{\lambda_2}{\lambda_1} = \dfrac{W + E_1}{W + E_2} \Longrightarrow W(\lambda_2 - \lambda_1) = E_1\lambda_1 - E_2\lambda_2 \Longrightarrow \boldsymbol{W} = \dfrac{\boldsymbol{E_1\lambda_1 - E_2\lambda_2}}{\boldsymbol{\lambda_2 - \lambda_1}} \end{cases}$$

A photon of wavelength 3310 Å falls on a photocathode and ejects an electron of energy 3 × 10^{-19} J. If the wavelength of the incident photon is changed to 5000 Å, the energy of the emitted electron is 0.972 × 10^{-19} J. Calculate the value of the Planck's constant, the threshold frequency and the workfunction of the photocathode.

$$h = \frac{\lambda_1\lambda_2(E_1 - E_2)}{c(\lambda_2 - \lambda_1)} \approx 6.6246 \times 10^{-34}\ \mathrm{Js}$$
$$W = \frac{E_1\lambda_1 - E_2\lambda_2}{\lambda_2 - \lambda_1} = 3 \times 10^{-19}\ \mathrm{J} \approx 1.87\ \mathrm{eV} = h\nu_\mathrm{th} \quad [\because W = h\nu_\mathrm{th}]$$
$$\Longrightarrow \nu_\mathrm{th} = \frac{3 \times 10^{-19}\ \mathrm{J}}{6.6246 \times 10^{-34}\ \mathrm{Js}} \approx 4.53 \times 10^{14}\ \mathrm{Hz} = 453\ \mathrm{THz}$$

A metallic surface when illuminated with light of wavelength 3333 Å emits electrons with energies upto 0.6 eV, and when illuminated with light of wavelength 2400 Å, it emits electrons with energies upto 2.04 eV. Calculate Planck's constant and the workfunction of the metal.

$$h = \frac{\lambda_1\lambda_2(E_1 - E_2)}{c(\lambda_2 - \lambda_1)} \approx 6.6 \times 10^{-34}\ \mathrm{Js}$$
$$W = \frac{E_1\lambda_1 - E_2\lambda_2}{\lambda_2 - \lambda_1} \approx 3.1\ \mathrm{eV}$$

Light of wavelength 4300 Å is incident on (a) Nickel surface of workfunction 5 eV and (b) Potassium surface of workfunction 2.3 eV. Find out whether electrons will be emitted, and if so, the maximum velocity of the emitted electrons in each case.

$$\frac{hc}{\lambda} = W + \frac{1}{2}mv^2 \Longrightarrow \frac{1}{2}mv^2 = \frac{hc}{\lambda} - W = \begin{cases} -3.39 \times 10^{-19}\ \mathrm{J} \quad [\text{no electron emission}] \\ 0.935 \times 10^{-19}\ \mathrm{J} \equiv 4.53 \times 10^5\ \mathrm{m/s} = v_\mathrm{max} \end{cases}$$

In the photoelectric effect, how is the maximum velocity of the photoelectron related to the stopping potential?

$$eV_0 = \frac{1}{2} m v_{\max}{}^2 \Rightarrow v_{\max} = \sqrt{\frac{2e}{m} V_0} \approx 593097\sqrt{V_0} \quad [\because m = m_\mathrm{e}]$$

A photon of wavelength 1400 Å is absorbed by cold mercury vapour and two other photons are emitted. If the wavelength of one of them is 1850 Å, what is the wavelength of the other photon?

Energy of the other photon, $E_2 = E_0 - E_1 \Rightarrow \frac{hc}{\lambda_2} = hc\left(\frac{1}{\lambda_0} - \frac{1}{\lambda_1}\right) \Rightarrow \lambda_2 = \left(\frac{1}{\lambda_0} - \frac{1}{\lambda_1}\right)^{-1} \approx 5756$ Å

A radio transmitter operates at a frequency of 1760 kHz and a power of 10 kW. How many photons does it emit?

Number of photons emitted, $N = \frac{\text{Energy emitted}}{\text{Energy of one photon}} = \frac{E}{h\nu} = \frac{10 \times 10^3\ \text{J}}{6.626 \times 10^{-34}\ \text{J-s} \times 1760 \times 10^3\ \text{s}^{-1}} \approx 8.575 \times 10^{30}$

The energy required to remove an electron from sodium metal is 2.3 eV. Does sodium exhibit the photovoltaic effect from an orange light having wavelength 2800 Å?

Energy of orange light, $E = hc/\lambda = 7.1 \times 10^{-19}\ \text{J} \approx 4.43\ \text{eV}$

$\because$ the energy is greater than workfunction (2 eV) of the metal, photoelectric effect is possible.

Calculate the number of photons from green light of mercury (4961 Å) required to do one Joule of work.

Number of photons required $N = \frac{E}{hc/\lambda} \approx \frac{1\ \text{J}}{4 \times 10^{-19}\ \text{J}} \approx 2.5 \times 10^{18}$

In a photocell, a copper surface was irradiated by light of wavelength 1849 Å, the stopping potential was found to be 2.72 V. Calculate the threshold frequency, the workfunction, the maximum energy of the photoelectrons.

Maximum energy of photoelectrons, $T_{\max} = eV_0 = 4.358 \times 10^{-19}\ \text{J} = 2.72\ \text{eV}$

$\because T_{\max} = E_{\text{incident}} - E_{\text{threshold}} = hc/\lambda - W \Rightarrow W = hc/\lambda - T_{\max} \approx 4\ \text{eV}$

Threshold frequency, $\nu_0 = \frac{W}{h} = 9.637 \times 10^{14}\ \text{Hz} = 963.7\ \text{THz}$

A superconducting tin has a critical temperature of 3.7 K at zero magnetic field (intensity) and a critical field of 0.0306 T at 0 K. Find the critical field at 2 K.

$$B_\mathrm{c}(T) = B_0(0)\left\{1 - \left(\frac{T}{T_c}\right)^2\right\} = (0.0306\ \text{T})\left\{1 - \left(\frac{2}{3.7}\right)^2\right\} \approx 0.02166\ \text{T}$$

A superconducting lead has a critical temperature of 7.26 K at zero magnetic field and a critical field (intensity) of 8×10^5 A/m at 0 K. Find the critical field at 5 K.

$$H_\mathrm{c}(T) = H_0(0)\left\{1 - \left(\frac{T}{T_c}\right)^2\right\} = (8 \times 10^5\ \text{T})\left\{1 - \left(\frac{5}{7.26}\right)^2\right\} \approx 4.2 \times 10^5\ \text{A/m}$$

The magnetic field intensity in the tin material is zero at 3.69 K and 3×10^5/4π A/m at 0 K. Calculate the temperature of the superconductor if the field intensity is measured at 2×10^5/4π A/m.

$$H_\mathrm{c}(T) = H_0(0)\left\{1 - \left(\frac{T}{T_c}\right)^2\right\} \Rightarrow 2 \times 10^5 = (3 \times 10^5)\left\{1 - \left(\frac{T}{3.69\ \text{K}}\right)^2\right\} \Rightarrow \left(\frac{T}{3.69\ \text{K}}\right)^2 = \frac{1}{3} \Rightarrow T \approx 2.13\ \text{K}$$

Calculate the critical current for a wire of lead having a diameter of 1 mm at 4.2 K. The critical temperature for lead is 7.18 K and H$_c$(0) = 6.5×10^4 A/m.

$$H_\mathrm{c}(T) = H_0(0)\left\{1 - \left(\frac{T}{T_c}\right)^2\right\} = (6.5 \times 10^4\ \text{A/m})\left\{1 - \left(\frac{4.2\ \text{K}}{7.18\ \text{K}}\right)^2\right\} \approx 4.276 \times 10^4\ \text{A/m}$$

$$I_\mathrm{c} = 2\pi r H_\mathrm{c} = \pi d H_\mathrm{c} \approx 134.33\ \text{A}$$

Calculate the critical current which can flow through a long thin superconducting wire of aluminium of diameter 1 mm. The critical field for aluminium is 7.9×10³ A/m.

$I_\mathrm{c} = 2\pi r H_\mathrm{c} = \pi d H_\mathrm{c} \approx 24.82\ \mathrm{A}$

The critical temperature for mercury with isotopic mass 199.5 is 4.185 K. Calculate its critical temperature when its isotopic mass changes to 203.4.

$$T_\mathrm{c} \propto M^{-\frac{1}{2}} \Rightarrow \frac{T_\mathrm{c2}}{T_\mathrm{c1}} = \sqrt{\frac{M_1}{M_2}} \Rightarrow T_\mathrm{c2} = T_\mathrm{c1}\sqrt{\frac{M_1}{M_2}} \approx 2.145\ \mathrm{K}$$

The penetration depths for lead are 396 Å and 1730 Å at 3 K and 7.1 K respectively. Calculate the critical temperature for lead.

$$\lambda_\mathrm{c}(T) = \lambda(0)\left\{1 - \left(\frac{T}{T_c}\right)^4\right\}^{-\frac{1}{2}} \Rightarrow \frac{\lambda_\mathrm{c}(T_1)}{\lambda_\mathrm{c}(T_2)} = \frac{\left\{1 - \left(\frac{T_1}{T_c}\right)^4\right\}^{-\frac{1}{2}}}{\left\{1 - \left(\frac{T_2}{T_c}\right)^4\right\}^{-\frac{1}{2}}} \Rightarrow \left(\frac{\lambda_\mathrm{c1}}{\lambda_\mathrm{c2}}\right)^{-2} = \frac{\left\{1 - \left(\frac{T_1}{T_c}\right)^4\right\}}{\left\{1 - \left(\frac{T_2}{T_c}\right)^4\right\}} \Rightarrow \left(\frac{\lambda_\mathrm{c2}}{\lambda_\mathrm{c1}}\right)^2 = \frac{{T_c}^4 - {T_1}^4}{{T_c}^4 - {T_2}^4}$$

$$\Rightarrow {T_c}^4\left\{\left(\frac{\lambda_\mathrm{c2}}{\lambda_\mathrm{c1}}\right)^2 - 1\right\} = {T_2}^4\left(\frac{\lambda_\mathrm{c2}}{\lambda_\mathrm{c1}}\right)^2 - {T_1}^4 \Rightarrow {T_c}^4 = \frac{{T_2}^4\left(\frac{\lambda_\mathrm{c2}}{\lambda_\mathrm{c1}}\right)^2 - {T_1}^4}{\left(\frac{\lambda_\mathrm{c2}}{\lambda_\mathrm{c1}}\right)^2 - 1} \approx 7.193\ \mathrm{K}$$

Néel temperature (Magnetic ordering temperature)

Temperature at which an antiferromagnetic material becomes paramagnetic.

Weak Equivalence principle

$$\begin{cases} \vec{F}_{12} = -\dfrac{G m_{gp1} m_{ga2}}{{r_{12}}^2}\hat{r}_{12} \\ \vec{F}_{21} = +\dfrac{G m_{ga1} m_{gp2}}{{r_{12}}^2}\hat{r}_{12} \end{cases} \begin{bmatrix} m_{ga} = \text{active gravitational mass} \\ m_{gp} = \text{passive gravitational mass} \end{bmatrix}$$

$$\Rightarrow \frac{G m_{gp1} m_{ga2}}{{r_{12}}^2} = \frac{G m_{ga1} m_{gp2}}{{r_{12}}^2} \quad [\because \vec{F}_{12} = -\vec{F}_{21}]$$

$$\Rightarrow m_{gp1} m_{ga2} = m_{ga1} m_{gp2} \Rightarrow \frac{m_{gp1}}{m_{ga1}} = \frac{m_{gp2}}{m_{ga2}} = k_g \text{ (say)}$$

where k_g is same for all massive bodies.

Empirically, $k_g = 1$, i.e. $m_{ga} = m_{gp}$

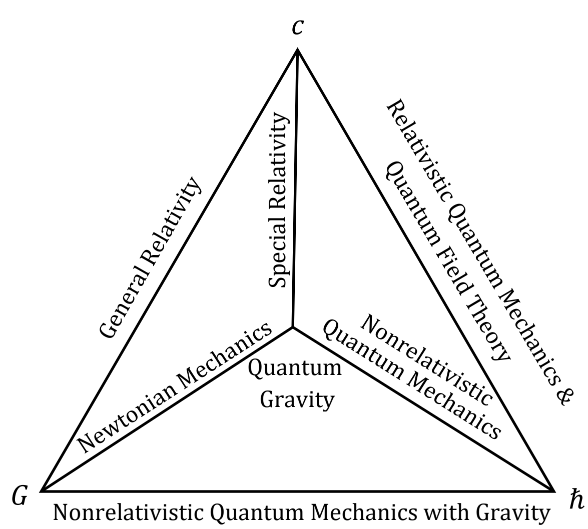
$$\begin{cases} \vec{F}_{10} = -\dfrac{G m_{gp1} m_{ga0}}{{r_{10}}^2}\hat{r}_{10} \\ \vec{F}_{20} = -\dfrac{G m_{gp2} m_{ga0}}{{r_{20}}^2}\hat{r}_{20} \end{cases}$$

$$\begin{cases} \vec{a}_{10} = \dfrac{\vec{F}_{12}}{m_{i1}} = -\dfrac{G m_{gp1} m_{ga0}}{{r_{10}}^2 m_{i1}}\hat{r}_{10} \\ \vec{a}_{20} = \dfrac{\vec{F}_{21}}{m_{i2}} = -\dfrac{G m_{gp2} m_{ga0}}{{r_{20}}^2 m_{i2}}\hat{r}_{20} \end{cases} \quad [m_i = \text{inertial mass}]$$

$$\Rightarrow \frac{m_{gp1} m_{ga0}}{m_{i1}} = \frac{m_{gp2} m_{ga0}}{m_{i2}} \quad \begin{bmatrix} \text{for } r_{10} = r_{20}, a_{10} \text{ must equal } a_{20} \\ \text{(Weak Equivalence principle)} \end{bmatrix} \Rightarrow \frac{m_{gp1}}{m_{i1}} = \frac{m_{gp2}}{m_{i2}} = k_i \text{ (say)}$$

where k_i is same for all massive bodies. Empirically, $k_i = 1$, i.e. $m_{gp} = m_i$. Thus, $m_{ga} = m_{gp} = m_i$.

The equivalence of gravitational mass and inertial mass is due to the Weak Equivalence principle.



Habitable Zone (HZ) defined by presence of liquid water

$L_* = 4\pi d^2 \sigma T^4 \Longrightarrow d = \sqrt{\dfrac{L_*}{4\pi\sigma T^4}}$ [roughly, ignoring albedo and exposure]

$d_{\max} = \sqrt{\dfrac{L_*}{4\pi\sigma {T_{\min}}^4}} = \sqrt{\dfrac{L_*}{4\pi\sigma(273.15\text{ K})^4}}$ [$\because T_{\min} = 273.15$ K, the freezing point of water]

For solar system, $d_{\max} = \sqrt{\dfrac{L_\odot}{4\pi\sigma(273.15\text{ K})^4}} = \sqrt{\dfrac{3.828 \times 10^{26}\text{ W}}{4\pi\sigma(273.15\text{ K})^4}} \approx 3.11 \times 10^{11}\text{ m} \approx 2.07\text{ AU}$

$d_{\max} = \sqrt{\dfrac{L_*}{4\pi\sigma {T_{\min}}^4}} = \sqrt{\dfrac{L_*}{4\pi\sigma(273.15\text{ K})^4}}$ [$\because T_{\min} = 373.15$ K, the boiling point of water]

For solar system, $d_{\min} = \sqrt{\dfrac{L_\odot}{4\pi\sigma(373.15\text{ K})^4}} = \sqrt{\dfrac{3.828 \times 10^{26}\text{ W}}{4\pi\sigma(373.15\text{ K})^4}} \approx 1.66 \times 10^{11}\text{ m} \approx 1.11\text{ AU}$

Note that Earth's distance from Sun is 1 AU (0.98 AU – 1.02 AU) which is outside the HZ.

Conversely, $T^4 = \dfrac{L_*}{4\pi d^2 \sigma} \Longrightarrow T = \sqrt[4]{\dfrac{L_*}{4\pi d^2\sigma}} = \left(\dfrac{R_*}{d}\right)\sqrt[4]{\dfrac{L_*}{4\pi {R_*}^2\sigma}} = \left(\dfrac{R_*}{d}\right) T_{\text{eff}_*}$ $\left[\text{where } T_{\text{eff}_*} = \sqrt[4]{\dfrac{L_*}{4\pi {R_*}^2\sigma}}\right]$

For Earth, $T = \sqrt[4]{\dfrac{L_\odot}{4\pi d^2\sigma}} = \sqrt[4]{\dfrac{3.828 \times 10^{26}\text{ W}}{4\pi(1.5 \times 10^{11}\text{ m})^2\sigma}} \approx 393\text{ K} = 120°\text{C}$

For Mars, $T = \sqrt[4]{\dfrac{L_\odot}{4\pi d^2\sigma}} = \sqrt[4]{\dfrac{3.828 \times 10^{26}\text{ W}}{4\pi(2.28 \times 10^{11}\text{ m})^2\sigma}} \approx 319\text{ K} = 46°\text{C}$

a better Approach for better estimate:

albedo, $a = \dfrac{\text{reflected power}}{\text{incident power}} \Longrightarrow \dfrac{\text{absorbed power}}{\text{incident power}} = 1 - a$

For a light-coloured (reflective) body, $a \approx 0.7$–0.9

For a dark-coloured (absorptive) body, $a \approx 0.1$–0.3

Area of the planetary surface exposed to Sun $= \dfrac{1}{2} \times 4\pi r^2 = 2\pi r^2$

But, the effective surface area for normal incidence of incoming radiation $= \displaystyle\int_0^{\frac{\pi}{2}} \cos\theta\,(2\pi r \sin\theta)(r\,d\theta) = \pi r^2$

$\therefore L_{\text{abs}} = \dfrac{L_*}{4\pi d^2}(\pi r^2)(1-a) = L_*\left(\dfrac{r^2}{4d^2}\right)(1-a)$

Also, $L_{\text{rad}} = 4\pi r^2 \sigma T^4$

$\because L_{\text{abs}} = L_{\text{rad}}$ [assuming the planet to be a blackbody, otherwise emissivity needs to be taken into account]

$\therefore L_*\left(\dfrac{r^2}{4d^2}\right)(1-a) = 4\pi r^2\sigma T^4 \Longrightarrow L_* = \dfrac{16\pi d^2\sigma T^4}{1-a} = \left(\dfrac{4}{1-a}\right)4\pi d^2\sigma T^4$ [compare with the previous formula]

$\Longrightarrow d = \sqrt{\dfrac{L_*}{4\pi\sigma T^4}\left(\dfrac{1-a}{4}\right)}$ and $T = \sqrt[4]{\dfrac{L_*}{4\pi d^2\sigma}\left(\dfrac{1-a}{4}\right)} = \sqrt[4]{1-a}\left(\dfrac{R_*}{2d}\right)T_{\text{eff}_*}$

$d_{\max} = \sqrt{\dfrac{L_*}{4\pi\sigma {T_{\min}}^4}\left(\dfrac{1-a_{\min}}{4}\right)} = \sqrt{\dfrac{L_\odot}{4\pi\sigma(273.15\text{ K})^4}\left(\dfrac{1-0.1}{4}\right)}$ [for solar system] $= 1.47 \times 10^{11}\text{ m} \approx 0.98\text{ AU}$

$d_{\min} = \sqrt{\dfrac{L_*}{4\pi\sigma {T_{\max}}^4}\left(\dfrac{1-a_{\max}}{4}\right)} = \sqrt{\dfrac{L_\odot}{4\pi\sigma(373.15\text{ K})^4}\left(\dfrac{1-0.9}{4}\right)}$ [for solar system] $= 2.63 \times 10^{11}\text{ m} \approx 0.18\text{ AU}$

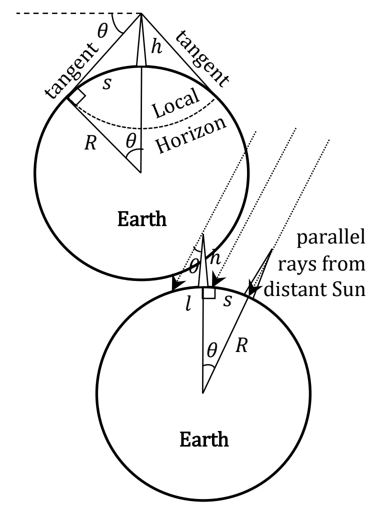
For Earth, $a = 0.3$, so, $T = \sqrt[4]{\frac{L_\odot}{4\pi d^2 \sigma}\left(\frac{1-0.3}{4}\right)} \approx 254\text{ K} \approx -19℃$

This is much below the freezing point of water. Still Earth harbours liquid water. This is because Earth has an atmospheric blanket to keep it warm enough (greenhouse effect due to the greenhouse gases such as H_2O, O_3, CO_2, CH_4, NO, CFC, etc.) for water to be liquid and the conditions to be conducive for life.

Measurement of the Earth's radius

Direct method:

$R = \frac{s}{\theta}$



Distance of Horizon method:

$$(R+h)\cos\theta = R \Longrightarrow \sec\theta = 1 + \frac{h}{R} \Longrightarrow \sec^2\theta = \left(1+\frac{h}{R}\right)^2 \Longrightarrow 1 + \tan^2\theta \approx 1 + \frac{2h}{R}$$

$$\Longrightarrow \tan^2\theta \approx \frac{2h}{R} \Longrightarrow \theta \approx \tan^{-1}\sqrt{\frac{2h}{R}} \Longrightarrow \theta \approx \sqrt{\frac{2h}{R}} \Longrightarrow \frac{s}{R} \approx \sqrt{\frac{2h}{R}} \Longrightarrow R \approx \frac{s^2}{2h}$$

Alternatively, $\sqrt{(R+h)^2 - R^2} \approx \sqrt{s^2 + h^2} \Longrightarrow 2hR + h^2 \approx s^2 + h^2 \Longrightarrow R \approx \frac{s^2}{2h}$

Or, $s^2 \approx \left(\sqrt{(R+h)^2 - R^2}\right)^2 - h^2 = (R+h)^2 - R^2 - h^2 = 2hR \Longrightarrow R = \frac{s^2}{2h}$

Also, $\sec\theta = \frac{\sqrt{(R+h)^2 - R^2}}{s} = \frac{R+h}{R} \Longrightarrow \sqrt{\frac{2hR}{s^2}} = 1 + \frac{h}{R} \approx 1 \Longrightarrow R \approx \frac{s^2}{2h}$

Alternatively, $\frac{h + 2R\sin^2\frac{\theta}{2}}{\tan\theta} = R\sin\theta \Longrightarrow \frac{h + 2R\left(\frac{\theta}{2}\right)^2}{\theta} \approx R\theta \Longrightarrow h + \frac{R\theta^2}{2} \approx R\theta^2$

$$\Longrightarrow h \approx \frac{R\theta^2}{2} \Longrightarrow h \approx \frac{s^2}{2R} \quad \left[\because \theta = \frac{s}{R}\right] \Longrightarrow R \approx \frac{s^2}{2h}$$

Or, $\begin{cases} \sin\theta \approx \theta \Longrightarrow \frac{\sqrt{(R+h)^2 - R^2}}{R+h} = \frac{s}{R} \Longrightarrow \frac{\sqrt{2hR}}{s} = 1 + \frac{h}{R} \approx 1 \Longrightarrow R = \frac{s^2}{2h} \\ \tan\theta \approx \theta \Longrightarrow \frac{\sqrt{(R+h)^2 - R^2}}{R} = \frac{s}{R} \Longrightarrow \frac{\sqrt{2hR}}{s} = 1 \Longrightarrow R = \frac{s^2}{2h} \end{cases}$

Angle of Dip method (Al-Biruni's method):

$$(R+h)\cos\theta = R \Longrightarrow \sec\theta = 1 + \frac{h}{R} \Longrightarrow R = \frac{h}{\sec\theta - 1}$$

Shadow method (Eratosthenes' method):

$$\frac{l}{h} = \tan\theta \approx \sin\theta = \frac{s}{R} \Longrightarrow R = \frac{sh}{l}$$

Double Sunset method

$$\frac{\theta}{2\pi} = \frac{t\text{ (in seconds)}}{24 \times 3600\text{ s}} \Longrightarrow \theta \approx \frac{t}{13751\text{ s}}$$

$$d^2 + R^2 = (R+h)^2 = R^2 + h^2 + 2Rh \Longrightarrow d^2 = 2Rh\left(1 + \frac{h}{2R}\right) \approx 2Rh$$

$$\Longrightarrow (R\tan\theta)^2 \approx 2Rh \quad [\because d = R\tan\theta] \Longrightarrow R \approx \frac{2h}{\tan^2\theta} \Longrightarrow R \approx \frac{2h}{\theta^2}$$

Mass of celestial object using Kepler's 3rd law

If T is the time period, and a is the length of semi-major axis of the orbit, then with using Kepler's 3rd law,
Mass of a planet can be found from T, a of the moons (satellites) orbiting it.
Mass of a star can be found from T, a of the planets orbiting it.
Mass of a black hole can be found from T, a of the planets orbiting it.
If the orbit is circular, a is the orbital radius.

Assuming Moon's orbit to be circular, estimate the mass of Earth. Orbital period of Moon is 27 days and average distance of Moon from Earth is 384,400 km.

$$T^2 = \frac{4\pi^2}{GM}a^3 \Rightarrow M = \frac{4\pi^2}{GT^2}a^3 \Rightarrow M_\oplus = \frac{4\pi^2}{G{T_{\mathrm{Moon}}}^2}{a_{\mathrm{Moon}}}^3 = \frac{4\pi^2(3.844 \times 10^8\ \mathrm{m})^3}{G(27 \times 24 \times 3600\ \mathrm{s})^2} \approx 6.2 \times 10^{24}\ \mathrm{kg}$$

Assuming Earth's orbit to be circular, estimate the mass of Sun.

$$T^2 = \frac{4\pi^2}{GM}a^3 \Rightarrow M = \frac{4\pi^2}{GT^2}a^3 \Rightarrow M_\odot = \frac{4\pi^2}{G{T_\oplus}^2}{a_\oplus}^3 = \frac{4\pi^2(1\ \mathrm{AU})^3}{G(1\ \mathrm{year})^2} = \frac{4\pi^2(1.5 \times 10^{11}\ \mathrm{m})^3}{G(365.25 \times 24 \times 3600\ \mathrm{s})^2} \approx 2.0 \times 10^{30}\ \mathrm{kg}$$

At the center of our Milky Way galaxy, star S13 orbits an unseen object in a near-circular orbit of radius 1750 AU. If the orbital period is 36 years, estimate the mass of the central object.

$$T^2 = \frac{4\pi^2}{GM}a^3 \Rightarrow M_{\mathrm{BH}} = \frac{4\pi^2}{GT^2}a^3 = \frac{4\pi^2(1750 \times 1.5 \times 10^{11}\ \mathrm{m})^3}{G(36 \times 365.25 \times 24 \times 3600\ \mathrm{s})^2} \approx 8.3 \times 10^{36}\ \mathrm{kg} \approx 4.15 \times 10^6 M_\odot$$

Orbital radius of exoplanet using Kepler's 3rd law
If the mass of the host star is known from its luminosity, and the orbital period of the exoplanet is known, the orbital radius of the exoplanet can be estimated using Kepler's 3rd law (assuming the orbit of the planet to be circular).
Mass of exoplanet using Kepler's 3rd law and conservation of linear momentum
If the mass of the host star is known from its luminosity, and the orbital period of the exoplanet is known from the radial velocity (vs time) graph, the orbital radius of the exoplanet can be estimated using Kepler's 3rd law (assuming that it is the only planet in the system) followed by an application of conservation of linear momentum.

Exoplanet HD 28185b orbits the star HD 28185 of mass M = 1.24 M⊙. The star is 138 ly from Earth in the constellation Eridanus. Its radial velocity plot shows peak radial velocity of 170 m/s and a period of 383 days. Find the mass of HD 28185b.

$$a^3 = \frac{GM}{4\pi^2}T^2 \Rightarrow a = \sqrt[3]{\frac{GM}{4\pi^2}T^2} = \sqrt[3]{\frac{(6.67 \times 10^{-11}\ \mathrm{Nm^2/kg^2})(1.24 \times 2.0 \times 10^{30}\ \mathrm{kg})(383 \times 24 \times 3600\ \mathrm{s})^2}{4\pi^2}}$$
$$\approx 1.66 \times 10^{11}\ \mathrm{m} \approx 1.11\ \mathrm{AU}$$

$$M_* v_* = M_{\mathrm{p}} v_{\mathrm{p}} \Rightarrow M_{\mathrm{p}} = \frac{M_* v_*}{v_{\mathrm{p}}} = \frac{M_* v_* T}{2\pi a} \quad \left[\because v_{\mathrm{p}} = \frac{2\pi a}{T}\right] = \frac{(1.24 \times 2.0 \times 10^{30}\ \mathrm{kg})(170\ \mathrm{m/s})(383 \times 24 \times 3600\ \mathrm{s})}{2\pi(1.66 \times 10^{11}\ \mathrm{m})}$$
$$\approx 1.34 \times 10^{28}\ \mathrm{kg} \approx 2240 M_\oplus \approx 7 M_{\mathrm{Jupiter}}$$

Estimate the radial velocity of Sun due to Earth and due to Jupiter having known the masses of Sun, Earth and Jupiter, and the radii of Earth's and Jupiter's (5.2 AU) near-circular orbits around Earth.

$$M_\odot v_{\odot,\oplus} = M_\oplus v_\oplus \Rightarrow v_{\odot,\oplus} = \frac{M_\oplus}{M_\odot}\frac{2\pi a_\oplus}{T_\oplus} = \frac{(5.97 \times 10^{24}\ \mathrm{kg}) \times 2\pi(1.5 \times 10^{11}\ \mathrm{m})}{(2.0 \times 10^{30}\ \mathrm{kg})(365.25 \times 24 \times 3600\ \mathrm{s})} = 0.09\ \mathrm{m/s}$$

$$M_\odot v_{\odot,\mathrm{Jup}} = M_{\mathrm{Jup}} v_{\mathrm{Jup}} \Rightarrow v_{\odot,\oplus} = \frac{M_{\mathrm{Jup}}}{M_\odot}\frac{2\pi a_{\mathrm{Jup}}}{T_{\mathrm{Jup}}} = \frac{(1.9 \times 10^{27}\ \mathrm{kg}) \times 2\pi(5.2 \times 1.5 \times 10^{11}\ \mathrm{m})}{(2.0 \times 10^{30}\ \mathrm{kg})(11.86 \times 365.25 \times 24 \times 3600\ \mathrm{s})} = 12.44\ \mathrm{m/s}$$

Calculate the location of barycenter (centre of mass) of Earth-Sun system taking Sun's centre as origin.

$$a_{\mathrm{CM}} = \frac{M_\oplus a_\oplus}{M_\odot + M_\oplus} \approx \frac{M_\oplus a_\oplus}{M_\odot} = \frac{(5.97\times10^{24}\ \mathrm{kg})(1.5\times10^{11}\ \mathrm{m})}{(2.0\times10^{30}\ \mathrm{kg})} \approx 447.55\ \mathrm{km\ (from\ Sun's\ centre)}$$

Calculate the location of barycenter (centre of mass) of Earth-Moon system taking Earth's centre as origin.

$$a_{\mathrm{CM}} = \frac{M_{\mathrm{Moon}} a_{\mathrm{Moon}}}{M_\oplus + M_{\mathrm{Moon}}} \approx \frac{M_\oplus a_\oplus}{M_\odot} = \frac{(7.35\times10^{22}\ \mathrm{kg})(3.844\times10^{8}\ \mathrm{m})}{5.97\times10^{24}\ \mathrm{kg} + 7.35\times10^{22}\ \mathrm{kg}} \approx 4675\ \mathrm{km\ (from\ Earth's\ centre)}$$

Stellar Energy Generation: Proton-Proton Chain Reaction (p-p chain)

$$\left.\begin{array}{l} \left\{\begin{array}{l} \textbf{(99.76\%):}\ \left.\begin{array}{c} {}^1_1\mathrm{H} + {}^1_1\mathrm{H} + \gamma \rightleftharpoons {}^2_2\mathrm{He} \\ -0.7344\ \mathrm{MeV} \\ {}^2_2\mathrm{He} \to {}^2_1\mathrm{H} + \mathrm{e}^+ + \nu_\mathrm{e} \\ +\mathbf{1.1564\ MeV} \end{array}\right\} {}^1_1\mathrm{H} + {}^1_1\mathrm{H} \to {}^2_1\mathrm{H} + \mathrm{e}^+ + \nu_\mathrm{e} + 0.422\ \mathrm{MeV} \\ \qquad\qquad \mathrm{or} \\ \textbf{pep reaction (0.24\%):}\ {}^1_1\mathrm{H} + \mathrm{e}^- + {}^1_1\mathrm{H} \to {}^2_1\mathrm{H} + \nu_\mathrm{e} + \mathbf{1.444\ MeV} \end{array}\right. \\ {}^2_1\mathrm{H} + {}^1_1\mathrm{H} \to {}^3_2\mathrm{He} + \gamma + 5.49\ \mathrm{MeV} \end{array}\right\} \mathbf{3\ {}^1_1H \to {}^3_2He + e^+ + \nu_e + \gamma + 5.91\ MeV}$$

PP-I branch (dominant at 10-14 MK):
83.30%

$$\mathbf{{}^3_2He + {}^3_2He \to {}^4_2He + 2\ {}^1_1H + \gamma + 12.862\ MeV}$$

PP-II branch (dominant at 14-23 MK)
16.68%

$$\left.\begin{array}{l} \mathbf{{}^3_2He + {}^4_2He \to {}^7_4Be + \gamma + 1.5854\ MeV} \\ \left.\begin{array}{l} {}^7_4\mathrm{Be} + \mathrm{e}^- \to {}^7_3\mathrm{Li} + \nu_\mathrm{e} + \mathbf{0.866\ MeV} \\ \qquad \mathrm{or} \\ \left\{\begin{array}{l} {}^7_4\mathrm{Be} + \mathrm{e}^- \to {}^7_3\mathrm{Li}^* + \nu_\mathrm{e} + \mathbf{0.383\ MeV} \\ {}^7_3\mathrm{Li}^* \to {}^7_3\mathrm{Li} + \gamma + 0.483\ \mathrm{MeV} \end{array}\right. \\ {}^7_3\mathrm{Li} + {}^1_1\mathrm{H} \to 2\ {}^4_2\mathrm{He} + \gamma + 17.3454\ \mathrm{MeV} \end{array}\right\} \mathbf{{}^7_4Be + e^- + {}^1_1H \to 2\ {}^4_2He + \nu_e + \gamma} \end{array}\right\} \mathbf{{}^3_2He + e^- + {}^1_1H \to {}^4_2He + \nu_e + \gamma}$$

PP-III branch (dominant at > 23 MK)
0.02%

$$\left.\begin{array}{l} \mathbf{{}^3_2He + {}^4_2He \to {}^7_4Be + \gamma + 1.5854\ MeV} \\ \left.\begin{array}{l} {}^7_4\mathrm{Be} + {}^1_1\mathrm{H} \to {}^8_5\mathrm{B} + \gamma + 0.138\ \mathrm{MeV} \\ {}^8_5\mathrm{B} \to {}^8_4\mathrm{Be}^* + \mathrm{e}^+ + \nu_\mathrm{e} + \mathbf{14.06\ MeV} \\ {}^8_4\mathrm{Be}^* \to 2\ {}^4_2\mathrm{He} + \gamma + 2.9918\ \mathrm{MeV} \end{array}\right\} \mathbf{{}^7_4Be + {}^1_1H \to 2\ {}^4_2He + e^+ + \nu_e + \gamma} \end{array}\right\} \mathbf{{}^3_2He + {}^1_1H \to {}^4_2He + e^+ + \nu_e + \gamma}$$

PP-IV (Hep) branch:
0.00002%

$${}^3_2\mathrm{He} + {}^1_1\mathrm{H} \to {}^4_2\mathrm{He} + \mathrm{e}^+ + \nu_\mathrm{e} + \gamma + \mathbf{18.774\ MeV}$$

Beta annihilation: $\mathrm{e}^+ + \mathrm{e}^- \to 2\gamma + 1.022\ \mathrm{MeV}$

$$\therefore \left\{\begin{array}{c} \text{Energy released in PP-I chain} = (5.9157\times2 + 12.862 + 1.022\times2)\ \mathrm{MeV} \approx 26.737\ \mathrm{MeV} \\ \text{Energy released in PP-II and PP-II chains} = \text{Energy released in PP-IV chain} \quad \text{[see the net equations]} \\ \text{Energy released in PP-IV chain} = (5.9157 + 18.774 + 1.022\times2)\ \mathrm{MeV} \approx 26.734\ \mathrm{MeV} \end{array}\right.$$

The first step i.e. diproton (He-2) formation is the limiting reaction (extremely slow) since the subsequent positron emission of the diproton to deuteron is extremely rare.

Helium burning process (Helium fusion / Triple Alpha process) near **1×10^8 K** (≡ 10 keV)

${}^4_2\mathrm{He} + {}^4_2\mathrm{He} \rightleftharpoons {}^8_4\mathrm{Be} - 92.8\ \mathrm{keV}$
${}^8_4\mathrm{Be} + {}^4_2\mathrm{He} \to {}^{12}_6\mathrm{C}^* + 7.367\ \mathrm{MeV}$
${}^{12}_6\mathrm{C}^* \to {}^{12}_6\mathrm{C} + \gamma$

Heavy element burning processes

Mass: > 8 $M_\odot$ (at birth) ; **Temperature:** > 5×10^8 K (≡ 40 keV) ; **Density:** > 3×10^9 kg/m^3
Carbon burning process (Carbon fusion) near **6×10^8 K** (≡ 50 keV)

$$^{12}_{6}C + ^{12}_{6}C \rightarrow \begin{cases} ^{20}_{10}Ne + ^{4}_{2}He + 4.617\ MeV \\ ^{23}_{11}Na + ^{1}_{1}H + 2.241\ MeV \\ ^{23}_{12}Mg + ^{1}_{0}n - 2.599\ MeV \\ ^{24}_{12}Mg + \gamma + 13.933\ MeV \\ ^{16}_{8}O + 2\ ^{4}_{2}He - 0.113\ MeV \end{cases}$$

Neon burning process (Neon fusion) near **1×10^9 K** (≡ 90 keV)

Photodisintegration: $^{20}_{10}Ne + \gamma \rightarrow ^{16}_{8}O + ^{4}_{2}He$

$^{20}_{10}Ne + ^{4}_{2}He \rightarrow ^{24}_{12}Mg + \gamma + 9.32\ MeV$

$$\begin{Bmatrix} ^{20}_{10}Ne + ^{1}_{0}n \rightarrow ^{21}_{10}Ne + \gamma \\ ^{21}_{10}Ne + ^{4}_{2}He \rightarrow ^{24}_{12}Mg + ^{1}_{0}n \end{Bmatrix}$$

Oxygen burning process (Oxygen fusion) near **2×10^9 K** (≡ 170 keV)

$$^{16}_{8}O + ^{16}_{8}O \rightarrow \begin{cases} ^{28}_{14}Si + ^{4}_{2}He + 9.593\ MeV \\ ^{31}_{15}P + ^{1}_{1}H + 7.676\ MeV \\ ^{31}_{16}S + ^{1}_{0}n + 1.459\ MeV \\ ^{30}_{14}Si + 2\ ^{1}_{1}H + 0.381\ MeV \\ ^{30}_{15}P + ^{2}_{1}H - 2.409\ MeV \\ ^{32}_{16}S + \gamma + 16.539\ MeV \\ ^{28}_{14}Si + ^{4}_{2}He + ???\ MeV \\ ^{24}_{12}Mg + 2\ ^{4}_{2}He - 0.393\ MeV \end{cases}$$

Silicon burning process (Silicon fusion) near **3×10^9 K** (≡ 250 keV)

Photodisintegration: $^{28}_{14}Si + \gamma \rightarrow ^{24}_{12}Mg + ^{4}_{2}He$

$^{28}_{14}Si + ^{4}_{2}He \rightarrow ^{32}_{16}S + \gamma + 6.95\ MeV$

$^{32}_{16}S + ^{4}_{2}He \rightarrow ^{36}_{18}Ar + \gamma + 6.64\ MeV$

$^{36}_{18}Ar + ^{4}_{2}He \rightarrow ^{40}_{20}Ca + \gamma + 7.04\ MeV$

$^{40}_{18}Ca + ^{4}_{2}He \rightarrow ^{44}_{22}Ti + \gamma + 5.13\ MeV$

$^{44}_{20}Ti + ^{4}_{2}He \rightarrow ^{48}_{24}Cr + \gamma + 7.70\ MeV$

$^{48}_{24}Cr + ^{4}_{2}He \rightarrow ^{52}_{26}Fe + \gamma + 7.94\ MeV$

$^{52}_{26}Fe + ^{4}_{2}He \rightarrow ^{56}_{28}Ni + \gamma + 8.00\ MeV$

Nickel-56 decay: $^{56}_{28}Ni \rightarrow ^{56}_{26}Fe + 2e^{+} + 2\nu_e$

Alpha process (Alpha ladder)

$^{12}_{6}C + ^{4}_{2}He \rightarrow ^{16}_{8}O + \gamma + 7.16\ MeV$

$^{16}_{8}O + ^{4}_{2}He \rightarrow ^{20}_{10}Ne + \gamma + 4.73\ MeV$

$^{20}_{10}Ne + ^{4}_{2}He \rightarrow ^{24}_{12}Mg + \gamma + 9.32\ MeV$

$^{24}_{14}Mg + ^{4}_{2}He \rightarrow ^{28}_{14}Si + \gamma + 9.98\ MeV$

$^{28}_{14}Si + ^{4}_{2}He \rightarrow ^{32}_{16}S + \gamma + 6.95\ MeV$

$^{32}_{16}S + ^{4}_{2}He \rightarrow ^{36}_{18}Ar + \gamma + 6.64\ MeV$

$^{36}_{18}Ar + ^{4}_{2}He \rightarrow ^{40}_{20}Ca + \gamma + 7.04\ MeV$

$^{40}_{18}Ca + ^{4}_{2}He \rightarrow ^{44}_{22}Ti + \gamma + 5.13\ MeV$

$^{44}_{20}Ti + ^{4}_{2}He \rightarrow ^{48}_{24}Cr + \gamma + 7.70\ MeV$

$^{48}_{24}Cr + ^{4}_{2}He \rightarrow ^{52}_{26}Fe + \gamma + 7.94\ MeV$

$^{52}_{26}Fe + ^{4}_{2}He \rightarrow ^{56}_{28}Ni + \gamma + 8.00\ MeV$

n-p reaction

A nuclear reaction in which the entering of neutron into the nucleus is immediately followed by the exit of a proton. Most n-p reactions have threshold neutron energies (thereby exothermic) below which the reaction is not feasible.

$^{32}_{16}S + ^{1}_{0}n \rightarrow ^{32}_{15}P + ^{1}_{1}H$ (endothermic)

$^{14}_{7}N + ^{1}_{0}n \rightarrow ^{14}_{6}C + ^{1}_{1}H$ (exothermic)

Fermi's Theory of Beta Decay

Based on Pauli's neutrino hypothesis, Enrico Fermi in 1934 developed a theory for beta decay (energy and momentum distribution of beta particles) using the result of Dirac's time-dependent perturbation theory. Rate of transition from an initial state i to a final state f is given by Fermi's golden rule: $w = \frac{2\pi}{\hbar}\left|H'_{fi}\right|^2 \rho(E)$ where is the density of final states, and H'_{if} is the matrix element of the harmonic perturbing interaction H' causing the transition, given by $H'_{fi} = \int \psi_f{}^* H \psi_i \, d\tau$

$$\begin{cases} X(A,Z) \to Y(A,Z+1) + e^- + \bar{\nu} \Rightarrow X(A,Z) + \nu \to Y(A,Z+1) + e^- \\ X(A,Z) \to Y(A,Z-1) + e^+ + \nu \Rightarrow X(A,Z) + \bar{\nu} \to Y(A,Z-1) + e^+ \end{cases} \Rightarrow \begin{cases} \psi_i = u_i \phi_\beta \\ \psi_f = u_f \phi_\nu \end{cases}$$

ψ_i is the initial state wavefunction ; ψ_f is the final state wavefunction

u_i is the initial state nuclear wavefunction ; u_f is the final state nuclear wavefunction

ϕ_β and ϕ_ν are the leptonic wavefunctions

Note: Electron, Positron, Neutrino, Antineutrino are all Leptons.

$H = F(\vec{r}_n)\mathcal{O}_n$

Hydrogen atom: Radial equation

$$\boldsymbol{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2\mu}{\hbar^2}\left(E - \frac{l(l+1)\hbar^2}{2\mu} + \frac{kZe^2}{r}\right)R = 0}$$

$$\Rightarrow \frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \frac{2\mu}{\hbar^2}\left(E - \frac{l(l+1)\hbar^2}{2\mu r^2} + \frac{kZe^2}{r}\right)R = 0 \quad \left[\because \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{1}{r^2}\left(r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr}\right)\right]$$

$$\left[\begin{array}{l} \text{Let } \boldsymbol{\rho} = \sqrt{\boldsymbol{\frac{-8\mu E}{\hbar^2}}}\,\boldsymbol{r} \quad \left[\begin{array}{c} E \text{ is negative} \\ \text{for bound states.} \end{array}\right] \text{ then, } \frac{dR}{dr} = \frac{dR}{d\rho}\frac{d\rho}{dr} = \frac{dR}{d\rho}\sqrt{\frac{-8\mu E}{\hbar^2}} = \frac{\rho}{r}\frac{dR}{d\rho} \\ \frac{d^2R}{dr^2} = \frac{d}{dr}\left(\frac{dR}{dr}\right) = \frac{d\rho}{dr}\frac{d}{d\rho}\left(\frac{dR}{dr}\right) = \sqrt{\frac{-8\mu E}{\hbar^2}}\frac{d}{d\rho}\left(\frac{dR}{d\rho}\sqrt{\frac{-8\mu E}{\hbar^2}}\right) = \left(\frac{-8\mu E}{\hbar^2}\right)\frac{d^2R}{d\rho^2} = \frac{\rho^2}{r^2}\frac{d^2R}{d\rho^2} \end{array}\right]$$

$$\Rightarrow \frac{\rho^2}{r^2}\frac{d^2R}{d\rho^2} + \frac{2\rho}{r^2}\frac{dR}{d\rho} + \frac{2\mu}{\hbar^2}\left(E - \frac{l(l+1)\hbar^2}{2\mu r^2} + \frac{kZe^2}{r}\right)R = 0$$

$$\Rightarrow \frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{dR}{d\rho} + \frac{2\mu r^2}{\hbar^2\rho^2}\left(E - \frac{l(l+1)\hbar^2}{2\mu r^2} + \frac{kZe^2}{r}\right)R = 0 \Rightarrow \frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{dR}{d\rho} + \left(-\frac{1}{4} - \frac{l(l+1)\hbar^2}{\rho^2} + \frac{\lambda}{\rho}\right)R = 0$$

$$\left[\text{where } \frac{\lambda}{\rho} = \frac{2\mu r}{\hbar^2\rho^2}(kZe^2) \text{ so that } \lambda = \frac{2\mu r}{\hbar^2\rho}(kZe^2) = \frac{2\mu r}{\hbar^2\rho}(kZe^2) = \frac{2\mu r}{\hbar^2}(kZe^2)\sqrt{\frac{\hbar^2}{-8\mu E}} \Rightarrow \boldsymbol{\lambda} = \boldsymbol{\frac{kZe^2}{\hbar}\sqrt{\frac{\mu}{-2E}}}\right]$$

When $r \to \infty, \rho \to \infty$ so that $\frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{dR}{d\rho} + \left(-\frac{1}{4} - \frac{l(l+1)\hbar^2}{\rho^2} + \frac{\lambda}{\rho}\right)R = 0$ reduces to $\frac{d^2R}{d\rho^2} - \frac{1}{4}R = 0$ whose solutions are $R = e^{-\rho/2}$ and $e^{\rho/2}$ of which $e^{\rho/2}$ is not feasible as $e^{\rho/2} \to \infty$ as $\rho \to \infty$ $\therefore R = e^{-\rho/2}$ is the asymptotic solution of the radial equation. The exact solution is therefore, $\boldsymbol{R = F(\rho)e^{-\rho/2}}$

$$\therefore \frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{dR}{d\rho} + \left(-\frac{1}{4} - \frac{l(l+1)\hbar^2}{\rho^2} + \frac{\lambda}{\rho}\right) = 0 \Rightarrow e^{-\rho/2}\frac{d^2F}{d\rho^2} + e^{-\rho/2}\left(\frac{2}{\rho} - 1\right)\frac{dF}{d\rho} + \left(-\frac{1}{\rho} - \frac{l(l+1)}{\rho^2} + \frac{\lambda}{\rho}\right)Fe^{-\rho/2} = 0$$

$$\left[\begin{array}{l} \frac{dR}{d\rho} = \frac{d}{d\rho}\left(Fe^{-\rho/2}\right) = e^{-\rho/2}\left(\frac{dF}{d\rho} - \frac{F}{2}\right) \Rightarrow \frac{2}{\rho}\frac{dR}{d\rho} = \frac{2}{\rho}e^{-\rho/2}\left(\frac{dF}{d\rho} - \frac{F}{2}\right) \\ \frac{d^2R}{d\rho^2} = \frac{d}{d\rho}\frac{dR}{d\rho} = e^{-\rho/2}\left(\frac{d^2F}{d\rho^2} - \frac{1}{2}\frac{dF}{d\rho}\right) - \frac{e^{-\rho/2}}{2}\left(\frac{dF}{d\rho} - \frac{F}{2}\right) = e^{-\rho/2}\left(\frac{d^2F}{d\rho^2} - \frac{dF}{d\rho} + \frac{F}{4}\right) \end{array}\right]$$

$$\Rightarrow \boldsymbol{\rho^2\frac{d^2F}{d\rho^2} + \rho(2-\rho)\frac{dF}{d\rho} + (-\rho - l(l+1) + \lambda\rho)F = 0}$$

When $\rho = 0,\ l(l+1)F(0) = 0$ $\left[\begin{array}{l} \text{which is true for any } l \\ \text{whether } l = 0 \text{ or } l \neq 0 \end{array}\right] \Rightarrow F(0) = 0 \Rightarrow F(\rho) = \sum_{q=0}^{\infty} a_q \rho^{c+q}$ [where $c \geq 1$]

so that $\dfrac{dF}{d\rho} = \sum_{q=0}^{\infty}(c+q)a_q\rho^{c+q-1}$ and $\dfrac{d^2F}{d\rho^2} = \sum_{q=0}^{\infty}(c+q)(c+q-1)a_q\rho^{c+q-2}$

$$\therefore \sum_{q=0}^{\infty}(c+q)(c+q-1)a_q\rho^{c+q} + (2-\rho)\sum_{q=0}^{\infty}(c+q)a_q\rho^{c+q} + (-\rho - l(l+1) + \lambda\rho)\sum_{q=0}^{\infty}a_q\rho^{c+q} = 0$$

$$\Rightarrow \{-(c+q) - 1 + \lambda\}\sum_{q=0}^{\infty}a_q\rho^{c+q+1} + \sum_{q=0}^{\infty}\{(c+q)(c+q-1) + 2(c+q) - l(l+1)\}a_q\rho^{c+q} = 0$$

which is valid for all values of ρ only if the coefficient of each power of ρ vanishes separately. For ρ^c term, $a_0\{c(c-1) + 2c - l(l+1)\} = 0 \Rightarrow c(c-1) + 2c - l(l+1)$ $[\because a_0 \neq 0] \Rightarrow c(c+1) - l(l+1) = 0$
$\Rightarrow c^2 - l^2 + c - l = 0 \Rightarrow (c-l)(c+l+1) = 0 \Rightarrow c = l$ or $c = -(l+1)$ of which $c = -(l+1)$ is not feasible $\because$ the first term of $F(\rho)$, $a_0/\rho^{(l+1)} \to \infty$ as $\rho \to \infty$ $\therefore c = l$ is the only acceptable value. Setting the coefficient of ρ^{c+q+1} i.e. ρ^{l+q+1} to zero,
$\{-(l+q) - 1 + \lambda\}a_q + \{(l+q+1)(l+q) + 2(l+q+1) - l(l+1)\}a_{q+1} = 0$

$$\Rightarrow a_{q+1} = \frac{(l+q+1) - \lambda}{(l+q+1)(l+q+2) - l(l+1)}a_q = \boldsymbol{a_{q+1} = \frac{(l+q+1)-\lambda}{(q+1)(q+2l+2)}a_q} \quad \text{[recursion relation]}$$

$$\begin{bmatrix} (l+q+1)(l+q+2) - l(l+1) = l^2 + l(2q+3) + (q+1)(q+2) - l^2 - l \\ = 2l(q+1) + (q+1)(q+2) = (q+1)(q+2l+2) \end{bmatrix}$$

