Introductory problem set for source modelling using analytical methods

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Below we list a set of problems (elementary level) related to modelling of inspiralling compact binaries (binary systems with neutron star and/or BHs as components) as a warm-up exercise for those who would like to do a project in analytical modelling of GW sources. Although, most of the problems listed below can be done by hand up to a large extent, we suggest you to write Mathematica codes for solving them which will be useful while you would be working on the project this summer.

- **Energy loss from inspiralling compact binary systems:** According to the General Relativity (GR) binary systems spiral inwards due to radiation reaction effects, as they lose energy and angular momentum in the form of outgoing gravitational waves (GWs). The leading order contribution to the energy loss is quadrupolar in nature and was first computed by Einstein (known as Quadrupole Formula for energy loss). Peters and Mathews [1] first obtained the energy loss formula for two point particles moving in Keplerian orbit and is given by Eq. (15) of Ref. [1]. Further, average rate at which the system was loosing energy was obtained by averaging Eq. (15) over the orbit of elliptical motion and the result was combined given as Eq. (16)-(17).

  1. We would like you to reproduce Eq. (15)-(17) of Ref. [1] following the method presented there.
  2. Plot the *enhancement function* as a function of eccentricity of the system given by Eq. (17).

- **Secular evolution of orbital elements:** Peters [2] computed the rate at which orbital parameters of the binary change (for semi-major axis (a) and eccentricity (e) of the binary) based on the energy and orbital angular momentum losses and the corresponding expressions were given by Eq. (5.6) and (5.7). Next, these two differential equations were used to obtain variation of the semi-major axis (a) as a function of the eccentricity (e) of the system and the result is given by Eq. (5.11) there.

  1. We would like you to reproduce Eq. (5.6) and (5.7) given there using the formulae presented in Sec. V of that paper.
  2. Further, using these two equation reproduce Eq. (5.11) of the same paper which gives variation of the semi-major axis (a) as a function of eccentricity (e).
  3. Use Eq. (5.11) to reproduce the Fig. (1) of the paper.
  4. Starting From Eq. (5.11) find expressions for a-e relation in the limit when $e \to 0$, and $e \to 1$.

- **Loss of linear momentum from ICBs and estimates of recoil velocities:** In addition to the fact that gravitational waves carry energy and angular momentum of the source, they also can carry linear momentum if source has some sort of asymmetry such as the one associated with mass (binary composed of components with different masses). Mishra, Arun and Iyer [3] computed the loss rate of linear momentum from ICBs moving in circular orbits during the *inspiral* phase of binary's evolution and the expression is given by Eq. (3.23) of [3]. This was further used to compute the recoil velocity accumulated during the inspiral phase and the related analytical expression is given by Eq. (4.6) of [3].

  1. We would like you to reproduce Eq. (4.6) of Ref. [3] starting from Eq. (3.23) there.
  2. Plot numerical value of recoil velocity as a function of the parameter ν appearing there (a plot similar to the one shown in Fig. 1 of the same work). Here ν is a dimensionless mass parameter and takes values in the range [0, 0.25]. Choose $x = 1/6$ when plotting the result.

Note: In case, you have difficulties retrieving the references mentioned above, please write to `chandra@icts.res.in` and we shall share the soft copy of the reference with you. Although, all the inputs you would require for the above exercise are there in respective reference, in case you feel some of the desired inputs are missing then write to the same email address and we would

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help you in that context. Finally, we would like you to have your solutions in the form of the Mathematica notebooks. Figures can be displayed in the Mathematica notebooks only.

[1] P. Peters and J. Mathews, "Gravitational Radiation from Point Masses in a Keplerian Orbit", Phys. Rev. **131**, 435 (1963) PM63.
[2] P. Peters, "Gravitational Radiation and the Motion of Two Point Masses", Phys. Rev. **136**, B1224 (1964) Pe64.
[3] C. K. Mishra, K. G. Arun and B. R. Iyer, "2.5PN linear momentum flux from inspiralling compact binaries in quasicircular orbits and associated recoil: Nonspinning case", arXiv:1111.2704.

Numerical methods for Physics and Astrophysics: Lab exercises

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I. LAB 1

A. Finite differencing, convergence, error estimates

We derived the following finite-differencing approximants for the derivative of a function $f(x)$:

$$\begin{aligned} \textit{Forward differencing}: \quad & f'(x) \simeq \tfrac{f(x+h)-f(x)}{h} + O(h) && (1.1)\\ \textit{Backward differencing}: \quad & f'(x) \simeq \tfrac{f(x)-f(x-h)}{h} + O(h) && (1.2)\\ \textit{Central differencing}: \quad & f'(x) \simeq \tfrac{f(x+h)-f(x-h)}{2h} + O(h^2) && (1.3)\end{aligned}$$

Problems:

1. Write a Python function to compute derivatives using these three finite differencing methods. Compute the derivative of the function $f(x) = e^x \sin(x)$ over the range $x = [0, 2\pi]$. Plot the numerically computed derivative $f'_{(h)}(x)$ for three different values of h.
2. Plot the error $\Delta f'_{(h)}(x) := |f'_{(h)}(x) - f'(x)|$ for three different values of h, and estimate the order of convergence of each finite-difference approximation.
3. Reduce the step size h successively. At what value of h does the round off error dominate the error budget?
4. Derive a central differencing approximant for the second derivative $f''(x)$ using the Taylor expansions of $f(x+h)$ and $f(x-h)$.

B. Richardson extrapolation

We have seen that, if the order of the error in the numerical estimate of a function $f(x)$ is known, Richardson extrapolation provides a powerful way of improving the accuracy of the estimate. If we have two numerical estimates $f_h(x)$ and $f_{h/2}(x)$ each having an error of $O\,h^k$, a better estimate is given by

$$f(x) \simeq \frac{2^k\, f_{h/2}(x) - f_h(x)}{2^k - 1} + O(h^l), \tag{1.4}$$

where l is the next-to-leading-order error term (for e.g., $l = k + 2$ for central differencing, while $l = k + 1$ for forward/backward differencing).

Problems:

1. Gravitational-waves (GWs) have two independent polarization states – called "plus" and "cross" states. GW signals from the coalescence of black-hole binaries, in the simplest case, are circularly polarized:

$$h_+(t) \; = A(t)\, \cos\varphi(t), \quad h_\times(t) \; = A(t)\, \sin\varphi(t). \tag{1.5}$$

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Download the data file [1] containing $h_+(t)$ and $h_\times(t)$. (This is the reduced form of the data produced by a numerical-relativity simulation of black-hole binaries performed by the Caltech-Cornell-CITA collaboration and is publicly available at [2]). Compute the phase evolution $\varphi(t)$, the frequency evolution $\omega(t) := d\varphi(t)/dt$ and the rate of change of frequency $\dot{\omega}(t) := d\omega(t)/dt$ using second-order central difference approximation.

2. Estimate the order of convergence of the numerical computation of $\omega(t)$ and $\dot{\omega}(t)$.

3. Perform an extrapolation of $\omega(t)$ and $\dot{\omega}(t)$ to the next order using estimates of two different time-resolutions.

4. Derive an explicit expression for $f'(x)$ with error $O(h^4)$ using Richardson extrapolation. This is the fourth-order finite differencing approximant for the derivative, which we will use later.

II. LAB 2

A. Ordinary differential equations: Calculation of gravitational waves from inspiralling compact binaries

Problems:

The time evolution of the orbital phase $\varphi(t)$ of a binary of black holes evolving under the gravitational radiation reaction can be computed, in the post-Newtonian approximation, by solving the following coupled ODEs:

$$\frac{dv}{dt} = -\frac{\mathcal{F}(v)}{dE(v)/dv}, \qquad \frac{d\varphi}{dt} = \frac{v^3}{m}, \tag{2.1}$$

where $E(v)$ is the binding energy of the orbit, $\mathcal{F}(v)$ is the energy flux of radiated gravitational waves, $m := m_1 + m_2$ is the total mass of the binary, $v = (m\omega)^{1/3}$, ω being the orbital frequency. (Here we use geometric units, in which $G = c = 1$. This means that in all the expressions m has to be replaced by Gm/c^3.)

The binding energy and gravitational-wave flux are given as post-Newtonian expansions in terms of the small parameter v

$$E(v) = -\frac{1}{2}\mu v^2 \left[1 + O(v^2)\right], \quad \mathcal{F}(v) = \frac{32}{5}\left(\frac{\mu}{m}\right)^2 v^{10}\left[1 + O(v^2)\right], \tag{2.2}$$

where $\mu := m_1 m_2/m$ is the reduced mass of the system.

1. Compute v as a function of t by solving the first equation in Eq. (2.1) using Scipy's `odeint` routine. Assume the following parameters $m_1 = m_2 = 5M_\odot$, $v_0 = 0.3$, $\varphi_0 = 0$. Plot $v(t)$.

2. Solve the coupled system in Eq. (2.1) to compute v and φ. Compute and plot the two gravitational-wave polarizations:

$$h_+(t) = 4\frac{\mu}{m}v^2\,\cos\varphi(t), \quad h_\times(t) = 4\frac{\mu}{m}v^2\,\sin\varphi(t). \tag{2.3}$$

3. Repeat the same calculation using the adaptive Runge-Kutta method by Dormand & Prince using Scipy's `integrate.ode` module (choose method = "dopri5"). This method is very similar to the Runge-Kutta-Fehlberg method that we learned in the class.

III. LAB 3

A. Ordinary differential equations: Structure of a relativistic, spherically symmetric star

The interior structure of a relativistic, spherically symmetric star is described by a metric that has the line element

$$ds^2 = -e^{2\Phi(r)}\,c^2dt^2 + \left(1 - \frac{2Gm(r)}{rc^2}\right)^{-1}\,dr^2 + r^2\,d\Omega^2, \tag{3.1}$$

where $m(r)$ is called the *mass function* (which encapsulates the gravitational mass inside the radius r), $e^{2\Phi(r)}$ is the *lapse function* (which relates the proper time with the coordinate time). Outside the "surface" of the star (in vacuum), the spacetime is described by the Schwarzschild metric; the lapse function becomes

$$\Phi(r) = \frac{1}{2}\ln\left(1 - \frac{2Gm_\star}{rc^2}\right) \tag{3.2}$$

where $m_\star$ is the total (gravitational) mass of the star. The structure can be computed by solving the following set of ordinary differential equations, derived by Tolman, Oppenheimer and Volkoff (TOV).

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r), \tag{3.3}$$

$$\frac{dP(r)}{dr} = -\frac{G(\rho + P(r)/c^2)}{r^2}\left[m(r) + \frac{4\pi r^3 P(r)}{c^2}\right]\left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1}, \tag{3.4}$$

$$\frac{d\Phi(r)}{dr} = \frac{Gm(r) + 4\pi G r^3 P(r)/c^2}{c^2 r[r - 2Gm(r)/c^2]}. \tag{3.5}$$

The TOV equations have to be supplemented by an *equation of state* $P = P(\rho)$, that relates the pressure P to the energy density ρ. We assume the equating of state to be of polytropic form

$$P(r) = K\,\rho(r)^\gamma. \tag{3.6}$$

We also need to specify initial conditions for the variables m, P, Φ. The following conditions can be used

$$m(r=0) = 0, \quad P(r=0) = P_c = P(\rho_c), \quad \Phi(r_\star) = \frac{1}{2}\ln\left(1 - \frac{2Gm_\star}{r_\star c^2}\right) \tag{3.7}$$

Problems:

1. Compute the structure, i.e., $m(r)$ and $P(r)$, of a neutron star with central density $\rho_c = 5 \times 10^{17}$ kg/m^3 by solving Eqs. (3.3) and (3.4). Assume a polytropic equation of state with $\gamma = 5/3$ and $K = 5380.3$ (SI units). What is the mass $m_\star$ and radius $r_\star$ of the neutron star? (Useful tip: You will need to start the integration at $r = \Delta r$, where Δr is a small number. You can assume $m(r = \Delta r) := 4/3\,\pi\rho_c(\Delta r)^3$).
2. Compute the lapse function $e^{2\Phi(r)}$ by solving Eq.(3.5) starting from $r = r_\star$ to $r = 0$. On top of that, plot the lapse function for a Schwarzschild black hole (see Eq.3.2) from $r = r_s$ to $r = 2r_\star$, where $r_s \equiv 2Gm_\star/c^2$ is the Schwarzschild radius of the star. This exterior solution should match the interior solution at $r = r_\star$.

IV. LAB 4

A. Non-linear ordinary differential equations showing chaotic behavior: Lorenz equations

The Lorenz equations were originally developed as a simplified mathematical model for atmospheric convection by Edward Lorenz. This was the first set of equations where deterministic chaos was observed. The equations are given by

$$\frac{dx(t)}{dt} = \sigma[y(t) - x(t)], \quad \frac{dy(t)}{dt} = x(t)[\rho - z(t)] - y(t), \quad \frac{dz(t)}{dt} = x(t)y(t) - \beta z(t), \tag{4.1}$$

where ρ, σ and β are parameters of the system.

1. Problems:

1. Solve the Lorenz system for $\rho = 28, \sigma = 10$ and $\beta = 8/3$ with the following initial conditions $x(t=0) = y(t=0) = z(t=0) = 1$. Plot $x(t)$, $y(t)$ and $z(t)$ for $t = 0...100$. Is the solution deterministic or stochastic?
2. Repeat the calculation with same parameters except for a tiny change in the initial condition for x: i.e., $x(t=0) = 1 + 10^{-9}$. Plot $x(t=0) = y(t=0) = z(t=0) = 1$ on top of the earlier estimate. Explain the result.
3. Make a 3D plot of x, y, z. You should see the famous butterfly shaped structure now!
4. Optional exercise: Make an animation of the above [1].

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[1] You can either use the matplotlib animation package or convert a number of PNG files to a gif animation using ImageMagick.

B. Stochastic ordinary differential equations: Langevin equation

The random motion of a particle in a fluid due to collisions with the molecules of the fluid, called the Brownian motion, is described by the Langevin equation:

$$m\frac{d^2\mathbf{x}}{dt^2} = -\lambda\frac{d\mathbf{x}}{dt} + \boldsymbol{\eta}(t), \tag{4.2}$$

where m is the mass of the particle, $\mathbf{x}$ its position vector, λ a damping coefficient, and $\boldsymbol{\eta}(t)$ (called the *noise term*) describes the stochastic processes affecting the system.

1. Problems:

1. Using the Euler-Maruyama method, compute the $1-d$ Brownian motion trajectories generated by the Langevin equation $dx/dt = \eta(t)$, where $\eta(t)$ is Gaussian noise with zero mean and unit variance. Plot $x(t)$ for $t \in [0, 100]$ assuming $x(t=0) = 0$. Is the solution deterministic or stochastic?
2. Generalize the code so that it can deal with arbitrary number of particles. Plot $x(t)$ for 1000 particles on a single plot. Compute the average displacement $\bar{x}(t)$ of the particles (from $x(t=0)$) as a function of t and plot it against t. What is the relation between $\bar{x}(t)$ and t?
3. Using the `hist` function, plot the probability distribution $P(x)$ of $x(t)$ at $t = 10, 50, 100$.

V. LAB 5

A. Two-point boundary value problems: The shooting method

Solve the TOV equations described in Sec. III A as a two-point boundary value problem using the Shooting method. Use the Newton-Raphson method for root finding. The boundary conditions are given in Eq. (3.7).

