Math 308 Midterm 2
Version 4
March 19, 2018
Luis Goddyn

NAME (printed) : SOLUTION KEY
(First Name) Family (Last) Name

SFU Computing ID :

Student Number :

Signature :

(1) Do NOT open this test booklet until told to do so.

(2) Do ALL your work in this test booklet. If you run out of space, then use the back of the page.

(3) SHOW ALL YOUR WORK.

(4) The value of each question is shown below and on the question.

(5) Exam Duration: 45 minutes

(6) Although it is not required, a simple 4-function calculator may be used.

(7) After the exam starts, verify that you have all 6 pages (including this cover page).

Question	1	2	3	4	5	TOTAL
Score						
Value	10	10	10	10	10	50

(1) Consider the following payoff matrix for a matrix game payed by Ron (rows) and Coleen (columns).

$$A = \begin{bmatrix} 0 & 3 & 0 & 5 \\ 4 & -1 & 4 & 2 \\ 0 & 1 & 1 & -3 \\ 3 & -4 & 2 & 2 \end{bmatrix}$$

(a) **[4 pts]** Reduce matrix A as far as possible using domination.

Solution:
Row 4 is dominated by row 3; delete it. Then column 3 dominates column 1; delete it.
Then Row 3 is dominated by row 1; delete it. Then column 3 dominates column 2; delete it.

$$\begin{bmatrix} 0 & 3 & 0 & 5 \\ 4 & -1 & 4 & 2 \\ 0 & 1 & 1 & -3 \\ \cancel{3} & \cancel{-4} & \cancel{2} & \cancel{2} \end{bmatrix} \to \begin{bmatrix} 0 & 3 & \cancel{0} & 5 \\ 4 & -1 & \cancel{4} & 2 \\ 0 & 1 & \cancel{1} & -3 \end{bmatrix} \to \begin{bmatrix} 0 & 3 & 5 \\ 4 & -1 & 2 \\ \cancel{0} & \cancel{1} & \cancel{-3} \end{bmatrix} \to \begin{bmatrix} 0 & 3 & \cancel{5} \\ 4 & -1 & \cancel{2} \end{bmatrix} \to \begin{bmatrix} 0 & 3 \\ 4 & -1 \end{bmatrix}$$

(b) **[4 pts]** Set up an initial Game Tableau, either for matrix A or for the reduced matrix you found part (a). When labeling the four sides of the dual Tucker tableau, use $v, q_1, q_2 \dots q_n, t_1, t_2, \dots, t_m$ for Coleen's variables, and use $u, p_1, p_2, \dots, p_n, s_1, s_2, \dots, s_m$ for Ron variables.

Solution:

$$\begin{array}{c|ccc|c|l} & v & q_1 & q_2 & -1 & \\ \hline u & 0 & -1 & -1^{*} & -1 & = -0 \\ p_1 & -1^{**} & 0 & 3^{\circ} & 0 & = -t_1 \\ p_2 & -1 & 4^{\circ} & -1 & 0 & = -t_2 \\ \hline -1 & -1 & 0 & 0 & 0 & = f \\ \hline & = 0 & = s_1 & = s_2 & = g & \end{array}$$

(c) **[2 pts]** Mark with asterisks (* and **) on your tableau of part (b), the pivot entries of the first two pivots which guarantees that the resulting tableau corresponds is a feasible non-canonical tableau. Do not perform the pivots!

Solution:
As shown above. The circles (3°, 4°) mark the maximum entry of each column of the payoff matrix, and 3° is the smallest of these.

(2) Ron and Coleen play the following game with two coins:

- Ron antes **$1** and Coleen antes **$1** into a pot.
- Ron flips a coin. He looks at the coin, but hides it from Coleen. Then Ron chooses:
  - If Ron **Folds**, then Coleen takes the all the money in the pot and the game is finished.
  - If Ron **Bets**, then he adds **$5** to the pot.
- Coleen flips another coin, and also chooses:
  - If Coleen **Folds**, then Ron takes the pot, and the game is finished.
  - If Coleen **Sees**, then she adds **$1** to the pot.
- Payoff:
  - If both coins are Heads, then Ron takes the pot.
  - If both coins are Tails, then Coleen takes the pot.
  - If the coins differ, then the pot is divided equally between Ron and Coleen ($4 each).