For large k, $a_{q+1} = \dfrac{(l+q+1)-\lambda}{(q+1)(q+2l+2)}a_q \Rightarrow a_{q+1} \approx \dfrac{a_q}{q} \Rightarrow a_q \approx \dfrac{1}{q!} \Rightarrow F(\rho) = \sum_{q=0}^{\infty}a_q\rho^{l+q} \approx \rho^l\sum_{q=0}^{\infty}\dfrac{\rho^q}{q!} = \rho^l e^\rho$

Then $R = \rho^l e^{-\rho/2} \approx \rho^l e^\rho e^{-\rho/2} = \rho^l e^{\rho/2}$ which is not feasible as $\rho^l e^{\rho/2} \to \infty$ as $\rho \to \infty$ $\therefore$ the series must terminate after a certain value of q, say q'. For this to happen, $a_{q'+1} = 0 \Rightarrow (l+q'+1) - \lambda \Rightarrow \lambda = l + q' + 1$

[where $q' = 0, 1, 2, 3, \ldots$] Let $n = l + q' + 1 = \lambda = \dfrac{kZe^2}{\hbar}\sqrt{\dfrac{\mu}{-2E_n}} \Rightarrow \boldsymbol{E_n = -\dfrac{k^2Z^2e^4\mu}{2n^2\hbar^2}}$ $[n = 1, 2, 3, \ldots]$

$\because n = l + q' + 1$ and $q' = 0, 1, 2, 3, \ldots$ $\therefore n \geq l + 1 \Rightarrow l \leq n - 1 \Rightarrow l_{\max} = n - 1 \Rightarrow l = 0, 1, 2, 3, \ldots (n-1)$
$\boldsymbol{n}$ is called the **principal quantum number**, whereas, $\boldsymbol{l}$ is called the **orbital** (or **azimuthal**) **quantum number.**

So, $F(\rho) = \rho^l L(\rho)$ $\begin{bmatrix} L(\rho) \text{ is the finite series} \\ \text{that's terminated at } q. \end{bmatrix}$ $\therefore \rho^2\dfrac{d^2F}{d\rho^2} + \rho(2-\rho)\dfrac{dF}{d\rho} + (-\rho - l(l+1) + \lambda\rho)F = 0$

$$\Rightarrow \left\{l(l-1)\rho^l L + 2l\rho^{l+1}\frac{dL}{d\rho} + \rho^{l+2}\frac{d^2L}{d\rho^2}\right\} + (2-\rho)\left\{l\rho^l L + \rho^{l+1}\frac{dL}{d\rho}\right\} + (-\rho - l(l+1) + \lambda\rho)\rho^l L = 0$$

$$\begin{bmatrix} \dfrac{dF}{d\rho} = l\rho^{l-1}L + \rho^l\dfrac{dL}{d\rho} \\ \dfrac{d^2F}{d\rho^2} = l(l-1)\rho^{l-2}L + 2l\rho^{l-1}\dfrac{dL}{d\rho} + \rho^l\dfrac{d^2L}{d\rho^2} \end{bmatrix} \Rightarrow \rho^2\frac{d^2L}{d\rho^2} + \rho(2-\rho+2l)\frac{dL}{d\rho} + \big(-\rho - 2l + \lambda\rho + l(2-\rho)\big)L = 0$$

$$\Rightarrow \rho\frac{d^2L}{d\rho^2} + (2l+2-\rho)\frac{dL}{d\rho} + (n-l-1)L = 0 \quad [\because n = \lambda]$$

This is identifiable as the Associated Laguerre's equation: $\rho\dfrac{d^2L(\rho)}{d\rho^2} + (\alpha+1-\rho)\dfrac{dL(\rho)}{d\rho} + \beta L(\rho) = 0$

whose solutions are the Associated Laguerre polynomials of order α and degree β where $\begin{cases} \alpha = 2l+1 \\ \beta = n - l - 1 = \\ (n+l) - (2l+1) \end{cases}$

$$\begin{bmatrix} \because L_\beta^\alpha(\rho) = \displaystyle\sum_{q=0}^{\beta}(-1)^q\frac{{}^{(\beta+\alpha)}C_{(\beta-q)}}{q!}\rho^q = \sum_{q=0}^{\beta}(-1)^q\frac{(\beta+\alpha)!}{(\beta-q)!\,(\alpha+q)!\,q!}\rho^q \\ \therefore L_{n-l-1}^{2l+1}(\rho) = \displaystyle\sum_{q=0}^{n-l-1}(-1)^q\frac{{}^{(n+l)}C_{(n-l-1-q)}}{q!}\rho^q = \sum_{q=0}^{n-l-1}(-1)^q\frac{(n+l)!}{(n-l-1-q)!\,(2l+1+q)!\,q!}\rho^q \end{bmatrix}$$

$\therefore R = F(\rho)e^{-\rho/2} = e^{-\rho/2}\rho^l L_{n-l-1}^{2l+1}(\rho) \Rightarrow \boldsymbol{R_{nl}(r) = Ne^{-\rho/2}\rho^l L_{n-l-1}^{2l+1}(\rho)}$ $\left[\begin{matrix}\text{where } \rho \equiv \rho(r) \text{ and} \\ N \text{ is normalization factor}\end{matrix}\right]$

Normalization: $\iiint\limits_{0\,0\,0}^{2\pi\,\pi\,\infty} R_{nl}^2(r) Y_{lm}^2(\theta,\phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi = 1 \Rightarrow \iint\limits_{0\,0}^{2\pi\,\pi} Y_{lm}^2(\theta,\phi) \sin\theta\, d\theta\, d\phi \int_0^\infty R_{nl}^2(r)\, r^2\, dr = 1$

$\Rightarrow \int_0^\infty R_{nl}^2(r)\, r^2\, dr = 1$ [$\because$ Spherical harmonic, $Y_{lm}(\theta,\phi)$, is normalized.]

$$\Rightarrow \left(\frac{\hbar^2}{-8\mu E}\right)^{\frac{3}{2}} \int_0^\infty R_{nl}^2(\rho)\, \rho^2\, d\rho = 1 \quad \left[\because \rho = \sqrt{\frac{-8\mu E}{\hbar^2}}\, r\right] \Rightarrow \left(\frac{\hbar^2}{-8\mu E}\right)^{\frac{3}{2}} N^2 \int_0^\infty \left[L_{n-l-1}^{2l+1}(\rho)\right]^2 e^{-\rho} \rho^{2l+2}\, d\rho = 1$$

$$\Rightarrow \left(\frac{\hbar^2}{-8\mu E}\right)^{\frac{3}{2}} \frac{2n(n+l)!}{(n-l-1)!} N^2 = 1 \Rightarrow N = \pm\sqrt{\frac{(n-l-1)!}{2n(n+l)!}\left(\frac{2Z}{na_0}\right)^3} \quad \left[\because E_n = -\frac{k^2 Z^2 e^4 \mu}{2n^2\hbar^2} = -\frac{\hbar^2 Z^2}{2n^2 a_{\mathrm{H}}{}^2 \mu}\right]$$

$$\left[\because \int_0^\infty \left[L_{n-l-1}^{2l+1}(\rho)\right]^2 e^{-\rho}\rho^{2l+2}\, d\rho = \int_0^\infty \left[L_\beta^\alpha(\rho)\right]^2 e^{-\rho}\rho^{\alpha+1}\, d\rho = \frac{(\alpha+\beta)!}{\beta!}(2\beta+\alpha+1) = \frac{(n+l)!}{(n-l-1)!}(2n)\right]$$

$$\left[\rho = \sqrt{\frac{-8\mu E}{\hbar^2}}\, r = \frac{2Zk\mu e^2}{n\hbar^2} r = \frac{2Z}{na_{\mathrm{H}}} r \text{ where } a_{\mathrm{H}} = \frac{\hbar^2}{k\mu e^2} \approx \frac{\hbar^2}{km_{\mathrm{e}}e^2} = a_0 \;\because \mu \approx m_{\mathrm{e}}\right]$$

$$\therefore \boldsymbol{R_{nl}(r)} = \sqrt{\frac{\boldsymbol{(n-l-1)!}}{\boldsymbol{(n+l)!}}\left(\frac{\boldsymbol{Z}}{\boldsymbol{a_0}}\right)} \frac{\boldsymbol{e^{-\frac{Z}{na_0}r}}}{\boldsymbol{n^2 r}} \left(\frac{\boldsymbol{2Z}}{\boldsymbol{na_0}}\boldsymbol{r}\right)^{\boldsymbol{l+1}} \boldsymbol{L_{n-l-1}^{2l+1}}\left(\frac{\boldsymbol{2Z}}{\boldsymbol{na_0}}\boldsymbol{r}\right)$$

Deuteron

A sample of an element is placed in a magnetic field of 0.3 T. How far apart are the Zeeman components (splitting) of a spectral line of wavelength 4500 Å?

$$E = \frac{hc}{\lambda} \Rightarrow dE = -\frac{hc}{\lambda^2}d\lambda \Rightarrow |d\lambda| = \frac{\lambda^2}{hc}|dE| = \frac{eB}{4\pi m_{\mathrm{e}}c}\lambda^2 \quad \left[\because |dE| = \frac{e\hbar}{2m_{\mathrm{e}}}B\right] \approx 2.836 \times 10^{-12}\ \mathrm{m} \approx 0.0284\ \text{Å}$$

Determine the normal Zeeman effect on the Cadmium red line of 6438 Å when the atoms are placed in a magnetic field of 0.009 T.

$$E = \frac{hc}{\lambda} \Rightarrow dE = -\frac{hc}{\lambda^2}d\lambda \Rightarrow |d\lambda| = \frac{\lambda^2}{hc}|dE| = \frac{eB}{4\pi m_{\mathrm{e}}c}\lambda^2 \quad \left[\because |dE| = \frac{e\hbar}{2m_{\mathrm{e}}}B\right] \approx 1.74 \times 10^{-13}\ \mathrm{m} \approx 0.0017\ \text{Å}$$

The Calcium line of wavelength λ = 4226.73 Å (P → S) exhibits normal Zeeman splitting when placed in uniform magnetic field of 4 Wb/m² (4 T). Calculate the wavelengths of the three components of normal Zeeman pattern and the separation between them.

$$E = \frac{hc}{\lambda} \Rightarrow dE = -\frac{hc}{\lambda^2}d\lambda \Rightarrow |d\lambda| = \frac{\lambda^2}{hc}|dE| = \frac{eB}{4\pi m_{\mathrm{e}}c}\lambda^2 \quad \left[\because |dE| = \frac{e\hbar}{2m_{\mathrm{e}}}B\right] \approx 3.34 \times 10^{-11}\ \mathrm{m} \approx 0.33\ \text{Å}$$

$$\therefore \text{Wavelengths of the three Zeeman components are } \begin{cases} 4226.73\ \text{Å} - 0.33\ \text{Å} = 4226.40\ \text{Å} \\ 4226.73\ \text{Å} \\ 4226.73\ \text{Å} + 0.33\ \text{Å} = 4227.06\ \text{Å} \end{cases}$$

What magnetic flux density *B* is required to observe the normal Zeeman effect if a spectrometer can resolve spectral lines separated by 0.5 Å at 5000 Å?

$$E = \frac{hc}{\lambda} \Rightarrow dE = -\frac{hc}{\lambda^2}d\lambda \Rightarrow |d\lambda| = \frac{\lambda^2}{hc}|dE| = \frac{eB}{4\pi m_{\mathrm{e}}c}\lambda^2 \quad \left[\because |dE| = \frac{e\hbar}{2m_{\mathrm{e}}}B\right] \Rightarrow B = \frac{4\pi m_{\mathrm{e}}c}{e}\frac{|d\lambda|}{\lambda^2} = 4.284\ \mathrm{T}$$

In a normal Zeeman experiment, the Calcium 4226 Å line splits into 3 lines separated by 0.25 Å in a magnetic field of 3 T. Determine *e/m* for the electron from these data.

$$E = \frac{hc}{\lambda} \Rightarrow dE = -\frac{hc}{\lambda^2}d\lambda \Rightarrow |d\lambda| = \frac{\lambda^2}{hc}|dE| = \frac{eB}{4\pi m_e c}\lambda^2 \quad \left[\because |dE| = \frac{e\hbar}{2m_e}B\right] \Rightarrow \frac{e}{m_e} = \frac{4\pi c}{B}\frac{|d\lambda|}{\lambda^2} = 1.758 \times 10^{11}\ \text{C/kg}$$

The ground state of Chlorine is $^2P_{3/2}$. Find its magnetic moment. In how many substates will the ground state split in a weak magnetic field?

$$\text{Magnetic moment, } \mu_J = g_J\sqrt{J(J+1)}\,\mu_B \quad \text{where Lande's } g\text{-factor, } g_J = 1 + \frac{J(J+1) - L(L+1) + S(S+1)}{2J(J+1)}$$

$$= 1 + \frac{\frac{3}{2}\left(\frac{3}{2}+1\right) - 1(1+1) + \frac{1}{2}\left(\frac{1}{2}+1\right)}{2\cdot\frac{3}{2}\left(\frac{3}{2}+1\right)} = 1 + \frac{\frac{15}{4} - 2 + \frac{3}{4}}{\frac{15}{2}} = 1 + \frac{\frac{5}{2}}{\frac{15}{2}} = 1 + \frac{1}{3} = \frac{4}{3} \quad \left[\because {}^2P_{3/2} \Rightarrow \begin{cases} J = \frac{3}{2} \\ L = 1 \\ S = \frac{1}{2} \end{cases}\right]$$

$$\mu_J = g_J\sqrt{J(J+1)}\,\mu_B = \frac{4}{3}\sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right)}\,\mu_B = \frac{4}{3}\sqrt{\frac{15}{4}}\,\mu_B = \frac{2}{3}\sqrt{15}\,\mu_B$$

$$\therefore \text{Number of substates in which the ground state will split up} = 2J + 1 = 2\cdot\frac{3}{2} + 1 = 4$$

Calculate the Lande's *g*-factors and the total magnetic moments of atoms in the states $^2D_{3/2}$, $^2D_{5/2}$ and $^2F_{7/2}$.

$$\text{Lande's } g\text{-factor, } g_J = 1 + \frac{J(J+1) - L(L+1) + S(S+1)}{2J(J+1)} = \begin{cases} 4/5 \text{ for } {}^2D_{3/2} \text{ state} \\ 6/5 \text{ for } {}^2D_{5/2} \text{ state} \\ 8/7 \text{ for } {}^2F_{7/2} \text{ state} \end{cases} \Rightarrow \mu_J = g_J\sqrt{J(J+1)}\,\mu_B$$

$$= \begin{cases} (2/5)\sqrt{15}\,\mu_B \text{ for } {}^2D_{3/2} \text{ state} \\ (3/5)\sqrt{35}\,\mu_B \text{ for } {}^2D_{5/2} \text{ state} \\ (12/7)\sqrt{7}\,\mu_B \text{ for } {}^2F_{7/2} \text{ state} \end{cases} \quad \left[\because {}^2D_{3/2} \Rightarrow \begin{cases} J = \frac{3}{2} \\ L = 2 \\ S = \frac{1}{2} \end{cases};\ {}^2D_{5/2} \Rightarrow \begin{cases} J = \frac{5}{2} \\ L = 2 \\ S = \frac{1}{2} \end{cases};\ {}^2F_{3/2} \Rightarrow \begin{cases} J = \frac{7}{2} \\ L = 3 \\ S = \frac{1}{2} \end{cases}\right]$$

An electron in a circular orbit has an angular momentum √2 ħ in a field of 0.5 T. What is its Larmor frequency?

$$\omega_L = \frac{eB}{2m_e} = \frac{\mu_B}{\hbar}B \Rightarrow \nu_L = \frac{\omega_L}{2\pi} = \frac{\mu_B}{h}B \approx 7 \times 10^9\ \text{Hz}$$

Classical Big Bang

$$r_1 = r_0 + dr\ ;\ r_2 = r_1 + dr\ ;\ r_3 = r_2 + dr\ ;\ \cdots\ ;\ R = r_n = r_{n-1} + dr$$

$$dr = \frac{R - r_0}{n} \quad \text{and} \quad r_0 \geq dr$$

$$a_0 = \frac{F_0}{4\pi {r_0}^2 \rho\, dr}$$

$$t_0 = \sqrt{\frac{2\,dr}{a_0}}$$

$$F_1 = F_0 \Rightarrow a_1 = \frac{F_0}{4\pi {r_1}^2 \rho\, dr} = \frac{F_0}{4\pi (r_0 + dr)^2 \rho\, dr} = \frac{F_0}{4\pi {r_0}^2 \rho\, dr\left(1 + \frac{dr}{r_0}\right)^2} = \frac{a_0}{\left(1 + \frac{dr}{r_0}\right)^2}$$

$$t_1 = \sqrt{\frac{2\,dr}{a_1}} = \sqrt{\frac{2\,dr}{a_0}\left(1 + \frac{dr}{r_0}\right)^2} = \sqrt{\frac{2\,dr}{a_0}}\left(1 + \frac{dr}{r_0}\right)$$

$$a_2=\frac{F_0}{4\pi r_2{}^2\rho\,dr}=\frac{F_0}{4\pi(r_0+2\,dr)^2\rho\,dr}=\frac{F_0}{4\pi r_0{}^2\rho\,dr\left(1+\frac{2\,dr}{r_0}\right)^2}=\frac{a_0}{\left(1+\frac{2\,dr}{r_0}\right)^2}$$

$$t_2=\sqrt{\frac{2\,dr}{a_2}}=\sqrt{\frac{2\,dr}{a_0}\left(1+\frac{2\,dr}{r_0}\right)^2}=\sqrt{\frac{2\,dr}{a_0}}\left(1+\frac{2\,dr}{r_0}\right)$$

$$\therefore a_n=\frac{a_0}{\left(1+\frac{n\,dr}{r_0}\right)^2}=\frac{a_0}{\left(1+\frac{R-r_0}{r_0}\right)^2}=\frac{a_0}{\left(2+\frac{R}{r_0}\right)^2}$$

and $t_{n-1}=\sqrt{\frac{2\,dr}{a_0}}\left(1+\frac{(n-1)\,dr}{r_0}\right)$

$$t=t_0+t_1+t_2+\cdots+t_{n-1}=\sqrt{\frac{2\,dr}{a_0}}\left\{1+\left(1+\frac{dr}{r_0}\right)+\left(1+\frac{2\,dr}{r_0}\right)+\cdots+\left(1+\frac{(n-1)\,dr}{r_0}\right)\right\}$$

$$=\sqrt{\frac{2(R-r_0)}{na_0}}\left\{n+\frac{n(n-1)}{2}\frac{(R-r_0)}{nr_0}\right\}=\sqrt{\frac{(8\pi r_0{}^2\rho\,dr)n(R-r_0)}{F_0}}\left\{1+\frac{(n-1)}{n}\frac{(R-r_0)}{2r_0}\right\}$$

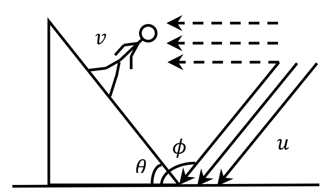
$$=\sqrt{\frac{(8\pi r_0{}^2\rho)(R-r_0)^2}{F_0}}\left\{1+\frac{(R-r_0)}{2r_0}\right\}\quad[\because n\approx n-1]=r_0(R-r_0)\sqrt{\frac{8\pi\rho}{F_0}}\left\{1+\frac{(R-r_0)}{2r_0}\right\}$$

$$\approx r_0R\sqrt{\frac{8\pi\rho}{F_0}}\left\{\frac{R}{2r_0}\right\}\quad[\because R\gg r_0]=R^2\sqrt{\frac{2\pi\rho}{F_0}}$$

$$\Rightarrow\frac{R}{c}=R^2\sqrt{\frac{2\pi\rho}{F_0}}\quad\left[\because t=\frac{R-r_0}{c}\approx\frac{R}{c}\right]\Rightarrow\begin{cases}c=\frac{1}{R}\sqrt{\frac{F_0}{2\pi\rho}}\\F_0=R^2c^2(2\pi\rho)\end{cases}$$

If $R=R_{\text{neutron}}\approx 0.875$ fm, and $\rho_{\text{proton}}=\frac{m_{\text{neutron}}}{\frac{4}{3}\pi R_{\text{neutron}}{}^3}\approx 5.96\times10^{-17}\ \text{kg/m}^3$, then $F_0\approx 2.577\times10^{-29}$ N

Same amount of force, when applied externally, can cause the proton to accelerate with $a_{\text{neutron}}\approx 0.0154\ \text{m/s}^2$



A man is moving down an incline of 60° with a speed *v*. He meets rain with the same speed but horizontally. What is the speed of the rain?

Let the angle of the incline be θ, and the angle between the rain and the ground be ϕ in the same sense as θ.

$$\begin{bmatrix}\text{As per the scenario, } u_{\text{horizontal}}=v_{\text{rain, man}}-v_{\text{horizontal}}=v-v\cos\theta\\ \Rightarrow 2v\sin^2\frac{\theta}{2}=u\cos(\pi-\phi)=u\sin\left(\phi-\frac{\pi}{2}\right)\Rightarrow u=-\frac{2v\sin^2\frac{\theta}{2}}{\cos\phi}\end{bmatrix}$$

$\vec{v}_{\text{man, ground}}=\vec{v}=v\cos\theta\,\hat{\imath}-v\cos\left(\frac{\pi}{2}-\theta\right)\hat{\jmath}$ [man moving **down**] $=v(\cos\theta\,\hat{\imath}-\sin\theta\,\hat{\jmath})$

$\vec{v}_{\text{rain, ground}}=\vec{u}=-u\cos(\pi-\phi)\,\hat{\imath}-u\cos\left(\phi-\frac{\pi}{2}\right)\hat{\jmath}$ [rain falling **down**] $=u(\cos\phi\,\hat{\imath}-\sin\phi\,\hat{\jmath})$

$\vec{v}_{\text{rain, man}}=-v\,\hat{\imath}$ [rain horizontally meeting the man with the same speed as him]

$\vec{v}_{\text{rain, ground}}=\vec{v}_{\text{rain, man}}+\vec{v}_{\text{man, ground}}=-v\,\hat{\imath}+v(\cos\theta\,\hat{\imath}-\sin\theta\,\hat{\jmath})\Rightarrow\vec{u}=-v(1-\cos\theta)\hat{\imath}-v\sin\theta\,\hat{\jmath}$

$$\therefore\left|\vec{v}_{\text{rain, ground}}\right|=u=v\sqrt{(1-\cos\theta)^2+\sin^2\theta}=v\sqrt{(1+\cos^2\theta-2\cos\theta)+\sin^2\theta}=v\sqrt{2(1-\cos\theta)}=2v\sin\frac{\theta}{2}$$

$\because\vec{u}=u\cos\phi\,\hat{\imath}-u\sin\phi\,\hat{\jmath}=-v(1-\cos\theta)\hat{\imath}-v\sin\theta\,\hat{\jmath}$

$$\therefore u\sin\phi=v\sin\theta\Rightarrow\sin\phi=\frac{v}{u}\sin\theta=\frac{\sin\theta}{2\sin\frac{\theta}{2}}=\cos\frac{\theta}{2}=\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\Rightarrow\phi=\pi-\left(\frac{\pi}{2}-\frac{\theta}{2}\right)=\frac{\pi}{2}+\frac{\theta}{2}$$

Also, $u\cos\phi = -v(1-\cos\theta) \Rightarrow \cos\phi = -\frac{v}{u}(1-\cos\theta) = \frac{-2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}} = -\sin\frac{\theta}{2} = \cos\left(\frac{\pi}{2}+\frac{\theta}{2}\right) \Rightarrow \phi = \frac{\pi}{2}+\frac{\theta}{2}$

If $\theta = 60° = \frac{\pi}{3}$ then $u = 2v\sin\frac{\pi}{6} = v$ and $\phi = 120°$

Calculate the angle made by $s = \frac{1}{2}$ vector with the *z*-axis (direction of external field).

$$\frac{\sqrt{3}}{2}\hbar\cos\theta = \frac{1}{2}\hbar \quad \left[\because S = \sqrt{s(s+1)}\hbar = \sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)}\hbar = \frac{\sqrt{3}}{2}\hbar\right] \Rightarrow \cos\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \cos^{-1}\frac{1}{\sqrt{3}} \approx 54.73°$$

What is the orbital magnetic moment associated with the 3s electronic state?

$$\vec{\mu}_{\mathrm{l}} = -g_{\mathrm{l}}\mu_{\mathrm{B}}\frac{\vec{L}}{\hbar} \Rightarrow \mu_{\mathrm{l}} = \mu_{\mathrm{B}}\sqrt{l(l+1)} \quad \begin{bmatrix}\because L = \sqrt{l(l+1)}\hbar \\ \text{and, } g_{\mathrm{l}} = -1\end{bmatrix} \Rightarrow \mu_{\mathrm{l}} = \mu_{\mathrm{B}} \quad [\because l = 0]$$

He-Ne Laser

$$\begin{cases}\mathrm{He}(2^1S_0) + \mathrm{Ne(1s)} \rightarrow \mathrm{He}(1^1S_0) + \mathrm{Ne(3s)} \\ \mathrm{He}(2^3S_1) + \mathrm{Ne(1s)} \rightarrow \mathrm{He}(1^1S_0) + \mathrm{Ne(2s)}\end{cases}$$

A typical optical fibre has a numerical aperture of 0.2. Estimate the acceptance angle of the fibre.

Acceptance angle, $\theta_a = \sin^{-1}(\mathrm{NA}) = \sin^{-1}(\mathrm{NA}) \approx 11.537° \quad \begin{bmatrix}\text{assuming air medium} \\ \text{i.e. } n = 1\end{bmatrix}$

Calculate the numerical aperture and the acceptance angle of an optical fibre with n_{core} = 1.45 and n_{cladding} = 1.4. Also calculate the critical angle of the core-cladding interface.

Numerical aperture, $\mathrm{NA} = \sqrt{{n_{\text{core}}}^2 - {n_{\text{cladding}}}^2} = 0.3775$

Acceptance angle, $\theta_a = \sin^{-1}(\mathrm{NA}) = \sin^{-1}(\mathrm{NA}) \approx 22.18° \quad \begin{bmatrix}\text{assuming air medium} \\ \text{i.e. } n = 1\end{bmatrix}$

Critical angle, $\theta_c = \sin^{-1}\frac{n_{\text{cladding}}}{n_{\text{core}}} = 74.91° \approx 75°$

Find the number of modes that can be propagated in a fibre of 100 µm diameter with core index 1.53 and cladding index 1.5 if the wavelength is 1 µm.

$\mathrm{NA} = \sqrt{{n_{\text{core}}}^2 - {n_{\text{cladding}}}^2} \approx 0.3$

V-number, $V = \frac{(2\pi r)(\mathrm{NA})}{\lambda} \approx 94.718$

Number of modes (for step-index fibre), $N = \frac{V^2}{2} \approx 4486$

Number of modes (for graded-index fibre), $N = \frac{V^2}{4} \approx 2243$

The series limit of Balmer series is 3646 Å. Calculate the value of Rydberg constant.

$$\frac{1}{\lambda} = R_\infty\left(\frac{1}{{n_1}^2} - \frac{1}{{n_2}^2}\right) \Rightarrow \frac{1}{3646 \times 10^{-10}\ \mathrm{m}} = R_\infty\left(\frac{1}{2^2} - \frac{1}{\infty^2}\right) \Rightarrow R_\infty \approx 1.0971 \times 10^7\ \mathrm{m}^{-1}$$

The wavelength of H_α line of the Balmer series is 6563 Å, calculate the wavelength of H_β line and the series limit.

$$\frac{1}{\lambda}=R_\infty\left(\frac{1}{{n_1}^2}-\frac{1}{{n_2}^2}\right)\Longrightarrow\begin{cases}\dfrac{1}{\lambda_\alpha}=R_\infty\left(\dfrac{1}{2^2}-\dfrac{1}{3^2}\right)=\dfrac{5}{36}R_\infty\Longrightarrow R_\infty=\dfrac{36}{5\lambda_{1\text{st}}}\\ \dfrac{1}{\lambda_\beta}=R_\infty\left(\dfrac{1}{2^2}-\dfrac{1}{4^2}\right)=\dfrac{3}{16}R_\infty\Longrightarrow\lambda_{4\text{th}}=\dfrac{16}{3R_\infty}=\dfrac{20}{27}\lambda_{1\text{st}}=4861.5\text{ Å}\\ \dfrac{1}{\lambda_{\text{limit}}}=R_\infty\left(\dfrac{1}{2^2}-\dfrac{1}{\infty^2}\right)=\dfrac{1}{4}R_\infty\Longrightarrow\lambda_{\text{limit}}=\dfrac{4}{R_\infty}=\dfrac{5}{9}\lambda_{1\text{st}}=3646\text{ Å}\end{cases}$$

If the wavelength of the first line of the Lyman series is 1215 Å, calculate the wavelength of the fourth line and the series limit.

$$\frac{1}{\lambda}=R_\infty\left(\frac{1}{{n_1}^2}-\frac{1}{{n_2}^2}\right)\Longrightarrow\begin{cases}\dfrac{1}{\lambda_{1\text{st}}}=R_\infty\left(\dfrac{1}{1^2}-\dfrac{1}{2^2}\right)=\dfrac{3}{4}R_\infty\Longrightarrow R_\infty=\dfrac{4}{3\lambda_{1\text{st}}}\\ \dfrac{1}{\lambda_{4\text{th}}}=R_\infty\left(\dfrac{1}{1^2}-\dfrac{1}{5^2}\right)=\dfrac{24}{25}R_\infty\Longrightarrow\lambda_{4\text{th}}=\dfrac{25}{24R_\infty}=\dfrac{25}{32}\lambda_{1\text{st}}=949\text{ Å}\\ \dfrac{1}{\lambda_{\text{limit}}}=R_\infty\left(\dfrac{1}{1^2}-\dfrac{1}{\infty^2}\right)=R_\infty\Longrightarrow\lambda_{\text{limit}}=\dfrac{1}{R_\infty}=\dfrac{3}{4}\lambda_{1\text{st}}=911\text{ Å}\end{cases}$$

Hydrogen atom in the ground state is excited by means of monochromatic radiation of wavelength 975 Å. How many different lines are possible in the resulting spectrum? Calculate the longest wavelength among them. (Ionization energy for Hydrogen is 13.6 eV)

Energy of the incident radiation, $E_{\text{incident}}=\dfrac{hc}{\lambda_{\text{incident}}}=\dfrac{hc}{975\times10^{-10}\text{ m}}\approx2.03738\times10^{-18}\text{ J}\approx12.7\text{ eV}$

$\because E_{\text{incident}}<E_{\text{ionization}}$ $\quad\therefore$ radiation cannot cause ionization, i.e. $n_{\max}<\infty$

$$\begin{cases}\text{Energy of the } n_{\max}\text{ level, } E_{n_{\max}}=\dfrac{-13.6\text{ eV}}{{n_{\max}}^2}\\ \text{Energy of the ground } (n=1)\text{ level, } E_{n_{\max}}=-13.6\text{ eV}\end{cases}\Longrightarrow\Delta E=13.6\text{ eV}\left(1-\frac{1}{{n_{\max}}^2}\right)$$

$\Longrightarrow12.7\text{ eV}=13.6\text{ eV}\left(1-\dfrac{1}{{n_{\max}}^2}\right)\quad[\because\Delta E=E_{\text{incident}}=12.7\text{ eV}]\Longrightarrow n_{\max}\approx3.89>3$

$\therefore$ Number of lines possible in the resulting spectrum, $N={}^{n_{\max}}C_2={}^3C_2=3$

The transition for the longest wavelength is $2\longrightarrow3$. $\quad\therefore\dfrac{1}{\lambda}=R_\infty\left(\dfrac{1}{2^2}-\dfrac{1}{3^2}\right)\approx6561\text{ Å}$ [visible]

A spectroscopic examination of light from a certain star shows that the apparent wavelength of a certain line is 5001 Å, whereas the observed wavelength of the same line produced by a terrestrial source is 5000 Å. In what direction and with what speed is the star moving relative to Earth?

$$\nu_{\text{observed}}=\nu_{\text{actual}}\gamma\left(1-\frac{v}{c}\right)\Longrightarrow\boldsymbol{\nu}=\boldsymbol{\nu_0}\left(\mathbf{1}-\frac{\boldsymbol{v}}{\boldsymbol{c}}\right)\quad\begin{bmatrix}\because\gamma=\dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}\approx1\text{ as } v\ll c\\ v\text{ assumed away from observer}\end{bmatrix}$$

$\Longrightarrow v=c\left(1-\dfrac{\nu}{\nu_0}\right)=c\left(1-\dfrac{\lambda_0}{\lambda}\right)=c\left(1-\dfrac{5000}{5001}\right)\approx6\times10^4\text{ m/s}$ [The star is moving away from Earth.]

<u>Fine structure of Hydrogen</u>

Relativistic effect

$$T=\sqrt{p^2c^2+m^2c^4}-mc^2=mc^2\left(1+\frac{p^2}{m^2c^2}\right)^{\frac{1}{2}}-mc^2\approx mc^2\left(1+\frac{p^2}{2m^2c^2}-\frac{1}{8}\left(\frac{p^2}{m^2c^2}\right)^2\right)-mc^2$$

$$=\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}=\frac{p^2}{2m}\left(1-\frac{p^2}{4m^2c^2}\right)\Longrightarrow\boldsymbol{H'}=T-T_0=-\frac{\boldsymbol{p^4}}{\boldsymbol{8m^3c^2}}\quad\left[T_0=T_{\text{classical}}=\frac{p^2}{2m}\right]$$

$$E_{nlm}^{(1)}=\Delta E_{nlm}=\langle n,l,m|H'|n,l,m\rangle=-\frac{1}{8m^3c^2}\langle n,l,m|p^4|n,l,m\rangle=-\frac{1}{2mc^2}\langle n,l,m|(E-V)^2|n,l,m\rangle$$

$$\left[\because \text{Schrödinger equation: } \frac{p^2}{2m}\psi_{nlm} = (E-V)\psi_{nlm} \Rightarrow p^2 = 2m(E-V) \Rightarrow p^4 = 4m^2(E-V)^2\right]$$

$$\Rightarrow \Delta E_{nlm} = -\frac{1}{2mc^2}\langle n,l,m|(E^2+V^2-2EV)|n,l,m\rangle \quad \left[V = -\frac{Ze^2}{4\pi\varepsilon_0 r} \text{ is the Coulomb potential energy}\right]$$

$$= -\frac{1}{2mc^2}\{E_n{}^2 + \langle n,l,m|V^2|n,l,m\rangle - 2E_n\langle n,l,m|V|n,l,m\rangle\} = -\frac{1}{2mc^2}\left\{E_n{}^2 + \left(\frac{Ze^2}{4\pi\varepsilon_0}\right)^2\left\langle\frac{1}{r^2}\right\rangle + 2E_n\left(\frac{Ze^2}{4\pi\varepsilon_0}\right)\left\langle\frac{1}{r}\right\rangle\right\}$$

$$= -\frac{1}{2mc^2}\left\{E_n{}^2 + \left(\frac{Ze^2}{4\pi\varepsilon_0}\right)^2\frac{Z^3}{\left(l+\frac{1}{2}\right)n^3 a_0{}^2} + 2E_n\left(\frac{Ze^2}{4\pi\varepsilon_0}\right)\frac{Z^2}{n^2 a_0}\right\} \quad \left[\because \left\langle\frac{1}{r}\right\rangle = \frac{Z^2}{n^2 a_0} \quad \text{and} \quad \left\langle\frac{1}{r^2}\right\rangle = \frac{Z^3}{\left(l+\frac{1}{2}\right)n^3 a_0{}^2}\right]$$

$$= -\frac{1}{2mc^2}\left\{E_n{}^2 + \frac{4E_0{}^2 Z^5}{\left(l+\frac{1}{2}\right)n^3} - \frac{4Z^3 E_n E_0}{n^2}\right\} \quad \left[\begin{array}{l}\because E_n = \frac{E_0 Z^2}{n^2}\ ; \\ E_0 = -\frac{e^2}{8\pi\varepsilon_0 a_0}\end{array}\right] = -\frac{1}{2mc^2}\left\{E_n{}^2 + \frac{4E_n{}^2 Zn}{\left(l+\frac{1}{2}\right)} - 4ZE_n{}^2\right\}$$

$$= -\frac{E_n{}^2}{2mc^2}\left\{(1-4Z) + \frac{4Zn}{\left(l+\frac{1}{2}\right)}\right\} \Rightarrow \Delta \boldsymbol{E_{nlm}} = \frac{\boldsymbol{2E_n{}^2}}{\boldsymbol{mc^2}}\left\{\left(\boldsymbol{Z}-\frac{\mathbf{1}}{\mathbf{4}}\right) - \frac{\boldsymbol{Zn}}{\left(\boldsymbol{l}+\frac{\mathbf{1}}{\mathbf{2}}\right)}\right\}$$

Spin-Orbit coupling

$$\vec{E} = \frac{Ze\vec{r}}{4\pi\varepsilon_o r^3}$$

$$\vec{B} = -\frac{\vec{v}\times\vec{E}}{c^2}$$

$$\mu_S = -\frac{e}{m}\vec{S}$$

$$H' = -\vec{\mu}\cdot\vec{B} = -\left(-\frac{e}{m}\vec{S}\right)\cdot\left(-\frac{\vec{v}\times\vec{E}}{c^2}\right) = -\frac{e}{mc^2}\{\vec{S}\cdot(\vec{v}\times\vec{E})\} = -\frac{e^2}{4\pi\varepsilon_o mc^2 r^3}\{\vec{S}\cdot(\vec{v}\times\vec{r})\} = \frac{e^2}{4\pi\varepsilon_o mc^2 r^3}\{\vec{S}\cdot\vec{L}\}$$

$$[\because \vec{L} = \vec{r}\times\vec{v}] = \frac{Ze^2(\vec{L}\cdot\vec{S})}{4\pi\varepsilon_o mc^2}\left(\frac{1}{r^3}\right) \quad \therefore \text{However, } \boldsymbol{H'} = \frac{\boldsymbol{e^2(\vec{L}\cdot\vec{S})}}{\boldsymbol{8\pi\varepsilon_o mc^2}}\left(\frac{\mathbf{1}}{\boldsymbol{r^3}}\right) \quad [\text{due to } \textbf{Thomas precession}]$$

$$E_{nlm}^{(1)} = \Delta E_{nlm} = \langle n,l,m|H'|n,l,m\rangle = \frac{Ze^2(\vec{L}\cdot\vec{S})}{8\pi\varepsilon_o mc^2}\left\langle n,l,m\left|\frac{1}{r^3}\right|n,l,m\right\rangle = \frac{Ze^2(\vec{L}\cdot\vec{S})}{8\pi\varepsilon_o mc^2}\left\langle\frac{1}{r^3}\right\rangle$$

$$= \frac{Ze^2(\vec{L}\cdot\vec{S})}{8\pi\varepsilon_o mc^2}\left\{\frac{Z^4}{l\left(l+\frac{1}{2}\right)(l+1)n^3 a_0{}^3}\right\} \quad \left[\because \left\langle\frac{1}{r^3}\right\rangle = \frac{Z^4}{l\left(l+\frac{1}{2}\right)(l+1)n^3 a_0{}^3}\right] = \frac{Z^5 e^2(\vec{L}\cdot\vec{S})}{8\pi\varepsilon_o mc^2 a_0{}^3\left(l\left(l+\frac{1}{2}\right)(l+1)n^3\right)}$$

$$\Rightarrow \Delta\boldsymbol{E_{nlm}} = \boldsymbol{a}(\boldsymbol{\vec{L}}\cdot\boldsymbol{\vec{S}}) \quad \left[\text{where } a = \frac{Z^5 e^2}{8\pi\varepsilon_o mc^2 a_0{}^3\left(l\left(l+\frac{1}{2}\right)(l+1)n^3\right)}\right] \Rightarrow \Delta\boldsymbol{E_{nlm}} = \frac{\boldsymbol{a}}{\mathbf{2}}(\boldsymbol{J^2}-\boldsymbol{L^2}-\boldsymbol{S^2})$$

$$\left[\begin{array}{l}\because \vec{J} = \vec{L}+\vec{S} \Rightarrow \vec{J}\cdot\vec{J} = (\vec{L}+\vec{S})\cdot(\vec{L}+\vec{S}) \Rightarrow J^2 = L^2+S^2+2\vec{L}\cdot\vec{S} \Rightarrow \vec{L}\cdot\vec{S} = \frac{1}{2}(J^2-L^2-S^2) \\ = \frac{\hbar}{2}\big(j(j+1)-l(l+1)-s(s+1)\big) = \frac{\hbar}{2}\left(\left(l\pm\frac{1}{2}\right)\left(\left(l\pm\frac{1}{2}\right)+1\right)-l(l+1)-\frac{3}{4}\right) \quad \left[\begin{array}{l}\because s = \frac{1}{2} \text{ and} \\ j = l+s = l\pm\frac{1}{2}\end{array}\right]\end{array}\right]$$

Interaction energy in LS coupling (lighter/low-Z atom with 2 valence electrons)*

Spin-orbit interaction energy, $H = a\vec{L}\cdot\vec{S}$

$\because$ there are 4 angular momenta, viz. s_1, s_2, l_1, l_2 $\quad\therefore$ there are six $[{}^4C_2 = 6]$ possible interactions:

$$\begin{cases} \begin{cases} H_1 = a_1(\vec{l}_1 \cdot \vec{l}_2) = a_1 l_1 l_2 \cos(\vec{l}_1 \cdot \vec{l}_2) \\ H_2 = a_2(\vec{s}_1 \cdot \vec{s}_2) = a_2 s_1 s_2 \cos(\vec{s}_1 \cdot \vec{s}_2) \end{cases} & \text{[dominant]} \\ \begin{cases} H_3 = a_3(\vec{l}_1 \cdot \vec{s}_1) = a_3 l_1 s_1 \cos(\vec{l}_1 \cdot \vec{s}_1) \\ H_4 = a_4(\vec{l}_2 \cdot \vec{s}_2) = a_4 l_2 s_2 \cos(\vec{l}_2 \cdot \vec{s}_2) \end{cases} & \text{[reduced]} \\ \begin{cases} H_5 = a_5(\vec{l}_1 \cdot \vec{s}_2) = a_5 l_1 s_2 \cos(\vec{l}_1 \cdot \vec{s}_2) \\ H_6 = a_6(\vec{l}_2 \cdot \vec{s}_1) = a_6 l_2 s_1 \cos(\vec{l}_2 \cdot \vec{s}_1) \end{cases} & \text{[negligible]} \end{cases}$$

$$\begin{cases} \vec{L} = \vec{l}_1 + \vec{l}_2 \Rightarrow L^2 = {l_1}^2 + {l_2}^2 + 2l_1 l_2 \cos(\vec{l}_1 \cdot \vec{l}_2) \Rightarrow l_1 l_2 \cos(\vec{l}_1 \cdot \vec{l}_2) = L^2 - {l_1}^2 - {l_2}^2 \Rightarrow \boldsymbol{H_1 = \frac{a_1}{2}(L^2 - {l_1}^2 - {l_2}^2)} \\ \vec{S} = \vec{s}_1 + \vec{s}_2 \Rightarrow S^2 = {s_1}^2 + {s_2}^2 + 2s_1 s_2 \cos(\vec{s}_1 \cdot \vec{s}_2) \Rightarrow s_1 s_2 \cos(\vec{s}_1 \cdot \vec{s}_2) = S^2 - {s_1}^2 - {s_2}^2 \Rightarrow \boldsymbol{H_2 = \frac{a_2}{2}(S^2 - {s_1}^2 - {s_2}^2)} \end{cases}$$

$$\vec{S} = \vec{s}_1 + \vec{s}_2 \Rightarrow \begin{cases} \vec{s}_2 = \vec{S} - \vec{s}_1 \Rightarrow {s_2}^2 = S^2 + {s_1}^2 - 2s_1 S \cos(\vec{s}_1 \cdot \vec{S}) \Rightarrow \cos(\vec{s}_1 \cdot \vec{S}) = \dfrac{S^2 + {s_1}^2 - {s_2}^2}{2s_1 S} \\ \vec{s}_1 = \vec{S} - \vec{s}_2 \Rightarrow {s_1}^2 = S^2 + {s_2}^2 - 2s_1 S \cos(\vec{s}_2 \cdot \vec{S}) \Rightarrow \cos(\vec{s}_2 \cdot \vec{S}) = \dfrac{S^2 + {s_2}^2 - {s_1}^2}{2s_2 S} \end{cases}$$

$$\vec{L} = \vec{l}_1 + \vec{l}_2 \Rightarrow \begin{cases} \vec{l}_2 = \vec{L} - \vec{l}_1 \Rightarrow {l_2}^2 = L^2 + {l_1}^2 - 2l_1 L \cos(\vec{l}_1 \cdot \vec{L}) \Rightarrow \cos(\vec{l}_1 \cdot \vec{L}) = \dfrac{L^2 + {l_1}^2 - {l_2}^2}{2l_1 L} \\ \vec{l}_1 = \vec{L} - \vec{l}_2 \Rightarrow {l_1}^2 = L^2 + {l_2}^2 - 2l_1 L \cos(\vec{l}_2 \cdot \vec{L}) \Rightarrow \cos(\vec{l}_2 \cdot \vec{L}) = \dfrac{L^2 + {l_2}^2 - {l_1}^2}{2l_2 L} \end{cases}$$

Time averaged, $\cos(\vec{l}_1 \cdot \vec{l}_2) = \cos(l_1 \cdot \vec{L}) \cos(\vec{L} \cdot \vec{L}) \cos(\vec{L} \cdot \vec{l}_2) = \cos(l_1 \cdot \vec{L}) \cos(\vec{L} \cdot \vec{l}_2)$

Time averaged, $\cos(\vec{s}_1 \cdot \vec{s}_2) = \cos(s_1 \cdot \vec{S}) \cos(\vec{S} \cdot \vec{S}) \cos(\vec{S} \cdot \vec{s}_2) = \cos(s_1 \cdot \vec{S}) \cos(\vec{S} \cdot \vec{s}_2)$

$$\therefore \begin{cases} \boldsymbol{H_1} = a_1 l_1 l_2 \cos(l_1 \cdot \vec{L}) \cos(\vec{L} \cdot \vec{l}_2) = a_1 l_1 l_2 \left(\dfrac{L^2 + {l_1}^2 - {l_2}^2}{2l_1 L}\right)\left(\dfrac{L^2 + {l_2}^2 - {l_1}^2}{2l_2 L}\right) = \boldsymbol{\dfrac{a_1}{4L^2}\left(L^4 - ({l_1}^2 - {l_2}^2)^2\right)} \\ \boldsymbol{H_2} = a_2 s_1 s_2 \cos(s_1 \cdot \vec{S}) \cos(\vec{S} \cdot \vec{s}_2) = a_2 s_1 s_2 \left(\dfrac{S^2 + {s_1}^2 - {s_2}^2}{2s_1 S}\right)\left(\dfrac{S^2 + {s_2}^2 - {s_1}^2}{2s_2 S}\right) = \boldsymbol{\dfrac{a_2}{4S^2}(S^4 - ({s_1}^2 - {s_2}^2)^2)} \end{cases}$$

Time averaged, $\cos(\vec{l}_1 \cdot \vec{s}_1) = \cos(\vec{l}_1 \cdot \vec{L}) \cos(\vec{L} \cdot \vec{S}) \cos(\vec{S} \cdot \vec{s}_1)$

Time averaged, $\cos(\vec{l}_2 \cdot \vec{s}_2) = \cos(\vec{l}_2 \cdot \vec{L}) \cos(\vec{L} \cdot \vec{S}) \cos(\vec{S} \cdot \vec{s}_2)$

$$\therefore \begin{cases} H_3 = a_3 l_1 s_1 \cos(\vec{l}_1 \cdot \vec{L}) \cos(\vec{L} \cdot \vec{S}) \cos(\vec{S} \cdot \vec{s}_1) \\ H_4 = a_4 l_2 s_2 \cos(\vec{l}_2 \cdot \vec{L}) \cos(\vec{L} \cdot \vec{S}) \cos(\vec{S} \cdot \vec{s}_2) \end{cases}$$

$$\therefore \begin{cases} \boldsymbol{H_3} = a_3 l_1 s_1 \left(\dfrac{L^2 + {l_1}^2 - {l_2}^2}{2l_1 L}\right)\left(\dfrac{S^2 + {s_1}^2 - {s_2}^2}{2s_1 S}\right) \cos(\vec{L} \cdot \vec{S}) = \boldsymbol{\dfrac{a_3}{4LS}(L^2 + {l_1}^2 - {l_2}^2)(S^2 + {s_1}^2 - {s_2}^2) \cos(\vec{L} \cdot \vec{S})} \\ \boldsymbol{H_4} = a_4 l_2 s_2 \left(\dfrac{L^2 + {l_2}^2 - {l_1}^2}{2l_2 L}\right)\left(\dfrac{S^2 + {s_2}^2 - {s_1}^2}{2s_2 S}\right) \cos(\vec{L} \cdot \vec{S}) = \boldsymbol{\dfrac{a_4}{4LS}(L^2 + {l_2}^2 - {l_1}^2)(S^2 + {s_2}^2 - {s_1}^2) \cos(\vec{L} \cdot \vec{S})} \end{cases}$$

Time averaged, $\cos(\vec{l}_1 \cdot \vec{s}_2) = \cos(\vec{l}_1 \cdot \vec{L}) \cos(\vec{L} \cdot \vec{S}) \cos(\vec{S} \cdot \vec{s}_2)$

Time averaged, $\cos(\vec{l}_2 \cdot \vec{s}_1) = \cos(\vec{l}_2 \cdot \vec{L}) \cos(\vec{L} \cdot \vec{S}) \cos(\vec{S} \cdot \vec{s}_1)$

$$\therefore \begin{cases} H_5 = a_5 l_1 s_2 \cos(\vec{l}_1 \cdot \vec{L}) \cos(\vec{L} \cdot \vec{S}) \cos(\vec{S} \cdot \vec{s}_2) \\ H_6 = a_6 l_2 s_1 \cos(\vec{l}_2 \cdot \vec{L}) \cos(\vec{L} \cdot \vec{S}) \cos(\vec{S} \cdot \vec{s}_1) \end{cases}$$

$$\therefore \begin{cases} \boldsymbol{H_5} = a_5 l_1 s_2 \left(\dfrac{L^2 + {l_1}^2 - {l_2}^2}{2l_1 L}\right)\left(\dfrac{S^2 + {s_2}^2 - {s_1}^2}{2s_2 S}\right) \cos(\vec{L} \cdot \vec{S}) = \boldsymbol{\dfrac{a_5}{4LS}(L^2 + {l_1}^2 - {l_2}^2)(S^2 + {s_2}^2 - {s_1}^2) \cos(\vec{L} \cdot \vec{S})} \\ \boldsymbol{H_6} = a_6 l_2 s_1 \left(\dfrac{L^2 + {l_2}^2 - {l_1}^2}{2l_2 L}\right)\left(\dfrac{S^2 + {s_1}^2 - {s_2}^2}{2s_1 S}\right) \cos(\vec{L} \cdot \vec{S}) = \boldsymbol{\dfrac{a_6}{4LS}(L^2 + {l_2}^2 - {l_1}^2)(S^2 + {s_1}^2 - {s_2}^2) \cos(\vec{L} \cdot \vec{S})} \end{cases}$$

$\therefore \boldsymbol{H = H_1 + H_2 + H_3 + H_4 + H_5 + H_6}$

Interaction energy in JJ coupling (heavier/high-Z atom with 2 valence electrons)***

