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GRUPA 312 CC

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Parțial Algebră

1) Fie $P_1(x) = 1+3X-5X^2$; $P_2(x)=7+42X-53X^2$; $P_3(x)=-7-56X+65X^2 \in \mathbb{R}_2[X]$ și subspațiul:

$$W=\{P\in \mathbb{R}_2[x] \mid P(0)+3P'(0)-5P''(0)=0, -7P'(0)+6P''(0)=0\}$$

a) Arătăm că P_1, P_2, P_3 = liniar dependente:

Fie ~~$\lambda_1, \lambda_2, \lambda$~~ $\alpha_1,\alpha_2,\alpha_3 \in \mathbb{R}$, a.î.: $\alpha_1 P_1+\alpha_2 P_2+\alpha_3 P_3=0 \Leftrightarrow$

$$\Leftrightarrow \begin{cases} \alpha_1+7\alpha_2-7\alpha_3=0 \\ 3\alpha_1+42\alpha_2-56\alpha_3=0 \\ -5\alpha_1-53\alpha_2+65\alpha_3=0 \end{cases}$$

$$\cancel{\left(\begin{array}{ccc|c} 1 & 7 & 7 & 0 \\ 3 & 42 & -56 & 0 \\ 5 & -53 & 65 & 0 \end{array}\right) \xrightarrow[L_3+5L_1]{L_2-3L_1} \left(\begin{array}{ccc|c} 1 & 7 & 7 & 0 \\ 0 & 21 & -77 & \\ 0 & 167 & & \end{array}\right)}$$

Cum $\alpha_3=-3$ (din ipoteză) $\Rightarrow \begin{cases} \alpha_1+7\alpha_2=-21 \\ 3\alpha_1+42\alpha_2=-168 \\ -5\alpha_1-53\alpha_2=195 \end{cases}$

$\begin{cases} \alpha_1+7\alpha_2=-21 \quad |\cdot(-3) \\ 3\alpha_1+42\alpha_2=-168 \end{cases} \Rightarrow \begin{cases} -3\alpha_1-21\alpha_2=63 \\ 3\alpha_1+42\alpha_2=-168 \quad + \end{cases}$

$21\alpha_2=-105 \Rightarrow \boxed{\alpha_2=-5}$

$\alpha_1-35=-21 \Rightarrow \boxed{\alpha_1=14}$

Cum $\alpha_1,\alpha_2,\alpha_3 \neq 0 \Rightarrow P_1,P_2,P_3$ = lin. dependente $\Rightarrow$

$\Rightarrow \alpha_1P_1+\alpha_2P_2+\alpha_3P_3=0 \Leftrightarrow \boxed{14P_1-5P_2-3P_3=0}$

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b) Analizăm fiecare relație din W. Pentru asta considerăm un polinom $P \in \mathbb{R}_2[X]$, $P(X) = a_0 + a_1X + a_2X^2$, $a_0, a_1, a_2 \in \mathbb{R}$

$$P'(X) = a_1 + 2a_2X \Rightarrow P'(0) = a_1$$

$$P''(X) = 2a_2 \Rightarrow P''(0) = 2a_2$$

$$P(0) = a_0$$

$$P(0) + 3P'(0) - 5P''(0) = 0 \Leftrightarrow a_0 + 3a_1 - 10a_2 = 0 \Rightarrow a_0 = \frac{7}{7}10a_2 - \frac{36}{7}a_2 = \frac{34}{7}a_2$$

$$-7P'(0) + 6P''(0) = 0 \Leftrightarrow -7a_1 + 12a_2 = 0 \Rightarrow a_1 = \frac{12}{7} \cdot a_2$$

Deci: $P(X) = a_0 + a_1X + a_2X^2 = \frac{34}{7} \cdot a_2 + \frac{12}{7}a_2X + a_2X^2 =$

$$= \frac{a_2}{7}(34 + 12X + 7X^2) \Rightarrow W = Sp\{34 + 12X + 7X^2\} \Rightarrow$$