VI. PARTIAL DIFFERENTIAL EQUATIONS

A few explicit first order numerical methods for solving the advection equation $\frac{\partial u}{\partial t} = -v\frac{\partial u}{\partial x}$ are given below:

$$\text{Forward time centered space(FTCS)}: \quad u_j^{n+1} = \frac{-v\Delta t}{2\Delta x}\left[u_{j+1}^n - u_{j-1}^n\right] + u_j^n \tag{6.1}$$

$$\text{Upwind method (for } v > 0): \quad u_j^{n+1} = \frac{-v\Delta t}{\Delta x}\left[u_j^n - u_{j-1}^n\right] + u_j^n \tag{6.2}$$

$$\text{Upwind method (for } v < 0): \quad u_j^{n+1} = \frac{-v\Delta t}{\Delta x}\left[u_{j+1}^n - u_j^n\right] + u_j^n \tag{6.3}$$

1. Problems:

1. Solve the advection equation with $v = 1$ and initial condition $u(t = 0, x) := \exp[-(x - x_0)^2]$ (Gaussian pulse) where $x \in [0, 25]$ and $x_0 = 5$ using the FTCS scheme. Is the evolution stable?
2. Repeat the calculation with upwind methods using Courant factor $\lambda := |v\Delta t|/\Delta x = 1/2, 1, 2$. Which of the evolutions are stable?
3. Plot the order of convergence of $u(x, t)$ as a function of x at $t = 10$ (use $\lambda = 1$).

VII. FOURIER AND SPECTRAL METHODS

A. Fast Fourier transform (FFT)

1. Problems:

1. Hercules X-1 is a high-mass X-ray binary (HMXB) system having a magnetized spinning neutron star which happens to be an X-ray pulsar. Here [3] you are given the data of an RXTE [4] observation after cleaning and pre-processing. The file contains two columns, (i) time (in seconds) and (ii) count-rate of X-ray photons (i.e., number of photons detected per unit time). Plot the count-rate as function of time. This is called lightcurve in X-ray astronomy. Compute the FFT of the given time-series and plot its absolute value against the frequency. Are you able to find any periodic signal(s)? The lowest frequency signal corresponds to the spin frequency of the neutron star. What is the spin period of this pulsar? Can you identify other periodic signals and how they are related to the lowest frequency signal? [2]

B. Power spectrum estimation using FFT

1. Problems:

1. A sample data set from the LIGO gravitational-wave observatory can be downloaded from here [5]. This contains 256 seconds of LIGO data from 2005, sampled at a rate of 4096 Hz. Compute the power spectral density of the data using Welch's modified periodogram method.

C. Time-frequency signal detection methods

1. Problems

1. 4U 1636–536 is a low-mass X-ray binary (LMXB) system consisting of a rapidly spinning weakly magnetized neutron star which does not exhibit any coherent X-ray pulsation like Hercules X-1. However, X-ray radiation from this source shows signals of high frequency (500–1000 Hz) quasi-periodic oscillations (QPOs). Here [6] you are given another data of RXTE observation. This contains the arrival times of photons. Identify the frequency span of this QPO using a time-frequency spectrogram or a PSD. Note: Firstly, you need to sample the data with a fixed sampling rate, say 4096 Hz.

D. Comouting correlations using FFT: Matched filtering

In the case a known signal $h(t)$ buried in stationary Gaussian, white noise, the optimal technique for signal extraction is the *matched filtering*, which involves cross-correlating the data with a *template* of the signal. The correlation function between two time series $x(t)$ and $\hat{h}(t)$ (both assumed to be real-valued) for a time shift τ is defined as:

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\,\hat{h}(t+\tau)\,dt. \tag{7.1}$$

Above, $\widehat{h(t)} := h(t)/||h||$, where the norm $||h||$ of the template is defined by

$$||h||^2 = \int_0^{t_c} |h(t)|^2/\sigma^2\,dt,$$

where σ^2 is the variance of the noise. The optimal signal-to-noise ratio (SNR) is obtained when the template exactly matches with the signal. i.e., $\mathrm{SNR}_{\mathrm{opt}} = ||h||$. If the SNR is greater than a predetermined threshold (which corresponds to an acceptably small false alarm probability), a detection can be claimed.

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[2] This problem and data are graciously provided by Dr. Arunava Mukherjee (IUCAA).

1. Problems

1. You are given a time-series data set $d(t)$ here [7]. This contains a simulated gravitational-wave signal from a black hole binary buried in zero-mean, Gaussian white noise $n(t)$ with standard deviation $\sigma = 10^{-21}$. i.e.,

$$d(t) = n(t) + m\, h_+(t)/D_L, \tag{7.2}$$

where $h_+(t)$ is given by Eq. (2.3), $m = m_1 + m_2$ is the total mass of the binary, and D_L is the (unknown) luminosity distance to the binary. Detect the location of the signal in the data by maximizing the correlation of the waveform templates computed in Eq. (2.3) with the data.:

$$R_{\rm max} = \max{}_{m,\mu,\tau}\, R(\tau), \tag{7.3}$$

where $R(\tau)$ is given by Eq. (7.1). We have the prior information that the parameters of the signal are in the following range: $5 < m/M_\odot < 15$, $0.2 < \mu/m < 0.25$. Estimate the parameters m, μ of the signal (parameters that maximize the correlation).

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[1] URL http://home.icts.res.in/~ajith/Downloads/nr_data.gz.
[2] URL http://www.black-holes.org/waveforms/.
[3] URL http://home.icts.res.in/~ajith/Downloads/extracted_lightcurve_HerX-1.dat.gz.
[4] RXTE is an X-ray timing satellite, URL https://heasarc.gsfc.nasa.gov/docs/xte/rxte.html.
[5] Download the file L1-STRAIN_4096Hz-815045078-256.txt.gz from, URL http://www.ligo.org/science/GRB051103/index.php.
[6] URL http://home.icts.res.in/~ajith/Downloads/4U1636-536_LC-extract.dat.gz.
[7] URL home.icts.res.in/~ajith/Downloads/mock_gw_data.dat.gz.

Solutions to the INTRODUCTORY PROBLEM SET FOR SOURCE MODELLING USING ANALYTICAL METHODS

Energy loss from inspiralling compact binary systems

$$\begin{cases} d_1 = \left(\dfrac{m_2}{m_1+m_2}\right) d = \dfrac{m_2}{M} d \\ d_2 = \left(\dfrac{m_1}{m_1+m_2}\right) d = \dfrac{m_1}{M} d \end{cases} \text{ and } T = \frac{2\pi a^{\frac{3}{2}}}{(GM)^{\frac{1}{2}}} \quad [\text{where } M = m_1 + m_2]$$

$$Q_{xx} = \mu d^2 \cos^2\psi$$

$$Q_{yy} = \mu d^2 \sin^2\psi$$

$$Q_{xy} = Q_{yx} = \mu d^2 \sin\psi\cos\psi = \frac{1}{2}\mu d^2 \sin 2\psi$$

$$\boldsymbol{d = \frac{a(1-e^2)}{1+e\cos\psi}} \quad \begin{bmatrix}\text{Orbit equation for} \\ \text{Kepler motion}\end{bmatrix} \Rightarrow \dot{d} = \frac{-a(1-e^2)}{(1+e\cos\psi)^2}(-e\sin\psi)\dot{\psi} = \frac{a(1-e^2)e\sin\psi}{(1+e\cos\psi)^2}\frac{\{GMa(1-e^2)\}^{\frac{1}{2}}}{d^2}$$

$$\Rightarrow d\dot{d} = \frac{a(1-e^2)e\sin\psi}{(1+e\cos\psi)^2}\frac{\{GMa(1-e^2)\}^{\frac{1}{2}}}{\left\{\dfrac{a(1-e^2)}{1+e\cos\psi}\right\}} = \{GMa(1-e^2)\}^{\frac{1}{2}}\frac{e\sin\psi}{(1+e\cos\psi)}$$

$$\text{angular velocity, } \boldsymbol{\dot{\psi} = \frac{\{GMa(1-e^2)\}^{\frac{1}{2}}}{d^2}} = \frac{\{GMa(1-e^2)\}^{\frac{1}{2}}}{\left\{\dfrac{a(1-e^2)}{1+e\cos\psi}\right\}^2} = \boldsymbol{\frac{(GM)^{\frac{1}{2}}}{\{a(1-e^2)\}^{\frac{3}{2}}}(1+e\cos\psi)^2 = \frac{2\pi(1+e\cos\psi)^2}{T(1-e^2)^{\frac{3}{2}}}}$$

$$\frac{dQ_{xx}}{dt} = -2\mu d^2\dot{\psi}\cos\psi\sin\psi + 2\mu d\dot{d}\cos^2\psi = -\mu d^2\dot{\psi}\sin 2\psi + 2\mu d\dot{d}\cos^2\psi$$

$$\Rightarrow \frac{dQ_{xx}}{dt} = \mu\{GMa(1-e^2)\}^{\frac{1}{2}}\sin 2\psi\left\{-1 + \frac{e\cos\psi}{(1+e\cos\psi)}\right\} = -\mu\{GMa(1-e^2)\}^{\frac{1}{2}}\sin 2\psi\left\{\frac{1}{1+e\cos\psi}\right\}$$

$$\Rightarrow \frac{d^2Q_{xx}}{dt^2} = -\mu\{GMa(1-e^2)\}^{\frac{1}{2}}\dot{\psi}\left\{\frac{2\cos 2\psi}{1+e\cos\psi} - \frac{\sin 2\psi\,(-e\sin\psi)}{(1+e\cos\psi)^2}\right\}$$

$$\Rightarrow \frac{d^2Q_{xx}}{dt^2} = -\mu\{GMa(1-e^2)\}^{\frac{1}{2}}\frac{\dot{\psi}}{(1+e\cos\psi)^2}\{2\cos 2\psi\,(1+e\cos\psi) + \sin 2\psi\,(e\sin\psi)\}$$

$$\Rightarrow \frac{d^2Q_{xx}}{dt^2} = -\mu\{GMa(1-e^2)\}^{\frac{1}{2}}\frac{\dot{\psi}}{(1+e\cos\psi)^2}(2\cos 2\psi + e\cos\psi\cos 2\psi + e\cos\psi)$$

$$\Rightarrow \frac{d^2Q_{xx}}{dt^2} = -\mu\{GMa(1-e^2)\}^{\frac{1}{2}}\frac{(GM)^{\frac{1}{2}}}{\{a(1-e^2)\}^{\frac{3}{2}}}(2\cos 2\psi + 2e\cos^3\psi) = -2\mu\frac{GM}{a(1-e^2)}(\cos 2\psi + e\cos^3\psi)$$

$$\Rightarrow \frac{d^3Q_{xx}}{dt^3} = 2\mu\frac{GM}{a(1-e^2)}\dot{\psi}(2\sin 2\psi + 3e\sin\psi\cos^2\psi) = \frac{2\mu(GM)^{\frac{3}{2}}}{\{a(1-e^2)\}^{\frac{5}{2}}}(1+e\cos\psi)^2(2\sin 2\psi + 3e\sin\psi\cos^2\psi)$$

$$\Rightarrow \boldsymbol{\frac{d^3Q_{xx}}{dt^3} = \beta(1+e\cos\psi)^2(2\sin 2\psi + 3e\sin\psi\cos^2\psi)}$$

$$\left[\text{where } \beta = \frac{2\mu(GM)^{\frac{3}{2}}}{\{a(1-e^2)\}^{\frac{5}{2}}} \Rightarrow \beta^2 = \frac{4\mu^2(GM)^3}{\{a(1-e^2)\}^5} = \frac{4G^3{m_1}^2{m_2}^2(m_1+m_2)}{\{a(1-e^2)\}^5}\right]$$

$$\frac{dQ_{yy}}{dt} = 2\mu d^2\dot{\psi}\sin\psi\cos\psi + 2\mu d\dot{d}\sin^2\psi = \mu d^2\dot{\psi}\sin 2\psi + 2\mu d\dot{d}(1-\cos^2\psi)$$

$$\Rightarrow \frac{dQ_{yy}}{dt} = \mu\{GMa(1-e^2)\}^{\frac{1}{2}}\left\{\sin 2\psi + \frac{2e\sin\psi\,(1-\cos^2\psi)}{(1+e\cos\psi)}\right\} = \mu\{GMa(1-e^2)\}^{\frac{1}{2}}\left\{\frac{\sin 2\psi + 2e\sin\psi}{(1+e\cos\psi)}\right\}$$

$$\Rightarrow \frac{d^2Q_{yy}}{dt^2} = \mu\{GMa(1-e^2)\}^{\frac{1}{2}}\dot{\psi}\left\{\frac{2(\cos 2\psi + e\cos\psi)}{1+e\cos\psi} - \frac{(\sin 2\psi + 2e\sin\psi)(-e\sin\psi)}{(1+e\cos\psi)^2}\right\}$$

$$\Rightarrow \frac{d^2Q_{yy}}{dt^2} = \mu\{GMa(1-e^2)\}^{\frac{1}{2}} \frac{\dot{\psi}}{(1+e\cos\psi)^2}\{2(\cos 2\psi + e\cos\psi)(1+e\cos\psi) + (\sin 2\psi + 2e\sin\psi)(e\sin\psi)\}$$

$$\Rightarrow \frac{d^2Q_{yy}}{dt^2} = \mu\{GMa(1-e^2)\}^{\frac{1}{2}} \frac{\dot{\psi}}{(1+e\cos\psi)^2}\{(2\cos 2\psi + 2e\cos\psi + 2e\cos 2\psi\cos\psi + 2e^2\cos^2\psi) + (e\sin 2\psi\sin\psi + 2e^2\sin^2\psi)\}$$

$$\Rightarrow \frac{d^2Q_{yy}}{dt^2} = \mu\{GMa(1-e^2)\}^{\frac{1}{2}} \frac{(GM)^{\frac{1}{2}}}{\{a(1-e^2)\}^{\frac{3}{2}}}(2\cos 2\psi + 3e\cos\psi + e\cos 2\psi\cos\psi + 2e^2)$$

$$\Rightarrow \frac{d^3Q_{yy}}{dt^3} = -\mu\frac{GM}{a(1-e^2)}\dot{\psi}(4\sin 2\psi + 3e\sin\psi + e\cos 2\psi\sin\psi + 2e\sin 2\psi\cos\psi)$$

$$\Rightarrow \frac{d^3Q_{yy}}{dt^3} = -\mu\frac{GM}{a(1-e^2)}\dot{\psi}\{4\sin 2\psi + 3e\sin\psi + e\sin\psi\,(\cos 2\psi + 4\cos^2\psi)\}$$

$$\Rightarrow \frac{d^3Q_{yy}}{dt^3} = -\mu\frac{GM}{a(1-e^2)}\frac{(GM)^{\frac{1}{2}}}{\{a(1-e^2)\}^{\frac{3}{2}}}(1+e\cos\psi)^2\{4\sin 2\psi + 3e\sin\psi + e\sin\psi\,(6\cos^2\psi - 1)\}$$

$$\Rightarrow \frac{d^3Q_{yy}}{dt^3} = -\frac{\mu(GM)^{\frac{3}{2}}}{\{a(1-e^2)\}^{\frac{5}{2}}}(1+e\cos\psi)^2\{4\sin 2\psi + 2e\sin\psi + 6e\sin\psi\cos^2\psi\}$$

$$\Rightarrow \frac{d^3Q_{yy}}{dt^3} = -\frac{2\mu(GM)^{\frac{3}{2}}}{\{a(1-e^2)\}^{\frac{5}{2}}}(1+e\cos\psi)^2\{2\sin 2\psi + e\sin\psi\,(1+3\cos^2\psi)\}$$