Spin-orbit interaction energy, $H = a\vec{L} \cdot \vec{S}$

$\because$ there are 4 angular momenta, viz. s_1, s_2, l_1, l_2 $\therefore$ there are six [${}^4C_2 = 6$] possible interactions:

$$\begin{cases} \begin{cases} H_1 = a_1\left(\vec{l}_1 \cdot \vec{l}_2\right) = a_1 l_1 l_2 \cos\left(\vec{l}_1 \cdot \vec{l}_2\right) \\ H_2 = a_2(\vec{s}_1 \cdot \vec{s}_2) = a_2 s_1 s_2 \cos(\vec{s}_1 \cdot \vec{s}_2) \end{cases} & \text{[reduced]} \\ \begin{cases} H_3 = a_3\left(\vec{l}_1 \cdot \vec{s}_1\right) = a_3 l_1 s_1 \cos\left(\vec{l}_1 \cdot \vec{s}_1\right) \\ H_4 = a_4\left(\vec{l}_2 \cdot \vec{s}_2\right) = a_4 l_2 s_2 \cos\left(\vec{l}_2 \cdot \vec{s}_2\right) \end{cases} & \text{[dominant]} \\ \begin{cases} H_5 = a_5\left(\vec{l}_1 \cdot \vec{s}_2\right) = a_5 l_1 s_2 \cos\left(\vec{l}_1 \cdot \vec{s}_2\right) \\ H_6 = a_6\left(\vec{l}_2 \cdot \vec{s}_1\right) = a_6 l_2 s_1 \cos\left(\vec{l}_2 \cdot \vec{s}_1\right) \end{cases} & \text{[negligible]} \end{cases}$$

$$\begin{cases} \vec{j}_1 = \vec{l}_1 + \vec{s}_1 \Rightarrow {j_1}^2 = {l_1}^2 + {s_1}^2 + 2 l_1 s_1 \cos\left(\vec{l}_1 \cdot \vec{s}_1\right) \Rightarrow l_1 l_2 \cos\left(\vec{l}_1 \cdot \vec{s}_1\right) = {j_1}^2 - {l_1}^2 - {s_1}^2 \Rightarrow \boldsymbol{H_3 = \frac{a_3}{2}\left({j_1}^2 - {l_1}^2 - {s_1}^2\right)} \\ \vec{j}_2 = \vec{l}_2 + \vec{s}_2 \Rightarrow {j_2}^2 = {l_2}^2 + {s_2}^2 + 2 l_2 s_2 \cos\left(\vec{l}_2 \cdot \vec{s}_2\right) \Rightarrow l_2 s_2 \cos\left(\vec{l}_2 \cdot \vec{s}_2\right) = {j_2}^2 - {l_2}^2 - {s_2}^2 \Rightarrow \boldsymbol{H_4 = \frac{a_4}{2}\left({j_2}^2 - {l_2}^2 - {s_2}^2\right)} \end{cases}$$

$$\vec{j}_1 = \vec{l}_1 + \vec{s}_1 \Rightarrow \begin{cases} \vec{s}_1 = \vec{j}_1 - \vec{l}_1 \Rightarrow {s_1}^2 = {j_1}^2 + {l_1}^2 - 2 j_1 l_1 \cos\left(\vec{l}_1 \cdot \vec{j}_1\right) \Rightarrow \cos\left(\vec{l}_1 \cdot \vec{j}_1\right) = \dfrac{{j_1}^2 + {l_1}^2 - {s_1}^2}{2 j_1 l_1} \\ \vec{l}_1 = \vec{j}_1 - \vec{s}_1 \Rightarrow {l_1}^2 = {j_1}^2 + {s_1}^2 - 2 j_1 s_1 \cos(\vec{s}_1 \cdot \vec{j}_1) \Rightarrow \cos(\vec{s}_1 \cdot \vec{j}_1) = \dfrac{{j_1}^2 + {s_1}^2 - {l_1}^2}{2 j_1 s_1} \end{cases}$$

$$\vec{j}_2 = \vec{l}_2 + \vec{s}_2 \Rightarrow \begin{cases} \vec{s}_2 = \vec{j}_2 - \vec{l}_2 \Rightarrow {s_2}^2 = {j_2}^2 + {l_2}^2 - 2 j_2 l_2 \cos\left(\vec{l}_2 \cdot \vec{j}_2\right) \Rightarrow \cos\left(\vec{l}_2 \cdot \vec{j}_2\right) = \dfrac{{j_2}^2 + {l_2}^2 - {s_2}^2}{2 j_2 l_2} \\ \vec{l}_2 = \vec{j}_2 - \vec{s}_2 \Rightarrow {l_2}^2 = {j_2}^2 + {s_2}^2 - 2 j_2 s_2 \cos(\vec{s}_2 \cdot \vec{j}_2) \Rightarrow \cos(\vec{s}_2 \cdot \vec{j}_2) = \dfrac{{j_2}^2 + {s_2}^2 - {l_2}^2}{2 j_2 s_2} \end{cases}$$

Time averaged, $\cos\left(\vec{l}_1 \cdot \vec{s}_1\right) = \cos\left(l_1 \cdot \vec{j}_1\right)\cos\left(\vec{j}_1 \cdot \vec{j}_1\right)\cos\left(\vec{s}_1 \cdot \vec{j}_1\right) = \cos\left(l_1 \cdot \vec{j}_1\right)\cos\left(\vec{s}_1 \cdot \vec{j}_1\right)$

Time averaged, $\cos\left(\vec{l}_2 \cdot \vec{s}_2\right) = \cos\left(l_2 \cdot \vec{j}_2\right)\cos\left(\vec{j}_2 \cdot \vec{j}_2\right)\cos\left(\vec{s}_2 \cdot \vec{j}_2\right) = \cos\left(l_2 \cdot \vec{j}_2\right)\cos\left(\vec{s}_2 \cdot \vec{j}_2\right)$

$$\therefore \begin{cases} \boldsymbol{H_3} = a_3 l_1 s_1 \cos\left(l_1 \cdot \vec{j}_1\right)\cos\left(\vec{s}_1 \cdot \vec{j}_1\right) = a_1 l_1 l_2 \left(\dfrac{{j_1}^2 + {l_1}^2 - {s_1}^2}{2 j_1 l_1}\right)\left(\dfrac{{j_1}^2 + {s_1}^2 - {l_1}^2}{2 j_1 s_1}\right) = \boldsymbol{\dfrac{a_3}{4{j_1}^2}\left({j_1}^4 - \left({l_1}^2 - {s_1}^2\right)^2\right)} \\ \boldsymbol{H_4} = a_4 l_2 s_2 \cos\left(l_2 \cdot \vec{j}_2\right)\cos\left(\vec{s}_2 \cdot \vec{j}_2\right) = a_2 s_1 s_2 \left(\dfrac{{j_2}^2 + {l_2}^2 - {s_2}^2}{2 j_2 l_2}\right)\left(\dfrac{{j_2}^2 + {s_2}^2 - {l_2}^2}{2 j_2 s_2}\right) = \boldsymbol{\dfrac{a_4}{4{j_2}^2}\left({j_2}^4 - \left({l_2}^2 - {s_2}^2\right)^2\right)} \end{cases}$$

Time averaged, $\cos\left(\vec{l}_1 \cdot \vec{l}_2\right) = \cos\left(\vec{l}_1 \cdot \vec{j}_1\right)\cos\left(\vec{j}_1 \cdot \vec{j}_2\right)\cos\left(\vec{j}_2 \cdot \vec{l}_2\right)$

Time averaged, $\cos(\vec{s}_1 \cdot \vec{s}_2) = \cos\left(\vec{s}_1 \cdot \vec{j}_1\right)\cos\left(\vec{j}_1 \cdot \vec{j}_2\right)\cos\left(\vec{j}_2 \cdot \vec{s}_2\right)$

$$\therefore \begin{cases} H_1 = a_1 l_1 l_2 \cos\left(\vec{l}_1 \cdot \vec{j}_1\right)\cos\left(\vec{j}_1 \cdot \vec{j}_2\right)\cos\left(\vec{j}_2 \cdot \vec{l}_2\right) \\ H_2 = a_2 s_1 s_2 \cos\left(\vec{s}_1 \cdot \vec{j}_1\right)\cos\left(\vec{j}_1 \cdot \vec{j}_2\right)\cos\left(\vec{j}_2 \cdot \vec{s}_2\right) \end{cases}$$

$$\therefore \begin{cases} \boldsymbol{H_1} = a_1 l_1 l_2 \left(\dfrac{{j_1}^2 + {l_1}^2 - {s_1}^2}{2 j_1 l_1}\right)\left(\dfrac{{j_2}^2 + {l_2}^2 - {s_2}^2}{2 j_2 l_2}\right)\cos\left(\vec{j}_1 \cdot \vec{j}_2\right) = \boldsymbol{\dfrac{a_3}{4 j_1 j_2}\left({j_1}^2 + {l_1}^2 - {s_1}^2\right)\left({j_2}^2 + {l_2}^2 - {s_2}^2\right)\cos\left(\vec{j}_1 \cdot \vec{j}_2\right)} \\ \boldsymbol{H_2} = a_4 l_2 s_2 \left(\dfrac{{j_1}^2 + {s_1}^2 - {l_1}^2}{2 j_1 s_1}\right)\left(\dfrac{{j_2}^2 + {s_2}^2 - {l_2}^2}{2 j_2 s_2}\right)\cos\left(\vec{j}_1 \cdot \vec{j}_2\right) = \boldsymbol{\dfrac{a_4}{4 j_1 j_2}\left({j_1}^2 + {s_1}^2 - {l_1}^2\right)\left({j_2}^2 + {s_2}^2 - {l_2}^2\right)\cos\left(\vec{j}_1 \cdot \vec{j}_2\right)} \end{cases}$$

Time averaged, $\cos\left(\vec{l}_1 \cdot \vec{s}_2\right) = \cos\left(\vec{l}_1 \cdot \vec{j}_1\right)\cos\left(\vec{j}_1 \cdot \vec{j}_2\right)\cos\left(\vec{j}_2 \cdot \vec{s}_2\right)$

Time averaged, $\cos\left(\vec{l}_2 \cdot \vec{s}_1\right) = \cos\left(\vec{l}_2 \cdot \vec{j}_2\right)\cos\left(\vec{j}_2 \cdot \vec{j}_1\right)\cos\left(\vec{j}_1 \cdot \vec{s}_1\right)$

$$\therefore \begin{cases} H_5 = a_5 l_1 s_2 \cos\left(\vec{l}_1 \cdot \vec{j}_1\right)\cos\left(\vec{j}_1 \cdot \vec{j}_2\right)\cos\left(\vec{j}_2 \cdot \vec{s}_2\right) \\ H_6 = a_6 l_2 s_1 \cos\left(\vec{l}_2 \cdot \vec{j}_2\right)\cos\left(\vec{j}_2 \cdot \vec{j}_1\right)\cos\left(\vec{j}_1 \cdot \vec{s}_1\right) \end{cases}$$

$$\therefore \begin{cases} \boldsymbol{H_5} = a_5 l_1 s_2 \left(\dfrac{{j_1}^2 + {l_1}^2 - {s_1}^2}{2 j_1 l_1}\right)\left(\dfrac{{j_2}^2 + {s_2}^2 - {l_2}^2}{2 j_2 s_2}\right)\cos\left(\vec{j}_1 \cdot \vec{j}_2\right) = \boldsymbol{\dfrac{a_5}{4 j_1 j_2}\left({j_1}^2 + {l_1}^2 - {s_1}^2\right)\left({j_2}^2 + {s_2}^2 - {l_2}^2\right)\cos\left(\vec{j}_1 \cdot \vec{j}_2\right)} \\ \boldsymbol{H_6} = a_6 l_2 s_1 \left(\dfrac{{j_2}^2 + {l_2}^2 - {s_2}^2}{2 j_2 l_2}\right)\left(\dfrac{{j_1}^2 + {s_1}^2 - {l_1}^2}{2 j_1 s_1}\right)\cos\left(\vec{j}_1 \cdot \vec{j}_2\right) = \boldsymbol{\dfrac{a_6}{4 j_1 j_2}\left({j_2}^2 + {l_2}^2 - {s_2}^2\right)\left({j_1}^2 + {s_1}^2 - {l_1}^2\right)\cos\left(\vec{j}_1 \cdot \vec{j}_2\right)} \end{cases}$$

$$\therefore \boldsymbol{H = H_1 + H_2 + H_3 + H_4 + H_5 + H_6}$$

The spin-orbit interaction in an atom is given by $H = a\vec{L} \cdot \vec{S}$ where $\vec{L}$ and $\vec{S}$ denote the orbital and spin angular momenta, respectively, of the electron. What is the splitting between $^2P_{3/2}$ and $^2P_{1/2}$?

$$H = a\vec{L}\cdot\vec{S} = \frac{a}{2}(J^2 - L^2 - S^2) \quad \begin{bmatrix} \because \vec{J} = \vec{L} + \vec{S} \Rightarrow \vec{J}\cdot\vec{J} = (\vec{L}+\vec{S})\cdot(\vec{L}+\vec{S}) \\ \Rightarrow J^2 = L^2 + S^2 + 2\vec{L}\cdot\vec{S} \Rightarrow \vec{L}\cdot\vec{S} = \frac{1}{2}(J^2 - L^2 - S^2) \end{bmatrix}$$

$\because L = 1$ and $S = \dfrac{1}{2}$ for both $^2P_{3/2}$ and $^2P_{1/2}$ $\quad \therefore \Delta H = H(^2P_{3/2}) - H(^2P_{1/2}) = \dfrac{a}{2}\{J^2(^2P_{3/2}) - J^2(^2P_{1/2})\}$

$$\because J^2 = J(J+1)\hbar^2 = \begin{cases} \frac{15}{4}\hbar^2 & \text{for } ^2P_{3/2} \\ \frac{3}{4}\hbar^2 & \text{for } ^2P_{1/2} \end{cases} \quad \therefore \Delta H = \frac{3a}{2}\hbar^2$$

If the spin-orbit interaction is given by $H = a\vec{L}\cdot\vec{S}$, calculate the value of *a* for the states with L = 1 and *S* = 1/2. (Given: wavelengths of sodium D lines are 5890 Å and 5896 Å)

$$H = a\vec{L}\cdot\vec{S} = \frac{a}{2}(J^2 - L^2 - S^2) = \frac{a}{2}\{J(J+1) - L(L+1) - S(S+1)\}\hbar^2$$

$\because L = 1$ and $S = \dfrac{1}{2}$ for both states $\quad \therefore \Delta H = a\left\{(\vec{L}\cdot\vec{S})_{J=3/2} - (\vec{L}\cdot\vec{S})_{J=1/2}\right\} = \dfrac{a}{2}\left\{\dfrac{3}{2}\left(\dfrac{3}{2}+1\right) - \dfrac{1}{2}\left(\dfrac{1}{2}+1\right)\right\} = \dfrac{3a}{2}\hbar^2$

$$\left[\because J = L \pm S = 1 \pm \frac{1}{2} = \frac{3}{2} \text{ or } \frac{1}{2}\right]$$

$$E = \frac{hc}{\lambda} \Rightarrow \Delta E = \frac{hc}{\lambda^2}\Delta\lambda = \frac{hc}{(5893\ \text{Å})^2}(6\ \text{Å}) \approx 1727.74hc$$

$$\therefore \Delta H = \Delta E \Rightarrow \frac{3a}{2}\hbar^2 = 1727.74hc \Rightarrow a = \frac{2}{3} \times 4\pi^2 \times \frac{1727.74c}{h} = 3.3 \times 10^{27}\ (\text{eV-s}^2)^{-1}$$

Landé interval rule

$$H = a\vec{L}\cdot\vec{S} = \frac{a}{2}(J^2 - L^2 - S^2) = \frac{a}{2}\{J(J+1) - L(L+1) - S(S+1)\}$$

$$\therefore \text{Landé interval, } (\Delta \boldsymbol{H})_{\boldsymbol{J+1\to J}} = H(J+1, L, S) - H(J, L, S) = \frac{a}{2}\{(J+1)(J+2) - J(J+1)\} = \boldsymbol{a(J+1)}$$

$$\text{or, } (\Delta \boldsymbol{H})_{\boldsymbol{J\to J-1}} = \boldsymbol{aJ} \quad [\text{Landé interval rule}]$$

Estimate the strength of magnetic field produced by the electron's orbital motion which results in the two sodium D lines (5890 Å and 5896 Å).

$$E = \frac{hc}{\lambda} \Rightarrow \Delta E = \frac{hc}{\lambda^2}\Delta\lambda = \frac{hc}{(5893\ \text{Å})^2}(6\ \text{Å}) \approx 1727.74hc$$

$$E_{\text{mag}} = -\vec{\mu}\cdot\vec{B} = -\mu_z B_z = -(g_s\mu_B m_s)B_z \ [\because \boldsymbol{\mu_z = g_s\mu_B m_s}] \Rightarrow \Delta E_{\text{mag}} = (g_s\mu_B B_z)\Delta m_s = 2\mu_B B_z \quad \begin{bmatrix} \because m_s = \pm\frac{1}{2} \\ \text{and, } g_s \approx 2 \end{bmatrix}$$

$$\therefore \Delta E_{\text{mag}} = \Delta E \Rightarrow \frac{e\hbar}{m}B_z = 1727.74hc \quad \left[\because \mu_B = \frac{e\hbar}{2m}\right] \Rightarrow B_z = \frac{2\pi m}{e} \times 1727.74c \approx 18.5\ \text{T}$$

The spin-orbit effect splits the 3P → 3S transition into two lines, 5890 Å corresponding to $^2P_{3/2} \to {}^2S_{1/2}$ and 5896 Å corresponding to $^2P_{1/2} \to {}^2S_{1/2}$. Calculate by using these wavelengths, the effective magnetic induction experienced by an outer electron in the Sodium atom as a result of its orbital motion.

$$E = \frac{hc}{\lambda} \Rightarrow dE = -\frac{hc}{\lambda^2}d\lambda \Rightarrow |d\lambda| = \frac{\lambda^2}{hc}|dE| = \frac{eB}{2\pi m_e c}\lambda^2 \quad \begin{bmatrix} \because |dE| = \frac{e\hbar}{m_e}B \\ \lambda = 5893\ \text{Å} \end{bmatrix} \Rightarrow B = \frac{2\pi m_e c}{e}\frac{|d\lambda|}{\lambda^2} \approx 18.5\ \text{T}$$

Consider a 'd' electron in a one electron system. Calculate the values of possible angles between L and S.

$$\cos\theta = \frac{j(j+1) - l(l+1) - s(s+1)}{2\sqrt{l(l+1)}\sqrt{s(s+1)}} = \begin{cases} \cos\theta = \dfrac{\sqrt{2}}{3} \Longrightarrow \theta \approx 62^\circ & \text{for } {}^2D_{5/2} \text{ state, i.e. } l = 2 \text{ and } j = \dfrac{5}{2} \\ \cos\theta = -\dfrac{1}{\sqrt{2}} \Longrightarrow \theta = 135^\circ & \text{for } {}^2D_{3/2} \text{ state, i.e. } l = 2 \text{ and } j = \dfrac{3}{2} \end{cases}$$

Equivalent electrons

Electrons having the same values of *n* and *l* (quantum numbers) in an atom are called equivalent electrons.

Spectral term

Energy levels corresponding to definite values of L and S are called (spectral) terms denoted as ${}^{2S+1}L$ where $(2S+1)$ is the multiplicity of the term, and the total orbital angular momentum quantum number $L \equiv S, P, D, F, G, H$ for $L = 0, 1, 2, 3, 4, 5$ respectively. Allowed values of L are $|l_1 - l_2| \le L \le (l_1 + l_2)$.
Similarly, allowed values of S are $|s_1 - s_2| \le S \le (s_1 + s_2)$. $\because s_2 = s_2 = 1/2, \therefore S = 0, 1$.
Allowed values of J are $|L - S| \le J \le (L + S)$. Thus, the number of possible values of J is $2S + 1$ (multiplicity) if $L \ge S$ and $2L + 1$ if $L \le S$. $\therefore \sum_{J=|L-S|}^{L+S} (2J+1) = (2L+1)(2S+1)$ which is the degeneracy attached to ${}^{2S+1}L$.
Allowed values of M_L are $-\boldsymbol{L} \le \boldsymbol{M_L} \le \boldsymbol{L}$. Similarly, allowed values of M_S are $-\boldsymbol{S} \le \boldsymbol{M_S} \le \boldsymbol{S}$.

For a filled subshell, i.e. a configuration containing the maximum number $2(2l+1)$ electrons, there is only one possible term, $\boldsymbol{{}^1S}$ which is a consequence of $M_L = \sum_i m_{l_i} = 0$ and $M_S = \sum_i m_{s_i} = 0 \;\therefore L = 0$ and $S = 0$
For an incomplete subshell (containing the 'optically active' electrons), three cases need to be considered:

Electrons belonging to different subshells (non-equivalent electrons)

Configuration: $\boldsymbol{n}\mathbf{p}\,\boldsymbol{n'}\mathbf{p}$ (two non-equivalent p electrons)
Possible terms: $\boldsymbol{{}^1S}, \boldsymbol{{}^1P}, \boldsymbol{{}^1D}, \boldsymbol{{}^3S}, \boldsymbol{{}^3P}, \boldsymbol{{}^3D}$ $[\because l_1 = l_2 = 1$ and $s_1 = s_2 = 1/2 \Longrightarrow L = 0, 1, 2$ and $S = 0, 1]$

Configuration: $\boldsymbol{n}\mathbf{p}\,\boldsymbol{n'}\mathbf{d}$ (two non-equivalent p and d electrons)
Possible terms: $\boldsymbol{{}^1P}, \boldsymbol{{}^1D}, \boldsymbol{{}^1F}, \boldsymbol{{}^3P}, \boldsymbol{{}^3D}, \boldsymbol{{}^3F}$ $[\because l_1 = 1, l_2 = 2$ and $s_1 = s_2 = 1/2 \Longrightarrow L = 1, 2, 3$ and $S = 0, 1]$

Electrons belonging to same subshell (equivalent electrons)

Configuration: $\boldsymbol{n}\mathbf{s}^{\mathbf{1}}$ (one s electron)
Number of possibilities: ${}^{1\times 2}C_1 = {}^2C_1 = 2$
Possible term: $\boldsymbol{{}^2S}$ [for $L = 0$, corresponding to $(M_L = 0, M_S = \pm 1/2)$, 2 possibilities]

Configuration: $\boldsymbol{n}\mathbf{s}^{\mathbf{2}}$ (two equivalent s electrons) or $\boldsymbol{n}\mathbf{p}^{\mathbf{6}}$ (six equivalent p electrons)
Number of possibilities: ${}^{1\times 2}C_2 = {}^{3\times 2}C_6 = {}^2C_2 = {}^6C_6 = 1$
Possible term: $\boldsymbol{{}^1S}$ [for $L = 0$, corresponding to $(M_L = 0, M_S = 0)$, 1 possibility]

Configuration: $\boldsymbol{n}\mathbf{p}^{\mathbf{1}}$ (one p electron) or $\boldsymbol{n}\mathbf{p}^{\mathbf{5}}$ (five equivalent p electrons)
Number of possibilities: ${}^{3\times 2}C_1 = {}^{3\times 2}C_5 = {}^6C_1 = {}^6C_5 = 6$
Possible term: $\boldsymbol{{}^2P}$ [for $L = 1$, corresponding to $(M_L = 0, \pm 1, M_S = \pm 1/2)$, 6 possibilities]

Configuration: $\boldsymbol{n}\mathbf{p}^{\mathbf{2}}$ (two equivalent p electrons) or $\boldsymbol{n}\mathbf{p}^{\mathbf{4}}$ (four equivalent p electrons)
Number of possibilities: ${}^{3\times 2}C_2 = {}^{3\times 2}C_4 = {}^6C_2 = {}^6C_4 = 15$

$$\text{Possible terms: } \begin{cases} \boldsymbol{{}^1S} & [\text{for } L = 0, \text{ corresponding to } (M_L = 0, M_S = 0), 1 \text{ possibility}] \\ \boldsymbol{{}^3P} & [\text{for } L = 1, \text{ corresponding to } (M_L = 0, \pm 1, M_S = 0, \pm 1), 9 \text{ possibilities}] \\ \boldsymbol{{}^1D} & [\text{for } L = 2, \text{ corresponding to } (M_L = 0, \pm 1, \pm 2, M_S = 0), 5 \text{ possibilities}] \end{cases}$$

Configuration: $\boldsymbol{n}\mathbf{p}^{\mathbf{3}}$ (three equivalent p electrons)
Number of possibilities: ${}^{3\times2}C_3 = {}^{6}C_3 = 20$

$$\text{Possible terms: }\begin{cases} {}^{\mathbf{4}}\boldsymbol{S} & [\text{for } L = 0, \text{corresponding to } (M_L = 0, M_S = 0, \pm 1/2, \pm 3/2), 6 \text{ possibilities}] \\ {}^{\mathbf{2}}\boldsymbol{P} & [\text{for } L = 1, \text{corresponding to } (M_L = 0, \pm 1, M_S = 0, \pm 1/2), 9 \text{ possibilities}] \\ {}^{\mathbf{2}}\boldsymbol{D} & [\text{for } L = 2, \text{corresponding to } (M_L = 0, \pm 1, \pm 2, M_S = 0), 5 \text{ possibilities}] \end{cases}$$

Equivalent and non-equivalent electrons

Write down the normal electronic configuration of lithium, beryllium, boron, carbon, nitrogen, oxygen, fluorine neon and obtain the spectral terms arising from it.

Electronic configuration of lithium ($Z = 3$): $1s^2 2s^1$
Spectral terms: ${}^2S_{1/2}$

Electronic configuration of beryllium ($Z = 4$): $1s^2 2s^2$
Electronic configuration of neon ($Z = 10$): $1s^2 2s^2 2p^6$
Spectral terms: ${}^1S_0$

Electronic configuration of boron ($Z = 5$): $1s^2 2s^2 2p^1$
Electronic configuration of fluorine ($Z = 9$): $1s^2 2s^2 2p^5$
Spectral terms: ${}^2P_{3/2}, {}^2P_{1/2}$

Electronic configuration of carbon ($Z = 6$): $1s^2 2s^2 2p^2$
Electronic configuration of oxygen ($Z = 8$): $1s^2 2s^2 2p^4$
Spectral terms: ${}^1S_0, {}^3P_0, {}^3P_1, {}^3P_2, {}^1D_2$

Electronic configuration of nitrogen ($Z = 7$): $1s^2 2s^2 2p^3$
Spectral terms: ${}^4S_{3/2}, {}^2P_{3/2}, {}^2P_{1/2}, {}^2D_{5/2}, {}^2D_{3/2}$

Doppler half intensity width of spectral line

$$\Delta\nu_{\text{FWHM}} = 1.665\frac{\nu_0}{c}\sqrt{\frac{2kT}{m}} \Longrightarrow \Delta\lambda_{\text{FWHM}} = 1.665\frac{\nu_0}{c}\left(\frac{c}{{\nu_0}^2}\right)\sqrt{\frac{2kT}{m}}\begin{bmatrix} \because \lambda = \dfrac{c}{\nu} \Longrightarrow \\ \Delta\lambda = -\dfrac{c}{\nu^2}\Delta\nu \end{bmatrix} = \frac{1.665}{\nu_0}\sqrt{\frac{2kT}{m}} = 1.665\frac{\lambda_0}{c}\sqrt{\frac{2kT}{m}}$$

Calculate the Doppler half intensity width of sodium D lines of 5893 Å corresponding to a temperature of 500 K.

$$\Delta\lambda_{\text{FWHM}} = 1.665\frac{\lambda_0}{c}\sqrt{\frac{2kT}{m}} \approx 0.02\ \text{Å} \quad \begin{bmatrix} m = 23u \\ \text{for sodium} \end{bmatrix}$$

Calculate the Doppler half intensity breath of mercury green D lines of 5461 Å if light source is at 1000 K.

$$\Delta\lambda_{\text{FWHM}} = 1.665\frac{\lambda_0}{c}\sqrt{\frac{2kT}{m}} \approx 0.00873\ \text{Å} \quad \begin{bmatrix} m = 200.6u \\ \text{for mercury} \end{bmatrix}$$

Find the approximate magnitude of Doppler broadening of an argon glow tube whose temperature is 200 K. Assume a wavelength of 0.5 μm for the radiation.

$$\Delta\lambda_{\text{FWHM}} = 1.665\frac{\lambda_0}{c}\sqrt{\frac{2kT}{m}} \approx 0.008\ \text{Å} \quad \begin{bmatrix} m = 40u \\ \text{for argon} \end{bmatrix}$$

The average lifetime of an excited atomic state is 10^{-8} s. Find the width of the line if the wavelength of the spectral line associated with the decay of this state is 6000 Å.

$$\Delta E\cdot\Delta t=\frac{\hbar}{2}\Longrightarrow\Delta\nu\cdot\Delta t=\frac{1}{4\pi}\Longrightarrow\Delta\nu=\frac{1}{4\pi\Delta t}\Longrightarrow\Delta\lambda=\frac{\lambda^2}{c}\frac{1}{4\pi\Delta t}\quad\left[\because\lambda=\frac{c}{\nu}\Longrightarrow\Delta\lambda=\frac{c}{\nu^2}\Delta\nu=\frac{\lambda^2}{c}\Delta\nu\right]\approx 0.0001\text{ Å}$$

Boltzmann equation

$$\begin{cases}N_1=N_0g_1e^{-E_1/kT}\\N_2=N_0g_2e^{-E_2/kT}\end{cases}\Longrightarrow\frac{N_2}{N_1}=\frac{g_2e^{-E_2/kT}}{g_1e^{-E_1/kT}}\Longrightarrow\boldsymbol{\frac{N_2}{N_1}=\frac{g_2}{g_1}e^{-(E_2-E_1)/kT}}$$

For a gas of neutral Hydrogen atoms, at what temperature will equal number of atoms have electrons in the ground state (n = 1) and in the first excited state (n = 2)? [Degeneracy of the nth energy level is $g_n = 2n^2$. **]**

$$\frac{N_2}{N_1}=\frac{g_2}{g_1}e^{-(E_2-E_1)/kT}\Longrightarrow 1=\frac{2(2)^2}{2(1)^2}e^{-(-13.6\text{ eV})(1/2^2-1/1^2)/kT}\quad\left[\text{for Hydrogen atom, }E_n=-\frac{13.6\text{ eV}}{n^2}\right]$$

$$\Longrightarrow\ln\frac{1}{4}=-\frac{3}{4}\left(\frac{13.6\text{ eV}}{kT}\right)\Longrightarrow\frac{8}{3}\ln 2=\frac{13.6\text{ eV}}{kT}\Longrightarrow T=\frac{5.1\text{ eV}}{k\ln 2}\approx 85383\text{ K}\approx 8.54\times10^4\text{ K}$$

Saha equation (Saha-Langmuir equation) and Milne-Fowler improvement

$$\boldsymbol{\frac{N_{i+1}}{N_i}=\left(\frac{g_s}{\lambda^3 n_e}\right)\frac{g_{i+1}}{g_i}e^{-(E_{i+1}-E_i)/kT}}\quad\left[\begin{array}{c}\text{spin degeneracy factor of electron, }g_s=2\\ \text{thermal de Broglie wavelength of electron, }\lambda=\sqrt{\dfrac{h^2}{2\pi m_e kT}}\\ n_e\text{ is the electron density (concentration)}\\ (i+1)\text{-th state ionization energy, }\chi_i=E_{i+1}-E_i\end{array}\right]$$

$$\Longrightarrow\frac{N_{i+1}}{N_i}=\frac{2}{n_e}\left(\frac{2\pi m_e kT}{h^2}\right)^{\frac{3}{2}}\frac{g_{i+1}}{g_i}e^{-\chi_i/kT}=\frac{2kT}{P_e}\left(\frac{2\pi m_e kT}{h^2}\right)^{\frac{3}{2}}\frac{g_{i+1}}{g_i}e^{-\chi_i/kT}\quad[\because P_eV=N_ekT\Longrightarrow P_e=n_ekT]$$

If only one level of ionization is important (such as Hydrogen), $n_{II}=n_e\ and\ n_{II}=n-n_e$

$$\therefore\frac{N_{II}}{N_I}=\frac{n_{II}}{n_I}=\frac{n_e}{n-n_e}=\frac{2}{n_e}\left(\frac{2\pi m_e kT}{h^2}\right)^{\frac{3}{2}}\frac{g_{II}}{g_I}e^{-\chi_i/kT}\Longrightarrow\frac{{n_e}^2}{n-n_e}=2\left(\frac{2\pi m_e kT}{h^2}\right)^{\frac{3}{2}}\frac{g_{II}}{g_I}e^{-\chi_i/kT}$$

Saha equation only holds for weakly ionized plasmas for which the Debye length $\left[\lambda_D\approx\sqrt{\dfrac{kT_e}{n_e e^2/\varepsilon_0}}\right]$ is large.

Further, it can only be applied to a gas in thermodynamic equilibrium, so that the Maxwell-Boltzmann law of velocity distribution is obeyed. Moreover, the density should not be too great (less than roughly 1 kg/m^3), otherwise the presence of neighboring ions would distort an atom's orbitals and lower its ionization energy.

What is the degree of ionization in a stellar atmosphere composed only of pure Hydrogen? Evaluate for T = 5000 K, 8300 K, 9600 K and 11300 K. Assume constant P_e = 20 N/m^2.

$$\frac{N_{II}}{N_I}=\frac{2kT}{P_e}\left(\frac{2\pi m_e kT}{h^2}\right)^{\frac{3}{2}}\frac{g_{II}}{g_I}e^{-\chi_i/kT}\quad\left[\begin{array}{c}\chi_i=13.6\text{ eV}\\ g_{II}=1\,;\ g_I=2(1)^2=2\end{array}\right]=\frac{kT}{P_e}\left(\frac{2\pi m_e kT}{h^2}\right)^{\frac{3}{2}}e^{-\chi_i/kT}$$

$$\Longrightarrow\frac{N_{II}}{N_I}\approx\begin{cases}5.8\times10^{-8}\text{ at 5000 K}\\5.8\times10^{-2}\text{ at 8300 K}\\1.1\text{ at 9600 K}\\19.5\text{ at 11300 K}\end{cases}\quad\therefore\frac{N_{II}}{N}=\frac{N_{II}}{N_I+N_{II}}=\frac{N_{II}/N_I}{1+N_{II}/N_I}\approx\begin{cases}5.8\times10^{-8}\approx0\%\text{ at 5000 K}\\5.5\times10^{-2}\approx5.5\%\text{ at 8300 K}\\0.524\approx52.4\%\text{ at 9600 K}\\0.95=95\%\text{ at 11300 K}\end{cases}$$

Thus, the ionization of Hydrogen occurs within a temperature interval of roughly 3000 K.

Strength of Balmer lines

$$\frac{N_2}{N}=\frac{N_2}{N_\mathrm{I}}\cdot\frac{N_\mathrm{I}}{N}\approx\frac{N_2}{N_1+N_2}\cdot\frac{N_\mathrm{I}}{N_\mathrm{I}+N_\mathrm{II}}\quad\begin{bmatrix}\because N_\mathrm{I}\approx N_1+N_2\text{ i.e. almost all}\\ \text{neutral H atoms are either in}\\ \text{ground state or first excited state}\end{bmatrix}=\left(\frac{N_2/N_1}{1+N_2/N_1}\right)\left(\frac{1}{1+N_\mathrm{II}/N_\mathrm{I}}\right)$$

$$\Rightarrow\frac{N_2}{N}=\frac{\left(\frac{4e^{-10.2\text{ eV}/kT}}{1+4e^{-10.2\text{ eV}/kT}}\right)}{1+\frac{kT}{P_\mathrm{e}}\left(\frac{2\pi m_\mathrm{e}kT}{h^2}\right)^{\frac{3}{2}}e^{-13.6\text{ eV}/kT}}\approx\frac{\left(\frac{4e^{-118365.96/T}}{1+4e^{-118365.96/T}}\right)}{1+0.001667\,T^{\frac{5}{2}}e^{-157821.28/T}}$$

From the graph, $\left(\frac{N_2}{N}\right)_{\max}\approx 8.73777\times10^{-6}$ at $T=9873$ K i.e. in this set-up, Hydrogen gas produces the most intense Balmer lines at 9873 K. The strength of the Balmer lines diminishes at higher temperatures due to the rapid ionization of Hydrogen above 9873 K.

<u>Temperatures in Astronomy</u>

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Kinetic temperature (Maxwell-Boltzmann distribution law)

Color temperature (UBV photometry) obtained by fitting the shape of a star's continuous spectrum to the Planck function.

Effective temperature / Radiation temperature (Stefan-Boltzmann law)

Bolometric temperature

Brightness temperature

Excitation temperature (Boltzmann equation)

Ionization temperature (Saha Equation)

Antenna temperature

Spin temperature

<u>Equations of Stellar Structure</u>

Equation of Mass Continuity (Mass Conservation)

$$dM=4\pi r^2\rho(r)\,dr\Rightarrow\boldsymbol{\frac{dM}{dr}=4\pi r^2\rho(r)}\Rightarrow\boldsymbol{\frac{dr}{dM}=\frac{1}{4\pi r(M)^2\rho(M)}}$$

$$\boldsymbol{M(r)}=\int_0^r dM=\int_0^r 4\pi r^2\rho(r)\,dr=\boldsymbol{4\pi\int_0^r r^2\rho(r)\,dr}$$

Equation of Hydrostatic Equilibrium (HE) / Archimedes-Newton equation

Force Approach

Consider a Newtonian self-gravitating, spherically symmetric fluid and a spherical shell therein of mass dm.
Force due to fluid pressure $-$ Gravitational force by the interior mass $=$ Acceleration of the shell

$$\Rightarrow A(r)\{P-(P+dP)\}-\frac{GM(r)\,dM}{r^2}=dM\left(\frac{d^2r}{dt^2}\right)\quad[\because A(r)=4\pi r^2\,;\ P(r)=P\,;\ P(r+dr)=P+dP]$$

$$\Rightarrow A(r)\{-dP\}-\frac{GM(r)\{\rho(r)(A(r)\,dr)\}}{r^2}=\{\rho(r)(A(r)\,dr)\}\left(\frac{d^2r}{dt^2}\right)\Rightarrow\boldsymbol{\frac{dP}{dr}=-\frac{GM(r)\rho(r)}{r^2}-\rho(r)\left(\frac{d^2r}{dt^2}\right)}$$

For hydrostatic equilibrium, $\frac{d^2r}{dt^2}=0\quad\therefore\boldsymbol{\frac{dP}{dr}=-\frac{GM(r)\rho(r)}{r^2}}\quad\left[\frac{dP}{dr}\text{ is the pressure gradient.}\right]$

This is a state of static or dynamic equilibrium i.e. the shell may still grow or shrink but with a constant rate.

Alternatively: Buoyancy $=$ Weight of displaced fluid $\begin{bmatrix}\text{Archimedes'}\\ \text{principle}\end{bmatrix}\Rightarrow-(dP)A(r)=(dM)g(r)=(\rho\,dV)g(r)$

$$\Rightarrow-(dP)A(r)=(\rho A(r)\,dr)\frac{GM(r)}{r^2}\Rightarrow\boldsymbol{\frac{dP}{dr}=-\frac{GM(r)\rho(r)}{r^2}}$$

Work Approach

Work done by fluid pressure + Work done by Gravity = Change in Kinetic energy of the shell

$$\Rightarrow -A(r)(dP)\,dr - \frac{GM(r)\,dM}{r^2}\,dr = \frac{1}{2}(dM)\frac{d}{dt}\left(\frac{dr}{dt}\right)^2 dt = (dM)\left(\frac{dr}{dt}\right)\frac{d^2r}{dt^2}dt = (dM)\frac{d^2r}{dt^2}dr$$

$$\Rightarrow \boldsymbol{\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} - \rho(r)\left(\frac{d^2r}{dt^2}\right)} \quad [\because dM = \rho(r)\,dV = \rho(r)(A(r)\,dr)]$$

$$\Rightarrow \frac{dP}{dM} = \frac{1}{4\pi r^2\rho(r)}\left\{-\frac{GM(r)\rho(r)}{r^2} - \rho(r)\left(\frac{d^2r}{dt^2}\right)\right\} \quad \left[\because \frac{dr}{dM} = \frac{1}{4\pi r^2\rho(r)} \text{ by Continuity equation}\right]$$

$$\Rightarrow \boldsymbol{\frac{dP}{dM} = -\frac{GM}{4\pi r(M)^4} - \frac{1}{4\pi r(M)^2}\left(\frac{d^2r}{dt^2}\right)}$$

For hydrostatic equilibrium, $\frac{d^2r}{dt^2} = 0 \quad \therefore \boldsymbol{\frac{dP}{dM} = -\frac{GM(r)}{4\pi r(M)^4}}$ $\left[\frac{dP}{dM} \text{ is the pressure gradient in the mass coordinate.}\right]$

Pressure Approach

Pressure from the fluid layer on top is $P + dP$ and that from the bottom is P.

Pressure due to gravity $= -\frac{GM(r)\,dM}{r^2 dA} = -\frac{GM(r)\{\rho(r)\,dr\}}{r^2}$ $[\because dM = \rho(r)\,dV = \rho(r)(dA\,dr)]$

$\therefore$ for hydrostatic equilibrium, $dP = -\frac{GM(r)\rho(r)\,dr}{r^2} \Rightarrow \boldsymbol{\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}}$

$\Rightarrow \boldsymbol{\vec{\nabla}P = \rho\vec{g}}$ $\left[\because \vec{g} = -\frac{GM(r)\rho(r)}{r^2}\right] \Rightarrow \boldsymbol{\vec{\nabla}P = -\rho\vec{\nabla}\Phi}$ $[\because \vec{g} = \vec{\nabla}\Phi]$ [which is the expression arrived while deriving from **Navier-Stokes equation**]

wherein Pressure, P and Density, ρ are scalar functions of r whereas Gravity, $\vec{g}$ is a vector function of r.

Take divergence on both sides to get $\nabla^2 P = \rho(\vec{\nabla}\cdot\vec{g}) + \vec{g}\cdot\vec{\nabla}\rho = -4\pi G\rho^2 + \vec{g}\cdot\vec{\nabla}\rho$ $\left[\because \text{ by Gauss law, } \vec{\nabla}\cdot\vec{g} = 4\pi G\rho\right]$

If the shell is accelerating outwards, a pseudo force (acting inwards) has to be introduced alongside gravity.

Pressure due to pseudo force $= -\frac{1}{dA}\left\{dM\left(\frac{d^2r}{dt^2}\right)\right\} = -\{\rho(r)\,dr\}\left(\frac{d^2r}{dt^2}\right)$ $[\because dM = \rho(r)\,dV = \rho(r)(dA\,dr)]$

$$\therefore dP = -\frac{GM(r)\rho(r)\,dr}{r^2} - \rho(r)\,dr\left(\frac{d^2r}{dt^2}\right) \Rightarrow \boldsymbol{\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} - \rho(r)\left(\frac{d^2r}{dt^2}\right)}$$

Gravitational self-energy (binding energy), $\boldsymbol{U_G = -\frac{3}{5}\frac{GM^2}{R}} = -\frac{3}{5}\frac{GM^2}{\left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}} = -\frac{3}{5}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} GM^2 V^{-\frac{1}{3}}$ $\left[\because V = \frac{4\pi}{3}R^3\right]$

Gravitational Pressure, $P_G = -\frac{\partial U_G}{\partial V} = \frac{3}{5}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} GM^2\left(-\frac{1}{3}V^{-\frac{4}{3}}\right) = \frac{1}{3}\frac{U_G}{V} \Rightarrow \boldsymbol{P_G V = \frac{1}{3}U_G \Rightarrow U_G = 3P_G V}$

[Thus, the gravitational **pressure, $\boldsymbol{P_G}$,** of "gravitational radiation" is **one-third** of its **energy-density, $\boldsymbol{U_G/V}$** .] This applies for photons as well. It holds true for any ultra-relativistic particle. For non-relativistic particle such as electrons, monoatomic ideal gas, the pressure is two-third of the energy density.

Calculate the density of Hydrogen, Deuterium, Tritium, Helium-3 and Helium-4 atoms.

$$\rho_{\mathrm{H}} = \frac{1.00794\ \mathrm{u}}{\frac{4\pi}{3}(53\ \mathrm{pm})^3} \approx 2684\ \mathrm{kg/m^3}$$

$$\rho_{\mathrm{D}} = \frac{2.0141\ \mathrm{u}}{\frac{4\pi}{3}(53\ \mathrm{pm})^3} \approx 5363\ \mathrm{kg/m^3}$$

$$\rho_{\mathrm{T}} = \frac{3.01605\ \mathrm{u}}{\frac{4\pi}{3}(53\ \mathrm{pm})^3} \approx 8031\ \mathrm{kg/m^3}$$

$$\rho_{\text{He-3}} = \frac{3.01603\ \text{u}}{\frac{4\pi}{3}(31\ \text{pm})^3} \approx 40134\ \text{kg/m}^3$$
$$\rho_{\text{He-4}} = \frac{4.0026\ \text{u}}{\frac{4\pi}{3}(31\ \text{pm})^3} \approx 53262\ \text{kg/m}^3$$

Determine the gravitational potential energy of a sphere of uniform density with mass *M* and radius *R*.

$$U_G = -\int_0^R \frac{GM(r)}{r}(4\pi r^2 \rho\, dr) = -4\pi G \int_0^R \left(\frac{Mr^3}{R^3}\right) r \left(\frac{M}{\frac{4\pi}{3}R^3}\right) dr = -\frac{3GM^2}{R^6}\int_0^R r^4\, dr = -\frac{3GM^2}{R^6}\left(\frac{R^5}{5}\right) = -\frac{3}{5}\frac{GM^2}{R}$$

Assuming it to be a sphere, calculate the average density of Sun.

$$\begin{cases} M_\odot = 1.989 \times 10^{30}\ \text{kg} \\ R_\odot = 6.957 \times 10^{8}\ \text{m} \end{cases} \Rightarrow \bar{\rho}_\odot = \frac{M_\odot}{\frac{4}{3}\pi {R_\odot}^3} \approx 1410.2\ \text{kg/m}^3 \approx \begin{cases} \sqrt{2} \times \text{density of water} \\ \frac{4}{3} \times \text{density of human body} \end{cases}$$

Assuming surface pressure to be zero, obtain a crude estimate of the pressure at the center of the Sun.

$$\frac{dP}{dr} \approx \frac{P_s - P_c}{R_s - R_c} \approx \frac{0 - P_c}{R_\odot - 0} = -\frac{P_c}{R_\odot}$$

$$\therefore \frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \quad \begin{bmatrix}\text{Hydrostatic} \\ \text{equilibrium}\end{bmatrix} \Rightarrow \frac{P_c}{R_\odot} = \frac{G(M_\odot/2)\bar{\rho}_\odot}{(R_\odot/2)^2} \Rightarrow \boldsymbol{P_c} = \frac{2GM_\odot\bar{\rho}_\odot}{R_\odot} \approx \mathbf{5.38 \times 10^{14}\ N/m^2}$$

Using Virial theorem, estimate the temperature inside Sun.

$$2K + U = 0 \Rightarrow K = -\frac{1}{2}U \Rightarrow \frac{3}{2}Nk_\text{B}T = \frac{1}{2}\left(\frac{3}{5}\frac{G{M_\odot}^2}{R_\odot}\right) \Rightarrow T = \frac{1}{5}\frac{G{M_\odot}^2}{Nk_\text{B}R_\odot} = \frac{1}{10}\frac{Gm_\text{H}M_\odot}{k_\text{B}R_\odot} \Rightarrow \boldsymbol{T}_{\textbf{virial}} \approx \mathbf{2.3 \times 10^6\ K}$$

$$\begin{bmatrix} N = \frac{2M_\odot}{\bar{m}} \approx \frac{2M_\odot}{m_\text{H}} \text{ as interior is composed mainly of H}^+ \text{ and e}^- \\ \text{Alternatively, } N = \frac{M_\odot}{\bar{m}} \approx \frac{2M_\odot}{m_\text{H}} \text{ as, } \bar{m} = \frac{m_p + m_e}{2} \approx \frac{m_\text{H}}{2} \end{bmatrix}$$

Estimate the temperature at the center of the Sun assuming that the central density is twice the mean density.

Assuming ideal gas with $\rho_c \approx 2\bar{\rho}$, $T_c = \frac{P_cV}{Nk_\text{B}} = \frac{m_\text{H}P_cV}{2k_\text{B}M} = \frac{m_\text{H}P_c}{2k_\text{B}\rho_c} = \frac{m_\text{H}P_c}{4k_\text{B}\bar{\rho}} \Rightarrow \boldsymbol{T_c} \approx \mathbf{11.56 \times 10^6\ K}$

Estimate the temperature at the center of the Sun assuming a Helium-rich core.

