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GRUPA 311CC

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(1.) $u = \begin{pmatrix} 5 \\ 1 \\ 2 \\ 3 \end{pmatrix}$

40/40 !!

Subiectul 1

$w = \begin{pmatrix} a \\ -2 \\ 0 \\ -1 \end{pmatrix}$

$v = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 2 \end{pmatrix}$

1) u, v, w sunt liniari independenți $\Rightarrow$ ecuația $\alpha u + \beta w + \gamma v = 0$ are doar soluția $\alpha = \beta = \gamma = 0$

$$\alpha \begin{pmatrix} 5 \\ 1 \\ 2 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -1 \\ 2 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} a \\ -2 \\ 0 \\ -1 \end{pmatrix} = 0 \Leftrightarrow$$

$$\begin{cases} 5\alpha + \beta + a\gamma = 0 \\ \alpha - \beta - 2\gamma = 0 \\ 2\alpha + 2\beta = 0 \\ 3\alpha + 2\beta - \gamma = 0 \end{cases}$$

Rezolvăm sistemul folosind metoda lui Gauss.

$$\left(\begin{array}{ccc|c} 5 & 1 & a & 0 \\ 1 & -1 & -2 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 2 & -1 & 0 \end{array}\right) \xrightarrow{L_3 = L_3 - 2L_2} \left(\begin{array}{ccc|c} 5 & 1 & a & 0 \\ 1 & -1 & -2 & 0 \\ 0 & 4 & 4 & 0 \\ 3 & 2 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{L_4 = L_4 - 3L_2} \left(\begin{array}{ccc|c} 5 & 1 & a & 0 \\ 1 & -1 & -2 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 5 & 5 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 5 & 1 & a & 0 \end{array}\right)$$

$\Rightarrow \xrightarrow{L_1 = L_1 + L_2}$ $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 5 & 1 & a & 0 \end{array}\right) \Rightarrow$ $\alpha = \gamma$, $\beta = -\gamma$, $5\alpha + \beta + a\gamma = 0$ 2/

$\Rightarrow 5\gamma - \gamma + a\gamma = 0 \Rightarrow$

$(4+a)\gamma = 0$

Cum sistemul trebuie să aibă ca soluție doar $(0,0,0)$ ~~pt. care acesta să f~~ pt. ca u, v, w să fie liniar independenți $\Rightarrow$ $a \neq -4$ și $\gamma = 0 = \beta = \alpha$

$\Rightarrow$ Pentru $a \in \mathbb{R} - \{4\}$ u, v, w sunt liniar independenți. $\Rightarrow$ pentru $\boxed{a=4}$ vectorii dați sunt liniari dependenți.

2. Pentru $a = 4 \Rightarrow$

$$Sp(u,v,w) = \cancel{Sp\left\{\begin{pmatrix}1\\0\\5\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\4\end{pmatrix}\right\}}\ Sp\left\{\begin{pmatrix}5\\1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\-1\\2\\2\end{pmatrix}, \begin{pmatrix}-4\\-2\\0\\-1\end{pmatrix}\right\}$$

$$\begin{pmatrix}5\\1\\2\\3\end{pmatrix} = \begin{pmatrix}1\\-1\\2\\2\end{pmatrix} - \begin{pmatrix}-4\\-2\\0\\-1\end{pmatrix} \Rightarrow$$

$$Sp(u,v,w) = Sp\underbrace{\left\{\begin{pmatrix}1\\-1\\2\\2\end{pmatrix}, \begin{pmatrix}-4\\-2\\0\\-1\end{pmatrix}\right\}}_{\text{lin. independenți}}$$

Completăm cu încă 2 vectori din $\mathbb{R}^4$ pt. a forma o bază.

Fie $B = \left\{ \underbrace{\begin{pmatrix}1\\-1\\2\\2\end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix}-4\\-2\\0\\-1\end{pmatrix}}_{u_2}, \underbrace{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}_{u_3}, \underbrace{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}_{u_4} \right\}$

Cum $\det A = -2$, unde $A = (u_1 | u_2 | u_3 | u_4)$, deci $\operatorname{rang} A = 3 \Rightarrow u_1, u_2, u_3, u_4$ sunt lin. independenți și cum $\dim_{\mathbb{R}} \mathbb{R}^4 = 4 \Rightarrow B$ este o bază în $\mathbb{R}^4$

$$\begin{pmatrix} -1 \\ -5 \\ 6 \\ 5 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ -1 \\ 2 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} -4 \\ -2 \\ 0 \\ -1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \exists!$$

