Definition 1.1 Point $\boldsymbol{x}^*$ is said to be *regular* for a given set of equality constraints $\{a_i(\boldsymbol{x}) = 0, i = 1,$ $2, \ldots, p\}$ if the p vectors $\{\nabla a_i(\boldsymbol{x}^*), i = 1, 2...., p\}$ are linearly independent.

1.3.1 Optimality Conditions in Simple Cases

Definition 1.2 Let $\boldsymbol{x}$ be a feasible point for problem (1.1). Vector $\boldsymbol{d} \in R^n$ is said to be a *feasible direction* at $\boldsymbol{x}$ if there exists $\hat{\alpha} > 0$ such that $\boldsymbol{x} + \alpha \boldsymbol{d}$ remains feasible for all $\alpha \in [0, \hat{\alpha}]$.

Theorem 1.1 If $f(\boldsymbol{x}) \in C^1$ and $\boldsymbol{x}^*$ is a local minimizer of problem (1.1), then

$$\nabla f(\boldsymbol{x}^*)^T \boldsymbol{d} \geq 0 \quad \text{for every feasible direction } \boldsymbol{d} \text{ at } \boldsymbol{x}^* \tag{1.11}$$

Proof We examine Taylor's expansion of $f(\boldsymbol{x})$ at $\boldsymbol{x} = \boldsymbol{x}^* + \alpha \boldsymbol{d}$:

$$f(\boldsymbol{x}) = f(\boldsymbol{x}^*) + \alpha \nabla f(\boldsymbol{x}^*)^T \boldsymbol{d} + o(\alpha \| \boldsymbol{d} \|_2)$$

Hence

$$\alpha \nabla f(\boldsymbol{x}^*)^T \boldsymbol{d} + o(\alpha \| \boldsymbol{d} \|_2) = f(\boldsymbol{x}) - f(\boldsymbol{x}^*) \geq 0$$

which implies (1.11). ■

Example 1.2 Let $\boldsymbol{x}^*$ be a local minimizer of the equality constrained problem

$$\begin{aligned} &\text{minimize} \quad f(\boldsymbol{x}) \\ &\text{subject to:} \;\; \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{aligned} \tag{1.12}$$

where $\boldsymbol{A} \in R^{p \times n}$ with $p < n$. The feasible region is defined by $\mathcal{R} = \{\boldsymbol{x}: \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$. Obviously, $\boldsymbol{d}$ is a feasible direction at a feasible point $\boldsymbol{x}$ if and only if $\boldsymbol{A}\boldsymbol{d} = \boldsymbol{0}$, namely $\boldsymbol{d}$ belongs to the null space of $\boldsymbol{A}$. By applying Theorem 1.1, we obtain

$$\nabla f(\boldsymbol{x}^*)^T \boldsymbol{d} \geq 0 \quad \text{for } \boldsymbol{d} \in \mathcal{N}(\boldsymbol{A}) \tag{1.13}$$

where $\mathcal{N}(\boldsymbol{A})$ denotes the null space of $\boldsymbol{A}$. Since $\boldsymbol{d} \in \mathcal{N}(\boldsymbol{A})$ implies $-\boldsymbol{d} \in \mathcal{N}(\boldsymbol{A})$, (1.13) implies that $\nabla f(\boldsymbol{x}^*)^T \boldsymbol{d} = 0$ for $\boldsymbol{d} \in \mathcal{N}(\boldsymbol{A})$. From linear algebra, we conclude that

$$\nabla f(\boldsymbol{x}^*) \in \mathcal{N}^{\perp}(\boldsymbol{A}) = \mathcal{R}(\boldsymbol{A}^T) \tag{1.14}$$

This is to say that gradient $\nabla f(\boldsymbol{x}^*)$ is in the range of matrix $\boldsymbol{A}^T$. Therefore, there exists $\boldsymbol{\lambda} \in R^p$ (known as *Lagrange multiplier* for equality constraints) such that $\nabla f(\boldsymbol{x}^*) = -\boldsymbol{A}^T \boldsymbol{\lambda}$, namely,

$$\nabla f(\boldsymbol{x}^*) + \boldsymbol{A}^T \boldsymbol{\lambda} = \boldsymbol{0} \tag{1.15}$$

■

1.3.2 Equality Constraints

We now consider the constrained problem

$$\begin{aligned} &\text{minimize} \quad f(\boldsymbol{x}) \\ &\text{subject to:} \ a_i(\boldsymbol{x}) = 0 \ \text{ for } \ 1 \le i \le p \end{aligned} \tag{1.16}$$

Let $\boldsymbol{x}^*$ be a local minimizer of (1.16) and $\boldsymbol{s}$ be a feasible vector at $\boldsymbol{x}^*$. Thus we have $a_i(\boldsymbol{x}^*) = 0$ and $a_i(\boldsymbol{x}^* + \boldsymbol{s}) = 0$. The Taylor expansion of $a_i(\boldsymbol{x}^* + \boldsymbol{s})$ gives

$$0 = a_i(\boldsymbol{x}^* + \boldsymbol{s}) = a_i(\boldsymbol{x}^*) + \boldsymbol{s}^T \nabla a_i(\boldsymbol{x}^*) + o(\| \boldsymbol{s} \|) = \boldsymbol{s}^T \nabla a_i(\boldsymbol{x}^*) + o(\| \boldsymbol{s} \|)$$

which implies that

$$\boldsymbol{s}^T \nabla a_i(\boldsymbol{x}^*) = 0 \ \text{ for } \ 1 \le i \le p \tag{1.17}$$

In other words, $\boldsymbol{s}$ is feasible if and only if it is orthogonal to the gradients of the constraint functions. Now we project the gradient $\nabla f(\boldsymbol{x}^*)$ onto the space spanned by $\{\nabla a_1(\boldsymbol{x}^*), \nabla a_2(\boldsymbol{x}^*),$ $\ldots, \nabla a_p(\boldsymbol{x}^*)\}$. Let the projection be given by $-\sum_{i=1}^{p} \lambda_i^* \nabla a_i(\boldsymbol{x}^*)$, then $\nabla f(\boldsymbol{x}^*)$ can be expressed as

$$\nabla f(\boldsymbol{x}^*) = -\sum_{i=1}^{p} \lambda_i^* \nabla a_i(\boldsymbol{x}^*) + \boldsymbol{d} \tag{1.18}$$

where $\boldsymbol{d}$ is orthogonal to $\nabla a_i(\boldsymbol{x}^*)$ for i = 1, 2, …, p. Hence $\boldsymbol{s} = -\boldsymbol{d}$ is a feasible direction. From (1.17) and (1.18), it follows that $\boldsymbol{s}^T \nabla f(\boldsymbol{x}^*) = -\|\boldsymbol{d}\|^2$ which means that for $\boldsymbol{x}^*$ to be a local minimizer $\boldsymbol{d}$ must be zero (otherwise $\boldsymbol{s} = -\boldsymbol{d}$ would be a nontrivial feasible direction from $\boldsymbol{x}^*$ along which $f(\boldsymbol{x})$ would decrease, a contradiction with the assumption of $\boldsymbol{x}^*$ being a minimizer). This leads (1.18) to

$$\nabla f(\boldsymbol{x}^*) = -\sum_{i=1}^{p} \lambda_i^* \nabla a_i(\boldsymbol{x}^*) \tag{1.19}$$

The concept of Lagrange multiplier may also be understood from a different perspective. To see this we introduce the *Lagrangian* of problem (1.16) as

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) + \sum_{i=1}^{p} \lambda_i a_i(\boldsymbol{x}) \tag{1.20}$$

where $\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_p \end{bmatrix}^T$. In terms of the Lagrangian, (1.19) becomes

$$\nabla_x L(\boldsymbol{x},\boldsymbol{\lambda}) = \mathbf{0} \ \text{ for } \ \{\boldsymbol{x},\boldsymbol{\lambda}\} = \{\boldsymbol{x}^*,\boldsymbol{\lambda}^*\} \tag{1.21}$$

Eq. (1.21) offers n equations while the equality constraints in (1.16) contain p equations. Note that the number of equations available, $n + p$, matches the number of unknowns in $\boldsymbol{x}$ and $\boldsymbol{\lambda}$. Moreover, if we introduce the gradient operator ∇ as

$$\nabla = \begin{bmatrix} \nabla_x \\ \nabla_\lambda \end{bmatrix}$$

then these equations can be expressed as

$$\nabla L(\boldsymbol{x},\boldsymbol{\lambda}) = \mathbf{0} \ \text{ for } \ \{\boldsymbol{x},\boldsymbol{\lambda}\} = \{\boldsymbol{x}^*,\boldsymbol{\lambda}^*\} \tag{1.22}$$

In summary, the Lagrangian incorporates the constraints into a modified objective function in a way such that constrained minimizer $\boldsymbol{x}^*$ is connected to an unconstrained minimizer $\{\boldsymbol{x}^*,\boldsymbol{\lambda}^*\}$ for the augmented objective function $L(\boldsymbol{x},\boldsymbol{\lambda})$ where the augmentation is achieved with the p Lagrange multipliers.

Example 1.3 Solve the convex quadratic problem

$$\begin{aligned} &\text{minimize} \quad f(\boldsymbol{x}) = \tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{p} \\ &\text{subject to:} \ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$

where $\boldsymbol{H}$ is positive definite and $\boldsymbol{A} \in R^{p\times n}$ with $p \le n$ has full row rank.

Solution The Lagrangian of the problem is given by $L(\boldsymbol{x},\boldsymbol{\lambda}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{p} + \boldsymbol{\lambda}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})$. Applying (1.22), we obtain

$$\nabla L(\boldsymbol{x},\boldsymbol{\lambda}) = \begin{bmatrix} \boldsymbol{H}\boldsymbol{x} + \boldsymbol{p} + \boldsymbol{A}^T\boldsymbol{\lambda} \\ \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} \boldsymbol{H} & \boldsymbol{A}^T \\ \boldsymbol{A} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \boldsymbol{p} \\ -\boldsymbol{b} \end{bmatrix} = \mathbf{0}$$

Because $\boldsymbol{H} \succ \mathbf{0}$ and $\boldsymbol{A}$ has full row rank, one can show that matrix

$$\begin{bmatrix} \boldsymbol{H} & \boldsymbol{A}^T \\ \boldsymbol{A} & \mathbf{0} \end{bmatrix}$$

is nonsingular (try to prove this fact as an exercise!). Therefore,

$$\begin{bmatrix} \boldsymbol{x}^* \\ \boldsymbol{\lambda}^* \end{bmatrix} = \begin{bmatrix} \boldsymbol{H} & \boldsymbol{A}^T \\ \boldsymbol{A} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} -\boldsymbol{p} \\ \boldsymbol{b} \end{bmatrix}$$

It follows that

$$\begin{aligned} \boldsymbol{x}^* &= -\boldsymbol{H}^{-1}(\boldsymbol{A}^T\boldsymbol{\lambda}^* + \boldsymbol{p}) \\ \boldsymbol{\lambda}^* &= -(\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{A}^T)^{-1}(\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{p} + \boldsymbol{b}) \end{aligned} \tag{1.23}$$

1.3.3 Inequality Constraints

First, a concept concerning inequality constraints. Suppose $\boldsymbol{x}^*$ is a minimizer of problem (1.1) and let $I(\boldsymbol{x}^*)$ be an index set $I(\boldsymbol{x}^*) = \{j_1, j_2, \ldots, j_K\} \subseteq \{1, 2, \ldots, q\}$ for *active constraints* at $\boldsymbol{x}^*$, i.e., $c_j(\boldsymbol{x}^*) = 0$ for $j = j_1, j_2, \ldots, j_K$. As an example, Figure 1.2 shows a feasible region defined by R = $\{\boldsymbol{x} : c_j(\boldsymbol{x}) \le 0, \text{ for } j = 1,2,3\}$ and a point $\bar{\boldsymbol{x}}$ at which constraint $c_3(\boldsymbol{x}) \le 0$ is active because $c_3(\bar{\boldsymbol{x}}) = 0$. We see that a point lies on the boundary of the feasible region if at the point some constraints become active.

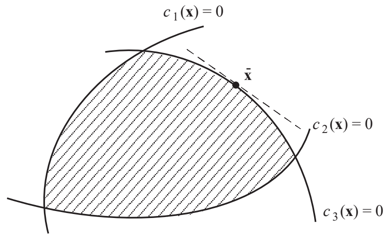

Figure 1.2

Now let us consider the general constrained problem (1.1):

$$\text{minimize} \quad f(\boldsymbol{x}) \tag{1.1a}$$
$$\text{subject to:} \quad a_i(\boldsymbol{x}) = 0 \quad \text{for } i = 1,2,\ldots,p \tag{1.1b}$$
$$c_j(\boldsymbol{x}) \le 0 \quad \text{for } j = 1,2,\ldots,q \tag{1.1c}$$

and let $\boldsymbol{x}^*$ be a local minimizer of (1.1) at which there are K active inequality constraints, namely,

$$c_j(\boldsymbol{x}^*) = 0 \quad \text{for} \quad j \in I(\boldsymbol{x}^*) = \{j_1, j_2, \ldots, j_K\} \tag{1.24}$$

The K active constraints at $\boldsymbol{x}^*$ act like K additional equality constraints. Consequently, the equation in (1.19) need to be modified to

$$\nabla f(\boldsymbol{x}^*) = -\sum_{i=1}^{p} \lambda_i^* \nabla a_i(\boldsymbol{x}^*) - \sum_{k=1}^{K} \mu_{j_k}^* \nabla c_{j_k}(\boldsymbol{x}^*) \tag{1.25}$$

where $\mu_{j_k}^*$ are called Lagrange multipliers of the active inequality constraints.

An important property of the Lagrange multipliers associated with inequality constraints is that they are nonnegative, i.e.

$$\mu_{j_k}^* \ge 0 \quad \text{for} \quad 1 \le k \le K \tag{1.26}$$

This property is illustrated in Fig. 1.3 for the case where the minimizer is in the interior of the feasible region (case (a)) and the case where the minimizer is on the boundary of the feasible region (case (b)).

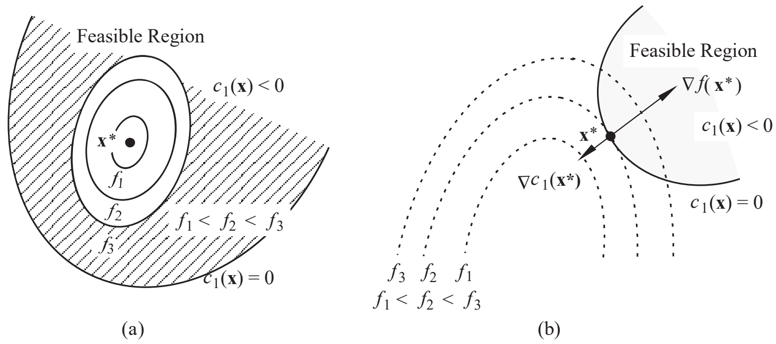

Figure 1.3

1.3.4 Karush-Kuhn-Tucker (KKT) Conditions

The KKT conditions are first-order necessary conditions that a local minimizer of problem (1.1) must satisfy.

Theorem 1.2 ***Karush-Kuhn-Tucker conditions*** *If $\boldsymbol{x}^*$ is a local minimizer of problem* (1.1) *and is regular for the constraints that are active at $\boldsymbol{x}^*$, then*

(*a*) $$a_i(\boldsymbol{x}^*) = 0 \;\text{ for }\; i = 1, 2, \ldots, p$$

(*b*) $$c_j(\boldsymbol{x}^*) \le 0 \;\text{ for }\; i = 1, 2, \ldots, q$$

(*c*) *There exist Lagrange multipliers* λ_i^* for $1 \le i \le p$ and μ_i^* for $1 \le i \le q$ *such that*

$$\nabla f(\boldsymbol{x}^*) + \sum_{i=1}^{p} \lambda_i^* \nabla a_i(\boldsymbol{x}^*) + \sum_{j=1}^{q} \mu_j^* \nabla c_j(\boldsymbol{x}^*) = \mathbf{0} \tag{1.27}$$

(*d*) *Complementarity conditions*

$$\lambda_i^* a_i(\boldsymbol{x}^*) = 0 \;\text{ for }\; 1 \le i \le p \tag{1.28a}$$

$$\mu_j^* c_j(\boldsymbol{x}^*) = 0 \;\text{ for }\; 1 \le j \le q \tag{1.28b}$$

(*e*) $$\mu_j^* \ge 0 \;\text{ for }\; 1 \le j \le q \tag{1.29}$$

Definition 1.3 The Lagrangian for the general constrained problem (1.1) is defined by

$$L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\boldsymbol{x}) + \sum_{i=1}^{p} \lambda_i a_i(\boldsymbol{x}) + \sum_{j=1}^{q} \mu_j c_j(\boldsymbol{x}) \tag{1.30}$$

Note that the condition in (1.27) can be expressed in terms of Lagrangian as

$$\nabla_x L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \mathbf{0}$$

which is a set of n equations. These in combination with the p equations in condition (*a*) and q

equations in (1.28b) form a system of ($n + p + q$) equations for the same number of "unknowns" in $\boldsymbol{x}, \boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. However, these equations in many cases are nonlinear and solving nonlinear system of equations is often no easier than solving the optimization problem itself. As a matter of fact, nonlinear equations are often solved by optimization techniques such as nonlinear least squares these days.