$$\Rightarrow \boldsymbol{\frac{d^3Q_{yy}}{dt^3} = -\beta(1+e\cos\psi)^2\{2\sin 2\psi + e\sin\psi\,(1+3\cos^2\psi)\}}$$

$$\frac{dQ_{xy}}{dt} = \mu d^2\dot{\psi}\cos 2\psi + \mu d\dot{d}\sin 2\psi$$

$$\Rightarrow \frac{dQ_{xy}}{dt} = \mu\{GMa(1-e^2)\}^{\frac{1}{2}}\left\{\cos 2\psi + \frac{e\sin\psi\sin 2\psi}{(1+e\cos\psi)}\right\} = \mu\{GMa(1-e^2)\}^{\frac{1}{2}}\left\{\frac{\cos 2\psi + e\cos\psi}{(1+e\cos\psi)}\right\}$$

$$\Rightarrow \frac{d^2Q_{xy}}{dt^2} = \mu\{GMa(1-e^2)\}^{\frac{1}{2}}\dot{\psi}\left\{-\frac{2\sin 2\psi + e\sin\psi}{1+e\cos\psi} - \frac{(\cos 2\psi + e\cos\psi)(-e\sin\psi)}{(1+e\cos\psi)^2}\right\}$$

$$\Rightarrow \frac{d^2Q_{xy}}{dt^2} = -\mu\{GMa(1-e^2)\}^{\frac{1}{2}}\frac{\dot{\psi}}{(1+e\cos\psi)^2}\{(2\sin 2\psi + e\sin\psi)(1+e\cos\psi) - (\cos 2\psi + e\cos\psi)(e\sin\psi)\}$$

$$\Rightarrow \frac{d^2Q_{xy}}{dt^2} = -\mu\{GMa(1-e^2)\}^{\frac{1}{2}}\frac{\dot{\psi}}{(1+e\cos\psi)^2}\{(2\sin 2\psi + e\sin\psi + 2e\sin 2\psi\cos\psi + e^2\sin\psi\cos\psi) - (e\cos 2\psi\sin\psi + e^2\cos\psi\sin\psi)\}$$

$$\Rightarrow \frac{d^2Q_{xy}}{dt^2} = -\mu\{GMa(1-e^2)\}^{\frac{1}{2}}\frac{(GM)^{\frac{1}{2}}}{\{a(1-e^2)\}^{\frac{3}{2}}}\{2\sin 2\psi + e\sin\psi + 2e\sin 2\psi\cos\psi - e\cos 2\psi\sin\psi\}$$

$$\Rightarrow \frac{d^2Q_{xy}}{dt^2} = -\mu\frac{GM}{a(1-e^2)}\{2\sin 2\psi + 2e\sin^3\psi + 2e\sin 2\psi\cos\psi\}$$

$$\Rightarrow \frac{d^2Q_{xy}}{dt^2} = -2\mu\frac{GM}{a(1-e^2)}\{\sin 2\psi + e\sin^3\psi + e\sin 2\psi\cos\psi\}$$

$$\Rightarrow \frac{d^3Q_{xy}}{dt^3} = -2\mu\frac{GM}{a(1-e^2)}\dot{\psi}\{2\cos 2\psi + 3e\sin^2\psi\cos\psi + 2e\cos 2\psi\cos\psi - e\sin 2\psi\sin\psi\}$$

$$\Rightarrow \frac{d^3Q_{xy}}{dt^3} = -2\mu\frac{GM}{a(1-e^2)}\dot{\psi}\{2\cos 2\psi + 3e\sin^2\psi\cos\psi + 2e\cos\psi\,(\cos 2\psi - \sin^2\psi)\}$$

$$\Rightarrow \frac{d^3Q_{xy}}{dt^3} = -2\mu\frac{GM}{a(1-e^2)}\dot{\psi}\{2\cos 2\psi + 3e\sin^2\psi\cos\psi + 2e\cos\psi\,(2\cos^2\psi - 1 - 1 + \cos^2\psi)\}$$

$$\Longrightarrow \frac{d^3Q_{xy}}{dt^3} = -2\mu\frac{GM}{a(1-e^2)}\dot{\psi}\{2\cos 2\psi + 3e(1-\cos^2\psi)\cos\psi + 2e\cos\psi\,(3\cos^2\psi - 2)\}$$
$$\Longrightarrow \frac{d^3Q_{xy}}{dt^3} = -2\mu\frac{GM}{a(1-e^2)}\dot{\psi}\{2\cos 2\psi + 3e\cos\psi - 3e\cos^3\psi + 6e\cos^3\psi - 4e\cos\psi\}$$
$$\Longrightarrow \frac{d^3Q_{xy}}{dt^3} = -2\mu\frac{GM}{a(1-e^2)}\frac{(GM)^{\frac{1}{2}}}{\{a(1-e^2)\}^{\frac{3}{2}}}(1+e\cos\psi)^2\{2\cos 2\psi - e\cos\psi + 3e\cos^3\psi\}$$

$$\Longrightarrow \frac{d^3Q_{xy}}{dt^3} = -\frac{2\mu(GM)^{\frac{3}{2}}}{\{a(1-e^2)\}^{\frac{5}{2}}}(1+e\cos\psi)^2\{2\cos 2\psi - e\cos\psi\,(1-3\cos^2\psi)\}$$
$$\Longrightarrow \boldsymbol{\frac{d^3Q_{xy}}{dt^3} = -\beta(1+e\cos\psi)^2\{2\cos 2\psi - e\cos\psi\,(1-3\cos^2\psi)\}}$$

$$\therefore \begin{cases} \dfrac{d^3Q_{xx}}{dt^3} = \beta(1+e\cos\psi)^2\{2\sin 2\psi + 3e\sin\psi\cos^2\psi\} \\ \dfrac{d^3Q_{yy}}{dt^3} = -\beta(1+e\cos\psi)^2\{2\sin 2\psi + e\sin\psi\,(1+3\cos^2\psi)\} \\ \dfrac{d^3Q_{xy}}{dt^3} = \dfrac{d^3Q_{yx}}{dt^3} = -\beta(1+e\cos\psi)^2\{2\cos 2\psi - e\cos\psi\,(1-3\cos^2\psi)\} \end{cases}$$

$$\boldsymbol{P = \frac{G}{5c^5}\left\{\frac{d^3Q_{ij}}{dt^3}\frac{d^3Q_{ij}}{dt^3} - \frac{1}{3}\frac{d^3Q_{ii}}{dt^3}\frac{d^3Q_{jj}}{dt^3}\right\}}$$
$$= \frac{G}{5c^5}\left\{\left(\frac{d^3Q_{xx}}{dt^3}\frac{d^3Q_{xx}}{dt^3} - \frac{1}{3}\frac{d^3Q_{xx}}{dt^3}\frac{d^3Q_{xx}}{dt^3}\right) + \left(\frac{d^3Q_{xy}}{dt^3}\frac{d^3Q_{xy}}{dt^3} - \frac{1}{3}\frac{d^3Q_{xx}}{dt^3}\frac{d^3Q_{yy}}{dt^3}\right) + \left(\frac{d^3Q_{yx}}{dt^3}\frac{d^3Q_{yx}}{dt^3} - \frac{1}{3}\frac{d^3Q_{yy}}{dt^3}\frac{d^3Q_{xx}}{dt^3}\right) + \left(\frac{d^3Q_{yy}}{dt^3}\frac{d^3Q_{yy}}{dt^3} - \frac{1}{3}\frac{d^3Q_{yy}}{dt^3}\frac{d^3Q_{yy}}{dt^3}\right)\right\}$$
$$= \frac{2G}{5c^5}\left\{\frac{1}{3}\left(\frac{d^3Q_{xx}}{dt^3}\right)^2 + \left(\frac{d^3Q_{xy}}{dt^3}\right)^2 - \frac{1}{3}\frac{d^3Q_{xx}}{dt^3}\frac{d^3Q_{yy}}{dt^3} + \frac{1}{3}\left(\frac{d^3Q_{yy}}{dt^3}\right)^2\right\}$$
$$\boldsymbol{= \frac{2G}{15c^5}\left\{3\left(\frac{d^3Q_{xy}}{dt^3}\right)^2 - \frac{d^3Q_{xx}}{dt^3}\frac{d^3Q_{yy}}{dt^3} + \left(\frac{d^3Q_{xx}}{dt^3}\right)^2 + \left(\frac{d^3Q_{yy}}{dt^3}\right)^2\right\}}$$

$$\left(\frac{d^3Q_{xy}}{dt^3}\right)^2 = \beta^2(1+e\cos\psi)^4\{2\cos 2\psi - e\cos\psi\,(1-3\cos^2\psi)\}^2$$
$$= \beta^2(1+e\cos\psi)^4\{4\cos^2 2\psi + e^2\cos^2\psi\,(1-3\cos^2\psi)^2 - 4e\cos 2\psi\cos\psi\,(1-3\cos^2\psi)\}$$
$$\frac{d^3Q_{xx}}{dt^3}\frac{d^3Q_{yy}}{dt^3} = -\beta^2(1+e\cos\psi)^4\{2\sin 2\psi + e\sin\psi\,(3\cos^2\psi)\}\{2\sin 2\psi + e\sin\psi\,(1+3\cos^2\psi)\}$$
$$= -\beta^2(1+e\cos\psi)^4\{4\sin^2 2\psi + e^2\sin^2\psi\,(3\cos^2\psi)(1+3\cos^2\psi) + 2e\sin 2\psi\sin\psi\,(1+6\cos^2\psi)\}$$
$$\boldsymbol{\left(\frac{d^3Q_{xx}}{dt^3}\right)^2 + \left(\frac{d^3Q_{yy}}{dt^3}\right)^2} = \beta^2(1+e\cos\psi)^4\{\{2\sin 2\psi + 3e\sin\psi\cos^2\psi\}^2 + \{2\sin 2\psi + e\sin\psi\,(1+3\cos^2\psi)\}^2\}$$
$$= \beta^2(1+e\cos\psi)^4\{8\sin^2 2\psi + 9e^2\sin^2\psi\cos^4\psi + 12e\sin\psi\sin 2\psi\cos^2\psi + e^2\sin^2\psi\,(1+3\cos^2\psi)^2 + 4e\sin 2\psi\sin\psi\,(1+3\cos^2\psi)\}$$
$$= \beta^2(1+e\cos\psi)^4\{8\sin^2 2\psi + 18e^2\sin^2\psi\cos^4\psi + 24e\sin\psi\sin 2\psi\cos^2\psi + e^2\sin^2\psi + 6e^2\sin^2\psi\cos^2\psi + 4e\sin 2\psi\sin\psi\}$$

$$\therefore \boldsymbol{3\left(\frac{d^3Q_{xy}}{dt^3}\right)^2 - \frac{d^3Q_{xx}}{dt^3}\frac{d^3Q_{yy}}{dt^3}} = \beta^2(1+e\cos\psi)^4\{(12\cos^2 2\psi + 4\sin^2 2\psi) + 3e^2\cos^2\psi\,(1-3\cos^2\psi)^2 + 3e^2\sin^2\psi\cos^2\psi\,(1+3\cos^2\psi) - 12e\cos 2\psi\cos\psi\,(1-3\cos^2\psi) + 2e\sin 2\psi\sin\psi\,(1+6\cos^2\psi)\}$$
$$= \beta^2(1+e\cos\psi)^4\{(12\cos^2 2\psi + 4\sin^2 2\psi) + 3e^2\cos^2\psi\,(1+9\cos^4\psi - 6\cos^2\psi + \sin^2\psi + 3\cos^2\psi\sin^2\psi) - 2e\{6\cos 2\psi\cos\psi - 6\cos 2\psi\cos^3\psi - \sin 2\psi\sin\psi - 6\cos^2\psi\sin 2\psi\sin\psi\}\}$$

$$= \beta^2(1+e\cos\psi)^4\{(12-12\sin^2 2\psi+4\sin^2 2\psi)+3e^2\cos^2\psi\,(1+9\cos^4\psi-3\cos^2\psi+\sin^2\psi-3\cos^4\psi)$$
$$-2e\{6\cos 2\psi\cos\psi-18(1-2\sin^2\psi)\cos^3\psi-\sin 2\psi\sin\psi-12\cos^3\psi\sin^2\psi\}\}$$
$$= \beta^2(1+e\cos\psi)^4\{(12-8\sin^2 2\psi)+3e^2\cos^2\psi\,(6\cos^4\psi-2\cos^2\psi+2\sin^2\psi)$$
$$-2e\{6\cos 2\psi\cos\psi-18\cos^3\psi-\sin 2\psi\sin\psi+24\cos^3\psi\sin^2\psi\}\}$$

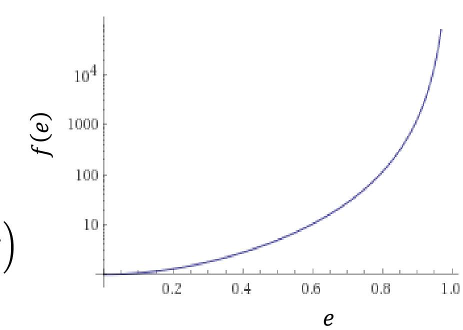
$$\therefore \boldsymbol{3\left(\frac{d^3Q_{xy}}{dt^3}\right)^2-\frac{d^3Q_{xx}}{dt^3}\frac{d^3Q_{yy}}{dt^3}+\left(\frac{d^3Q_{xx}}{dt^3}\right)^2+\left(\frac{d^3Q_{yy}}{dt^3}\right)^2}$$
$$= \beta^2(1+e\cos\psi)^4\{12+3e^2\cos^2\psi\,(6\cos^2\psi-2\cos^2\psi+4\sin^2\psi)-2e\{6\cos 2\psi\cos\psi-18\cos^3\psi\}$$
$$+e^2\sin^2\psi+6e\sin 2\psi\sin\psi\}$$
$$= \beta^2(1+e\cos\psi)^4\{12+12e^2\cos^2\psi-12e\cos\psi\,\{\cos 2\psi-3\cos^2\psi\}+e^2\sin^2\psi+12e\cos\psi\sin^2\psi\}$$
$$= \beta^2(1+e\cos\psi)^4\{12+12e^2\cos^2\psi-12e\cos\psi\,\{1+\cos^2\psi\}+e^2\sin^2\psi+12e\cos\psi\,(1-\sin^2\psi)\}$$
$$= \beta^2(1+e\cos\psi)^4\{12+12e^2\cos^2\psi+24e\cos\psi+e^2\sin^2\psi\}$$
$$= \beta^2(1+e\cos\psi)^4\{12(1+e\cos\psi)^2+e^2\sin^2\psi\}$$
$$\therefore P=\frac{2G}{15c^5}\beta^2(1+e\cos\psi)^4\{12(1+e\cos\psi)^2+e^2\sin^2\psi\}$$
$$\Longrightarrow \boldsymbol{P=\frac{8}{15}\frac{G^4}{c^5}\frac{m_1{}^2m_2{}^2(m_1+m_2)}{a^5(1-e^2)^5}(1+e\cos\psi)^4\{12(1+e\cos\psi)^2+e^2\sin^2\psi\}}$$