$\Rightarrow \boxed{\dim_{\mathbb{R}_2[X]} W = 1}$; Notăm $P = 34 + 12X + 7X^2$

c) ~~Deoarece~~ P_1, P_2, P_3 = lin. dependente și $14P_1 - 5P_2 - 3P_3 = 0 \Rightarrow$

$\Rightarrow P_1 = \frac{5}{14}P_2 + \frac{3}{14} \cdot P_3 \Rightarrow B = Sp\{P_1, P_2, P_3\} = Sp\{P_2, P_3\}$, care este o bază deoarece P_2, P_3 = lin. independente.

Deci $W + B = Sp\{P_2, P_3, P\} = \mathbb{R}_2[X] \Leftrightarrow P_2, P_3, P$ = lin. independenți.

Studiem liniar independența lui P_2, P_3, P: Fie $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ a.î.: $\lambda_1P_2 + \lambda_2P_3 + \lambda_3P = 0$ $\Leftrightarrow$

$$\Leftrightarrow \begin{cases} 7\lambda_1 - 7\lambda_2 + 34\lambda_3 = 0 \\ 42\lambda_1 - 56\lambda_2 + 12\lambda_3 = 0 \\ -53\lambda_1 + 65\lambda_2 + 7\lambda_3 = 0 \end{cases}$$

Luăm coordonatele fiecărui polinom în parte și le scriem în baza canonică:

$$[P_2]_{B_c} = \begin{pmatrix} 7 \\ 42 \\ -53 \end{pmatrix}$$

$$[P_3]_{B_c} = \begin{pmatrix} -7 \\ -56 \\ 65 \end{pmatrix}$$

$$[P]_{B_c} = \begin{pmatrix} 34 \\ 12 \\ 7 \end{pmatrix}$$

Calculăm : $\begin{vmatrix} 7 & -7 & 34 \\ 42 & -56 & 12 \\ -53 & 65 & 7 \end{vmatrix} = \cancel{-49\cdot 56 + 42\cdot 65\cdot 34} \begin{vmatrix} -4 & 2 & 53 \\ -11 & -9 & 19 \\ -53 & 65 & 7 \end{vmatrix} =$

$= 36\cdot 7 - 11\cdot 65\cdot 53 - 53\cdot 2\cdot 19 - 53^2\cdot 9 + 19\cdot 65\cdot 4 + 7\cdot 2\cdot 11 =$

$= -59844 \neq 0 \Rightarrow P_2, P_3, P =$ liniar independenți $\Rightarrow$

$\Rightarrow W + B = \mathbb{R}_2[X]$, deci W este un complement al lui $B = Sp\{P_2, P_3\}$

2) a) Se consideră $A = \begin{pmatrix} -6 & -1 \\ -8 & -4 \end{pmatrix}$ și aplicația liniară:

$f: \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R}),\ f(X) = AX - XA$

a) $\operatorname{Ker}(f) = \{ B \in \mathcal{M}_2(\mathbb{R}) \mid f(B) = O_2 \}$

$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow f(B) = AB - BA = O_2 \Rightarrow AB = BA \Leftrightarrow$

$\Leftrightarrow \begin{pmatrix} -6 & -1 \\ -8 & -4 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} -6 & -1 \\ -8 & -4 \end{pmatrix} \Leftrightarrow$

$\Leftrightarrow \begin{pmatrix} -6a - c & -6b - d \\ -8a - 4c & -8b - 4d \end{pmatrix} = \begin{pmatrix} -6a - 8b & -a - 4b \\ -6c - 8d & -c - 4d \end{pmatrix} \Rightarrow$