Example 1.4 Solve the constrained minimization problem

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) = x_1^2 + x_2^2 - 14x_1 - 6x_2 \\ \text{subject to:} \quad & c_1(\boldsymbol{x}) = -2 + x_1 + x_2 \le 0 \\ & c_2(\boldsymbol{x}) = -3 + x_1 + 2x_2 \le 0 \end{aligned}$$

by applying the KKT conditions.

Solution The KKT conditions imply that

$$\begin{aligned} & 2x_1 - 14 + \mu_1 + \mu_2 = 0 \\ & 2x_2 - 6 + \mu_1 + 2\mu_2 = 0 \\ & \mu_1(-2 + x_1 + x_2) = 0 \\ & \mu_2(-3 + x_1 + 2x_2) = 0 \\ & \mu_1 \ge 0 \\ & \mu_2 \ge 0 \end{aligned}$$

One way to find the solution in this simple case is to consider all possible cases with regard to active constraints and verify the nonnegativity of the $\mu_j's$ obtained.

Case 1: *No active constraints*. If there are no active constraints, we have $\mu_1^* = \mu_2^* = 0$, which leads to

$$\boldsymbol{x}^* = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

Obviously, this $\boldsymbol{x}^*$ violates both constraints and it is not a solution.

Case 2: *One constraint is active.* If only the first constraint is active, then we have $\mu_2^* = 0$, and

$$\begin{aligned} & 2x_1 - 14 + \mu_1 = 0 \\ & 2x_2 - 6 + \mu_1 = 0 \\ & -2 + x_1 + x_2 = 0 \end{aligned}$$

Solving this system of equations, we obtain

$$\boldsymbol{x}^* = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \text{and} \quad \mu_1^* = 8$$

Since $\boldsymbol{x}^*$ also satisfies the second constraint, $\boldsymbol{x}^* = [3 \quad -1]^T$ and $\boldsymbol{\mu}^* = [8 \quad 0]^T$ satisfy the KKT conditions.

If only the second constraint is active, then $\mu_1^* = 0$ and the KKT conditions become

$$
\begin{aligned}
2x_1 - 14 + \mu_2 &= 0 \\
2x_2 - 6 + 2\mu_2 &= 0 \\
-3 + x_1 + 2x_2 &= 0
\end{aligned}
$$

The solution of this system of equations is given by

$$\boldsymbol{x}^* = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \text{ and } \mu_2^* = 4$$

As $\boldsymbol{x}^*$ violates the first constraint, the above $\boldsymbol{x}^*$ and $\boldsymbol{\mu}^*$ do not satisfy the KKT conditions.

Case 3: *Both constraints are active*. If both constraints are active, we have

$$
\begin{aligned}
2x_1 - 14 + \mu_1 + \mu_2 &= 0 \\
2x_2 - 6 + \mu_1 + 2\mu_2 &= 0 \\
-2 + x_1 + x_2 &= 0 \\
-3 + x_1 + 2x_2 &= 0
\end{aligned}
$$

The solution to this system of equations is given by

$$\boldsymbol{x}^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \boldsymbol{\mu}^* = \begin{bmatrix} 20 \\ -8 \end{bmatrix}$$

Since $\mu_2^* < 0$, this is not a solution of the optimization problem.

Therefore, the only candidate for a minimizer of the problem is given by

$$\boldsymbol{x}^* = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \text{ and } \boldsymbol{\mu}^* = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

As can be observed in Fig. 1.4, the above point is actually the global minimizer.

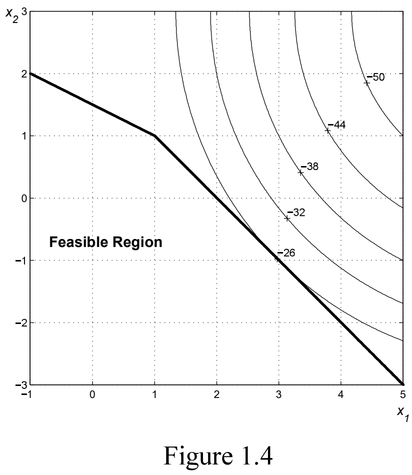

Figure 1.4

Chapter 2 Convexity and Duality

A distinguishing idea which dominates many issues in optimization theory is convexity. Before we proceed, we quote the following from R. T. Rockafellar "Lagrange multipliers and optimality", *SIAM Review*, vol. 35, no. 2, pp. 183-238, June 1993: "Convexity is a large subject which can hardly be addressed here, but much of the impetus for its growth in recent decades has come from applications in optimization. An important reason is the fact that when a convex function is minimized over a convex set every locally optimal solution is global. Also, first-order necessary conditions for optimality turn out to be sufficient. A variety of other properties conducive to computation and interpretation of solutions ride on convexity as well. In fact the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity. Even for problems that aren't themselves of convex type, convexity may enter, for instance, in setting up subproblems as part of an iterative numerical scheme."

2.0 Notation

$\mathbf{R}$: the set of all real numbers.

$\mathbf{R}_+$: the set of all nonnegative real numbers.

$\mathbf{R}_{++}$: the set of all positive real numbers.

$\mathbf{R}^p$: the set of p-dimensional real-valued (column) vectors.

$\mathbf{R}^{p\times q}$: the set of real-valued matrices of size p by q.

$\mathbf{S}^n$: the set of n by n symmetric matrices.

$\mathbf{S}_+^n$: the set of n by n symmetric positive semidefinite matrices.

$\mathbf{S}_{++}^n$: the set of n by n symmetric positive definite matrices.

$\mathrm{dom}\, f$: domain where function $f(\boldsymbol{x})$ is defined.

2.1 Convex Sets and Convex Functions

Definition 2.1 A set $\mathcal{R}_c \subset E^n$ is said to be *convex* if for every pair of points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ in $\mathcal{R}_c$ and for every real number α in the range $0 < \alpha < 1$, the point $\boldsymbol{x} = \alpha\boldsymbol{x}_1 + (1-\alpha)\boldsymbol{x}_2$ is located in $\mathcal{R}_c$.

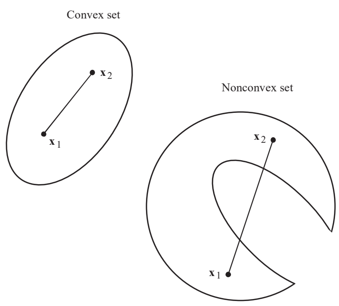

Figure 2.1 Convex and nonconvex sets.

Definition 2.2 A function $f(\boldsymbol{x})$ defined over a convex set $\mathcal{R}_C$ is said to be convex if for every pair of points in $\mathcal{R}_C$ and every real number α in the range $0 < \alpha < 1$, the following inequality holds

$$f[\alpha \boldsymbol{x}_1 + (1-\alpha)\boldsymbol{x}_2] \le \alpha f(\boldsymbol{x}_1) + (1-\alpha) f(\boldsymbol{x}_2) \tag{2.1}$$

Geometrically, (2.1) says that $f(\boldsymbol{x})$ is convex if and only if the function's graph always lies underneath (or coincides with) the corresponding line segment, see Fig. 2.2.

- $f(\boldsymbol{x})$ is said to be *strictly* convex if (2.1) holds strictly, i.e.,

$$f[\alpha \boldsymbol{x}_1 + (1-\alpha)\boldsymbol{x}_2] < \alpha f(\boldsymbol{x}_1) + (1-\alpha) f(\boldsymbol{x}_2)$$

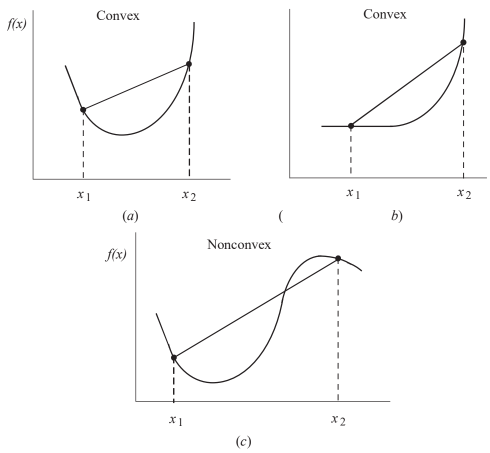

Figure 2.2 Convex and nonconvex functions

Definition 2.3 A function $f(\boldsymbol{x})$ is said to be concave if for every pair of points and every real number α in the range $0 < \alpha < 1$, the following inequality holds

$$f[\alpha \boldsymbol{x}_1 + (1-\alpha)\boldsymbol{x}_2] \ge \alpha f(\boldsymbol{x}_1) + (1-\alpha) f(\boldsymbol{x}_2) \tag{2.2}$$

- $f(\boldsymbol{x})$ is concave if and only if the function's graph always lies above (or coincides with) the corresponding line segment.
- $f(\boldsymbol{x})$ is said to be *strictly* concave if the inequality in (2.2) holds strictly.
- $f(\boldsymbol{x})$ is (strictly) convex if and only if $-f(\boldsymbol{x})$ is (strictly) concave, and vice versa.

Definition 2.4 The extended-value extension of a convex function $f(\boldsymbol{x})$ is defined as

$$\tilde{f}(\boldsymbol{x}) = \begin{cases} f(\boldsymbol{x}) & \boldsymbol{x} \in \operatorname{dom} f \\ +\infty & \boldsymbol{x} \notin \operatorname{dom} f \end{cases} \tag{2.3}$$

Examples of convex functions of one variable:

- affine function $f(x) = ax + b$
- exponential function $f(x) = e^{\alpha x}$ for any $\alpha \in \mathrm{R}$
- power function $f(x) = x^{\alpha}$ on R_{++} for $\alpha \geq 1$ or $\alpha \leq 0$
- negative entropy $f(x) = x \log x$ on R_{++}

Examples of concave functions of one variable:

- affine function $f(x) = ax + b$
- logrithmic function $f(x) = \log x$ on R_{++}
- power function $f(x) = x^{\alpha}$ on R_{++} for $0 \leq \alpha \leq 1$

Examples of multivariable convex functions:

- affine function $f(\boldsymbol{x}) = \boldsymbol{a}^T \boldsymbol{x} + b$
- norm $f(\boldsymbol{x}) = \| \boldsymbol{x} \|_p = \left(| x_1 |^p + | x_2 |^p + \cdots + | x_n |^p \right)^{1/p}$ for $p \geq 1$, and $\| \boldsymbol{x} \|_{\infty} = \max_{1 \leq i \leq n} \{ | x_i | \}$

2.2 Properties of Convex Functions

(1) ***Convexity of linear combination of convex functions*** If $f_1(\boldsymbol{x})$ and $f_2(\boldsymbol{x})$ are convex and a and b are nonnegative scalars, then $a f_1(\boldsymbol{x}) + b f_2(\boldsymbol{x})$ is convex.

(2) ***Relation between a convex function and convex sets*** If $f(\boldsymbol{x})$ is convex on a convex set $\mathcal{R}_c$, then the set $\mathcal{S}_K = \{ \boldsymbol{x} : \boldsymbol{x} \in \mathcal{R}_c, f(\boldsymbol{x}) \leq K \}$ is convex for every real number K. See Fig. 2.3 for an illustration of this property.

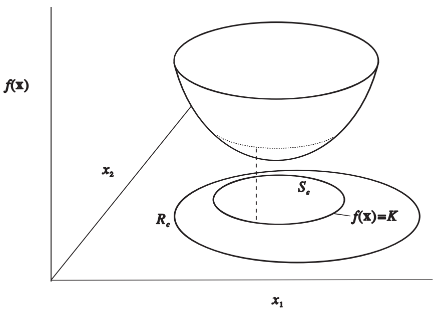

Figure 2.3

Convex functions can be characterized in different ways. The next definition of convexity turns out to be very useful.

(3) ***Property of convex function relating to gradient***

Suppose $f(\boldsymbol{x})$ is a smooth (C^1) function. Then $f(\boldsymbol{x})$ is convex over $\mathcal{R}_c$ if and only if

$$f(\boldsymbol{x}_1) \geq f(\boldsymbol{x}) + \nabla^T f(\boldsymbol{x})(\boldsymbol{x}_1 - \boldsymbol{x}) \tag{2.4}$$

for all $\boldsymbol{x}$ and $\boldsymbol{x}_1$ in $\mathcal{R}_c$. See Fig. 2.4 for an illustration of this property.

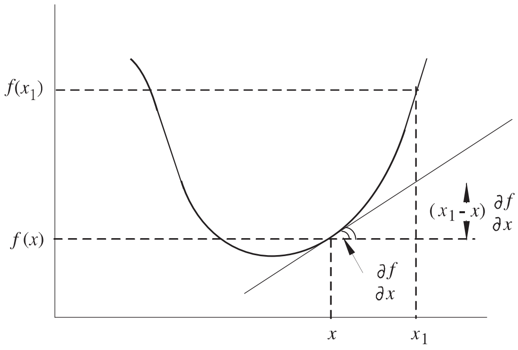

Fig. 2.4

(4) ***Property of convex functions relating to the Hessian*** A function $f(\boldsymbol{x}) \in C^2$ is convex over a convex $\mathcal{R}_c$ if and only if the Hessian $\nabla^2 f(\boldsymbol{x})$ is positive semidefinite over $\mathcal{R}_c$.

To prove property 4, we use Taylor expansion to write

$$f(\boldsymbol{x}_1) = f(\boldsymbol{x}) + \nabla^T f(\boldsymbol{x})(\boldsymbol{x}_1 - \boldsymbol{x}) + \tfrac{1}{2}(\boldsymbol{x}_1 - \boldsymbol{x})^T \nabla^2 f(\boldsymbol{x})(\boldsymbol{x}_1 - \boldsymbol{x}) + o(\|\boldsymbol{x}_1 - \boldsymbol{x}\|^2)$$

for all $\boldsymbol{x}$ and $\boldsymbol{x}_1$ in $\mathcal{R}_c$. Hence

$$f(\boldsymbol{x}_1) - f(\boldsymbol{x}) - \nabla^T f(\boldsymbol{x})(\boldsymbol{x}_1 - \boldsymbol{x}) = \tfrac{1}{2}(\boldsymbol{x}_1 - \boldsymbol{x})^T \nabla^2 f(\boldsymbol{x})(\boldsymbol{x}_1 - \boldsymbol{x}) + o(\|\boldsymbol{x}_1 - \boldsymbol{x}\|^2)$$

which implies that the left-hand side of the above equation is nonnegative if and only if $\nabla^2 f(\boldsymbol{x})$ is positive semidefinite. But from property 3 the left-hand side of the above equation is nonnegative if and only if $f(\boldsymbol{x})$ is convex. ■

(5) ***From an optimization perspective, what* (2.4) *tells us?*** (2.4) is a characterization of a function being convex, but it also provides a global linear lower bound of a convex that is tightest at a given point in the function’s (convex) domain. The observation made below offers yet another way to appreciate the importance of convexity.

Let $\boldsymbol{x}$ be a point in the domain (let us assume it is the entire space R^n) of a smooth convex function $f(\boldsymbol{x})$ with gradient $\nabla f(\boldsymbol{x}) \neq \boldsymbol{0}$ (this is reasonable to assume. In fact, if $\nabla f(\boldsymbol{x}) = \boldsymbol{0}$, then $\boldsymbol{x}$ would be a global minimizer of $f(\boldsymbol{x})$, hence not much left for us to do, see the next property). The hyperplane that contains point $\boldsymbol{x}$ and with gradient $\nabla f(\boldsymbol{x})$ as its normal can be described as a set of $\boldsymbol{x}_1$, each satisfies

$$\mathcal{P}_x: \qquad \nabla^T f(\boldsymbol{x})(\boldsymbol{x}_1 - \boldsymbol{x}) = 0$$

and hyperplane $\mathcal{P}_x$ cuts (divides) the entire domain into two parts, see Fig. 2.5. Let us examine the part of the domain that contains gradient $\nabla f(\boldsymbol{x})$ (point $\boldsymbol{x} + \nabla f(\boldsymbol{x})$ to be precise). From the figure it is immediate that at each point $\boldsymbol{x}_1$ in this part (excluding hyperplane $\mathcal{P}_x$ itself) satisfies

$$\nabla^T f(\boldsymbol{x})(\boldsymbol{x}_1 - \boldsymbol{x}) > 0$$

which in conjunction with (2.4) gives

$$f(\boldsymbol{x}_1) \geq f(\boldsymbol{x}) + \nabla^T f(\boldsymbol{x})(\boldsymbol{x}_1 - \boldsymbol{x}) > f(\boldsymbol{x})$$

Therefore, this part of the domain contains *no* minimizers. In other words, the minimizers are all contained in the other part of the domain. We see that (2.4) is of help in reducing a region in searching a minimizer.