$$\langle P\rangle=\frac{1}{T}\int_0^T P\,dt=\frac{1}{T}\frac{\{a(1-e^2)\}^{\frac{3}{2}}}{(GM)^{\frac{1}{2}}}\int_0^{2\pi}P\,d\psi=\frac{(1-e^2)^{\frac{3}{2}}}{2\pi}\int_0^{2\pi}P\frac{d\psi}{(1+e\cos\psi)^2}\quad\left[\because\frac{d\psi}{dt}=\frac{2\pi(1+e\cos\psi)^2}{T(1-e^2)^{\frac{3}{2}}}\right]$$
$$=\frac{(1-e^2)^{\frac{3}{2}}}{2\pi}\int_0^{2\pi}P\frac{d\psi}{(1+e\cos\psi)^2}=\frac{8}{15}\frac{G^4}{c^5}\frac{m_1{}^2m_2{}^2(m_1+m_2)}{2\pi a^5(1-e^2)^{\frac{7}{2}}}\int_0^{2\pi}(1+e\cos\psi)^2\{12(1+e\cos\psi)^2+e^2\sin^2\psi\}\,d\psi$$
$$=\frac{8}{15}\frac{G^4}{c^5}\frac{m_1{}^2m_2{}^2(m_1+m_2)}{a^5(1-e^2)^{\frac{7}{2}}}\langle(1+e\cos\psi)^2\{12(1+e\cos\psi)^2+e^2\sin^2\psi\}\rangle$$

$$(1+e\cos\psi)^2\{12(1+e\cos\psi)^2+e^2\sin^2\psi\}$$
$$=(1+2e\cos\psi+e^2\cos^2\psi)\{12+24e\cos\psi+12e^2\cos^2\psi+e^2\sin^2\psi\}$$
$$=(12+24e\cos\psi+12e^2\cos^2\psi+e^2\sin^2\psi)+(24e\cos\psi+48e^2\cos^2\psi+24e^3\cos^3\psi+2e^3\sin^2\psi\cos\psi)$$
$$+(12e^2\cos^2\psi+24e^3\cos^3\psi+12e^4\cos^4\psi+e^4\sin^2\psi\cos^2\psi)$$
$$=(12+36e\cos\psi+72e^2\cos^2\psi+48e^3\cos^3\psi+12e^4\cos^4\psi)$$
$$+(e^2\sin^2\psi+2e^3\sin^2\psi\cos\psi+e^4\sin^2\psi\cos^2\psi)$$

$$\Longrightarrow\langle(1+e\cos\psi)^2\{12(1+e\cos\psi)^2+e^2\sin^2\psi\}\rangle$$
$$=\langle 12+72e^2\cos^2\psi+12e^4\cos^4\psi\rangle+\langle e^2\sin^2\psi+e^4\sin^2\psi\cos^2\psi\rangle$$
$$=\left(12+72e^2\left(\frac{1}{2}\right)+12e^4\left(\frac{3}{8}\right)\right)+\left(e^2\left(\frac{1}{2}\right)+e^4\left(\frac{1}{8}\right)+e^6\left(\frac{1}{16}\right)\right)=\left(12+e^2\left(\frac{73}{2}\right)+e^4\left(\frac{37}{8}\right)\right)$$
$$\therefore \boldsymbol{\langle P\rangle=\frac{32}{5}\frac{G^4}{c^5}\frac{m_1{}^2m_2{}^2(m_1+m_2)}{a^5(1-e^2)^5}\left(1+e^2\left(\frac{73}{24}\right)+e^4\left(\frac{37}{96}\right)\right)}$$

For elliptical orbit,
$$\langle P\rangle=\frac{32}{5}\frac{G^4}{c^5}\frac{m_1{}^2m_2{}^2(m_1+m_2)}{a^5(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)$$
For circular orbit, $e=0$, so that
$$\langle P\rangle_{e=0}=\frac{32}{5}\frac{G^4}{c^5}\frac{m_1{}^2m_2{}^2(m_1+m_2)}{a^5}$$
enhancement factor, $f(e)=\dfrac{\langle P\rangle}{\langle P\rangle_{e=0}}=\dfrac{1}{(1-e^2)^{7/2}}\left(1+\dfrac{73}{24}e^2+\dfrac{37}{96}e^4\right)$



Secular evolution of orbital elements

For Kepler's orbit, $a = \dfrac{-Gm_1m_2}{2E} \Rightarrow \dfrac{da}{dt} = \dfrac{Gm_1m_2}{2E^2}\dfrac{dE}{dt} \Rightarrow \dot{a} = \dfrac{Gm_1m_2}{2E^2}P$

$$\langle \dot{L} \rangle = -\frac{32}{5}\frac{G^{7/2}}{c^5}\frac{{m_1}^2{m_2}^2(m_1+m_2)^{1/2}}{a^{7/2}(1-e^2)^2}\left(1+\frac{7}{8}e^2\right)$$

$$\langle \dot{a} \rangle = \frac{1}{T}\int_0^T \dot{a}\,dt = \frac{Gm_1m_2}{2T}\int_0^T \frac{P}{E^2}dt$$

$$= \frac{Gm_1m_2}{2T}\int_0^T \frac{P}{\left\{\frac{-Gm_1m_2}{2a}\right\}^2}dt = \frac{-2a^2}{Gm_1m_2T}\int_0^T P\,dt = \frac{-2a^2}{Gm_1m_2}\langle P \rangle = -\frac{64}{5}\frac{G^3}{c^5}\frac{m_1m_2(m_1+m_2)}{a^3(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)$$

$$L^2 = \frac{G{m_1}^2{m_2}^2}{(m_1+m_2)}a(1-e^2) \Rightarrow e^2 = 1 - \frac{(m_1+m_2)}{G{m_1}^2{m_2}^2a}L^2 \Rightarrow 2e\frac{de}{dt} = \frac{(m_1+m_2)}{G{m_1}^2{m_2}^2}\left\{-\frac{2L}{a}\left(\frac{dL}{dt}\right)+\frac{L^2}{a^2}\left(\frac{da}{dt}\right)\right\}$$

$$\Rightarrow \frac{de}{dt} = \frac{1}{2}\frac{(m_1+m_2)}{G{m_1}^2{m_2}^2e}\left\{\frac{L^2}{a^2}\left(\frac{da}{dt}\right)-\frac{2L}{a}\left(\frac{dL}{dt}\right)\right\}$$

$$\langle \dot{e} \rangle = \frac{1}{T}\int_0^T \dot{e}\,dt = \frac{1}{2}\frac{(m_1+m_2)}{G{m_1}^2{m_2}^2e}\frac{1}{T}\int_0^T \left\{\frac{L^2}{a^2}\left(\frac{da}{dt}\right)-\frac{L}{a}\left(\frac{dL}{dt}\right)\right\}dt = \frac{(m_1+m_2)}{G{m_1}^2{m_2}^2e}\left\{\frac{L^2}{a^2}\langle \dot{a} \rangle - \frac{2L}{a}\langle \dot{L} \rangle\right\}$$

$$= \frac{1}{2}\frac{(m_1+m_2)}{G{m_1}^2{m_2}^2e}\left\{\frac{1}{a^2}\frac{G{m_1}^2{m_2}^2}{(m_1+m_2)}a(1-e^2)\langle \dot{a} \rangle - \frac{2}{a}\left\{\frac{G{m_1}^2{m_2}^2}{(m_1+m_2)}a(1-e^2)\right\}^{\frac{1}{2}}\langle \dot{L} \rangle\right\}$$

$$= \frac{1}{2}\frac{1}{ae}\left\{(1-e^2)\langle \dot{a} \rangle - 2\left\{\frac{G{m_1}^2{m_2}^2}{(m_1+m_2)}\right\}^{-\frac{1}{2}}\{a(1-e^2)\}^{\frac{1}{2}}\langle \dot{L} \rangle\right\}$$

$$= \frac{1}{2}\frac{1}{ae}\left\{-(1-e^2)\frac{64}{5}\frac{G^3}{c^5}\frac{m_1m_2(m_1+m_2)}{a^3(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)\right.$$
$$\left. + \left\{\frac{G{m_1}^2{m_2}^2}{(m_1+m_2)}\right\}^{-\frac{1}{2}}\{a(1-e^2)\}^{\frac{1}{2}}\frac{64}{5}\frac{G^{7/2}}{c^5}\frac{{m_1}^2{m_2}^2(m_1+m_2)^{1/2}}{a^{7/2}(1-e^2)^2}\left(1+\frac{7}{8}e^2\right)\right\}$$

$$= \frac{32}{5}\frac{G^3}{c^5}\frac{m_1m_2(m_1+m_2)}{a^3(1-e^2)^{5/2}}\frac{1}{ae}\left\{-\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)+(1-e^2)\left(1+\frac{7}{8}e^2\right)\right\}$$
$$= \frac{32}{5}\frac{G^3}{c^5}\frac{m_1m_2(m_1+m_2)}{a^4(1-e^2)^{5/2}}\frac{1}{e}\left\{-\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)+\left(1-\frac{1}{8}e^2-\frac{7}{8}e^4\right)\right\}$$
$$= \frac{32}{5}\frac{G^3}{c^5}\frac{m_1m_2(m_1+m_2)}{a^4(1-e^2)^{5/2}}\frac{1}{e}\left(-\frac{76}{24}e^2-\frac{121}{96}e^4\right) = -\frac{304}{15}\frac{G^3}{c^5}\frac{m_1m_2(m_1+m_2)}{a^4(1-e^2)^{5/2}}e\left(1+\frac{121}{304}e^2\right)$$

$$\left\langle\frac{da}{de}\right\rangle = \frac{\langle \dot{a} \rangle}{\langle \dot{e} \rangle} = \frac{-\frac{64}{5}\frac{G^3}{c^5}\frac{m_1m_2(m_1+m_2)}{a^3(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)}{-\frac{304}{15}\frac{G^3}{c^5}\frac{m_1m_2(m_1+m_2)}{a^4(1-e^2)^{5/2}}e\left(1+\frac{121}{304}e^2\right)} = \frac{12a\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)}{19(1-e^2)e\left(1+\frac{121}{304}e^2\right)}$$

$$\Rightarrow \frac{da}{a} = \frac{12}{19}\int \frac{\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)}{(1-e^2)e\left(1+\frac{121}{304}e^2\right)}de \Rightarrow \boldsymbol{a = C\exp\left\{\frac{12}{19}\int \frac{\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)}{(1-e^2)e\left(1+\frac{121}{304}e^2\right)}de\right\}}$$

$$\int \frac{(1+ue^2+ve^4)}{(1-e^2)e(1+we^2)}de = \int\left\{\left(\frac{u+v+1}{w+1}\right)\frac{e}{(1-e^2)}+\frac{1}{e}+\left(\frac{uw-v-w^2}{w+1}\right)\frac{e}{(1+we^2)}\right\}de$$
$$= -\frac{1}{2}\left(\frac{u+v+1}{w+1}\right)\ln(1-e^2)+\ln e+\frac{1}{2w}\left(\frac{uw-v-w^2}{w+1}\right)\ln(1+we^2)$$

$$= \frac{145}{242}\ln\left(1+\frac{121}{304}e^2\right)+\ln e-\frac{19}{12}\ln(1-e^2) \quad \left[\because u=\frac{73}{24}\ ;\ v=\frac{37}{96}\ ;\ w=\frac{121}{304}\right]$$

$$\left[\begin{array}{c} \dfrac{(1+ue^2+ve^4)}{(1-e^2)e(1+we^2)}=\dfrac{A_1e+A_2}{(1-e^2)}+\dfrac{A_3}{e}+\dfrac{A_4e+A_5}{(1+we^2)}=\left(\dfrac{u+v+1}{w+1}\right)\dfrac{e}{(1-e^2)}+\dfrac{1}{e}+\left(\dfrac{uw-v-w^2}{w+1}\right)\dfrac{e}{(1+we^2)} \\ (A_1e+A_2)e(1+we^2)+A_3(1-e^2)(1+we^2)+(A_4e+A_5)e(1-e^2)=(1+ue^2+ve^4) \\ \Rightarrow (A_1e^2+A_2e+A_1we^4+A_2we^3)+(A_3-A_3e^2+A_3we^2-A_3we^4)+(A_4e^2+A_5e-A_4e^4-A_5e^3)=(1+ue^2+ve^4) \\ \Rightarrow A_3+(A_2+A_5)e+(A_1-A_3+A_3w+A_4)e^2+(A_2w-A_5)e^3+(A_1w-A_3w-A_4)e^4=(1+ue^2+ve^4) \\ \Rightarrow A_3=1\ ;\ A_2=A_5=0\ ;\ (A_1-1+w+A_4)=u\ ;\ (A_1w-w-A_4)=v \Rightarrow A_1=\dfrac{u+v+1}{w+1}\ ;\ A_4=\dfrac{uw-v+w}{w+1}-w \end{array}\right]$$

$$\therefore a(e)=C\exp\left\{\frac{12}{19}\int\frac{\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)}{(1-e^2)e\left(1+\frac{121}{304}e^2\right)}de\right\}=C\exp\left\{\frac{12}{19}\left\{\frac{145}{242}\ln\left(1+\frac{121}{304}e^2\right)-\frac{19}{12}\ln(1-e^2)+\ln e\right\}\right\}$$

$$=C\exp\left\{\frac{870}{2299}\ln\left(1+\frac{121}{304}e^2\right)-\ln(1-e^2)+\frac{12}{19}\ln e\right\}=C\exp\left\{\ln\left(1+\frac{121}{304}e^2\right)^{\frac{870}{2299}}-\ln(1-e^2)+\ln e^{\frac{12}{19}}\right\}$$

$$\Rightarrow \boldsymbol{a(e)=\frac{Ce^{\frac{12}{19}}}{(1-e^2)}\left(1+\frac{121}{304}e^2\right)^{\frac{870}{2299}}}$$

As $e\longrightarrow 0, e^2\longrightarrow 0$

$$\therefore a(e)\approx Ce^{\frac{12}{19}}(1+e^2)\left(1+\frac{121}{304}\times\frac{870}{2299}e^2\right)\approx Ce^{\frac{12}{19}}(1+e^2)(1+1.0518e^2)\approx Ce^{\frac{12}{19}}(1+2.0518e^2)\approx Ce^{\frac{12}{19}}$$

As $e\longrightarrow 1, (1-e^2)\longrightarrow 0$

$$\therefore a(e)=C\frac{(1-(1-e^2))^{\frac{6}{19}}}{(1-e^2)}\left(1+\frac{121}{304}(1-(1-e^2))\right)^{\frac{870}{2299}}\approx C\frac{\left(1-\frac{6}{19}(1-e^2)\right)}{(1-e^2)}\left(\frac{425}{304}-\frac{121}{304}(1-e^2)\right)^{\frac{870}{2299}}$$

$$\approx C\frac{(1-0.31579(1-e^2))}{(1-e^2)}\left(\frac{425}{304}\right)^{\frac{870}{2299}}\left(1-\frac{121}{425}(1-e^2)\right)^{\frac{870}{2299}}$$

$$\approx C\frac{(1-0.31579(1-e^2))}{(1-e^2)}\left(\frac{425}{304}\right)^{\frac{870}{2299}}\left(1-\frac{121}{425}\times\frac{870}{2299}(1-e^2)\right)$$

$$\approx C\frac{(1-0.31579(1-e^2))}{(1-e^2)}\left(\frac{425}{304}\right)^{\frac{870}{2299}}(1-0.10774(1-e^2))$$

$$\approx \frac{C}{(1-e^2)}\left(\frac{425}{304}\right)^{\frac{870}{2299}}(1-0.42353(1-e^2))\approx\frac{C}{(1-e^2)}\left(\frac{425}{304}\right)^{\frac{870}{2299}}\approx\frac{1.1352C}{(1-e^2)}$$