$\Rightarrow \begin{cases} 6a + c = 6a + 8b \Rightarrow \boxed{c = 8b} \\ 8b + 4d = c + 4d \\ 6c + 8d = 8a + 4c \Rightarrow 8(a-d) = 2c \Leftrightarrow 8(a-d) = 16b \Rightarrow a - d = 2b \\ 6b + d = a + 4b \end{cases}$

Deci $c = 8b$,
$a - d = 2b \Rightarrow a = 2b + d$

$$B = \begin{pmatrix} 2b+d & b \\ 8b & d \end{pmatrix} = \begin{pmatrix} 2b & b \\ 8b & 0 \end{pmatrix} + \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} =$$

$$= b\begin{pmatrix} 2 & 1 \\ 8 & 0 \end{pmatrix} + d\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \operatorname{Ker} f = Sp\left\{\begin{pmatrix} 2 & 1 \\ 8 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} \Rightarrow$$

$\Rightarrow \dim_{\mathbb{R}} \operatorname{Ker}(f) = 2$

$\operatorname{Im}(f) = \{ D \in M_2(\mathbb{R}) \mid \cancel{f(C)} = \exists C \in M_2(\mathbb{R})\ \text{a.î.}\ f(C) = D \}$

$$C = \begin{pmatrix} x & y \\ z & t \end{pmatrix};\ D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$f(C) = D \Leftrightarrow AC - CA = D \Leftrightarrow (-1)\cdot\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix} + \begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix} =$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Leftrightarrow -\begin{pmatrix} 6x+z & 6y+t \\ 8x+4z & 8y+4t \end{pmatrix} + \begin{pmatrix} 6x+8y & x+4y \\ 6z+8t & z+4t \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} 8y - z & x - 2y - t \\ -8x + 2z + 8t & z - 8y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} 8y - z = a \\ x - 2y - t = b \\ -8x + 2z + 8t = c \\ z - 8y = d \end{cases} \Rightarrow a + d = 0 \text{ (prin adunarea relației 1 cu relația 4)} \Rightarrow d = -a$$

$$\left(\begin{array}{cccc|c} 0 & 8 & -1 & 0 & a \\ 1 & -2 & 0 & -1 & b \\ -8 & 0 & 2 & 8 & c \\ 0 & 0 & 1 & -8 & d \end{array}\right) \xrightarrow{L_2 \leftrightarrow L_1} \left(\begin{array}{cccc|c} 1 & -2 & 0 & -1 & b \\ 0 & 8 & -1 & 0 & a \\ -8 & 0 & 2 & 8 & c \\ 0 & 0 & 1 & -8 & d \end{array}\right) \xrightarrow{L_3 + 8L_1}$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & -2 & 0 & -1 & b \\ 0 & 8 & -1 & 0 & a \\ 0 & -16 & 2 & 0 & c+8a \\ 0 & 0 & 1 & -8 & d \end{array}\right) \xrightarrow{L_3+2L_2} \left(\begin{array}{cccc|c} 1 & -2 & 0 & -1 & b \\ 0 & 8 & -1 & 0 & a \\ 0 & 0 & 0 & 0 & c+10a \\ 0 & 0 & 1 & -8 & d \end{array}\right)$$

Avem

$$\begin{cases} c+10a=0 \Rightarrow c=-10a \\ 8y-z=a \\ x-2y-t=b \\ z-8t=d \end{cases}$$

Deci $D=\begin{pmatrix} a & b \\ -10a & -a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ -10 & -1 \end{pmatrix} + b\cdot\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow$

$\Rightarrow \operatorname{Im}(f) = Sp\left\{\begin{pmatrix} 1 & 0 \\ -10 & -1 \end{pmatrix}; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\} \Rightarrow \boxed{\dim_{\mathbb{R}} \operatorname{Im} f = 2}$

ii) verificați dacă $E_{12}, E_{21}, E_{11}-E_{22}, E_{11}+E_{22}$ = bază în $M_2(\mathbb{R})$ și

det. $M_f = ?$

~~$E_{12}, E_{21},$~~ $A_1 = E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $A_2 = E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$; $A_3 = E_{11}-E_{22} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;