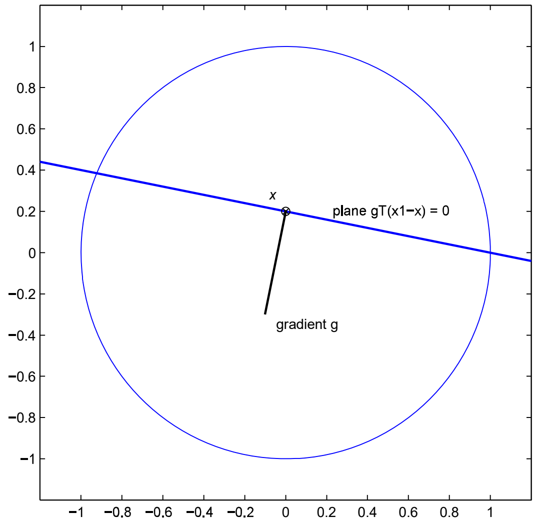

Figure 2.5

(6) *Global solution of an unconstrained convex minimization problem*

Let $f(\boldsymbol{x})$ be a smooth convex function. If $\nabla f(\boldsymbol{x}^*) = \boldsymbol{0}$, then $\boldsymbol{x}^*$ is a global minimizer of $f(\boldsymbol{x})$. This property is an immediate consequence of (2.4):

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*) + \nabla^T f(\boldsymbol{x}^*)(\boldsymbol{x} - \boldsymbol{x}^*) = f(\boldsymbol{x}^*)$$

(7) *Globalness and convexity of minimizers in convex programming (CP) problems.* Recall a constrained problem is a CP problem if it assumes the form

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to:} & \boldsymbol{a}_i^T \boldsymbol{x} = b_i \quad \text{for } 1 \leq i \leq p \\ & c_j(\boldsymbol{x}) \leq 0 \quad \text{for } 1 \leq j \leq q \end{array}$$

where $f(\boldsymbol{x})$ and $c_j(\boldsymbol{x})$ are convex. That is, a CP problem minimizes a *convex objective* *function* over *a convex feasible region*. We can state that

(a) $\boldsymbol{x}^*$ is a minimizer if and only if

$$\nabla f(\boldsymbol{x}^*)^T(\boldsymbol{x}-\boldsymbol{x}^*) \ge 0 \quad \text{for any feasible } \boldsymbol{x} \tag{2.5}$$

Proof The necessity of (2.5) immediately follows from Theorem 1.1, and the sufficiency of (2.5) follows from (2.4), namely,

$$f(\boldsymbol{x}) \ge f(\boldsymbol{x}^*) + \nabla^T f(\boldsymbol{x}^*)(\boldsymbol{x}-\boldsymbol{x}^*)$$

which in conjuction with (2.5) gives

$$f(\boldsymbol{x}) - f(\boldsymbol{x}^*) \ge \nabla^T f(\boldsymbol{x}^*)(\boldsymbol{x}-\boldsymbol{x}^*) \ge 0$$

hence $f(\boldsymbol{x}) \ge f(\boldsymbol{x}^*)$. ■

(b) If $\boldsymbol{x}^*$ is a local minimizer, then $\boldsymbol{x}^*$ is also a global minimizer.

(c) The set of minimizers of a CP problem is a convex set.

(d) If the objective function is strictly convex, then the minimizer is unique.

(8) ***Sufficiency of KKT Conditions for Convex Problems*** If $\boldsymbol{x}^*$ is a regular point of the constraints in (1.44) and satisfies the KKT conditions, then it is a global minimizer.

Proof For a feasible point $\bar{\boldsymbol{x}}$ with $\bar{\boldsymbol{x}} \ne \boldsymbol{x}^*$, we have $a_i(\bar{\boldsymbol{x}}) = 0$ for $1 \le i \le p$ and $c_j(\bar{\boldsymbol{x}}) \le 0$ for $1 \le j \le q$. Thus we can write

$$f(\bar{\boldsymbol{x}}) \ge f(\bar{\boldsymbol{x}}) + \sum_{j=1}^{q} \mu_j^* c_j(\bar{\boldsymbol{x}}).$$

Since $f(\boldsymbol{x})$ and $c_j(\boldsymbol{x})$ are convex, we have

$$f(\bar{\boldsymbol{x}}) \ge f(\boldsymbol{x}^*) + \nabla^T f(\boldsymbol{x}^*)(\bar{\boldsymbol{x}}-\boldsymbol{x}^*) \quad \text{and} \quad c_j(\bar{\boldsymbol{x}}) \ge c_j(\boldsymbol{x}^*) + \nabla^T c_j(\boldsymbol{x}^*)(\bar{\boldsymbol{x}}-\boldsymbol{x}^*)$$

Hence

$$f(\bar{\boldsymbol{x}}) \ge f(\boldsymbol{x}^*) + \nabla^T f(\boldsymbol{x}^*)(\bar{\boldsymbol{x}}-\boldsymbol{x}^*) + \sum_{j=1}^{q} \mu_j^* \nabla^T c_j(\boldsymbol{x}^*)(\bar{\boldsymbol{x}}-\boldsymbol{x}^*) + \sum_{j=1}^{q} \mu_j^* c_j(\boldsymbol{x}^*)$$

In the light of the complementarity conditions, the last term in the above inequality vanishes, hence we have

$$f(\bar{\boldsymbol{x}}) \ge f(\boldsymbol{x}^*) + \left[\nabla f(\boldsymbol{x}^*) + \sum_{j=1}^{q} \mu_j^* \nabla c_j(\boldsymbol{x}^*) \right]^T (\bar{\boldsymbol{x}}-\boldsymbol{x}^*) \tag{2.6}$$

Since $a_i(\bar{\boldsymbol{x}}) = a_i(\boldsymbol{x}^*) = 0$, we can write

$$0 = a_i(\bar{\boldsymbol{x}}) - a_i(\boldsymbol{x}^*) = \boldsymbol{a}_i^T(\bar{\boldsymbol{x}}-\boldsymbol{x}^*) = \nabla^T a_i(\boldsymbol{x}^*)(\bar{\boldsymbol{x}}-\boldsymbol{x}^*)$$

Multiplying the above by λ_i^* and adding it to (2.6) for $1 \le i \le p$ gives

$$f(\bar{\boldsymbol{x}}) \ge f(\boldsymbol{x}^*) + \left[\nabla f(\boldsymbol{x}^*) + \sum_{i=1}^{p} \lambda_i^* \nabla a_i(\boldsymbol{x}^*) + \sum_{j=1}^{q} \mu_j^* \nabla c_j(\boldsymbol{x}^*) \right]^T (\bar{\boldsymbol{x}} - \boldsymbol{x}^*)$$

By the KKT conditions, the last term in the above inequality is zero, which leads to $f(\bar{\boldsymbol{x}}) \ge f(\boldsymbol{x}^*)$. This shows that $\boldsymbol{x}^*$ is a global minimizer. ■

2.3 How to Verify Convexity of a Function?

(1) By definitions. See Eqs. (2.1) and (2.4).

(2) Check if its Hessian is positive semidefinite, see Perperty (4) in Sec. 2.2.

(3) Check the function's convexity along lines: $f(\boldsymbol{x})$ is convex if and only if $g(t) = f(\boldsymbol{x} + t\boldsymbol{v})$ is convex in t in $\text{dom } g = \{t : \boldsymbol{x} + t\boldsymbol{v} \in \text{dom } f)$ for any $\boldsymbol{x} \in \text{dom } f$.

- Example: $f(\boldsymbol{X}) = -\log\det(\boldsymbol{X})$ on $\text{dom} f = \mathrm{S}_{++}^n$ is convex.

(4) Composition with affine function: if $f(\boldsymbol{x})$ is convex, then so is $f(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b})$.

- Examples:
  (i) log barrier for linear inequalities, i.e.,

$f(\boldsymbol{x}) = -\sum_{i=1}^{m} \log(b_i - \boldsymbol{a}_i^T \boldsymbol{x})$, $\text{dom} f = \{\boldsymbol{x} : \boldsymbol{a}_i^T \boldsymbol{x} < b_i, i = 1, 2, ..., m\}$ is convex;

  (ii) any norm of affine function $f(\boldsymbol{x}) = \|\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\|$ is convex.

(5) If all $f_i(\boldsymbol{x})$ are convex, then $f(\boldsymbol{x}) = \max\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \dots, f_m(\boldsymbol{x})\}$ is convex.

- Example: piecewise-linear function $f(\boldsymbol{x}) = \max_{1 \le i \le m}\{\boldsymbol{a}_i^T \boldsymbol{x} + b_i\}$ is convex.

(6) If $g(\boldsymbol{x})$ is convex, $h(\boldsymbol{x})$ is convex and $\tilde{h}(\boldsymbol{x})$ is nondecreasing, then $f(\boldsymbol{x}) = h(g(\boldsymbol{x}))$ is convex. (here $\tilde{h}(\boldsymbol{x})$ denotes the extended-value extension of $h(\boldsymbol{x})$, see Definition 2.4)

- Example: if $f(\boldsymbol{x})$ is convex, then so is $e^{f(x)}$.

(7) If $g(\boldsymbol{x})$ is concave, $h(\boldsymbol{x})$ is convex and $\tilde{h}(\boldsymbol{x})$ is nonincreasing, then $f(\boldsymbol{x}) = h(g(\boldsymbol{x}))$ is convex.

- Examples:
  (i) if $f(\boldsymbol{x})$ is concave and positive, then $1/f(\boldsymbol{x})$ is convex.

(ii) if $f(\boldsymbol{x})$ is concave and positive, then $-\log f(\boldsymbol{x})$ is convex.

(8) A vector version of property (6): If all $g_i(\boldsymbol{x})$ are convex and $h(\boldsymbol{x}) = h(x_1, x_2, ..., x_m)$ is convex and $\tilde{h}(\boldsymbol{x})$ is nondecreasing in each variable, then $f(\boldsymbol{x}) = h(g_1(\boldsymbol{x}), g_2(\boldsymbol{x}), ..., g_m(\boldsymbol{x}))$ is convex.

- Example: if all $f_i(\boldsymbol{x})$ are convex, then $\log \sum_{i=1}^{m} e^{f_i(\boldsymbol{x})}$ is convex (Prob. 2.5).

(9) A vector version of property (7): If all $g_i(\boldsymbol{x})$ are concave and $h(\boldsymbol{x}) = h(x_1, x_2, ..., x_m)$ is convex and $\tilde{h}(\boldsymbol{x})$ is nonincreasing in each variable, then $f(\boldsymbol{x}) = h(g_1(\boldsymbol{x}), g_2(\boldsymbol{x}), ..., g_m(\boldsymbol{x}))$ is convex.

- Example: if all $f_i(\boldsymbol{x})$ are concave and positive, then $-\sum_{i=1}^{m} \log f_i(\boldsymbol{x})$ is convex.

2.4 Duality

The concept of duality as applied to optimization is essentially a problem transformation that leads to an indirect but sometimes more efficient solution method. In a duality-based method, the original problem, which is referred to as the *primal* problem, is transformed into a problem in which the parameters are the Lagrange multipliers of the primal. The transformed problem is called the *dual* problem. In the case where the number of inequality constraints is much greater than the dimension of $\boldsymbol{x}$, solving the dual problem to find the Lagrange multipliers and then finding $\boldsymbol{x}^*$ for the primal problem becomes an attractive alternative.

2.4.1 The Lagrange Dual of a Convex Programming Problem

In this section we introduce another concept known as the Lagrange dual which turns out to be more useful in the study of convex programming. To this end we need a related concept called Lagrange dual function. Consider the general convex programming (CP) problem

$$\begin{aligned} &\text{minimize} \quad f(\boldsymbol{x}) \\ &\text{subject to:} \ \boldsymbol{a}_i^T \boldsymbol{x} = b_i \quad \text{for } 1 \le i \le p \\ &\qquad\qquad c_j(\boldsymbol{x}) \le 0 \quad \text{for } 1 \le j \le q \end{aligned} \tag{2.7}$$

and recall its Lagrangian

$$L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\boldsymbol{x}) + \sum_{i=1}^{p} \lambda_i \left(\boldsymbol{a}_i^T \boldsymbol{x} - b_i \right) + \sum_{j=1}^{q} \mu_i c_j(\boldsymbol{x})$$

Definition 2.5 The *Lagrange dual function* of problem is defined as

$$q(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \inf_{\boldsymbol{x}} L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \tag{2.8}$$

for $\boldsymbol{\lambda} \in R^p$ and $\boldsymbol{\mu} \in R^q$ with $\boldsymbol{\mu} \ge \boldsymbol{0}$. Note that the Lagrangian is *convex* with respect to $\boldsymbol{x}$. If $L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ is unbounded below for some $\{\boldsymbol{\lambda}, \ \boldsymbol{\mu}\}$, then the value of $q(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is assigned to $-\infty$.

Property 1 $q(\boldsymbol{\lambda},\boldsymbol{\mu})$ is a concave function with respect to $\{\boldsymbol{\lambda},\ \boldsymbol{\mu}\}$. The property follows from that fact that for $\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2 \in R^p$ and $\boldsymbol{\mu}_1,\boldsymbol{\mu}_2 \in R^q$ with $\boldsymbol{\mu}_1,\boldsymbol{\mu}_2 \geq \mathbf{0}$ and for $t \in (0,1)$, we have

$$q(t\boldsymbol{\lambda}_1+(1-t)\boldsymbol{\lambda}_2, t\boldsymbol{\mu}_1+(1-t)\boldsymbol{\mu}_2) = \inf_{\boldsymbol{x}} L(\boldsymbol{x}, t\boldsymbol{\lambda}_1+(1-t)\boldsymbol{\lambda}_1, t\boldsymbol{\mu}_1+(1-t)\boldsymbol{\mu}_2)$$

$$= \inf_{\boldsymbol{x}}\left[(t+1-t)f(\boldsymbol{x}) + \sum_{i=1}^{p}(t\lambda_{1,i}+(1-t)\lambda_{2,i})(a_i^T\boldsymbol{x}-b_i) + \sum_{j=1}^{q}(t\mu_{1,j}+(1-t)\mu_{2,j})c_j(\boldsymbol{x})\right]$$

$$\geq t\cdot\inf_{\boldsymbol{x}}\left[f(\boldsymbol{x})+\sum_{i=1}^{p}\lambda_{1,i}(a_i^T\boldsymbol{x}-b_i)+\sum_{j=1}^{q}\mu_{1,j}c_j(\boldsymbol{x})\right]+(1-t)\cdot\inf_{\boldsymbol{x}}\left[f(\boldsymbol{x})+\sum_{i=1}^{p}\lambda_{2,i}(a_i^T\boldsymbol{x}-b_i)+\sum_{j=1}^{q}\mu_{2,j}c_j(\boldsymbol{x})\right]$$

$$= t\cdot q(\boldsymbol{\lambda}_1,\boldsymbol{\mu}_1)+(1-t)\cdot q(\boldsymbol{\lambda}_2,\boldsymbol{\mu}_2)$$

Definition 2.6 The *Lagrange dual problem* with respect to problem (2.7) is defined as

$$\begin{aligned} &\underset{\boldsymbol{\lambda},\boldsymbol{\mu}}{\text{maximize}} \quad q(\boldsymbol{\lambda},\boldsymbol{\mu}) \\ &\text{subject to:}\ \ \boldsymbol{\mu} \geq \mathbf{0} \end{aligned} \tag{2.9}$$

Property 2 For any $\boldsymbol{x}$ feasible for problem (2.7) and $\{\boldsymbol{\lambda},\ \boldsymbol{\mu}\}$ feasible for problem (2.9), we have

$$f(\boldsymbol{x}) \geq q(\boldsymbol{\lambda},\boldsymbol{\mu}) \tag{2.10}$$

This is because

$$L(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\mu}) = f(\boldsymbol{x})+\sum_{i=1}^{p}\lambda_i\left(\boldsymbol{a}_i^T\boldsymbol{x}-b_i\right)+\sum_{j=1}^{q}\mu_i c_j(\boldsymbol{x}) = f(\boldsymbol{x})+\sum_{j=1}^{q}\mu_i c_j(\boldsymbol{x}) \leq f(\boldsymbol{x})$$

thus

$$q(\boldsymbol{\lambda},\boldsymbol{\mu}) = \inf_{\boldsymbol{x}} L(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\mu}) \leq L(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\mu}) \leq f(\boldsymbol{x})$$

We call the convex minimization problem in (2.7) the *primal problem* and the concave maximization problem in (2.9) the *dual problem*. From (2.10), it is natural to introduce a *duality gap* between the primal and dual objectives as

$$\delta(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\mu}) = f(\boldsymbol{x}) - q(\boldsymbol{\lambda},\boldsymbol{\mu}) \tag{2.11}$$

It follows that for feasible $\{\boldsymbol{x},\boldsymbol{\lambda},\ \boldsymbol{\mu}\}$ the duality gap is always nonnegative.

Property 3 Let $\boldsymbol{x}^*$ be a solution of the primal problem in (2.7). Then the dual function at any feasible $\{\boldsymbol{\lambda},\ \boldsymbol{\mu}\}$ serves as a lower bound of the optimal value of the primal objective, $f(\boldsymbol{x}^*)$, namely,

$$f(\boldsymbol{x}^*) \geq q(\boldsymbol{\lambda},\boldsymbol{\mu}) \tag{2.12}$$

This property follows immediately from (2.10) by taking the minimum of $f(\boldsymbol{x})$ on its left-hand side.

- Now a question naturally arises: what will be the tightest lower bound the dual function $q(\boldsymbol{\lambda}, \boldsymbol{\mu})$ can offer? From (2.12), obviously the answer is found by maximizing the dual function $q(\boldsymbol{\lambda}, \boldsymbol{\mu})$ on the right-hand side of (2.12) subject to $\boldsymbol{\mu} \geq \mathbf{0}$. This is exactly the Lagrange dual problem as formulated in (2.9). Therefore, if we denote the solution of (2.9) by $(\boldsymbol{\lambda}^*,\ \boldsymbol{\mu}^*)$, then we have

$$f(\boldsymbol{x}^*) \geq q(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) \tag{2.13}$$

Based on (2.13), we now introduce the concept of *strong* and *weak* duality as follows.