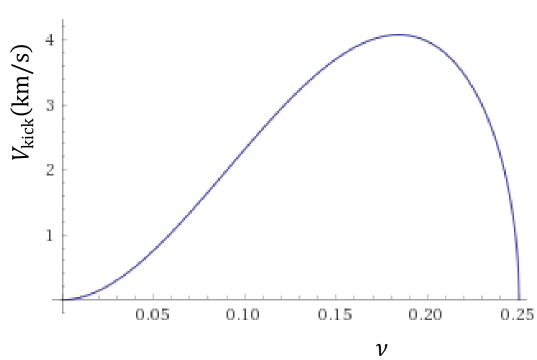


figure for third problem

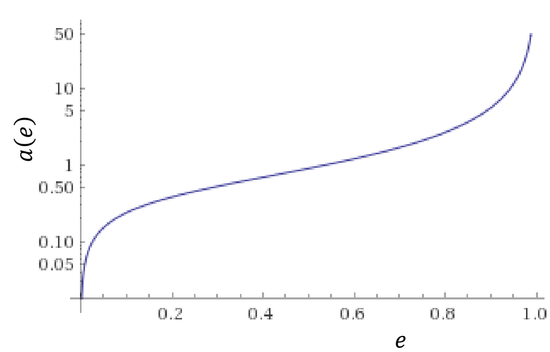


figure for second problem

*acknowledgement – Classical Mechanics by R Douglas Gregory

Loss of linear momentum from ICBs and estimates of recoil velocities

dimensionless symmetric mass ratio, $\nu = \dfrac{\mu}{m} = \dfrac{m_1 m_2}{(m_1+m_2)^2} \quad [m = m_1 + m_2]$

time period, $T = \dfrac{2\pi a^{\frac{3}{2}}}{(Gm)^{\frac{1}{2}}} \Longrightarrow \omega = \dfrac{2\pi}{T} = \dfrac{(Gm)^{\frac{1}{2}}}{a^{\frac{3}{2}}}$

dimensionless parameter, $x = \dfrac{(Gm\omega)^{\frac{2}{3}}}{c^2} = \dfrac{(Gm)^{\frac{2}{3}}}{c^2}\left(\dfrac{(Gm)^{\frac{1}{2}}}{a^{\frac{3}{2}}}\right)^{\frac{2}{3}} = \dfrac{(Gm)^{\frac{2}{3}}}{c^2}\dfrac{(Gm)^{\frac{1}{3}}}{a} = \dfrac{Gm}{c^2 a}$

$$\boldsymbol{\mathcal{F}_P^i = -\frac{464}{105}\frac{c^4}{G}x^{\frac{11}{2}}\nu^2(1-4\nu)^{\frac{1}{2}}\left\{f(\nu,x)\hat{\lambda}_i + g(\nu,x)\hat{n}_i + \mathcal{O}\left(\frac{1}{c^6}\right)\right\}}$$

$$f(\nu,x) = 1 + \left(-\frac{452}{87} - \frac{1139}{522}\nu\right)x + \frac{309}{58}\pi x^{\frac{3}{2}} + \left(-\frac{71345}{22968} + \frac{36761}{2088}\nu + \frac{147101}{68904}\nu^2\right)x^2 + \left(-\frac{2663}{116} - \frac{2185}{87}\nu\right)\pi x^{\frac{5}{2}}$$

$$\approx 1 + (-5.1954 - 2.182\nu)x + 5.3276\pi x^{\frac{3}{2}} + (-3.1063 + 17.606\nu + 2.135\nu^2)x^2 + (-22.957 - 25.115\nu)\pi x^{\frac{5}{2}}$$

$$g(\nu,x) = \left\{-\frac{106187}{50460} + \frac{32835}{841}\log 2 - \frac{77625}{3364}\log 3 + \left(\frac{32698}{12615} - \frac{109740}{841}\log 2 + \frac{66645}{841}\log 3\right)\nu\right\}x^{\frac{5}{2}}$$

$$\approx \{-1.361 + 1.121\nu\}x^{\frac{5}{2}}$$

$$\frac{dP^i}{dt} = -\mathcal{F}_P^i \Longrightarrow \Delta P^i = \int_{-\infty}^{t}\mathcal{F}_P^i\,dt' = -\frac{464}{105}\frac{c^4}{G}x^{\frac{11}{2}}\nu^2(1-4\nu)^{\frac{1}{2}}\int_{-\infty}^{t}\left\{f(\nu,x)\hat{\lambda}_i + g(\nu,x)\hat{n}_i + \mathcal{O}\left(\frac{1}{c^6}\right)\right\}dt'$$

$$= -\frac{464}{105}\frac{c^4}{G}x^{\frac{11}{2}}\nu^2(1-4\nu)^{\frac{1}{2}}\left\{f(\nu,x)\int_{-\infty}^{t}\hat{\lambda}_i\,dt' + g(\nu,x)\int_{-\infty}^{t}\hat{n}_i\,dt' + \mathcal{O}\left(\frac{1}{c^6}\right)\right\}$$

$$= -\frac{464}{105}\frac{c^4}{G}x^{\frac{11}{2}}\nu^2(1-4\nu)^{\frac{1}{2}}\left\{f(\nu,x)\left(\frac{\hat{n}_i}{\omega}\right) + g(\nu,x)\left(-\frac{\hat{\lambda}_i}{\omega}\right) + \mathcal{O}\left(\frac{1}{c^6}\right)\right\}$$

$$= -\frac{464}{105}\frac{c^4}{G}x^{\frac{11}{2}}\nu^2(1-4\nu)^{\frac{1}{2}}\frac{1}{\omega}\left\{f(\nu,x)\hat{n}_i - g(\nu,x)\hat{\lambda}_i + \mathcal{O}\left(\frac{1}{c^6}\right)\right\}$$

$$= -\frac{464}{105}\frac{c^4}{G}x^{\frac{11}{2}}\nu^2(1-4\nu)^{\frac{1}{2}}\frac{a^{\frac{3}{2}}}{(Gm)^{\frac{1}{2}}}\left\{f(\nu,x)\hat{n}_i - g(\nu,x)\hat{\lambda}_i + \mathcal{O}\left(\frac{1}{c^6}\right)\right\}$$

$$\Delta V^i = \frac{\Delta P^i}{m} = -\frac{464}{105}c^4x^4\nu^2(1-4\nu)^{\frac{1}{2}}\frac{x^{\frac{3}{2}}a^{\frac{3}{2}}}{(Gm)^{\frac{3}{2}}}\left\{f(\nu,x)\hat{n}_i - g(\nu,x)\hat{\lambda}_i + \mathcal{O}\left(\frac{1}{c^6}\right)\right\}$$

$$= -\frac{464}{105}c^4x^4\nu^2(1-4\nu)^{\frac{1}{2}}\frac{\left(\frac{Gm}{c^2a}\right)^{\frac{3}{2}}a^{\frac{3}{2}}}{(Gm)^{\frac{3}{2}}}\left\{f(\nu,x)\hat{n}_i - g(\nu,x)\hat{\lambda}_i + \mathcal{O}\left(\frac{1}{c^6}\right)\right\}$$

$$= -\frac{464}{105}cx^4\nu^2(1-4\nu)^{\frac{1}{2}}\left\{f(\nu,x)\hat{n}_i - g(\nu,x)\hat{\lambda}_i + \mathcal{O}\left(\frac{1}{c^6}\right)\right\}$$

$$\boldsymbol{V^i_{\text{recoil}}} = -\Delta V^i = \boldsymbol{\frac{464}{105}cx^4\nu^2(1-4\nu)^{\frac{1}{2}}\left\{f(\nu,x)\hat{\lambda}_i - g(\nu,x)\hat{n}_i + \mathcal{O}\left(\frac{1}{c^6}\right)\right\}}$$

$$\begin{cases} \dfrac{d\hat{n}^i}{dt} = \omega\hat{\lambda}^i \Longrightarrow dt = \dfrac{d\hat{n}^i}{\omega\hat{\lambda}^i} \\ \dfrac{d\hat{\lambda}^i}{dt} = -\omega\hat{n}^i \Longrightarrow dt = -\dfrac{d\hat{\lambda}^i}{\omega\hat{n}^i} \end{cases} \Longrightarrow \begin{cases} \displaystyle\int_{-\infty}^{t}\hat{n}^i\,dt' = \int_0^{\hat{\lambda}^i}\hat{n}^i\left(-\frac{d\hat{\lambda}^{i\prime}}{\omega\hat{n}^i}\right) = -\frac{1}{\omega}\int_0^{\hat{\lambda}^i}d\hat{\lambda}^{i\prime} = -\frac{1}{\omega}\hat{\lambda}^i \\ \displaystyle\int_{-\infty}^{t}\hat{\lambda}^i\,dt' = \int_0^{\hat{n}^i}\hat{\lambda}^i\left(\frac{d\hat{n}^{i\prime}}{\omega\hat{\lambda}^i}\right) = \frac{1}{\omega}\int_0^{\hat{n}^i}d\hat{n}^{i\prime} = \frac{1}{\omega}\hat{n}^i \end{cases}$$

Recoil velocity due to the inspiral phase, $\left|V^i_{\text{recoil}}\right| \approx \dfrac{464}{105}cx^4\nu^2(1-4\nu)^{\frac{1}{2}}\sqrt{\{f(\nu,x)\}^2 + \{g(\nu,x)\}^2}$

For $x = \dfrac{1}{6}$, $\left|V^i_{\text{recoil}}\right| \approx 0.00341c\nu^2(1-4\nu)^{\frac{1}{2}}\sqrt{\{f(\nu,x)\}^2 + \{g(\nu,x)\}^2}\Big|_{x=1/6}$

**all graphs generated using WolframAlpha*

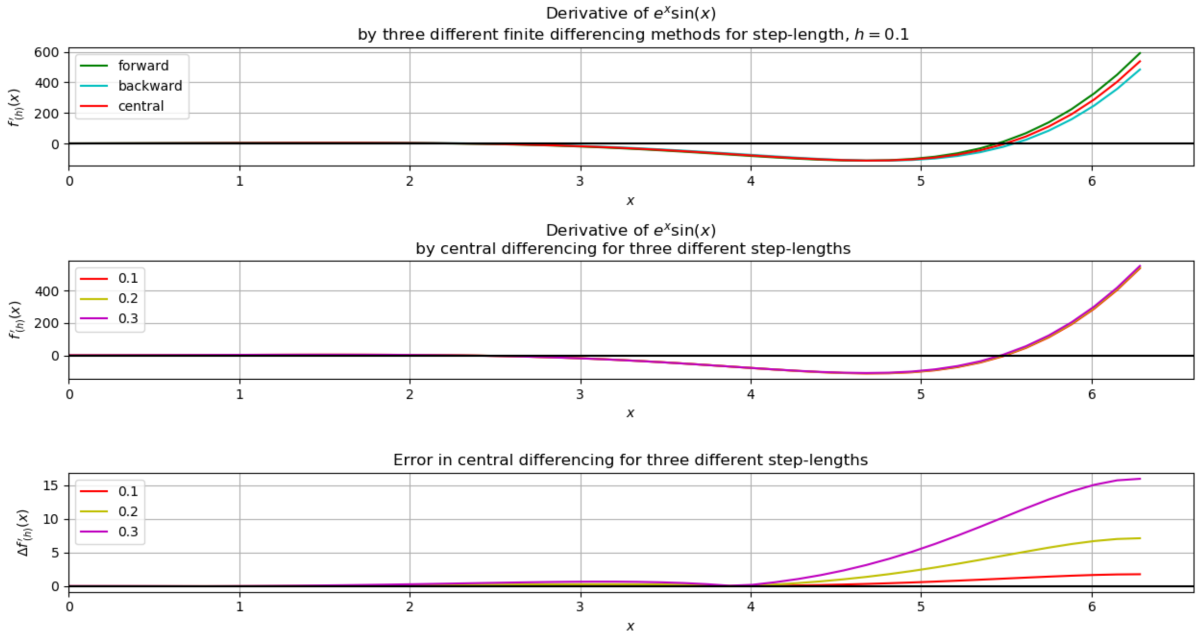
Solutions to the Numerical methods Lab Exercises

Pinaki Roy

This document contains solutions to the problems given in this document.

Lab 1

Derivatives using Finite Differencing Methods




```
import numpy as np
import matplotlib.pyplot as plot

x = np.linspace(0,2*np.pi,48)
y = np.exp(x)*np.sin(x)
dy = np.exp(x)*(np.cos(x)+np.sin(x))

def fdiff(f,x,h):
    h = float(h)
    return (f(x+h)-f(x))/h
def bdiff(f,x,h):
    h = float(h)
    return (f(x)-f(x-h))/h
def cdiff(f,x,h):
    h = float(h)
    return (f(x+h)-f(x-h))/(2*h)

f = lambda x: np.exp(x)*np.sin(x)
yf=fdiff(f,x,.1)
yb=bdiff(f,x,.1)
```

```
yc1=cdiff(f,x,.1)
yc2=cdiff(f,x,.2)
yc3=cdiff(f,x,.3)

eyc1=abs(yc1-dy)
eyc2=abs(yc2-dy)
eyc3=abs(yc3-dy)

plot.subplot(3,1,1)
plot.grid(True,which='both')
#plot.plot(x,y,color='b')
#plot.plot(x,dy,color='b')
plot.plot(x,yf,color='g',label='forward')
plot.plot(x,yb,color='c',label='backward')
plot.plot(x,yc1,color='r',label='central')
plot.axhline(y=0,color='k')
plot.xlabel('$x$')
plot.ylabel('$f\,\'_{(h)}(x)$')
plot.xlim(xmin=0)
plot.legend()
plot.title('Derivative of $e^x\sin(x)$ \n by three different
           finite differencing methods for step-length, $h=0.1$')

plot.subplot(3,1,2)
plot.grid(True,which='both')
plot.plot(x,yc1,color='r',label='0.1')
plot.plot(x,yc2,color='y',label='0.2')
plot.plot(x,yc3,color='m',label='0.3')
plot.axhline(y=0,color='k')
plot.xlabel('$x$')
plot.ylabel('$f\,\'_{(h)}(x)$')
plot.xlim(xmin=0)
plot.legend()
plot.title('Derivative of $e^x\sin(x)$ \n by central differencing
            for three different step-lengths')

plot.subplot(3,1,3)
plot.grid(True,which='both')
plot.plot(x,eyc1,color='r',label='0.1')
plot.plot(x,eyc2,color='y',label='0.2')
plot.plot(x,eyc3,color='m',label='0.3')
plot.axhline(y=0,color='k')
plot.xlabel('$x$')
plot.ylabel('$\Delta f\,\'_{(h)}(x)$')
plot.xlim(xmin=0)
plot.legend()
plot.title('Error in central differencing for three different step-lengths')

plot.subplots_adjust(wspace=None, hspace=0.8)
plot.show()
```

Order of Convergence of each Finite Difference Approximation

$f(x) = e^x \sin x$
$f'(x) = e^x(\sin x + \cos x)$
At x=0, $f'(x) = 1$

Forward Difference: $f'(x) = \dfrac{f(x+h) - f(x)}{h} = 1.10333, 1.213276, 1.329702$ for $h = 0.1, 0.2, 0.3$ respectively, so that the corresponding errors are $\varepsilon = 0.10333, 0.213276, 0.329702$
Assuming that error, $\varepsilon \propto h^n \implies \ln\varepsilon = n\,\ln h + C$, one obtains $n \approx (1.05 + 1.07)/2 = 1.06$

