Dumitrache Adrian - George 11 / XII / 2021

314 CC

Parțial - ALGAED

II) $A = \begin{pmatrix} -5 & -1 \\ -8 & 4 \end{pmatrix}$; $f: M_2(\mathbb{R}) \to M_2(\mathbb{R})$, $f(X) = AX - XA$; f aplic. liniară

1) $\mathrm{Ker}(f) = \{X \in M_2(\mathbb{R}) \mid f(X) = 0_2\}$

Fie $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$f(X) = 0$

$$\begin{pmatrix} -5 & -1 \\ -8 & 4 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -5 & -1 \\ -8 & 4 \end{pmatrix} = 0_2$$

$$\begin{pmatrix} -5a - c & -5b - d \\ -8a + 4c & -8b + 4d \end{pmatrix} - \begin{pmatrix} -5a - 8b & -a + 4b \\ -5c - 8d & -c + 4d \end{pmatrix} = 0_2$$

$$\begin{pmatrix} 8b - c & a - 9b - d \\ -8a + 9c + 8d & -8b + c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} 8b - c = 0 \\ a - 9b - d = 0 \\ -8a + 9c + 8d = 0 \\ -8b + c = 0 \end{cases} \Rightarrow \begin{cases} c = 8b \\ a - 9b - d = 0 \\ -8a + 9c + 8d = 0 \\ c = 8b \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} c = 8b \\ \cancel{-8a+} \\ \end{cases} \begin{cases} c = 8b \\ a - 9b - d = 0 \\ -8a + 72b + 8d = 0 \end{cases} \Rightarrow \begin{cases} c = 8b \\ a - 9b - d = 0 \\ a - 9b + d = 0 \end{cases} \Rightarrow d = 0$$

$a - 9b - d = 0$

$a - 9b = 0$

$a = 9b$

Deci, $X \in Ker(f) = \begin{pmatrix} 9b & b \\ 8b & 0 \end{pmatrix} = Sp\left\{\begin{pmatrix} 9 & 1 \\ 8 & 0 \end{pmatrix}\right\}$,

care este o bază pentru $Ker(f)$, fiind formată dintr-un singur vector $\Rightarrow \dim_{\mathbb{R}} Ker(f) = 1$

I) $P_1(X) = 1+X+5X^2$

$P_2(X) = 3+45X+57X^2$

$P_3(X) = 7-35X-7X^2 \in \mathbb{R}_2[X]$

$$W = \{ P \in \mathbb{R}_2[X] \mid P(0)+P'(0)+5P''(0)=0,\ 6P'(0)+6P''(0)=0 \}$$

1) $\lambda_1 \cdot P_1(X) + \lambda_2 P_2(X) + \lambda_3 P_3(X) = 0$

$$X^2(5\lambda_1+57\lambda_2-7\lambda_3)+X(\lambda_1+45\lambda_2-35\lambda_3)+\lambda_1+3\lambda_2+7\lambda_3=0$$

$$\Rightarrow \begin{cases} 5\lambda_1+57\lambda_2-7\lambda_3=0 \\ \lambda_1+45\lambda_2-3\lambda_3=0 \\ \lambda_1+3\lambda_2+7\lambda_3=0 \end{cases}$$

$$\Delta = \begin{vmatrix} 5 & 57 & -7 \\ 1 & 45 & -3 \\ 1 & 3 & 7 \end{vmatrix} = \cancel{\begin{vmatrix} 4 & 12 & -10 \\ 1 & 45 & -3 \\ 1 & 3 & 7 \end{vmatrix}} =$$

$$= 5\cdot45\cdot7 + 57(-3)\cdot1 + (-7)\cdot1\cdot3 - 1\cdot45\cdot(-7) - 3(-3)\cdot5$$

$$-7\cdot1\cdot57 = 1344 \neq 0 \Rightarrow \text{lin inden}$$

$$\lambda_3 = 7 \Leftrightarrow \begin{cases} 5\lambda_1+57\lambda_2=49 \\ \lambda_1+45\lambda_2=21 \\ \lambda_1+3\lambda_2 = -49 \end{cases} \underset{L_1=L_1-5L_3}{\Leftrightarrow} \begin{cases} 42\lambda_2=294 \\ \lambda_1+45\lambda_2=21 \\ \lambda_1+3\lambda_2=-49 \end{cases}$$