Definition 2.7 Let $\boldsymbol{x}^*$ and $(\boldsymbol{\lambda}^*,\ \boldsymbol{\mu}^*)$ be solutions of primal problem (2.7) and dual problem (2.9), respectively. We say strong duality holds if $f(\boldsymbol{x}^*) = q(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$, i.e., the optimal duality gap is zero; and a weak duality holds if $f(\boldsymbol{x}^*) > q(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$.

- It can be shown that if the primal problem is strictly feasible, i.e., there exists $\boldsymbol{x}$ satisfying

$$\begin{aligned} &\boldsymbol{a}_i^T \boldsymbol{x} = b_i \quad \text{for } 1 \leq i \leq p \\ &c_j(\boldsymbol{x}) < 0 \quad \text{for } 1 \leq j \leq q \end{aligned}$$

(this is to say that the interior of the feasible region of problem (2.7) is nonempty), then strong duality holds, i.e., the optimal duality gap is zero.

Example 2.1 Find the Lagrange dual of the LP problem

$$\begin{aligned} &\text{minimize} \quad \boldsymbol{c}^T \boldsymbol{x} \\ &\text{subject to:} \ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \mathbf{0} \end{aligned} \tag{1.3}$$

Solution We write $\boldsymbol{x} \geq \mathbf{0}$ as $-\boldsymbol{x} \leq \mathbf{0}$ hence the Lagrangian of the LP problem is given by

$$L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \boldsymbol{c}^T \boldsymbol{x} + (\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})^T \boldsymbol{\lambda} - \boldsymbol{x}^T \boldsymbol{\mu}$$

thus

$$q(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \inf_{\boldsymbol{x}} \left\{ \boldsymbol{c}^T \boldsymbol{x} + (\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})^T \boldsymbol{\lambda} - \boldsymbol{x}^T \boldsymbol{\mu} \right\} = \inf_{\boldsymbol{x}} \left\{ (\boldsymbol{c} + \boldsymbol{A}^T \boldsymbol{\lambda} - \boldsymbol{\mu})^T \boldsymbol{x} - \boldsymbol{b}^T \boldsymbol{\lambda} \right\} \tag{2.14}$$

For given $\{\boldsymbol{\lambda}, \boldsymbol{\mu}\}$ such that $\boldsymbol{c} + \boldsymbol{A}^T \boldsymbol{\lambda} - \boldsymbol{\mu} \neq \mathbf{0}$, from (2.14) we have $q(\boldsymbol{\lambda}, \boldsymbol{\mu}) = -\infty$. Therefore, to deal with a well-defined dual function $q(\boldsymbol{\lambda}, \boldsymbol{\mu})$ we assume $\boldsymbol{c} + \boldsymbol{A}^T \boldsymbol{\lambda} - \boldsymbol{\mu} = \mathbf{0}$ which leads to

$$q(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \inf_{\boldsymbol{x}} (-\boldsymbol{b}^T \boldsymbol{\lambda}) = -\boldsymbol{b}^T \boldsymbol{\lambda}$$

and the Lagrange dual of (1.3) is given by

$$\begin{aligned}&\underset{\boldsymbol{\lambda},\boldsymbol{\mu}}{\text{maximize}} \quad -\boldsymbol{b}^T\boldsymbol{\lambda}\\&\text{subject to:} \quad \boldsymbol{\mu}\ge\mathbf{0}\end{aligned}$$

Since $\boldsymbol{c}+\boldsymbol{A}^T\boldsymbol{\lambda}-\boldsymbol{\mu}=\mathbf{0}$, $\boldsymbol{\mu}=\boldsymbol{c}+\boldsymbol{A}^T\boldsymbol{\lambda}$ so the above problem becomes

$$\begin{aligned}&\underset{\boldsymbol{\lambda}}{\text{maximize}} \quad -\boldsymbol{b}^T\boldsymbol{\lambda}\\&\text{subject to:} \quad -\boldsymbol{c}-\boldsymbol{A}^T\boldsymbol{\lambda}\le\mathbf{0}\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\underset{\boldsymbol{\lambda}}{\text{minimize}} \quad \boldsymbol{b}^T\boldsymbol{\lambda}\\&\text{subject to:} \quad (-\boldsymbol{A}^T)\boldsymbol{\lambda}\le\boldsymbol{c}\end{aligned}$$

■

Example 2.2 Find the Lagrange dual of the QP problem

$$\begin{aligned}&\text{minimize} \quad \tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x}+\boldsymbol{p}^T\boldsymbol{x}\\&\text{subject to:} \quad \boldsymbol{A}\boldsymbol{x}\le\boldsymbol{b}\end{aligned} \tag{1.5}$$

where $\boldsymbol{H}$ is positive definite.

Solution The Lagrangian of the QP problem is given by

$$L(\boldsymbol{x},\boldsymbol{\mu})=\tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x}+\boldsymbol{p}^T\boldsymbol{x}+\boldsymbol{\mu}^T(\boldsymbol{A}\boldsymbol{x}-\boldsymbol{b})$$

Hence

$$q(\boldsymbol{\mu})=\inf_{\boldsymbol{x}}\left\{\tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x}+\boldsymbol{p}^T\boldsymbol{x}+\boldsymbol{\mu}^T(\boldsymbol{A}\boldsymbol{x}-\boldsymbol{b})\right\} \tag{2.15}$$

where the infimum is attained at $\boldsymbol{x}=-\boldsymbol{H}^{-1}(\boldsymbol{p}+\boldsymbol{A}^T\boldsymbol{\mu})$. By substituting this solution into (2.15), we obtain

$$q(\boldsymbol{\mu})=-\tfrac{1}{2}\boldsymbol{\mu}^T\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{A}^T\boldsymbol{\mu}-\boldsymbol{\mu}^T(\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{p}+\boldsymbol{b})-\tfrac{1}{2}\boldsymbol{p}^T\boldsymbol{H}^{-1}\boldsymbol{p} \tag{2.16}$$

If we let $\boldsymbol{P}=\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{A}^T$, $\boldsymbol{t}=\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{p}+\boldsymbol{b}$ and neglect the constant term in (2.16), the dual function becomes $q(\boldsymbol{\mu})=-\tfrac{1}{2}\boldsymbol{\mu}^T\boldsymbol{P}\boldsymbol{\mu}-\boldsymbol{\mu}^T\boldsymbol{t}$, hence the Lagrange dual of (1.5) is given by

$$\begin{aligned}&\text{minimize} \quad \tfrac{1}{2}\boldsymbol{\mu}^T\boldsymbol{P}\boldsymbol{\mu}+\boldsymbol{\mu}^T\boldsymbol{t}\\&\text{subject to:} \quad \boldsymbol{\mu}\ge\mathbf{0}\end{aligned} \tag{2.17}$$

Note that by definition matrix $\boldsymbol{P}$ in (2.17) is positive definite, therefore the dual problem is also a convex QP problem, but with simpler constraints in comparison with the primal problem in (1.5). In addition, if the number of constraints involved in the primal problem is smaller than n, so is the size of the dual problem. ■

Chapter 3 LP, QP, SDP, SOCP Problems and Software

In this chapter we examine several specific classes of convex constrained problems. For each class, examples are provided to illustrate its usefulness by explaining how practical problems can be formulated as that type of optimization. A popular CP solver known as **CVX** is introduced to facilitate computing numerical solution of CP problems.

3.1 Linear Programming

3.1.1 Examples

Example 3.1 *Transportation problem*

Quantities $q_1, q_2, \ldots, q_m$ of a certain product are produced by m manufacturing divisions of a company, which are at distinct locations. The product is to be shipped to n destinations that require quantities $b_1, b_2, \ldots, b_n$. Assume that the cost of shipping a unit from manufacturing division i to destination j is $c_{i,j}$ with $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Let $x_{i,j}$ to be the quantity shipped from division i to destination j so as to minimize the total cost of transportation. There are several constraints on variables $x_{i,j}$. First, each division can provide only a fixed quantity of the product, hence

$$\sum_{j=1}^{n} x_{i,j} = q_i \quad \text{for } i = 1, 2, \ldots, m$$

Second, the quantity to be shipped to a specific destination has to meet the need of that destination and so

$$\sum_{i=1}^{m} x_{i,j} = b_j \quad \text{for } j = 1, 2, \ldots, n$$

In addition, the variables $x_{i,j}$ are nonnegative and thus, we have

$$x_{i,j} \ge 0 \quad \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

The transportation problem is to find quantities $\{x_{i,j}\}$ that minimizes $C = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j} x_{i,j}$ subject to the above constraints. If we let

$$\boldsymbol{c} = \begin{bmatrix} c_{11} & \cdots & c_{1n} & c_{21} & \cdots & c_{2n} & \cdots & c_{m1} & \cdots & c_{mn} \end{bmatrix}^T$$

$$\boldsymbol{x} = \begin{bmatrix} x_{11} & \cdots & x_{1n} & x_{21} & \cdots & x_{2n} & \cdots & x_{m1} & \cdots & x_{mn} \end{bmatrix}^T$$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} q_1 & \cdots & q_m & b_1 & \cdots & b_n \end{bmatrix}^T$$

then the transportation problem can be stated as

$$\begin{aligned} &\text{minimize} \quad C = \boldsymbol{c}^T \boldsymbol{x} \\ &\text{subject to:} \ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ &\qquad\qquad\quad \boldsymbol{x} \ge \boldsymbol{0} \end{aligned}$$

Since both the objective function and constraints are linear, the problem is known as a *linear programming (LP) problem*. ■

Example 3.2 *Chebyshev centre of a polyhedron*

A polyhedron is a set defined by

$$\mathcal{P} = \{\boldsymbol{x} : \boldsymbol{a}_i^T \boldsymbol{x} \le b_i, \ i = 1, 2, \ldots, q\}$$

The Chebeshev centre of $\mathcal{P}$ is the centre $\boldsymbol{x}_c$ of the largest ball $\mathcal{B} = \{\boldsymbol{x}_c + \boldsymbol{u} : \|\boldsymbol{u}\|_2 \le r\}$ that is inscribed in $\mathcal{P}$.

To find the largest inscribed ball, the centre $\boldsymbol{x}_c$ and radius r of ball $\mathcal{B}$ are treated as *variables* such that r is to be maximized among the balls inscribed in $\mathcal{P}$. Note that

$$\boldsymbol{a}_i^T \boldsymbol{x} \le b_i \ \text{ for } \boldsymbol{x} \in \mathcal{B} \text{ for } i = 1, 2, \ldots, q$$

$$\Updownarrow$$

$$\sup_{\|\boldsymbol{u}\|_2 \le r} \{\boldsymbol{a}_i^T (\boldsymbol{x}_c + \boldsymbol{u}) \le b_i\} \ \text{ for } i = 1, 2, \ldots, q$$

$$\Updownarrow$$

$$\boldsymbol{a}_i^T \boldsymbol{x}_c + \sup_{\|\boldsymbol{u}\|_2 \le r} \{\boldsymbol{a}_i^T \boldsymbol{u}\} \le b_i \ \text{ for } i = 1, 2, \ldots, q$$

$$\Updownarrow$$

$$\boldsymbol{a}_i^T \boldsymbol{x}_c + r \|\boldsymbol{a}_i\|_2 \le b_i \ \text{ for } i = 1, 2, \ldots, q$$

Therefore, the problem at hand can be formulated as

$$\begin{aligned} &\text{minimize} \quad -r \\ &\text{subject to:} \ \boldsymbol{a}_i^T \boldsymbol{x}_c + r \|\boldsymbol{a}_i\|_2 \le b_i \ \text{ for } i = 1, 2, \ldots, q \end{aligned}$$

which is obviously an LP problem. ■

Example 3.3 *Piecewise-linear minimization*

It is well known that the piecewise linear function $f(\boldsymbol{x}) = \max_{1 \le i \le q} \{\boldsymbol{a}_i^T \boldsymbol{x} + b_i\}$ itself is convex, and is a simple model to approximate sophisticated convex functions. Here we consider minimizing such a function, i.e.,

$$\text{minimize} \ \max_{1 \le i \le q} \{\boldsymbol{a}_i^T \boldsymbol{x} + b_i\}$$

By introducing an auxiliary scalar variable t, the above problem can be cast as an LP problem:

$$\begin{aligned} &\text{minimize} \quad t \\ &\text{subject to:} \ \boldsymbol{a}_i^T \boldsymbol{x} + b_i \le t \ \text{ for } \ i = 1, 2, \ldots, q \end{aligned}$$

■

3.1.2 Primal and Dual of LP Problems

Consider the standard LP problem

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to:} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \tag{3.1}$$

where matrix $\boldsymbol{A} \in R^{p\times n}$ is of full row rank as the *primal* problem. From Chapter 2, the *dual* problem to (3.1) is given by

$$\begin{array}{ll} \underset{\boldsymbol{\lambda},\boldsymbol{\mu}}{\text{maximize}} & -\boldsymbol{b}^T\boldsymbol{\lambda} \\ \text{subject to:} & -\boldsymbol{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu} = \boldsymbol{c} \\ & \boldsymbol{\mu} \geq \boldsymbol{0} \end{array} \tag{3.2}$$

It is known from the KKT conditions that $\boldsymbol{x}^*$ is a minimizer of the problem in (3.1) if and only if there exist $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^*$ with $\boldsymbol{\mu}^* \geq \boldsymbol{0}$ such that

$$-\boldsymbol{A}^T\boldsymbol{\lambda}^* + \boldsymbol{\mu}^* = \boldsymbol{c} \tag{3.3a}$$

$$\boldsymbol{A}\boldsymbol{x}^* = \boldsymbol{b} \tag{3.3b}$$

$$x_i^* \cdot \mu_i^* = 0 \quad \text{for } 1 \leq i \leq n \tag{3.3c}$$

$$\boldsymbol{x}^* \geq \boldsymbol{0}, \quad \boldsymbol{\mu}^* \geq \boldsymbol{0} \tag{3.3d}$$

It follows that $\boldsymbol{x}^*$ is a feasible point of (3.1) and $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ is a feasible pair of (3.2). Actually, $\boldsymbol{x}^*$ solves problem (3.1) and $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ solves the dual problem (3.2). Therefore we shall call $\{\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*\}$ satisfying (3.3) a primal-dual solution.

Let $\{\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*\}$ satisfies (3.3), then $\boldsymbol{x}^*$ is a solution of (3.1) and $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ are the Lagrange multipliers of (3.1). Furthermore, $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ is a solution of (3.2) and $\boldsymbol{x}^*$ may be interpreted as the Lagrange multiplier of the dual problem (3.2).

The *duality gap*, defined as the difference between the cost of the primal and the cost of the dual, is given by

$$\delta(\boldsymbol{x}, \boldsymbol{\lambda}) = \boldsymbol{c}^T\boldsymbol{x} - \left(-\boldsymbol{b}^T\boldsymbol{\lambda}\right) \tag{3.4}$$

which is always nonnegative, namely,

$$\delta(\boldsymbol{x}, \boldsymbol{\lambda}) \geq 0 \tag{3.5}$$

This is because from (3.1) and (3.2) we have

$$\boldsymbol{c}^T\boldsymbol{x} = \left(-\boldsymbol{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu}\right)^T \boldsymbol{x} = -\boldsymbol{\lambda}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{\mu}^T\boldsymbol{x} \ge -\boldsymbol{b}^T\boldsymbol{\lambda} \tag{3.6}$$

Furthermore, at the primal-dual solution $\{\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*\}$, the duality gap is reduced to its minimum – zero, i.e.,

$$\delta(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) = 0 \tag{3.7}$$

This follows from a line of argument similar to (3.6) in conjunction with the complementarity condition (3.3c):

$$\boldsymbol{c}^T\boldsymbol{x}^* = \left(-\boldsymbol{A}^T\boldsymbol{\lambda}^* + \boldsymbol{\mu}^*\right)^T \boldsymbol{x}^* = -\boldsymbol{\lambda}^{*T}\boldsymbol{A}\boldsymbol{x}^* + \boldsymbol{\mu}^{*T}\boldsymbol{x}^* = -\boldsymbol{b}^T\boldsymbol{\lambda}^*$$

· Another important concept related to primal-dual solutions is *central path*. By (3.3), set $\{\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}\}$ is a primal-dual solution if it satisfies the conditions

$$\begin{aligned} &\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} && \text{with } \boldsymbol{x} \ge \boldsymbol{0} \\ &-\boldsymbol{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu} = \boldsymbol{c} && \text{with } \boldsymbol{\mu} \ge \boldsymbol{0} \\ &\boldsymbol{X}\boldsymbol{\mu} = \boldsymbol{0} && \end{aligned} \tag{3.8}$$

where $\boldsymbol{X} = \text{diag}\{x_1, x_2, \ldots, x_n\}$.

3.2 Quadratic Programming

General quadratic programming (QP) problems assume the form

$$\begin{aligned} &\text{minimize} \quad \tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} + \boldsymbol{p}^T\boldsymbol{x} + \kappa \\ &\text{subject to:} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ &\qquad\qquad\quad\ \ \boldsymbol{C}\boldsymbol{x} \le \boldsymbol{d} \end{aligned} \tag{3.9a-c}$$

The QP problem is convex if matrix $\boldsymbol{H}$ is positive semidefinite.