$A_4 = E_{11}+E_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Studiem lin. ind matricelor A_1, A_2, A_3, A_4. Fie $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$, a.î.:

$\lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3 = O_2 \Rightarrow \begin{cases} \lambda_3 + \lambda_4 = 0 \\ \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_4 - \lambda_3 = 0 \Rightarrow \lambda_4 = \lambda_3 \Rightarrow 2\lambda_3 = 0 \Rightarrow \lambda_3 = 0 \Rightarrow \lambda_4 = 0, \end{cases}$

$\Rightarrow A_1, A_2, A_3, A_4$ = lin. ind. $\Rightarrow B = Sp\{A_1, A_2, A_3, A_4\}$ = bază

$M^{B}(f) = ?$

$\cancel{f(E_{1,1})}$ $f(E_{1,2}) = A \cdot E_{12} - E_{12} \cdot A = -\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}$

$= -\begin{pmatrix} 0 & 6 \\ 0 & 8 \end{pmatrix} + \begin{pmatrix} 8 & 4 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 8 & -2 \\ 0 & -8 \end{pmatrix}$

c) $Ker(f) \oplus Im f \overset{?}{=} \mathcal{M}_2(\mathbb{R})$?

$Ker(f) = Sp\left\{ \begin{pmatrix} 2 & 1 \\ 8 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$Im(f) = Sp\left\{ \begin{pmatrix} 1 & 0 \\ -10 & -1 \end{pmatrix}; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

$Im(f) + ker f = Sp\left\{ \underbrace{\begin{pmatrix} 2 & 1 \\ 8 & 0 \end{pmatrix}}_{B_1} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{B_2}; \underbrace{\begin{pmatrix} 1 & 0 \\ -10 & -1 \end{pmatrix}}_{B_3}; \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{B_4} \right\}$

Studiem lin. ind. lui B_1, B_2, B_3, B_4: Fie $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ a.î.:

$\alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3 + \alpha_4 B_4 = O_2 \Rightarrow \begin{cases} 2\alpha_1 + \alpha_2 + \alpha_3 = 0 \\ \alpha_1 + \alpha_4 = 0 \Rightarrow \alpha_4 = \frac{5}{4}\alpha_3 \\ 8\alpha_1 - 10\alpha_3 = 0 \Rightarrow \alpha_1 = -\frac{5}{4}\alpha_3 \\ \alpha_2 - \alpha_3 = 0 \Rightarrow \alpha_2 = \alpha_3 \Rightarrow \end{cases}$

$2\alpha_1 + \alpha_2 + \alpha_3 = 0 \Leftrightarrow -\frac{5}{2}\alpha_3 + \alpha_3 + \alpha_3 = 0 \Rightarrow \alpha_3 = 0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_4 = 0 \Rightarrow$

$\Rightarrow B_1, B_2, B_3, B_4 =$ lin. ind. $\Rightarrow Ker(f) \cap Im f = \{O_2\} \Rightarrow Ker(f) \oplus Im(f) =$

$=$ suma directă și deoarece $B_1, B_2, B_3, B_4 =$ lin. ind. $\Rightarrow$

$\Rightarrow ker(f) \oplus Im(f) = \mathcal{M}_2(\mathbb{R})$

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4) $U = Sp\left\{ v_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix};\ v_2 = \begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix},\ v_3 = \begin{pmatrix} -5 \\ 14 \\ 2 \end{pmatrix} \right\}$ și $w = \begin{pmatrix} 12 \\ 0 \\ 12 \end{pmatrix}$.