Assuming ideal gas with $V = \frac{4\pi}{3}{a_0}^3$ and $N = 1$ $\begin{bmatrix} a_\text{He} \approx 30\ \text{pm} = 3 \times 10^{-11}\ \text{m} \\ \text{i.e. the center is composed} \\ \text{of one Helium atom.} \end{bmatrix}$ $\boldsymbol{T_c} = \frac{P_cV}{k_\text{B}} \approx \mathbf{4.4 \times 10^6\ K}$

Luminosity gradient equation (Energy conservation)

Contribution to the total luminosity of star due to the shell of mass dM, $\boldsymbol{dL = \epsilon\, dM}$ where $\epsilon = \epsilon_\text{nuclear} + \epsilon_\text{gravity}$ is the total energy released per unit mass per unit time. $\because dM = 4\pi r^2 \rho\, dr$ $\therefore \boldsymbol{\frac{dL}{dM} = \epsilon(M)} \Rightarrow \boldsymbol{\frac{dL}{dr} = 4\pi r^2 \rho(r)\epsilon(r)}$ where $L(r)$ is the **interior luminosity** due to all the energy generated in the stellar interior out to the radius r.

$$\left[\epsilon_\text{gravity} = \frac{1}{dM\, dt}\left(-\frac{GM(r)\, dM}{r^2}\right) = -\frac{GM(r)}{r^2}\frac{dr}{dt}\right]$$

Relations between various stellar quantities (useful for main-sequence stars)

$$\frac{P}{R} \approx \frac{GM\rho}{R^2} \begin{bmatrix}\text{Hydrostatic} \\ \text{equilibrium} \\ \dfrac{dP}{dr} \approx -\dfrac{P}{R}\end{bmatrix} \Rightarrow P \propto \frac{M\rho}{R} \Rightarrow \boldsymbol{P \propto \frac{M^2}{R^4}} \quad \left[\because \rho \approx \frac{M}{R^3}\right]$$

$$P = K\rho T \begin{bmatrix}\text{Ideal gas} \\ \text{EoS}\end{bmatrix} \Rightarrow P \propto \rho T \Rightarrow P \propto \frac{M}{R^3} T \Rightarrow K'' = \left(\frac{R}{M}\right) T \begin{bmatrix}\because \dfrac{M}{R^3} = \dfrac{M^2}{R^4}\left(\dfrac{R}{M}\right) \propto P\left(\dfrac{R}{M}\right) \\ \Rightarrow \dfrac{M}{R^3} = K'P\left(\dfrac{R}{M}\right)\end{bmatrix} \Rightarrow \boldsymbol{T \propto \frac{M}{R}} \Rightarrow T \propto C$$

where $C = \dfrac{M}{R}$ [estimate of the mass gradient] is the referred as the compactness of the star.

$$\frac{T}{R} \approx \frac{3\kappa\rho}{16\sigma T^3} \cdot \frac{L}{4\pi R^2} \begin{bmatrix}\text{Radiative} \\ \text{equilibrium} \\ \dfrac{dT}{dr} \approx -\dfrac{T}{R}\end{bmatrix} \Rightarrow \frac{T}{R} \propto \frac{M}{R^3 T^3}\frac{L}{R^2} \Rightarrow L \propto \frac{R^4 T^4}{M} \Rightarrow \boldsymbol{L \propto M^3} \quad \left[\because T \propto \frac{M}{R}\right] \begin{bmatrix}\textbf{Eddington's} \\ \textbf{Mass-Luminosity} \\ \textbf{relation}\end{bmatrix}$$

$$L = 4\pi\sigma R^2 {T_{\text{eff}}}^4 \text{ [Stefan-Boltzmann law]} \Rightarrow L \propto R^2 {T_{\text{eff}}}^4 \Rightarrow L \propto R^2 T^4 \begin{bmatrix}\text{assuming } T_{\text{eff}} \propto T \\ \text{(linear temperature gradient)}\end{bmatrix}$$

$$\Rightarrow L \propto M^2 T^2 \left[\because T \propto \frac{M}{R}\right] \Rightarrow L^3 \propto M^6 T^6 \Rightarrow L^3 \propto L^2 T^6 \; [\because L \propto M^3] \Rightarrow L \propto T^6 \Rightarrow \boldsymbol{L \propto {T_{\text{eff}}}^6} \text{ [assuming } T \propto T_{\text{eff}}]$$

$$L = \frac{dE}{dt} \approx \frac{E}{\tau} \propto \frac{M}{\tau} \quad [E \approx E_{\text{nuclear}} = fMc^2 \propto M] \Rightarrow \text{(main-sequence) lifetime, } \tau \propto \frac{M}{L} \Rightarrow \boldsymbol{\tau \propto M^{-2}} \quad [\because L \propto M^3]$$

<u>**Ideal Gas laws**</u>

Avogadro's law
Equal volumes of all gases, at the same temperature and pressure, have the same number of molecules.
$\boldsymbol{V \propto N}$
Boyle's law
The absolute pressure exerted by a given mass of an ideal gas is inversely proportional to the volume it occupies, if the temperature and the amount of gas remain unchanged.
$\boldsymbol{P \propto \frac{1}{V}}$
Charles' law
At constant pressure, the temperature of a dry (ideal) gas changes proportionately with the volume.
$\boldsymbol{V \propto T}$
Gay-Lussac's law
The pressure of a gas of fixed mass and fixed volume, is directly proportional to the gas's absolute temperature.
$\boldsymbol{P \propto T}$

Gas Pressure

For ideal gas, $\boldsymbol{PV = Nk_{\text{B}}T} \Rightarrow P = \dfrac{N}{V} k_{\text{B}} T = \dfrac{\rho}{\bar{m}} k_{\text{B}} T \begin{bmatrix}\because \text{Concentration, } n = \dfrac{N}{V} = \dfrac{N}{m/\rho} = \dfrac{\rho}{m/N} = \dfrac{\rho}{\bar{m}} \\ \text{where } \bar{m} = \text{average mass of a gas particle}\end{bmatrix} \Rightarrow \boldsymbol{P = nk_{\text{B}}T}$

"Mean molecular mass", $\mu = \dfrac{\bar{m}}{m_{\text{H}}}$ is the average molecular mass expressed in mass of hydrogen atom, m_{H}

$\therefore$ Thermal Pressure (or, Gas Pressure), $\boldsymbol{P = \dfrac{\rho k_{\text{B}} T}{\mu m_{\text{H}}}} \Rightarrow \dfrac{P}{\rho} = \dfrac{k_{\text{B}} T}{\mu m_{\text{H}}}$

$\therefore$ Ideal gas equation of state is $\boldsymbol{P \propto \rho T}$

$\because$ isothermal sound speed, $c_{sT} = \sqrt{\dfrac{P}{\rho}} = \sqrt{\dfrac{k_{\text{B}} T}{\mu m_{\text{H}}}} \quad \therefore \boldsymbol{P = \rho {c_{sT}}^2}$

i.e. in isothermal condition, $\boldsymbol{P \propto \rho}$

Mean molecular weight and Mass-fraction

For a neutral gas, $\bar{m}_n = \dfrac{\sum_j N_j m_j}{\sum_j N_j} \Longrightarrow \mu_n = \dfrac{\sum_j N_j A_j}{\sum_j N_j}$ $\begin{bmatrix} N_j = \text{total number of atoms of type } j \\ m_j = \text{mass of atom of type } j \\ A_j = \dfrac{m_j}{m_\mathrm{H}} \approx \text{atomic mass of atom of type } j \end{bmatrix}$

For an ionized gas, $\bar{m}_i \approx \dfrac{\sum_j N_j m_j}{\sum_j N_j (1 + z_j)}$ $\begin{bmatrix} \text{ignoring mass of electron} \\ z_j = \text{atomic number of ions of type } j \\ = \text{number of free electrons obtained} \\ \text{on complete ionization of atom of type } j \end{bmatrix} \Longrightarrow \mu_i \approx \dfrac{\sum_j N_j A_j}{\sum_j N_j (1 + z_j)}$

$$\therefore \frac{1}{\bar{m}_n} = \frac{1}{\mu_n m_\mathrm{H}} = \frac{\sum_j N_j}{\sum_j N_j m_j} = \frac{\sum_j \frac{N_j}{m_j} \cdot m_j}{\sum_j N_j m_j} = \sum_j \frac{1}{m_j} \left(\frac{N_j m_j}{\sum_k N_k m_k} \right) = \sum_j \frac{X_j}{m_j} = \sum_j \frac{X_j}{A_j m_\mathrm{H}} \quad \begin{bmatrix} X_j = \dfrac{N_j m_j}{\sum_k N_k m_k} \\ \text{is the } \textbf{mass-fraction} \\ \text{of atom of type } j \end{bmatrix}$$

$$\Longrightarrow \boldsymbol{\frac{1}{\mu_n} = \sum_j \frac{1}{A_j} \cdot X_j \approx X + \frac{1}{4} Y + \left\langle \frac{1}{A} \right\rangle_n Z} \quad \begin{bmatrix} X_\mathrm{H} = X \,;\, Y_\mathrm{He} = Y \,;\, X_{A>4} = Z \text{ and } X + Y + Z = 1 \\ \text{ignoring Deuterium, Tritium, Helium-3 abundances} \\ \left\langle \dfrac{1}{A} \right\rangle_n \text{ is the weighted average of all metals} \\ \text{(elements heavier than Helium-4) in the gas.} \end{bmatrix}$$

Similarly, $\boldsymbol{\dfrac{1}{\mu_i} = \sum_j \dfrac{1 + z_j}{A_j} \cdot X_j} \approx \left(\dfrac{1+1}{1} \right) X + \left(\dfrac{1+2}{4} \right) Y + \left\langle \dfrac{1+z}{A} \right\rangle_i Z = \boldsymbol{2X + \dfrac{3}{4} Y + \left\langle \dfrac{1+z}{A} \right\rangle_i Z}$

For elements much heavier than Helium, $\boldsymbol{1 + z_j \approx z_j}$ and $\boldsymbol{A_j \approx 2 z_j}$ $\begin{bmatrix} \because m_\mathrm{p} \approx m_\mathrm{n} \\ \text{and } N \approx Z \end{bmatrix}$ $\therefore \boldsymbol{\left\langle \dfrac{1+z}{A} \right\rangle_i \approx \dfrac{1}{2} = 0.5}$

For Sun's surface, $X = 0.74$, $Y = 0.25$, $Z = 0.01$ and $\boldsymbol{\left\langle \dfrac{1}{A} \right\rangle_n \approx \dfrac{1}{15.5}}$ so that $\begin{cases} \dfrac{1}{\mu_n} \approx 0.8 \Longrightarrow \boldsymbol{\mu_n \approx 1.25} \\ \dfrac{1}{\mu_i} \approx 1.6725 \Longrightarrow \boldsymbol{\mu_i \approx 0.6} \end{cases}$

For Sun's centre, $X = 0.64$, $Y = 0.34$, $Z = 0.02$ and $\boldsymbol{\left\langle \dfrac{1}{A} \right\rangle_n \approx \dfrac{1}{15.5}}$ so that $\begin{cases} \dfrac{1}{\mu_n} \approx 0.73 \Longrightarrow \boldsymbol{\mu_n \approx 1.38} \\ \dfrac{1}{\mu_i} \approx 1.545 \Longrightarrow \boldsymbol{\mu_i \approx 0.65} \end{cases}$

Speed of sound

Adiabatic speed of sound, $v_s = \sqrt{\dfrac{\gamma P}{\rho}}$ [Laplace's formula] $= \sqrt{\dfrac{\gamma k_\mathrm{B} T}{\mu m_\mathrm{H}}}$

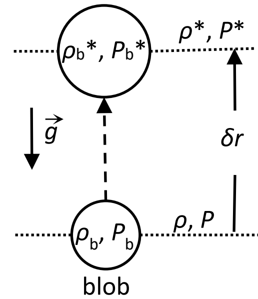
Isothermal speed of sound ($\gamma = 1$), $v_{sT} = \sqrt{\dfrac{P}{\rho}}$ [Newton's formula] $= \sqrt{\dfrac{k_\mathrm{B} T}{\mu m_\mathrm{H}}}$

Convective instability and Schwarzschild criterion

Convection occurs when the Schwarzschild stability criterion is violated which happens when the temperature gradient becomes larger than that which would exist if the star were adiabatic.

Initially, the blob is at mechanical and thermal equilibrium with the ambient medium, so that, $P_b = P$ [mechanical equilibrium] ; $T_b = T$ [thermal equilibrium] $\Longrightarrow \rho_b = \rho$

Post-displacement, the blob needs to establish a new mechanical equilibrium, so that, ${P_b}^* = P^* \begin{bmatrix}\text{mechanical (pressure) equilibrium is} \\ \text{established on speed of sound timescale, } \tau_s\end{bmatrix}$ while ${T_b}^* \neq T^* \Longrightarrow$ ${\rho_b}^* \neq \rho^* \begin{bmatrix}\text{thermal (temperature) equilibrium is established} \\ \text{much slower on conduction timescale, } \tau_c\end{bmatrix}$ If $\tau_s \ll \tau_c$ the blob's displacement may be treated as adiabatic so that the adiabatic EoS applies.



NOTE: fluid inside the blob is only adiabatic, whereas, outside it is ideal adiabatic.

$$\therefore \begin{cases} P_b V_b^{\gamma} = K_1 \Longrightarrow P_b = K\rho_b^{\gamma} \\ {P_b}^* {V_b}^{*\gamma} = K_1 \Longrightarrow {P_b}^* = K{\rho_b}^{*\gamma} \end{cases} \Longrightarrow \frac{P_b}{{P_b}^*} = \left(\frac{\rho_b}{{\rho_b}^*}\right)^{\gamma} \Longrightarrow {\rho_b}^* = \rho_b\left(\frac{{P_b}^*}{P_b}\right)^{\frac{1}{\gamma}} = \rho_b\left(\frac{P^*}{P}\right)^{\frac{1}{\gamma}}$$

$$\because P^* = P + \frac{dP}{dr}\delta r \Longrightarrow \frac{P^*}{P} = \left(1 + \frac{1}{P}\frac{dP}{dr}\delta r\right) \quad \therefore {\rho_b}^* = \rho_b\left(1 + \frac{1}{P}\frac{dP}{dr}\delta r\right)^{\frac{1}{\gamma}} \approx \rho_b\left(1 + \frac{1}{\gamma P}\frac{dP}{dr}\delta r\right) = \rho + \frac{1}{{c_s}^2}\frac{dP}{dr}\delta r$$

$\begin{bmatrix}\because \rho_b = \rho \\ c_s = \sqrt{\dfrac{\gamma P}{\rho}}\end{bmatrix}$ Further, $\because {\rho_b}^* = \rho_b + \dfrac{d\rho_b}{dr}\delta r \quad \therefore \boldsymbol{\dfrac{d\rho_b}{dr} = \dfrac{1}{{c_s}^2}\dfrac{dP}{dr}}$ Now, consider a stratified medium (through which the blob is displaced) in which $\dfrac{d\rho}{dr} < 0$ and $\dfrac{dP}{dr} < 0$. Then for convective stability (blob cannot rise), the displacement must not make the blob more buoyant, i.e. $\boldsymbol{\rho^* < {\rho_b}^*} \Longrightarrow \dfrac{d\rho}{dr} < \dfrac{d\rho_b}{dr} \quad \left[\because \rho^* = \rho + \dfrac{d\rho}{dr}\delta r\right]$

$$\Longrightarrow \boldsymbol{\frac{d\rho}{dr} < \frac{1}{{c_s}^2}\frac{dP}{dr}} \quad \begin{bmatrix}\textbf{Schwarzschild criterion} \\ \textbf{for convective stability}\end{bmatrix} \Longrightarrow \frac{\rho}{P}\frac{dP}{dr} - \frac{\rho}{T}\frac{dT}{dr} < \frac{\rho}{\gamma P}\frac{dP}{dr} \Longrightarrow -\frac{1}{T}\frac{dT}{dr} < -\left(1 - \frac{1}{\gamma}\right)\frac{1}{P}\frac{dP}{dr}$$

$$\left[\because \boldsymbol{PV = nRT} \Longrightarrow \frac{Pm}{\rho} = nRT \Longrightarrow \boldsymbol{\rho = K'\frac{P}{T}} \Longrightarrow \frac{d\rho}{dr} = \frac{K'}{T}\frac{dP}{dr} - \frac{K'P}{T^2}\frac{dT}{dr} = \frac{\rho}{P}\frac{dP}{dr} - \frac{\rho}{T}\frac{dT}{dr} \quad \left[\because \frac{K'}{T} = \frac{\rho}{P}\right]\right]$$

$$\Longrightarrow \begin{cases} \boldsymbol{\left|\frac{dT}{dr}\right| < \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\left|\frac{dP}{dr}\right|} \quad \left[\because \frac{dT}{dr} < 0 \text{ and } \frac{dP}{dr} < 0\right] \\ -\frac{P}{T}\frac{dT/dr}{dP/dr} > -\left(1 - \frac{1}{\gamma}\right) \quad \left[\because \frac{dP}{dr} < 0\right] \Longrightarrow \frac{P}{T}\frac{dT}{dP} < \left(1 - \frac{1}{\gamma}\right) \Longrightarrow \boldsymbol{\left(\frac{P}{T}\frac{dT}{dP}\right)_{\textbf{stellar}} < \left(\frac{P}{T}\frac{dT}{dP}\right)_{\textbf{adiabatic}}} \end{cases}$$

$$\left[\begin{array}{l} \because \begin{cases} \boldsymbol{P = K\rho^{\gamma}} \\ \boldsymbol{\rho = K'\frac{P}{T}} \end{cases} \Longrightarrow P = K\left(K'\frac{P}{T}\right)^{\gamma} \Longrightarrow T^{\gamma} = \left(KK'^{\gamma}\right)P^{\gamma-1} \Longrightarrow \gamma T^{\gamma-1}\frac{dT}{dP} = \left(KK'^{\gamma}\right)(\gamma - 1)P^{\gamma-2} \\ \Longrightarrow \frac{dT}{dP} = \left(\frac{\gamma - 1}{\gamma}\right)\frac{P^{\gamma-2}}{T^{\gamma-1}}\left(\frac{P}{\rho^{\gamma}}\right)\left(\frac{\rho T}{P}\right)^{\gamma} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P} \Longrightarrow \boldsymbol{\frac{P}{T}\frac{dT}{dP} = \left(1 - \frac{1}{\gamma}\right)} \text{ for ideal adiabatic fluid} \end{array}\right]$$

$$\Longrightarrow \boldsymbol{\left(\frac{d(\ln T)}{d(\ln P)}\right)_{\textbf{stellar}} < \left(\frac{d(\ln T)}{d(\ln P)}\right)_{\textbf{adiabatic}}} \quad \left[\because \frac{P}{T}\frac{dT}{dP} = \frac{d(\ln T)}{d(\ln P)}\right]$$

If $\gamma = \dfrac{5}{3}$ $\begin{bmatrix}\textbf{monoatomic} \\ \textbf{ideal gas}\end{bmatrix}$ then $\left(\dfrac{P}{T}\dfrac{dT}{dP}\right)_{\text{adiabatic}} = \left(1 - \dfrac{1}{\gamma}\right) = 0.4$

Adiabatic convection

For ideal adiabatic fluid, $\dfrac{P}{T}\dfrac{dT}{dP} = \left(1 - \dfrac{1}{\gamma}\right) \Longrightarrow \dfrac{dP}{dr} = \dfrac{P}{T}\dfrac{dT}{dr}\left(1 - \dfrac{1}{\gamma}\right)^{-1}$

$$\therefore \frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \quad \begin{bmatrix}\text{Hydrostatic} \\ \text{equilibrium}\end{bmatrix} \Longrightarrow \begin{cases} \frac{P}{T}\frac{dT}{dr}\left(1 - \frac{1}{\gamma}\right)^{-1} = -\frac{GM(r)}{r^2}\left(\frac{P}{T}\frac{\mu m_H}{k_B}\right) \\ \frac{P}{T}\frac{dT}{dr}\left(1 - \frac{1}{\gamma}\right)^{-1} = -\frac{GM(r)\rho(r)}{r^2} \Longrightarrow \frac{dr}{dM}\frac{dT}{dr} = -\frac{GM(r)T(r)}{4\pi r^4 P(r)} \end{cases}$$

$$\begin{bmatrix} \because P = \dfrac{\rho k_B T}{\mu m_H} \Rightarrow \rho = \dfrac{P}{T}\dfrac{\mu m_H}{k_B}\ \text{[ideal gas]} \\ \dfrac{dr}{dM} = \dfrac{1}{4\pi r^2 \rho(r)}\ \text{[continuity equation]} \end{bmatrix} \Rightarrow \begin{cases} \boldsymbol{\dfrac{dT}{dr} = -\left(1-\dfrac{1}{\gamma}\right)\dfrac{\mu m_H}{k_B}\dfrac{GM(r)}{r^2}} \\ \boldsymbol{\dfrac{dT}{dM} = -\dfrac{GMT(M)}{4\pi r(M)^4 P(M)}} \end{cases}$$

Radiation Transport

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\varepsilon_0 m_e c^2}\right)^2 \approx 6.65246 \times 10^{-29}\ \text{kg/m}^2$$

Assuming that the gas is all Hydrogen so that there is only one electron per atom, $n_e \approx \dfrac{\rho}{m_H}$

Equation of Radiative Equilibrium (RE)

Radiation pressure gradient, $\dfrac{dP_{\text{rad}}}{dr} = -\dfrac{\bar{\kappa}\rho}{c}F_{\text{rad}}$ [F_{rad} is the outward radiative flux.]

$$\Rightarrow \frac{4}{3}aT^3\frac{dT}{dr} = -\frac{\bar{\kappa}\rho}{c}F_R \quad \left[\because P_R = \frac{1}{3}aT^4 \Rightarrow \frac{dP_R}{dr} = \frac{4}{3}aT^3\frac{dT}{dr}\right] \Rightarrow \frac{dT}{dr} = -\frac{3\bar{\kappa}(r)\rho(r)}{4acT^3(r)}\frac{L(r)}{4\pi r^2} \quad \left[\because F_{\text{rad}} = \frac{L(r)}{4\pi r^2}\right]$$

$$\Rightarrow \boldsymbol{\frac{dT}{dr} = -\frac{3\bar{\kappa}(r)\rho(r)}{16\sigma T^3(r)}\frac{L(r)}{4\pi r^2}} \quad \left[\because a = \frac{4\sigma}{c}\right] \Rightarrow \boldsymbol{\frac{dT}{dM} = -\frac{3\bar{\kappa}(M)L(M)}{256\pi^2\sigma r(M)^4 T^3(M)}} \quad \begin{bmatrix} \because \dfrac{dr}{dM} = \dfrac{1}{4\pi r^2\rho(r)} \\ \Rightarrow \rho(r) = \dfrac{1}{4\pi r^2}\dfrac{dM}{dr} \\ \text{by Continuity equation} \end{bmatrix}$$

From the equation of radiative equilibrium, estimate the central temperature of Sun.

$$\frac{dT}{dr} \approx \frac{T_s - T_c}{R_s - R_c} \approx \frac{T_{\text{eff}} - T_c}{R_\odot}$$

$$\therefore \frac{dT}{dr} = -\frac{3\bar{\kappa}\rho(r)\, L(r)}{64\pi\sigma r^2 T^3(r)} \quad \begin{bmatrix}\text{Radiative} \\ \text{equilibrium}\end{bmatrix} \Rightarrow \frac{T_{\text{eff}} - T_c}{R_\odot} \approx -\frac{3(\bar{\kappa}/2)\bar{\rho}_\odot(L_\odot/2)}{64\pi\sigma(R_\odot/2)^2(T_c/2)^3} \Rightarrow {T_c}^4 \approx \frac{3\bar{\kappa}\bar{\rho}_\odot L_\odot}{8\pi\sigma R_\odot} \quad [T_c \gg T_{\text{eff}}]$$

$$\Rightarrow \boldsymbol{T_c \approx 2.73 \times 10^6\ \text{K}} \quad \begin{bmatrix} \bar{\rho} = 1410.2\ \text{kg/m}^3\ ;\ \bar{\kappa} = 0.034 \\ R_\odot = 6.957 \times 10^8\ \text{m}\ ;\ L_\odot = 3.828 \times 10^{26}\ \text{W} \end{bmatrix}$$

Eddington Luminosity Limit

At the surface (outer layer)of a star, $\dfrac{dP}{dr} \approx \dfrac{dP_R}{dr} = -\dfrac{\bar{\kappa}\rho(R)}{c}F_R = -\dfrac{\bar{\kappa}\rho(R)}{c}\dfrac{L(R)}{4\pi r^2} \Rightarrow \dfrac{dP}{dr} \approx -\dfrac{\bar{\kappa}\rho(R)L}{4\pi c r^2}$ $[L(R) = L]$

$$\because \frac{dP}{dr} \approx -\frac{GM\rho(R)}{r^2} \quad \begin{bmatrix}\text{Hydrostatic} \\ \text{equilibrium} \\ M(R) = M\end{bmatrix} \Rightarrow -\frac{GM\rho(R)}{r^2} \approx -\frac{\bar{\kappa}\rho(R)L}{4\pi c r^2} \Rightarrow \boldsymbol{L_{\text{Ed}} = \frac{4\pi cG}{\bar{\kappa}}M} \quad \textbf{[Eddington limit]}$$

L_{Ed} is the maximum radiative luminosity a star of mass M can have while remaining in hydrostatic equilibrium.
If the luminosity exceeds L_{Ed}, mass loss occurs driven by the radiation pressure.
For hot massive stars, scattering is mainly due to **electron-scattering**, $\therefore \bar{\kappa} \approx \bar{\kappa}_{es} = 0.02(1+X)\ \text{m}^2\ \text{kg}^{-1}$

For a typical main-sequence star (like Sun), $X = 0.7$ so that $\bar{\kappa} = 0.034$ $\therefore L_{\text{Ed}} = \dfrac{4\pi cG}{\bar{\kappa}}M \approx (7.4\ \text{W/kg})M$

$$\Rightarrow L_{\text{Ed}} = (7.4 \times 1.989 \times 10^{30}\ \text{W})\frac{M}{M_\odot} = (1.47 \times 10^{31}\ \text{W})\frac{M}{M_\odot} \Rightarrow \boldsymbol{\frac{L_{\text{Ed}}}{L_\odot} = 3.845 \times 10^4 \frac{M}{M_\odot}}$$

Stellar equations and Boundary conditions

$$\begin{cases} \dfrac{dM}{dr} = 4\pi \boldsymbol{r^2}\rho(r) \\ \dfrac{dP}{dr} = -\dfrac{GM(r)\rho(r)}{\boldsymbol{r^2}} \\ \dfrac{dL}{dr} = 4\pi \boldsymbol{r^2}\rho(r)\epsilon(r) \\ \dfrac{dT}{dr} = \begin{cases} -\dfrac{3\bar{\kappa}(r)\rho(r)L(r)}{64\pi\sigma \boldsymbol{r^2} T^3(r)} & \text{[radiation]} \\ -\left(1-\dfrac{1}{\gamma}\right)\dfrac{\mu m_{\mathrm{H}}}{k_{\mathrm{B}}}\dfrac{GM(r)}{\boldsymbol{r^2}} & \begin{bmatrix}\text{adiabatic} \\ \text{convection}\end{bmatrix} \end{cases} \end{cases}$$

$$\begin{bmatrix} \text{Four coupled first-order} \\ \text{differential equations} \\ \text{(left: in radial coordinate} \\ \text{right: in mass coordinate)} \\ \text{describing } \textbf{equilibrium} \\ \text{stellar structure.} \\ \\ \begin{cases} \text{Boundary conditions:} \\ M(r=0)=0 \\ M(r=R)=M \\ P(r=R)=0 \\ L(r=0)=0 \end{cases} \end{bmatrix}$$

$$\begin{cases} \dfrac{dr}{dM} = \dfrac{1}{4\pi r(M)^2\rho(M)} \\ \dfrac{dP}{dM} = -\dfrac{GM(r)}{4\pi r(M)^4} \\ \dfrac{dL}{dM} = \epsilon(M) \\ \dfrac{dT}{dM} = \begin{cases} -\dfrac{3\bar{\kappa}(M)L(M)}{256\pi^2\sigma r(M)^4 T^3(M)} \\ -\dfrac{GMT(M)}{4\pi r(M)^4 P(M)} \end{cases} \end{cases}$$

Parker Wind Model (for Sun-like stars)

Expansion of the solar corona produces the solar wind which is a result of the corona's high temperature and high thermal conductivity of the ionized gas (plasma). Plasma's ability to conduct heat ensures **isothermality** of the corona. Assume (for simplicity)that the gas is completely ionized and composed entirely of Hydrogen, so that the number density of protons, $n(r) = \dfrac{\rho(r)}{m_{\mathrm{H}}} \Longrightarrow \rho(r) = m_{\mathrm{H}} n(r)$

$$\frac{dP}{dr} \approx -\frac{GM_{\odot}\rho(r)}{r^2} \quad \begin{bmatrix}\text{Hydrostatic} \\ \text{equilibrium} \\ M(r)\approx M_{\odot}\end{bmatrix} = -\frac{GM_{\odot}m_{\mathrm{H}}n(r)}{r^2} \quad \left[\because P = \frac{\rho k_{\mathrm{B}}T}{\mu m_{\mathrm{H}}} = \frac{nk_{\mathrm{B}}T}{\mu}\right] \Longrightarrow \frac{k_{\mathrm{B}}T}{\mu}\frac{dn}{dr} = -\frac{GM_{\odot}m_{\mathrm{H}}n(r)}{r^2}$$

$$\Longrightarrow \frac{dn}{dr} = -\left(\frac{GM_{\odot}m_{\mathrm{H}}}{2k_{\mathrm{B}}T}\right)\frac{n(r)}{r^2} \quad \left[\because \mu \approx \frac{1}{2} \text{ for ionized hydrogen}\right] \Longrightarrow \int_{n}^{n_0}\frac{dn}{n} = -\left(\frac{GM_{\odot}m_{\mathrm{H}}}{2k_{\mathrm{B}}T}\right)\int_{r}^{r_0}\frac{dr}{r^2}$$

$$\Longrightarrow \ln\frac{n}{n_0} = -\frac{GM_{\odot}m_{\mathrm{H}}}{2k_{\mathrm{B}}Tr_0}\left(1-\frac{r_0}{r}\right) \Longrightarrow \boldsymbol{n(r) = n_0 e^{-\lambda\left(1-\frac{r_0}{r}\right)}} \quad \left[\lambda = \frac{GM_{\odot}m_{\mathrm{H}}}{2k_{\mathrm{B}}Tr_0}\right] \Longrightarrow \boldsymbol{P(r) = P_0 e^{-\lambda\left(1-\frac{r_0}{r}\right)}} \quad \left[\because P = \frac{nk_{\mathrm{B}}T}{\mu}\right]$$

Differential equation for Sinc function (Spherical Bessel equation)

$$\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) + \lambda y x^2 = 0 \Longleftrightarrow \frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) + \lambda y = 0 \Longrightarrow \frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \lambda y = 0 \Longleftrightarrow \boldsymbol{x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \lambda xy = 0}$$

Frobenius (Power series) method: Let $y = \sum_{r=0}^{\infty} a_r x^{k+r}$ $[a_0 \neq 0]$

$$\therefore x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \lambda xy = 0 \Longrightarrow \sum_{r=0}^{\infty}(k+r)(k+r-1)a_r x^{k+r-1} + 2\sum_{r=0}^{\infty}(k+r)a_r x^{k+r-1} + \lambda\sum_{r=0}^{\infty}a_r x^{k+r+1} = 0$$

$$\Longrightarrow \sum_{r=0}^{\infty}(k+r)(k+r+1)a_r x^{k+r-1} + \lambda\sum_{r=0}^{\infty}a_r x^{k+r+1} = 0$$

Equating to zero the coefficient of lowest power of x i.e. x^{k-1}: $k(k+1)a_0 \Longrightarrow k=0$ or $k=-1$ $[\because a_0 \neq 0]$

Equating to zero the coefficient of next lowest power of x i.e. x^k: $(k+1)(k+2)a_1 = 0 \Longrightarrow \begin{cases} a_1 = 0 \text{ for } k=0 \\ a_1 \neq 0 \text{ for } k=-1 \end{cases}$

Equating to zero the coefficient of a higher power of x i.e. $(k+r+2)(k+r+3)a_{r+2} + \lambda a_r = 0$

$$\Longrightarrow a_{r+2} = \frac{-\lambda}{(k+r+2)(k+r+3)}a_r = \begin{cases} \dfrac{-\lambda}{(r+2)(r+3)}a_r \text{ for } k=0 \\ \dfrac{-\lambda}{(r+1)(r+2)}a_r \text{ for } k=-1 \end{cases}$$

$$\text{For } k=0, \begin{cases} \ldots + a_7 = a_5 = a_3 = a_1 = 0 \\ a_2 = \dfrac{-\lambda}{2\cdot 3}a_0 = \dfrac{-\lambda}{3!}a_0\ ;\ a_4 = \dfrac{-\lambda}{4\cdot 5}a_2 = \dfrac{\lambda^2}{5!}a_0\ ;\ a_6 = \dfrac{-\lambda}{6\cdot 7}a_4 = \dfrac{-\lambda^3}{7!}a_0 \end{cases}$$

$$y_1=\sum_{r=0}^{\infty}a_rx^r=a_0-\frac{\lambda x^2}{3!}a_0+\frac{\lambda^2x^4}{5!}a_0-\frac{\lambda^3x^6}{7!}a_0+\cdots=\frac{a_0}{\sqrt{\lambda}x}\left(\sqrt{\lambda}x-\frac{\left(\sqrt{\lambda}x\right)^3}{3!}+\frac{\left(\sqrt{\lambda}x\right)^5}{5!}-\frac{\left(\sqrt{\lambda}x\right)^7}{7!}+\cdots\right)=\frac{a_0}{\sqrt{\lambda}x}\sin\sqrt{\lambda}x$$

For $k=-1$, $\begin{cases} a_2=\dfrac{-\lambda}{2}a_0=\dfrac{-\lambda}{2!}a_0\ ;\ a_4=\dfrac{-\lambda}{3\cdot4}a_2=\dfrac{\lambda^2}{4!}a_0\ ;\ a_6=\dfrac{-\lambda}{5\cdot6}a_4=\dfrac{-\lambda^3}{6!}a_0 \\ a_3=\dfrac{-\lambda}{2\cdot3}a_1=\dfrac{-\lambda}{3!}a_1\ ;\ a_5=\dfrac{-\lambda}{4\cdot5}a_2=\dfrac{\lambda^2}{5!}a_1\ ;\ a_6=\dfrac{-\lambda}{6\cdot7}a_4=\dfrac{-\lambda^3}{7!}a_1 \end{cases}$

$$y_2=\sum_{r=0}^{\infty}a_rx^{r-1}=\frac{1}{x}\sum_{r=0}^{\infty}a_rx^r=\frac{1}{x}\left(a_0-\frac{\lambda x^2}{2!}a_0+\frac{\lambda^2x^4}{4!}a_0-\frac{\lambda^3x^6}{6!}a_0+\cdots\right)+\frac{1}{x}\left(a_1x-\frac{\lambda x^3}{3!}a_1+\frac{\lambda^2x^5}{5!}a_1-\frac{\lambda^3x^7}{7!}a_1+\cdots\right)$$

$$=\frac{a_0}{x}\left(1-\frac{\left(\sqrt{\lambda}x\right)^2}{2!}+\frac{\left(\sqrt{\lambda}x\right)^4}{4!}-\frac{\left(\sqrt{\lambda}x\right)^6}{6!}+\cdots\right)+\frac{a_1}{\sqrt{\lambda}x}\left(\sqrt{\lambda}x-\frac{\left(\sqrt{\lambda}x\right)^3}{3!}+\frac{\left(\sqrt{\lambda}x\right)^5}{5!}-\frac{\left(\sqrt{\lambda}x\right)^7}{7!}+\cdots\right)$$

$$=\frac{a_0}{\sqrt{\lambda}x}\cos\sqrt{\lambda}x+\frac{a_1}{\sqrt{\lambda}x}\sin\sqrt{\lambda}x=\frac{a_0}{\sqrt{\lambda}x}\cos\sqrt{\lambda}x+\frac{a_1}{a_0}y_1$$ [a_1 may be set to zero as it does not accompany a solution other than y_1.]

$$\therefore y=Ay_1+By_2=\frac{Aa_0+Ba_1}{\sqrt{\lambda}x}\sin\sqrt{\lambda}x+\frac{Ba_0}{\sqrt{\lambda}x}\cos\sqrt{\lambda}x\,y\Rightarrow \boldsymbol{y=C_1\,\mathrm{sinc}\sqrt{\lambda}x+C_2\frac{\cos\sqrt{\lambda}x}{\sqrt{\lambda}x}}$$

Alternatively, $y_3=Cy_1\ln x+y_2$ so that $\dfrac{d^2y_3}{dx^2}+\dfrac{2}{x}\dfrac{dy_3}{dx}+\lambda y_3=0$

$$\Rightarrow\left(C\frac{d^2y_1}{dx^2}\ln x+\frac{C}{x}\frac{dy_1}{dx}+\frac{C}{x}\frac{dy_1}{dx}-\frac{Cy_1}{x^2}+\frac{d^2y_2}{dx^2}\right)+\frac{2}{x}\left(C\frac{dy_1}{dx}\ln x+\frac{Cy_1}{x}+\frac{dy_2}{dx}\right)+\lambda(Cy_1\ln x+y_2)=0$$

$$\Rightarrow C\ln x\left(\frac{d^2y_1}{dx^2}+\frac{2}{x}\frac{dy_1}{dx}+\lambda y_1\right)+C\left(\frac{2}{x}\frac{dy_1}{dx}+\frac{y_1}{x^2}\right)+\left(\frac{d^2y_2}{dx^2}+\frac{2}{x}\frac{dy_2}{dx}+\lambda y_2\right)=0$$

$$\Rightarrow C\left(\frac{2}{x}\frac{dy_1}{dx}+\frac{y_1}{x^2}\right)=0$$ [$\because$ Both y_1 and y_2 are solutions.] $\Rightarrow C=0\Rightarrow y_3=y_2$

Polytropic process

For a **barotropic fluid**, density is a function of pressure only, i.e. $\boldsymbol{\rho=f(P)\Rightarrow P=h(\rho)}$ $[h=f^{-1}]$

If $h(\rho)=K\rho^{\gamma'}$ i.e. $P=K\rho^{\gamma'}$ it describes a **polytropic fluid.** When $\gamma'=\gamma$, it describes an **adiabatic fluid.**

$\boldsymbol{P=K\rho^{\gamma'}}$ [Polytropic equation of state] $\Rightarrow \boldsymbol{Pv^{\gamma'}=K}$ [specific volume, $v=\dfrac{V}{m}$] $\Rightarrow \boldsymbol{PV^{\gamma'}=Km^{\gamma'}}\Rightarrow \boldsymbol{PV^{\gamma'}=K_1}$

[when shell mass, $m=dM$ is static] Polytropic index, γ' or $\Gamma=1+\dfrac{1}{n}\Rightarrow \boldsymbol{n=\dfrac{1}{\gamma'-1}}$

This arises from the relation, $PV=(\gamma'-1)U\Rightarrow \boldsymbol{u=nP}$ [Energy-density, $u=U/V$] $\Rightarrow \boldsymbol{n=u/P}$

For ideal gas, $\boldsymbol{PV=n'RT}$ [n' = amount of gas as number of moles and, $R=N_Ak_{\mathrm{B}}$] $\therefore \begin{cases}\dfrac{V^{\gamma'}}{V}=\dfrac{K_1}{n'RT}\Rightarrow \boldsymbol{TV^{\gamma'-1}=K_2}\quad \left[K_2=\dfrac{K_1}{n'R}=\dfrac{Km^{\gamma'}}{n'R}\right] \\ \dfrac{P}{P^{\gamma'}}=\dfrac{K_1}{(n'RT)^{\gamma'}}\Rightarrow P^{\gamma'-1}=K_3T^{\gamma'}\Rightarrow \boldsymbol{P=K_3T^{\gamma'/(\gamma'-1)}}\end{cases}$

Also, $PV=n'RT\Rightarrow P=\dfrac{n'RT}{m}\rho=K_4\rho T$ $\left[K_4=\dfrac{n'R}{m}\right]$ $\therefore 1=\dfrac{K\rho^{\gamma'}}{K_4\rho T}\Rightarrow \boldsymbol{T=K_5\rho^{\gamma'-1}}$ $\left[K_5=\dfrac{K}{K_4}=\dfrac{Km}{n'R}\right]$

Note: For a given shell (spherical layer) of fluid, $\boldsymbol{m}$ and $\boldsymbol{n'}$ are constants due to the equal rates of outflux and influx.

Lane-Emden equation for spherical polytropes

$\boldsymbol{\dfrac{dP}{dr}=-\dfrac{GM\rho}{r^2}}$ [Hydrostatic equilibrium] $\Rightarrow \dfrac{r^2}{\rho}\dfrac{dP}{dr}=-GM\Rightarrow \dfrac{d}{dr}\left(\dfrac{r^2}{\rho}\dfrac{dP}{dr}\right)=-G\dfrac{dM}{dr}=-G(4\pi r^2\rho)$ $\left[\because \dfrac{dM}{dr}=4\pi r^2\rho\right]$

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right)=-4\pi G\rho\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2K\gamma'\rho^{\gamma'-2}\frac{d\rho}{dr}\right)=-4\pi G\rho\Rightarrow \boldsymbol{\frac{K\gamma'}{r^2}\frac{d}{dr}\left(r^2\rho^{\gamma'-2}\frac{d\rho}{dr}\right)=-4\pi G\rho}$$

$\left[\because \text{For a polytropic fluid, } P(\rho)=K\rho^{\gamma'}\ [K \text{ and } \gamma' \text{ are constants.}]\Rightarrow \dfrac{dP}{dr}=\left(\dfrac{dP}{d\rho}\right)\dfrac{d\rho}{dr}=K\gamma'\rho^{\gamma'-1}\dfrac{d\rho}{dr}\right]$

$$\Rightarrow\left(1+\frac{1}{n}\right)\frac{K}{r^2}\frac{d}{dr}\left(r^2\rho^{-\left(1-\frac{1}{n}\right)}\frac{d\rho}{dr}\right)=-4\pi G\rho\ \left[\begin{array}{c}\text{Putting polytropic index, } \gamma'=1+\dfrac{1}{n} \text{ where } n \text{ is a constant.}\\ \text{For adiabatic process, } \gamma'=\gamma=\dfrac{C_P}{C_V} \text{ i.e. ratio of specific heats.}\end{array}\right]$$

$$\Rightarrow\left(1+\frac{1}{n}\right)\frac{K}{r^2}\frac{d}{dr}\left(r^2{\rho_c}^{-\left(1-\frac{1}{n}\right)}\theta^{1-n}\frac{d\theta^n}{dr}\right)=-4\pi G\theta^n\ \left[\begin{array}{c}\text{Putting } \rho=\rho_c\theta^n \text{ where } \theta\equiv\theta(r) \text{ is a dimensionless}\\ \text{density scaling function such that } 0\le\theta\le 1\end{array}\right]$$

$$\Rightarrow\left(1+\frac{1}{n}\right){\rho_c}^{-\left(1-\frac{1}{n}\right)}\frac{K}{r^2}\frac{d}{dr}\left(r^2 n\frac{d\theta}{dr}\right)=-4\pi G\theta^n\ \left[\because\frac{d\theta^n}{dr}=n\theta^{n-1}\frac{d\theta}{dr}\right]$$

$$\Rightarrow\left(\frac{K(n+1)}{4\pi G}{\rho_c}^{-\left(1-\frac{1}{n}\right)}\right)\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right)=-\theta^n\Rightarrow\boldsymbol{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)=-\theta^n}\quad\text{which is the } \textbf{Lane-Emden} \text{ equation.}$$

$$\left[\begin{array}{c}\text{Putting } \lambda^2=\left(\dfrac{K(n+1)}{4\pi G}{\rho_c}^{-\left(1-\frac{1}{n}\right)}\right) \text{ and } r=\lambda\xi \text{ where } \xi \text{ is a dimensionless independent variable that replaces}\\ r \text{ while } \lambda \text{ has the unit of distance. This equation is to be solved for } \theta(\xi) \text{ which for a specific } n \text{ then gives } \rho(r)\\ \text{as } \rho=\rho_c\theta^n \text{ and } \xi=r/\lambda \text{ that leads to } P(r) \text{ since } P=K\rho^\gamma=K\rho^{\left(1+\frac{1}{n}\right)} \text{ and further to } T(r).\end{array}\right]$$

For isothermal case ($\gamma=1$), $n=\dfrac{1}{\gamma-1}=\infty$ $\therefore$ **Lane-Emden equation is not applicable for isotropic fluid.**

Solving the Lane-Emden equation: Boundary conditions

$\theta=1$ at $\xi=0$ so that $\rho=\rho_c$ at $r=0$

$\dfrac{d\theta}{d\xi}=0$ at $\xi=0$

In spherically symmetric cases, the Lane–Emden equation is integrable (and thus solvable) for only three values of the polytropic index *n* viz. 0, 1 and 5. Hence there are only three analytic solutions.

Solving the Lane-Emden equation: for *n* = 0

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)=-1\Rightarrow\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)=-\xi^2\Rightarrow\xi^2\frac{d\theta}{d\xi}=-\frac{1}{3}\xi^3+C_1\Rightarrow\frac{d\theta}{d\xi}=-\frac{1}{3}\xi+\frac{C_1}{\xi}=-\frac{1}{3}\xi\ \left[\because\left.\frac{d\theta}{d\xi}\right|_{\xi=0}=0\right]$$

$$\Rightarrow\theta=-\frac{1}{6}\xi^2+C_0\Rightarrow\boldsymbol{\theta=1-\frac{1}{6}\xi^2}\ \left[\because\theta|_{\xi=0}=1\right]\Rightarrow\text{When } \xi=\sqrt{6} \text{ i.e. } r=\lambda\xi=\infty\ \ [\because\lambda=\infty], \theta=0$$

Then $\rho=\rho_c\theta^n=\rho_c$ $[\because\theta^0=1]$ everywhere, even at the boundary $[\because 0^0=1]$ which lies at infinity! Clearly, this cannot be a physical solution unless $\rho_c=0$ which nullifies the mass of the star.

Solving the Lane-Emden equation: for *n* = 1

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)=-\theta\Rightarrow\left(\xi^2\frac{d^2\theta}{d\xi^2}+2\xi\frac{d\theta}{d\xi}\right)+\theta\xi^2=0\Rightarrow\frac{d^2\theta}{d\xi^2}+\frac{2}{\xi}\frac{d\theta}{d\xi}+\theta=0\ \text{ whose solution is known to be}$$

$$\theta=C_0\frac{\sin\xi}{\xi}+C_1\frac{\cos\xi}{\xi}\Rightarrow\boldsymbol{\theta=\frac{\sin\xi}{\xi}}\ \ \left[\because\theta|_{\xi=0}=1\Rightarrow C_0+C_1\cdot\infty=1\Rightarrow C_0=1\text{ and }C_1=0\right]$$

$$\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\theta\xi^2=0 \text{ is spherical Bessel differential equation } \frac{d}{dx}\left(x^2\frac{dy}{dx}\right)+\{\lambda x^2-l(l+1)\}y=0 \text{ with}\begin{cases}l=0\\ \lambda=1\end{cases}$$

whose solution is $y=Aj_l(\sqrt{\lambda}x)+By_l(\sqrt{\lambda}x)=Aj_0(x)+By_0(x)=C_0\dfrac{\sin x}{x}+C_1\dfrac{\cos x}{x}$

$n=1$ case (i.e. is relevant in modelling planetary interiors (such as Jupiter and Saturn) and brown dwarfs.

Transformations of Lane-Emden equation

Put $\theta=\dfrac{\chi}{\xi}\left[\begin{array}{c}\xi \text{ is a dimensionless}\\ \text{independent variable.}\end{array}\right]$ so that $\dfrac{d\theta}{d\xi}=\dfrac{1}{\xi}\dfrac{d\chi}{d\xi}-\dfrac{\chi}{\xi^2}$

$$\therefore\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)=-\theta^n\Rightarrow\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi\frac{d\chi}{d\xi}-\chi\right)=-\left(\frac{\chi}{\xi}\right)^n\Rightarrow\frac{1}{\xi^2}\left(\frac{d\chi}{d\xi}+\xi\frac{d^2\chi}{d\xi^2}-\frac{d\chi}{d\xi}\right)=-\frac{\chi^n}{\xi^n}\Rightarrow\boldsymbol{\frac{d^2\chi}{d\xi^2}=-\frac{\chi^n}{\xi^{n-1}}}$$

When $n=1$, $\dfrac{d^2\chi}{d\xi^2}=-\chi\Rightarrow\chi=C_0\sin\xi+C_1\cos\xi=C_0\sin\xi\ \ \left[\because\theta|_{\xi=0}=1\Rightarrow\chi|_{\xi=0}=\xi=0\Rightarrow C_1=0\right]$

$$\Rightarrow \boldsymbol{\theta} = C_0\frac{\sin\xi}{\xi} = \frac{\boldsymbol{\sin\xi}}{\boldsymbol{\xi}} \quad \left[\because \theta|_{\xi=0} = 1 \Rightarrow C_0 = 1\right]$$

Kelvin's Transformation of Lane-Emden equation

Put $\xi = \dfrac{1}{x}$ so that $\dfrac{d}{d\xi} = \dfrac{dx}{d\xi}\dfrac{d}{dx} = -x^2\dfrac{d}{dx}$

$$\therefore \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \Rightarrow x^2\left(-x^2\frac{d}{dx}\right)\left(-\frac{d\theta}{dx}\right) = -\theta^n \Rightarrow \boldsymbol{x^4\frac{d^2\theta}{dx^2} = -\theta^n} \Rightarrow \frac{d^2\theta}{dx^2} + \frac{\theta^n}{x^4} = 0$$

Solving the Lane-Emden equation: Singular solution

To ascertain whether Lane-Emden equation has a solution of the form $\theta = ax^\omega$, substitute it into the equation obtained after Kelvin's transformation. $\therefore x^4\dfrac{d^2\theta}{dx^2} = -\theta^n \Rightarrow ax^4\omega(\omega-1)x^{\omega-2} = -a^nx^{n\omega}$

$$\Rightarrow \omega(\omega-1)x^{\omega+2} = -a^{n-1}x^{n\omega} \Rightarrow \begin{cases} \omega+2 = n\omega \Rightarrow \omega(n-1) = 2 \Rightarrow \omega = \dfrac{2}{n-1} \\ a^{n-1} = -\omega(\omega-1) = \dfrac{-2}{n-1}\left(\dfrac{2}{n-1}-1\right) \Rightarrow a = \left\{\dfrac{2(n-3)}{(n-1)^2}\right\}^{\frac{1}{n-1}} \end{cases}$$

$$\because x = \frac{1}{\xi} \quad \therefore \theta = ax^\omega = \frac{a}{\xi^\omega} = \frac{1}{\xi^{\frac{2}{n-1}}}\left\{\frac{2(n-3)}{(n-1)^2}\right\}^{\frac{1}{n-1}}$$

For $n = 0, \omega = -2,\ a = (-6)^{-1} = -\dfrac{1}{6} \quad \therefore \theta = -\dfrac{\xi^2}{6}$

For $n = 2, \omega = 2,\ a = -2 \quad \therefore \theta = -\dfrac{2}{\xi^2}$

Emden's transformations

To get the particular solution $\theta = Ax^\omega z(x)$, substitute it into the equation post Kelvin's transformation.

$$\therefore x^4\frac{d^2\theta}{dx^2} = -\theta^n \Rightarrow Ax^4\left\{x^\omega\frac{d^2z}{dx^2} + 2\omega x^{\omega-1}\frac{dz}{dx} + \omega(\omega-1)x^{\omega-2}z\right\} = -A^nx^{n\omega}z^n$$

$$\left[\because \frac{d^2\theta}{dx^2} = A\left\{x^\omega\frac{d^2z}{dx^2} + 2\omega x^{\omega-1}\frac{dz}{dx} + \omega(\omega-1)x^{\omega-2}z\right\} \quad \text{[Leibniz' rule]}\right]$$

$$\Rightarrow x^4\left\{x^2\frac{d^2z}{dx^2} + 2\omega x\frac{dz}{dx} + \omega(\omega-1)z\right\} = -A^{n-1}x^{n\omega-(\omega-2)}z^n = -A^{n-1}x^{(n-1)\omega+2}z^n = -A^{n-1}x^4z^n \quad \left[\because \omega = \frac{2}{n-1}\right]$$

$$\Rightarrow x^2\frac{d^2z}{dx^2} + 2\omega x\frac{dz}{dx} + \omega(\omega-1)z + A^{n-1}z^n = 0$$

Put $x = e^q$ i.e. $q = \ln x$ to eliminate the powers of x so that $\begin{cases} \dfrac{dz}{dx} = \dfrac{dq}{dx}\dfrac{dz}{dq} = \dfrac{1}{x}\dfrac{dz}{dq} = e^{-q}\dfrac{dz}{dq} \\ \dfrac{d^2z}{dx^2} = \dfrac{d}{dx}\left(\dfrac{1}{x}\dfrac{dz}{dq}\right) = \dfrac{dq}{dx}\dfrac{d}{dq}\left(e^{-q}\dfrac{dz}{dq}\right) = e^{-2q}\left(\dfrac{d^2z}{dq^2} - \dfrac{dz}{dq}\right) \end{cases}$

$$\therefore \left(\frac{d^2z}{dq^2} - \frac{dz}{dq}\right) + 2\omega\frac{dz}{dq} + \omega(\omega-1)z + A^{n-1}z^n = 0 \Rightarrow \boldsymbol{\frac{d^2z}{dq^2} + (2\omega-1)\frac{dz}{dq} + \omega(\omega-1)z + A^{n-1}z^n = 0}$$

which is the condition for z to satisfy so that $\theta = Ax^\omega z$ is a solution of Lane-Emden equation.