Example 3.4 *Unconstrained least-squares (LS) problem*

$$\text{minimize} \quad \tfrac{1}{2}\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2^2$$

where $\boldsymbol{A} \in R^{p\times n}$ with $p > n$. The objective function is quadratic and convex because

$$\tfrac{1}{2}\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2^2 = \tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}^T\boldsymbol{A}\boldsymbol{x} + \tfrac{1}{2}\|\boldsymbol{b}\|_2^2$$

If $\boldsymbol{A}$ has full (column) rank, the LS problem has unique and global solution $\boldsymbol{x}^* = \left(\boldsymbol{A}^T\boldsymbol{A}\right)^{-1}\boldsymbol{A}^T\boldsymbol{b}$.

■

Example 3.5 *QP with equality constraints*

Consider convex QP problem

$$\begin{aligned} &\text{minimize} \quad \tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} + \boldsymbol{p}^T\boldsymbol{x} + \kappa \\ &\text{subject to:} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{aligned} \tag{3.10a-b}$$

where $\boldsymbol{A} \in R^{p\times n}$. An effective approach to solve (3.10) is to use the QR decomposition of $\boldsymbol{A}^T$, i.e.,

$$\boldsymbol{A}^T = \boldsymbol{Q}\begin{bmatrix}\boldsymbol{R}\\ \boldsymbol{0}\end{bmatrix} \tag{3.11}$$

where $\boldsymbol{Q}$ is an $n\times n$ orthogonal and $\boldsymbol{R}$ is a $p\times p$ upper triangular matrix (see Appendix). Using (3.11), the constraints in (3.10b) can be expressed as

$$\boldsymbol{R}^T\hat{\boldsymbol{x}}_1 = \boldsymbol{b}$$

where $\hat{\boldsymbol{x}}_1$ is the vector composed of the first p elements of $\hat{\boldsymbol{x}} = \boldsymbol{Q}^T\boldsymbol{x}$. If we denote

$$\hat{\boldsymbol{x}} = \begin{bmatrix}\hat{\boldsymbol{x}}_1\\ \boldsymbol{\phi}\end{bmatrix} \text{ and } \boldsymbol{Q} = [\boldsymbol{Q}_1 \quad \boldsymbol{Q}_2]$$

with $\boldsymbol{\phi} \in R^{(n-p)\times 1}$, $\boldsymbol{Q}_1 \in R^{n\times p}$, and $\boldsymbol{Q}_2 \in R^{n\times(n-p)}$, then we obtain

$$\boldsymbol{x} = \boldsymbol{Q}_2\boldsymbol{\phi} + \boldsymbol{Q}_1\boldsymbol{R}^{-T}\boldsymbol{b} \tag{3.12}$$

The parameterized solutions in (3.12) can be used to convert the problem in (3.10) into a unconstrained problem which leads to the unique global minimizer of the problem in (3.10) as

$$\boldsymbol{x}^* = \boldsymbol{Q}_2\boldsymbol{\phi}^* + \boldsymbol{Q}_1\boldsymbol{R}^{-T}\boldsymbol{b} \tag{3.13a}$$

where $\boldsymbol{\phi}^*$ is the solution of the linear system

$$\left(\boldsymbol{Q}_2^T\boldsymbol{H}\boldsymbol{Q}_2\right)\boldsymbol{\phi} = -\boldsymbol{Q}_2^T\left(\boldsymbol{H}\boldsymbol{Q}_1\boldsymbol{R}^{-T}\boldsymbol{b} + \boldsymbol{p}\right) \tag{3.13b}$$

Alternatively, problem (3.10) can also be solved by using the first-order necessary conditions described in Theorem 1.1, which are given by

$$\begin{aligned}&\boldsymbol{H}\boldsymbol{x}^* + \boldsymbol{p} + \boldsymbol{A}^T\boldsymbol{\lambda}^* = \boldsymbol{0}\\ &\boldsymbol{A}\boldsymbol{x}^* - \boldsymbol{b} = \boldsymbol{0}\end{aligned}$$

i.e.,

$$\begin{bmatrix}\boldsymbol{H} & \boldsymbol{A}^T\\ \boldsymbol{A} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}^*\\ \boldsymbol{\lambda}^*\end{bmatrix} = \begin{bmatrix}-\boldsymbol{p}\\ \boldsymbol{b}\end{bmatrix} \tag{3.14}$$

see Example 1.3. If $\boldsymbol{H}$ is positive definite and $\boldsymbol{A}$ has full row rank, then the system matrix in (3.14) is nonsingular (explain why?) and the solution $\boldsymbol{x}^*$ from (3.14) is the unique global minimizer of the problem in (3.10). Rather than solving linear system (3.14) of size ($n + p$), the solution $\boldsymbol{x}^*$ and Lagrange multipliers $\boldsymbol{\lambda}^*$ can be expressed as

$$\boldsymbol{\lambda}^* = -(\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{A}^T)^{-1}(\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{p} + \boldsymbol{b}) \tag{3.15a}$$

$$x^* = -H^{-1}(A^T \lambda^* + p) \tag{3.15b}$$

■

Example 3.6 *LP with random cost*

Consider an LP problem of the form

$$\begin{aligned} &\text{minimize} \quad c^T x \\ &\text{subject to:} \ \ Ax = b, \ x \geq 0 \end{aligned}$$

where c is a *random* vector with mean $\bar{c}$ and covariance $\Sigma = E[(c - \bar{c})(c - \bar{c})^T]$. Consequently, the objective $c^T x$ is *random* with mean $\bar{c}^T x$ (representing expected cost) and variance $x^T \Sigma x$ (representing risk). A deterministic way to deal with such LP problems is to examine the convex QP problem

$$\begin{aligned} &\text{minimize} \quad \bar{c}^T x + \gamma x^T \Sigma x \\ &\text{subject to:} \ \ Ax = b, \ x \geq 0 \end{aligned}$$

instead, where $\gamma > 0$ is a parameter that controls the trade-off between expected cost and variance (risk). ■

3.3 Semidefinite Programming (SDP)

Semidefinite programming (SDP) is a branch of convex programming that has been a subject of intensive research since the early 1990's. The continued interest in SDP has been motivated mainly by two reasons. First, many important classes of optimization problems such as LP and convex QP problems can be viewed as SDP problems, and many CP problems of practical usefulness that are neither LP nor QP problems can also be formulated as SDP problems. Second, several interior-point methods that have proven efficient for LP and convex QP problems have been extended to SDP.

SDP refers to a class of convex problems of the form

$$\begin{aligned} &\text{minimize} \quad c^T x \\ &\text{subject to:} \ \ F(x) \succeq 0 \end{aligned} \tag{3.16a-b}$$

where

$$F(x) = F_0 + \sum_{i=1}^{p} x_i F_i$$

with $F_i \in S^n$ for $0 \leq i \leq p$. Note that the positive semidefinite constraint on matrix $F(x)$ in (3.16b) is dependent on vector x in an *affine* manner. In the literature, the type of problems described by (3.16) are often referred to as *convex optimization* problems with *linear matrix inequality* (LMI) constraints, and have found many applications in science and engineering. Since algorithm 35 iterations to converge to a solution $\boldsymbol{\theta}^*$. The SNR of the deconvolved signal was found to be 27.2745 dB which is practically the same as that obtained by Algorithm 4.7. As expected, Algorithm 4.9 was considerably faster: the average CPU time it required was approximately 20% of that required by Algorithm 4.7. ■

4.5 Alternating Direction Methods

Alternating direction methods have become increasingly important because of their ability to deal with large scale convex problems. This section presents two representative classes of alternating direction methods known as *alternating direction methods of multipliers* and *alternating minimization algorithms*.

4.5.1 Alternating direction method of multipliers

The alternating direction methods of multipliers (ADMM) [8] are aimed at solving the class of convex problems

$$\begin{aligned} &\text{minimize } \ f(\boldsymbol{x}) + h(\boldsymbol{y}) \\ &\text{subject to: } \boldsymbol{Ax} + \boldsymbol{By} = \boldsymbol{c} \end{aligned} \tag{4.76a-b}$$

where $\boldsymbol{x} \in R^n$ and $\boldsymbol{y} \in R^m$ are variables, $\boldsymbol{A} \in R^{p\times n}$, $\boldsymbol{B} \in R^{p\times m}$, $\boldsymbol{c} \in R^{p\times 1}$, and $f(\boldsymbol{x})$ and $h(\boldsymbol{y})$ are convex functions. Note that in Eq. (4.76) the variables in both objective function and constraint are split into two parts, each involves only one set of variables. By definition, the Lagrangian for the problem in Eq. (4.76) is given by

$$L(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) + h(\boldsymbol{y}) + \boldsymbol{\lambda}^T(\boldsymbol{Ax} + \boldsymbol{By} - \boldsymbol{c})$$

If both $f(\boldsymbol{x})$ and $h(\boldsymbol{y})$ are differentiable, the KKT conditions for the problem in Eq. (4.76) are given by

$$\begin{aligned} &\boldsymbol{Ax} + \boldsymbol{By} = \boldsymbol{c} \\ &\nabla f(\boldsymbol{x}) + \boldsymbol{A}^T\boldsymbol{\lambda} = \boldsymbol{0} \\ &\nabla h(\boldsymbol{y}) + \boldsymbol{B}^T\boldsymbol{\lambda} = \boldsymbol{0} \end{aligned} \tag{4.77a-c}$$

The Lagrange dual of Eq. (4.76) assumes the form

$$\text{maximize } \ q(\boldsymbol{\lambda}) \tag{4.78}$$

where

$$q(\boldsymbol{\lambda}) = \inf_{\boldsymbol{x},\boldsymbol{y}} \left[f(\boldsymbol{x}) + h(\boldsymbol{y}) + \boldsymbol{\lambda}^T(\boldsymbol{Ax} + \boldsymbol{By} - \boldsymbol{c}) \right]$$

which in conjunction with Eq. (4.21) leads to

$$\begin{aligned} q(\boldsymbol{\lambda}) &= \inf_{\boldsymbol{x}} \left[f(\boldsymbol{x}) + \boldsymbol{\lambda}^T\boldsymbol{Ax} \right] + \inf_{\boldsymbol{y}} \left[h(\boldsymbol{y}) + \boldsymbol{\lambda}^T\boldsymbol{By} \right] - \boldsymbol{\lambda}^T\boldsymbol{c} \\ &= -\sup_{\boldsymbol{x}} \left[(-\boldsymbol{A}^T\boldsymbol{\lambda})^T\boldsymbol{x} - f(\boldsymbol{x}) \right] - \sup_{\boldsymbol{y}} \left[(-\boldsymbol{B}^T\boldsymbol{\lambda})^T\boldsymbol{y} - h(\boldsymbol{y}) \right] - \boldsymbol{\lambda}^T\boldsymbol{c} \end{aligned} \tag{4.79}$$

$$= -f^*(-\boldsymbol{A}^T\boldsymbol{\lambda}) - h^*(-\boldsymbol{B}^T\boldsymbol{\lambda}) - \boldsymbol{c}^T\boldsymbol{\lambda}$$

By the properties that $\boldsymbol{u} = \nabla f(\boldsymbol{v})$ if and only if $\boldsymbol{v} = \nabla f^*(\boldsymbol{u})$ and that $\nabla f^*(-\boldsymbol{A}^T\boldsymbol{\lambda}) = -\boldsymbol{A}\nabla f^*(\boldsymbol{\lambda})$ (see Sec. 4.2.4), Eq. (4.79) implies that

$$\nabla q(\boldsymbol{\lambda}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} - \boldsymbol{c} \tag{4.80}$$

where $\{\boldsymbol{x}, \boldsymbol{y}\}$ minimizes $L(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda})$ for a given $\boldsymbol{\lambda}$.

If in addition we assume that $f(\boldsymbol{x})$ and $h(\boldsymbol{y})$ are strictly convex, a solution of the problem in Eq. (4.76) can be found by minimizing the Lagranging $L(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}^*)$ with respect to primal variables $\boldsymbol{x}$ and $\boldsymbol{y}$, where $\boldsymbol{\lambda}^*$ maximizes the dual function $q(\boldsymbol{\lambda})$ in Eq. (4.79). This in conjunction with Eq. (4.80) suggests *dual ascent* iterations for the problem in Eq. (4.76) as follows:

$$\begin{aligned}
\boldsymbol{x}_{k+1} &= \arg\min_{\boldsymbol{x}} L(\boldsymbol{x}, \boldsymbol{y}_k, \boldsymbol{\lambda}_k) = \arg\min_{\boldsymbol{x}} \left[f(\boldsymbol{x}) + \boldsymbol{\lambda}_k^T \boldsymbol{A}\boldsymbol{x} \right] \\
\boldsymbol{y}_{k+1} &= \arg\min_{\boldsymbol{y}} L(\boldsymbol{x}_k, \boldsymbol{y}, \boldsymbol{\lambda}_k) = \arg\min_{\boldsymbol{y}} \left[h(\boldsymbol{y}) + \boldsymbol{\lambda}_k^T \boldsymbol{B}\boldsymbol{y} \right] \\
\boldsymbol{\lambda}_{k+1} &= \boldsymbol{\lambda}_k + \alpha_k (\boldsymbol{A}\boldsymbol{x}_{k+1} + \boldsymbol{B}\boldsymbol{y}_{k+1} - \boldsymbol{c})
\end{aligned} \tag{4.81a-c}$$

where scalar $\alpha_k > 0$ is chosen to maximize $q(\boldsymbol{\lambda})$ (see Eq. (4.78)) along the direction $\boldsymbol{A}\boldsymbol{x}_{k+1} + \boldsymbol{B}\boldsymbol{y}_{k+1} - \boldsymbol{c}$.

It is well known that convex problems of form in Eq. (4.76) with less restrictive $f(\boldsymbol{x})$ and $h(\boldsymbol{y})$ and data matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ can be handled by augmented dual based on the *augmented Lagrangian* [8]

$$L_\alpha(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) + h(\boldsymbol{y}) + \boldsymbol{\lambda}^T(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} - \boldsymbol{c}) + \tfrac{\alpha}{2} \| \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} - \boldsymbol{c} \|_2^2 \tag{4.82}$$

which includes the conventional Lagrangian $L(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda})$ as a special case when parameter α is set to zero. The introduction of augmented Lagrangian may be understood by considering the following [8]: if we modify the objective function in Eq. (4.76) by adding a penalty term $\tfrac{\alpha}{2} \| \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} - \boldsymbol{c} \|_2^2$ for violation of the equality constraint, namely,

$$\begin{aligned}
&\text{minimize} \quad f(\boldsymbol{x}) + h(\boldsymbol{y}) + \tfrac{\alpha}{2} \| \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} - \boldsymbol{c} \|_2^2 \\
&\text{subject to:} \ \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} = \boldsymbol{c}
\end{aligned} \tag{4.83}$$

then the conventional Lagrangian of problem (4.83) is exactly equal to $L_\alpha(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda})$ in Eq. (4.82).

Associated with $L_\alpha(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda})$, the augmented dual problem is given by

$$\text{maximize} \quad q_\alpha(\boldsymbol{\lambda})$$

where

$$q_\alpha(\boldsymbol{\lambda}) = \inf_{\boldsymbol{x},\boldsymbol{y}} \left[f(\boldsymbol{x}) + h(\boldsymbol{y}) + \boldsymbol{\lambda}^T(\boldsymbol{Ax} + \boldsymbol{By} - \boldsymbol{c}) + \tfrac{\alpha}{2} \| \boldsymbol{Ax} + \boldsymbol{By} - \boldsymbol{c} \|_2^2 \right]$$

Unlike the dual ascent iterations in Eq. (4.81) where the minimization of the Lagrangian with respect to variables {$\boldsymbol{x}$, $\boldsymbol{y}$} is split into two separate steps with reduced problem size, the augmented Lagrangian are no longer separable in variables $\boldsymbol{x}$ and $\boldsymbol{y}$ because of the presence of the penalty term. In ADMM iterations, this issue is addressed by *alternating* updates of the primal variables $\boldsymbol{x}$ and $\boldsymbol{y}$, namely,

$$\begin{aligned}
\boldsymbol{x}_{k+1} &= \arg\min_{\boldsymbol{x}} \left[f(\boldsymbol{x}) + \boldsymbol{\lambda}_k^T \boldsymbol{Ax} + \tfrac{\alpha}{2} \| \boldsymbol{Ax} + \boldsymbol{By}_k - \boldsymbol{c} \|_2^2 \right] \\
\boldsymbol{y}_{k+1} &= \arg\min_{\boldsymbol{y}} \left[h(\boldsymbol{y}) + \boldsymbol{\lambda}_k^T \boldsymbol{By} + \tfrac{\alpha}{2} \| \boldsymbol{Ax}_{k+1} + \boldsymbol{By} - \boldsymbol{c} \|_2^2 \right] \\
\boldsymbol{\lambda}_{k+1} &= \boldsymbol{\lambda}_k + \alpha(\boldsymbol{Ax}_{k+1} + \boldsymbol{By}_{k+1} - \boldsymbol{c})
\end{aligned} \tag{4.84a-c}$$