a) $$\cos(\sphericalangle(v_1, v_2)) = \frac{\langle v_1, v_2 \rangle}{\|v_1\| \|v_2\|} = \frac{-6-12}{\sqrt{9} \cdot \sqrt{9+36}} = \frac{-18}{3\sqrt{45}} =$$

$$= -\frac{18}{3 \cdot 3\sqrt{5}} = -\frac{2}{\sqrt{5}} \Leftrightarrow \boxed{\cos(\sphericalangle(v_1, v_2)) = -\frac{2}{\sqrt{5}}}$$

b) $\langle v_1, v_2 \rangle = -18 \Rightarrow v_1 \not\perp v_2$

Aplicăm algoritmul Gram-Schmidt:

$$u_1 = v_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$$u_2 = v_2 - pr_{u_1} v_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} \cdot u_1 = v_2 - \frac{-18}{9} \cdot u_1 =$$

$$= v_2 + 2u_1 = v_2 + 2v_1 = \begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \Leftrightarrow u_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$u_3 = v_3 - pr_{u_1} v_3 - pr_{u_2} v_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} \cdot u_2 =$$

$$= v_3 - \frac{-36}{9} \cdot u_1 - \frac{27}{9} \cdot u_2 = v_3 + 4v_1 - 3u_2 = \begin{pmatrix} -5 \\ 14 \\ 2 \end{pmatrix} + \begin{pmatrix} 8 \\ -8 \\ 4 \end{pmatrix} -$$

$$- \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix} = 0$$

Completăm la o bază ortogonală astfel: Căutăm $u_3 \perp u_1$ și $u_3 \perp u_2 \Rightarrow$

$$\Leftrightarrow \begin{cases} \langle u_3, u_1 \rangle = 0 \\ \langle u_3, u_2 \rangle = 0 \end{cases}$$

$$u_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{cases} \langle u_3, u_1 \rangle = 0 \Leftrightarrow 2a - 2b + c = 0 \\ \langle u_3, u_2 \rangle = 0 \Leftrightarrow a + 2b + 2c = 0 \ + \end{cases}$$

$$a + c = 0 \Rightarrow \boxed{c = -a} \Rightarrow 2b = -a - 2c = -a + 2a = a \Rightarrow$$

$$\Rightarrow b = \frac{a}{2}$$

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$u_3 = \begin{pmatrix} a \\ \frac{a}{2} \\ -a \end{pmatrix}$

Pt. $a=2 \Rightarrow u_3 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

Formăm o bază ortonormată:

$e_1 = \frac{1}{\|u_1\|} \cdot u_1 = \frac{1}{3} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$e_2 = \frac{1}{\|u_2\|} \cdot u_2 = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$e_3 = \frac{1}{\|u_3\|} \cdot u_3 = \frac{1}{3} \cdot \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

$\{e_1, e_2, e_3\}$ = bază ortonormată

$pr_U u = pr_{e_1} u + pr_{e_2} u + pr_{e_3} u = \frac{\langle u, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1 + \frac{\langle u, e_2 \rangle}{\langle e_2, e_2 \rangle} \cdot e_2 +$

$+ \frac{\langle u, e_3 \rangle}{\langle e_3, e_3 \rangle} \cdot e_3 = \frac{1}{3}(24+12) \cdot e_1 + \frac{1}{3} \cdot (12+24) \cdot e_2 + \frac{1}{3}(24-24) e_3$

$= \cancel{4e_1 + 4e_2 =} \; 12(e_1+e_2) = 4 \cdot \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 16 \\ 0 \\ 12 \end{pmatrix} \Leftarrow$

$\Leftarrow \boxed{pr_U u = \begin{pmatrix} 16 \\ 0 \\ 12 \end{pmatrix}}$

c) $\sum_{i=1}^{3} \|u - e_i\|^2 \geq 3\|u\|^2 - 2\sqrt{3} \cdot \|u\| + 3$

$\cancel{\|u-e\|}$ $|\langle u, v \rangle| \leq \|u\| \cdot \|v\|, \; \forall u, v \in \mathbb{R}^3$