If $A = 1$ [i.e. $\theta = x^\omega z$], then $\dfrac{d^2z}{dq^2} + (2\omega-1)\dfrac{dz}{dq} + \omega(\omega-1)z + z^n = 0$

If $A = a$ [i.e. $\theta = ax^\omega z$], then $\dfrac{d^2z}{dq^2} + (2\omega-1)\dfrac{dz}{dq} + \omega(\omega-1)z(1-z^{n-1}) = 0 \quad [\because a^{n-1} = -\omega(\omega-1)]$

Solving the Lane-Emden equation: for *n* = 5

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^5 \Rightarrow \frac{d^2z}{dq^2} + (2\omega-1)\frac{dz}{dq} + \omega(\omega-1)z(1-z^4) = 0 \ \left[\text{with } n = 5 \text{ and } A = a = \frac{\pm 1}{\sqrt{2}} \text{ for } n = 5\right]$$

$$\Rightarrow \frac{d^2z}{dq^2} = \frac{1}{4}z(1-z^4) \quad \left[\because \omega = \frac{2}{n-1}\right] \Rightarrow \frac{dz}{dq}\left(\frac{d^2z}{dq^2}\right) = \frac{1}{4}z(1-z^4)\frac{dz}{dq} \Rightarrow \frac{1}{2}\frac{d}{dq}\left(\frac{dz}{dq}\right)^2 = \frac{1}{4}z(1-z^4)\frac{dz}{dq}$$

$$\Rightarrow 2\left(\frac{dz}{dq}\right)^2 = \int z(1-z^4)\,dz = \int (z-z^5)\,dz = \frac{z^2}{2} - \frac{z^6}{6} + C \Rightarrow \frac{dz}{dq} = \pm\sqrt{\frac{z^2}{4} - \frac{z^6}{12} + C}$$

When $C = 0$, $\frac{dz}{dq} = \frac{z}{2}\sqrt{1-\frac{z^4}{3}} \Rightarrow \int \frac{dz}{z\sqrt{1-\frac{z^4}{3}}} = \frac{1}{2}\int dq \Rightarrow \frac{1}{2}q + \frac{1}{2}C_0{}' = \int \frac{dz}{z\sqrt{1-\frac{z^4}{3}}} = \frac{1}{4}\ln\left(\frac{\sqrt{1-\frac{z^4}{3}}-1}{\sqrt{1-\frac{z^4}{3}}+1}\right)$

$$\Rightarrow \frac{\sqrt{1-\frac{z^4}{3}}-1}{\sqrt{1-\frac{z^4}{3}}+1} = e^{2q+2C_0{}'} = C_0e^{2q} \Rightarrow \sqrt{1-\frac{z^4}{3}} = \frac{C_0e^{2q}-1}{C_0e^{2q}+1} \Rightarrow \frac{z^4}{3} = 1 - \left(\frac{C_0e^{2q}-1}{C_0e^{2q}+1}\right)^2 \Rightarrow z^4 = \frac{12C_0e^{2q}}{(C_0e^{2q}+1)^2}$$

$$\Rightarrow z^4 = \frac{12C_0e^{2q}}{(C_0e^{2q}+1)^2} = \frac{12C_0\xi^{-2}}{(C_0\xi^{-2}+1)^2} = \frac{12C_0\xi^{-2}}{C_0{}^2\xi^{-4}\left(1+C_0{}^{-1}\xi^2\right)^2} = \frac{12C_0{}^{-1}\xi^2}{\left(1+C_0{}^{-1}\xi^2\right)^2} \quad \left[q = \ln x = \ln\frac{1}{\xi} = -\ln\xi\right]$$

$$\Rightarrow \pm\left\{\frac{12C_1\xi^2}{(1+C_1\xi^2)^2}\right\}^{\frac{1}{4}} = z = \sqrt{2}\theta\xi^{\frac{1}{2}} \quad \left[\because \theta = Ax^{\omega}z = \frac{x^{\frac{1}{2}}}{\pm\sqrt{2}}z = \frac{z}{\pm\sqrt{2}\xi^{\frac{1}{2}}} \Rightarrow z = \pm\sqrt{2}\theta\xi^{\frac{1}{2}}\right] \Rightarrow \boldsymbol{\theta} = \left\{\frac{\mathbf{3}\boldsymbol{C_1}}{(\mathbf{1}+\boldsymbol{C_1\xi^2})^{\mathbf{2}}}\right\}^{\frac{1}{4}}$$

$\Rightarrow \boldsymbol{\theta} = \dfrac{\mathbf{1}}{\sqrt{\mathbf{1}+\frac{\boldsymbol{\xi^2}}{\mathbf{3}}}} \quad \left[\because \theta|_{\xi=0} = 1 \Rightarrow \{3C_1\}^{\frac{1}{4}} = 1 \Rightarrow C_1 = \frac{1}{3}\right]$ which is the **Lane-Emden function** for $n = 5$

The solution to the Lane-Emden equation (for a specified n) which satisfies the boundary conditions is called the **Lane-Emden function** (for that n). For $n = 0, 1$ and 5, they are, respectively, $\mathbf{1} - \frac{\mathbf{1}}{\mathbf{6}}\boldsymbol{\xi}^{\mathbf{2}}$, $\frac{\sin\boldsymbol{\xi}}{\boldsymbol{\xi}}$ and $\frac{\mathbf{1}}{\sqrt{\mathbf{1}+\frac{\boldsymbol{\xi^2}}{\mathbf{3}}}}$

Chandrasekhar equation (isothermal Lane-Emden equation) / Isothermal equation for spherical polytropes

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \quad \begin{bmatrix}\text{Hydrostatic}\\ \text{equilibrium}\end{bmatrix} \Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \Rightarrow \frac{k_BT}{\mu m_H}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d\rho}{dr}\right) = -4\pi G\rho \quad \begin{bmatrix}\because \boldsymbol{P} = \frac{\boldsymbol{k_BT}}{\boldsymbol{\mu m_H}}\boldsymbol{\rho}\\ \text{and } \frac{dT}{dr} = 0\end{bmatrix}$$

$$\Rightarrow \frac{k_BT}{\mu m_H}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d(\ln\rho)}{dr}\right) = -4\pi G\rho \Rightarrow \frac{k_BT}{\mu m_H}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = 4\pi G\rho_ce^{-\theta} \Rightarrow \left(\frac{k_BT\rho_c{}^{-1}}{4\pi G\mu m_H}\right)\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = e^{-\theta}$$

$$\begin{bmatrix}\text{Putting } \rho = \rho_ce^{-\theta} \Rightarrow \ln\rho = \ln\rho_c - \theta \text{ where } \theta \equiv \theta(r)\\ \text{is a dimensionless density scaling function such that } 0 \le \theta \le 1\end{bmatrix} \Rightarrow \frac{\mathbf{1}}{\boldsymbol{\xi^2}}\frac{\boldsymbol{d}}{\boldsymbol{d\xi}}\left(\boldsymbol{\xi^2}\frac{\boldsymbol{d\theta}}{\boldsymbol{d\xi}}\right) = \boldsymbol{e^{-\theta}} \quad \begin{bmatrix}\textbf{Chandrasekhar}\\ \textbf{equation}\end{bmatrix}$$

$$\left[\text{Putting } \lambda^2 = \left(\frac{k_BT\rho_c{}^{-1}}{4\pi G\mu m_H}\right) \text{ and } r = \lambda\xi \Rightarrow \xi = r/\lambda \text{ so that } \frac{d}{dr} = \frac{d\xi}{dr}\frac{d}{d\xi} = \frac{1}{\lambda}\frac{d}{d\xi}\right]$$

Note that Chandrasekhar equation is valid even when radiation pressure is included, i.e. $\boldsymbol{P} = \frac{\boldsymbol{k_BT}}{\boldsymbol{\mu m_H}}\boldsymbol{\rho} + \frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{aT^4}$

Solving the Chandrasekhar equation: Boundary conditions

$\theta = 0$ at $\xi = 0$ so that $\rho = \rho_c$ at $r = 0$

$\frac{d\theta}{d\xi} = 0$ at $\xi = 0$

Solving the Chandrasekhar equation: asymptotic (near-zero) solution

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = 1 \quad [\theta \to 0] \Rightarrow \frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = \xi^2 \Rightarrow \xi^2\frac{d\theta}{d\xi} = \frac{1}{3}\xi^3 + C_1 \Rightarrow \frac{d\theta}{d\xi} = \frac{1}{3}\xi + \frac{C_1}{\xi} = \frac{1}{3}\xi \quad \left[\because \left.\frac{d\theta}{d\xi}\right|_{\xi=0} = 0\right]$$

$$\Rightarrow \theta = \frac{1}{6}\xi^2 + C_0 \Rightarrow \boldsymbol{\theta = \frac{1}{6}\xi^2} \quad [\because \theta|_{\xi=0} = 0] \Rightarrow \theta = \frac{r^2}{6\lambda^2} \Rightarrow \rho(r) = \rho_c e^{-\frac{r^2}{6\lambda^2}} \quad \begin{bmatrix}\text{This is the } \textbf{near-zero} \text{ solution} \\ \text{of Chandrasekhar equation.}\end{bmatrix}$$

$\therefore$ Let the actual solution be $\theta = \dfrac{\xi^2}{6}\displaystyle\sum_{k=0}^{\infty} a_k \xi^k = \frac{1}{6}\sum_{k=0}^{\infty} a_k \xi^{k+2} \quad [a_0 \neq 0 \text{ at } \xi = 0]$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = e^{-\theta} \Rightarrow \frac{1}{6}\sum_{k=0}^{\infty} a_k(k+2)(k+3)\xi^k = \sum_{j=0}^{\infty}(-1)^j\frac{\theta^j}{j!} \quad \left[\because e^{-\theta} = \sum_{j=0}^{\infty}(-1)^j\frac{\theta^j}{j!}\right]$$

$$\left[\frac{d\theta}{d\xi} = \frac{1}{6}\sum_{k=0}^{\infty} a_k(k+2)\xi^{k+1} \Rightarrow \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = \frac{1}{\xi^2}\frac{d}{d\xi}\left(\frac{1}{6}\sum_{k=0}^{\infty} a_k(k+2)\xi^{k+3}\right) = \frac{1}{6\xi^2}\sum_{k=0}^{\infty} a_k(k+2)(k+3)\xi^{k+2}\right]$$

$$\Rightarrow \frac{1}{6}\sum_{k=0}^{\infty} a_k(k+2)(k+3)\xi^k = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(\frac{1}{6}\sum_{k=0}^{\infty} a_k{}'\xi^{k+2}\right)^j \Rightarrow \sum_{k=0}^{\infty} a_k(k+2)(k+3)\xi^k = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,6^{j-1}}\left(\sum_{k=0}^{\infty} a_k{}'\xi^{k+2}\right)^j$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k(k+2)(k+3)\xi^k = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,6^{j-1}}(a_0{}'\xi^2 + a_1{}'\xi^4 + a_2{}'\xi^6 + \cdots)^j \Rightarrow \begin{cases} a_k = 0 \ \forall \text{ odd } k \\ a_0 = 1 = a_0{}' \\ a_2 = \dfrac{-1}{4\cdot 5}a_0{}' = \dfrac{-1}{20} = a_1{}' \\ a_4 = \dfrac{1}{6\cdot 7}\left(\dfrac{a_0{}'^2}{12} - a_1{}'\right) = \dfrac{1}{315} = a_2{}' \\ a_6 = \dfrac{1}{8\cdot 9}\left(-\dfrac{a_0{}'^3}{216} + \dfrac{a_0{}'a_1{}'}{6} - a_2{}'\right) \\ = \dfrac{-61}{272160} \approx \dfrac{-1}{4461.64} \approx -\dfrac{-1}{4460} = a_3{}' \end{cases}$$

$$a_8 = \frac{1}{10\cdot 11}\left(\frac{a_0{}'^4}{5184} - \frac{a_0{}'^2 a_1{}'}{72} + \frac{2a_0{}'a_2{}' + a_1{}'^2}{12} - a_3{}'\right) = \frac{629}{37422000} \approx \frac{1}{59494.44} \approx \frac{1}{59495} \approx a_4{}'$$

$$a_8 = \frac{1}{12\cdot 13}\left(-\frac{a_0{}'^5}{155520} + \frac{a_0{}'^3 a_1{}'}{1296} - \frac{a_0{}'^2 a_2{}' + a_0{}'a_1{}'^2}{72} + \frac{a_0{}'a_3{}' + a_1{}'a_2{}'}{6} - a_4{}'\right) = -\frac{2869}{2189187000} \approx -\frac{1}{763049} \approx a_5{}'$$

$$\boldsymbol{\theta} = \frac{\xi^2}{6}\sum_{k=0}^{\infty} a_k{}'\xi^k = \frac{\xi^2}{6}(a_0{}'\xi^2 + a_1{}'\xi^4 + a_2{}'\xi^6 + \cdots) = \boldsymbol{\frac{\xi^2}{6} - \frac{\xi^4}{120} + \frac{\xi^6}{1890} - \frac{\xi^8}{26760} + \frac{\xi^{10}}{356970} - \frac{\xi^{12}}{4578294}} + \cdots$$

This solution is valid only when $\xi \ll 1$. The curve rises continuously for series with odd number of terms.

$$\frac{d\theta}{d\xi} = 0 \Rightarrow \boldsymbol{\frac{\xi}{3} - \frac{\xi^3}{30} + \frac{\xi^5}{315} - \frac{\xi^7}{3345} + \frac{\xi^9}{35697} - \frac{\xi^{11}}{381524.5}} + \cdots = \boldsymbol{0} \Rightarrow \begin{cases} \xi \approx 3.16 \text{ with LHS upto second term} \\ \xi \approx 3.21 \text{ with LHS upto fourth term} \\ \xi \approx 3.23 \text{ with LHS upto sixth term} \end{cases}$$

When even number of terms are taken, $\theta(\xi)$ sharply drops after 3.

Transformations of Chandrasekhar equation

Put $\theta = \dfrac{\chi}{\xi}$ $\begin{bmatrix}\xi \text{ is a dimensionless} \\ \text{independent variable.}\end{bmatrix}$ so that $\dfrac{d\theta}{d\xi} = \dfrac{1}{\xi}\dfrac{d\chi}{d\xi} - \dfrac{\chi}{\xi^2}$

$$\therefore \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = e^{-\theta} \Rightarrow \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi\frac{d\chi}{d\xi} - \chi\right) = e^{-\chi/\xi} \Rightarrow \frac{1}{\xi^2}\left(\frac{d\chi}{d\xi} + \xi\frac{d^2\chi}{d\xi^2} - \frac{d\chi}{d\xi}\right) = e^{-\chi/\xi} \Rightarrow \boldsymbol{\frac{d^2\chi}{d\xi^2} = \xi e^{-\chi/\xi}}$$

Kelvin's Transformation applied to Chandrasekhar equation

Put $\xi = \dfrac{1}{x}$ so that $\dfrac{d}{d\xi} = \dfrac{dx}{d\xi}\dfrac{d}{dx} = -x^2\dfrac{d}{dx}$

$$\therefore \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = e^{-\theta} \Rightarrow x^2\left(-x^2\frac{d}{dx}\right)\left(-\frac{d\theta}{dx}\right) = e^{-\theta} \Rightarrow \boldsymbol{x^4\frac{d^2\theta}{dx^2} = e^{-\theta}}$$

Solving the Chandrasekhar equation: Singular solution

To ascertain if Chandrasekhar equation has a solution of the form $\theta = \ln(ax^{\omega})$, substitute into the equation obtained after Kelvin's transformation. $\therefore x^4\dfrac{d^2\theta}{dx^2}=e^{-\theta}\Rightarrow -\omega x^2=\dfrac{1}{ax^\omega}\Rightarrow \omega=-2\Rightarrow a=-\dfrac{1}{\omega}=\dfrac{1}{2}$

$\left[\because \dfrac{d^2\theta}{dx^2}=\dfrac{d}{dx}\left(\dfrac{a\omega x^{\omega-1}}{ax^\omega}\right)=\dfrac{d}{dx}\left(\dfrac{\omega}{x}\right)=\dfrac{-\omega}{x^2}\right]\quad \therefore \theta=\ln\left(ax^\omega\right)=\ln\left(\dfrac{1}{2x^2}\right)\Rightarrow \boldsymbol{e^{-\theta}=2x^2}\quad \because x=\dfrac{1}{\xi}\quad \therefore \boldsymbol{\theta=\ln\left(\dfrac{\xi^2}{2}\right)}$

Note that this solution **does not satisfy** the boundary conditions at $\xi=0$. It is valid only for $\boldsymbol{\xi\gg 1}$, so that

$\dfrac{\boldsymbol{\rho}}{\boldsymbol{\rho_c}}=\dfrac{\mathbf{2}}{\boldsymbol{\xi^2}}\Rightarrow \boldsymbol{\rho(r)=\dfrac{c_{sT}{}^2}{2\pi G r^2}}\quad \left[\because \xi=\left(\dfrac{4\pi G\rho_C}{c_{sT}{}^2}\right)^{\frac{1}{2}}r\right]$ This is the density in singular isothermal spherical cloud.

Emden's transformation applied to Chandrasekhar equation

To get the particular solution $e^{-\theta}=Ax^2e^{z(x)}$, substitute it into the equation post Kelvin's transformation.

$\therefore x^4\dfrac{d^2\theta}{dx^2}=e^{-\theta}\Rightarrow x^4\left(\dfrac{2}{Ax^2}-\dfrac{d^2z}{dx^2}\right)=Ax^2e^z\Rightarrow \boldsymbol{Ax^2\dfrac{d^2z}{dx^2}+A^2e^z-2=0}$

$\left[\theta=-\ln\left(Ax^2\right)-z\Rightarrow \dfrac{d^2\theta}{dx^2}=\dfrac{d}{dx}\left\{\dfrac{-1}{Ax^2}(2x)-\dfrac{dz}{dx}\right\}=\dfrac{2}{Ax^2}-\dfrac{d^2z}{dx^2}\right]$

Put $x=e^q$ i.e. $q=\ln x$ to eliminate the power of x so that $\begin{cases}\dfrac{dz}{dx}=\dfrac{dq}{dx}\dfrac{dz}{dq}=\dfrac{1}{x}\dfrac{dz}{dq}=e^{-q}\dfrac{dz}{dq}\\ \dfrac{d^2z}{dx^2}=\dfrac{d}{dx}\left(\dfrac{1}{x}\dfrac{dz}{dq}\right)=\dfrac{dq}{dx}\dfrac{d}{dq}\left(e^{-q}\dfrac{dz}{dq}\right)=e^{-2q}\left(\dfrac{d^2z}{dq^2}-\dfrac{dz}{dq}\right)\end{cases}$

$\therefore Ax^2\dfrac{d^2z}{dx^2}+A^2e^z-2=0\Rightarrow A\left(\dfrac{d^2z}{dq^2}-\dfrac{dz}{dq}\right)+A^2e^z-2=0\Rightarrow \boldsymbol{A\dfrac{d^2z}{dq^2}-A\dfrac{dz}{dq}+A^2e^z-2=0}$

which is the condition for z to satisfy so that $\theta=Ax^2e^z$ is a solution of Chandrasekhar equation.

If $A=1$ [i.e. $\theta=x^2e^z$], then $\dfrac{d^2z}{dq^2}-\dfrac{dz}{dq}+e^z-2=0$

If $A=a=2$ [i.e. $\theta=2x^2e^z$], then $\dfrac{d^2z}{dq^2}-\dfrac{dz}{dq}+2e^z-1=0$

Evaluate the integral

$\displaystyle\int_0^\infty \frac{x^s}{e^x-1}dx=\int_0^\infty x^se^{-x}\left(\frac{1}{1-e^{-x}}\right)dx=\int_0^\infty x^se^{-x}\sum_{n=0}^\infty e^{-nx}\,dx\quad\left[\because \text{geometric series, }\sum_{n=0}^\infty e^{-nx}=\frac{1}{1-e^{-x}}\text{ as }e^{-1}<1\right]$

$\displaystyle=\sum_{m=1}^\infty\int_0^\infty x^se^{-mx}dx=\sum_{n=0}^\infty\frac{1}{m^{s+1}}\int_0^\infty y^se^{-y}dy\left[\begin{array}{c}\text{Putting } y=mx\\ \Rightarrow x=\dfrac{y}{m}\Rightarrow dx=\dfrac{dy}{m}\end{array}\right]=s!\sum_{n=0}^\infty\frac{1}{m^{s+1}}=s!\sum_{m=1}^\infty\frac{1}{m^{s+1}}=s!\,\zeta(s+1)$

$\displaystyle\left[\because \int_0^\infty y^{r-1}e^{-y}dy=\Gamma(r)=(r-1)!\ \text{ and }\ \sum_{m=1}^\infty\frac{1}{m^r}=\zeta(r)\right]\quad \therefore \boldsymbol{\int_0^\infty\frac{x^s}{e^x-1}dx=s!\,\zeta(s+1)}$

$\displaystyle\Rightarrow\begin{cases}\displaystyle\int_0^\infty\frac{x^2}{e^x-1}dx=2\,\zeta(3)\approx 2.404\quad[\because \zeta(3)\approx 1.202]\\ \displaystyle\int_0^\infty\frac{x^3}{e^x-1}dx=6\,\zeta(4)=\frac{\pi^4}{15}\quad\left[\because \zeta(4)=\frac{\pi^4}{90}\right]\end{cases}$

Gauss' law for gravity

Differential form: $\vec{\nabla}\cdot\vec{g}=-4\pi G\rho\quad\left[\begin{array}{c}\text{where }\vec{g}\text{ is the gravitational field}\\ \text{and }\rho\text{ is the mass density}\end{array}\right]$

Integral form: $\displaystyle\oiint_S\vec{g}\cdot d\vec{S}=-4\pi GM\quad\left[\begin{array}{c}\text{where }d\vec{S}\text{ is the outward area vector}\\ \text{and }M\text{ is the total mass enclosed}\end{array}\right]$

Poisson's equation for gravity

$\vec{\nabla} \cdot \vec{g} = -4\pi G\rho \Rightarrow \vec{\nabla} \cdot \left(-\vec{\nabla}\Phi\right) = -4\pi G\rho \Rightarrow \boldsymbol{\nabla^2 \Phi = 4\pi G\rho}$

For spherically symmetric system: $\dfrac{1}{r^2}\dfrac{d}{dr}\left(r^2\dfrac{d\Phi}{dr}\right) = 4\pi G\rho(r)$

Hydrostatic self-gravitating sphere

$$\frac{1}{\rho(r)}\vec{\nabla}P = -\vec{\nabla}\Phi \quad \begin{bmatrix}\text{Hydrostatic}\\ \text{equilibrium}\end{bmatrix} \Rightarrow \frac{1}{\rho(r)}\frac{dP}{dr} = -\frac{d\Phi}{dr} \quad \begin{bmatrix}\text{for spherically}\\ \text{symmetric mass}\end{bmatrix} \Rightarrow \frac{1}{\rho(r)}\frac{d(\rho c_{sT}{}^2)}{dr} = -\frac{d\Phi}{dr}$$

$$\Rightarrow \frac{1}{\rho(r)}\frac{d\rho}{dr} = -\frac{1}{c_{sT}{}^2}\frac{d\Phi}{dr} \Rightarrow \frac{d(\ln\rho)}{dr} = -\frac{1}{c_{sT}{}^2}\frac{d\Phi}{dr} \Rightarrow \int_{\rho_C}^{\rho} d(\ln\rho) = -\frac{1}{c_{sT}{}^2}\int_0^{\Phi} d\Phi \quad \begin{bmatrix}\because \rho(r=0) = \rho_C\\ \Phi(r=0) = 0\end{bmatrix}$$

$$\Rightarrow \ln\frac{\rho}{\rho_C} = -\frac{1}{c_{sT}{}^2}\Phi \Rightarrow \boldsymbol{\rho(r) = \rho_C \exp\left(-\frac{\Phi}{c_{sT}{}^2}\right)} \Rightarrow \rho(r) = \rho_C\exp(-\psi) \quad \left[\text{where } \psi = \frac{\Phi}{c_{sT}{}^2}\right]$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\rho(r) \quad \begin{bmatrix}\text{Poisson's}\\ \text{equation}\end{bmatrix} \Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\rho_C\exp\left(-\frac{\Phi}{c_{sT}{}^2}\right)$$

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = \frac{4\pi G\rho_C}{c_{sT}{}^2}\exp(-\psi) \Rightarrow \boldsymbol{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right) = \exp(-\psi)} \quad \left[\xi = \left(\frac{4\pi G\rho_C}{c_{sT}{}^2}\right)^{\frac{1}{2}} r\right]$$

which is the isothermal Lane-Emden equation a.k.a. **Chandrasekhar equation.**

Jeans criterion for gravitational collapse of spherical molecular cloud

For a stable, gravitationally-bound system, $2K + U = 0$ $\begin{bmatrix}\textbf{Virial Theorem}\\ \text{Kinetic and Potential energy}\\ \text{terms are averaged over time.}\end{bmatrix}$

$$K = \frac{3}{2}Nk_{\mathrm{B}}T = \frac{3M_c k_{\mathrm{B}}T}{2\mu m_{\mathrm{H}}} \quad \begin{bmatrix}\because \text{Total number of particles, } N = \dfrac{M_c}{\mu m_{\mathrm{H}}}\ ;\ M_c = \text{mass of molecular cloud}\\ \mu = \text{mean molecular mass}\ ;\ m_{\mathrm{H}} = \text{mass of Hydrogen atom}\end{bmatrix}$$

Gravitational potential energy, $U = -\dfrac{3}{5}\dfrac{GM_c{}^2}{R_c}$ $\quad [R_c = \text{radius of molecular cloud}]$

For gravitational collapse, $\boldsymbol{2K + U < 0} \Rightarrow \dfrac{3M_c k_{\mathrm{B}}T}{\mu m_{\mathrm{H}}} - \dfrac{3}{5}\dfrac{GM_c{}^2}{R_c} < 0 \Rightarrow \dfrac{k_{\mathrm{B}}T}{\mu m_{\mathrm{H}}} < \dfrac{1}{5}\dfrac{GM_c}{R_c} \Rightarrow \boldsymbol{M_c > \left(\dfrac{5k_{\mathrm{B}}T}{G\mu m_{\mathrm{H}}}\right)R_c}$

$$\Rightarrow M_c > \left(\frac{5k_{\mathrm{B}}T}{G\mu m_{\mathrm{H}}}\right)\left(\frac{3M_c}{4\pi\rho_c}\right)^{\frac{1}{3}} \quad \left[\because \text{Volume, } V = \frac{4\pi}{3}R_c{}^3 \Rightarrow R_c = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} = \left(\frac{3M_c}{4\pi\rho_c}\right)^{\frac{1}{3}}\right] \Rightarrow M_c{}^{\frac{2}{3}} > \left(\frac{5k_{\mathrm{B}}T}{G\mu m_{\mathrm{H}}}\right)\left(\frac{3}{4\pi\rho_c}\right)^{\frac{1}{3}}$$

$$\Rightarrow \boldsymbol{M_c > \left(\frac{5k_{\mathrm{B}}T}{G\mu m_{\mathrm{H}}}\right)^{\frac{3}{2}}\left(\frac{3}{4\pi\rho_c}\right)^{\frac{1}{2}}} \Rightarrow \boldsymbol{M_c > M_J} \quad \begin{bmatrix}\textbf{Jeans criterion}\text{ for gravitational collapse of molecular cloud.}\\ \boldsymbol{M_J = \left(\dfrac{5k_{\mathrm{B}}T}{G\mu m_{\mathrm{H}}}\right)^{\frac{3}{2}}\left(\dfrac{3}{4\pi\rho_c}\right)^{\frac{1}{2}} = \dfrac{c_J v_{sT}{}^4}{G^{3/2}P_c{}^{1/2}}} \text{ is called } \textbf{Jeans mass.}\end{bmatrix}$$

$$\left[c_J = \left(\frac{125}{4\pi}\right)^{\frac{1}{2}} \approx 5.463\ ;\ \text{isothermal sound speed, } v_{sT} = \sqrt{\frac{P_c}{\rho_c}} \Rightarrow \rho_c = \frac{P_c}{v_{sT}{}^2}\ ;\ P_c = \frac{\rho_c k_{\mathrm{B}}T}{\mu m_{\mathrm{H}}} \Rightarrow \frac{k_{\mathrm{B}}T}{\mu m_{\mathrm{H}}} = \frac{P_c}{\rho_c} = v_{sT}{}^2\right]$$

Also, $M_c > \left(\dfrac{5k_{\mathrm{B}}T}{G\mu m_{\mathrm{H}}}\right)R_c \Rightarrow \rho_c\left(\dfrac{4\pi}{3}R_c{}^3\right) > \left(\dfrac{5k_{\mathrm{B}}T}{G\mu m_{\mathrm{H}}}\right)R_c \quad \left[\because M_c = \rho_c V = \rho_c\left(\dfrac{4\pi}{3}R_c{}^3\right)\right] \Rightarrow R_c{}^2 > \dfrac{15k_{\mathrm{B}}T}{4\pi G\mu m_{\mathrm{H}}\rho_c}$

$$\Rightarrow \boldsymbol{R_c > \left(\frac{15k_{\mathrm{B}}T}{4\pi G\mu m_{\mathrm{H}}\rho_c}\right)^{\frac{1}{2}}} \Rightarrow \boldsymbol{R_c > R_J} \quad \begin{bmatrix}\textbf{Jeans criterion}\text{ for gravitational collapse of molecular cloud.}\\ \boldsymbol{R_J}\text{ or } \boldsymbol{\lambda_J = \left(\dfrac{15k_{\mathrm{B}}T}{4\pi G\mu m_{\mathrm{H}}\rho_c}\right)^{\frac{1}{2}}} \text{ is called } \textbf{Jeans length.}\end{bmatrix}$$

For a typical diffuse hydrogen cloud, T = 50 K and n = 5 × 10^8 m^{-3}. If we assume the cloud is entirely composed of H I, $\rho_0 = m_{\mathrm{H}}n_{\mathrm{H}}$ = 8.4 × 10^{-19} kg m^{-3}. Taking μ = 1, find minimum mass to cause the cloud to collapse spontaneously.

$$M_J = \left(\frac{5k_B T}{G\mu m_H}\right)^{\frac{3}{2}}\left(\frac{3}{4\pi\rho_0}\right)^{\frac{1}{2}} = \left(\frac{5k_B T n_H}{G\mu\rho_0}\right)^{\frac{3}{2}}\left(\frac{3}{4\pi\rho_0}\right)^{\frac{1}{2}} \approx 2.88\times10^{33}\text{ kg} \approx 1448\, M_\odot$$

∵ this value exceeds the estimated believed to be contained in H I clouds,
∴ diffuse hydrogen clouds are stable against gravitational collapse.

For a dense core of a giant molecular cloud, typical temperature and number density are T = 10 K and n_{H_2} = 10^{10} m^{-3}. Assuming that dense clouds are predominantly molecular hydrogen, $\rho_0 = 2m_H n_{H_2} = 3\times10^{-17}$ kg m^{-3} and $\mu \approx 2$, determine the minimum mass to cause the cloud to collapse spontaneously.

$$M_J = \left(\frac{5k_B T}{G\mu m_H}\right)^{\frac{3}{2}}\left(\frac{3}{4\pi\rho_0}\right)^{\frac{1}{2}} = \left(\frac{5k_B T n_{H_2}}{G\rho_0}\right)^{\frac{3}{2}}\left(\frac{3}{4\pi\rho_0}\right)^{\frac{1}{2}} \approx 1.8\times10^{31}\text{ kg} \approx 9\, M_\odot$$

∵ this value is characteristic of the masses of dense cores,
∴ dense cores of GMCs are unstable to gravitational collapse.

Bonnor-Ebert mass* ** (Refer Chandrasekhar Pg 156-164, Stahler & Palla Pg 244-245)

$$2K + U = 3P_{\text{ext}}V \Longrightarrow \frac{3M_c k_B T}{\mu m_H} - \frac{3}{5}\frac{GM_c^2}{R_c} = 3P_{\text{ext}}\left(\frac{4\pi}{3}R_c^3\right) \Longrightarrow \frac{1}{5}\frac{GM_c^2}{R_c^2} = P_{\text{ext}}\left(4\pi R_c^2\right) \Longrightarrow M_c^2 = \frac{20\pi P_{\text{ext}} R_c^4}{G}$$

If $R_c = R_J$ and $P_c = P_{\text{ext}}$, M_{BE}^2

$$= \frac{20\pi P_{\text{ext}}}{G}\left(\frac{15}{4\pi G P_{\text{ext}}}\right)^2 v_{sT}^8 \quad \left[R_J = \left(\frac{15k_B T}{4\pi G\mu m_H\rho_c}\right)^{\frac{1}{2}} = \left(\frac{15 v_{sT}^2}{4\pi G(P_c/v_{sT}^2)}\right)^{\frac{1}{2}} = \left(\frac{15}{4\pi G P_c}\right)^{\frac{1}{2}} v_{sT}^2\right]$$

$$\Longrightarrow M_{BE} = \frac{c_{BE} v_{sT}^4}{G^{3/2}P_{\text{ext}}^{1/2}} \quad \left[c_{BE} = \sqrt{20\pi}\left(\frac{15}{4\pi}\right) = 7.5\sqrt{\frac{5}{\pi}}\right]$$

$$M = \int_0^{r_0} 4\pi r^2\rho(r)\,dr = 4\pi\rho_C\left(\frac{c_{sT}^2}{4\pi G\rho_C}\right)^{\frac{3}{2}}\int_0^{\xi_0}\xi^2 e^{-\psi}\,d\xi \quad \left[\because \rho(r) = \rho_C e^{-\psi} \text{ and } \xi = \left(\frac{4\pi G\rho_C}{c_{sT}^2}\right)^{\frac{1}{2}} r\right]$$

$$\text{Now, } \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right) = e^{-\psi} \quad \begin{bmatrix}\text{Chandrasekhar}\\ \text{equation}\end{bmatrix} \Longrightarrow \left(\xi^2\frac{d\psi}{d\xi}\right)_{\xi_0} = \int_0^{\xi_0}\xi^2 e^{-\psi}\,d\xi$$

$$\therefore M = 4\pi\rho_C\left(\frac{c_{sT}^2}{4\pi G\rho_C}\right)^{\frac{3}{2}}\left(\xi^2\frac{d\psi}{d\xi}\right)_{\xi_0} = (4\pi\rho_C)^{-\frac{1}{2}}\left(\frac{c_{sT}^2}{G}\right)^{\frac{3}{2}}\left(\xi^2\frac{d\psi}{d\xi}\right)_{\xi_0} = \left(4\pi\frac{(c_{sT}^2\rho_C)}{c_{sT}^2}\right)^{-\frac{1}{2}}\left(\frac{c_{sT}^2}{G}\right)^{\frac{3}{2}}\left(\xi^2\frac{d\psi}{d\xi}\right)_{\xi_0}$$

$$\left[\because \frac{P_0}{\rho_0} = \frac{P_c}{\rho_c} = c_{sT}^2\right] = \left(4\pi\frac{P_0\rho_C}{\rho_0 c_{sT}^2}\right)^{-\frac{1}{2}}\left(\frac{c_{sT}^2}{G}\right)^{\frac{3}{2}}\left(\xi^2\frac{d\psi}{d\xi}\right)_{\xi_0} = \frac{1}{\sqrt{4\pi}}\frac{c_{sT}^4}{G^{\frac{3}{2}}P_0^{\frac{1}{2}}}\left(\frac{\rho_0}{\rho_c}\right)^{\frac{1}{2}}\left(\xi^2\frac{d\psi}{d\xi}\right)_{\xi_0} \Longrightarrow \boldsymbol{M = \frac{m_{BE}c_{sT}^4}{G^{\frac{3}{2}}P_0^{\frac{1}{2}}}}$$

$$\left[\text{where dimensionless mass, } m_{BE} = \frac{1}{\sqrt{4\pi}}\left(\frac{\rho_0}{\rho_c}\right)^{\frac{1}{2}}\left(\xi^2\frac{d\psi}{d\xi}\right)_{\xi_0}\right]$$

$$\rho_0 = \rho(r = r_0) = \rho_C e^{-\psi(\xi_0)}$$

$$\frac{d\xi_0}{d\left(\frac{\rho_c}{\rho_0}\right)} = \frac{d\xi_0}{d(e^{\psi(\xi_0)})} = \left(\frac{d(e^{\psi(\xi_0)})}{d\xi_0}\right)^{-1} = e^{-\psi(\xi_0)}\left(\frac{d\psi(\xi_0)}{d\xi_0}\right)^{-1} = e^{-\psi(\xi_0)}\left(\frac{d\psi_0}{d\xi_0}\right)^{-1}$$

$$\psi(\xi) = \frac{\xi^2}{6} - \frac{\xi^4}{120} + \frac{\xi^6}{1890} - \frac{\xi^8}{26760} + \frac{\xi^{10}}{356970} - \frac{\xi^{12}}{4578294}\cdots \Longrightarrow \frac{d\psi}{d\xi} = \frac{\xi}{3} - \frac{\xi^3}{30} + \frac{\xi^5}{315} - \frac{\xi^7}{3345} + \frac{\xi^9}{35697} - \frac{\xi^{11}}{381524.5}\cdots$$

$$\Longrightarrow \left(\xi^2\frac{d\psi}{d\xi}\right)_{\xi_0} = \xi_0^2\left(\frac{\xi_0}{3} - \frac{\xi_0^3}{30} + \frac{\xi_0^5}{315} - \frac{\xi_0^7}{3345} + \cdots\right) = \frac{\xi_0^3}{3}\left(1 - \frac{\xi_0^2}{10} + \frac{\xi_0^4}{105} - \frac{\xi_0^6}{1115} + \frac{\xi_0^8}{11899} - \cdots\right) = \xi_0^2\frac{d\psi_0}{d\xi_0}$$

$$m_{BE} = \frac{1}{\sqrt{4\pi}}\left(\frac{\rho_c}{\rho_0}\right)^{-\frac{1}{2}}\xi_0^2\frac{d\psi_0}{d\xi_0}$$

$$\frac{dm_{\mathrm{BE}}}{d\left(\frac{\rho_c}{\rho_0}\right)}=0\Rightarrow -\frac{1}{2}\left(\frac{\rho_c}{\rho_0}\right)^{-\frac{3}{2}}{\xi_0}^2\frac{d\psi_0}{d\xi_0}+\left(\frac{\rho_c}{\rho_0}\right)^{-\frac{1}{2}}2\xi_0\frac{d\psi_0}{d\xi_0}\frac{d\xi_0}{d\left(\frac{\rho_c}{\rho_0}\right)}+\left(\frac{\rho_c}{\rho_0}\right)^{-\frac{1}{2}}{\xi_0}^2\frac{d^2\psi_0}{d{\xi_0}^2}\frac{d\xi_0}{d\left(\frac{\rho_c}{\rho_0}\right)}=0$$

$$\Rightarrow -\frac{1}{2}\left(\frac{\rho_c}{\rho_0}\right)^{-\frac{3}{2}}{\xi_0}^2\frac{d\psi_0}{d\xi_0}+\left(\frac{\rho_c}{\rho_0}\right)^{-\frac{1}{2}}2\xi_0\frac{d\psi_0}{d\xi_0}\left\{e^{-\psi(\xi_0)}\left(\frac{d\psi_0}{d\xi_0}\right)^{-1}\right\}+\left(\frac{\rho_c}{\rho_0}\right)^{-\frac{1}{2}}{\xi_0}^2\frac{d^2\psi_0}{d{\xi_0}^2}\left\{e^{-\psi(\xi_0)}\left(\frac{d\psi_0}{d\xi_0}\right)^{-1}\right\}=0$$

$$\Rightarrow -\frac{1}{2}\left(e^{\psi(\xi_0)}\right)^{-\frac{3}{2}}{\xi_0}^2\frac{d\psi_0}{d\xi_0}+\left(e^{\psi(\xi_0)}\right)^{-\frac{1}{2}}2\xi_0 e^{-\psi(\xi_0)}+\left(e^{\psi(\xi_0)}\right)^{-\frac{1}{2}}{\xi_0}^2\frac{d}{d\xi_0}\left(\ln\frac{d\psi_0}{d\xi_0}\right)e^{-\psi(\xi_0)}=0$$

$$\Rightarrow -\frac{1}{2}{\xi_0}^2\frac{d\psi_0}{d\xi_0}+2\xi_0+{\xi_0}^2\frac{d}{d\xi_0}\left(\ln\frac{d\psi_0}{d\xi_0}\right)=0\Rightarrow 2\frac{d}{d\xi_0}\left(\ln\frac{d\psi_0}{d\xi_0}\right)-\frac{d\psi_0}{d\xi_0}+\frac{4}{\xi_0}=0$$

$$\Rightarrow 2\left(\frac{d\psi_0}{d\xi_0}\right)^{-1}\frac{d^2\psi_0}{d{\xi_0}^2}-\frac{d\psi_0}{d\xi_0}+\frac{4}{\xi_0}=0\Rightarrow 2\frac{d^2\psi_0}{d{\xi_0}^2}-\left(\frac{d\psi_0}{d\xi_0}\right)^2+\frac{4}{\xi_0}\left(\frac{d\psi_0}{d\xi_0}\right)=0$$

$$\Rightarrow 2\left(\frac{1}{3}-\frac{{\xi_0}^2}{10}+\frac{{\xi_0}^4}{63}-\frac{{\xi_0}^6}{478}+\cdots\right)-\frac{{\xi_0}^2}{9}\left(1-\frac{{\xi_0}^2}{10}+\frac{{\xi_0}^4}{105}-\frac{{\xi_0}^6}{1115}+\cdots\right)^2+\frac{4}{3}\left(1-\frac{{\xi_0}^2}{10}+\frac{{\xi_0}^4}{105}-\frac{{\xi_0}^6}{1115}+\cdots\right)=0$$

$$(m_{\mathrm{BE}})_{\max}=\frac{1}{\sqrt{4\pi}}\left(\frac{\rho_0}{\rho_c}\right)^{\frac{1}{2}}\frac{{\xi_0}^3}{3}\left(1-\frac{{\xi_0}^2}{10}+\frac{{\xi_0}^4}{105}-\frac{{\xi_0}^6}{1134}+\cdots\right)=\frac{1}{\sqrt{4\pi}}\left(e^{-\psi(\xi_0)}\right)^{\frac{1}{2}}\frac{{\xi_0}^3}{3}\left(1-\frac{{\xi_0}^2}{10}+\frac{{\xi_0}^4}{105}-\frac{{\xi_0}^6}{1134}+\cdots\right)$$

$$=\frac{1}{\sqrt{4\pi}}e^{-\frac{1}{2}\left(\frac{{\xi_0}^2}{6}-\frac{{\xi_0}^4}{120}+\frac{{\xi_0}^6}{1890}-\frac{{\xi_0}^8}{27216}+\cdots\right)}\frac{{\xi_0}^3}{3}\left(1-\frac{{\xi_0}^2}{10}+\frac{{\xi_0}^4}{105}-\frac{{\xi_0}^6}{1134}+\cdots\right)$$

Random Walk

Photons diffuse through the stellar material in haphazard manner called a random walk. If d is the net vector displacement after a large number N of randomly directed steps, each of length l (mean free path), then,

$$\vec{d}=\vec{l}_1+\vec{l}_2+\vec{l}_3+\cdots+\vec{l}_N\Rightarrow \vec{d}\cdot\vec{d}=\left(\vec{l}_1+\vec{l}_2+\vec{l}_3+\cdots+\vec{l}_N\right)\cdot\left(\vec{l}_1+\vec{l}_2+\vec{l}_3+\cdots+\vec{l}_N\right)=\sum_{i=1}^{N}\sum_{j=1}^{N}\vec{l}_i\cdot\vec{l}_j$$

$$\Rightarrow d^2=Nl^2+l^2\{(\cos\theta_{12}+\cos\theta_{13}+\cdots+\cos\theta_{1N})+(\cos\theta_{21}+\cos\theta_{23}+\cdots+\cos\theta_{2N})+\cdots$$

$$+\left(\cos\theta_{N1}+\cos\theta_{N2}+\cdots+\cos\theta_{N,N-1}\right)\}=Nl^2+l^2\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\cos\theta_{ij}\Rightarrow d^2=Nl^2\Rightarrow \boldsymbol{d=l\sqrt{N}}$$

$\because$ the optical depth at a point is roughly the number of photon mean free paths from that point to the surface (measured along a light ray's straight path), $\boldsymbol{d=\tau_\nu l}\Rightarrow l\sqrt{N}=\tau_\nu l\Rightarrow \boldsymbol{\tau_\nu=\sqrt{N}}\Rightarrow N={\tau_\nu}^2$

Total distance travelled, $D=Nl\Rightarrow D={\tau_\nu}^2 l\Rightarrow dD=2l\tau_\nu d\tau_\nu\Rightarrow ds=2l\tau_\nu(-\chi_\nu\, ds)\Rightarrow l=\dfrac{1}{2\chi_\nu\tau_\nu}$

$$\boldsymbol{d=\tau_\nu l}\Rightarrow ds=-l\,d\tau_\nu\quad[\because dd=ds]\Rightarrow l=\frac{ds}{d\tau_\nu}=\frac{1}{\chi_\nu}\quad[\because d\tau_\nu=-\chi_\nu(s)\,ds]\Rightarrow \boldsymbol{l=\frac{1}{\chi_\nu}=\frac{1}{\kappa_\nu\rho}}$$

Spectral Radiance

It is the power emitted per unit steradian (solid angle) per unit area per unit frequency (or wavelength).

Specific energy density, $\dfrac{\partial u_R}{\partial\nu}=\boldsymbol{u_\nu=\dfrac{4\pi}{c}B_\nu}\Rightarrow$ Spectral radiance, $\dfrac{\partial B_R}{\partial\nu}=\boldsymbol{B_\nu=\dfrac{c}{4\pi}u_\nu}$

Specific energy density per unit steradian $=\dfrac{u_\nu}{4\pi}$

$\therefore$ Energy flow at speed $c=$ Spectral radiance, $B_\nu=\left(\dfrac{u_\nu}{4\pi}\right)c=\dfrac{c}{4\pi}u_\nu$

Specific and Mean Intensities

Specific energy, $E_\nu=\dfrac{\partial E_R}{\partial\nu}$

Specific intensity, $I_\nu = \dfrac{\partial I}{\partial \nu} = \dfrac{E_\nu}{dt\ dS\ d\Omega} = \dfrac{E_\nu}{dt\ (dA\cos\theta)\ d\Omega}$ $[d\Omega = \sin\theta\ d\theta\ d\phi]$

Mean intensity (Zeroth moment of specific intensity), $J_\nu = \langle I_\nu \rangle = \dfrac{1}{4\pi}\int_\Omega I_\nu\, d\Omega = \dfrac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\nu \sin\theta\ d\theta\ d\phi$

Specific radiative flux (First moment of specific intensity) $= \int_\Omega I_\nu \cos\theta\ d\Omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\nu \cos\theta\sin\theta\ d\theta\ d\phi$

For isotropic radiation, $F_\nu = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\nu \cos\theta\sin\theta\ d\theta\ d\phi = 2\pi I_\nu \int_{\theta=0}^{\pi}\cos\theta\sin\theta\ d\theta = 0$

Astrophysical flux, $F_{\nu,\mathrm{AP}} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} I_\nu \cos\theta\sin\theta\ d\theta\ d\phi = \dfrac{2\pi}{2} I_\nu \int_{\theta=0}^{\pi/2}\sin 2\theta\ d\theta = \pi I_\nu\left[\int_{\theta=0}^{\pi/2}\sin 2\theta\ d\theta\right] = \pi I_\nu$

Eddington flux, $H_\nu =$

Observed flux, $F_{\nu,\mathrm{obs}} = \dfrac{\pi R^2}{\pi D^2}F_{\nu,\mathrm{AP}} = \dfrac{R^2}{D^2}F_{\nu,\mathrm{AP}}$ $\left[\because F_{\nu,\mathrm{obs}}(\pi D^2) = F_{\nu,\mathrm{AP}}(\pi R^2) = \dfrac{L_\nu}{4}\right]$

Second moment of specific intensity, $P_\nu = \int_\Omega I_\nu \cos^2\theta\ d\Omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\nu \cos^2\theta\sin\theta\ d\theta\ d\phi$

Radiation Pressure

Consider a radiation confined in a 3D box.

$dp_\nu\ d\nu = \{p_\nu(z,\mathrm{final}) - p_\nu(z,\mathrm{initial})\} = \left\{\dfrac{E_\nu\cos\theta}{c} - \left(-\dfrac{E_\nu\cos\theta}{c}\right)\right\} = \dfrac{2E_\nu\cos\theta}{c}d\nu = \dfrac{2}{c}I_\nu\cos^2\theta\ dt\ dA\ d\Omega\ d\nu$

$[\because I_\nu dt\ (dA\cos\theta)\ d\Omega = E_\nu] \Longrightarrow \dfrac{dp_\nu}{dt\ dA}d\nu = \dfrac{2}{c}I_\nu\cos^2\theta\ d\Omega\, d\nu \Longrightarrow dP_{\mathrm{rad}} = \dfrac{2}{c}I_\nu\cos^2\theta\ d\Omega\, d\nu$

$\Longrightarrow P_{\mathrm{rad},\nu}\ d\nu = \dfrac{2}{c}d\nu\int_{\Omega/2} I_\nu\cos^2\theta\ d\Omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} I_\nu\cos^2\theta\sin\theta\ d\theta\ d\phi$

$\Longrightarrow P_{\mathrm{rad},\nu}\ d\nu = \dfrac{4\pi}{c}I_\nu\ d\nu\int_{\theta=0}^{\pi/2}\cos^2\theta\sin\theta\ d\theta = \dfrac{4\pi}{3c}I_\nu\ d\nu$ [for isotropic radiation]

(Total) Radiation pressure, $P_{\mathrm{rad}} = \dfrac{4\pi}{3c}\int_0^\infty I_\nu\ d\nu$

For blackbody radiation, $P_{\mathrm{rad}} = \dfrac{4\pi}{3c}\int_0^\infty B_\nu(T)\ d\nu = \dfrac{8\pi h}{3c^4}\int_0^\infty \dfrac{\nu^3 d\nu}{e^{h\nu/kT}-1}d\nu = \dfrac{1}{3}aT^4$

Specific Energy Density

For an isotropic radiation field (same intensity in all directions), $\langle I_\nu\rangle = I_\nu$

Blackbody radiation is isotropic so that $\langle I_\nu\rangle = I_\nu = B_\nu(T) = \dfrac{2}{c^3}\dfrac{h\nu^3 d\nu}{e^{h\nu/kT}-1}$

$u_\nu\ d\nu = \dfrac{4\pi}{c}\langle I_\nu\rangle\ d\nu$

For an isotropic radiation field, $u_\nu\ d\nu = \dfrac{4\pi}{c}I_\nu\ d\nu$

For blackbody radiation, $u_\nu\ d\nu = \dfrac{4\pi}{c}B_\nu\ d\nu = \dfrac{8\pi}{c^4}\dfrac{h\nu^3 d\nu}{e^{h\nu/kT}-1}$

(Total) Energy density, $u_R = \int_0^\infty u_\nu \, d\nu = \frac{8\pi h}{c^4} \int_0^\infty \frac{\nu^3 d\nu}{e^{h\nu/kT} - 1} = 8\pi \frac{(kT)^4}{(hc)^3} \left(\frac{\pi^4}{15} \right) = aT^4$

Efficiency of the medium to absorb/scatter (hijack) a photon is given by the photon mean free path, $l_{\nu,\text{free}}$

Extinction coefficient, or, (monochromatic) **opacity**, $\boldsymbol{\chi_\nu} = \frac{\mathbf{1}}{\boldsymbol{l_{\nu,\text{free}}}}$ so that $\boldsymbol{l_{\nu,\text{free}}} = \frac{\mathbf{1}}{\boldsymbol{\chi_\nu}}$

Caution: Mass opacity (or, mass extinction coefficient), $\boldsymbol{\kappa_\nu} = \frac{\boldsymbol{\chi_\nu}}{\boldsymbol{\rho}}$ so that $\boldsymbol{\chi_\nu} = \boldsymbol{\kappa_\nu \rho}$ and $\boldsymbol{l_{\nu,\text{free}}} = \frac{\mathbf{1}}{\boldsymbol{\kappa_\nu \rho}}$

$\chi_\nu = \kappa_\nu \rho = \frac{1}{l_\nu}$ can be regarded as the fraction of photons lost (scattered/absorbed) per unit path length.

Pure extinction: $dI_\nu = -\chi_\nu(s) I_\nu \, ds \Rightarrow \int_{I_{\nu,0}}^{I_\nu} \frac{dI_\nu}{I_\nu} = -\int_0^s \chi_\nu(s) \, ds \Rightarrow \boldsymbol{I_\nu = I_{\nu,0} e^{-\int_0^s \chi_\nu \, ds}} \Rightarrow \boldsymbol{I_\nu = I_{\nu,0} e^{-\int_0^s \kappa_\nu \rho \, ds}}$

$\therefore$ the intensity declines exponentially, and falls by a factor of $\frac{1}{e}$ over a characteristic distance of $l_\nu = \frac{1}{\chi_\nu} = \frac{1}{\kappa_\nu \rho}$

$d(\ln I_\nu) = \frac{dI_\nu}{I_\nu} \Rightarrow \boldsymbol{d\tau_\nu = -\chi_\nu(s) \, ds} = -\kappa_\nu(s)\rho(s) \, ds \Rightarrow \tau_{\nu,s} - \tau_{\nu,0} = -\int_0^s \chi_\nu(s) \, ds$

For convenience, $\tau_\nu = 0$ for the outermost layer after which the light travels unimpeded to the observer,
i.e. $\tau_{\nu,s} = 0 \Rightarrow$ **Optical depth,** $\boldsymbol{\tau_\nu} = \tau_{\nu,0} = \int_0^s \chi_\nu(s) \, ds = \int_0^s \kappa_\nu(s)\rho(s) \, ds = \int_0^s \frac{\boldsymbol{ds}}{\boldsymbol{l_\nu(s)}}$

$\therefore \boldsymbol{I_\nu = I_{\nu,0} e^{-\tau_\nu}}$

The number of mean free path lengths the photons travel through a medium is called **optical depth,** $\boldsymbol{\tau}$. A medium with $\tau \gg 1$ is called **optically thick**, whereas, a medium with $\tau \ll 1$ is called **optically thin.**

$\boldsymbol{\kappa_\nu = \kappa_{\nu,\text{bb}} + \kappa_{\nu,\text{bf}} + \kappa_{\nu,\text{ff}} + \kappa_{\text{es}} + \kappa_{\text{H}^-}}$

Rosseland mean opacity, $\bar{\kappa} = \dfrac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B_\nu(T)}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu(T)}{\partial T} d\nu}$

Kramers opacity law: $\bar{\kappa} = \frac{\kappa_0 \rho}{T^{3.5}}$

$\bar{\kappa} = \overline{\kappa_{\text{bb}} + \kappa_{\text{bf}} + \kappa_{\text{ff}} + \kappa_{\text{es}} + \kappa_{\text{H}^-}}$

$\bar{\kappa}_{\text{bf}} = 4.34 \times 10^{21} \frac{g_{\text{bf}}}{g} Z(1 + X) \frac{\rho}{T^{3.5}} \text{ m}^2/\text{kg}$

$\bar{\kappa}_{\text{bf}} = 3.68 \times 10^{18} g_{\text{ff}}(1 - Z)(1 + X) \frac{\rho}{T^{3.5}} \text{ m}^2/\text{kg}$

$\bar{\kappa}_{\text{es}} \approx 0.02(1 + X) \text{ m}^2/\text{kg}$

$\kappa_{\text{H}^-} \approx 7.9 \times 10^{-34} \frac{Z}{0.02} \rho^{0.5} T^9 \text{ m}^2/\text{kg}$

where the **Gaunt factors** $\bar{\kappa}_{\text{bf}}$ and $\bar{\kappa}_{\text{bf}}$ are quantum mechanical correction terms, g is the **guillotine factor** which describes the cut-off of an atom's contribution to the opacity after it has been ionized. Typically, $1 < g < 100$.

Radiative Transfer equation

$dI_\nu(\hat{n}, s) = -\chi_\nu I_\nu(\hat{n}, s) + \eta_\nu \, ds \quad \begin{bmatrix} s \text{ is the length travelled in the direction of ray} \\ \eta_\nu(s) \text{ is the emissivity} \\ \chi_\nu(s) \text{ is the extinction coefficient} \end{bmatrix}$

$\frac{dI_\nu(\hat{n}, s)}{ds} = \eta_\nu(s) - \chi_\nu(s) I_\nu(\hat{n}, s)$

$\Rightarrow \frac{\boldsymbol{dI_\nu(\hat{n}, s)}}{\boldsymbol{ds}} = \boldsymbol{\chi_\nu(s)\{S_\nu(s) - I_\nu(\hat{n}, s)\}} \quad \left[\text{Source function, } S_\nu = \frac{\eta_\nu}{\chi_\nu} = \frac{j_\nu \rho}{\kappa_\nu \rho} = \frac{j_\nu}{\kappa_\nu} \right]$

$$\Rightarrow \frac{1}{\chi_\nu(s)}\frac{dI_\nu(\hat{n},s)}{ds} = S_\nu(s) - I_\nu(\hat{n},s) \Rightarrow \boldsymbol{\frac{dI_\nu(\hat{n},s)}{d\tau_\nu} = I_\nu(\hat{n},s) - S_\nu(s)} \quad [\because d\tau_\nu = -\chi_\nu(s)\,ds]$$

$$\begin{cases} \text{If the intensity of the light does not vary i.e.} \dfrac{dI_\nu}{ds} = 0 \text{ then } I_\nu = S_\nu \\ \text{If } I_\nu > S_\nu \text{ then } \dfrac{dI_\nu}{ds} < 0 \text{ i.e. the intensity decreases with distance.} \\ \text{If } I_\nu < S_\nu \text{ then } \dfrac{dI_\nu}{ds} > 0 \text{ i.e. the intensity increases with distance.} \end{cases}$$

In case of LTE, by Kirchhoff's law, $\dfrac{dI_\nu(\hat{n},s)}{ds} = \eta_\nu(s)\{B_\nu(T(s)) - I_\nu(\hat{n},s)\}$

where Planck function, $B_\nu(T) = \dfrac{2h\nu^3}{c^2}\dfrac{1}{e^{h\nu/kT}-1}$

If speed of photon is so large that it passes through the object of interest in a time much less than that the object can change its properties, the steady state equation works well.

$$\left\{\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right\} I_\nu(\hat{n},s,t) = \eta_\nu(s) - \chi_\nu(s) I_\nu(\hat{n},s,t)$$
$$\Rightarrow \left\{\frac{1}{c}\frac{\partial}{\partial t} + \hat{n}\cdot\vec{\nabla}\right\} I_\nu(\hat{n},s,t) = \eta_\nu(s) - \chi_\nu(s) I_\nu(\hat{n},s,t) \quad \left[\because \frac{\partial}{\partial s} = \hat{n}\cdot\vec{\nabla}\right]$$

Radiation pressure gradient, $\dfrac{dP_{\text{rad}}}{dr} = -\dfrac{\bar{\kappa}\rho}{c}F_{\text{rad}}$ [F_{rad} is the outward radiative flux.]