(a) [**5 pts**] Suppose that Ron uses the pure strategy **BF** (that is, he always **Bets** if he flips Heads, and he always **Folds** if he flips Tails). Also suppose that Coleen uses the pure strategy **S** (that is, she **Sees** every time that Ron **Bets**). Calculate the average payoff for Ron under these two strategies.

Solution:

Ron's coin	Coleen's coin	Ron's action	Coleen's action	Ron's payoff	Probability
H	H	Bet	See	1+1	1/4
H	T	Bet	See	4-(1+5)	1/4
T	N/A	Fold	N/A	-1	1/2

Ron's expected payoff is the inner product of the last two columns

$$\frac{1+1}{4}+\frac{4-(1+5)}{4}+\frac{-1}{2}=\frac{1}{2}-\frac{1}{2}-\frac{1}{2}=-\frac{1}{2}.$$

(b) [**2+3 pts**] If you compute the entire payoff matrix and game tableau and apply linear programming, then you arrive at the optimum tableau shown here. (Ron is the row player).

$$\begin{array}{c|cc|c|l} & t_{BB} & t_{BF} & -1 & \\ \hline s_F & -4/9 & 4/9 & 1/3 & =-q_F \\ p_{FB} & 5/9 & -14/9 & 7/3 & =-t_{FB} \\ p_{FF} & -4/9 & -5/9 & 7/3 & =-t_{FF} \\ s_S & 4/9 & -4/9 & 2/3 & =-q_S \\ \hline -1 & -4/9 & -5/9 & -7/3 & =f \\ \hline & =p_{BB} & =p_{BF} & =g & \end{array}$$

(i) Suppose that after 100 games, both Ron and Coleen have $0 net winnings. Is Rob or Coleen likely to be the more skillful player? (Circle one).
Explain your choice.

Ron **Coleen**

Solution:
The game has Von Neumann value $-v=-7/3$ so Ron can expect to lose more than $2 per game with optimal play. Since he won $0 over many rounds, Ron is probably the more skillful player.

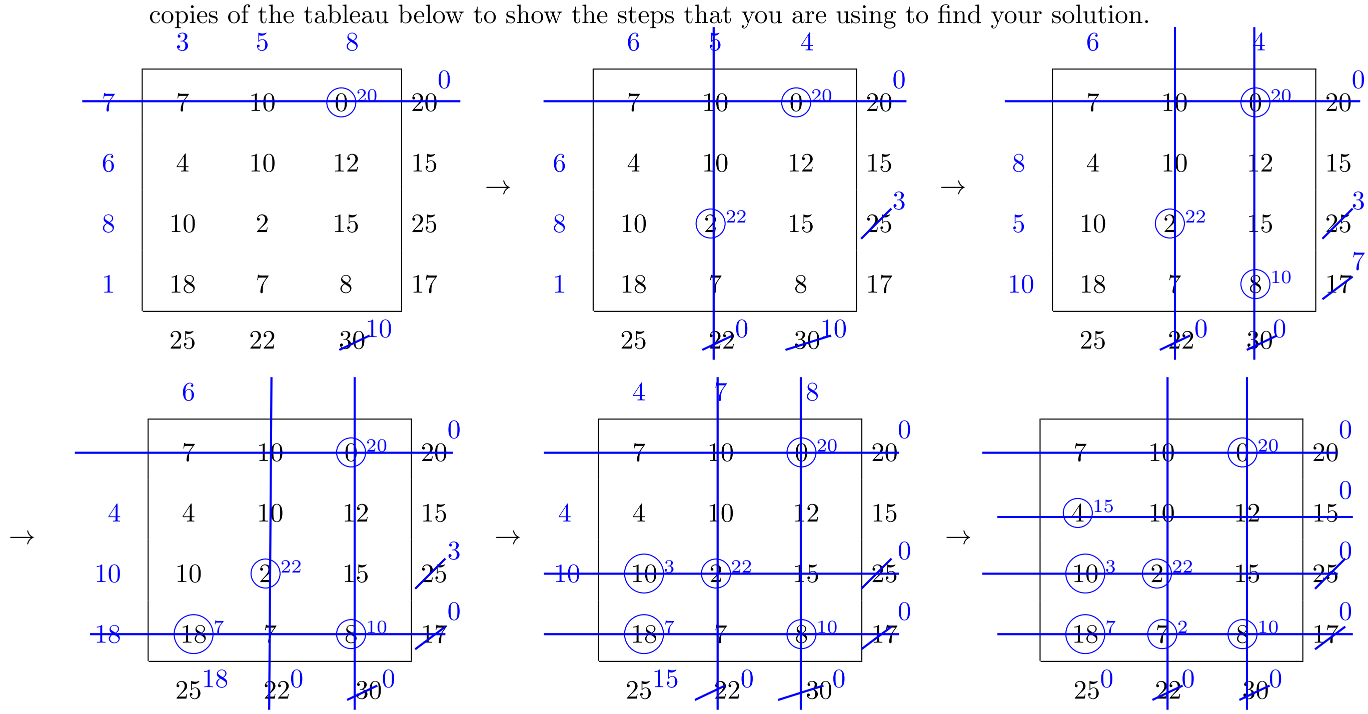
(ii) Write out some instructions which describe Ron's optimal strategy. Use plain English, as if Ron does not know any mathematics, but he knows how to play poker.