$\Rightarrow x_2 = 7 \Rightarrow x_1 + 3 \cdot 7 = \cancel{-54}\ -49$

$x_1 = -49 - 21 = -60$

$\cancel{\Rightarrow \{w\} - w}$

2) $P(x) = ax^2 + bx + c$; $P'(x) = 2ax + b$; $P''(x) = 2a$

$P(0) + P'(0) + P''(0) = 0$

$c + b + 2a = 0$

$6P'(0) + 6P''(0) = 0$

$\cancel{12ax +}$

$6b + 12a = 0$

$$\left.\begin{matrix} a + b = 0 \\ 2a + b + c = 0 \end{matrix}\right\} \Rightarrow a + c = 0$$

Astfel, $b = -a$, $c = -a \Rightarrow P(x) = ax^2 - ax - a$

$\Rightarrow W = \{x^2 - x - 1\} \Rightarrow \dim_{\cancel{\mathbb{R}_2[x]}\ \mathbb{R}} W = 1$

II) cont.

$$\operatorname{Im}(f) = \left\{ B \in \mathcal{M}_2(\mathbb{R}) \;\middle|\; \exists\, X \in \mathcal{M}_2(\mathbb{R}) \; f(X) = B \right\}$$

$$B = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

$$\Rightarrow \begin{cases} 8b - c = x \\ -a - 9b - d = y \\ -8a + 9c + 8d = z \\ -8b + c = \cancel{0}\, t \end{cases}$$

$L_1, L_2 \Rightarrow x = -t$

~~$L_2 + L_3 \Rightarrow 9(-a - b + c$~~

$$\Rightarrow \begin{cases} 8b - c = x \\ a - 9b - d = y \\ -8a + 9c + 8d = z \end{cases}$$

$$\Delta_1 = \begin{vmatrix} 0 & 8 & -1 \\ 1 & -9 & 0 \\ -8 & 0 & 9 \end{vmatrix} = +1 \cdot (-9) \cdot (-8) - 9 \cdot 8 = 0$$

$$\Delta_2 = \begin{vmatrix} 0 & 8 & 0 \\ 1 & -9 & -1 \\ -8 & 0 & 8 \end{vmatrix} = 0 + 0 + 64 - 64 = 0$$

$$\Delta_3 = \begin{vmatrix} 8 & -1 & 0 \\ -9 & 0 & -1 \\ 0 & 9 & 8 \end{vmatrix} = 72 - 72 = 0$$

$$\Delta_4 = \begin{vmatrix} 0 & 8 \\ 1 & -9 \end{vmatrix} = -8 \neq 0 \Rightarrow r_g = 2$$

$$\Delta_c = \begin{vmatrix} 0 & 8 & x \\ 1 & -5 & y \\ -8 & 0 & z \end{vmatrix} = -64y - 72x + 8z$$

$\Delta_c = 0$

$-72x - 64y + 8z = 0$

$z = 9x + 8y \quad ; \quad y = \frac{z}{8} - \frac{9}{8}y$

$$\Rightarrow B = \begin{pmatrix} x & \frac{z}{8} - \frac{9}{8}y \\ 9x + 8y & -x \end{pmatrix} = x\begin{pmatrix} 1 & 0 \\ 9 & -1 \end{pmatrix} + z\begin{pmatrix} 0 & \frac{1}{8} \\ 0 & 0 \end{pmatrix}$$

$$+ y\begin{pmatrix} 0 & -\frac{9}{8} \\ 8 & 0 \end{pmatrix} = Sp_R\left\{ \begin{pmatrix} 1 & 0 \\ 9 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{9}{8} \\ 8 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{8} \\ 0 & 0 \end{pmatrix} \right\}$$

$$\Rightarrow \overset{\text{bază} \Rightarrow}{\dim_R C(A) = 3}$$

IV)

1) $\cos \sphericalangle (v_1, v_2) = \dfrac{\langle v_1, v_2 \rangle}{\|v_1\| \cdot \|v_2\|} = \dfrac{9}{\sqrt{9}\,\sqrt{18}} = \dfrac{9}{9\sqrt{2}} =$

$= \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

2) $u_1 = v_1$

$u_2 = v_2 - pr_{u_1} v_2 = v_2 - \dfrac{\langle u_1, v_2 \rangle}{\langle u_1, u_1 \rangle} \cdot u_1 =$

$$= \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} - \frac{9}{9} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