Note that parameter α from the quadratic penalty term is used in Eq. (4.84c) to update Lagrange multiplier $\boldsymbol{\lambda}_k$, thereby eliminating a line search step to compute α_k as required in Eq. (4.81c). To justify Eq. (4.84), note that $\boldsymbol{y}_{k+1}$ minimizes $h(\boldsymbol{y}) + \boldsymbol{\lambda}_k^T \boldsymbol{By} + \tfrac{\alpha}{2} \| \boldsymbol{Ax}_{k+1} + \boldsymbol{By} - \boldsymbol{c} \|_2^2$, hence

$$\begin{aligned}
\boldsymbol{0} &= \nabla h(\boldsymbol{y}_{k+1}) + \boldsymbol{B}^T \boldsymbol{\lambda}_k + \alpha \boldsymbol{B}^T(\boldsymbol{Ax}_{k+1} + \boldsymbol{By}_{k+1} - \boldsymbol{c}) \\
&= \nabla h(\boldsymbol{y}_{k+1}) + \boldsymbol{B}^T \left[\boldsymbol{\lambda}_k + \alpha(\boldsymbol{Ax}_{k+1} + \boldsymbol{By}_{k+1} - \boldsymbol{c}) \right]
\end{aligned}$$

which in conjunction with Eq. (4.84c) leads to

$$\nabla h(\boldsymbol{y}_{k+1}) + \boldsymbol{B}^T \boldsymbol{\lambda}_{k+1} = 0$$

Therefore, the KKT condition in Eq. (4.77c) is satisfied by ADMM iterations. In addition, since $\boldsymbol{x}_{k+1}$ minimizes $f(\boldsymbol{x}) + \boldsymbol{\lambda}_k^T \boldsymbol{Ax} + \tfrac{\alpha}{2} \| \boldsymbol{Ax} + \boldsymbol{By}_k - \boldsymbol{c} \|_2^2$, we have

$$\begin{aligned}
\boldsymbol{0} &= \nabla f(\boldsymbol{x}_{k+1}) + \boldsymbol{A}^T \boldsymbol{\lambda}_k + \alpha \boldsymbol{A}^T(\boldsymbol{Ax}_{k+1} + \boldsymbol{By}_k - \boldsymbol{c}) \\
&= \nabla f(\boldsymbol{x}_{k+1}) + \boldsymbol{A}^T \left[\boldsymbol{\lambda}_k + \alpha(\boldsymbol{Ax}_{k+1} + \boldsymbol{By}_k - \boldsymbol{c}) \right] \\
&= \nabla f(\boldsymbol{x}_{k+1}) + \boldsymbol{A}^T \boldsymbol{\lambda}_{k+1} - \alpha \boldsymbol{A}^T \boldsymbol{B}(\boldsymbol{y}_{k+1} - \boldsymbol{y}_k)
\end{aligned}$$

i.e.,

$$\nabla f(\boldsymbol{x}_{k+1}) + \boldsymbol{A}^T \boldsymbol{\lambda}_{k+1} = \alpha \boldsymbol{A}^T \boldsymbol{B}(\boldsymbol{y}_{k+1} - \boldsymbol{y}_k) \tag{4.85}$$

On comparing Eq. (4.85) with Eq. (4.77b), a *dual residual* in the kth iteration can be defined as

$$\boldsymbol{d}_k = \alpha \boldsymbol{A}^T \boldsymbol{B}(\boldsymbol{y}_{k+1} - \boldsymbol{y}_k) \tag{4.86}$$

From (4.77a), a *primal residual* in the kth iteration is defined as

$$\boldsymbol{r}_k = \boldsymbol{A}\boldsymbol{x}_{k+1} + \boldsymbol{B}\boldsymbol{y}_{k+1} - \boldsymbol{c} \tag{4.87}$$

Together, $\{\boldsymbol{r}_k, \boldsymbol{d}_k\}$ measures closeness of the kth ADMM iteration $\{\boldsymbol{x}_k, \boldsymbol{y}_k, \boldsymbol{\lambda}_k\}$ to the solution of the problem in Eq. (4.76), thus a reasonable criteria for terminating ADMM iterations is when

$$\|\boldsymbol{r}_k\| < \varepsilon_p \text{ and } \|\boldsymbol{d}_k\| < \varepsilon_d \tag{4.88}$$

where ε_p and ε_d are prescribed tolerances for primal and dual residuals, respectively.

Convergence of the ADMM iterations in Eq. (4.84) have been investigated under various assumptions, see [8] and [9] and the references cited therein. If both $f(\boldsymbol{x})$ and $h(\boldsymbol{y})$ are strongly convex with parameters m_f and m_h, respectively, and parameter α is chosen to satisfy

$$\alpha^3 \le \frac{m_f m_h^2}{\rho(\boldsymbol{A}^T\boldsymbol{A})\rho(\boldsymbol{B}^T\boldsymbol{B})^2}$$

where $\rho(\boldsymbol{M})$ denotes the largest eigenvalue of symmetric matrix $\boldsymbol{M}$, then both primal and dual residuals vanish at rate $O(1/k)$ [9], namely,

$$\|\boldsymbol{r}_k\|_2 \le O(1/k) \text{ and } \|\boldsymbol{d}_k\|_2 \le O(1/k)$$

We now summarize the method for solving the problem in Eq. (4.76) as an algorithm.

Algorithm 4.10 ADMM for the problem in Eq. (4.76)

Step 1 Input parameter $\alpha > 0$, $\boldsymbol{y}_0$, $\boldsymbol{\lambda}_0$, and tolerance $\varepsilon_p > 0$, $\varepsilon_d > 0$.

Set $k = 0$.

Step 2 Compute { $\{\boldsymbol{x}_{k+1}, \boldsymbol{y}_{k+1}, \boldsymbol{\lambda}_{k+1}\}$ using Eq. (4.84).

Step 3 Compute $\boldsymbol{d}_k$ and $\boldsymbol{r}_k$ using Eqs. (4.86) and (4.87), respectively.

Step 4 If the conditions in Eq. (4. 88) are satisfied, output ($\boldsymbol{x}_{k+1}$, $\boldsymbol{y}_{k+1}$) as solution and stop; Otherwise, set $k = k + 1$ and repeat from Step 2.

Several variants of ADMM are available, one of them is that of the *scaled form* ADMM [8]. By letting

$$\boldsymbol{r} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} - \boldsymbol{c} \text{ and } \boldsymbol{\nu} = \boldsymbol{\lambda}/\alpha,$$

we write the augmented Lagrangian as

$$
\begin{aligned}
L_\alpha(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\lambda}) &= f(\boldsymbol{x}) + h(\boldsymbol{y}) + \boldsymbol{\lambda}^T\boldsymbol{r} + \tfrac{\alpha}{2}\|\boldsymbol{r}\|_2^2 \\
&= f(\boldsymbol{x}) + h(\boldsymbol{y}) + \tfrac{\alpha}{2}\|\boldsymbol{r}+\boldsymbol{\nu}\|_2^2 - \tfrac{\alpha}{2}\|\boldsymbol{\nu}\|_2^2 \\
&= f(\boldsymbol{x}) + h(\boldsymbol{y}) + \tfrac{\alpha}{2}\|\boldsymbol{Ax}+\boldsymbol{By}-\boldsymbol{c}+\boldsymbol{\nu}\|_2^2 - \tfrac{\alpha}{2}\|\boldsymbol{\nu}\|_2^2
\end{aligned}
$$

Consequently, the scaled ADMM algorithm can be outlined as follows.

Algorithm 4.11 Scaled ADMM for the problem in Eq. (4.76)

Step 1 Input parameter $\alpha > 0$, y_0, $\boldsymbol{\nu}_0$, and tolerance $\varepsilon_p > 0$, $\varepsilon_d > 0$.

Set $k = 0$.

Step 2 Compute

$$
\begin{aligned}
\boldsymbol{x}_{k+1} &= \arg\min_{\boldsymbol{x}}\left[f(\boldsymbol{x}) + \tfrac{\alpha}{2}\|\boldsymbol{Ax}+\boldsymbol{By}_k-\boldsymbol{c}+\boldsymbol{\nu}_k\|_2^2 \right] \\
\boldsymbol{y}_{k+1} &= \arg\min_{\boldsymbol{y}}\left[h(\boldsymbol{y}) + \tfrac{\alpha}{2}\|\boldsymbol{Ax}_{k+1}+\boldsymbol{By}-\boldsymbol{c}+\boldsymbol{\nu}_k\|_2^2 \right] \\
\boldsymbol{\nu}_{k+1} &= \boldsymbol{\nu}_k + \boldsymbol{Ax}_{k+1} + \boldsymbol{By}_{k+1} - \boldsymbol{c}
\end{aligned}
\tag{4.89}
$$

Step 3 Compute $\boldsymbol{d}_k$ and $\boldsymbol{r}_k$ using Eqs. (4.86) and (4.87), respectively.

Step 4 If the conditions in Eq. (4. 88) are satisfied, output $(\boldsymbol{x}_{k+1}, \boldsymbol{y}_{k+1})$ as solution and stop; Otherwise, set $k = k + 1$ and repeat from Step 2.

A variant of scaled ADMM is the *over-relaxed* ADMM in that the term $\boldsymbol{Ax}_{k+1}$ in the $\boldsymbol{y}$- and $\boldsymbol{\nu}$-updates is replaced by $\tau\boldsymbol{Ax}_{k+1} - (1-\tau)(\boldsymbol{By}_k - \boldsymbol{c})$ with $\tau \in (0, 2]$. Thus the over-relaxed ADMM iterations assume the form

$$
\begin{aligned}
\boldsymbol{x}_{k+1} &= \arg\min_{\boldsymbol{x}}\left[f(\boldsymbol{x}) + \tfrac{\alpha}{2}\|\boldsymbol{Ax}+\boldsymbol{By}_k-\boldsymbol{c}+\boldsymbol{\nu}_k\|_2^2 \right] \\
\boldsymbol{y}_{k+1} &= \arg\min_{\boldsymbol{y}}\left[h(\boldsymbol{y}) + \tfrac{\alpha}{2}\|\tau\boldsymbol{Ax}_{k+1}-(1-\tau)\boldsymbol{By}_k+\boldsymbol{By}-\tau\boldsymbol{c}+\boldsymbol{\nu}_k\|_2^2 \right] \\
\boldsymbol{\nu}_{k+1} &= \boldsymbol{\nu}_k + \tau\boldsymbol{Ax}_{k+1} - (1-\tau)\boldsymbol{By}_k + \boldsymbol{By}_{k+1} - \tau\boldsymbol{c}
\end{aligned}
\tag{4.90}
$$

In addition, the idea from Nesterov's accelerated gradient descent algorithm [2] has been extended to ADMM [9]. If both functions $f(\boldsymbol{x})$ and $h(\boldsymbol{y})$ are strongly convex, then a fast ADMM outlined below is shown to converge [9].

Algorithm 4.12 Accelerated ADMM for the problem in Eq. (4.76)

Step 1 Input parameter $\alpha > 0$, $\hat{y}_0, \hat{\boldsymbol{\lambda}}_0$, and tolerance $\varepsilon_p > 0$, $\varepsilon_d > 0$.

Set $\boldsymbol{y}_{-1} = \hat{\boldsymbol{y}}_0, \boldsymbol{\lambda}_{-1} = \hat{\boldsymbol{\lambda}}_0$, $t_0 = 1$, and $k = 0$.

Step 2 Compute

$$
\begin{aligned}
\boldsymbol{x}_k &= \arg\min_{\boldsymbol{x}} \left[f(\boldsymbol{x}) + \hat{\boldsymbol{\lambda}}_k^T \boldsymbol{A}\boldsymbol{x} + \tfrac{\alpha}{2} \| \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\hat{\boldsymbol{y}}_k - \boldsymbol{c} \|_2^2 \right] \\
\boldsymbol{y}_k &= \arg\min_{\boldsymbol{y}} \left[h(\boldsymbol{y}) + \hat{\boldsymbol{\lambda}}_k^T \boldsymbol{B}\boldsymbol{y} + \tfrac{\alpha}{2} \| \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{y} - \boldsymbol{c} \|_2^2 \right] \\
\boldsymbol{\lambda}_k &= \hat{\boldsymbol{\lambda}}_k + \alpha(\boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{y}_k - \boldsymbol{c}) \\
t_{k+1} &= \tfrac{1}{2}\left(1 + \sqrt{1 + 4t_k^2}\right) \\
\hat{\boldsymbol{y}}_{k+1} &= \boldsymbol{y}_k + \tfrac{t_k - 1}{t_{k+1}} (\boldsymbol{y}_k - \boldsymbol{y}_{k-1}) \\
\hat{\boldsymbol{\lambda}}_{k+1} &= \boldsymbol{\lambda}_k + \tfrac{t_k - 1}{t_{k+1}} (\boldsymbol{\lambda}_k - \boldsymbol{\lambda}_{k-1})
\end{aligned}
\tag{4.91}
$$

Step 3 If $\| \alpha \boldsymbol{A}^T \boldsymbol{B}(\boldsymbol{y}_k - \boldsymbol{y}_{k-1}) \|_2 < \varepsilon_d$ and $\| \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{y}_k - \boldsymbol{c} \|_2 < \varepsilon_p$, output $(\boldsymbol{x}_k, \boldsymbol{y}_k)$ as solution and stop; Otherwise, set $k = k + 1$ and repeat from Step 2.

Example 4.8

(a) Apply Algorithm 4.10 to solve the l_1-l_2 minimization problem

$$
\text{minimize} \quad \tfrac{1}{2} \| \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \|_2^2 + \mu \| \boldsymbol{x} \|_1 \tag{4.92}
$$

(b) Apply the results from part (a) to solve the deconvolution problem in Example 4.7.

Solution (a) The problem in Eq. (4.92) can be formulated as [8]

$$
\begin{aligned}
&\text{minimize} \quad \tfrac{1}{2} \| \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \|_2^2 + \mu \| \boldsymbol{y} \|_1 \\
&\text{subject to:} \quad \boldsymbol{x} - \boldsymbol{y} = \boldsymbol{0}
\end{aligned}
$$

which fits into the formulation in Eq. (4.76) with $f(\boldsymbol{x}) = \tfrac{1}{2} \| \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \|_2^2$ and $h(\boldsymbol{y}) = \mu \| \boldsymbol{y} \|_1$. The scaled ADMM iterations in this case assume the form

$$
\begin{aligned}
\boldsymbol{x}_{k+1} &= \arg\min_{\boldsymbol{x}} \left[\tfrac{1}{2} \| \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \|_2^2 + \tfrac{\alpha}{2} \| \boldsymbol{x} - (\boldsymbol{y}_k - \boldsymbol{\lambda}_k / \alpha) \|_2^2 \right] \\
\boldsymbol{y}_{k+1} &= \arg\min_{\boldsymbol{y}} \left[\mu \| \boldsymbol{y} \|_1 + \tfrac{\alpha}{2} \| \boldsymbol{y} - (\boldsymbol{x}_{k+1} + \boldsymbol{\lambda}_k / \alpha) \|_2^2 \right] \\
\boldsymbol{\lambda}_{k+1} &= \boldsymbol{\lambda}_k + \alpha(\boldsymbol{x}_{k+1} - \boldsymbol{y}_{k+1})
\end{aligned}
$$

Evidently, updating $\boldsymbol{x}_k$ amounts to minimizing a convex quadratic function and updating $\boldsymbol{y}_k$ can be done by soft-shrinkage of $\boldsymbol{x}_{k+1} + \boldsymbol{\lambda}_k / \alpha$ by μ / α, see Eq. (4.67). Thus we have

$$
\begin{aligned}
\boldsymbol{x}_{k+1} &= (\boldsymbol{A}^T \boldsymbol{A} + \alpha \boldsymbol{I})^{-1} (\boldsymbol{A}^T \boldsymbol{b} + \alpha \boldsymbol{y}_k - \boldsymbol{\lambda}_k) \\
\boldsymbol{y}_{k+1} &= \mathcal{S}_{\mu/\alpha}(\boldsymbol{x}_{k+1} + \boldsymbol{\lambda}_k / \alpha) \\
\boldsymbol{\lambda}_{k+1} &= \boldsymbol{\lambda}_k + \alpha(\boldsymbol{x}_{k+1} - \boldsymbol{y}_{k+1})
\end{aligned}
\tag{4.93}
$$

where operator $\mathcal{S}_{\mu/\alpha}$ is defined by Eq. (4.68). Note that matrix $(\boldsymbol{A}^T \boldsymbol{A} + \alpha \boldsymbol{I})^{-1}$ as well as vector $\boldsymbol{A}^T \boldsymbol{b}$ in Eq. (4.93) are independent of iterations, hence they need to be computed only once.