Solution:
If Ron's coin is heads, then Ron should always bet. If Ron's coin is tails then he should bet with probability 4/9, and fold with probability 5/9.

	M_1	M_2	M_3	
W_1	7	10	0	20
W_2	4	10	12	15
W_3	10	2	15	~~20~~ 25
W_4	15	7	8	17
	~~20~~ 25	20	30	

(3) The following transportation tableau represents a transportation problem with four warehouses and three markets. The cost c_{ij} of transporting one item from W_i to M_j appears in cell (i, j).

(a) **[5 points]** Apply the Vogel Advanced-start Method to find an initial feasible solution. Use the copies of the tableau below to show the steps that you are using to find your solution.



(b) **[2+2+1 points]** The first transportation tableau below shows a basic feasible solution $\mathbf{x}_1$ (the superscripts) that is probably different from the one that you obtained in part (a). Money can be saved if we transport k items directly from W_1 to M_1, and adjust the values of some of the basic variables in $\mathbf{x}_1$ by $\pm k$ to obtain another feasible solution $\mathbf{x}_2$.

(i) Indicate $\mathbf{x}_2$ by writing superscripts on the circled entries in Tableau 2.

$\mathbf{x}_1$:

7	10	0^{20}	20
4^{15}	10	12	15
10^{10}	2^{15}	15	20
18	7^{7}	8^{10}	17
25	22	30	

$\mathbf{x}_2$:

7^{k}	10	0^{20-k}	20
4^{15}	10	12	15
10^{10-k}	2^{15+k}	15	20
18	7^{7-k}	8^{10+k}	17
20	22	30	

(ii) How much money will be saved for each item that is shipped directly from W_1 to M_1 in this way?

Solution:
A unit increase in k costs $7 - 0 + 8 - 7 + 2 - 10 = 0$. No money is saved by increasing k.

(iii) What is the maximum value of k such that $\mathbf{x}_2$ is feasible?

Solution:
The maximum value of k is $\min\{10, 7, 20\} = 7$. The flow on cell (4,3) can not become negative!