$\Rightarrow$ Aplicăm metoda lui Gauss de rezolvare

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & -4 & -1 \\ 0 & 1 & -1 & -2 & -5 \\ 0 & 0 & 2 & 0 & 6 \\ 0 & 0 & 2 & -1 & 5 \end{array}\right) \xrightarrow{L_4 = L_4 - L_3} \begin{array}{c} \begin{array}{cccc} \alpha_3 & \alpha_4 & \alpha_1 & \alpha_2 \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 1 & -4 & -1 \\ 0 & 1 & -1 & -2 & -5 \\ 0 & 0 & 2 & 0 & 6 \\ 0 & 0 & 0 & -1 & -1 \end{array}\right) \end{array}$$

$$\Rightarrow \begin{cases} \alpha_2 = 1 \\ \alpha_1 = 3 \\ \alpha_4 - \alpha_1 - 2\alpha_2 = -5 \Rightarrow \alpha_4 = -5 + 3 + 2 = 0 \\ \alpha_3 = -1 + 4\alpha_2 - \alpha_1 = 0 \end{cases}$$

$$\Rightarrow \underbrace{\begin{pmatrix} -1 \\ -5 \\ 6 \\ 5 \end{pmatrix}}_{v} = 3 \begin{pmatrix} 1 \\ -1 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} -4 \\ -2 \\ 0 \\ -1 \end{pmatrix} \Rightarrow [v]_B = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Subiectul 2

$U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x+y+z=0,\ x+2y+3z=0 \right\}$

1. Determinăm o bază pentru U.

$\begin{cases} x+y+z=0 \\ x+2y+3z=0 \end{cases} \Rightarrow \begin{cases} x=-y-z \\ -y-z+2y+3z=0 \end{cases} \Rightarrow \begin{cases} x=-y-z \\ y+2z=0 \end{cases} \Rightarrow$

$\begin{cases} x=-y-z \\ y=-2z \end{cases} \Rightarrow \begin{cases} x=z \\ y=-2z \end{cases} \Rightarrow U = Sp\left\{ \underbrace{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}_{u_1} \right\}$

Cum $\dim_{\mathbb{R}} U = 1 \Rightarrow B' = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$ este bază pt. U și bază ortogonală de asemenea.

$\|u_1\| = \sqrt{1+4+1} = \sqrt{6} \Rightarrow g_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$B'' = \left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$ este bază ortonormată pentru U.

2. Fie $B = \left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \overbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}^{g_2}, \overbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}^{g_3} \right\}$ rang $A=3$

Cum $\det A = 1$, unde $A = (u_1 | g_2 | g_3) \Rightarrow$ u_1, g_2, g_3 lin. independenți $\Rightarrow g_1, g_2, g_3$ lin. independenți și $\dim_{\mathbb{R}} \mathbb{R}^3 = 3 \Rightarrow B$ bază în $\mathbb{R}^3$

Aplicăm algoritmul Gram - Schmidt pentru obținerea unei baze ortogonale.

$w_1 = u_1$

$w_2 = g_2 - (pr_{w_1} g_2) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{\langle w_1, g_2 \rangle}{\langle w_1, w_1 \rangle} \cdot w_1 =$

$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1+0+0}{1+4+1} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/6 \\ 1/3 \\ -1/6 \end{pmatrix}$

$$w_3 = g_3 - (\underset{w_1}{pr}\, g_3 + \underset{w_2}{pr}\, g_3) = \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{\langle g_3, w_1\rangle}{\langle w_1, w_1\rangle}\cdot w_1 -$$

$$\frac{\langle g_3, w_2\rangle}{\langle w_2, w_2\rangle}\cdot w_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{0-2+0}{1+4+1}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix} -$$

$$\frac{\frac{1}{3}}{\frac{25}{36}+\frac{1}{9}+\frac{1}{36}}\cdot\begin{pmatrix}5/6\\1/3\\-1/6\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\-2\\1\end{pmatrix} - \frac{\frac{1}{3}}{\frac{5}{6}\frac{\cancel{36}}{\cancel{36}}}\begin{pmatrix}5/6\\1/3\\-1/6\end{pmatrix} =$$

$$= \begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\-2\\1\end{pmatrix} - \frac{2}{5}\begin{pmatrix}5/6\\1/3\\-1/6\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\-2\\1\end{pmatrix} - \begin{pmatrix}1/3\\2/15\\-1/15\end{pmatrix}$$

$$= \begin{pmatrix}0\\1/5\\2/5\end{pmatrix} \Rightarrow B_w = \{w_1, w_2, w_3\} = \left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}, \begin{pmatrix}5/6\\1/3\\-1/6\end{pmatrix}, \begin{pmatrix}0\\1/5\\2/5\end{pmatrix}\right\}$$

este o bază ortogonală.