Backward Difference: $f'(x) = \dfrac{f(x) - f(x-h)}{h} = 0.90333, 0.813283, 0.729756$ for $h = 0.1, 0.2, 0.3$ respectively, so that the corresponding errors are $\varepsilon = 0.09667, 0.186717, 0.270244$
Assuming that error, $\varepsilon \propto h^n \implies \ln\varepsilon = n\,\ln h + C$, one obtains $n \approx (0.95 + 0.91)/2 = 0.93$

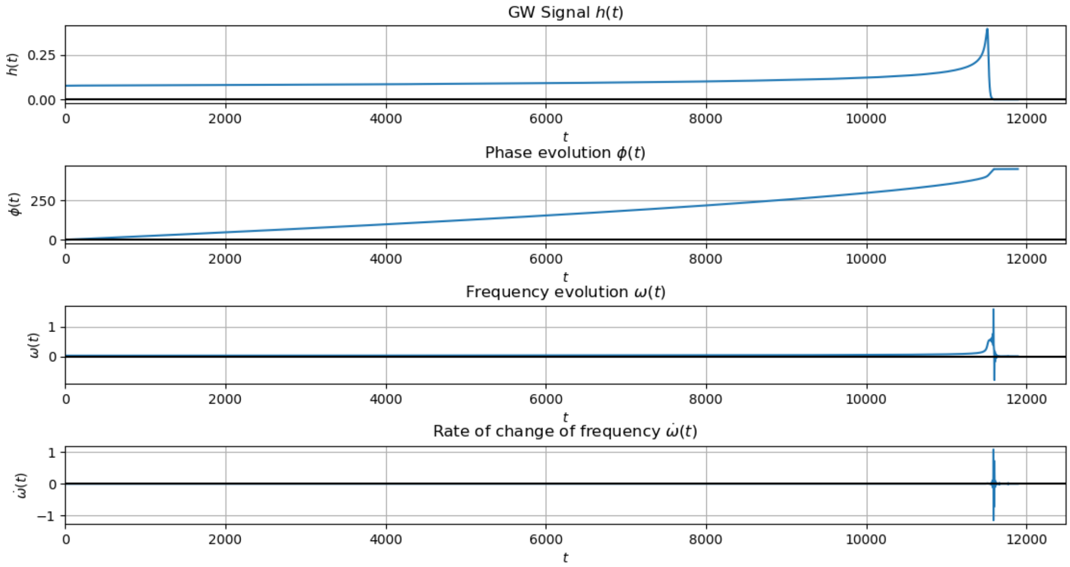
Central Difference: $f'(x) = \dfrac{f(x+h) - f(x-h)}{2h} = 1.00333, 1.01328, 1.02973$ for $h = 0.1, 0.2, 0.3$ respectively, so that the corresponding errors are $\varepsilon = 0.00333, 0.01328, 0.02973$
Assuming that error, $\varepsilon \propto h^n \implies \ln\varepsilon = n\,\ln h + C$, one obtains $n \approx (2.002 + 1.784)/2 = 1.893$

Central Differencing Approximant for the Second Derivative

$$f(x+h) = f(x) + hf'(x) + h^2\frac{f''(x)}{2!} + h^3\frac{f'''(x)}{3!} + h^4\frac{f''''(x)}{4!}$$
$$f(x-h) = f(x) - hf'(x) + h^2\frac{f''(x)}{2!} - h^3\frac{f'''(x)}{3!} + h^4\frac{f''''(x)}{4!}$$
$$\therefore\ f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + h^4\frac{f''''(x)}{12}$$
$$\implies f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - h^2\frac{f''''(x)}{12}$$
$$\implies f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)$$

Mock GW Signal Analysis

```
import math
import matplotlib.pyplot as plot

htime=[]
hplus=[]
hcross=[]
hamp=[]
hphase=[]
hdphase=[]
hddphase=[]
k=0
turn=0
for line in open("nr_data.dat"):
    columns = line.split()
    htime.append(float(columns[0]))
    hplus.append(float(columns[1]))
    hcross.append(float(columns[2]))
    k+=1
```




```
lastph=0
for i in range(0,k):
    hamp.append(math.sqrt(hplus[i]**2+hcross[i]**2))
    phase=math.atan2(hcross[i],hplus[i])
    if len(hphase)>1 and hphase[-1]>turn*2*math.pi:
        if phase*hphase[-1]<0:
            turn+=1
    hphase.append(turn*2*math.pi+phase)

for i in range(0,k):
    if i==0: #forward difference
        hdphase.append((hphase[i+1]-hphase[i])/
        (htime[i+1]-htime[i]))
        hddphase.append((hphase[i+2]-2*hphase[i+1]+hphase[i])/
        ((htime[i+1]-htime[i])**2))
    elif i==k-1: #backward difference
        hdphase.append((hphase[i]-hphase[i-1])/
        (htime[i]-htime[i-1]))
        hddphase.append((hphase[i]-2*hphase[i-1]+hphase[i-2])/
        ((htime[i]-htime[i-1])**2))
    else: #central difference
        #print i
        hdphase.append((hphase[i+1]-hphase[i-1])/
        (2*(htime[i+1]-htime[i])))
        hddphase.append((hphase[i+1]-2*hphase[i]+hphase[i-1])/
        ((htime[i+1]-htime[i])**2))

plot.subplot(4,1,1)
plot.grid(True,which='both')
plot.plot(htime,hamp)
plot.axhline(y=0,color='k')
plot.xlabel('$t$')
plot.ylabel('$h(t)$')
plot.xlim(xmin=0)
#plot.legend()
plot.title('GW Signal $h(t)$')
```

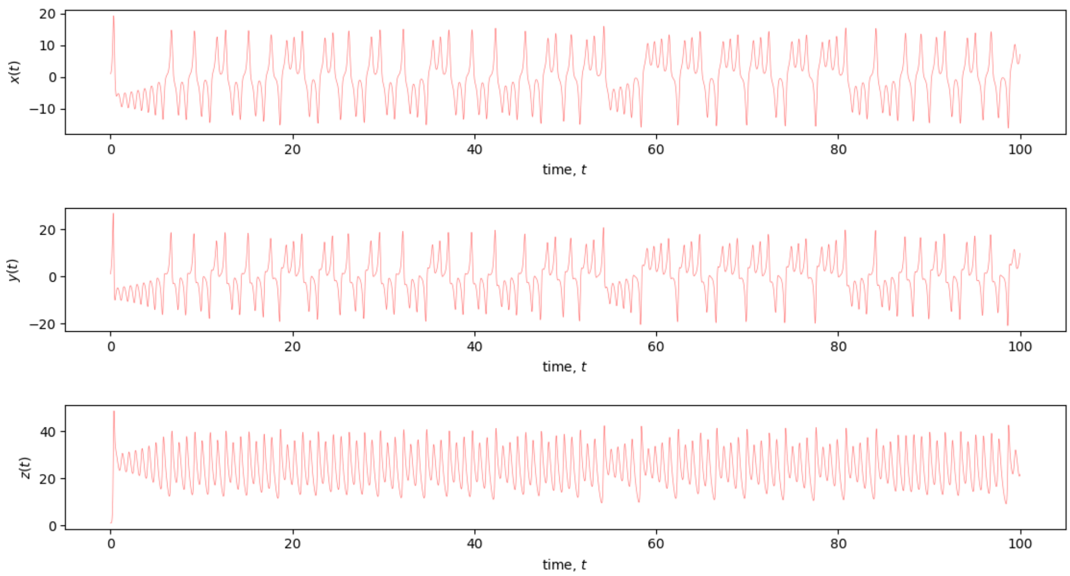
```
plot.subplot(4,1,2)
plot.grid(True,which='both')
plot.plot(htime,hphase)
plot.axhline(y=0,color='k')
plot.xlabel('$t$')
plot.ylabel('$\phi(t)$')
plot.xlim(xmin=0)
#plot.legend()
plot.title('Phase evolution $\phi(t)$')

plot.subplot(4,1,3)
plot.grid(True,which='both')
plot.plot(htime,hdphase)
plot.axhline(y=0,color='k')
plot.xlabel('$t$')
plot.ylabel('$\omega(t)$')
plot.xlim(xmin=0)
#plot.legend()
plot.title('Frequency evolution $\omega(t)$')

plot.subplot(4,1,4)
plot.grid(True,which='both')
plot.plot(htime,hddphase)
plot.axhline(y=0,color='k')
plot.xlabel('$t$')
plot.ylabel('$\dot{\omega}(t)$')
plot.xlim(xmin=0)
#plot.legend()
plot.title('Rate of change of frequency $\dot{\omega}(t)$')

plot.subplots_adjust(wspace=None, hspace=0.8)
plot.show()
```

Lab 4

Non-linear ordinary differential equations showing chaotic behavior: Lorenz equations

Problem 1

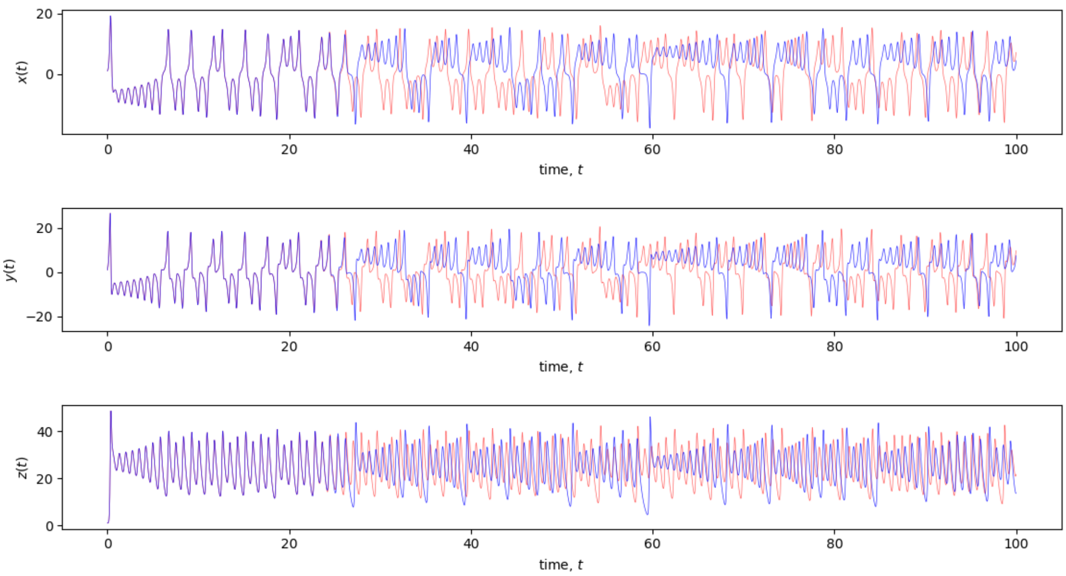
```
from scipy.integrate import odeint
from numpy import arange
from pylab import *
import matplotlib.pyplot as plot

def Lorenz(state, t):
    x, y, z = state
    d_x = 10 * (y - x)
    d_y = x * (28 - z) - y
    d_z = x*y - (8/3)*z
    return [d_x, d_y, d_z]

t = arange(0, 100, .01)
init_state = [1, 1, 1]
state = odeint(Lorenz, init_state, t)

plot.subplot(3,1,1)
plot.xlabel('time, $t$')
```




```
plot.ylabel('$x(t)$')
plot.plot(t, state[:, 0], 'r-', alpha=0.5, linewidth=.5)

plot.subplot(3,1,2)
plot.xlabel('time, $t$')
plot.ylabel('$y(t)$')
plot.plot(t, state[:, 1], 'r-', alpha=0.5, linewidth=.5)

plot.subplot(3,1,3)
plot.xlabel('time, $t$')
plot.ylabel('$z(t)$')
plot.plot(t, state[:, 2], 'r-', alpha=0.5, linewidth=.5)

plot.subplots_adjust(wspace=None, hspace=0.6)
plot.show()
```

Problem 2

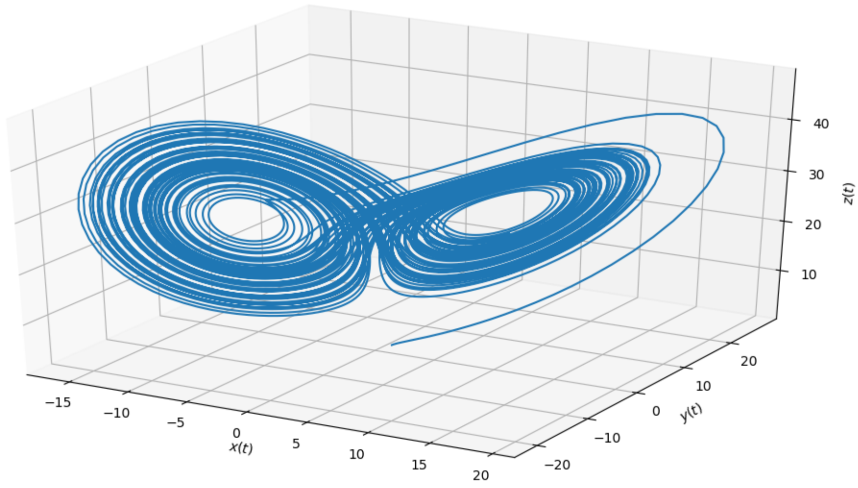
```
from scipy.integrate import odeint
from numpy import arange
from pylab import *
import matplotlib.pyplot as plot

def Lorenz(state, t):
    x, y, z = state
    d_x = 10 * (y - x)
    d_y = x * (28 - z) - y
    d_z = x*y - (8/3)*z
    return [d_x, d_y, d_z]

t = arange(0, 100, .01)
init_state1 = [1, 1, 1]
state1 = odeint(Lorenz, init_state1, t)
init_state2 = [1+10e-9, 1, 1]
state2 = odeint(Lorenz, init_state2, t)

plot.subplot(3,1,1)
```




```
plot.xlabel('time, $t$')
plot.ylabel('$x(t)$')
plot.plot(t, state1[:, 0], 'r-', alpha=0.6, linewidth=.5)
plot.plot(t, state2[:, 0], 'b-', alpha=0.8, linewidth=.5)

plot.subplot(3,1,2)
plot.xlabel('time, $t$')
plot.ylabel('$y(t)$')
plot.plot(t, state1[:, 1], 'r-', alpha=0.6, linewidth=.5)
plot.plot(t, state2[:, 1], 'b-', alpha=0.8, linewidth=.5)

plot.subplot(3,1,3)
plot.xlabel('time, $t$')
plot.ylabel('$z(t)$')
plot.plot(t, state1[:, 2], 'r-', alpha=0.6, linewidth=.5)
plot.plot(t, state2[:, 2], 'b-', alpha=0.8, linewidth=.5)

plot.subplots_adjust(wspace=None, hspace=0.6)
plot.show()
```

Problem 3

```
import matplotlib.animation as animation
from scipy.integrate import odeint
from numpy import arange
from pylab import *
import matplotlib.pyplot as plot
from mpl_toolkits import mplot3d

def Lorenz(state, t):
    x, y, z = state
    d_x = 10 * (y - x)
    d_y = x * (28 - z) - y
    d_z = x*y - (8/3)*z
    return [d_x, d_y, d_z]

t = arange(0, 100, .01)
```