$\|u - e_i\|^2 = \|u - e_i\| \cdot \|u - e_i\| \geq |\langle u - e_i, u - e_i \rangle| =$

$= |\langle u, u - e_i \rangle - \langle e_i, u - e_i \rangle| = |\langle u, u \rangle - \langle u, e_i \rangle - \langle e_i, u \rangle +$

$+ \langle e_i, e_i \rangle| =$

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$= \|u\|^2 - 2\langle u, q_i\rangle + \cancel{8}\ \|q_i\|^2 = \|u\|^2 - 2\langle u, q_i\rangle + 1$

Însumăm:

$\sum_{i=1}^{3} \|u-q_i\|^2 \geq \cancel{3\|u\|^2 - 2\langle u,q_i\rangle + 3}\ 3\cdot\|u\|^2 - 2\cdot\sum_{i=1}^{3}\langle u,q_i\rangle + 3$

$u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$q_1 = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \langle u,q_i\rangle \leq \|u\|\cdot\frac{1}{\sqrt{3}} \Rightarrow -\langle u,q_i\rangle \geq -\frac{1}{\sqrt{3}}\|u\|$

Deci $\sum_{i=1}^{3}\|u-q_i\|^2 \geq 3\|u\|^2 - 2\sqrt{3}\cdot\|u\| + 3$

3) a) $A = \begin{array}{c|ccc} & AR & S & AN \\ \hline AR & 0{,}75 & 0{,}25 & 0{,}20 \\ S & 0{,}05 & 0{,}25 & 0{,}55 \\ AN & 0{,}2 & 0{,}5 & 0{,}25 \end{array}$

$\overset{0{,}75}{AR} \xrightarrow{0{,}25} S \longrightarrow AN$

c) $u(n+1) = A\cdot u(n) \Rightarrow$

$\cancel{n=1}\ n=0 \Rightarrow u(1) = A\cdot u(0)$

$n=1 \Rightarrow u(2) = A\cdot u(1) = A^2 u(0)$

Inductiv avem: $u(n) = A^n\cdot(u_0)$, cu A determinată la a)

$u(2) = A^2\cdot u(0)$, $u(0) = \begin{pmatrix} 1 \\ 0{,}25 \\ 0{,}2 \end{pmatrix} \Rightarrow u(2) = \begin{pmatrix} 0{,}75 & 0{,}25 & 0{,}2 \\ 0{,}05 & 0{,}25 & 0{,}55 \\ 0{,}2 & 0{,}5 & 0{,}25 \end{pmatrix}\cdot\begin{pmatrix} 0{,}75 & 0{,}25 & 0{,}2 \\ 0{,}05 & 0{,}25 & 0{,}55 \\ 0{,}2 & 0{,}5 & 0{,}25 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0{,}25 \\ 0{,}2 \end{pmatrix} = \ldots$

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$$P_A(x) = \begin{vmatrix} 0,75-x & 0,25 & 0,20 \\ 0,05 & 0,25-x & 0,55 \\ 0,2 & 0,5 & 0,25-x \end{vmatrix} = 0, \begin{cases} \lambda_1 = \dots \\ \lambda_2 = \dots \\ \lambda_3 = \dots \end{cases}$$

Aflăm $V_{\lambda_1}, V_{\lambda_2}, V_{\lambda_3}$, unde $V_\lambda = \{u \in \mathbb{R}^3 \mid Au = \lambda \cdot u\}$

Formăm $P = (u_1, u_2, u_3)$, unde $u_1, u_2, u_3 \in V_{\lambda_1}, V_{\lambda_2}, V_{\lambda_3}$

$A = P \cdot D \cdot P^{-1}$, cu $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \Rightarrow A^n = P \cdot D^n \cdot P^{-1} \Rightarrow$

$\Rightarrow u(30) = A^{30} \cdot u(0) = P D^{30} \cdot P^{-1} \cdot u(0)$, cu $u(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$