General solution of the Radiative Transfer equation

$$\boldsymbol{\frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu} \Rightarrow \frac{dI_\nu}{d\tau_\nu} - I_\nu = -S_\nu \Rightarrow e^{-\tau_\nu}\frac{dI_\nu}{d\tau_\nu} - e^{-\tau_\nu}I_\nu = -e^{-\tau_\nu}S_\nu \Rightarrow \frac{d}{d\tau_\nu}(I_\nu e^{-\tau_\nu}) = -e^{-\tau_\nu}S_\nu$$
$$\Rightarrow \int_0^{\tau_{\nu,0}} d(I_\nu e^{-\tau_\nu}) = -\int_0^{\tau_{\nu,0}} e^{-\tau_\nu}S_\nu\,d\tau_\nu \Rightarrow I_{\nu,0}e^{-\tau_{\nu,0}} - I_\nu(0) = \int_{\tau_{\nu,0}}^0 S_\nu e^{-\tau_\nu}\,d\tau_\nu$$
$$\Rightarrow \boldsymbol{I_\nu(0) = I_{\nu,0}e^{-\tau_{\nu,0}} - \int_{\tau_{\nu,0}}^0 S_\nu e^{-\tau_\nu}\,d\tau_\nu}$$

Interpretation: The emergent intensity on LHS is the sum of two positive contributions. The first term on RHS is the initial intensity of the ray, reduced by absorption along the path to the surface. The second term, also positive, represents the emission at every point along the path, attenuated by the absorption between the point of emission and the surface.

If the atmosphere's radius of curvature is much larger than its thickness (thin atmosphere), it may be consi--dered as a plane parallel slab. Assume the z-axis to be in the vertical upwards direction, with $z = 0$ at the top of the plane parallel stellar atmosphere.

Vertical optical depth, $\tau_{\nu,\text{v}}(z) = \displaystyle\int_z^0 \chi_\nu\,dz$ [by definition]

$$\tau_\nu(s) = \tau_{\nu,0} = \int_0^s \chi_\nu\,ds = \sec\theta \int_0^s \chi_\nu\,dz \Rightarrow \tau_{\nu,\text{v}}(z) = \int_z^0 \chi_\nu\,dz \quad \left[\begin{matrix}\because dz = ds\cos\theta \\ \Rightarrow ds = dz\sec\theta\end{matrix}\right]$$

Eddington Approximation

To determine the temperature structure of plane-parallel gray atmosphere, it is imperative to assume a description of the angular description of the intensity. Owing to Eddington, the intensity of the radiation field is assigned one value, I_{out}, in the $+z$ direction (outward) and another value I_{in} in the $-z$ direction (inward). Both I_{in} and I_{out} vary with depth in the atmosphere. At the top of the atmosphere, $I_{\text{in}} = 0$ where $\tau_v = 0$.

$$\langle I\rangle = \frac{1}{2}(I_{\text{out}} + I_{\text{in}})$$

$F_{\text{rad}} = \pi(I_{\text{out}} - I_{\text{in}})$ $[\because F_{\nu,\text{AP}} = \pi I_\nu]$

Limb darkening

$I_\nu(0) = I_{\nu,0}e^{-\tau_{\nu,0}} - \int_{\tau_{\nu,0}}^{0} S_\nu e^{-\tau_\nu}\,d\tau_\nu = I_{\nu,0}e^{-\tau_{\nu,\text{v},0}\sec\theta} - \int_{\tau_{\nu,\text{v},0}\sec\theta}^{0} S_\nu \sec\theta\, e^{-\tau_{\nu,\text{v}}\sec\theta}\,d\tau_{\nu,\text{v}}$

$\Longrightarrow I_\nu(0) = -\int_{\infty}^{0} S_\nu \sec\theta\, e^{-\tau_{\nu,\text{v}}\sec\theta}\,d\tau_{\nu,\text{v}} = \int_{0}^{\infty} S_\nu \sec\theta\, e^{-\tau_{\nu,\text{v}}\sec\theta}\,d\tau_{\nu,\text{v}}$

Suppose the source function has the form, $S_\nu = a_\nu + b_\nu\tau_{\nu,\text{v}}$

Then, $I_\nu(0) = \int_{0}^{\infty} (a_\nu + b_\nu\tau_{\nu,\text{v}}) \sec\theta\, e^{-\tau_{\nu,\text{v}}\sec\theta}\,d\tau_{\nu,\text{v}}$

$\Longrightarrow I_\nu(0) = a_\nu\sec\theta\int_{0}^{\infty} e^{-\tau_{\nu,\text{v}}\sec\theta}\,d\tau_{\nu,\text{v}} + b_\nu\sec\theta\int_{0}^{\infty}\tau_{\nu,\text{v}}e^{-\tau_{\nu,\text{v}}\sec\theta}\,d\tau_{\nu,\text{v}} = a_\nu\sec\theta\left[\frac{e^{-\tau_{\nu,\text{v}}\sec\theta}}{-\sec\theta}\right]_0^\infty + b_\nu\frac{\Gamma(2)}{\sec\theta}$

$\Longrightarrow \boldsymbol{I_\nu(0) = a_\nu + b_\nu\cos\theta}$

The values of a_ν and b_ν for the solar source function can be ascertained by measuring the variation of specific intensity across the disk of the Sun. E.g. for a wavelength of 501 nm, $a_{501} = 1.04 \times 10^{13}\ \text{Wm}^{-3}\text{sr}^{-1}$ and $b_{501} = 3.52 \times 10^{13}\ \text{Wm}^{-3}\text{sr}^{-1}$ (Böhm-Vitense 1989).

$\begin{cases} \langle I\rangle = S \quad \text{[equilibrium gray atmosphere]} \\ \langle I\rangle = \dfrac{\sigma T^4}{\pi} \quad \text{[Eddington approximation]} \end{cases} \Longrightarrow S = \dfrac{\sigma T^4}{\pi} = \dfrac{3}{4}\dfrac{\sigma T_e^{\,4}}{\pi}\left(\tau_\text{v} + \dfrac{2}{3}\right) \quad \text{[LTE]} \Longrightarrow \boldsymbol{S = \dfrac{3\sigma}{4\pi}T_e^{\,4}\left(\tau_\text{v} + \dfrac{2}{3}\right)}$

$S_\nu = a_\nu + b_\nu\tau_{\nu,\text{v}} \Longrightarrow S = a + b\tau_\text{v}$ [form approximation] $\Longrightarrow$ emergent intensity, $I(0) = a + b\cos\theta$

$\therefore \boldsymbol{a = \dfrac{\sigma}{2\pi}T_e^{\,4}}$ and $\boldsymbol{b = \dfrac{3\sigma}{4\pi}T_e^{\,4}}$

$$\frac{\boldsymbol{I(\theta)}}{\boldsymbol{I(\theta=0)}} = \frac{a + b\cos\theta}{a+b} = \frac{\frac{\sigma}{2\pi}T_e^{\,4} + \frac{3\sigma}{4\pi}T_e^{\,4}\cos\theta}{\frac{\sigma}{2\pi}T_e^{\,4} + \frac{3\sigma}{4\pi}T_e^{\,4}} = \frac{\frac{1}{2} + \frac{3}{4}\cos\theta}{\frac{5}{4}} = \boldsymbol{\frac{2}{5} + \frac{3}{5}\cos\theta}$$

The photosphere is the surface layer of the Sun's atmosphere where the photons can escape into space. According to a model solar atmosphere, the temperature in one region of the photosphere varies from 5580 K to 5790 K over a distance of 25 km. Find the characteristic distance over which the temperature varies (temperature scale height *H_T*). How does it compare with the average distance traveled by an atom before hitting another atom (density of the photosphere at that level is 2.1 × 10⁻⁴ kg/m³, consisting primarily of neutral Hydrogen atoms in ground state)?

$T = \dfrac{T_1 + T_2}{2} = \dfrac{5790\ \text{K} + 5580\ \text{K}}{2} = 5685\ \text{K}$

$H_T = \dfrac{T}{|dT/dr|} = \dfrac{5685\ \text{K}}{(5790\ \text{K} - 5580\ \text{K})/25\ \text{km}} \approx 677\ \text{km}$

$n = \dfrac{\rho}{m_H} \approx 1.2547 \times 10^{23}\ \text{m}^{-3}$ and collision cross-section, $\sigma = \pi(2a_0)^2 \approx 3.52 \times 10^{-20}\ \text{m}^2$

Mean free path, $l = \dfrac{1}{n\sigma} = 2.2642 \times 10^{-4}\ \text{m}$

For solar photosphere, density, $\rho \approx 2.1 \times 10^{-4}\ \text{kg/m}^3$ and opacity (at 500 nm), $\kappa_{500} = 0.03\ \text{m}^2/\text{kg}$

$\therefore l_{500} = \dfrac{1}{\kappa_\nu\rho} \approx 159$ km which is comparable to the temperature scale height, $H_T \approx 677$ km implying that the photospheric photons do not see a constant temperature, and so LTE is not strictly valid in the photosphere.

According to one model of the Sun, the central density is 1.53×10^5 kg/m³ and the Rosseland mean opacity at the center is 0.217 m²/kg.

(a) Calculate the mean free path of a photon at the center of the Sun.

(b) Calculate the average time it would take for the photon to escape from the Sun if this mean free path remained constant for the photon's journey to the surface. (Ignore the fact that identifiable photons are constantly destroyed and created through absorption, scattering, and emission.)

Photon mean free path, $l = \dfrac{1}{\bar{\kappa}\rho} \approx 3 \times 10^{-5}\ \mathrm{m} = 30\ \mu\mathrm{m}$

Net displacement in a random walk, $d = l\sqrt{N} \Longrightarrow$ Number of steps, $N = \left(\dfrac{R_\odot}{l}\right)^2 \approx 5.38 \times 10^{26}$

Path length, $D = Nl = (5.38 \times 10^{26})(3 \times 10^{-5}\ \mathrm{m}) = 1.6 \times 10^{22}\ \mathrm{m}$

$\therefore$ Escape time for the photon $= \dfrac{D}{c} = \dfrac{1.6 \times 10^{22}\ \mathrm{m}}{3 \times 10^{8}\ \mathrm{m/s}} \approx 5.38 \times 10^{13}\ \mathrm{s} \approx 1.7 \times 10^{6}$ years

Solve the same problem with 162 g/cm³ as the central density and 1.16 cm²/g as the Rosseland mean opacity.

Photon mean free path, $l = \dfrac{1}{\bar{\kappa}\rho} \approx 5.32 \times 10^{-5}\ \mathrm{m} = 53.2\ \mu\mathrm{m}$ $\left[\begin{array}{c}\rho = 1.62 \times 10^{5}\ \mathrm{kg/m^3} \\ \bar{\kappa} = 0.116\ \mathrm{m^2/kg}\end{array}\right]$

Net displacement in a random walk, $d = l\sqrt{N} \Longrightarrow$ Number of steps, $N = \left(\dfrac{R_\odot}{l}\right)^2 \approx 1.71 \times 10^{26}$

Path length, $D = Nl = (1.71 \times 10^{26})(5.32 \times 10^{-5}\ \mathrm{m}) = 9.1 \times 10^{21}\ \mathrm{m}$

$\therefore$ Escape time for the photon $= \dfrac{D}{c} = \dfrac{9.1 \times 10^{21}\ \mathrm{m}}{3 \times 10^{8}\ \mathrm{m/s}} \approx 3.03 \times 10^{13}\ \mathrm{s} \approx 0.96 \times 10^{6}$ years

If the mean density of the Sun is 1.53 × 10⁵ kg/m³ and a mean free path of a typical photon is 0.003 cm, find the mean opacity. What is the estimation of the leakage time for a radiative diffusion? Compare it with the time for a photon to escape if it does not interact at all with matter and just flies straight through the Sun.

Photon mean free path, $l = \dfrac{1}{\bar{\kappa}\rho} \Longrightarrow \bar{\kappa} = \dfrac{1}{l\rho} \approx 0.218\ \mathrm{m^2/kg}$

Net displacement in a random walk, $d = l\sqrt{N} \Longrightarrow$ Number of steps, $N = \left(\dfrac{R_\odot}{l}\right)^2 \approx 5.38 \times 10^{26}$

Path length, $D = Nl = (5.38 \times 10^{26})(3 \times 10^{-5}\ \mathrm{m}) = 1.614 \times 10^{22}\ \mathrm{m}$

$\therefore$ Photon leakage time $= \dfrac{D}{c} = \dfrac{1.6 \times 10^{22}\ \mathrm{m}}{3 \times 10^{8}\ \mathrm{m/s}} \approx 5.38 \times 10^{13}\ \mathrm{s} \approx 1.7 \times 10^{6}$ years

Time taken by a non-interacting photon to escape $= \dfrac{R_\odot}{c} \approx 2.32\ \mathrm{s}$

Estimate the radiation energy density and total energy inside the Sun using typical interior temperature of 4.65 × 10⁶ K. Also estimate Sun's luminosity using photon leakage time of 1.7 × 10⁶ years. Compare with the listed value $L_\odot$ = 3.828 × 10²⁶ W.

Radiation energy density, $u_{\mathrm{rad}} = aT^4 = \dfrac{4\sigma}{c}T^4 \approx 3.54 \times 10^{11}\ \mathrm{J/m^3}$

Radiation energy, $U_{\mathrm{rad}} = u_{\mathrm{rad}}V = (aT^4)\left(\dfrac{4}{3}\pi {R_\odot}^3\right) = (3.83 \times 10^{13}\ \mathrm{J/m^3})(1.41 \times 10^{27}\ \mathrm{m^3}) \approx 5 \times 10^{38}\ \mathrm{J}$

Luminosity, $L_\odot = \dfrac{U_{\mathrm{rad}}}{t_{\mathrm{leak}age}} = \dfrac{5 \times 10^{38}\ \mathrm{J}}{(1.7 \times 10^{6} \times 365.25 \times 24 \times 3600)\ \mathrm{s}} \approx 9.3 \times 10^{24}\ \mathrm{W}$

$\dfrac{L_{\odot\mathrm{listed}}}{L_{\odot\mathrm{estimated}}} \approx \dfrac{3.828 \times 10^{26}\ \mathrm{W}}{9.3 \times 10^{24}\ \mathrm{W}} \approx 41$

Calculate the thermal plasma density at the centre of Sun with n_C = 10²⁵ cm⁻³ and T_C = 1.5 × 10⁷ K. How does plasma energy density compare with radiation energy density? How long would it take for Sun to turn off if all its nuclear reactions got stopped? (Given: photon leakage time = 5.39 × 10¹¹ s)

Plasma energy density, $u_{\text{plasma}} = \frac{3}{2}nkT \approx 3.1 \times 10^{15}\ \text{J/m}^3$

Radiation energy density, $u_{\text{rad}} = aT^4 = \frac{4\sigma}{c}T^4 \approx 3.83 \times 10^{13}\ \text{J/m}^3$

$\therefore \frac{u_{\text{plasma}}}{u_{\text{rad}}} \approx \frac{3.1 \times 10^{15}\ \text{J/m}^3}{3.83 \times 10^{13}\ \text{J/m}^3} \approx 80.94$

Plasma energy, $U_{\text{plasma}} = u_{\text{plasma}}V = \left(\frac{3}{2}nkT\right)\left(\frac{4}{3}\pi R_{\odot}{}^3\right) = (3.1 \times 10^{15}\ \text{J/m}^3)(1.41 \times 10^{27}\ \text{m}^3) \approx 4.371 \times 10^{42}\ \text{J}$

Radiation energy, $U_{\text{rad}} = u_{\text{rad}}V = (aT^4)\left(\frac{4}{3}\pi R_{\odot}{}^3\right) = (3.83 \times 10^{13}\ \text{J/m}^3)(1.41 \times 10^{27}\ \text{m}^3) \approx 5.386 \times 10^{40}\ \text{J}$

Luminosity, $L_{\odot} = \frac{U_{\text{rad}}}{t_{\text{leak}age}} = \frac{5.386 \times 10^{40}\ \text{J}}{5.39 \times 10^{11}\ \text{s}} \approx 1 \times 10^{29}\ \text{W}$

Turn-off time, $t_{\text{turnoff}} = \frac{U_{\text{plasma}}}{L_{\odot}} \approx 4.371 \times 10^{13}\ \text{s} \approx 1.385 \times 10^6\ \text{years}$

Using the results for the plane-parallel gray atmosphere in LTE, determine the ratio of the effective temperature of a star to its temperature at the top of the atmosphere. If T_e = 5777 K, what is the temperature at the top of the atmosphere?

$$T^4 = \frac{3}{4}T_e{}^4\left(\tau_v + \frac{2}{3}\right) \Rightarrow T^4 = \frac{1}{2}T_e{}^4 \quad \begin{bmatrix}\because \tau_v = 0 \text{ at the top} \\ \text{of the atmosphere.}\end{bmatrix} \Rightarrow \frac{T_e}{T} = \sqrt[4]{2} \approx 1.2 \Rightarrow T \approx 4858\ \text{K}$$

Using the Eddington approximation for a plane-parallel atmosphere, determine the values of I_{in} and I_{out} as functions of the vertical optical depth. At what depth is the radiation isotropic to within 1%?

$$\langle I \rangle = \frac{3\sigma}{4\pi}T_e{}^4\left(\tau_v + \frac{2}{3}\right) \Rightarrow \frac{1}{2}(I_{\text{out}} + I_{\text{in}}) = \frac{3\sigma}{4\pi}T_e{}^4\left(\tau_v + \frac{2}{3}\right) \Rightarrow I_{\text{out}} + I_{\text{in}} = \frac{3\sigma}{2\pi}T_e{}^4\left(\tau_v + \frac{2}{3}\right)$$

When $\tau_v = 0, I_{\text{in}} = 0$, so that $I_{\text{out}} = \frac{\sigma}{\pi}T_e{}^4 \Rightarrow I_{\text{out}} - I_{\text{in}} = \frac{\sigma}{\pi}T_e{}^4 = \frac{F_{\text{rad}}}{\pi} \quad [\because I_{\text{in}} = 0]$

$\because F_{\text{rad}}$ is a constant, $I_{\text{out}} - I_{\text{in}}$ is constant at any level of the atmosphere.

$$\therefore 2I_{\text{out}} = \frac{3\sigma}{2\pi}T_e{}^4\left(\tau_v + \frac{2}{3}\right) + \frac{\sigma}{\pi}T_e{}^4 = \frac{\sigma}{\pi}T_e{}^4\left(\frac{3}{2}\tau_v + 1\right) + \frac{\sigma}{\pi}T_e{}^4 \Rightarrow I_{\text{out}} = \frac{\sigma}{\pi}T_e{}^4\left(\frac{3}{4}\tau_v + 1\right)$$

and, $I_{\text{in}} = I_{\text{out}} - \frac{\sigma}{\pi}T_e{}^4 = \frac{\sigma}{\pi}T_e{}^4\left(\frac{3}{4}\tau_v + 1\right) - \frac{\sigma}{\pi}T_e{}^4 = \frac{3\sigma}{4\pi}T_e{}^4\tau_v$

$$\frac{I_{\text{out}}}{I_{\text{in}}} = 1.01 \Rightarrow \frac{\frac{3}{2}\tau_v + 2}{\frac{3}{2}\tau_v} = 1.01 \Rightarrow \frac{2}{\frac{3}{2}\tau_v} = 0.01 \Rightarrow \tau_v = \frac{4}{0.03} \approx 133.33$$

Alternatively, $\frac{I_{\text{out}} - I_{\text{in}}}{\frac{1}{2}(I_{\text{out}} + I_{\text{in}})} = 0.01 \Rightarrow \frac{\frac{\sigma}{\pi}T_e{}^4}{\frac{3\sigma}{4\pi}T_e{}^4\left(\tau_v + \frac{2}{3}\right)} = 0.01 \Rightarrow \frac{4}{3\tau_v + 2} = 0.01 \Rightarrow \tau_v = \frac{398}{3} \approx 132.67$

Suppose that we approximate hydrogen atoms as hard spheres with radii a = 0.5 Å. In a neutral atomic hydrogen cloud with density n_H = 30 cm^{-3} what is the mean free path for an H atom against scattering by other H atoms (assuming the other H atoms to be at rest)?

Mean free path, $l = \frac{1}{n\sigma} \approx 1.06 \times 10^{12}\ \text{m} \approx 7\ \text{AU} \quad [n = 30\ \text{cm}^{-3} = 30 \times 10^6\ \text{m}^{-3}]$

Consider the hydrogen atoms in the Sun's photosphere, where the temperature is 5777 K and the number density of hydrogen atoms is 1.5 × 10^{23} m^{-3}. Estimate the pressure broadening of the Hα line.

$$\Delta t \approx \frac{l}{v} = \frac{1}{n\sigma\sqrt{2kT/m}}$$

$$\Delta\lambda = \frac{\lambda^2}{\pi c}\left(\frac{1}{\Delta t}\right) \approx \frac{\lambda^2}{c}\left(\frac{n\sigma}{\pi}\right)\sqrt{\frac{2kT}{m}} \approx 2.36 \times 10^{-4}\ \text{Å} \quad \left[\begin{matrix} \because T = 5777\ \text{K}\ ;\ \lambda = 6563\ \text{Å}\ ;\ \sigma = \pi(2a_0)^2 \\ m = 1.0079\text{u}\ ;\ n = 1.25 \times 10^{-23}\ \text{m}^{-3} \end{matrix}\right]$$

Using the root-mean square speed, v_{rms}, estimate the mean free path of the nitrogen molecules at room temperature (300 K). What is the average time between collisions? Take the radius of a nitrogen molecule to be 0.1 nm and the density of air to be 1.2 kg/m³. A nitrogen molecule consists of 28 nucleons (14 protons and 14 neutrons), each of mass 1 u.

$$v_{\text{rms}} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3kT}{28\text{u}}} \approx 517\ \text{m/s}$$

number density, $n = \dfrac{\rho}{m} = 2.581 \times 10^{25}\ \text{m}^{-3}$

mean free path for molecules, $l = \dfrac{1}{n\sigma} = \dfrac{1}{n(2r)^2} \approx 3.1 \times 10^{-7}\ \text{m} = 3100\ \text{Å}$

$\therefore$ average time between collisions, $\Delta t = \dfrac{l}{v_{\text{rms}}} \approx 6 \times 10^{-10}\ \text{s} = 0.6\ \text{ns}$

Calculate how far you could see through Earth's atmosphere if it had the opacity of the solar photosphere (κ_{500} = 0.03 m²/kg). Use 1.2 kg/m³ as the density of Earth's atmosphere.

mean free path for photons, $l = \dfrac{1}{\kappa_{500}\rho} \approx 27.8\ \text{m}$

$\because$ the visibility is always up to an optical depth of about $\tau_{500} = \dfrac{2}{3}$ i.e. about $\dfrac{2}{3}$ of mean free path, l i.e. 18.52 m.

Estimate the escape velocity at the surface of a neutron star with mass 1.4 M⊙ and radius 9 km. Show that the effects of relativity must be included for an accurate description of a neutron star.

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}} \approx 1.437 \times 10^8\ \text{m/s} \approx 0.48c$$

Estimate the central temperature of the white dwarf Sirius B using radiative temperature gradient equations. (Given: density, ρ_{WD} = 3 × 10⁹ kg/m³ and radius, R_{WD} = 5.5 × 10⁶ m and κ = 0.02 m²/kg)

$$\frac{dT}{dr} \approx \frac{T_s - T_c}{R_s - R_c} \approx \frac{T_{\text{eff}} - T_c}{R_\odot}$$

$$\therefore \frac{dT}{dr} = -\frac{3\bar{\kappa}\rho(r)\,L(r)}{64\pi\sigma r^2 T^3(r)} \begin{bmatrix}\text{Radiative} \\ \text{equilibrium}\end{bmatrix} \Longrightarrow \frac{T_{\text{eff}} - T_c}{R_\odot} \approx -\frac{3(\bar{\kappa}/2)\bar{\rho}_{\text{WD}}(L_{\text{WD}}/2)}{64\pi\sigma(R_{\text{WD}}/2)^2(T_c/2)^3} \Longrightarrow {T_c}^4 \approx \frac{3\bar{\kappa}\bar{\rho}_{\text{WD}}L_{\text{WD}}}{8\pi\sigma R_{\text{WD}}}\ [T_c \gg T_{\text{eff}}]$$

$$\Longrightarrow \boldsymbol{T_c \approx 1.27 \times 10^8\ \text{K}} \quad \begin{bmatrix} \bar{\rho}_{\text{WD}} = 3 \times 10^9\ \text{kg/m}^3\ ;\ \bar{\kappa} = 0.02 \\ R_{\text{WD}} = 5.5 \times 10^6\ \text{m}\ ;\ L_{\text{WD}} \approx 0.03L_\odot \approx 1.1484 \times 10^{25}\ \text{W} \end{bmatrix}$$

Suppose that a B0 main sequence star with an absolute visual magnitude of M_V = –4.0 is observed to have an apparent visual magnitude of V = +8.2. Estimate the distance of the star (i) neglecting interstellar extinction (i.e. A_V = 0), (ii) assuming that extinction along the line of sight is 1 mag kpc⁻¹ (i.e. 10⁻³ mag pc⁻¹).

$V - M_{\text{V}} = 5\log d' - 5 \Longrightarrow d' = 10^{(V - M_{\text{V}} + 5)/5} \approx 2754.2\ \text{pc}$

$V - M_{\text{V}} = 5\log d' - 5 + A_{\text{V}} \Longrightarrow d' = 10^{(V - M_{\text{V}} - A_{\text{V}} + 5)/5} = 10^{(V - M_{\text{V}} - kd + 5)/5} \approx 1441.6\ \text{pc}$

Using Newtonian gravity, estimate the amount of energy required to move 10⁷ M⊙ from a position just above the event horizon of the SMBH (mass = 3.7 × 10⁶ M⊙) at the center of the galaxy to 3 kpc, the present location of the expanding arm. Compare the answer to the energy liberated in a typical Type II supernova.

Schwarzschild radius, $R_S = \dfrac{2GM_{BH}}{c^2} \approx 1.1 \times 10^{10}$ m

$\Delta U = \left(-\dfrac{GM_{BH}m}{r}\right) - \left(-\dfrac{GM_{BH}m}{R_S}\right) = GM_{BH}m\left(\dfrac{1}{R_S} - \dfrac{1}{r}\right) \approx 8.88 \times 10^{53}$ J which is several orders of magnitude greater than the typical energy (10^{44} J) liberated in a typical Type II supernova.

A gas cloud 0.3 pc from the center has a measured velocity of 260 km/s. If the cloud is in orbit about the center, estimate the amount of mass interior to the location of the gas.

$$\frac{mv^2}{r} = \frac{GM_r m}{r^2} \Rightarrow M_r = \frac{v^2 r}{G} \approx 9.377 \times 10^{36} \text{ kg} \approx 4.7 \times 10^6 M_\odot$$

Estimate the mass of the Milky Way galaxy interior to the solar galactocentric distance. Use R_0 = 8 kpc and Θ_0 = 220 km/s. Assume that the mass of the galaxy within the solar circle is much greater than the mass of a test particle orbiting along with the LSR, and that the bulk of the galaxy's mass is distributed spherically symmetrically.

Orbital period of LSR, $P_{LSR} = \dfrac{2\pi R_0}{\Theta_0} \approx 7.051 \times 10^{15}$ s ≈ 223.43 Myr

From Kepler's 3rd law, mass interior solar circle, $M_{LSR} = \dfrac{4\pi^2 {R_0}^3}{G{P_{LSR}}^2} \approx 1.79 \times 10^{41}$ kg $\approx 9 \times 10^{10}\ M_\odot$

Close to the galactic center, the star S2 has an orbital period of 15.2 yr, an orbital eccentricity of 0.87, and a perigalacticon distance of 1.8×10^{13} m = 120 au (17 light-hours). Estimate the mass interior to S2's orbit.

Semi-major axis of S2's orbit, $a_{S2} = \dfrac{r_p}{1-e} \approx 1.385 \times 10^6$ m $\approx 1.4 \times 10^6$ m

From Kepler's 3rd law, mass interior to S2's orbit, $M = \dfrac{4\pi^2 {a_{S2}}^3}{GP^2} \approx 6.824 \times 10^{36}$ kg $\approx 3.4 \times 10^6\ M_\odot$

Calculate the speed of S2 when it is closest to, and when it is farthest from Sgr A* whose estimated mass is $3.7 \times 10^6\ M_\odot$. Take orbital eccentricity of S2 to be 0.87 and its perigalacticon distance 1.8×10^6 m.

Speed at perigalacticon, $v_p = \sqrt{\dfrac{GM_{BH}}{a_{S2}}\left(\dfrac{1+e}{1-e}\right)} \approx 1883428\sqrt{\dfrac{1+e}{1-e}} \approx 7.14 \times 10^6$ m/s

Speed at apgalacticon, $v_A = \sqrt{\dfrac{GM_{BH}}{a_{S2}}\left(\dfrac{1+e}{1-e}\right)} \approx 1883428\sqrt{\dfrac{1-e}{1+e}} \approx 0.5 \times 10^6$ m/s

Using Newtonian gravity, estimate the Roche limit of an SMBH of mass $3.7 \times 10^6\ M_\odot$ (assume that a 1 $M_\odot$ main-sequence star is tidally disrupted). How does the answer compare with the black hole's Schwarzschild radius?

Radius of Black hole, $R_{BH} \approx$ Schwarzschild radius, $R_S = \dfrac{2GM_{BH}}{c^2} \approx 1.1 \times 10^{10}$ m $\approx 15.8 R_\odot \approx 0.07$ au

Mean density of SMBH, $\bar{\rho}_{BH} = \dfrac{M_{BH}}{\frac{4\pi}{3}{R_{BH}}^3} = \dfrac{3.7 \times 10^6 M_\odot}{\frac{4\pi}{3}(15.8 R_\odot)^3} \approx (938.06)\dfrac{M_\odot}{\frac{4\pi}{3}{R_\odot}^3} = (938.06)\bar{\rho}_\odot$

Roche limit, $r = f_R\left(\dfrac{\bar{\rho}_{BH}}{\bar{\rho}_\odot}\right)^{1/3} R_{BH} = 2.456(938.06)^{1/3} R_S \approx 2.456(9.79) R_S \approx 24 R_S \approx 379 R_\odot \approx 1.76$ au

Compute the lowest possible density of Sgr A* based on the data obtained from the orbit of S2. Assume a spherical mass distribution. (mass of BH = $3.7 \times 10^6\ M_\odot$; perigalacticon distance of S2 = 1.8×10^{13} m)
Assuming a radius of 1 au (roughly the current limit of resolution of the center of the Milky Way), estimate the density of Sgr A*.
In this case, maximum possible radius (for minimum density) of Sgr A* is equal to the closest approach of S2.

For $M_{\text{BH}} = r_p$, $\rho_{\text{BH}} = \dfrac{M_{\text{BH}}}{\frac{4\pi}{3}{r_p}^3} = 3.7 \times 10^{-4}\ \text{kg/m}^3$

For $M_{\text{BH}} = 1$ au, $\rho_{\text{BH}} = \dfrac{M_{\text{BH}}}{\frac{4\pi}{3}(1\ \text{au})^3} = 524.75\ \text{kg/m}^3$

If the accretion rate at the galactic center is 10^{-3} M⊙ yr^{-1} and if it has remained constant over the past 5 billion years, how much mass has fallen into the center over that period of time? Compare the mass with the estimated mass of a possible SMBH residing at the center of galaxy.

Mass accreted in 5 billion years, $\Delta M = 10^{-3} \times 5 \times 10^9 M_\odot = 5 \times 10^6 M_\odot$ which is roughly the mass estimated of SMBH at the center of our galaxy.

Photon gas inside a cavity and Stefan-Boltzmann law

$$f(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/kT} - 1} = \frac{1}{e^{h\nu/kT} - 1} = \quad [\because \epsilon = h\nu \text{ and } \mu = 0 \text{ for photons}]$$

$$g(\epsilon)\, d\epsilon = \frac{4\pi V}{h^3 c^3}\epsilon^2\, d\epsilon = \frac{4\pi V}{h^3 c^3} h^2 \nu^2 (h\, d\nu) \Rightarrow g(\nu)\, d\nu = \frac{4\pi V}{c^3}\nu^2\, d\nu$$

$$n(\epsilon)\, d\epsilon = g_s f(\epsilon) g(\epsilon)\, d\epsilon \Rightarrow n(\nu)\, d\nu = \frac{8\pi V}{c^3}\frac{\nu^2}{e^{h\nu/kT} - 1} d\nu \quad \begin{bmatrix} g_s = 2 \text{ for photons due to the} \\ \text{two polarization directions.} \end{bmatrix}$$

$$N = \int_0^\infty n(\nu)\, d\nu = \frac{8\pi V}{c^3}\int_0^\infty \frac{\nu^2\, d\nu}{e^{h\nu/kT} - 1} = 8\pi V\frac{(kT)^3}{(hc)^3}\int_0^\infty \frac{x^2\, dx}{e^x - 1}\left[x = \frac{h\nu}{kT} \Rightarrow \nu = \frac{kT}{h}x \Rightarrow d\nu = \frac{kT}{h}dx\right] = 16\pi V\frac{(kT)^3}{(hc)^3}\zeta(3)$$

$$dU_R = n(\nu)\, d\nu \cdot h\nu = \frac{8\pi V}{c^3}\frac{h\nu^3}{e^{h\nu/kT} - 1}d\nu \Rightarrow \frac{1}{V}\frac{dU_R}{d\nu} = \frac{du_R}{d\nu} = \frac{8\pi}{c^3}\frac{h\nu^3}{e^{h\nu/kT} - 1} \quad \text{[Planck's radiation law]}$$

Low-temperature, high frequency: When $h\nu \gg kT$ i.e. $\dfrac{h\nu}{kT} \gg 1$, $e^{h\nu/kT} - 1 \approx e^{h\nu/kT}$ $\quad \therefore\ \dfrac{du_R}{d\nu} = \dfrac{8\pi}{c^3}\dfrac{h\nu^3}{e^{h\nu/kT}}$

High-temperature, low frequency: When $h\nu \ll kT$ i.e. $\dfrac{h\nu}{kT} \ll 1$, $e^{h\nu/kT} - 1 \approx \left(1 + \dfrac{h\nu}{kT}\right) - 1 = \dfrac{h\nu}{kT}$

$$\therefore\ \frac{du_R}{d\nu} = \frac{8\pi}{c^3}\frac{h\nu^3}{h\nu/kT} = \frac{8\pi}{c^3}kT\nu^2 \Rightarrow \frac{c}{4\pi}\frac{du_R}{d\nu} = \frac{c}{4\pi}u_\nu = \boldsymbol{B_\nu} = \boldsymbol{\frac{2kT}{c^2}\nu^2}$$

$$du_R = \frac{8\pi}{c^3}kT\nu^2 d\nu = \frac{8\pi}{c^3}kT\left(\frac{c}{\lambda}\right)^2 d\left(\frac{c}{\lambda}\right) = -\frac{8\pi kT}{\lambda^4}d\lambda \Rightarrow \frac{du_R}{d\lambda} = -\frac{8\pi kT}{\lambda^4} \Rightarrow \frac{c}{4\pi}\frac{du_R}{d\lambda} = \frac{c}{4\pi}u_\lambda = \boldsymbol{B_\lambda} = -\boldsymbol{\frac{2ckT}{\lambda^4}}$$

$$U_R = \int_0^\infty dU_R = \frac{8\pi hV}{c^3}\int_0^\infty \frac{\nu^3\, d\nu}{e^{h\nu/kT} - 1} = 8\pi V\frac{(kT)^4}{(hc)^3}\int_0^\infty \frac{x^3\, dx}{e^x - 1}\ \left[x = \frac{h\nu}{kT} \Rightarrow \nu = \frac{kT}{h}x \Rightarrow d\nu = \frac{kT}{h}dx\right] = 8\pi V\frac{(kT)^4}{(hc)^3}\left(\frac{\pi^4}{15}\right)$$

$$\therefore \text{(Internal) Energy density, } u_R = \frac{U_R}{V} = \frac{8\pi^5 k^4}{15(hc)^3}T^4 = \boldsymbol{aT^4} \quad \begin{bmatrix} \text{where } a = \dfrac{8\pi^5 k^4}{15(hc)^3} \approx 7.566 \times 10^{-16}\ \text{J/m}^3\text{K}^4 \\ \text{is called } \textbf{radiation constant.} \end{bmatrix}$$

$$\text{Radiant flux (Power emitted per unit area), } F_R = \int_0^\infty F_\nu\, d\nu = \int_0^\infty \left(\int_0^{2\pi}\int_0^{\pi/2} (B_\nu \cos\theta)\sin\theta\, d\theta\, d\phi\right) d\nu$$

$$= 2\pi\int_0^\infty B_\nu\, d\nu\int_0^{\pi/2} \cos\theta\sin\theta\, d\theta = 2\pi\int_0^\infty B_\nu\, d\nu\int_0^{\pi/2} \sin 2\theta\, d\theta = 2\pi\int_0^\infty B_\nu\, d\nu\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2} = \pi\int_0^\infty B_\nu\, d\nu$$

$$= \pi\int_0^\infty \left(\frac{c}{4\pi}u_\nu\right)d\nu = \frac{c}{4}\int_0^\infty u_\nu\, d\nu = \boldsymbol{\frac{c}{4}u_R} = \frac{ca}{4}T^4 \Rightarrow \boldsymbol{F_R = \sigma T^4} \quad \begin{bmatrix} \textbf{Stefan-Boltzmann law} \\ \text{where } \sigma = \dfrac{ca}{4} \approx 5.67 \times 10^{-8}\ \text{W/m}^2\text{K}^4 \end{bmatrix}$$

$$\text{Internal Energy, } U_R = 48\pi V\frac{(kT)^4}{(hc)^3}\zeta(4) = 3 \cdot \frac{\zeta(4)}{\zeta(3)}N(kT) = 3 \cdot \frac{\zeta(4)}{\zeta(3)}n_0 RT \quad \begin{bmatrix} n_0 = \text{number of moles};\\ \text{gas constant, } R = N_A k \end{bmatrix}$$

$$\Rightarrow U_R = 2.7 n_0 RT \quad \left[\text{as } \frac{\zeta(4)}{\zeta(3)} \approx \frac{1.082323}{1.202057} \approx 0.9004\right]$$

$$N = 16\pi V\frac{(kT)^3}{(hc)^3}\zeta(3) \Rightarrow N^{\frac{4}{3}} = \left(16\pi\,\zeta(3)\right)^{\frac{4}{3}}\frac{(kT)^4}{(hc)^4}V^{\frac{4}{3}} \Rightarrow (kT)^4 = \frac{(hc)^4 N^{\frac{4}{3}}}{\left(16\pi\,\zeta(3)\right)^{\frac{4}{3}}}V^{-\frac{4}{3}}$$

$$\therefore U_R = 48\pi V\frac{(kT)^4}{(hc)^3}\zeta(4) = \frac{48\pi\,\zeta(4)}{\left(16\pi\,\zeta(3)\right)^{\frac{4}{3}}}(hc)N^{\frac{4}{3}}V^{-\frac{1}{3}} \approx \begin{cases} 0.68837(hc)N^{\frac{4}{3}}V^{-\frac{1}{3}} \\ \left(\frac{3}{4\pi}\right)^{\frac{1}{3}}\frac{48\pi\,\zeta(4)}{\left(16\pi\,\zeta(3)\right)^{\frac{4}{3}}}(hc)\frac{N^{\frac{4}{3}}}{R} \approx 0.427(hc)\frac{N^{\frac{4}{3}}}{R} \approx 2.683(\hbar c)\frac{N^{\frac{4}{3}}}{R} \end{cases}$$

$$\text{Radiation Pressure, } P_R = -\left(\frac{\partial U_R}{\partial V}\right)_{T,N} = \frac{1}{3}(0.68837\,hc)\frac{N^{\frac{4}{3}}}{V^{\frac{4}{3}}} = \begin{cases} \frac{1}{3}(0.68837\,hc)n^{\frac{4}{3}} \quad \left[\text{where } n = \frac{N}{V}\right] \\ \boldsymbol{\frac{1}{3}\frac{U_R}{V} \Rightarrow P_R V = \frac{1}{3}U_R = \frac{1}{3}aT^4 \Rightarrow U_R = 3P_R V} \end{cases}$$

[Thus, the (hydrostatic) **pressure, $\boldsymbol{P_R}$,** of isotropic radiation is **one-third** of its **energy-density, $\boldsymbol{U_R/V}$**.
Note that in this derivation the integral adopted to find N is analogous to the pressure-integral.]

$$\Rightarrow P_R V^{\frac{4}{3}} = \frac{N^{\frac{4}{3}}}{3}(0.68837\,hc) \Rightarrow \boldsymbol{\gamma' = \frac{4}{3}} \textbf{ for photon gas} \quad \left[\text{comparing with the polytropic equation, } PV^{\gamma'} = K\right]$$

$$= \frac{1}{3}(hc)\left(16\pi\frac{(kT)^3}{(hc)^3}\zeta(3)\right)^{\frac{4}{3}} = \frac{1}{3}\frac{k^4T^4}{(hc)^3}\left(16\pi\zeta(3)\right)^{\frac{4}{3}}$$

Tophon gas

Tophons are massless spin-0 (hypothetical) bosons.

$$n(\epsilon)\,d\epsilon = g_s f(\epsilon)g(\epsilon)\,d\epsilon \Rightarrow n(\nu)\,d\nu = \frac{4\pi V}{c^3}\frac{\nu^2}{e^{h\nu/kT}-1}d\nu \quad [g_s = 1 \text{ for tophons}]$$

$$N = \int_0^\infty n(\nu)\,d\nu = 8\pi V\frac{(kT)^3}{(hc)^3}\zeta(3) \Rightarrow N^{\frac{4}{3}} = \left(8\pi\,\zeta(3)\right)^{\frac{4}{3}}\frac{(kT)^4}{(hc)^4}V^{\frac{4}{3}} \Rightarrow (kT)^4 = \frac{(hc)^4N^{\frac{4}{3}}}{\left(8\pi\,\zeta(3)\right)^{\frac{4}{3}}}V^{-\frac{4}{3}}$$

$$dU_R = n(\nu)\,d\nu\cdot h\nu = \frac{4\pi V}{c^3}\frac{h\nu^3}{e^{h\nu/kT}-1}d\nu \quad \therefore U_R = \int_0^\infty dU_R = \frac{4\pi hV}{c^3}\int_0^\infty\frac{\nu^3\,d\nu}{e^{h\nu/kT}-1}$$

$$= 4\pi V\frac{(kT)^4}{(hc)^3}\left(\frac{\pi^4}{15}\right) = 24\pi V\frac{(kT)^4}{(hc)^3}\zeta(4) = \frac{24\pi\,\zeta(4)}{\left(8\pi\,\zeta(3)\right)^{\frac{4}{3}}}(hc)N^{\frac{4}{3}}V^{-\frac{1}{3}} = \begin{cases} 0.86729(hc)N^{\frac{4}{3}}V^{-\frac{1}{3}} \\ 3.38051(\hbar c)\frac{N^{\frac{4}{3}}}{R} \end{cases}$$

Gravitation-Radiation star (G-R star) and minimum mass of a Black Hole (BH)

G-R star is a hypothetical star (may be BH) in which the radiation pressure balances the gravitational pressure.

$$P_G = P_R \Rightarrow \frac{1}{3}\frac{U_G}{V} = \frac{1}{3}\frac{U_R}{V} \Rightarrow U_G = U_R \Rightarrow \frac{3}{5}\frac{GM^2}{R} = \left(2.683113N_\gamma^{\frac{4}{3}} + 1.43937N_e^{\frac{4}{3}} + 1.81349N_\nu^{\frac{4}{3}}\right)\frac{\hbar c}{R}$$

$$\left.\begin{matrix} n \to p^+ + e^- + \bar{\nu}_e \\ p^+ \to n + e^+ + \nu_e \\ \bar{\nu}_e + \nu_e \to \gamma + \gamma \\ e^- + e^+ \to \gamma + \gamma \end{matrix}\right\} \begin{cases} \text{If only photons constitute the radiation, } \frac{3}{5}GM^2 = 2.683113N_\gamma^{\frac{4}{3}}\hbar c \\ \text{If only tophons constitute the radiation, } \frac{3}{5}GM^2 = 3.38051N_\Gamma^{\frac{4}{3}}\hbar c \\ \text{If only electrons constitute the radiation, } \frac{3}{5}GM^2 = 1.43937N_e^{\frac{4}{3}}\hbar c \\ \text{If only neutrinos constitute the radiation, } \frac{3}{5}GM^2 = 1.81349N_\nu^{\frac{4}{3}}\hbar c \\ \text{If } N_\gamma = N_e = N_\nu = N\text{, then } \frac{3}{5}GM^2 = 5.935973N^{\frac{4}{3}}\hbar c \end{cases}$$

$$\therefore \begin{cases} \text{Number of photons to prevent collapse, } N_\gamma = 0.325188\left(\dfrac{M}{m_\mathrm{P}}\right)^{\frac{3}{2}} = 1.012757\times10^{11}M^{\frac{3}{2}} \\ \text{Number of tophons to prevent collapse, } N_\Gamma = 0.27345\left(\dfrac{M}{m_\mathrm{P}}\right)^{\frac{3}{2}} = 8.51626\times10^{10}M^{\frac{3}{2}} \\ \text{Number of electrons to prevent collapse, } N_e = 0.51878\left(\dfrac{M}{m_\mathrm{P}}\right)^{\frac{3}{2}} = 1.61568\times10^{11}M^{\frac{3}{2}} \\ \text{Number of neutrinos to prevent collapse, } N_\nu = 0.43624\left(\dfrac{M}{m_\mathrm{P}}\right)^{\frac{3}{2}} = 1.35862\times10^{11}M^{\frac{3}{2}} \\ \text{Number of ultra-relativistic particles to avoid collapse, } N = 0.1792646\left(\dfrac{M}{m_\mathrm{P}}\right)^{\frac{3}{2}} = 5.583\times10^{10}M^{\frac{3}{2}} \end{cases}$$

$$\begin{cases} \text{If } N_\gamma = 1,\ M = 4.6\times10^{-8}\text{ kg} \Rightarrow \boldsymbol{M = 2.1\, m_\mathrm{P}} \\ \text{If } N_\Gamma = 1,\ M = 5.2\times10^{-8}\text{ kg} \Rightarrow \boldsymbol{M = 2.4\, m_\mathrm{P}} \\ \text{If } N_e = 1,\ M = 3.4\times10^{-8}\text{ kg} \Rightarrow \boldsymbol{M = 1.5\, m_\mathrm{P}} \\ \text{If } N_\nu = 1,\ M = 3.8\times10^{-8}\text{ kg} \Rightarrow \boldsymbol{M = 1.7\, m_\mathrm{P}} \\ \text{If } N = 1,\ M = 6.8\times10^{-8}\text{ kg} \Rightarrow \boldsymbol{M = 3.1\, m_\mathrm{P}} \\ \text{If } N = 2,\ M = 1.1\times10^{-7}\text{ kg} \Rightarrow \boldsymbol{M = 5.0\, m_\mathrm{P}} \end{cases} \quad \text{[minimum Black Hole mass]}$$

According to Quantum theory, minimum BH mass is derived by equating Schwarzschild radius to reduced Compton wavelength, i.e. $\lambda\!\!\!^{-}_\mathrm{C} = R_\mathrm{S}$ $[\lambda_\mathrm{C} = 2\pi R_\mathrm{S}] \Rightarrow \dfrac{\hbar}{mc} = \dfrac{2GM}{c^2} \Rightarrow M = \dfrac{1}{\sqrt{2}}m_\mathrm{P}$ $\left[m_\mathrm{P} = \sqrt{\dfrac{\hbar c}{G}}\right] \Rightarrow \boldsymbol{M = 0.7m_\mathrm{P}}$

Note that the minimum mass of a gravitophoton BH (which represents the maximum mass whose self-gravity can be countered by the least radiation pressure) is 2.99 times the mass of a quantum mechanical BH. So three $0.7\boldsymbol{m_\mathrm{P}}$ BH in each of the three dimensions comprise a $2.1\boldsymbol{m_\mathrm{P}}$ BH. (Quantum mechanical BH is 2D, and therefore, non-physical.) Moreover, such tiny mass BHs might appear as massive particles with mass of the order of $\boldsymbol{m_\mathrm{P}}$.

If $M = M_\odot$ [solar mass Black Hole], $N = 2.1618\times10^{57} = 3.59\times10^{33}N_\mathrm{A} = 3.59\times10^{33}$ moles

Calculate electric (Coulomb) potential energy of a semi-classical atom.

A **semi-classical atom** consists of a uniformly-charged positive tiny sphere of radius R_0 and a uniformly-charged negative electron cloud extending from the first Bohr's orbit (n = 1) at radius R_1 to the atomic radius R (last orbit).