(b) By applying Eq. (4.93) to the data set described in Example 4.7 with y_0 = **0**, λ_0 = **0**, $\mu = 0.25$, and $\alpha = 0.11$, it took 32 ADMM iterations to yield a satisfactory estimation of the signal. The SNR of the estimated signal was found to be 27.2765 dB. The original, distorted, and recovered signals are depicted in Fig. 4.13. The profiles of the primal and dual residuals in terms of $\| \boldsymbol{r}_k \|_2$ and $\| \boldsymbol{d}_k \|_2$ are shown in Fig. 4.14. It is observed that both $\| \boldsymbol{r}_k \|_2$ and $\| \boldsymbol{d}_k \|_2$ fall below 5×10^{-3} after 32 iterations. On comparing with the proximal-point algorithms in Sec. 4.4, the average CPU time required by ADMM iterations was found be practically the same as that of Algorithm 14.9. ■

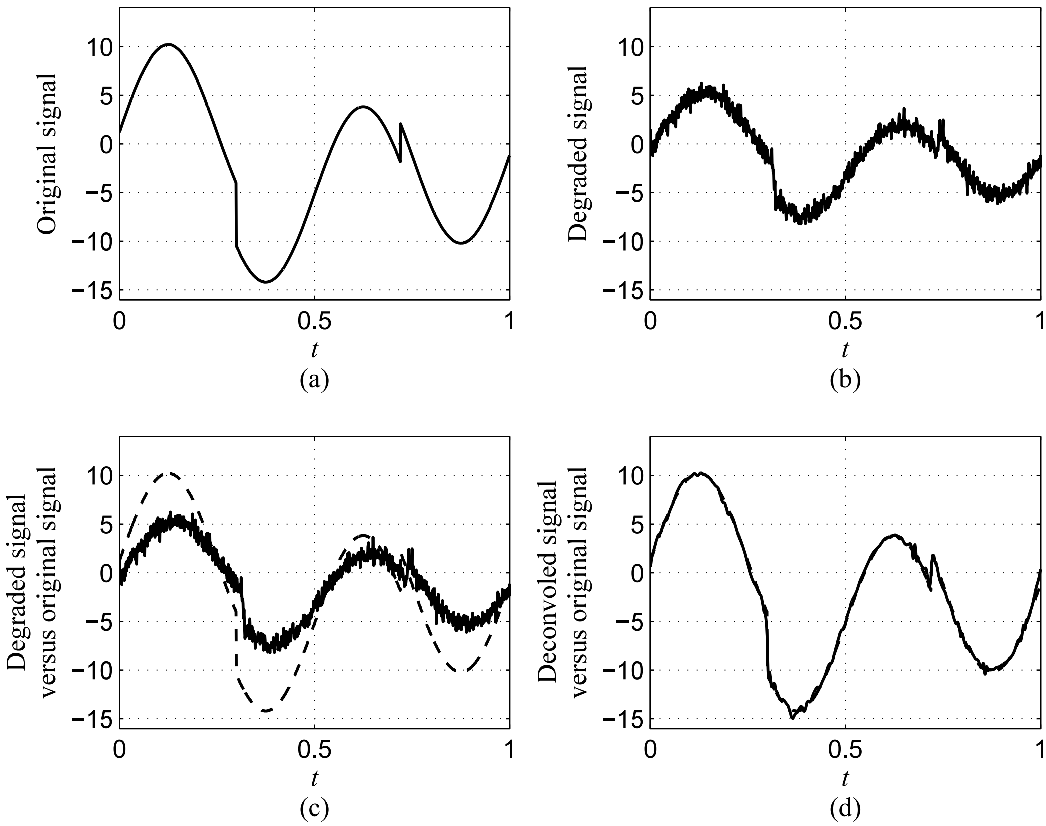

Figure 4.13. (a) Original signal heavisine (b) Distorted and noise-contaminated heavisine (c) Original (dashed) versus degraded (solid) heavisine, SNR = 5.7592 dB. (d) Original (dashed) versus reconstructed (solid) heavisine, SNR = 27.2765 dB.

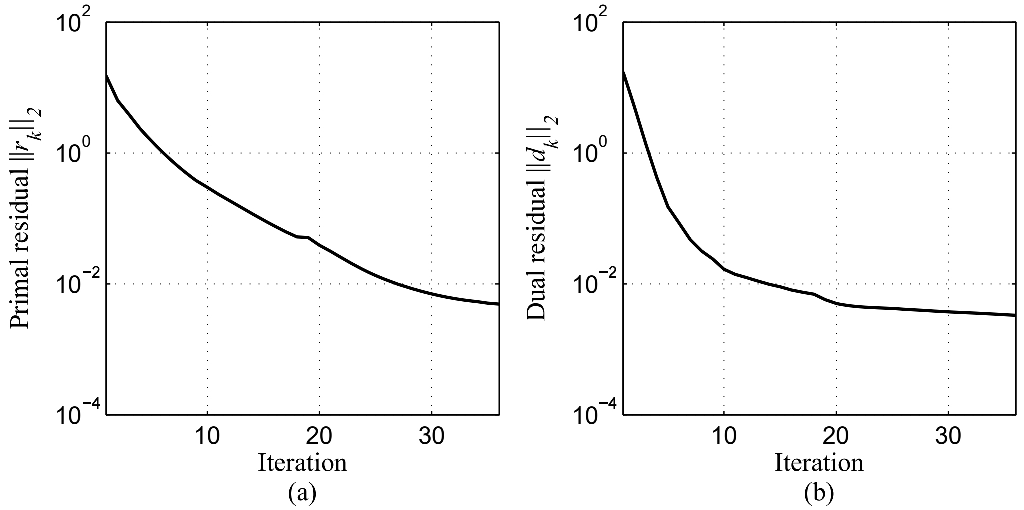

Figure 4.14. (a) Primal residual $\| \boldsymbol{r}_k \|_2$ versus iterations (b) Dual residual $\| \boldsymbol{d}_k \|_2$ versus iterations for Example 4.8.

4.5.2 ADMM for general convex optimization

Consider the constrained convex problem

$$\begin{aligned} &\text{minimize} \quad f(\boldsymbol{x}) \\ &\text{subject to: } \boldsymbol{x} \in C \end{aligned} \tag{4.94}$$

where $f(\boldsymbol{x})$ is a convex function and C is a convex set representing the feasible region of the problem. The problem in Eq. (4.94) can be formulated as

$$\text{minimize} \quad f(\boldsymbol{x}) + I_C(\boldsymbol{x}) \tag{4.95}$$

where $I_C(\boldsymbol{x})$ is the indicator function associated with set C:

$$I_C(\boldsymbol{x}) = \begin{cases} 0 & \text{if } \boldsymbol{x} \in C \\ +\infty & \text{otherwise} \end{cases}$$

The problem in Eq. (4.95) can be written as

$$\begin{aligned} &\text{minimize} \quad f(\boldsymbol{x}) + I_C(\boldsymbol{y}) \\ &\text{subject to: } \boldsymbol{x} - \boldsymbol{y} = \boldsymbol{0} \end{aligned} \tag{4.96}$$

That fits into the ADMM formulation in Eq. (4.76) [8]. The scaled ADMM iterations for Eq. (4.96) are given by

$$\begin{aligned} \boldsymbol{x}_{k+1} &= \arg\min_{\boldsymbol{x}} \left[f(\boldsymbol{x}) + \tfrac{\alpha}{2} \| \boldsymbol{x} - \boldsymbol{y}_k + \boldsymbol{v}_k \|_2^2 \right] \\ \boldsymbol{y}_{k+1} &= \arg\min_{\boldsymbol{y}} \left[I_C(\boldsymbol{y}) + \tfrac{\alpha}{2} \| \boldsymbol{y} - (\boldsymbol{x}_{k+1} + \boldsymbol{v}_k) \|_2^2 \right] \\ \boldsymbol{v}_{k+1} &= \boldsymbol{v}_k + \boldsymbol{x}_{k+1} - \boldsymbol{y}_{k+1} \end{aligned}$$

where the y-minimization is obtained by minimizing $\| \boldsymbol{y} - (\boldsymbol{x}_{k+1} + \boldsymbol{v}_k) \|_2$ subject to $\boldsymbol{y} \in C$. This means that $\boldsymbol{y}_{k+1}$ can be obtained by projecting $\boldsymbol{x}_{k+1} + \boldsymbol{v}_k$ onto set C. Therefore, the ADMM iterations become

$$\begin{aligned} \boldsymbol{x}_{k+1} &= \arg\min_{\boldsymbol{x}} \left[f(\boldsymbol{x}) + \tfrac{\alpha}{2} \| \boldsymbol{x} - \boldsymbol{y}_k + \boldsymbol{v}_k \|_2^2 \right] \\ \boldsymbol{y}_{k+1} &= P_C(\boldsymbol{x}_{k+1} + \boldsymbol{v}_k) \\ \boldsymbol{v}_{k+1} &= \boldsymbol{v}_k + \boldsymbol{x}_{k+1} - \boldsymbol{y}_{k+1} \end{aligned} \tag{4.97a-c}$$

where $P_C(\boldsymbol{z})$ denotes the projection of point $\boldsymbol{z}$ onto convex set C. The projection cab be accomplished by solving the convex problem

$$\begin{aligned} &\text{minimize} \quad \| \boldsymbol{y} - \boldsymbol{z} \|_2 \\ &\text{subject to:} \ \ \boldsymbol{y} \in C \end{aligned}$$

Example 4.9

Find a sparse solution of an underdetermined system of linear equation $\boldsymbol{Ax} = \boldsymbol{b}$ by solving the constrained convex problem

$$\begin{aligned} &\text{minimize} \ \ \| \boldsymbol{x} \|_1 \\ &\text{subject to:} \ \ \boldsymbol{Ax} = \boldsymbol{b} \end{aligned} \tag{4.98}$$

Solution The problem fits into the formulation in Eq. (4.94) with $f(\boldsymbol{x}) = \| \boldsymbol{x} \|_1$ and $C = \{ \boldsymbol{x} : \boldsymbol{Ax} = \boldsymbol{b} \}$. The $\boldsymbol{x}$-minimization step in Eq. (4.97a) becomes

$$\boldsymbol{x}_{k+1} = \arg\min_{\boldsymbol{x}} \left[\| \boldsymbol{x} \|_1 + \tfrac{\alpha}{2} \| \boldsymbol{x} - \boldsymbol{y}_k + \boldsymbol{v}_k \|_2^2 \right]$$

hence

$$\boldsymbol{x}_{k+1} = \mathcal{S}_{1/\alpha}(\boldsymbol{y}_k - \boldsymbol{v}_k) \tag{4.99}$$

where operator $\mathcal{S}$ is defined by (4.68). The $\boldsymbol{y}$-minimization step in Eq. (4.94b) is carried out by solving the simple convex QP problem

$$\begin{aligned} &\text{minimize} \quad \| \boldsymbol{y} - (\boldsymbol{x}_{k+1} + \boldsymbol{v}_k) \|_2 \\ &\text{subject to:} \quad \boldsymbol{Ay} = \boldsymbol{b} \end{aligned} \tag{4.100}$$

whose solution is given by

$$\boldsymbol{y}_{k+1} = \boldsymbol{A}^+ \boldsymbol{b} + \boldsymbol{P}_A(\boldsymbol{x}_{k+1} + \boldsymbol{v}_k) \tag{4.101}$$

where $\boldsymbol{A}^+ = \boldsymbol{A}^T(\boldsymbol{AA}^T)^{-1}$ and $\boldsymbol{P}_A = \boldsymbol{I} - \boldsymbol{A}^T(\boldsymbol{AA}^T)^{-1}\boldsymbol{A}$. To summarize, the ADMM iterations in Eq. (4.97) are realized using Eqs. (4.99), (4.101), and (4.97c).

For illustration purposes, above results were applied to the problem in Eq. (4.98) where $\boldsymbol{A}$ was a randomly generated matrix of size 20 by 50. A sparse column vector $\boldsymbol{x}_s$ of length 50 was produced by placing six randomly generated nonzero numbers in the zero vector of length 50 for six randomly selected coordinates, see Fig. 4.15a. Vector $\boldsymbol{b}$ was then generated as $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}_s$. In this way, $\boldsymbol{x}_s$ is a known sparse solution of the underdetermined system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. With $\alpha = 0.5$, it took 49 ADMM iterations to converge to a solution $\boldsymbol{x}^*$ which is depicted in Fig. 4.15b where the ADMM-based solution is found to well recover the true sparse solution $\boldsymbol{x}_s$. The l_2 reconstruction error of the solution is found to be $\|\boldsymbol{x}^* - \boldsymbol{x}_s\|_2 = 4.1282 \times 10^{-4}$. ■

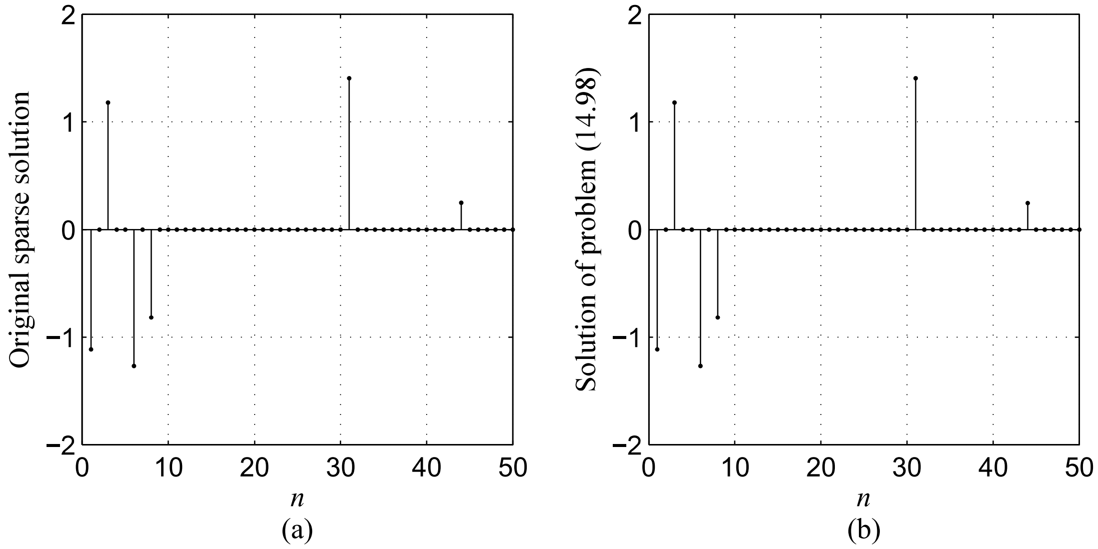

Figure 4.15. (a) Original sparse solution $\boldsymbol{x}_s$ and (b) Solution obtained by solving the problem in Eq. (4.98).

The primal and dual residuals are given by $\boldsymbol{r}_k = \boldsymbol{x}_k - \boldsymbol{y}_k$ and $\boldsymbol{d}_k = -\alpha(\boldsymbol{y}_k - \boldsymbol{y}_{k-1})$, respectively. The profiles of these residuals for the 49 ADMM iterations are shown in Fig. 4.16.

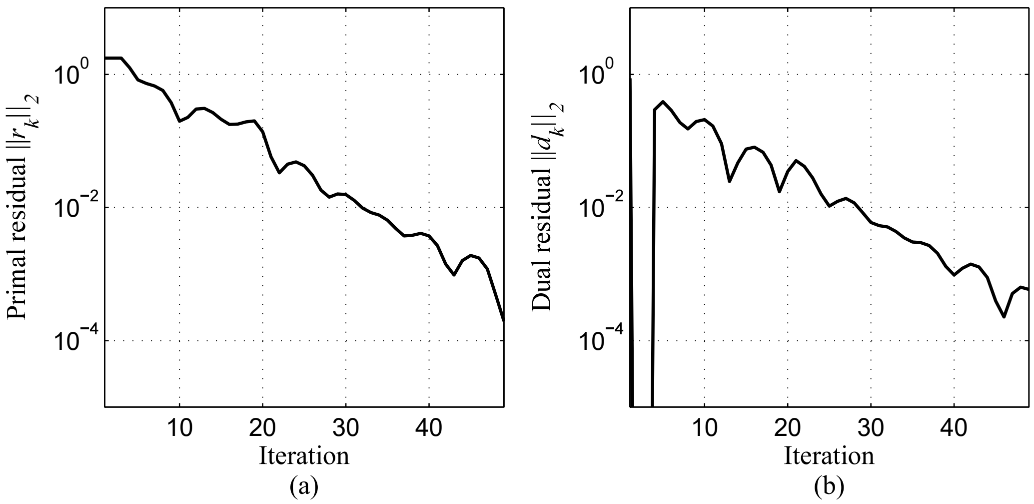

Figure 4.16. (a) Primal and (b) Dual residuals for the problem in Example 4.8.

Chapter 4 Algorithms for General Convex Problems

4.1 Introduction

In Chapter 3, we have addressed several special classes of convex problems that have found applications in engineering disciplines. There are however many problems in practice that do not fall into these classes. This chapter is devoted to studies of methods for general convex problems. These include Newton algorithms, proximal-point algorithms for composite convex functions, and alternating direction algorithms.

Several concepts and properties of convex functions that are needed in the development of these algorithms are introduced in Sec. 4.2. These include subgradients, convex functions with Lipschitz continuous gradients, strongly convex functions, conjugate functions, and proximal operators. Newton algorithms for unconstrained and constrained convex problems are addressed in Sec. 4.3. In Sec. 4.4, several algorithms for minimizing composite convex functions are studied. The l_1-l_2 minimization problem which finds applications in digital signal processing and machine learning is covered as a special case. Alternating direction methods have become increasingly important because of their ability to deal with large scale convex problems. In Sec. 4.5, we present two representative classes of alternating direction methods known as alternating direction methods of multipliers and alternating minimization algorithms.

4.2 Concepts and Properties of Convex Functions

The notion of convex functions and their elementary properties are addressed in Chapter 2. Here we introduce several additional concepts and properties of convex functions that are of use in the development of effective algorithms for convex problems.