(4) (a) **[4 pts]** Write the (noncanonical) dual linear program to (P) in the space provided. Use dual variables as suggested after each inequality.

$$(P): \quad \max \quad f = 2x_2 - 7$$
$$\begin{aligned} 3x_1 - 4x_2 &\le 4 \quad (y_1) \\ 2x_1 + 5x_2 &= 6 \quad (y_2) \\ -7x_1 &\le 9 \quad (y_3) \\ x_1, x_2 &\ge 0 \end{aligned}$$

$$(D): \min \quad g = 4y_1 + 6y_2 + 9y_3 - 7$$
$$\begin{aligned} 3y_1 + 2y_2 - 7y_3 &\ge 0 \\ -4y_1 + 5y_2 \quad &\ge 2 \\ y_1, y_3 &\ge 0 \end{aligned}$$

(b) **[2+4 pts]**

(i) Write out the payoff matrix for the game of Rock, Paper, Scissors, where the winner of each round wins one dollar.

Solution:

$$\begin{array}{c|ccc} & R & P & S \\ \hline R & 0 & -1 & 1 \\ P & 1 & 0 & -1 \\ S & -1 & 1 & 0 \end{array}$$

(ii) Suppose Ron and Coleen play Rock, Paper, Scissors. Ron's mixed strategy is to select R half of the time, and to select each of P and S one quarter of the time. Coleen's mixed strategy is to select S half of the time, and to select each of R and P one quarter of the time. What is Ron's expected payoff, when they use these strategies?

Solution:

If A is the payoff matrix and the strategies are $\mathbf{p} = (1/2, 1/4, 1/4)$ and $\mathbf{q} = (1/4, 1/4, 1/2)$, then Ron's expected payoff (in dollars) is

$$\begin{aligned} \mathbf{p}^{\mathrm{t}} A \mathbf{q} &= \begin{bmatrix} 1/2 & 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/4 \\ 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} -1/4 + 1/2 \\ 1/4 - 1/2 \\ -1/4 + 1/4 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1/4 \\ -1/4 \\ 0 \end{bmatrix} \\ &= 1/8 - 1/16 \\ &= 1/16 \end{aligned}$$

(5) This question regards the integer linear program labeled (ILP).

$$\text{(ILP)}\quad \text{Maximize } f(x_1,x_2) = 3x_1 - 2x_2$$
$$8x_1 - 4x_2 \le 9$$
$$-8x_1 + 11x_2 \le 5$$
$$x_1, x_2 \ge 0$$
$$x_1, x_2 \text{ integers}$$

(P_0)

$$\text{(ILP)}\quad \text{Maximize } f(x_1,x_2) = 3x_1 - 2x_2$$
$$8x_1 - 4x_2 \le 9$$
$$-8x_1 + 11x_2 \le 5$$
$$x_1, x_2 \ge 0$$

(a) **[2 pts]** Write the linear relaxation (P_0) of (ILP) in the above box.

(b) **[3+1 pts]** Shown below is the optimal tableau for (P_0) (with slack variables t_1 and t_2).

(i) Add a cutting plane to (P_0) by writing out the first tableau for the next linear program (P_1). Use the new slack variable t_3.

t_1	t_2	-1	
$11/28$	$1/14$	$13/4$	$= -x_1$
$2/7$	$1/7$	3	$= -x_2$
$-41/28$	$-5/14$	$-51/4$	$= f$

$\mapsto$

t_1	t_2	-1	
$11/28$	$1/14$	$13/4$	$= -x_1$
$2/7$	$1/7$	3	$= -x_2$
$-11/28^*$	$-1/14$	$-1/4$	$= -t_3$
$-41/28$	$-5/14$	$-51/4$	$= f$

Note: Only x_1 is fractional at this BFS and the fractional part of $13/4$ equals $1/4$.

(ii) On the tableau that you found in part (i), mark with an asterisk "*" the first pivot element that simplex algorithm will use when solving (P_1). Use Bland's anti-cycling rule with variables in the order x_1, x_2, t_1, t_2, t_3, if needed.
Note: Only t_1 precedes t_2 in the Bland ordering.

(c) **[1+3 pts]** The constraint set for (P_0) is plotted below on the (x_1, x_2)-grid.

(i) Mark the optimal solution to (P_0) with a hollow circle ○

(ii) Find in any way you want, the optimal solution to (ILP) and mark the optimum solution with a solid circle ●