Determinăm baza ortonormată.

$$\|w_1\| = \sqrt{6} \Rightarrow k_1 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

$$\|w_2\| = \sqrt{\frac{25}{36}+\frac{4}{36}+\frac{1}{36}} = \sqrt{\frac{30}{36}} = \sqrt{\frac{5}{6}} \Rightarrow k_2 = \sqrt{\frac{6}{5}}\cdot\begin{pmatrix}5/6\\1/3\\-1/6\end{pmatrix}$$

$$\|w_3\| = \sqrt{0+\frac{1}{25}+\frac{4}{25}} = \sqrt{\frac{5}{25}} = \sqrt{\frac{1}{5}} \Rightarrow k_3 = \sqrt{5}\begin{pmatrix}0\\1/5\\2/5\end{pmatrix}$$

$$\Rightarrow B_k = \{k_1, k_2, k_3\} = \left\{\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}, \sqrt{\frac{6}{5}}\begin{pmatrix}5/6\\1/3\\-1/6\end{pmatrix}, \sqrt{5}\begin{pmatrix}0\\1/5\\2/5\end{pmatrix}\right\}$$

este o bază ortonormată.

$$v = \begin{pmatrix}2\\2\\4\end{pmatrix} = \sqrt{6}\begin{pmatrix}1\\-2\\1\end{pmatrix} - 3\sqrt{30}\begin{pmatrix}5/6\\1/3\\-1/6\end{pmatrix} + \frac{1}{\sqrt{5}}\cdot 5\begin{pmatrix}0\\1/5\\2/5\end{pmatrix}$$

(am determinat întâi coordonatele în baza B_w și mai apoi în baza B_k) $\Rightarrow$

$$[v]_{B_K} = \begin{pmatrix} \sqrt{6} \\ -3\sqrt{30} \\ \sqrt{5} \end{pmatrix}.$$

Subiectul 3

3. $A = \begin{pmatrix} -7 & 12 & 6 \\ -4 & 7 & 4 \\ 0 & 0 & -1 \end{pmatrix}$

1. Determinăm polinomul caracteristic

$$P_A(x) = \det(A - xI_3) = \begin{vmatrix} -7-x & 12 & 6 \\ -4 & 7-x & 4 \\ 0 & 0 & -1-x \end{vmatrix} =$$

$$= -x^3 - x^2 + x + 1 = (x+1)^2 \cdot (1-x) \Rightarrow \begin{cases} \lambda_1 = -1 & a(\lambda_1) = 2 \\ \lambda_2 = 1 & a(\lambda_2) = 1 \end{cases}$$

Pentru $\lambda_1 = -1 \Rightarrow$ fie $v \in \mathbb{R}^3$ a.î. $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, x, y, z \in \mathbb{R}$

$$Av = -v \Rightarrow \begin{cases} -7x + 12y + 6z = -x \\ -4x + 7y + 4z = -y \\ \underline{-z = -z} \end{cases} \Rightarrow$$

$$\begin{cases} -6x + 12y + 6z = 0 \\ -4x + 8y + 4z = 0 \end{cases} \Rightarrow \begin{cases} -3x + 2y + 6z = 0 \\ -x + 2y + z = 0 \end{cases} \Rightarrow$$

$$x = 2y + z \Rightarrow V_{\lambda_1} = Sp\left\{ \underbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{\text{lin. indep.}} \right\} \Rightarrow$$

$$\dim_{\mathbb{R}} V_{\lambda_1} = 2 \Rightarrow g(\lambda_1) = a(\lambda_1) = 2 \quad (1)$$