4.2.1 Subgradients

Practical optimization problems involving non-differentiable objective functions and/or constraints are pervasive. The non-smoothness of the functions in an optimization problem may be encountered in several ways. For example, there are optimization problems with objective functions or constraints that are inherently non-differentiable. A simple example of the case is given by

$$\begin{array}{ll} \text{minimize} & \|\boldsymbol{x}\|_1 \\ \text{subject to:} & \|\boldsymbol{Ax}-\boldsymbol{b}\|_2 \le \varepsilon \end{array}$$

where the l_1-norm of variable $\boldsymbol{x}$ is minimized subject to an l_2-norm constraint. Obviously, the objective function $\|\boldsymbol{x}\|_1 = \sum_{i=1}^n |x_i|$ is continuous and convex, but *not differentiable*. There are also scenarios where the functions involved are differentiable, and it is the operation of these functions that yields non-smoothness. An example of the case is the objective function of the form

$$f(\boldsymbol{x}) = \max_{1\le j\le p}\{\varphi_j(\boldsymbol{x})\}$$

where $\varphi_j(\boldsymbol{x})$ are smooth convex functions, and the objective function is also convex but not necessarily differentiable, see Fig. 4.1 that depicts the objective function with three linear $\varphi_j(\boldsymbol{x})$.

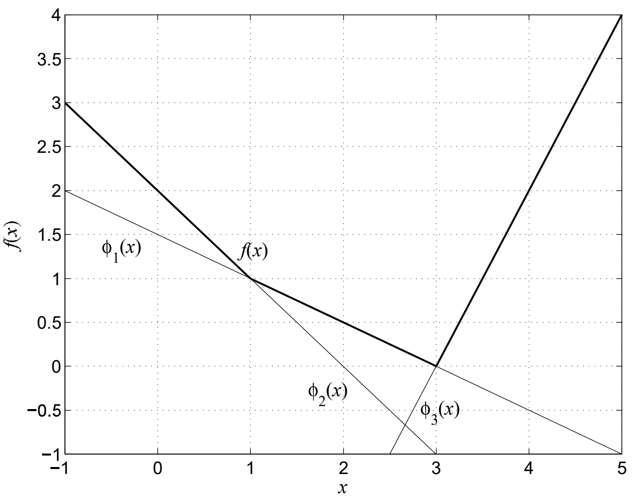

Figure 4.1. Pointwise maximum of three affine functions yields a piece-wise affine function which is convex but not differentiable.

Gradient plays an instrumental role in continuous optimization for functions that are differentiable. The concept of *subgradient* is a natural generalization of the concept of gradient that allows us to deal with optimization problems involving convex but non-differentiable functions [1]. Recall that the convexity of a differentiable function $f(\boldsymbol{x})$ can be characterized by

$$f(\tilde{\boldsymbol{x}}) \geq f(\boldsymbol{x}) + \nabla^T f(\boldsymbol{x})(\tilde{\boldsymbol{x}} - \boldsymbol{x}) \qquad \text{for } \boldsymbol{x}, \tilde{\boldsymbol{x}} \in \text{dom}(f) \tag{4.1}$$

where $\text{dom}(f)$ is the domain of function $f(\boldsymbol{x})$ that defines the set of points $\boldsymbol{x}$ where $f(\boldsymbol{x})$ assumes finite values. In geometric terms, Eq. (4.1) states that at any point $\boldsymbol{x}$ in the domain of a convex function $f(\boldsymbol{x})$, the tangent to the surface defined by $y = f(\boldsymbol{x})$ always lies below the surface.

Definition 4.1 If $f(\boldsymbol{x})$ is convex but not necessarily differentiable, then vector $\boldsymbol{g} \in R^n$ is said to be a *subgradient* of $f(\boldsymbol{x})$ at $\boldsymbol{x}$ if

$$f(\tilde{\boldsymbol{x}}) \geq f(\boldsymbol{x}) + \boldsymbol{g}^T(\tilde{\boldsymbol{x}} - \boldsymbol{x}) \quad \text{for } \tilde{\boldsymbol{x}} \in \text{dom}(f) \tag{4.2}$$

The subgradient of a convex function at point $\boldsymbol{x}$ where $f(\boldsymbol{x})$ is non-differentiable is *not* unique. The set of all subgradients at point $\boldsymbol{x}$ is called *subdifferential* of $f(\boldsymbol{x})$ and is denoted by $\partial f(\boldsymbol{x})$. ■

The right-hand side of Eq. (4.2) may be viewed as a linear lower bound of $f(\boldsymbol{x})$, and the subgradients at a point $\boldsymbol{x}$ where the convex function $f(\boldsymbol{x})$ is not differentiable correspond to different tangent lines at $\boldsymbol{x}$. This is illustrated in Fig. 4.2, where the two subgradients of $f(x)$ at x^* are given by $g_1 = \tan\theta_1$ and $g_2 = \tan\theta_2$. From the figure, it is obvious that any tangent line at x^* with a slop between g_2 and g_1 satisfies Eq. (4.2), therefore, any value $g \in [g_2, g_1]$ is a subgradient of $f(x)$ at x^*.

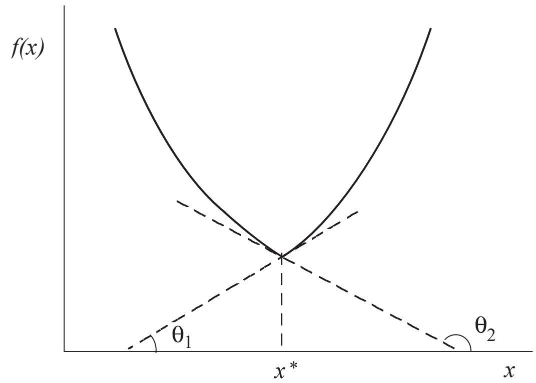

Figure 4.2. Two subgradients of $f(x)$ at x^* where $f(x)$ is not differentiable.

From Eq. (4.2), it follows that $f(\tilde{\boldsymbol{x}}) \geq f(\boldsymbol{x})$ as long as $\boldsymbol{g}^T(\tilde{\boldsymbol{x}} - \boldsymbol{x}) \geq 0$. Note that for a given point $\boldsymbol{x}$, $\boldsymbol{g}^T(\tilde{\boldsymbol{x}} - \boldsymbol{x}) = 0$ defines a hyperplane which passes through point $\boldsymbol{x}$ with $\boldsymbol{g}$ as its normal. This hyperplane divides space R^n into two parts with the hyperplane as boundary. In the part of the space where $\tilde{\boldsymbol{x}}$ satisfies $\boldsymbol{g}^T(\tilde{\boldsymbol{x}} - \boldsymbol{x}) > 0$, no minimizers exist because Eq. (4.2) in this case implies that $f(\tilde{\boldsymbol{x}}) > f(\boldsymbol{x})$. Consequently, a minimizer of $f(\boldsymbol{x})$ can only be found in the part of the space characterized by $\{\boldsymbol{x} : \boldsymbol{g}^T(\tilde{\boldsymbol{x}} - \boldsymbol{x}) \leq 0\}$. In this analysis, we see subgradient facilitates the construction of a cutting plane in the parameter space that reduces the search region significantly.

There are several important special cases in which the computation of a subgradient of a convex $f(\boldsymbol{x})$ can be readily carried out:

(a) If $f(\boldsymbol{x})$ is convex and differentiable at $\boldsymbol{x}$, then the subdifferential $\partial f(\boldsymbol{x})$ contains only one subgradient which is the same as the gradient $\nabla f(\boldsymbol{x})$;

(b) If $\alpha > 0$, a subgradient of $\alpha f(\boldsymbol{x})$ is given by $\alpha \boldsymbol{g}$ where $\boldsymbol{g}$ is a subgradient of $f(\boldsymbol{x})$;

(c) If $f(\boldsymbol{x}) = f_1(\boldsymbol{x}) + f_2(\boldsymbol{x}) + \cdots + f_r(\boldsymbol{x})$ where $f_i(\boldsymbol{x})$ for $1 \leq i \leq r$ are convex, then $\boldsymbol{g} = \boldsymbol{g}_1 + \boldsymbol{g}_2 + \cdots + \boldsymbol{g}_r$ is a subgradient of $f(\boldsymbol{x})$ where $\boldsymbol{g}_i$ is a subgradient of $f_i(\boldsymbol{x})$;

(d) Define function $f(\boldsymbol{x}) = \max\left[f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_r(\boldsymbol{x})\right]$ where $f_i(\boldsymbol{x})$ for i = 1, 2, …, r are convex. At a given point $\boldsymbol{x}$, there exists at least one index, say i^*, such that $f(\boldsymbol{x}) = f_{i^*}(\boldsymbol{x})$. Then a subgradient of $f_{i^*}(\boldsymbol{x})$ is a subgradient of $f(\boldsymbol{x})$.

(e) If $h(\boldsymbol{x}) = f(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b})$ where $f(\boldsymbol{x})$ is convex, then

$$\partial h(\boldsymbol{x}) = \boldsymbol{A}^T \partial f(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b})$$

Example 4.1 Verify the formula of subdifferential of function $f(x) = |x|$

$$\partial |x| = \begin{cases} 1 & \text{for } x > 0 \\ \text{any } g \in [-1, 1] & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases} \tag{4.3}$$

Solution Function $f(x) = |x|$ is convex because for $0 \le \alpha \le 1$ we have

$$f(\alpha x_1 + (1-\alpha)x_2) = |\alpha x_1 + (1-\alpha)x_2| \le \alpha |x_1| + (1-\alpha)|x_2| = \alpha f(x_1) + (1-\alpha) f(x_2)$$

Consider a point x > 0, we have $f(x)$ = x which is obviously differentiable. Hence the differential of $f(x)$ is equal to the derivative of $f(x)$, i.e., $f^{'}(x) = x' = 1$ which verifies the first line of the formula. Now consider a point x < 0, we have $f(x)$ = $-x$ which is differentiable. Hence the differential of $f(x)$ is equal to $f^{'}(x) = (-x)' = -1$ which verifies the third line of the formula. At x = 0, Eq. (4.2) is reduced to $|\tilde{x}| \ge g\tilde{x}$ which holds for any g between –1 and 1. This verifies the second line of the formula in Eq. (4.3). ■

The next two theorems concern optimization problems where the functions involved are convex but not necessarily differentiable. These theorems may be regarded as extensions of the well-known first-order optimality condition and KKT conditions to their non-differentiable counterparts that are studied earlier in Chapter 2 and Chapter 10, respectively.

Theorem 4.1 ***First-order optimality condition for non-differentiable unconstrained convex problems*** Point $\boldsymbol{x}^*$ is a global solution of the minimization problem

$$\underset{\boldsymbol{x} \in \text{dom}(f)}{\text{minimize}} \; f(\boldsymbol{x})$$

where $f(\boldsymbol{x})$ is convex, if and only if $\boldsymbol{0} \in \partial f(\boldsymbol{x}^*)$.

Proof Suppose $\mathbf{0} \in \partial f(\boldsymbol{x}^*)$. By letting $\boldsymbol{x} = \boldsymbol{x}^*$ and $\boldsymbol{g} = \mathbf{0}$ in Eq. (4.2), we obtain $f(\tilde{\boldsymbol{x}}) \geq f(\boldsymbol{x}^*)$ for all $\tilde{\boldsymbol{x}}$ in dom(f), hence $\boldsymbol{x}^*$ is a global minimizer of $f(\boldsymbol{x})$. Conversely, if $\boldsymbol{x}^*$ is a global minimizer, then $f(\tilde{\boldsymbol{x}}) \geq f(\boldsymbol{x}^*)$ for all $\tilde{\boldsymbol{x}} \in \text{dom}(f)$, hence $f(\tilde{\boldsymbol{x}}) \geq f(\boldsymbol{x}^*) + \mathbf{0}^T(\tilde{\boldsymbol{x}} - \boldsymbol{x}^*)$ for all $\tilde{\boldsymbol{x}} \in \text{dom}(f)$, which implies that $\mathbf{0} \in \partial f(\boldsymbol{x}^*)$. ■

Theorem 4.2 ***KKT conditions for non-differentiable constrained convex problems*** Consider constrained convex problem

$$\begin{aligned} &\text{minimize} \quad f(\boldsymbol{x}) \\ &\text{subject to:} \ \boldsymbol{a}_i^T \boldsymbol{x} = b_i \quad \text{for } 1 \leq i \leq p \\ &\qquad\qquad\quad c_j(\boldsymbol{x}) \leq 0 \quad \text{for } 1 \leq j \leq q \end{aligned} \tag{4.4a-c}$$

where $f(\boldsymbol{x})$ and $c_j(\boldsymbol{x})$ are convex but not necessarily differentiable. A regular point $\boldsymbol{x}^*$ is a solution of the problem in Eq. (4.4) if and only if

(a) $\boldsymbol{a}_i^T \boldsymbol{x}^* = b_i$ for $1 \leq i \leq p$.

(b) $c_j(\boldsymbol{x}^*) \leq 0$ for $1 \leq j \leq q$.

(c) there exist λ_i^* and μ_j^* such that $\mathbf{0} \in \partial f(\boldsymbol{x}^*) + \sum_{i=1}^{p} \lambda_i^* \boldsymbol{a}_i + \sum_{j=1}^{q} \mu_j^* \partial c_j(\boldsymbol{x}^*)$

(d) $\mu_j^* c_j(\boldsymbol{x}^*) = 0$ for $j = 1, 2, \ldots, q$.

(e) $\mu_j^* \geq 0$ for $j = 1, 2, \ldots, q$.

Proof Below we prove the sufficiency of these conditions for point $\boldsymbol{x}^*$ to be a global solution of the problem in Eq. (4.4), and leave the necessity part to the reader.

Suppose point $\boldsymbol{x}^*$ satisfies conditions (a) – (e). Conditions (a) and (b) imply that $\boldsymbol{x}^*$ is a feasible point of Eq. (4.4). Let $\boldsymbol{x}$ be an arbitrary feasible point for the problem in Eq. (4.4), below we show that $f(\boldsymbol{x}^*) \leq f(\boldsymbol{x})$.

Since both $\boldsymbol{x}$ and $\boldsymbol{x}^*$ are feasible, we can write

$$\sum_{i=1}^{p} \lambda_i^* \boldsymbol{a}_i^T (\boldsymbol{x} - \boldsymbol{x}^*) = \sum_{i=1}^{p} \lambda_i^* (\boldsymbol{a}_i^T \boldsymbol{x} - b_i) - \sum_{i=1}^{p} \lambda_i^* (\boldsymbol{a}_i^T \boldsymbol{x}^* - b_i) = 0 \tag{4.5}$$

Because $c_j(\boldsymbol{x}) \leq 0$ and $\mu_j^* \geq 0$ for j = 1, 2, …, q, we have

$$\sum_{j=1}^{q} \mu_j^* c_j(\boldsymbol{x}) \leq 0$$

which in conjunction with condition (d) implies that

$$\sum_{j=1}^{q} \mu_j^* \left(c_j(\boldsymbol{x}) - c_j(\boldsymbol{x}^*) \right) \le 0 \tag{4.6}$$

The convexity of functions $c_j(\boldsymbol{x})$ gives

$$c_j(\boldsymbol{x}) - c_j(\boldsymbol{x}^*) \ge \partial c_j(\boldsymbol{x}^*)^T (\boldsymbol{x} - \boldsymbol{x}^*)$$

which leads Eq. (4.6) to

$$\sum_{j=1}^{q} \mu_j^* \partial c_j(\boldsymbol{x}^*)^T (\boldsymbol{x} - \boldsymbol{x}^*) \le 0 \tag{4.7}$$

From Eqs. (4.5), (4.7), and the convexity of $f(\boldsymbol{x})$, we deduce

$$\begin{aligned}
& f(\boldsymbol{x}) - f(\boldsymbol{x}^*) \\
& \ge \partial f(\boldsymbol{x}^*)^T (\boldsymbol{x} - \boldsymbol{x}^*) + \sum_{i=1}^{p} \lambda_i^* \boldsymbol{a}_i^T (\boldsymbol{x} - \boldsymbol{x}^*) + \sum_{j=1}^{q} \mu_j^* \partial c_j(\boldsymbol{x}^*)^T (\boldsymbol{x} - \boldsymbol{x}^*) \\
& = \left[\partial f(\boldsymbol{x}^*) + \sum_{i=1}^{p} \lambda_i^* \boldsymbol{a}_i + \sum_{j=1}^{q} \mu_j^* \partial c_j(\boldsymbol{x}^*) \right]^T (\boldsymbol{x} - \boldsymbol{x}^*)
\end{aligned}$$

By condition (c), the expression in the last square bracket can be set to zero which leads to $f(\boldsymbol{x}^*) \le f(\boldsymbol{x})$. ■

4.2.2 Convex functions with Lipschitz-continuous gradients

A continuously differentiable function $f(\boldsymbol{x})$ is said to have Lipschitz continuous gradient [2] if

$$\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\|_2 \le L \|\boldsymbol{x} - \boldsymbol{y}\|_2 \tag{4.8}$$

for any $\boldsymbol{x}$ and $\boldsymbol{y} \in \mathrm{dom}(f)$, where $L > 0$ is called Lipschitz constant.

Example 4.2 Show that the gradient of $f(\boldsymbol{x}) = \frac{1}{2} \| \boldsymbol{Ax} - \boldsymbol{b} \|_2^2$ is Lipschitz continuous.

Solution We write function $f(\boldsymbol{x})$ as

$$f(\boldsymbol{x}) = \tfrac{1}{2} \boldsymbol{x}^T \boldsymbol{A}^T \boldsymbol{Ax} - \boldsymbol{x}^T \boldsymbol{A}^T \boldsymbol{b} + \tfrac{1}{2} \boldsymbol{b}^T \boldsymbol{b}$$

Hence the gradient of $f(\boldsymbol{x})$ is given by

$$\nabla f(\boldsymbol{x}) = \boldsymbol{A}^T \boldsymbol{Ax} - \boldsymbol{A}^T \boldsymbol{b}$$

and