$u_3 = v_3 - (pr_{u_1} v_3 + pr_{u_2} v_3) = v_3 - \left(\dfrac{\langle u_1, v_3 \rangle}{\langle u_1, u_1 \rangle} \cdot u_1 + \dfrac{\langle u_2, v_3 \rangle}{\langle u_2, u_2 \rangle} u_2 \right)$

$$\cancel{= \begin{pmatrix} -4 \\ -14 \\ -16 \end{pmatrix} - \begin{pmatrix} -5 \\ \\ \end{pmatrix}} \quad \begin{pmatrix} -4 \\ -14 \\ -16 \end{pmatrix} - \left(\frac{-54}{9} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \frac{8-28-16}{9} \cdot \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \right) =$$

$$= \begin{pmatrix} -4 \\ -14 \\ -16 \end{pmatrix} - \left(-6 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + (-4) \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \right) =$$

$$= \begin{pmatrix} -4 \\ -14 \\ -16 \end{pmatrix} - \begin{pmatrix} -12 \\ -6 \\ -12 \end{pmatrix} + \begin{pmatrix} -8 \\ 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad :C$$

Incorect, verificăm dacă putem elim v_3

Observăm că $v_3 = -2v_1 - 4v_3$, deci

$$U = Sp\left\{ \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} \right\}$$

$$\cancel{u_1 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}} \quad u_1 = v_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

$$g_1 = \frac{1}{\|u_1\|} \cdot u_1 = \frac{1}{\sqrt{9}} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$g_2 = \frac{1}{\|u_2\|} \cdot u_2 = \frac{1}{\sqrt{9}} \cdot \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

Deci $U = Sp\{g_1, g_2\}$ este o bază ortonormată

Fie $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, cum $\Delta = \begin{vmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 3 & 0 \end{vmatrix} = 3 - 6 = -3 \neq 0$

$\Rightarrow v_1, v_2, v_3$ sunt lin indep $\Rightarrow Sp\{v_1, v_2, v_3\} = R^3$

Il ortonormăm și pe v_3

$$u_3 = v_3 - (pr_{u_1} v_3 + pr_{u_2} v_3) =$$

$$= v_3 - \left(\frac{\langle u_1, v_3 \rangle}{\langle u_1, u_1 \rangle} \cdot u_1 + \frac{\langle u_2, v_3 \rangle}{\langle u_2, u_2 \rangle} \cdot u_2 \right) =$$

$$= v_3 - \left(\frac{2}{9} \cdot u_1 + \frac{-2}{9} u_2 \right) =$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \left(\frac{2}{9} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} - \frac{2}{9} \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \right) =$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{8}{9} \\ -\frac{2}{9} \\ \frac{2}{9} \end{pmatrix} = \begin{pmatrix} \frac{1}{9} \\ \frac{2}{9} \\ -\frac{2}{9} \end{pmatrix}$$

$$g_3 = \frac{1}{\|u_3\|} \cdot u_3 = \frac{1}{\sqrt{\frac{9}{81}}} \cdot u_3 = 3 u_3 = \begin{pmatrix} \frac{3}{9} \\ \frac{6}{9} \\ -\frac{6}{9} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$$

~~Sp~~ $\{ Sp\{g_1, g_2, g_3\} = \mathbb{R}^3$ bază ortonormată

$$w = \begin{pmatrix} 0 \\ -6 \\ -6 \end{pmatrix}$$

$$\begin{cases} \frac{2}{3} x - \frac{2}{3} y + \frac{1}{3} z = 0 \\ \frac{1}{3} x + \frac{2}{3} y + \frac{2}{3} z = -6 \\ \frac{2}{3} x + \frac{1}{3} y - \frac{2}{3} z = -6 \end{cases}$$

$\cancel{\overset{+}{\Rightarrow} x + z = -6}$

$\cancel{\overset{+}{\Rightarrow} x + y = -12}$

$\cancel{z - y = 6}$

$-$

$\cancel{\frac{2}{3} x + \frac{1}{3} y - \frac{2}{3} z = -6} \xrightarrow{L_2 - L_3}$

$\cancel{\frac{1}{3}(2x + y) - \frac{2}{3} z =}$