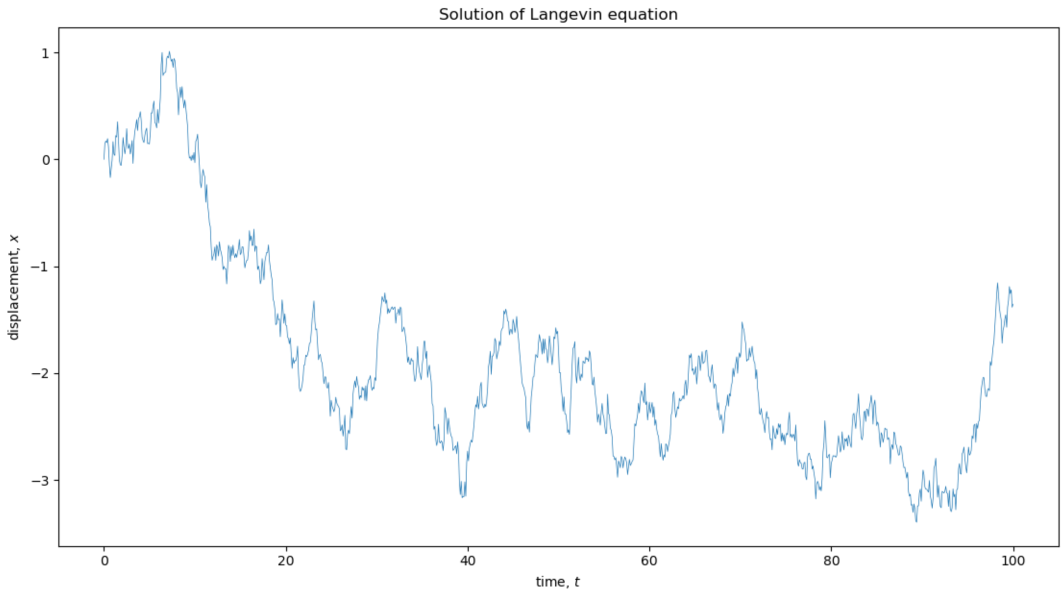



```
init_state = [1, 1, 1]
state = odeint(Lorenz, init_state, t)

fig = plot.figure()
ax = plot.axes(projection='3d')
ax.set_xlabel('$x(t)$')
ax.set_ylabel('$y(t)$')
ax.set_zlabel('$z(t)$')
ax.plot(state[:, 0], state[:, 1], state[:, 2])

#omit the above line and include the below block to animate
def animate(i):
    ax.set_title('time, $t$=%.1f' %i, loc='left')
    ax.plot(state[0:(100*i), 0], state[0:(100*i), 1],
    state[0:(100*i), 2], 'b-', linewidth=.25)
ani = animation.FuncAnimation(fig, animate, interval=1)

show()
```

Stochastic ordinary differential equations: Langevin equation

Problem 1

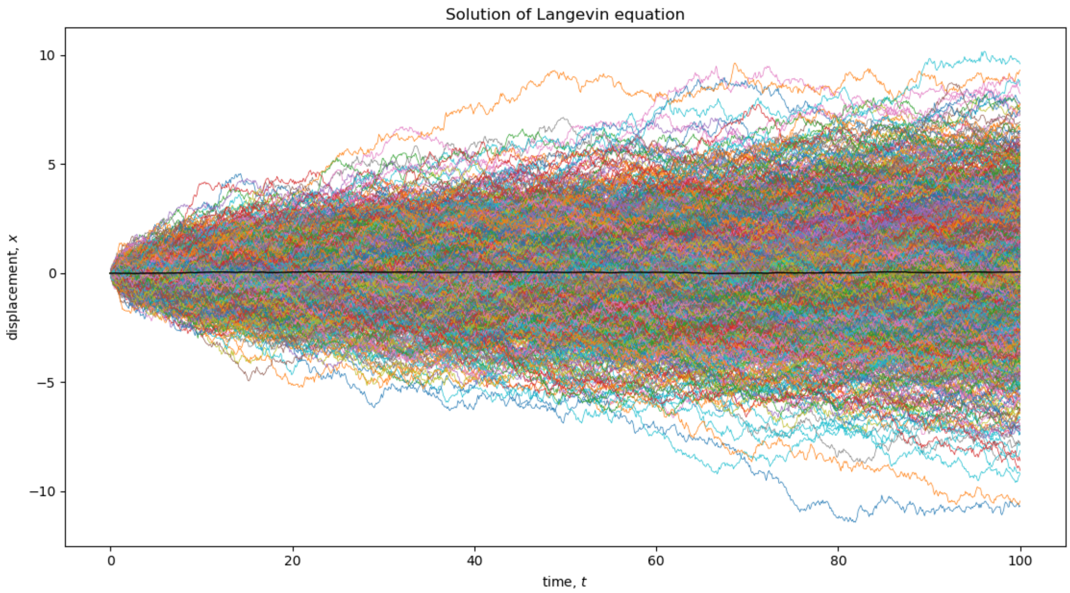
```
import numpy as np
from matplotlib import pyplot as plt

t0 = 0.0
x0 = 0.0
tf = 100.0
n = 1001 #no. of instants

deltat = (tf-t0)/(n-1)

t = np.linspace(t0,tf,n)
x = np.zeros([n])

noise = np.random.normal(0.,1.,n)
```




```
x[0] = x0
for i in range(1,n):
    x[i] = deltat*noise[i-1] + x[i-1]

#for i in range(n):
#     print(t[i],x[i])

plt.plot(t,x,linewidth=.5)

#print noise

plt.xlabel("time, $t$")
plt.ylabel("displacement, $x$")
plt.title("Solution of Langevin equation")
plt.show()
```

Problem 2

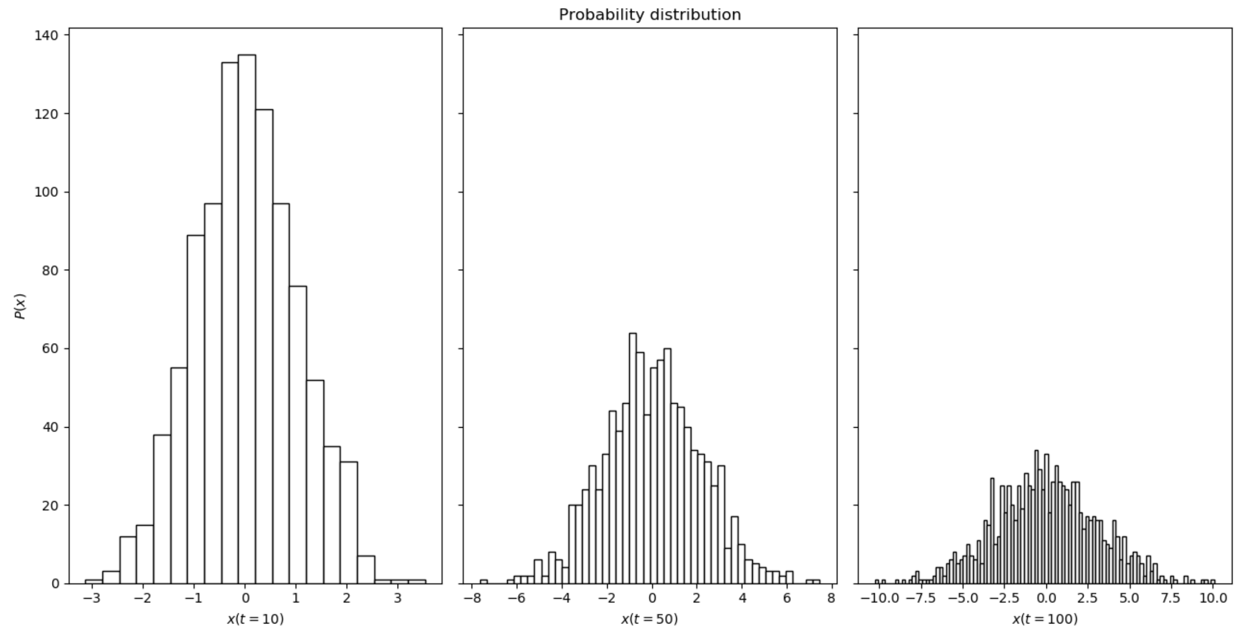
```
import numpy as np
from matplotlib import pyplot as plt

n = 1001 #no. of instants
p = 1000 #no. of particles
y = np.zeros([n])

for j in range(p):
    t0 = 0.0
    x0 = 0.0
    tf = 100.0

    deltat = (tf-t0)/(n-1)

    t = np.linspace(t0,tf,n)
    x = np.zeros([n])

    noise = np.random.normal(0.,1.,n)
```




```
    x[0] = x0
    for i in range(1,n):
        x[i] = deltat*noise[i-1] + x[i-1]
        y[i] = y[i] + x[i]

    #for i in range(n):
    #    print(t[i],x[i])

    plt.plot(t,x,linewidth=.5)

plt.plot(t,y/p,'k',linewidth=1) #mean displacement at an instant
plt.xlabel("time, $t$")
plt.ylabel("displacement, $x$")
plt.title("Solution of Langevin equation")
plt.show()
```

Problem 3

```
import numpy as np
from matplotlib import pyplot as plt

n = 1001 #no. of instants
p=1000 #no. of particles
y10 = np.zeros([p])
y50 = np.zeros([p])
y100 = np.zeros([p])

for j in range(p):
    t0 = 0.0
    x0 = 0.0
    tf = 100.0

    deltat = (tf-t0)/(n-1)

    t = np.linspace(t0,tf,n)
    x = np.zeros([n])
```




```
    noise = np.random.normal(0.,1.,n)

    x[0] = x0
    for i in range(1,n):
        x[i] = deltat*noise[i-1] + x[i-1]
        if i==100:
            y10[j] = x[i]
        if i==500:
            y50[j] = x[i]
        if i==1000:
            y100[j] = x[i]

fig, axs = plt.subplots(1, 3, sharey=True, tight_layout=True)

axs[0].hist(y10,20,color='w',ec='k')
axs[1].hist(y50,50,color='w',ec='k')
axs[2].hist(y100,100,color='w',ec='k')

axs[0].set_xlabel("$x(t=10)$")
axs[1].set_xlabel("$x(t=50)$")
axs[2].set_xlabel("$x(t=100)$")
axs[0].set_ylabel("$P(x)$")
axs[1].set_title("Probability distribution")

plt.show()
```

Lab 2

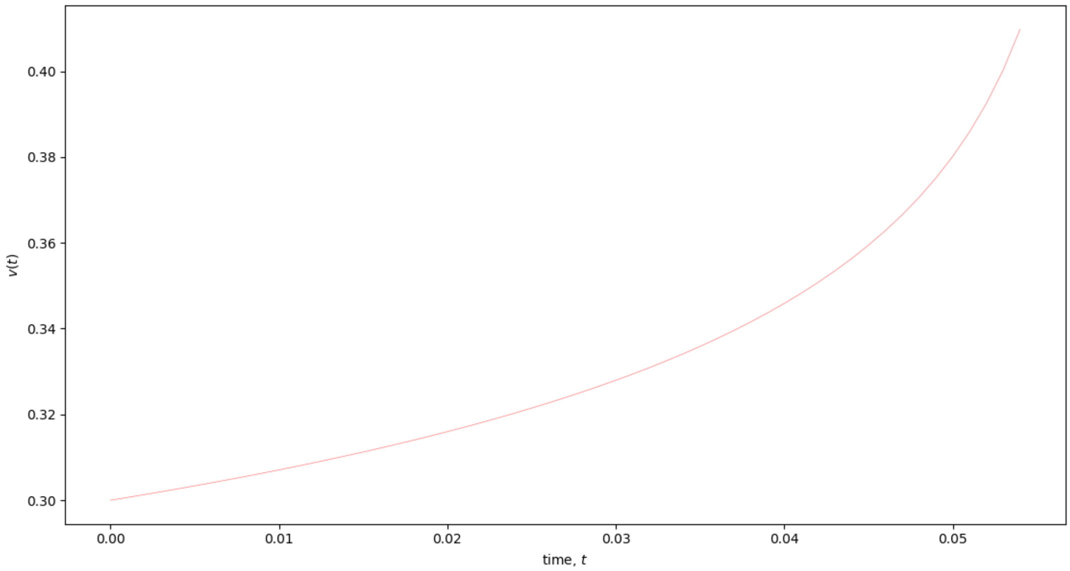
ODEs: Calculation of gravitational waves from inspiralling compact binaries

Problem 1

$$\frac{dv(t)}{dt} = -\frac{\mathcal{F}(v)}{dE(v)/dv} = \left(\frac{c^3}{G}\right)\frac{\frac{32}{5}\left(\frac{\mu}{m}\right)^2 v^{10}}{\mu v} \approx 32302\, v^9 \implies \frac{dv(t)}{v^9} = 32302\, dt \implies \frac{v^{-8}}{-8} = 32302\, t + C$$

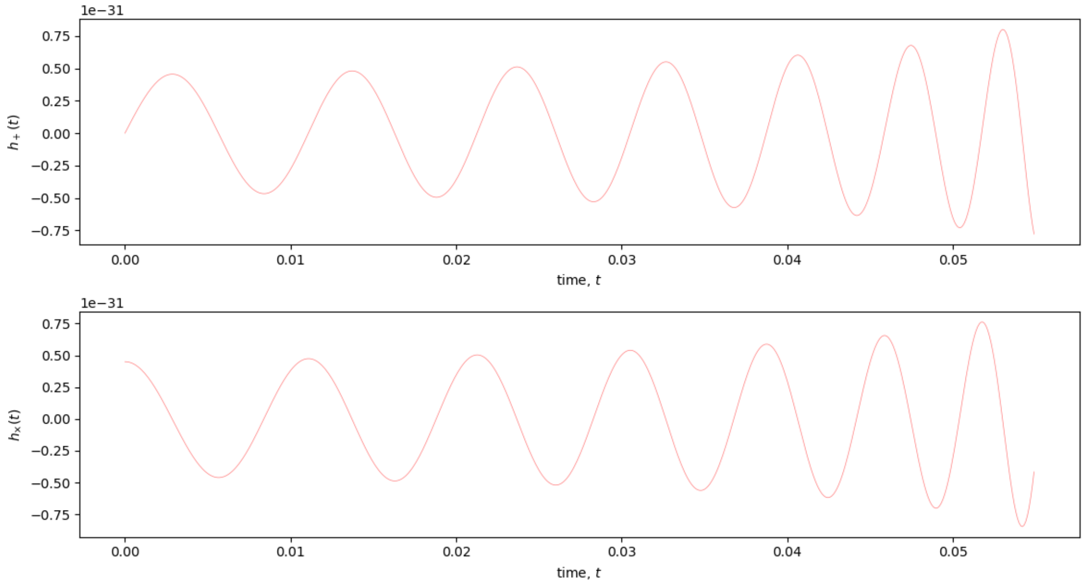
$$\because v = v_0 \text{ at } t = 0 \therefore C = \frac{v_0^{-8}}{-8} \text{ so that } t = \frac{v_0^{-8} - v^{-8}}{258416} \implies v(t) = (v_0^{-8} - 258416t)^{-1/8}$$

$$\implies v(t) = 1 \text{ at } t \approx 0.059$$




```
from scipy.integrate import odeint
import numpy as np
from numpy import arange
from pylab import *
import matplotlib.pyplot as plot

M_sun = 2.0e30
c0 = 3.0e8
G0 = 6.673e-11
k0 = (c0**3/G0)/M_sun

def bbh(state, t):
    v = state
    d_v = k0*(6.4*0.25*0.25)*(v**9)/2.5 #mu=2.5M_sun and dE/dv=-mu*v
    return d_v

t = arange(0, .055, .001) #t_final inferred analytically
init_state = 0.3
state = odeint(bbh, init_state, t)

plot.xlabel('time, $t$')
plot.ylabel('$v(t)$')
plot.plot(t, state, 'r-', alpha=0.5, linewidth=.5)
plot.show()
```