For the positively charged nucleus of radius R_0, $U_{C+} = \dfrac{3}{5}\left(\dfrac{1}{4\pi\varepsilon_0}\right)\dfrac{Q^2}{R_0} = \dfrac{3}{5}\left(\dfrac{1}{4\pi\varepsilon_0}\right)\dfrac{(Ze)^2}{R_0}$

For the negatively charged electron cloud of radius r, proceed carefully as follows:

$$dq = (4\pi r^2 dr)\rho = (4\pi r^2 dr)\left(\frac{-Q}{\frac{4\pi}{3}R^3 - \frac{4\pi}{3}{R_0}^3}\right) = \frac{-3Qr^2dr}{R^3 - {R_0}^3}$$

$$q(r) = Q + (-Q)\left(\frac{\frac{4\pi}{3}r^3 - \frac{4\pi}{3}{R_0}^3}{\frac{4\pi}{3}R^3 - \frac{4\pi}{3}{R_0}^3}\right) = Q\left(1 - \frac{r^3 - {R_0}^3}{R^3 - {R_0}^3}\right) = Q\left(\frac{R^3 - r^3}{R^3 - {R_0}^3}\right)$$

$$dW = V(r)\,dr = \frac{1}{4\pi\varepsilon_0}\frac{q(r)}{r}dq = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\left(\frac{R^3 - r^3}{R^3 - {R_0}^3}\right)\left(\frac{-3Qr^2dr}{R^3 - {R_0}^3}\right) = \frac{1}{4\pi\varepsilon_0}\frac{-3Q^2}{\left(R^3 - {R_0}^3\right)^2}(R^3 - r^3)r\,dr$$

$$\therefore U_{C-} = W = \int_{R_0}^{R_1} V(r)\,dr = \frac{-3Q^2}{4\pi\varepsilon_0\left(R^3 - {R_0}^3\right)^2}\int_{R_1}^{R}(R^3 - r^3)r\,dr = \frac{-3Q^2}{4\pi\varepsilon_0\left(R^3 - {R_0}^3\right)^2}\left\{\frac{R^3\left(R^2 - {R_1}^2\right)}{2} - \frac{\left(R^5 - {R_1}^5\right)}{5}\right\}$$

$$= \frac{-3Q^2}{40\pi\varepsilon_0\left(R^3 - {R_0}^3\right)^2}\left\{5R^3\left(R^2 - {R_1}^2\right) - 2\left(R^5 - {R_1}^5\right)\right\} = \frac{-3Q^2}{40\pi\varepsilon_0\left(R^3 - {R_0}^3\right)^2}\left\{3R^5 - 5R^3{R_1}^2 + 2{R_1}^5\right\}$$

Total Coulomb potential energy of the atom $= U_{C+} + U_{C-} = \dfrac{3Q^2}{20\pi\varepsilon_0}\left\{\dfrac{1}{R_0} - \dfrac{3}{2\left(R^3 - {R_0}^3\right)^2}\left(3R^5 - 5R^3{R_1}^2 + 2{R_1}^5\right)\right\}$

$$= \frac{3Q^2}{20\pi\varepsilon_0 R_0}\left\{1 - \frac{3R_0}{2R^6}\left(3R^5 - 5R^3{R_1}^2 + 2{R_1}^5\right)\right\} \quad [\because R \gg R_0] = \frac{3Q^2}{20\pi\varepsilon_0 R_0}\left\{1 - \frac{3R_0}{2R}\left(3 - \frac{5{R_1}^2}{R^2} + \frac{2{R_1}^5}{R^5}\right)\right\}$$

For Hydrogen and Helium atoms, $R = R_1 \quad \therefore U_C = \dfrac{3Q^2}{20\pi\varepsilon_0 R_0}\left\{1 - \dfrac{3R_0}{2R}(3 - 5 + 2)\right\} = \dfrac{3Q^2}{20\pi\varepsilon_0 R_0}$

Dynamical timescale (Free-fall timescale)

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} \Rightarrow 2\frac{dr}{dt}\left(\frac{d^2r}{dt^2}\right) = -\frac{2GM}{r^2}\frac{dr}{dt} \Rightarrow d\left(\frac{dr}{dt}\right)^2 = -2GM\frac{dr}{r^2} \Rightarrow \left(\frac{dr}{dt}\right)^2 = \frac{2GM}{r} + C_1 \quad \left[\begin{array}{c}\text{Mass, } M\text{, remains}\\ \text{same during gravi-}\\ \text{-tational collapse}\end{array}\right]$$

$$\Rightarrow \frac{dr}{dt} = \sqrt{\frac{2GM}{r} + C_1} = \sqrt{\frac{2GM}{r} - \frac{2GM}{R}} \quad \left[\because \left.\frac{dr}{dt}\right|_{r=R} = 0\right] = \sqrt{\frac{2GM}{R}}\sqrt{\frac{R}{r} - 1} \Rightarrow \sqrt{\frac{R}{2GM}}\frac{dr}{\sqrt{\frac{R}{r} - 1}} = dt$$

$$\Rightarrow t_{\text{ff}} = \sqrt{\frac{R}{2GM}}\int_R^0 \frac{dr}{\sqrt{\frac{R}{r} - 1}} \Rightarrow t_{\text{ff}} = \sqrt{\frac{R}{2GM}}\left[\left\{R\sin^{-1}\sqrt{\frac{r}{R}} - \sqrt{r(R - r)}\right\}\right]_R^0 \quad \left[\begin{array}{c}\text{Put } r = R\cos\theta\\ \text{to solve the integral.}\end{array}\right] = \sqrt{\frac{R}{2GM}}\left\{\frac{\pi}{2}R\right\}$$

$$\Rightarrow \boldsymbol{t_{\text{ff}} = \frac{\pi}{2}\sqrt{\frac{R^3}{2GM}}} \Rightarrow \boldsymbol{t_{\text{ff}} = \sqrt{\frac{3\pi}{32G\rho}}} \quad \left[\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} \Rightarrow \frac{R^3}{M} = \frac{3}{4\pi\rho}\right] \approx 0.66 \times 10^5\ (\text{kg-s}^2/\text{m}^3)^{\frac{1}{2}}\rho^{-\frac{1}{2}}$$

$\therefore$ For Sun, $t_{\text{ff}\odot} = \dfrac{\pi}{2}\sqrt{\dfrac{{R_\odot}^3}{2GM_\odot}} \approx 1769\text{ s} \approx 29.5\text{ min} \approx 30\text{ min}$

Assuming the interstellar cloud with sufficient mass to contract gravitationally, find the free-fall time of the cloud and the mass of the cloud if the density of the contracting cloud is ρ_0 = 10^{-15} kg/m^3 and the initial radius is 0.5 pc.

$$t_{\text{ff}} = \sqrt{\frac{3\pi}{32G\rho_c}} \approx 2.1 \times 10^{12}\text{ s} \approx 66572.7\text{ yr} \approx 66.57\text{ kyr}$$

$$M_c = \rho_c V = \rho_c\left(\frac{4\pi}{3}{R_c}^3\right) = \frac{4\pi}{3}\rho_c{R_c}^3 \approx 1.54 \times 10^{34}\text{ kg} \approx 7737\ M_\odot$$

How long would a test particle take to reach the center of a uniformly dense sphere of mass *M* from its surface?

$$\frac{d^2r}{dt^2} = -\frac{GM(r)}{r^2} = -\frac{G}{r^2}\left\{\frac{M}{\frac{4\pi}{3}R^3}\left(\frac{4\pi}{3}r^3\right)\right\} = -\frac{G}{r^2}\left(\frac{Mr^3}{R^3}\right) \Rightarrow \frac{d^2r}{dt^2} = -\frac{GM}{R^3}r \Rightarrow r = R\cos\sqrt{\frac{GM}{R^3}}t \quad \left[\begin{array}{c}\text{SHM}\\ \text{equation}\\ r = R\\ \text{at } t = 0\end{array}\right]$$

$$\Rightarrow \boldsymbol{t_{\text{fall}} = \frac{\pi}{2}\sqrt{\frac{R^3}{GM}} = \sqrt{2}\,t_{\text{ff}}} \quad \therefore \text{For Sun, } t_{\text{fall}\odot} = \sqrt{2}\,t_{\text{ff}\odot} \approx 2502\text{ s} \approx 41.7\text{ min} \approx 42\text{ min}$$

Kelvin-Helmholtz timescale (Thermal timescale)

Total mechanical energy of a star, $E = K + U = -\dfrac{1}{2}U + U \quad \left[\begin{array}{c}\because 2K + U = 0\\ \text{(Virial theorem)}\end{array}\right] = \dfrac{1}{2}U = -\dfrac{3}{10}\dfrac{GM^2}{R}$

$$\frac{dE}{dt} = -L \Rightarrow t_{\text{KH}} = -\frac{1}{L}\int_{-\frac{3}{10}\frac{GM^2}{R}}^{0} dE \quad \left[\begin{array}{c}\text{assuming a constant}\\ \text{luminosity throughout}\\ \text{the star's lifetime}\end{array}\right] \Rightarrow \boldsymbol{t_{\text{KH}} = \frac{3}{10}\frac{GM^2}{RL}}$$

$\therefore$ For Sun, $t_{\text{KH}\odot} = \dfrac{3}{10}\dfrac{G{M_\odot}^2}{R_\odot L_\odot} \approx 9.4 \times 10^6\text{ yrs} \approx 10^7\text{ yrs}$

Microscopic collision timescale (electron-photon timescale)

Mean free path between successive scatterings (collisions), $\bar{\lambda} = \dfrac{1}{n_e \sigma} \approx \dfrac{m_\mathrm{H}}{\bar{\rho}\sigma_\mathrm{T}}$ $\left[\begin{matrix} n_e \approx \dfrac{\bar{\rho}}{m_\mathrm{H}} \\ \sigma \approx \sigma_\mathrm{T} \end{matrix}\right]$

$\because \sigma_\mathrm{T} = \dfrac{8\pi}{3}\left(\dfrac{e^2}{m_e c^2}\right)^2 \approx 6.65 \times 10^{-29}\ \mathrm{m}^2$ [for electron] and $\bar{\rho}_\odot \approx 1410.2\ \mathrm{kg/m^3}$ $\therefore \bar{\lambda}_\odot \approx 0.018\ \mathrm{m} = 1.8\ \mathrm{cm}$

$\therefore$ Collision time, $\boldsymbol{t_{e\gamma} = \dfrac{\bar{\lambda}}{c}}$ $\therefore t_{e\gamma\odot} = \dfrac{\bar{\lambda}_\odot}{c} \approx 6 \times 10^{-11}\ \mathrm{s} = 0.6\ \mathrm{ps}$

Photodiffusion timescale

Photons escape by a random walk of timescale of step-length λ. The RMS distance after N steps is $s_N = \bar{\lambda}\sqrt{N}$
Number of steps to journey from center to surface is $\boldsymbol{N} = \left(s_N/\bar{\lambda}\right)^2 \approx \boldsymbol{(R/\bar{\lambda})^2}$ $\therefore N_\odot = \left(R_\odot/\bar{\lambda}_\odot\right)^2 \approx 1.5 \times 10^{21}$
$\therefore$ Photodiffusion time, $\boldsymbol{t_\mathrm{diff} = N t_{e\gamma}}$ $\therefore t_\mathrm{diff} = N_\odot t_{e\gamma\odot} \approx 9 \times 10^{10}\ \mathrm{s} \approx 2852\ \mathrm{yrs}$

Conversion fraction of Hydrogen fusion into Helium
Atomic mass approach

$m_\mathrm{H} = m_\mathrm{p} + m_\mathrm{e} - \dfrac{13.6\ \mathrm{eV}}{c^2} = 1.6735325 \times 10^{-27}\ \mathrm{kg} = 1.007825\ \mathrm{u}$

$\therefore$ Mass defect, $\Delta m = 4m_\mathrm{H} - m_\mathrm{He} = 4 \times 1.007825\ \mathrm{u} - 4.0026\ \mathrm{u} = 0.0287\ \mathrm{u}$

$\therefore$ Conversion fraction $= \dfrac{0.0287\ \mathrm{u}}{4 \times 1.007825\ \mathrm{u}} \approx 0.00712 = 0.712\% \approx 0.7\% = 0.007$

Nuclear mass approach

$m_\alpha = 2m_\mathrm{p} + 2m_\mathrm{n} - \dfrac{28.3\ \mathrm{MeV}}{c^2} = 6.644648 \times 10^{-27}\ \mathrm{kg} = 4.0015\ \mathrm{u}$

Mass defect, $\Delta m = 4 \times m_\mathrm{p} - m_\alpha = 4 \times 1.007276467\ \mathrm{u} - 4.0015\ \mathrm{u} = 0.0276\ \mathrm{u}$

$\therefore$ Conversion fraction $= \dfrac{0.0276\ \mathrm{u}}{4 \times 1.007276467\ \mathrm{u}} \approx 0.00685 = 0.685\% \approx 0.7\% = 0.007$

Nuclear timescale

Heat released on fusing ΔM mass of ${}^1_1\mathrm{H}$ into ${}^4_2\mathrm{He}$ is approximately $0.007\Delta M c^2$. Therefore, the time required to exhaust all the star's hydrogen (assuming the star was originally 100% hydrogen) at current luminosity is $\boldsymbol{t_\mathrm{nuc} = \dfrac{0.007Mc^2}{L}}$ $\therefore t_{\mathrm{nuc}\odot} = \dfrac{0.007M_\odot c^2}{L_\odot} \approx 10^{11}$ yrs However, the actual lifetime of the star is about 1/10-th of this (i.e. 10^{10} yrs) because its luminosity will increase rapidly (multifold) when it becomes a red giant.

Relativistic Force

$$\vec{p} = \gamma m\vec{v} = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} m\vec{v} \Longrightarrow \vec{F} = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} m\frac{d\vec{v}}{dt} + \frac{1}{2}m\vec{v}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left(\frac{2\vec{v}}{c^2} \cdot \frac{d\vec{v}}{dt}\right) = \gamma m\vec{a} + \gamma^3 m\frac{\vec{v}}{c^2}(\vec{v} \cdot \vec{a})$$

$$= \gamma m\vec{a} + \gamma^3 m\frac{\vec{v}}{c^2}(\vec{v} \cdot \vec{a}) = \gamma m\vec{a} + \gamma^3 m\frac{\vec{v}}{c^2}(\vec{v} \cdot \vec{a}_\parallel + \vec{v} \cdot \vec{a}_\perp) = \gamma m\vec{a} + \gamma^3 m\frac{\vec{v}}{c^2}(\vec{v} \cdot \vec{a}_\parallel) = \gamma m\vec{a} + \gamma^3 m\frac{v^2}{c^2}\vec{a}_\parallel$$

$$= \gamma m\vec{a}_\perp + \gamma^3 m\vec{a}_\parallel\left(\frac{1}{\gamma^2} + \frac{v^2}{c^2}\right) = \gamma m\vec{a}_\perp + \gamma^3 m\vec{a}_\parallel\left(\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2}{c^2}\right) \Longrightarrow \boldsymbol{\vec{F} = \gamma m\vec{a}_\perp + \gamma^3 m\vec{a}_\parallel}$$

$[\because \vec{v} \times (\vec{v} \times \vec{a}_\parallel) = \vec{v}(\vec{v} \cdot \vec{a}_\parallel) - \vec{a}_\parallel(\vec{v} \cdot \vec{v}) \Longrightarrow 0 = \vec{v}(\vec{v} \cdot \vec{a}_\parallel) - v^2\vec{a}_\parallel \Longrightarrow \vec{v}(\vec{v} \cdot \vec{a}_\parallel) = v^2\vec{a}_\parallel]$

$$\Longrightarrow \begin{cases} \vec{F} = m\vec{a}_\perp + m\vec{a}_\parallel = m\vec{a} & \text{when } v = 0 \\ \vec{F} = \infty\vec{a}_\perp + \infty^3\vec{a}_\parallel = \infty^3\vec{a}_\parallel & \text{when } v = c \end{cases}$$

Determine the Doppler shift for a photon of wavelength 632.8 nm emitted by a gas particle moving with a velocity of 10^3 m/s.

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c} \Longrightarrow \Delta\lambda = \frac{v}{c}\lambda = \frac{632.8 \times 10^{-9}\ \mathrm{m}}{1000\ \mathrm{m/s}} = 2.11 \times 10^{-12}\ \mathrm{m} = 2.11\ \mathrm{pm} = 0.002\ \mathrm{nm}$$

$$\nu = \frac{c}{\lambda} = \frac{c}{632.8 \times 10^{-9}\ \mathrm{m}} = \frac{c}{632.8 \times 10^{-9}\ \mathrm{m}} = 473.8\ \mathrm{GHz}$$

$$\nu = \frac{c}{\lambda} \Longrightarrow \Delta\nu = -\frac{c}{\lambda^2}\Delta\lambda = -\frac{c}{(632.8 \times 10^{-9}\ \mathrm{m})^2}(2.11 \times 10^{-12}\ \mathrm{m}) = 1.58\ \mathrm{GHz}$$

Luminous Blue Variables (LBVs)
Wolf-Rayet (WR) stars

<u>OPTICAL TELESCOPE</u>

<u>Refracting telescopes (Refractors)</u> use lenses (dioptrics)
Galilean telescope (GLT)
Keplerian telescope (KT)
Achromatic refractor (AR)
Apochromatic refractor (ApR)
<u>Reflecting telescopes (Reflectors)</u> use mirrors (catoptrics)
Gregorian telescope (GGT)
Newtonian telescope (NT)
Cassegrain telescope (CT)
Ritchey–Chrétien telescope (RCT)
Dall-Kirkham telescope (DKT)
Modified Dall-Kirkham telescope (MDKT)
Schiefspiegler telescope (ST) / Kutter telescope
Catadioptric Cassegrain (CCT)
  Schmidt-Cassegrain (SCT)
  Maksutov-Cassegrain (MCT)
  Argunov-Cassegrain (ACT)
  Klevtsov-Cassegrain (KCT)
Off-axis reflector (OAR)
  Herschelian telescope (HT)
  Schiefspiegler telescope (ST) / Kutter telescope
  Stevick-Paul telescope (SPT)
  Yolo telescope (YT)
<u>Catadioptric telescopes (Catadioptres)</u> use both lenses and mirrors
Maksutov telescope (MT)
Maksutov-Cassegrain (MCT)

Lamb shift

Casimir effect
Electrostatic force per unit area b/w two large parallel charged conducting static plates placed in the vacuum separated by a tiny distance, $\frac{F_Q}{A} = -\frac{\varepsilon_0 V^2}{2d^2}$
Casimir force per unit area b/w two large parallel uncharged conducting static plates placed in the vacuum separated by a tiny distance, $\frac{F_C}{A} = -\frac{\pi^2 \hbar c}{240 d^4}$

$\therefore$ To determine the equivalent Coulomb potential (Casimir potential) b/w the two plates, $\dfrac{\varepsilon_0 V^2}{2d^2} = \dfrac{\pi^2 \hbar c}{240 d^4}$

$\Longrightarrow \boldsymbol{V = \dfrac{\pi}{d}\sqrt{\dfrac{\hbar c}{120\varepsilon_0}}}$ If $d = 1\ \mu\text{m} = 1 \times 10^{-6}$ m, $V_C \approx 17.137$ mV

$\therefore$ Equivalent Coulomb charge (Casimir charge) on each plate, $Q_C = CV = \dfrac{\pi A}{d^2}\sqrt{\dfrac{\hbar c \varepsilon_0}{120}}$ $\left[\because C = \dfrac{\varepsilon_0 A}{d}\right]$

$\Longrightarrow \boldsymbol{\sigma_C} = \dfrac{Q_C}{A} = \boldsymbol{\dfrac{\pi}{d^2}\sqrt{\dfrac{\hbar c \varepsilon_0}{120}}}$ If $d = 1\ \mu\text{m} = 1 \times 10^{-6}$ m, $\sigma_C \approx 0.152\ \mu\text{C/m}^2$

Nuclear Reaction

$x + X \rightarrow Y + y$ where a is the projectile (incident/incoming particle), X is the target nucleus, Y is the residual nucleus, and b is the ejectile (emergent/outgoing particle). In shorthand, it is written as $X(x, y)Y$.

Conservation laws

1. Conservation of mass number (number of nucleons)
2. Conservation of atomic number (charge)
3. Conservation of energy (mass-energy)
4. Conservation of linear momentum
5. Conservation of angular momentum
6. Conservation of parity (except for weak interactions)
7. Conservation of isotopic spin (isospin)

Types of Nuclear Reactions

1. Elastic Scattering: Projectile (incident particle) is same as the Ejectile (emergent particle) without loss of energy and **angular** momentum, but with change in direction so that the target nucleus is the in the same (ground) state.
Representation: $X(x, x)X$
Example: ${}^{1}_{0}\text{n} + {}^{208}_{82}\text{Pb} \rightarrow {}^{208}_{82}\text{Pb} + {}^{1}_{0}\text{n}$

2. Inelastic Scattering: Projectile (incident particle) is same as the Ejectile (emergent particle) with less energy and different **angular** momentum so that the target nucleus is left in an excited state which soon returns to ground state by emitting gamma radiation.
Representation: $X(x, x)X^*$
Example: ${}^{1}_{0}\text{n} + {}^{208}_{82}\text{Pb} \rightarrow {}^{208}_{82}\text{Pb}^* \rightarrow {}^{208}_{82}\text{Pb} + {}^{1}_{0}\text{n} + \gamma$

3. Radiative capture: Projectile (incident particle) is **absorbed** by the target nucleus to form an excited compound nucleus which soon settles down to the ground state by emitting gamma radiation.
Representation: $X(x, \gamma)Y$
Example: ${}^{1}_{0}\text{H} + {}^{207}_{81}\text{Tl} \rightarrow {}^{208}_{82}\text{Pb}^* \rightarrow {}^{208}_{82}\text{Pb} + \gamma$

4. Disintegration process: Projectile (incident particle) is **absorbed** by the target nucleus and a different particle is emitted as Ejectile along with gamma radiation.
Representation: $X(x, y)Y$
Example: ${}^{4}_{2}\text{He} + {}^{14}_{7}\text{N} \rightarrow {}^{17}_{8}\text{O} + {}^{1}_{1}\text{H}$

5. Photodisintegration: When the target nucleus is bombarded with high-energy gamma rays, it is raised to an excited state and then subsequently disintegrates.

Representation: $X(\gamma, y)Y$
Example: $\gamma + {}_{4}^{9}\mathrm{Be} \rightarrow {}_{4}^{9}\mathrm{Be}^{*} \rightarrow {}_{4}^{8}\mathrm{Be} + {}_{0}^{1}\mathrm{n}$

6. Many-body reaction (Spallation): When the kinetic energy of the projectile is high, two or more particles can come out of the compound nucleus.
Representation: $X(x, y_1 y_2 y_3 \ldots)Y$
Example: $\begin{cases} {}_{1}^{1}\mathrm{H} + {}_{8}^{16}\mathrm{O} \rightarrow {}_{7}^{15}\mathrm{N} + 2\,{}_{1}^{1}\mathrm{H} \\ {}_{1}^{1}\mathrm{H} + {}_{8}^{16}\mathrm{O} \rightarrow {}_{7}^{14}\mathrm{N} + 3\,{}_{1}^{1}\mathrm{H} \\ {}_{1}^{1}\mathrm{H} + {}_{8}^{16}\mathrm{O} \rightarrow {}_{8}^{15}\mathrm{O} + {}_{1}^{1}\mathrm{H} + {}_{0}^{1}\mathrm{n} \end{cases}$

7. Heavy ion reaction: When the projectile is an ion heavier than α-particle, various types of products are produced.
Representation: $X(x, y)Y$
Example: $\begin{cases} {}_{8}^{16}\mathrm{O} + {}_{5}^{10}\mathrm{B} \rightarrow {}_{11}^{22}\mathrm{Na} + {}_{2}^{4}\mathrm{He} \\ {}_{7}^{14}\mathrm{N} + {}_{7}^{14}\mathrm{N} \rightarrow {}_{7}^{15}\mathrm{N} + {}_{7}^{13}\mathrm{N} \end{cases}$

8. Nuclear fission: When the target nucleus disintegrates into nuclei with comparable masses, the reaction is called nuclear fission.
Representation: $X(x, y_1 y_2 y_3 \ldots)Y$
Example: ${}_{0}^{1}\mathrm{n} + {}_{92}^{235}\mathrm{U} \rightarrow {}_{92}^{236}\mathrm{U}^{*} \rightarrow {}_{56}^{141}\mathrm{Ba} + {}_{36}^{92}\mathrm{Kr} + 3\,{}_{0}^{1}\mathrm{n}$

Elementary particle reaction

Direct nuclear reaction

Pick-up reaction: Projectile picks up one or more nucleons from the target nucleus.
Representation: $X(x, y)Y$
Example: ${}_{1}^{1}\mathrm{H} + {}_{29}^{64}\mathrm{Cu} \rightarrow {}_{29}^{63}\mathrm{Cu} + {}_{1}^{2}\mathrm{H}$
Stripping reaction: A composite projectile loses one or more nucleons to the target nucleus.
Representation: $X(x, y)Y$
Example: ${}_{1}^{2}\mathrm{H} + {}_{29}^{63}\mathrm{Cu} \rightarrow {}_{29}^{64}\mathrm{Cu} + {}_{1}^{1}\mathrm{H}$

Compound nuclear reaction

Energetics of nuclear reaction
During a nuclear reaction, energy is either evolved (exergic) or absorbed (endergic). The net energy evolved is called the Q-value (disintegration energy) of the reaction: $x + X \rightarrow Y + y$
$\therefore Q = E_Y + E_y - E_x - E_X$ where E denotes the kinetic energy.
By conservation of mass-energy, $M_x c^2 + E_x + M_X c^2 + E_X = M_Y c^2 + E_Y + M_y c^2 + E_y$
$\Longrightarrow E_Y + E_y - E_x - E_y = (M_x + M_X - M_Y - M_y)c^2 = B_y + B_Y - B_X - B_x$ where B denotes the binding energy.
[$\because$ the mass number and the atomic number are both conserved.] $\therefore \boldsymbol{Q = (M_x + M_X - M_Y - M_y)c^2}$
$\therefore Q > 0 \Longrightarrow M_x + M_X > M_Y + M_y$ for an exergic reaction, and $Q < 0 \Longrightarrow M_x + M_X < M_Y + M_y$ for an endergic reaction in which case the energy deficit has to be supplied through the kinetic energy of the projectile for the reaction to be feasible.

Threshold energy of an endergic reaction (non-relativistic case)
From the conservation of linear momentum along and perpendicular to the trajectory of the projectile,

$$\begin{cases} (p_x)_x = (p_Y)_x + (p_y)_x \\ (p_x)_y = 0 = (p_Y)_y + (p_y)_y \end{cases} \quad \begin{bmatrix} \text{Assuming the target nucleus to be at rest} \\ \text{and very heavy so that the recoil is negligible.} \end{bmatrix}$$

$$\Rightarrow \begin{cases} \sqrt{2M_xE_x} = \sqrt{2M_YE_Y}\cos\phi + \sqrt{2M_yE_y}\cos\theta \\ 0 = \sqrt{2M_YE_Y}\sin\phi + \sqrt{2M_yE_y}\sin\theta \end{cases} \quad [\because p = \sqrt{2ME} \text{ in the non-relativistic regime.}]$$

$$\Rightarrow \begin{cases} \sqrt{2M_xE_x} - \sqrt{2M_yE_y}\cos\theta = \sqrt{2M_YE_Y}\cos\phi \\ -\sqrt{2M_yE_y}\sin\theta = \sqrt{2M_YE_Y}\sin\phi \end{cases} \Rightarrow \begin{cases} 2M_xE_x + 2M_yE_y\cos^2\theta - 4\sqrt{M_YE_YM_yE_y}\cos\theta = 2M_YE_Y\cos^2\phi \\ 2M_yE_y\sin^2\theta = 2M_YE_Y\sin^2\phi \end{cases}$$

$$\Rightarrow 2M_xE_x + 2M_yE_y - 4\sqrt{M_xE_xM_yE_y}\cos\theta = 2M_YE_Y \Rightarrow \boldsymbol{E_Y = \frac{M_x}{M_Y}E_x + \frac{M_y}{M_Y}E_y - 2\sqrt{M_xE_xM_yE_y}\cos\theta}$$

$$\because Q = E_Y + E_y - E_x \quad \begin{bmatrix} \text{target nucleus, } X \text{ is} \\ \text{at rest so that } E_X = 0 \end{bmatrix} \quad \therefore Q = \left(\frac{M_x}{M_Y}E_x + \frac{M_y}{M_Y}E_y - \frac{2}{M_Y}\sqrt{M_xE_xM_yE_y}\cos\theta\right) + E_y - E_x$$

$$\Rightarrow \boldsymbol{Q = \left(1 + \frac{M_y}{M_Y}\right)E_y - \left(1 - \frac{M_x}{M_Y}\right)E_x - \frac{2}{M_Y}\sqrt{M_xE_xM_yE_y}\cos\theta}$$

$$\Rightarrow \left(1 + \frac{M_y}{M_Y}\right)E_y - \frac{2}{M_Y}\sqrt{M_xE_xM_yE_y}\cos\theta + \left\{-Q - \left(1 - \frac{M_x}{M_Y}\right)E_x\right\} = 0 \quad \left[\text{which is a quadratic in } \sqrt{E_y} = z \text{ (say)}\right]$$

$$\Rightarrow az^2 + bz + c = 0 \quad \left[\text{where } a = \left(1 + \frac{M_y}{M_Y}\right); b = -\frac{2}{M_Y}\sqrt{M_xE_xM_y}\cos\theta \; ; c = \left\{-Q - \left(1 - \frac{M_x}{M_Y}\right)E_x\right\}\right]$$

$$\Rightarrow z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \sqrt{E_y} = \frac{\frac{1}{M_Y}\sqrt{M_xE_xM_y}\cos\theta \pm \sqrt{\frac{1}{{M_Y}^2}M_xE_xM_y\cos^2\theta + \left(1 + \frac{M_y}{M_Y}\right)\left\{Q + \left(1 - \frac{M_x}{M_Y}\right)E_x\right\}}}{\left(1 + \frac{M_y}{M_Y}\right)}$$

$$\Rightarrow \sqrt{E_y} = \frac{\sqrt{M_xE_xM_y}\cos\theta \pm \sqrt{M_xE_xM_y\cos^2\theta + (M_Y + M_y)\{QM_Y + (M_Y - M_x)E_x\}}}{M_Y + M_y}$$

If $E_x = 0, b = 0$ and $c = -Q$ so that $z = \dfrac{\pm\sqrt{-4ac}}{2a} = \pm\sqrt{-\dfrac{c}{a}} \Rightarrow \sqrt{E_y} = \pm\sqrt{\dfrac{Q}{a}}$

For endergic reaction, $Q < 0 \Rightarrow \sqrt{E_y}$ is imaginary $\left[\because a = \left(1 + \dfrac{M_y}{M_Y}\right) > 0\right] \Rightarrow E_y$ is imaginary

$\Rightarrow$ endergic reaction is not possible with $E_x = 0 \Rightarrow$ a minimum $E_x = E_{\min}$ is required to initiate the reaction.

$\therefore$ When $E_x = E_{\min}, b^2 = 4ac \Rightarrow \dfrac{1}{{M_Y}^2}M_xE_{\min}M_y\cos^2\theta = \left(1 + \dfrac{M_y}{M_Y}\right)\left\{-Q - \left(1 - \dfrac{M_x}{M_Y}\right)E_{\min}\right\}$

$$\Rightarrow \frac{M_xM_y\cos^2\theta}{M_Y(M_Y + M_y)}E_{\min} = -Q - \left(1 - \frac{M_x}{M_Y}\right)E_{\min} \Rightarrow Q = \left\{-\frac{M_xM_y\cos^2\theta}{M_Y(M_Y + M_y)} - \left(1 - \frac{M_x}{M_Y}\right)\right\}E_{\min}$$

$$\Rightarrow Q = \left\{\frac{-M_xM_y\cos^2\theta - (M_Y - M_x)(M_Y + M_y)}{M_Y(M_Y + M_y)}\right\}E_{\min} = \left\{\frac{-M_xM_y\cos^2\theta - ({M_Y}^2 - M_xM_Y - M_xM_y + M_YM_y)}{M_Y(M_Y + M_y)}\right\}E_{\min}$$

$$\Rightarrow Q = \left\{\frac{M_xM_y\sin^2\theta - M_Y(M_Y - M_x + M_y)}{M_Y(M_Y + M_y)}\right\}E_{\min} = \left\{\frac{M_xM_y\sin^2\theta - M_Y\left(M_X - \frac{Q}{c^2}\right)}{M_Y(M_Y + M_y)}\right\}E_{\min}$$

$$\left[\because Q = (M_x + M_X - M_Y - M_y)c^2\right] \quad \therefore E_{\min} = \frac{Q(M_Y + M_y)}{\frac{M_xM_y}{M_Y}\sin^2\theta - \left(M_X - \frac{Q}{c^2}\right)} > 0 \quad [\because Q < 0 \text{ for endergic reaction.}]$$

When $\theta = 0$ i.e. y is emitted in the forward direction, $E_{\min}$ has the lowest value and is called the **threshold energy** for the endergic reaction. $\therefore E_{\text{th}} = -\dfrac{Q(M_Y + M_y)}{M_X - \frac{Q}{c^2}} \approx -\dfrac{Q(M_Y + M_y)}{M_X} \quad \left[\because M_X \gg \dfrac{Q}{c^2}\right] \Rightarrow E_{\text{th}} \approx -\dfrac{Q(M_X + M_x)}{M_X}$

$\left[\because M_Y + M_y = \frac{Q}{c^2} - M_x - M_X \approx -M_x - M_X\right] \therefore \boldsymbol{E}_{\text{th}} = -\boldsymbol{Q}\left(\mathbf{1} + \frac{\boldsymbol{M_x}}{\boldsymbol{M_X}}\right) \Rightarrow Q = \frac{-E_{\text{th}}}{\left(1 + \frac{M_x}{M_X}\right)}$

$\therefore$ by measuring the minmum energy E_{th} at which an endergic reaction is initiated, it is possible to determine the Q-value of the reaction. **If the projectile is a $\boldsymbol{\gamma}$-ray, $\boldsymbol{E}_{\text{th}} = -\boldsymbol{Q}$** $[\because M_x = 0]$

Calculate the Q-value (energy released) of the reaction: $^{1}_{1}\text{H} + ^{7}_{3}\text{Li} \rightarrow 2\ ^{4}_{2}\text{He} + \boldsymbol{Q}$
(Use M_H = 1.007825 u, M_{Li} = 7.016003 u, M_{He} = 4.002602 u)
$Q = M_{\text{H}} + M_{\text{Li}} - 2\ M_{\text{He}} = 0.018624\ u \approx 17.35\ \text{MeV}$ [i.e. the reaction is exothermic]

Calculate the Q-value (energy released) of the reaction: $^{59}_{27}\text{Co}(\boldsymbol{p}, \boldsymbol{n})^{59}_{28}\text{Ni}$. (Use M_H = 1.007825 u, M_{Co} = 58.93382 u, M_{Ni} = 58.934349 u, M_n = 1.008665 u)
$Q = M_{\text{Co}} + M_{\text{p}} - M_{\text{Ni}} - M_{\text{n}} = -0.00137\ u \approx -1.275\ \text{MeV}$ [i.e. the reaction is endothermic]

Q-value (energy released) of the reaction: $^{10}_{5}\text{B}(\boldsymbol{n}, \boldsymbol{\alpha})^{7}_{3}\text{Li}$ is −2.7945 MeV. Calculate the threshold energy. (Use M_n = 1.008665 u, M_N = 10.012939 u)
Threshold energy, $E_{\text{th}} = -Q\left(1 + \frac{M_{\text{projectile}}}{M_{\text{target}}}\right) = -Q\left(1 + \frac{M_{\text{n}}}{M_{\text{B}}}\right) = -3.076\ \text{MeV}$

Calculate the Q-value (energy released) of the reaction: $^{14}_{7}\text{N}(\boldsymbol{\alpha}, \boldsymbol{p})^{17}_{8}\text{O}$. (Use M_H = 1.00813 u, M_N = 14.00753 u, M_O = 17.00450 u, M_{He} = 4.00385 u) Also find the threshold energy.
$^{4}_{2}\text{He} + ^{14}_{7}\text{N} \rightarrow ^{17}_{8}\text{O} + ^{1}_{1}\text{H} + Q$
$Q = M_{\text{N}} + M_{\alpha} - M_{\text{p}} - M_{\text{O}} = -0.00125\ u \approx -1.1644\ \text{MeV}$ [i.e. the reaction is endothermic]
Threshold energy, $E_{\text{th}} = -Q\left(1 + \frac{M_{\text{projectile}}}{M_{\text{target}}}\right) = -Q\left(1 + \frac{M_{\alpha}}{M_{\text{N}}}\right) = 1.497\ \text{MeV}$

Calculate the Q-value (energy released) of the reaction: $^{14}_{7}\text{N}(\boldsymbol{n}, \boldsymbol{d})^{13}_{6}\text{C}$. (Use M_n = 1.008665 u, M_N = 14.003074 u, M_C = 13.003354 u, M_d = 2.014102 u) Hence find the threshold energy.
$^{1}_{0}\text{n} + ^{14}_{7}\text{N} \rightarrow ^{13}_{6}\text{C} + ^{2}_{1}\text{H} + Q$
$Q = M_{\text{N}} + M_{\text{n}} - M_{\text{d}} - M_{\text{O}} = -0.005717\ u \approx -5.3254\ \text{MeV}$ [i.e. the reaction is endothermic]
Threshold energy, $E_{\text{th}} = -Q\left(1 + \frac{M_{\text{projectile}}}{M_{\text{target}}}\right) = -Q\left(1 + \frac{M_{\text{n}}}{M_{\text{N}}}\right) = 5.709\ \text{MeV}$

Calculate the Q-value (energy released) of the reaction: $^{19}_{9}\text{F}(\boldsymbol{n}, \boldsymbol{p})^{19}_{8}\text{O}$. (Use M_n = 1.008665 u, M_F = 18.99840 u, M_O = 19.00358 u, M_p = 1.007825 u) Also find the lowest energy required to initiate the reaction.
$^{1}_{0}\text{n} + ^{19}_{9}\text{F} \rightarrow ^{19}_{8}\text{O} + ^{1}_{1}\text{H} + Q$
$Q = M_{\text{n}} + M_{\text{F}} - M_{\text{O}} - M_{\text{p}} = -0.00434\ u \approx -4.043\ \text{MeV}$ [i.e. the reaction is endothermic]
Threshold energy, $E_{\text{th}} = -Q\left(1 + \frac{M_{\text{projectile}}}{M_{\text{target}}}\right) = -Q\left(1 + \frac{M_{\text{n}}}{M_{\text{F}}}\right) = 4.258\ \text{MeV}$

A beam of 7.3 MeV α-particles is used for the reaction: $^{27}_{13}\text{Al}(\boldsymbol{\alpha}, \boldsymbol{p})^{30}_{14}\text{Si}$. The protons emitted at 0° are found to have an energy of 9.34 MeV. What is the Q-value of the reaction? (M_α = 4.002603 u, M_{Al} = 26.98154 u, M_{Si} = 29.97377 u, M_p = 1.007825 u) Check the result using the mass values.

$$Q = \left(1 + \frac{M_y}{M_Y}\right)E_y - \left(1 - \frac{M_x}{M_Y}\right)E_x - \frac{2}{M_Y}\sqrt{M_x E_x M_y E_y}\cos\theta$$

$$= \left(1 + \frac{M_{\text{p}}}{M_{\text{Si}}}\right)E_{\text{p}} - \left(1 - \frac{M_{\alpha}}{M_{\text{Si}}}\right)E_{\alpha} - \frac{2}{M_{\text{Si}}}\sqrt{M_{\alpha}E_{\alpha}M_{\text{p}}E_{\text{p}}}\quad [\cos\theta = \cos 0° = 1] = 2.2223\ \text{MeV}$$

$Q = M_{\alpha} + M_{\text{Al}} - M_{\text{Si}} - M_{\text{p}} = 0.002548\ u \approx 2.3735\ \text{MeV}$

Deuterons of 1.51 MeV are used for the reaction: $^{16}_{8}\mathrm{O}(d,\alpha)^{14}_{7}\mathrm{N}$. The α-particles emitted at 90° are found to have an energy of 3.427 MeV. Calculate the Q-value of the reaction. (M_d = 2.014102 u, M_O = 15.9949 u, M_N = 14.0031 u, M_α = 4.002603 u). Check the result from the masses of the nuclei.

$$Q=\left(1+\frac{M_y}{M_Y}\right)E_y-\left(1-\frac{M_x}{M_Y}\right)E_x-\frac{2}{M_Y}\sqrt{M_xE_xM_yE_y}\cos\theta$$
$$=\left(1+\frac{M_\alpha}{M_\mathrm{N}}\right)E_\alpha-\left(1-\frac{M_\mathrm{d}}{M_\mathrm{N}}\right)E_\mathrm{d}\quad[\cos\theta=\cos 90°=0]=3.114\ \mathrm{MeV}$$
$$Q=M_\mathrm{d}+M_\mathrm{O}-M_\mathrm{N}-M_\alpha=0.0033\,u\approx 3.073\ \mathrm{MeV}$$

The *Q*-values of the reactions: $^{2}_{1}\mathrm{H}(d,t)^{1}_{1}\mathrm{H}$ and $^{2}_{1}\mathrm{H}\left(d,{}^{3}_{2}\mathrm{He}\right)^{1}_{0}n$ are 4.032 MeV and 3.269 MeV respectively. The β-disintegration energy of $^{3}_{1}\mathrm{H}$ is known to be 0.019 MeV. Calculate the β-disintegration energy of the neutron from these data.

$$Q_1=M_\mathrm{d}+M_\mathrm{d}-M_\mathrm{p}-M_\mathrm{t}=2\,M_\mathrm{d}-M_\mathrm{p}-M_\mathrm{t}=4.032\ \mathrm{MeV}$$
$$Q_2=M_\mathrm{d}+M_\mathrm{d}-M_\mathrm{He}-M_\mathrm{n}=2\,M_\mathrm{d}-M_\mathrm{He}-M_\mathrm{n}=3.269\ \mathrm{MeV}$$
β-disintegration of tritium: $^{3}_{1}\mathrm{H}\rightarrow{}^{3}_{2}\mathrm{He}+\mathrm{e}^-+\bar{\nu}_\mathrm{e}+Q_3$ where $Q_3=M_\mathrm{t}-M_\mathrm{He}-M_\mathrm{e}=0.019\ \mathrm{MeV}$

β-disintegration of neutron: $^{1}_{0}\mathrm{n}\rightarrow{}^{1}_{1}\mathrm{H}+\mathrm{e}^-+\bar{\nu}_\mathrm{e}+Q_4$ where $Q_4=M_\mathrm{n}-M_\mathrm{H}-M_\mathrm{e}=Q_3+Q_1-Q_2=0.782\ \mathrm{MeV}$

If one gram of ^{235}U releases an energy of 2.29 × 10^4 kWh upon complete burning, calculate the amount of coal to be burnt to produce the same amount of energy. (Given: 1 atom of Carbon produced 4 eV of energy)

$$E=2.29\times10^4\ \mathrm{kWh}=2.29\times10^4\times10^3\times3600\ \mathrm{J}=8.244\times10^{10}\ \mathrm{J}=5.1455\times10^{29}\ \mathrm{eV}$$
$$\therefore\ \text{Number of Carbon atoms required to produce the same energy, } N=\frac{5.1455\times10^{29}\ \mathrm{eV}}{4\ \mathrm{eV}}\approx1.2864\times10^{29}$$
$$\therefore\ \text{Amount (weight)of coal to be burnt, } m=N\times12u\approx2563.3\ \mathrm{kg}$$

Fissile and Fertile materials

^{233}U, ^{235}U and ^{239}Pu are called **fissile** materials as they easily undergo fission with thermal neutrons. However ^{235}U occurs in nature with an abundance of 0.71% in natural Uranium, and other two do not exist in nature, but can be produced artificially (owing to their long half-lives **1.6 × 10^5 y** and **2.4 × 10^4 y**) from ^{232}Th and ^{238}U respectively. ^{232}Th and ^{238}U are called **fertile** materials as they can converted into thermally fissible isotopes. The first step in the conversion is **(n,γ)** reaction, followed by two successive **β decays**.

$$\begin{cases}{}^{1}_{0}\mathrm{n}+{}^{232}_{90}\mathrm{Th}\rightarrow{}^{233}_{90}\mathrm{Th}+\gamma\\ {}^{233}_{90}\mathrm{Th}\xrightarrow{\beta^-}{}^{233}_{91}\mathrm{Pa}\\ {}^{233}_{91}\mathrm{Pa}\xrightarrow{\beta^-}{}^{233}_{92}\mathrm{U}\end{cases}\qquad\qquad\begin{cases}{}^{1}_{0}\mathrm{n}+{}^{238}_{92}\mathrm{U}\rightarrow{}^{239}_{92}\mathrm{U}+\gamma\\ {}^{239}_{92}\mathrm{U}\xrightarrow{\beta^-}{}^{239}_{93}\mathrm{Np}\\ {}^{239}_{93}\mathrm{Np}\xrightarrow{\beta^-}{}^{239}_{94}\mathrm{Pu}\end{cases}$$

Sequence of Events in Nuclear Fission (Born-Wheeler theory of nuclear fission based on liquid-drop model)*

^{235}U captures a thermal neutron and forms ^{236}U* which undergoes violent oscillations due to excess energy from the neutron. The ^{236}U* becomes stretched and the inter-proton repulsive force tends to increase the deformation (electrical energy decreases with elongation) against the surface forces which tend to restore the original shape (surface energy is minimum for sphere). Finally the repulsive forces prevail and the heavy nucleus splits into two fragments and emitting several neutrons. $\mathbf{{}^{1}_{0}n+{}^{235}_{92}U\rightarrow{}^{236}_{92}U^*\rightarrow{}^{141}_{56}U+{}^{92}_{36}Kr+3\,{}^{1}_{0}n}$

Four-factor formula for neutron multiplication

Nuclear reactor is a device within which self-sustained fission chain reaction may be made to proceed in a controlled manner. The characteristic parameter which determines the course of the self-sustained chain reaction is the **multiplication factor, *k***.

In an experiment, the intensity of beta counts is reduced to 12064 when they are passed through Al absorber of thickness 0.01 cm. Calculate the mass absorption coefficient for beta particles in Al if the initial intensity is 24128 counts and density of Al is 2.7 g/cc.

$$I = I_0 e^{-\mu x} \Longrightarrow \mu = \frac{1}{x}\ln\frac{I_0}{I} = \frac{1}{0.01\text{ cm}}\ln\frac{24128}{12064} = \frac{\ln 2}{0.01\text{ cm}} \approx 69.315\text{ cm}^{-1} \;\therefore \mu_{\text{mass}} = \frac{\mu}{\rho} = \frac{69.315\text{ cm}^{-1}}{2.7\text{ g/cm}^3} = 25.67\text{ cm}^2/\text{g}$$

Calculate the range of beta particles with end-point energy (i) 0.746 MeV (ii) 0.540 MeV in Al (density = 2.7 g/cc).

Range, $R = 4.07\, {E_\text{m}}^{1.38}$ [empirical relation] where E_m is in MeV and R is in kg/m^2 when $E_\text{m} < 0.8$ MeV

$$\therefore \begin{cases}\text{For } E_\text{m} = 0.746\text{ MeV}, R \approx 2.716\text{ kg/m}^2 \approx 1.006\text{ mm} \\ \text{For } E_\text{m} = 0.540\text{ MeV}, R \approx 1.739\text{ kg/m}^2 \approx 0.644\text{ mm}\end{cases} \quad [\because \rho = 2.7\text{ g/cm}^3 = 2700\text{ kg/m}^3]$$

Conservation laws that govern elementary interactions

1. Conservation of energy
2. Conservation of linear momentum
3. Conservation of angular momentum
4. Conservation of parity (violated in weak interaction)
5. Conservation of charge
6. Conservation of baryon number: B = 1 for baryons, B = −1 for antibaryons, B = 0 for mesons and non-baryons.
7. Conservation of lepton number: L = 1 for leptons, L = −1 for antileptons, and L = 0 for non-leptons.
8. Conservation of isospin: I = 0 (singlet) for Λ-hyperons and Ω-hyperons, I = $^1/_2$ (doublet) for **nucleons**, K-mesons and Ξ-hyperons, I = 1 (triplet) for Σ-hyperons, I = $^3/_2$ (quartet) for Δ-baryons.
9. Conservation of strangeness (violated in weak interaction)

Why does a free neutron not decay into an electron and a positron?

It will violate the laws of conservation of mass and conservation of baryon number.

Which of these reactions is/are not allowed?

$\mathrm{p} + \mathrm{p} \to \mathrm{p} + \mathrm{p} + \bar{\mathrm{p}}$ [not allowed since baryon (nucleon) number and hence charge is not conserved]
$\mathrm{n} \to \mathrm{p} + \mathrm{e}^- + \bar{\nu}_\mathrm{e}$ [allowed]
$\mu^- \to \mathrm{e}^- + \bar{\nu}_\mathrm{e} + \nu_\mu$ [allowed]
$\pi^- \to \mu^- + \bar{\nu}_\mu$ [allowed]
$\Lambda^0 \to \pi^- + \pi^+$ [not allowed since baryon number and isospin are not conserved]
$\pi^- + \mathrm{p} \to \pi^0 + \mathrm{n}$ [allowed]
$\Lambda^0 + \mathrm{K}^0 \to \pi^- + \mathrm{p}$ [allowed]
$\pi^0 \to \gamma + \gamma + \gamma$ [allowed]
$\mathrm{K}^+ \to \pi^+ + \pi^0$ [not allowed since strangeness is not conserved]
$\eta^0 \to \mathrm{K}^0 + \gamma$ [not allowed since isospin and strangeness are not conserved]
$\mathrm{p} \to \mathrm{e}^+ + \gamma$ [not allowed since baryon (nucleon) number and lepton number are not conserved]

Baryon is a hadron containing three quarks. They are fermions.
Meson is a hadron containing a quark and an antiquark. They are bosons.
Hyperon is a baryon containing one or more strange quarks, but no charm, bottom or top quark.

Constitution of Pion (Pi meson)

$$\begin{cases}\pi^+ = \mathrm{u} + \bar{\mathrm{d}} \\ \pi^0 = \mathrm{u} + \bar{\mathrm{u}} = \mathrm{d} + \bar{\mathrm{d}} \\ \pi^- = \bar{\mathrm{u}} + \mathrm{d}\end{cases}$$

Constitution of Kaon (K meson)

$$\begin{cases} K^+ = u + \bar{s} \\ K^0 = d + \bar{s} = \bar{d} + s \\ K^- = \bar{u} + s \end{cases}$$

Constitution of Delta baryon

$$\begin{cases} \Delta^{++} = u + u + u \\ \Delta^+ = u + u + d \\ \Delta^0 = u + d + d \\ \Delta^- = d + d + d \end{cases}$$

Constitution of Xi hyperon (Xi baryon without charm, bottom or top quark)

$$\begin{cases} \Xi^0 = u + s + s \\ \Xi^- = d + s + s \end{cases} \quad [S = -(n_s - n_{\bar{s}}) = -2]$$

Constitution of Omega hyperon (Omega baryon without charm, bottom or top quark)

$$\Omega^- = s + s + s \quad [S = -(n_s - n_{\bar{s}}) = -3]$$

Constitution of Lambda hyperon (Lambda baryon without charm, bottom or top quark)

$$\Lambda^0 = u + d + s \quad [S = -(n_s - n_{\bar{s}}) = -1]$$

Constitution of Sigma hyperon (Sigma baryon without charm, bottom or top quark)

$$\begin{cases} \Sigma^+ = u + u + s \\ \Sigma^0 = u + d + s \\ \Sigma^0 = d + d + s \end{cases} \quad [S = -(n_s - n_{\bar{s}}) = -1]$$

<u>Types of Nuclear Radiation</u>

Charged particles: Heavy charged particles (e.g. protons, alpha, heavy ions, fission fragments)
Light charged particles (e.g. beta)

Neutral particles (e.g. gamma, neutrons, neutrinos)

The distance travelled by a charged particle in matter before losing all its energy is called its **Range**. The Range depends on the (initial kinetic) energy, type (mass & charge) of the charged particle and type of the medium.

$$\text{Range, } R = \int_0^R dx = \int_{E_0}^0 \frac{dE}{dE/dx} = \int_0^{E_0} \frac{dE}{S(E)} \quad \left[\because S(E) = -\frac{dE}{dx}\right] \Rightarrow R = \int_0^{v_0} \frac{v^2\,dE}{z^2 f(v)} \quad \left[\because S(E) = \frac{z^2}{v^2} f(v)\right]$$

$$\Rightarrow R = \frac{M}{z^2}\int_0^{v_0} \frac{v^3\,dv}{f(v)} \quad \left[\because E = \frac{1}{2}Mv^2\right] \approx \frac{M}{z^2\langle f(v)\rangle}\int_0^{v_0} v^3\,dv \quad \left[\begin{matrix} \because f(v) \text{ is a slow-varying function of } v. \\ \langle f(v)\rangle \text{ is the mean over the entire Range.} \end{matrix}\right]$$

$\Rightarrow \boldsymbol{R = \frac{M}{z^2\langle f(v)\rangle}\frac{{v_0}^4}{4}} \Rightarrow$ Ratios of ranges of α-particles, deuterons and protons (all-charged particles) of the same v_0 should be $R_\alpha : R_d : R_p = \left(\frac{M}{z^2}\right)_\alpha : \left(\frac{M}{z^2}\right)_d : \left(\frac{M}{z^2}\right)_p = \frac{4}{2^2} : \frac{2}{1^2} : \frac{1}{1^2} = 1 : 2 : 1$ whereas the ratios of E_0 for the same v_0 are $E_\alpha : E_d : E_p = M_\alpha : M_d : M_p = 4 : 2 : 1$

For α-particles, Geiger gave the **empirical** relationship: $R \propto {E_0}^{\frac{3}{2}} \Rightarrow \boldsymbol{R = a{E_0}^{\frac{3}{2}}} \approx (0.00318\text{ m})E^{\frac{3}{2}}$ [$\boldsymbol{E}$ **is in MeV**] which holds well for **3 MeV** $< \boldsymbol{E_0} <$ **7 MeV**. $\therefore R \propto v^3 \Rightarrow \boldsymbol{R = b{v_0}^3} \left[\because E_0 = \frac{1}{2}M_\alpha {v_0}^2\right]$ where $b = a\left(\frac{1}{2}M_\alpha\right)^{\frac{3}{2}}$

$$\therefore R = aE^{\frac{3}{2}} = (0.00318\text{ m})\left(\frac{1}{2}M_\alpha\right)^{\frac{3}{2}}(10^6 \times 1.6 \times 10^{-19})^{-\frac{3}{2}}v^3 \quad [\because E \text{ was in MeV}] = (9.52 \times 10^{-28}\text{ m})v^3 \quad [\boldsymbol{v} \textbf{ is in m/s}]$$

For $\boldsymbol{E} <$ **3 MeV**, the appropriate relationship is $R \propto E^{\frac{3}{4}} \Rightarrow R \propto v^{\frac{3}{2}}$

For $\boldsymbol{E} >$ **7 MeV**, the appropriate relationship is $R \propto E^2 \Rightarrow R \propto v^4$

Range straggling

For a given species of particles with the same initial energy (and therefore, velocity), the Range in a given material medium is narrowly spread about a mean value, as evident from the **intensity-distance** plot. Probable reasons are:

1. Statistical fluctuation in the number of collisions suffered by the different particles about a mean value in travelling through a given distance, i.e. different particles would have to move through different distances in order to lose their entire initial energy.
2. Statistical fluctuation in the energy lost per collision about a mean value.
3. Multiple scattering of the particles during collisions.
4. Density inhomogeneity of the absorbing medium.
5. As the moving particle slows down, it picks up electrons from the absorber. This is also a statistical process.