Pentru $\lambda_2 = 1 \Rightarrow$ fie $v \in \mathbb{R}^3$ a.î. $\left(v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, x, y, z \in \mathbb{R}\right)$

$$Av = v \Rightarrow \begin{cases} -7x + 12y + 6z = x \\ -4x + 7y + 4z = y \\ -z = z \end{cases} \Rightarrow \begin{cases} z = 0 \\ -8x + 12y = 0 \\ -4x + 6y = 0 \end{cases} \Rightarrow$$

$$\begin{cases} z = 0 \\ x = \frac{12y}{8} = \frac{3}{2}y \end{cases} \Rightarrow V_{\lambda_2} = Sp\left\{ \begin{pmatrix} 3/2 \\ 1 \\ 0 \end{pmatrix} \right\} = Sp\left\{ \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \right\}$$

$\Rightarrow g(\lambda_2) = a(\lambda_2) = 1$ (2) $\Rightarrow$

Din (1),(2) $\Rightarrow$ A este diagonalizabilă

și $A = PDP^{-1}$, unde

$D =$ matrice diagonală $= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$P = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\cancel{P^{-1} = \begin{pmatrix} 0,8 & -0,6 & -0,8 \\ 0 & 0 & 1 \\ -0,2 & 0,4 & 0,2 \end{pmatrix}} \qquad P^{-1} = \begin{pmatrix} 2 & -3 & -2 \\ 0 & 0 & 1 \\ -1 & 2 & 1 \end{pmatrix}$$

2. $A^{2020} = (PDP^{-1})^{2020} = PD^{2020}P^{-1} =$

$$= P \cdot \begin{pmatrix} (-1)^{2020} & 0 & 0 \\ 0 & (-1)^{2020} & 0 \\ 0 & 0 & 1^{2020} \end{pmatrix} P^{-1} = P \cdot I_3 \cdot P^{-1} = \boxed{I_3}$$

$$\Rightarrow \boxed{A^{2020} = I_3}$$

Subiectul IV

1. f este izomorfism $\Leftrightarrow$ M(f) este inversabilă $\Leftrightarrow a_n \neq 0$

Altă metodă:

"$\Rightarrow$" f izomorfism $\Rightarrow$ f bijectivă $\Rightarrow \begin{cases} \operatorname{Ker} f = \{0\} \\ \operatorname{Im} f = \mathbb{R}^4 \end{cases}$

$\cancel{[f(v)]_{B_c} = M(f) \cdot [v]_{B_c}}$

$$f(v) = M(f) \cdot v = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & \cdots & 0 & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_1x_1 + a_2x_2 + \ldots + a_nx_n \\ x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$f\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0 \Rightarrow \begin{cases} a_1x_1 + a_2x_2 + \dots + a_nx_n = 0 \\ x_1 = 0 \\ x_2 = 0 \\ \vdots \\ x_{n-1} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = x_2 = \dots = x_{n-1} = 0 \\ a_n \cdot x_n = 0 \end{cases}$$

$\Rightarrow \ker(f) = 0 \Rightarrow$ ec. $a_n \cdot x_n = 0$ are doar soluția $x_n = 0$

$\Rightarrow a_n \neq 0$

"$\Leftarrow$" Dacă $a_n \neq 0 \Rightarrow$

$$f\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix} a_1x_1 + \dots + a_nx_n \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow$$

$$\begin{cases} a_1x_1 + \dots + a_nx_n = 0 \\ x_1 = x_2 = \dots = x_{n-1} = 0 \end{cases} \overset{a_n \neq 0}{\Rightarrow} x_n = 0 \Rightarrow \operatorname{Ker}(f) = \{0\}$$

$$\dim_{\mathbb{R}} \operatorname{Im}(f) = \dim_{\mathbb{R}} \mathbb{R}^n - \dim_{\mathbb{R}} \ker(f) = \overset{n}{3} - 0 = \overset{n}{3} \Rightarrow$$

$\operatorname{Im}(f) = \mathbb{R}^n$ ($\dim_{\mathbb{R}} \operatorname{Im}(f) = \dim_{\mathbb{R}} \mathbb{R}^n$ și $\operatorname{Im}(f) \subseteq \mathbb{R}^n$)

$\Rightarrow f$ bijectivă $\Rightarrow f$ izomorfism

2. Fie $a_n = 0$ și $\{e_1, e_2, \dots, e_n\}$ baza canonică din $\mathbb{R}^n$

$$f(v) = M(f) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \dots & a_{n-1} & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \begin{pmatrix} a_1x_1 + a_2x_2 + \dots + a_{n-1}x_{n-1} \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix}$$