Problem 2

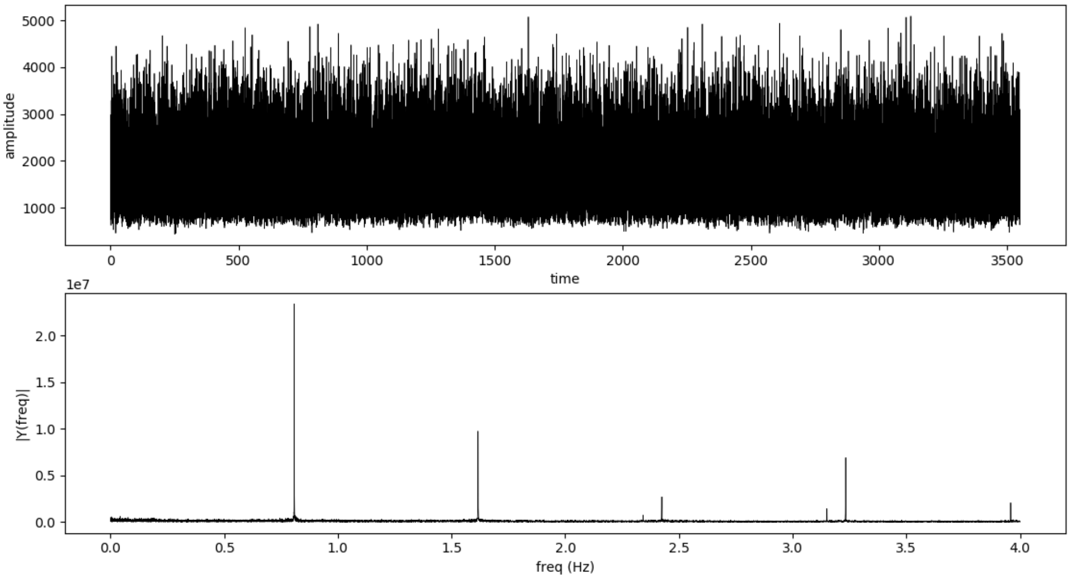



```
from scipy.integrate import odeint
import numpy as np
from numpy import arange
from pylab import *
import matplotlib.pyplot as plot

M_sun = 2.0e30
c0 = 3.0e8
G0 = 6.673e-11
k0 = (c0**3/G0)/M_sun

def bbh(state, t):
    v, phi = state
    d_v = k0*(6.4*0.25*0.25)*(v**9)/2.5 #mu=2.5M_sun and dE/dv=-mu*v
    d_phi = k0*(v**3)/10.0
    return [d_v, d_phi]

t = arange(0, .055, .0001)
init_state = [0.3, 0]
state = odeint(bbh, init_state, t)

h_plus=4*(0.25/M_sun)*state[:, 0]**2*np.sin(state[:, 1])
h_cross=4*(0.25/M_sun)*state[:, 0]**2*np.cos(state[:, 1])

plot.subplot(2,1,1)
plot.xlabel('time, $t$')
plot.ylabel('$v(t)$')
plot.plot(t, h_plus, 'r-', alpha=0.5, linewidth=.5)

plot.subplot(2,1,2)
plot.xlabel('time, $t$')
plot.ylabel('$\phi(t)$')
plot.plot(t, h_cross, 'r-', alpha=0.5, linewidth=.5)
```

```
plot.subplots_adjust(wspace=None,hspace=0.8)
plot.show()
```

Lab 7

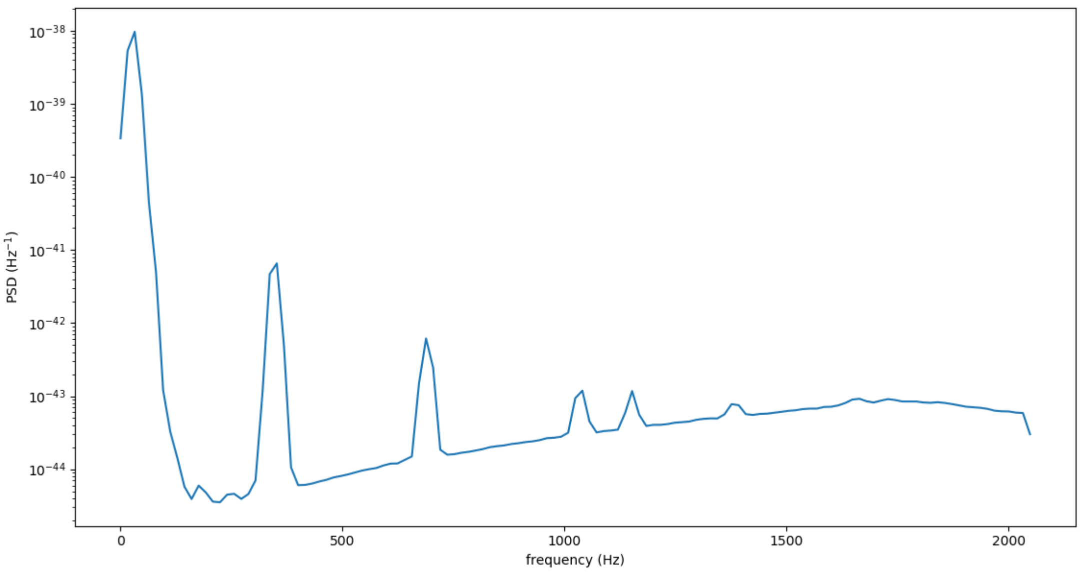
Fast Fourier transform (FFT)




```
import numpy as np
import matplotlib.pyplot as plt
from scipy.fftpack import fft

data=np.genfromtxt('lc_HerX-1.dat') # remove header row prior to this

t=data[:,0]
y=data[:,1]

Ts = 0.125 # sampling interval

plt.subplot(2,1,1)
plt.plot(t,y,'k-',linewidth=0.5)
plt.xlabel('time')
plt.ylabel('amplitude')

plt.subplot(2,1,2)
n = len(y) # length of the signal
f = np.linspace(0.0, 1/Ts, n)
fq = f[range(n/2)]
Y = fft(y) # fft computing
Y = 2*Y[range(n/2)]

plt.plot(fq[1:], abs(Y[1:]),'k-',linewidth=0.5) #excluding null frequency

plt.xlabel('freq (Hz)')
plt.ylabel('|Y(freq)|')
plt.show()
```

Spin frequency of the pulsar is 0.8 Hz (approx.) i.e. Spin period of the pulsar = 1.25 s (approx.)

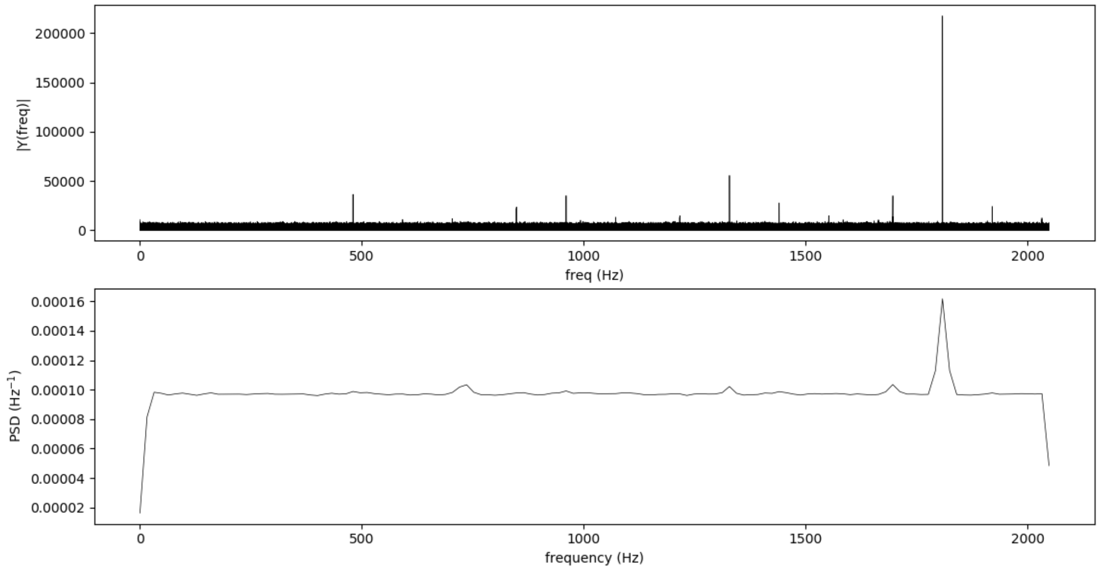
Power spectrum estimation using FFT




```
import numpy as np
from scipy import signal
import matplotlib.pyplot as plt

fs = 4096

y=np.genfromtxt('L1strain.txt')

f, psd = signal.welch(y, fs)
plt.semilogy(f, psd)
plt.xlabel('frequency (Hz)')
plt.ylabel('PSD (Hz$^{-1}$)')
plt.show()
```

Time-frequency signal detection methods

```
from scipy import signal
import numpy as np
import matplotlib.pyplot as plt
from scipy.fftpack import fft

data=np.genfromtxt('lc_4U 1636-536.dat') # remove header row prior to this

fs = 4096 #sampling rate

t = np.linspace(0.0, 1946, 1946*fs) #tf from data file
y = np.zeros([len(t)])

Ts = .000244141 #1/tf*fs

for i in range(len(data)):
```



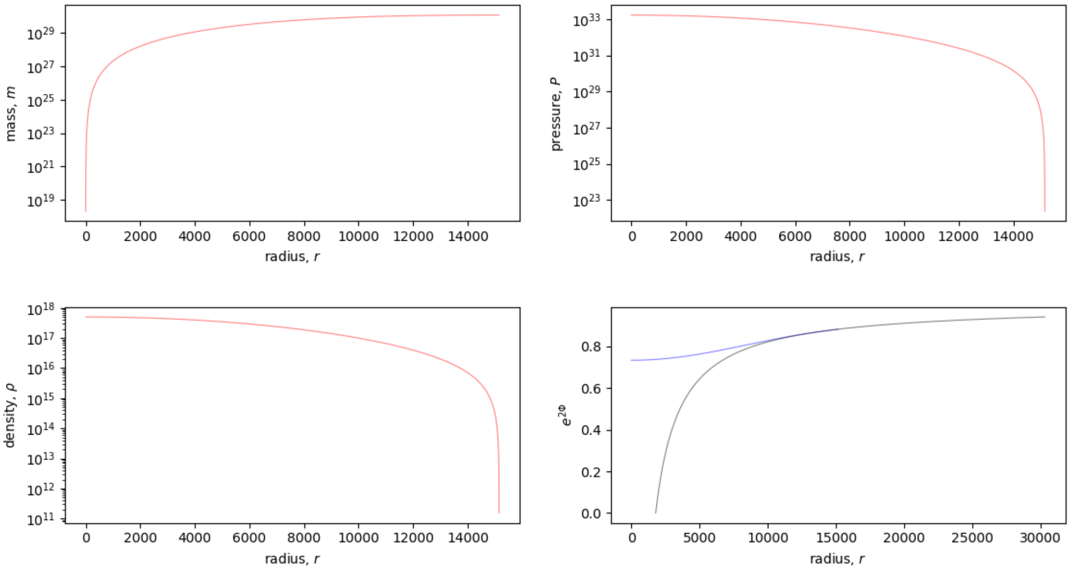

```
    stage=int(data[i]/Ts)
    y[stage]+=1

n = len(y) # length of the signal
f = np.linspace(0.0, 1/Ts, n)
fq = f[range(n/2)]
Y = fft(y) # fft computing
Y = 2*Y[range(n/2)]

plt.subplot(2,1,1)
plt.plot(fq[1:], abs(Y[1:]), 'k-', linewidth=0.5) #null frequency excluded
plt.xlabel('freq (Hz)')
plt.ylabel('|Y(freq)|')

plt.subplot(2,1,2)
F, psd = signal.welch(y, fs)
plt.plot(F, psd, 'k-', linewidth=0.5)
plt.xlabel('frequency (Hz)')
plt.ylabel('PSD (Hz$^{-1}$)')

plt.show()
```

The frequency span is about 125 Hz.

Lab 3

ODEs: Structure of a relativistic, spherically symmetric star

```
from scipy.integrate import odeint
import numpy as np
from numpy import arange
from pylab import *
import matplotlib.pyplot as plot

M_sun = 2.0e30
c0 = 3.0e8
G0 = 6.673e-11
```




```
def ns(state, r):
    m, P = state
    d_m = 4*np.pi*r**2*np.power(P/5380.3,0.6)
    d_P = -(G0/(r**2))*(np.power(P/5380.3,0.6)+P/(c0**2))*
    (m+4*np.pi*r**3*P/(c0**2))/(1-2*G0*m/(r*c0**2))
    return [d_m, d_P]

r = arange(1, 15157, 1)     #r_final inferred by iterative use of this code
init_state = [2.1e18, 1.7e33]
state = odeint(ns, init_state, r)

Phi = np.zeros([15156])
Phi[-1] = 0.5*np.log(1-2*G0*state[-1, 0]/(r[-1]*c0**2))

print 'Phi at r_star = ',Phi[-1]

for i in range(15155,0,-1):
    d_Phi = G0*(state[i, 0]+4*np.pi*r[i]**3*state[i, 1]/(c0**2))/
    ((r[i]*c0**2)*(r[i]-2*G0*state[i, 0]/(c0**2)))
    Phi[i-1] = Phi[i] - d_Phi

plot.subplot(2,2,1)
plot.xlabel(r'radius, $r$')
plot.ylabel(r'mass, $m$')
plot.semilogy(r, state[:, 0], 'r-', alpha=0.5, linewidth=.75)

plot.subplot(2,2,2)
plot.xlabel(r'radius, $r$')
plot.ylabel(r'pressure, $P$')
plot.semilogy(r, state[:, 1], 'r-', alpha=0.5, linewidth=.75)

rho = np.power(state[:, 1]/5380.3,0.6)

plot.subplot(2,2,3)
plot.xlabel(r'radius, $r$')
plot.ylabel(r'density, $\rho$')
```

```
plot.semilogy(r, rho, 'r-', alpha=0.5, linewidth=.75)

R = arange(1782, 30314, 1) #from Schwarzschild radius to twice the star radius
lapse = 0.5*np.log(1-2*G0*state[-1, 0]/(R*c0**2)) #for Schwarzschild BH

plot.subplot(2,2,4)
plot.xlabel(r'radius, $r$')
plot.ylabel(r'$e^{2\Phi}$')
plot.plot(r, np.exp(2*Phi), 'b-', alpha=0.5, linewidth=.75)
plot.plot(R, np.exp(2*lapse), 'k-', alpha=0.5, linewidth=.75)

print 'mass of the model star = ',state[-1, 0]/M_sun,'M_sun'

plot.subplots_adjust(wspace=None,hspace=0.4)
plot.show()
```

The mass of the model star comes to $0.6M_{sun}$ which is much below the Chandrasekhar limit. This is likely because of the non-relativistic equation of state used in the problem.