Protons of kinetic energy 1 MeV have the stopping power of 175 keV/mg/cm^2 in Aluminium. At what energies will muons, deuterons and alpha particles have the same stopping power in Aluminium? (Use $m_\mu \approx 207m_e$)

Stopping power, $S(E) \propto \dfrac{z^2}{v^2} \propto \dfrac{z^2 M}{E} \quad \left[\because E = \dfrac{1}{2}Mv^2\right] \Rightarrow E \propto \dfrac{z^2 M}{S(E)} \Rightarrow \boldsymbol{E \propto z^2 M} \quad [S(E) \text{ is same for all.}]$

$\therefore E_\mathrm{p} : E_\mu : E_\mathrm{d} : E_\alpha = (z^2M)_\mathrm{p} : (z^2M)_\mu : (z^2M)_\mathrm{d} : (z^2M)_\alpha = 1 : 0.113 : 2 : 16 \quad \left[\because M_\mu = 207M_\mathrm{e} = \dfrac{207}{1836}M_\mathrm{p}\right]$

For muons, $E = 0.113$ MeV ; For deuterons, $E = 2$ MeV ; For α-particles, $E = 16$ MeV

Calculate the kinetic energy of Compton electron when γ-ray of energy 662 keV is back-scattered and also at 90°.

$E_\mathrm{e} = E_\gamma - E_{\gamma'} = E_\gamma\left\{1 - \dfrac{1}{1+\alpha(1-\cos\theta)}\right\}$ where $\alpha = \dfrac{E_\gamma}{m_e c^2} \quad \therefore \alpha = 1.2955$ when $E_\gamma = 662$ keV

$$\begin{cases} \text{When } \theta = 180°, E_\mathrm{e} \approx 0.721526\, E_\gamma \approx 477.65 \text{ keV} \\ \text{When } \theta = 90°, E_\mathrm{e} \approx 0.564365\, E_\gamma \approx 373.61 \text{ keV} \end{cases}$$

<u>**Types (Modes) of β-decay**</u>

β⁻ decay (Electron emission) : $\begin{cases} {}^A_Z\mathrm{X} \to {}_{Z+1}^{\;\;A}\mathrm{Y} + \mathrm{e}^- + \bar{\nu}_\mathrm{e} \\ \mathrm{n} \to \mathrm{p}^+ + \mathrm{e}^- + \bar{\nu}_\mathrm{e} \\ \textbf{e.g. } {}^{137}_{55}\mathrm{Cs} \to {}^{137}_{56}\mathrm{Ba} + \mathrm{e}^- + \bar{\nu}_\mathrm{e} \end{cases}$

β⁺ decay (Positron emission) : $\begin{cases} {}^A_Z\mathrm{X} \to {}_{Z-1}^{\;\;A}\mathrm{Y} + \mathrm{e}^+ + \nu_\mathrm{e} \\ \mathrm{p} \to \mathrm{n} + \mathrm{e}^+ + \nu_\mathrm{e} \\ \textbf{e.g. } {}^{22}_{11}\mathrm{Na} \to {}^{22}_{10}\mathrm{Ne} + \mathrm{e}^+ + \nu_\mathrm{e} \end{cases}$

(Orbital) Electron Capture (EC) : $\begin{cases} {}^A_Z\mathrm{X} + \mathrm{e}^- \to {}_{Z-1}^{\;\;A}\mathrm{Y} + \nu_\mathrm{e} \\ \mathrm{p} + \mathrm{e}^- \to \mathrm{n} + \nu_\mathrm{e} \\ \textbf{e.g. } {}^{59}_{28}\mathrm{Ni} + \mathrm{e}^- \to {}^{59}_{27}\mathrm{Co} + \nu_\mathrm{e} \end{cases}$

<u>**Energetics of β-decay**</u>

Note: M_N stands for nuclear mass, and M stands for atomic mass in the following discussion.

β⁻ decay (Electron emission) : $Q_{\beta^-} = \{M_N(A,Z) - M_N(A,Z+1) - m_\mathrm{e}\}c^2$

$= \{M(A,Z) - Zm_\mathrm{e} - M(A,Z+1) + (Z+1)m_\mathrm{e} - m_\mathrm{e}\}c^2 \Rightarrow \boldsymbol{Q_{\beta^-} = \{M(A,Z) - M(A,Z+1)\}c^2}$

$\therefore Q_{\beta^-} > 0 \Rightarrow M(A,Z) > M(A,Z+1) \Rightarrow M_\text{parent atom} > M_\text{daughter atom}$

β⁺ decay (Positron emission) : $Q_{\beta^+} = \{M_N(A,Z) - M_N(A,Z-1) - m_\mathrm{e}\}c^2$

$= \{M(A,Z) - Zm_\mathrm{e} - M(A,Z-1) + (Z-1)m_\mathrm{e} - m_\mathrm{e}\}c^2 \Rightarrow \boldsymbol{Q_{\beta^+} = \{M(A,Z) - M(A,Z-1) - 2m_\mathrm{e}\}c^2}$

$\therefore Q_{\beta^-} > 0 \Rightarrow M(A,Z) > M(A,Z-1) + 2m_\mathrm{e} \Rightarrow M_\text{parent atom} > M_\text{daughter atom} + 2m_\mathrm{e}$

(Orbital) Electron Capture (EC) : $Q_\mathrm{EC} = \{M_N(A,Z) + m_\mathrm{e} - M_N(A,Z-1)\}c^2 - B_\mathrm{e} \quad [B_\mathrm{e} = \text{electron binding energy}]$

$= \{M(A,Z) - Zm_e + m_e - M(A,Z-1) + (Z-1)m_e\}c^2 - B_e \Longrightarrow \boldsymbol{Q_{EC} = \{M(A,Z) - M(A,Z-1)\}c^2 - B_e}$
$\therefore Q_{EC} > 0 \Longrightarrow M(A,Z) > M(A,Z+1) + B_e \Longrightarrow M_{\text{parent atom}} > M_{\text{daughter atom}} + B_e$

Gamma radiation
A high-energy photon (chargeless and massless) produced in nuclear transitions, energy ranging from 10 keV to 10 MeV, not deflected (being neutral) by electric or magnetic fields, much more penetrating than α and β particle. This was first realized by French scientist, PV Villard in 1900.

Energetics of γ-decay
A excited nucleus of mass M makes transition from the initial state E_i to the final state E_f through the emission of a photon. To conserve linear momentum, the nucleus recoils.

$$\begin{cases} \text{Conservation of energy: } E_i = E_f + E_\gamma + T_{\text{recoil}} \Longrightarrow \Delta E = E_\gamma + T_{\text{recoil}} = E_\gamma + \dfrac{{p_{\text{recoil}}}^2}{2M} \quad \left[\because T_{\text{recoil}} = \dfrac{{p_{\text{recoil}}}^2}{2M}\right] \\ \text{Conservation of linear momentum: } 0 = p_{\text{recoil}} + p_\gamma \Longrightarrow p_{\text{recoil}} = -p_\gamma \end{cases}$$

$$\Longrightarrow \Delta E = E_\gamma + \frac{{E_\gamma}^2}{2Mc^2} \quad [\because E_\gamma = p_\gamma c] \Longrightarrow \frac{{E_\gamma}^2}{2Mc^2} + E_\gamma - \Delta E = 0 \Longrightarrow \boldsymbol{E_\gamma = Mc^2\left\{-1 \pm \sqrt{1 + \frac{2\Delta E}{Mc^2}}\right\}}$$

$$\Longrightarrow E_\gamma \approx Mc^2\left\{-1 + \left(1 + \frac{1}{2}\cdot\frac{2\Delta E}{Mc^2} - \frac{1}{8}\cdot\left(\frac{2\Delta E}{Mc^2}\right)^2\right)\right\} \quad \begin{bmatrix}\because 2\Delta E \ll Mc^2 \\ \text{Binomial} \\ \text{expansion}\end{bmatrix} = Mc^2\left\{\frac{\Delta E}{Mc^2} - \frac{1}{2}\left(\frac{\Delta E}{Mc^2}\right)^2\right\} = \Delta E - \frac{(\Delta E)^2}{2Mc^2}$$

$$\Longrightarrow \boldsymbol{E_\gamma = \Delta E\left(1 - \frac{\Delta E}{2Mc^2}\right)} \Longrightarrow E_\gamma = \Delta E \quad \begin{bmatrix}\because \text{recoil correction is much smaller than experimental} \\ \text{uncertainty with which energies can be measured}\end{bmatrix}$$

Selection rules for radiative transitions (γ-emissions) in nuclei
1. In electric multipole transition (*EL*), parity is even when *L* is even and odd when *L* is odd so that $\Delta\Pi = (-1)^L$
2. In magnetic multipole transition (*ML*), parity is odd when *L* is even and even when *L* is odd so that $\Delta\Pi = (-1)^{L+1}$
3. Lower permitted multipole dominates.
4. For the same *L*, electric multipole emission is more probable (about 100 times) than magnetic multipole emission.
5. Emission of multipole (*L* + 1) is less probable than emission of multipole *L* by a factor of about 10^{-5}.

Predict the dominant γ-transitions in the following cases:
(i) $I^\Pi = \frac{5}{2}^+ \to I^\Pi = \frac{3}{2}^+$ (ii) $I^\Pi = \frac{5}{2}^+ \to I^\Pi = \frac{3}{2}^-$ (iii) $I^\Pi = 2^+ \to I^\Pi = 1^-$ (iv) $I^\Pi = \frac{3}{2}^- \to I^\Pi = \frac{1}{2}^+$
(i) $\Delta I = 1$ and $\Delta\Pi =$ No [possible L-values: $1, 2, 3, 4$; $M1, E2, M3, E4$ transitions ; $M1$ and $E2$ dominant]
(ii) $\Delta I = 3$ and $\Delta\Pi =$ Yes [possible L-values: $1, 2, 3, 4$; $E1, M2, E3, M4$ transitions ; $E1$ dominant]
(iii) $\Delta I = 1$ and $\Delta\Pi =$ Yes [possible L-values: $1, 2, 3$; $E1, M2, E3$ transitions ; $E1$ dominant]
(iv) $\Delta I = 1$ and $\Delta\Pi =$ Yes [possible L-values: $1, 2$; $E1, M2$ transitions ; $E1$ dominant]

Predict the multi-polarity of the following γ-transitions:
(i) $I^\Pi = \frac{5}{2}^+ \to I^\Pi = \frac{1}{2}^-$ (ii) $I^\Pi = 3^- \to I^\Pi = 0^+$
(i) $\Delta I = 2$ and $\Delta\Pi =$ Yes [possible L-values: $2, 3$; $M2$ and $E3$ transitions]
(ii) $\Delta I = 3$ and $\Delta\Pi =$ Yes [possible L-values: 3 ; $E3$ transition]

β-emitters may be classified into different categories based on their $f\tau$ (comparative half-life) values as
$\boldsymbol{\log f\tau = 3}$ **to** $\boldsymbol{4}$: super-allowed (allowed and favoured), β-transitions most probable
$\boldsymbol{\log f\tau = 4.5}$ **to** $\boldsymbol{5}$: allowed (allowed but not favoured), β-transitions less probable
$\boldsymbol{\log f\tau = 7}$ **to** $\boldsymbol{9}$: first forbidden transition, β-transitions much less probable

Selection rules in β-decay*

β-transitions are governed by certain selection rules depending on whether the transition is allowed or forbidden. A nuclear state is characterized by the total angular momentum (spin) $\vec{I} = \vec{L} + \vec{S}$ and the parity Π. When the transition takes place from an initial state $\vec{I_i}$ to a final state $\vec{I_f}$ the corresponding change in $\vec{I}$ is $\Delta\vec{I} = \Delta\vec{L} + \Delta\vec{S}$
Electron and neutrino wavefunctions ϕ_ν and ϕ_β can be respresented by plane waves if the Coulomb effect can be neglected for ϕ_β. Expanding the plane wave and neglecting the higher-order terms,

$$\phi = e^{i(\vec{k}\cdot\vec{r})} = e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{r})} = 1 + \frac{i}{\hbar}(\vec{p}\cdot\vec{r}) + \frac{1}{2}\left\{\frac{i}{\hbar}(\vec{p}\cdot\vec{r})\right\}^2 + \cdots \approx 1 + \frac{i}{\hbar}(\vec{p}\cdot\vec{r}) \equiv 1 + \frac{i}{\hbar}L_z = 1 + \frac{i}{\hbar}(l\hbar) = 1 + il$$

Pure allowed Fermi selection rule

${}^{14}_{8}\mathrm{O} \xrightarrow{\beta^+} {}^{14}_{7}\mathrm{N}^*$ (excited state) $[0^+$ to 0^+ ; so $\Delta I = 0$ and $\Delta\Pi = 0]$
${}^{34}_{17}\mathrm{Cl} \xrightarrow{\beta^+} {}^{34}_{16}\mathrm{S}$ (ground state) $[0^+$ to 0^+ ; so $\Delta I = 0$ and $\Delta\Pi = 0]$
${}^{10}_{6}\mathrm{C} \xrightarrow{\beta^+} {}^{10}_{5}\mathrm{B}^*$ (excited state) $[0^+$ to 0^+ ; so $\Delta I = 0$ and $\Delta\Pi = 0]$

Pure allowed G-T selection rule

${}^{14}_{8}\mathrm{O} \xrightarrow{\beta^+} {}^{14}_{7}\mathrm{N}$ (ground state) $[0^+$ to 1^+ ; so $\Delta I = 1$ and $\Delta\Pi = 0]$
${}^{6}_{2}\mathrm{He} \xrightarrow{\beta^-} {}^{6}_{3}\mathrm{Li}$ (ground state) $[0^+$ to 1^+ ; so $\Delta I = 1$ and $\Delta\Pi = 0]$
${}^{13}_{5}\mathrm{B} \xrightarrow{\beta^-} {}^{13}_{6}\mathrm{C}$ (ground state) $\left[\frac{3}{2}^-\text{ to }\frac{1}{2}^-\text{ ; so }\Delta I = 1\text{ and }\Delta\Pi = 0\right]$
${}^{230}_{91}\mathrm{Pa} \xrightarrow{\beta^+} {}^{230}_{90}\mathrm{Th}^*$ (excited state) $[2^-$ to 3^- ; so $\Delta I = 1$ and $\Delta\Pi = 0]$

Mixed selection rule

${}^{1}_{0}\mathrm{n} \xrightarrow{\beta^-} {}^{1}_{1}\mathrm{H}$ $\left[\frac{1}{2}^+\text{ to }\frac{1}{2}^+\text{ ; so }\Delta I = 0\text{ and }\Delta\Pi = 0\text{ ; also }I_i \neq 0\right]$
${}^{3}_{1}\mathrm{H} \xrightarrow{\beta^-} {}^{3}_{1}\mathrm{He}$ $\left[\frac{1}{2}^+\text{ to }\frac{1}{2}^+\text{ ; so }\Delta I = 0\text{ and }\Delta\Pi = 0\text{ ; also }I_i \neq 0\right]$
${}^{35}_{16}\mathrm{S} \xrightarrow{\beta^-} {}^{35}_{17}\mathrm{Cl}$ $\left[\frac{3}{2}^+\text{ to }\frac{3}{2}^+\text{ ; so }\Delta I = 0\text{ and }\Delta\Pi = 0\text{ ; also }I_i \neq 0\right]$

Two-cases of Lepton spin alignment:

$\boldsymbol{S = 0}$ when e-ν spins anti-align (anti-parallel) [**Fermi transition**]
$\boldsymbol{S = 1}$ when e-ν spins align (parallel) [**Gamow-Teller (G-T) transition**]

Type of Transition	Selection Rules	L_{ev}	ΔΠ	*fτ*
super-allowed	$\Delta I = 0, \pm 1$	0	no	$10^3 - 10^4$
allowed	$\Delta I = 0, \pm 1$	0	no	$2 \times 10^3 - 10^6$
1st forbidden	$\Delta I = 0, \pm 1$	1	yes	$10^6 - 10^8$
unique 1st forbidden	$\Delta I = \pm 2$	1	yes	$10^8 - 10^9$
2nd forbidden	$\Delta I = \pm 1, \pm 2$	2	no	$2 \times 10^{10} - 2 \times 10^{13}$
unique 2nd forbidden	$\Delta I = \pm 3$	2	no	10^{12}
3rd forbidden	$\Delta I = \pm 2, \pm 3$	3	yes	10^{18}
unique 3rd forbidden	$\Delta I = \pm 4$	3	yes	4×10^{15}
4th forbidden	$\Delta I = \pm 3, \pm 4$	4	no	10^{23}
unique 4th forbidden	$\Delta I = \pm 5$	4	no	10^{19}

Identify the type of beta transitions and the degree of forbiddenness in the following cases:

(i) $I^\Pi = \frac{3}{2}^+ \to I^\Pi = \frac{1}{2}^-$ (ii) $I^\Pi = \frac{7}{2}^- \to I^\Pi = \frac{3}{2}^-$
(i) $\Delta I = 1$ and $\Delta\Pi =$ Yes [Mixed transition; First forbidden]
(ii) $\Delta I = 2$ and $\Delta\Pi =$ No [Mixed transition; Second forbidden]

τ-θ puzzle (parity violation in weak interaction)

In mid-1950s two mesons were discovered in the cosmic rays and are called τ and θ mesons. They are identical having the same mass (493.667 MeV), same half-life (1.237×10^{-8} s) and same spin (0), differing only in the decay modes that indicated different spin-parity. Since, pions have spin-parity, $I^{\Pi} = 0^{-}$ and parity quantum numbers are multiplicative, and since these are weak decays and involve zero orbital angular momentum change, hence, it follows that τ^{+} have spin-parity 0^{+} and θ^{+} have spin--parity 0^{-}. Tsung-dao Lee and Chen-Ning Yang, two Chinese-American scientists proposed that these two were the same particles, but the different decay modes imply that parity was not conserved in decays involving weak interaction. Non-conservation of parity in weak-interaction was later experimentally proven by Chien-Shiung Wu and her team. The two particles are indeed the same and named Kaons or K-mesons (K^{+}).

$\tau^{+} \rightarrow \pi^{+} + \pi^{0}$

$\theta^{+} \rightarrow \begin{cases} \pi^{+} + \pi^{0} + \pi^{0} \\ \pi^{+} + \pi^{+} + \pi^{-} \end{cases}$

<u>Interaction of Beta particles (light charged particles) with matter</u>

Slow electrons — non-relativistic (KE much less than rest-mass energy). Their energy loss may be treated in the same way as heavy charged particles.

Fast electrons — relativistic (KE comparable with rest-mass energy). Beta particles are usually fast electrons.

Due to small mass and high speed of electrons or positrons,

1. They can transfer large fraction of their energy in a single collision.
2. They can rapidly change their direction after every collision.
3. Instead of the range, their actual path-length matters more.
4. After losing their KE, positrons annihilate with electrons to produce gamma rays.
5. Since the projectile electrons from the source and the orbital electrons of the medium are identical, quantum mechanical exchange effect needs to be taken into account.

Energy-loss mechanisms for electrons

a) Collisions with electrons (causing ionization/excitation of atoms) [important for slow electrons]
b) Radiation (Brehmstrahlung loss) in the EM field of nuclei of the medium [important for fast electrons]
c) Elastic scattering at low energies [important for slow electrons]

Stopping power of β-particles (KE loss per unit length, or Energy gradient)

$$\frac{dE}{dx} = \left(\frac{dE}{dx}\right)_{\text{ion}} + \left(\frac{dE}{dx}\right)_{\text{rad}}$$

$$\frac{\left(\frac{\boldsymbol{dE}}{\boldsymbol{dx}}\right)_{\textbf{rad}}}{\left(\frac{\boldsymbol{dE}}{\boldsymbol{dx}}\right)_{\textbf{ion}}} = \frac{\boldsymbol{EZ}}{\boldsymbol{1600 m_0 c^2}} \quad \begin{bmatrix} \text{where } E \text{ is in MeV} \\ \textbf{Empirical relation} \end{bmatrix}$$

$$\text{When } E = E_{\text{c}}, \left(\frac{dE}{dx}\right)_{\text{rad}} = \left(\frac{dE}{dx}\right)_{\text{ion}} \Rightarrow \boldsymbol{E_{\text{c}}} = \frac{\boldsymbol{1600 m_0 c^2}}{\boldsymbol{Z}} \approx \begin{cases} 62.9 \text{ MeV for Aluminium } [Z = 13] \\ 31.5 \text{ MeV for Iron } [Z = 26] \\ 28.2 \text{ MeV for Copper } [Z = 29] \\ 114.5 \text{ MeV for Air } [Z_{\text{eff}} \approx 7.14] \end{cases}$$

Critical energy of β-radiation is the energy at which rate of radiation loss equals the rate of ionization loss.

What is the critical energy at which the losses due to collusion and radiation are equal for fast electrons when they interact with (i) Lead target (ii) Silver target?

(i) $E_{\text{c}} = \dfrac{1600 m_0 c^2}{Z} \approx 10 \text{ MeV} \quad [\because Z = 82]$

(ii) $E_{\text{c}} = \dfrac{1600 m_0 c^2}{Z} \approx 17.4 \text{ MeV} \quad [\because Z = 47]$

The total cross-section for the scattering and absorption of neutrons of certain energy is 0.3 barns for Copper. Find the fraction of neutrons of that energy which penetrates 10 cm in Copper. At what distance will the neutron intensity drop to one-half of its initial value? (1 barn = 10^{-28} m² = 100 fm² and density of Copper = 8.96 g/cc)

$$\boldsymbol{I = I_0 e^{-\sigma n x}} \quad [n = \text{concentration of nuclei}] \Longrightarrow \frac{I}{I_0} = e^{-\sigma n x} = 0.775 \left[n = \frac{8.96 \times 10^{-3}}{63.5u}\text{cm}^{-3} \approx 8.5 \times 10^{22}\ \text{cm}^{-3}\right]$$

When $I = 0.5I_0$, $e^{-\sigma n x} = \frac{I}{I_0} = 0.5 \Longrightarrow \sigma n x = \ln 2 \Longrightarrow x = \frac{\ln 2}{\sigma n} \approx 8.153$ cm

Interaction of photons (γ-rays) with matter

Three important processes of interaction of a photon with matter are:

1. **Photoelectric interaction** (interacts, gets absorbed, orbital electron gets ejected)
2. **Compton scattering** (interacts, loses energy and changes direction)
3. **Pair production** (interacts, disappears and creates an electron-positron pair)

Photoelectric interaction

Kinetic energy of ejected electron, $E_\text{e} = E_\gamma - B_\text{e}$

Scattering cross section, $\sigma_\text{PE} \propto \boldsymbol{Z^5}\left(\frac{mc^2}{E_\gamma}\right)^{3.5}$

Compton scattering

Scattering cross section, σ_CE is independent of Z.

$$\begin{cases} \textbf{Conservation of energy: } E_\gamma + mc^2 = E_\gamma' + \gamma mc^2 \Longrightarrow E_\gamma + mc^2 = E_\gamma' + \dfrac{mc^2}{\sqrt{1-\beta^2}} \\ \textbf{Conservation of linear momentum: } \begin{cases} \dfrac{E_\gamma}{c} = \dfrac{E_\gamma'}{c}\cos\theta + \dfrac{mc\beta\cos\phi}{\sqrt{1-\beta^2}} \Longrightarrow \dfrac{mc\beta\cos\phi}{\sqrt{1-\beta^2}} = \dfrac{E_\gamma - E_\gamma'\cos\theta}{c} \\ 0 = \dfrac{E_\gamma'}{c}\sin\theta - \dfrac{mc\beta\sin\phi}{\sqrt{1-\beta^2}} \Longrightarrow \dfrac{mc\beta\sin\phi}{\sqrt{1-\beta^2}} = \dfrac{E_\gamma'}{c}\sin\theta \end{cases} \end{cases}$$

$$\begin{bmatrix} \text{Lorentz factor, } \gamma = \dfrac{1}{\sqrt{1-\beta^2}} = \dfrac{1}{\sqrt{1-v^2/c^2}} \\ p = \gamma mv = \dfrac{mv}{\sqrt{1-\beta^2}} = \dfrac{mc\beta}{\sqrt{1-\beta^2}} \end{bmatrix} \Longrightarrow \begin{cases} \dfrac{mc^2}{\sqrt{1-\beta^2}} = E_\gamma + mc^2 - E_\gamma' \\ \dfrac{m^2c^2\beta^2}{1-\beta^2} = \dfrac{\left(E_\gamma - E_\gamma'\cos\theta\right)^2}{c^2} + \dfrac{E_\gamma'^2\sin^2\theta}{c^2} \end{cases}$$

$$\Longrightarrow \begin{cases} \dfrac{m^2c^4}{1-\beta^2} = E_\gamma^2 + m^2c^4 + E_\gamma'^2 - 2E_\gamma E_\gamma' - 2E_\gamma' mc^2 + 2E_\gamma mc^2 \\ \dfrac{m^2c^4\beta^2}{1-\beta^2} = E_\gamma^2 + E_\gamma'^2 - 2E_\gamma E_\gamma'\cos\theta \end{cases} \quad \text{[next, subtract second from first]}$$

$$\Longrightarrow m^2c^4 = m^2c^4 - 2E_\gamma E_\gamma'(1-\cos\theta) + 2mc^2\left(E_\gamma - E_\gamma'\right) \Longrightarrow E_\gamma E_\gamma'(1-\cos\theta) = mc^2\left(E_\gamma - E_\gamma'\right)$$

$$\Longrightarrow \alpha E_\gamma'(1-\cos\theta) = E_\gamma - E_\gamma' \quad \left[\alpha = \frac{E_\gamma}{mc^2}\right] \Longrightarrow \boldsymbol{E_\gamma' = \frac{E_\gamma}{1+\alpha(1-\cos\theta)}}$$

$$\therefore \text{Change in KE of electron, } E_\text{e} = E_\gamma - E_\gamma' = E_\gamma\left\{1 - \frac{1}{1+\alpha(1-\cos\theta)}\right\} = E_\gamma\left\{\frac{\alpha(1-\cos\theta)}{1+\alpha(1-\cos\theta)}\right\}$$

Pair Production

$\gamma \rightarrow \text{e}^- + \text{e}^+$; $E_\text{min} = 2m = 2 \times 0.511$ MeV ≈ 1.02 MeV

If energy of photon, $E_\gamma = E_\text{min}$, the newly created particles are at rest and can recombine. So $E_\gamma > E_\text{min}$ for pair production so that the particles may have sufficient KE to move apart. $\therefore E_\gamma = 2mc^2 + E_{e^-} + E_{e^+}$

Absorption cross section, $\sigma_{PP} \propto \boldsymbol{Z^2}\left(A \ln\frac{1}{Z^{1/3}} + B\right)$

Properties of deuteron in ground state.

Deuteron is the only two-nucleon bound system (made up of a proton and a neutron) that exists in nature. Other two possible two-nucleon systems, diproton and dineutron, do not exist as bound systems.

1. Binding energy of deuteron is E_B = 2.2245 MeV. So the binding energy per nucleon (E_B/A) is 1.1122 MeV which is much less than the mean binding fraction (E_B/A) for nuclei with mass numbers more than 4. For α-particles, it 7.07 MeV. This indicates that deuteron is a weakly bound structure.
2. Spin (total angular momentum) of deuteron in its ground state (in unit of ħ) is I = 1
3. Measured magnetic moment of deuteron (in unit of μ_N) is μ = 0.8574
4. Deuteron possesses a small but finite electric quadrupole moment, Q = +0.282 e-fm^2
5. The parity of deuteron ground state is even.

Estimate the depth of potential well for the range of nuclear force from 2 fm to 6 fm in steps of 0.5 fm and draw the graph of potential depth versus range.

Compare and contrast nuclear and electromagnetic force.

Nuclear force	Electromagnetic force
Short range force	Long range force
Very strong	Much less strong
Charge-indepenendent	Charge-depenendent
Spin-dependent	Spin-independent
Saturative (interacts only with neighbouring nucleons)	Non-saturative (interacts globally)

<u>**Characteristics of nuclear force**</u>

1. **Strong and attractive** — Binding energy per nucleon is large (~ 8 MeV) and bound nuclei are found in nature
2. **Short and saturative** — Binding energy, $E_B \propto$ A, implies that each nucleon within the nucleus interacts only with its neigbouring nucleons, thereby the force is saturated. If every nucleon interacts with all other nucleons, then $E_B \propto A(A-1)/2 \approx A^2$. This square relation for binding energy is not observed.
3. **Charge independent** — Heisenberg in 1932 proposed that proton-proton nuclear force is equal to the neutron-neutron nuclear force i.e. $F_{nn} = F_{pp}$ which may be understood through the binding energy consideration of Tritium and Helium-3. Binding energy contribution due to nuclear force must be equal in both if it is charge-independent as $F_{nn} = F_{pp} \Rightarrow F_{nn} + F(^2H) = F_{pp} + F(^2H) \Rightarrow F(^3H) = F(^3He)$ which is just so as the observed binding energies of ^3H and ^3He are 8.48 MeV and 7.72 MeV respectively and the discrepancy of 0.76 MeV is due to Coulomb force in ^3He.
4. **Spin dependent** — Nuclear force is sensitive to the direction of spin and is stronger when the spins are parallel than when they are antiparallel (similar to magnetic force). Due to this and Pauli's exclusion principle, diproton and dineutron do not exist in nature. Spin dependency may be verified through the observed spin of deuteron in ground state which entirely exhibits the triplet state (S = 1) opposed to the singlet state (S = 0). Moreover, the scattering cross-section was found to be significantly different for the two states of nucleus.
5. **Exchange nature (Yukawa's meson theory)** — Yukawa in 1935 suggested the exchange nature of nuclear force in analogy with electromagnetic force wherein two charged particles interact via exchange of photons. The mediator for the nuclear force was called **meson** with lifetime equal to the time of exchange. The mass of meson may be estimated using the uncertainty principle assuming the particle velocity during exchange to be nearly the speed of light, c. $\Delta E = \hbar/2\Delta t = \hbar c/2\Delta x$. If this energy is all transformed into the rest-mass energy of meson, then $mc^2 = \hbar c/\Delta x \Rightarrow m = \hbar/(c\Delta x)$ which comes to around 112.2 MeV if Δx = 1 fm (≈ inter-nucleon separation).

A certain star, located 0.8 kpc from Earth, is found to be dimmer than expected at 550 nm by A_V = 1.1, where A_V is the amount of extinction (in magnitudes) measured through the visual wavelength filter. If Q_{550} = 1.5 and the dust grains are spherical with radii 0.2 µm, estimate the average density of material between the star and Earth.

$$\sigma_V = \sigma_g Q_V = (\pi a^2) Q_V \approx 1.885 \times 10^{-13}\ \mathrm{m}^2 \quad [a = 0.2\ \mu\mathrm{m}\ ;\ Q_V = 1.5]$$

$$A_V = 1.086\tau_V \Longrightarrow \tau_V = \frac{A_V}{1.086} = \frac{1.1}{1.086} \approx 1.0129$$

$$\tau_V = \sigma_V N_d \Longrightarrow N_d = \frac{\tau_V}{\sigma_V} = \frac{1.0129}{1.885 \times 10^{-13}\ \mathrm{m}^2} \approx 5.37 \times 10^{12}\ \mathrm{m}^{-2}$$

$$N_d = \bar{n}s \Longrightarrow \bar{n} = \frac{N_d}{s} = \frac{5.37 \times 10^{12}\ \mathrm{m}^{-2}}{0.8 \times 10^3\ \mathrm{pc}} = \frac{5.37 \times 10^{12}\ \mathrm{m}^{-2}}{2.47 \times 10^{19}\ \mathrm{m}} \approx 2.174 \times 10^{-7}\ \mathrm{m}^{-3}$$

At a certain time, a measurement establishes the position of an electron with an accuracy of ±10^{-11} m. Calculate the uncertainty in the electron's momentum and from this, the uncertainty in its position 1 sec later.

$$\Delta p = \frac{\hbar}{\Delta x} \Longrightarrow \Delta(mv) \approx 1.055 \times 10^{-23}\ \mathrm{kg\text{-}m/s}$$

$$\therefore \Delta v = \frac{\Delta(mv)}{m} \approx 1.16 \times 10^7\ \mathrm{m/s} \ll c$$

$$\therefore \Delta x' = t\Delta v = 1.16 \times 10^7\ \mathrm{m}$$

A surface is kept 30 cm away from a 60 W source of λ = 5 × 10^{-5} cm. Estimate (i) the average time interval between the arrival of two photons on the same atom (sectional area ~10^{-16} cm^2), and (ii) the mean lag between irradiation and ejection of the first electron from the surface (area 5 cm^2 ; efficiency 50%).

$$\text{Energy flux at the surface} = \frac{L}{4\pi d^2} = \frac{60\ \mathrm{W}}{4\pi (0.3\ \mathrm{m})^2} \approx 53\ \mathrm{W/m^2}$$

$$\text{Energy of photon} = \frac{hc}{\lambda} \approx 4 \times 10^{-21}\ \mathrm{J}$$

$$\text{Photon flux at the surface} = \frac{53}{4 \times 10^{-21}}\ /\mathrm{m^2/s} \approx 1.33 \times 10^{22}\ /\mathrm{m^2/s}$$

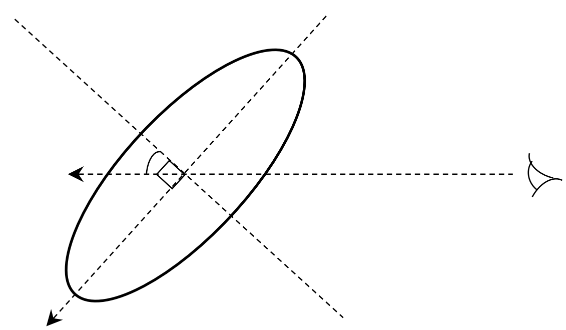
$$\text{Photon flux on the atom} = 1.33 \times 10^{22}\ /\mathrm{m^2/s} \times 10^{-20}\ \mathrm{m^2} = 1.33 \times 10^{-2}\ /\mathrm{s}$$

$$\therefore \text{Average time interval of arrival, } \Delta t = \frac{1}{1.33 \times 10^{-13}\ /\mathrm{s}} \approx 75\ \mathrm{s}$$

$$\text{Electron ejection rate} = 1.33 \times 10^{22}\ /\mathrm{m^2/s} \times 5 \times 10^{-4}\ \mathrm{m^2} \times 50\% = 3.3 \times 10^{22}\ /\mathrm{s}$$

$$\therefore \text{Mean time lag} = \frac{1}{3.3 \times 10^{22}\ /\mathrm{s}} \approx 3 \times 10^{-21}\ \mathrm{s}$$

Apparent radial velocity, $v' = v\cos(90° - i) = v\sin i$ where i is the inclination angle of the system (angle b/w the normal to the plane of the system and line of sight)



Escape velocity

For a massive particle to become free, $\frac{1}{2}mv^2 - \frac{GMm}{R} \geq 0$

At escape velocity, $\frac{1}{2}m{v_{\text{esc0}}}^2 = \frac{GMm}{R} \Rightarrow v_{\text{esc0}} = \sqrt{\frac{GM}{R}}$ [This is the escape velocity at the surface.]

At an height h, $\frac{1}{2}m{v_0}^2 - \frac{GMm}{R} = \frac{1}{2}mv^2 - \frac{GMm}{R+h} \Rightarrow v^2 = {v_0}^2 - 2GM\left\{\frac{1}{R+h} - \frac{1}{R}\right\} \approx {v_0}^2 - \frac{2GM}{R}\left\{-\frac{h}{R}\right\}$

At height h, $v_{\text{esc}} = \sqrt{\frac{GM}{R+h}} \Rightarrow {v_{\text{esc}}}^2 = \frac{GM}{R+h} \approx \frac{GM}{R}\left(1 - \frac{h}{R}\right)$

For the particle to escape, $v^2 \geq {v_{\text{esc}}}^2$

$\therefore {v_0}^2 + \frac{2GMh}{R^2} = {v_{\text{esc}}}^2 \Rightarrow {v_0}^2 = \frac{GM}{R}\left(1 - \frac{h}{R}\right) - \frac{2GMh}{R^2} = \frac{GM}{R}\left(1 - \frac{3h}{R}\right) \Rightarrow v_0 = v_{\text{esc0}}\sqrt{1 - \frac{3h}{R}} \approx v_{\text{esc0}}\left(1 - \frac{3h}{2R}\right)$

For gas molecules, $v_{rms} = \sqrt{\frac{3kT}{m}}$

Schwarzschild Radius

Schwarzschild radius, $R_{\text{s}} = \frac{2GM}{c^2} \approx (1.485 \times 10^{-27}\ \text{m/kg})M$

An object can be a black hole if $R < R_{\text{s}} \Rightarrow \frac{R}{M} < \frac{R_{\text{s}}}{M} \Rightarrow \boldsymbol{\frac{R}{M} < 1.485 \times 10^{-27}\ \textbf{m/kg}}$

For Sun, $R_s = \frac{2GM_{\odot}}{c^2} \approx 3\ \text{km}$

For Earth, $R_{\text{s}} = \frac{2GM_{\oplus}}{c^2} \approx 9\ \text{mm}$

For Moon, $R_{\text{s}} = \frac{2GM_{\text{Moon}}}{c^2} \approx 0.1\ \text{mm}$

For Proton or Neutron, $R_{\text{s}} \approx 2.48 \times 10^{-54}\ \text{m}$

For Electron, $R_{\text{s}} \approx 1.35 \times 10^{-57}\ \text{m}$

Mossbauer Spectroscopy

$E_{\text{rec}} = \frac{{p_\gamma}^2}{2m} = \frac{{E_\gamma}^2}{2mc^2} = \frac{(E_0 - E_{\text{rec}})^2}{2mc^2} \Rightarrow {E_{\text{rec}}}^2 - 2(E_0 + mc^2)E_{\text{rec}} + {E_0}^2 = 0$

$\Rightarrow E_{\text{rec}} = (E_0 + mc^2) \pm \sqrt{(E_0 + mc^2)^2 - {E_0}^2} = (E_0 + mc^2) \pm mc^2\sqrt{\frac{2E_0}{mc^2} + 1} = E_0 + mc^2\left(1 \pm \sqrt{1 + \frac{2E_0}{mc^2}}\right)$

$\Rightarrow E_{\text{rec}} \approx E_0 + mc^2\left\{1 - \left(1 + \frac{E_0}{mc^2} - \frac{1}{8}\left(\frac{2E_0}{mc^2}\right)^2\right)\right\} = E_0 + mc^2\left\{-\frac{E_0}{mc^2} + \frac{1}{2}\left(\frac{E_0}{mc^2}\right)^2\right\} \Rightarrow \boldsymbol{E_{\text{rec}} \approx \frac{{E_0}^2}{2mc^2}}$

Calculate the recoil velocity and energy of a free Mossbauer nucleus ^{119}Sn emitting a γ-ray of frequency 5.76 × 10^{18} Hz. What is Doppler shift of the photon to an external observer?

$E_{\text{rec}} = \frac{{E_0}^2}{2mc^2} = \frac{(h\nu)^2}{2mc^2} \approx 4.1 \times 10^{-22}\ \text{J} \approx 2.56\ \text{meV}$ $[\because m(^{119}\text{Sn}) = 119\text{u}]$

$\frac{1}{2}m{v_{\text{rec}}}^2 = \frac{{E_0}^2}{2mc^2} \Rightarrow {v_{\text{rec}}}^2 = \frac{{E_0}^2}{m^2c^2} \Rightarrow v_{\text{rec}} = \frac{E_0}{mc} = \frac{h\nu}{mc} \approx 64.4\ \text{m/s}$

$\frac{\Delta\nu}{\nu} = \frac{v_{\text{rec}}}{c} \Rightarrow \Delta\nu = \frac{v_{\text{rec}}}{c}\nu = \frac{\nu}{c}\left(\frac{h\nu}{mc}\right) = \frac{h}{m}\left(\frac{\nu}{c}\right)^2 \approx 1.24 \times 10^{12}\ \text{Hz} = 1.24\ \text{THz}$

Calculate the amount of power generated by solar cell of area 100 cm^2. Assume the solar radiation as 1000 W/m^2.and efficiency as 15%.

Power generated $= 0.15 \times (1000\ \text{W/m}^2) \times (100 \times 10^{-4}\ \text{m}^2) = 1.5\ \text{W}$

Calculate the declination angle for March 31 in a leap year.

For March 31, $n = 31 + 29 + 31 = 91$

Declination, $\delta = 23.45° \times \sin\left\{\frac{360}{365}(284 + n)\right\} \approx 4.017°$

Calculate the hour angle at 02:30 p.m.

Hour angle, $\omega = 15(\text{ST} - 12\text{ hours}) = 15(14.5 - 12\text{ hours}) \approx 37.5°$

Determine the sunset hour angle and day length for Lucknow (26°50' N latitude) for March 31.

For March 31, $n = 31 + 29 + 31 = 91$

Declination, $\delta = 23.45° \times \sin\left\{\frac{360}{365}(284 + n)\right\} \approx 4.017°$

Sunset hour angle, $\omega_s = \cos^{-1}(-\tan\phi\tan\delta) \approx 92°$

Determine the sunset hour angle and day length for Allahabad (24°25' N latitude) for January 1.

For January 1, $n = 1$

Declination, $\delta = 23.45° \times \sin\left\{\frac{360}{365}(284 + n)\right\} \approx -23°$

Sunset hour angle, $\omega_s = \cos^{-1}(-\tan\phi\tan\delta) \approx 79°$

$t_d = \frac{2}{15}\omega_s \approx 10.53\text{ hours}$

Determine the sunset hour angle and day length at a latitude of 35° N on February 14.

For February 14, $n = 45$

Declination, $\delta = 23.45° \times \sin\left\{\frac{360}{365}(284 + n)\right\} \approx -13.62°$

Sunset hour angle, $\omega_s = \cos^{-1}(-\tan\phi\tan\delta) \approx 80.23°$

$t_d = \frac{2}{15}\omega_s \approx 10.7\text{ hours}$

Calculate the number of daylight hours at Delhi on December 21 and June 21 in a leap year. (Given: *φ* = 28°35')

For December 21, $n = 366 - 10 = 356$

For June 21, $n = 31 + 29 + 31 + 30 + 31 + 21 = 173$

Declination, $\delta = 23.45° \times \sin\left\{\frac{360}{365}(284 + n)\right\} \approx \begin{cases} -23.445° \text{ for December 21} \\ \ \ 23.448° \text{ for June 21} \end{cases}$

$t_d = \frac{2}{15}\cos^{-1}(-\tan\phi\tan\delta) \approx \begin{cases} 10.18 \text{ hours for December 21} \\ 13.82 \text{ hours for June 21} \end{cases}$

Determine the local apparent time (LAT) corresponding to 1430 hours (IST) on July 1 at Mumbai (72°51' E longitude). Equation of time is –4'.

LAT = ST – 4' × (ST longitude – local longitude) + Equation of time correction

= 14.5 hours – 4' × (82.5 – 72.85) – 4' = 13 h 47 m 24 s

Determine the local apparent time (LAT) and declination at Ahmedabad (72°40' E longitude 23°00' N latitude) corresponding to 1430 hours (IST) on December 15. Equation of time is 5'13".

LAT = ST – 4' × (ST longitude – local longitude) + Equation of time correction

= 14.5 hours – 4' × (82.5 – 72.67) + 5'13" = 13 h 55 m 53 s

For December 15, $n = 365 - 16 = 349$

Declination, $\delta = 23.45° \times \sin\left\{\frac{360}{365}(284 + n)\right\} \approx -23.33523° \approx -23°20'6.83''$

A solar cell with area 4 cm^2 has I_{SC} = 0.17 A and V_{OC} = 0.6 V. It behaves as a constant current source with current 0.166 A up to a voltage of 0.5 V. Calculate the fill factor and efficiency. (Consider the incident solar radiation as 1000 W/m^2).

filling factor, $FF = \frac{V_\mathrm{m} I_\mathrm{m}}{V_\mathrm{OC} I_\mathrm{SC}} = \frac{P_\mathrm{m}}{V_\mathrm{OC} I_\mathrm{SC}} = 81.37\%$

efficiency, $\eta = \frac{V_\mathrm{OC} I_\mathrm{SC} \times FF}{\text{incident solar radiation} \times \text{area of solar cell}} = \frac{V_\mathrm{m} I_\mathrm{m}}{1000\ \mathrm{W/m^2} \times 4{\times}10^{-4}\ \mathrm{m^2}} = 20.75\%$

Design a solar PV system to run 2 CFLs (14 W each) and 1 fan (60 W) for 6 hours a day. (Assumptions: Operating factor (OF) = 0.75, Combined efficiency = 0.81, Inverter efficiency = 0.90, Depth of discharge = 0.80, Battery voltage = 12 V, Battery rating = 120 Ah, Solar panel power rating (PPR) = 40 W)

Load estimation: Power rating $= 2 \times 14\ \mathrm{W} + 60\ \mathrm{W} = 88\ \mathrm{W}$

Energy required in a day $= 88\ \mathrm{W} \times 6\ \text{hours} = 528\ \mathrm{Wh}$

Actual power output, $AP = OF \times PPR = 0.75 \times 40\ \mathrm{W} = 30\ \mathrm{W}$

Power available (to end user), $PA = \eta \times AP = 0.81 \times 30\ \mathrm{W} = 24.3\ \mathrm{W}$

PV panel requirement: Solar energy harnessed in a day (8 hrs of sunlight) $= 24.3\ \mathrm{W} \times 8\ \text{hours} = 194.4\ \mathrm{Wh}$

$\therefore$ Number of solar panels required $= \frac{\text{energy generated}}{\text{energy consumed}} = \frac{528\ \mathrm{Wh}}{194.4\ \mathrm{Wh}} \approx 2.7 < 3$

Battery banks: Total Ah required $= \frac{\text{Total Wh per day}}{\text{Inverter efficiency} \times \text{Discharge depth} \times \text{Battery voltage}} \approx 61.11\ \mathrm{Ah}$

Number of batteries required $= \frac{\text{Total Ah required}}{\text{Battery rating}} = \frac{61.11\ \mathrm{Ah}}{120\ \mathrm{Ah}} \approx 0.51 < 1$

Cost estimation: Cost of solar panels $=$ ₹ $8000 \times 3 =$ ₹ 24000

Cost of 120 Ah battery $=$ ₹ 9000

Cost of inverter $=$ ₹ 7000

Installation charge $= 5\%$ of Equipment cost $= 0.05 \times$ ₹ $40000 =$ ₹ 2000

Total cost $=$ ₹ 42000

How much energy is required to pump 1000 litres of water from a depth of 15 m? (Assume the frictional forces to be zero.)

$E = mgH = (\rho V)gH = (1000\ \mathrm{kg/m^3} \times 1\ \mathrm{m^3}) \times 9.8\ \mathrm{m/s^2} \times 15\ \mathrm{m} = 147000\ \mathrm{J} = 147\ \mathrm{kJ}$

Design a solar PV system for pumping 25000 litres of water everyday from a depth of about 10 m (take drawdown as 2 m). Estimate the cost of the system?

Daily water requirement = 25 m^3

Total vertical lift = (10 + 2) m = 12 m

Total dynamic head (TDH) = 12 m + 5% of 12 m (frictional losses) = 12.6 m

Hydraulic energy required per day = mgH = $(\rho V)gH$ = 3087000 J ≈ 857.5 Wh

Wattage of PV panel required (assuming 6 h of peak sunshine duration) = 857.5 Wh / 6 h ≈ 142.9 W

Effective wattage of PV panel required (considering system losses and OF) = 142.9 W / (η_{pump} × MF × OF)

= 142.9 W / (0.3 × 0.85 × 0.75) ≈ 747.3 W

Number of 75 W PV panel required = 747.3 W / 75 W ≈ 9.96 ≈ 10

Power rating of the DC motor = 747.3 W / 746 W/hp = 1 hp

Cost of PV panels (assuming ₹ 200 per Watt) = ₹ 15000 × 10 = ₹ 150000

Cost of 1 hp DC motor and pump = ₹ 5000

Total cost (including 5% installation cost) = ₹ 155000 + 5% of ₹ 155000 = ₹ 162750

Design a solar PV system to pump 10000 litres of water from a depth of 4 m (the drawdown will be about 2 m). The water needs to be discharged at about 5 m from the well. What would be the cost of the system?

Daily water requirement = 10 m^3

Total vertical lift = (4 + 2 + 5) m = 11 m

Total dynamic head (TDH) = 11 m + 5% of 11 m (frictional losses) = 11.55 m

Hydraulic energy required per day = $mgH = (\rho V)gH$ = 1131900 J ≈ 314.42 Wh

Wattage of PV panel required (assuming 6 h of peak sunshine duration) = 314.42 Wh / 6 h ≈ 52.4 W

Effective wattage of PV panel required (considering system losses and OF) = 52.4 W / (η_{pump} × MF × OF)

= 52.4 W / (0.3 × 0.85 × 0.75) ≈ 274 W

Number of 75 W PV panel required = 274 W / 75 W ≈ 3.65 ≈ 4

Power rating of the DC motor = 274 W / 746 hp/W = 0.37 hp

Cost of PV panels (assuming ₹ 200 per Watt) = ₹ 15000 × 4 = ₹ 60000

Cost of 0.5 hp DC motor and pump = ₹ 4000

Total cost (including 5% installation cost) = ₹ 64000 + 5% of ₹ 64000 = ₹ 67200

Fourier Transform

$$\mathcal{F}\{1\} = \int_{-\infty}^{\infty} 1 \cdot e^{-i\omega t}\, dt = 2\pi\delta(\omega) \quad \left[\because \delta(\omega - a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\omega-a)t}\, dt\right]$$

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-i\omega t}\, dt = e^{-i\omega t}\Big|_{t=0} = 1 \quad \left[\because \int_{-\infty}^{\infty} \delta(t-a)\cdot f(t)\, dt = f(a)\right]$$

$$\mathcal{F}\{e^{iat}\} = \int_{-\infty}^{\infty} e^{iat} \cdot e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} e^{-i(\omega-a)t}\, dt = 2\pi\delta(\omega - a)$$

$$\mathcal{F}\{\cos at\} = \int_{-\infty}^{\infty} \frac{e^{iat} + e^{-iat}}{2} \cdot e^{-i\omega t}\, dt = \frac{1}{2}(\mathcal{F}\{e^{iat}\} + \mathcal{F}\{e^{-iat}\}) = \pi\big(\delta(\omega - a) + \delta(\omega + a)\big)$$

$$\mathcal{F}\{\sin at\} = \int_{-\infty}^{\infty} \frac{e^{iat} - e^{-iat}}{2i} \cdot e^{-i\omega t}\, dt = \frac{1}{2i}(\mathcal{F}\{e^{iat}\} - \mathcal{F}\{e^{-iat}\}) = -i\pi\big(\delta(\omega - a) - \delta(\omega + a)\big)$$

$$\mathcal{F}\{t\} = \int_{-\infty}^{\infty} t \cdot e^{-i\omega t}\, dt = 2\pi i\delta'(\omega) \quad \left[\because \delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\, dt \Rightarrow \delta'(\omega) = \frac{-i}{2\pi}\int_{-\infty}^{\infty} te^{-i\omega t}\, dt\right]$$

$$\mathcal{F}\{t^n\} = \int_{-\infty}^{\infty} t^n \cdot e^{-i\omega t}\, dt = 2\pi i^n\delta^{(n)}(\omega) \quad \left[\begin{array}{c}\because \delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\, dt \Rightarrow \delta^{(n)}(\omega) = \frac{(-i)^n}{2\pi}\int_{-\infty}^{\infty} t^n e^{-i\omega t}\, dt \\ \text{Here, } n \in W\end{array}\right]$$

$$\mathcal{F}\left\{\frac{1}{t}\right\} = \int_{-\infty}^{\infty} \frac{1}{t} \cdot e^{-i\omega t}\, dt = -i\pi \operatorname{sgn}\omega$$

$$\mathcal{F}\{\operatorname{sgn} t\} = \int_{-\infty}^{\infty} \operatorname{sgn} t \cdot e^{-i\omega t}\, dt = -\frac{2i}{\omega}$$

Fourier Transform: Gaussian function

$$f(t) = e^{-\frac{t^2}{a^2}} \Rightarrow \mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} e^{-\frac{t^2}{a^2}} e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} e^{-\left(\frac{t^2}{a^2} + i\omega t\right)}\, dt = e^{\frac{a^2(i\omega)^2}{4}} \int_{-\infty}^{\infty} e^{-\left(\frac{t^2}{a^2} + i\omega t + \frac{a^2(i\omega)^2}{4}\right)}\, dt$$

$$= e^{-\frac{a^2\omega^2}{4}} \int_{-\infty}^{\infty} e^{-\left(\frac{t}{a} + \frac{a(i\omega)}{2}\right)^2}\, dt \Rightarrow F(\omega) = \left(a\sqrt{\pi}\right)e^{-\frac{a^2\omega^2}{4}} \text{ which is a Gaussian in the conjugate variable space.}$$