# Brownian Motion, Obstacles and Random Media

ALAIN-SOL SZNITMAN

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SZNITMAN

Brownian Motion, Obstacles and Random Media

This book is aimed at graduate students and researchers. It provides an account for the non-specialist of the circle of ideas, results and techniques, which grew out in the study of Brownian motion and random obstacles. This subject has a rich phenomenology which exhibits certain paradigms, emblematic of the theory of random media. It also brings into play diverse mathematical techniques such as stochastic processes, functional analysis, potential theory, first passage percolation.

In a first part, the book presents, in a concrete manner, background material related to the Feynman-Kac formula, potential theory, and eigenvalue estimates. In a second part, it discusses recent developments including the method of enlargement of obstacles, Lyapunov coefficients, and the pinning effect. The book also includes an overview of known results and connections with other areas of random media.

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Alain-Sol Sznitman

# Brownian Motion, Obstacles and Random Media

With 45 Figures

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*Alain-Sol Sznitman*
FIM
ETH Zurich
8092 Zurich
Switzerland

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# Preface

The principal purpose of this book is to provide an account of the circle of ideas, results and techniques, which emerged roughly over the last ten years in the study of *Brownian motion and random obstacles.* The accumulation of results in many separate sources eventually made it impractical, if not impossible, for the nonspecialist to gain access to the developments of the subject. This book is an attempt to remedy this situation.

Part of the thrill of the investigation of Brownian motion and random obstacles certainly stems from its many connections with various areas of mathematics, but also from the formal and mysterious physical heuristics which relate to it. In particular the loose concept of *pockets of low local eigenvalues* plays an important role in the study of Brownian motion and random obstacles, and also represents a paradigm which has natural resonances with several other areas of random media. This last feature has increasingly become clear over the last few years.

This book grew out of a series of graduate lectures at the ETH Zürich, which took place between the Fall '93 and the Spring '97, and surveyed a selection of topics from the theory of random media. I tried to write it at a level accessible to graduate students with a knowledge of stochastic processes and functional analysis. The first part entitled *Some Tools* makes the material in the book reasonably self-contained. Among other things, it includes proofs of a number of results which I found not always easy to track down through the literature, or which sometimes come as consequences of rather indirect methods. I generally tried to make the exposition concrete: this is maybe one of the many durable influences of my stay at the Courant Institute.

Of course the principal object of the book appears in the second part *Brownian motion and random obstacles.* Rather than attempting to present in details 'all that is known', I chose to explain some key techniques like the *method of enlargement of obstacles* or the *Lyapunov exponents* and discuss some of their applications. A particularly striking application of these methods, the *pinning effect*, is discussed extensively in Chapter 6. However another very interesting application, the *confinement property*, only comes in the last

Chapter 7 which is more of a survey nature, and describes several connections with other areas of random media.

At various stages, the successive versions of this book have greatly benefited from the remarks or comments of M. van den Berg, P. Carmona, J.D. Deuschel, L. Erdös, D. Ioffe, T. Komorowski, Y. Le Jan and L. Miclo. I am especially thankful to T. Povel, S. Sethuraman, M. Wüthrich and M. Zerner for their numerous comments. Marianne Pfister has not only wonderfully typed the book and composed its illustrations, but also has coped with the impatience or the erratic changes of the author. Erwin Bolthausen was a constant support during the writing of this book, during good times and bad times. Finally this project would simply not be without the support of my wife Véronique who was a first-hand innocent bystander of the uneven moods of the author.

Zürich, June 1998

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## Part II. BROWNIAN MOTION AND RANDOM OBSTACLES

# Introduction

The subject of random media has been the object of much interest and efforts over the last two decades. The presence of randomness in a medium is the source of a broad variety of effects, which escape the realm of behaviors displayed by constant or periodic media. The investigation of the consequences in the large of the introduction of randomness in the environment is a central question, if not a definition of the field of random media. This question represents a mathematical challenge, even when the conjectured resulting effects seem natural extensions of previously known behaviors.

The aim of the present book is to give an account of some of the recent developments in a specific chapter of random media: Brownian motion among Poissonian obstacles. This subject exhibits a rich phenomenology and displays paradigms present in several other areas of random media; it leads to the elaboration of novel mathematical tools and ideas, and has natural links to a variety of topics of disordered media such as trapping problems, random Schrödinger equations, Lifshitz tail effects, intermittency, first passage percolation, directed polymers ... All this is part of the motivation for writing this book and the thrill of the subject, which we shall now try to describe.

One typical class of random obstacles studied in this book are the soft Poissonian obstacles. They are random functions $V$ on $d$-dimensional space, obtained by translating a fixed nonnegative shape function $W$ at the points of a Poissonian cloud $\omega = \sum \delta_{x_i}$ on $\mathbb{R}^d$, with fixed intensity $\nu > 0$:

$$(1) \qquad V(x,\omega) = \sum_i W(x - x_i), \quad x \in \mathbb{R}^d, \quad \omega = \sum_i \delta_{x_i} \, .$$

The nonnegative functions $W$ are chosen bounded, measurable, compactly supported, nondegenerate, and $\omega$ runs over the set of cloud configurations, i.e. simple pure point Radon measures on $\mathbb{R}^d$.

Another class of random obstacles of interest are the hard Poissonian obstacles. They correspond to the deletion in $\mathbb{R}^d$ of all translates $x_i + C$ of a fixed model compact set $C$ at the point $x_i$ of the support of the Poissonian cloud $\omega = \sum_i \delta_{x_i}$. The compact set $C$ is assumed to fulfill a nondegeneracy condition (i.e. $C$ is nonpolar), and for the sake of this introduction, can be thought to be a closed ball of positive radius. Informally the hard obstacle situation relates to the soft Poissonian potential situation via the singular shape function $W = \infty \cdot 1_C$.

Soft Poissonian obstacles represent an archetype of translation invariant random fields, bounded from below, with good independence properties. Their most direct physical interpretation is perhaps expressed in terms of trapping problems. They can be viewed as random absorption rates for particles diffusing in a partially absorbing medium. In this light, the hard obstacles describe the random locations of totally absorbing traps. The problem of investigating long time survival to trapping for diffusing particles is widely portrayed in the literature of physical chemistry and disordered media physics, cf. Den Hollander-Weiss [HW94], Havlin-Ben Avraham [HA87]. It also closely relates to the themes developed in this book.

One of the central questions in the study of Brownian motion among Poissonian obstacles is the investigation of the large $t$ behavior of the respective path measures:

$$Q_{t,\omega} = \frac{1}{S_{t,\omega}} \exp\Big\{ - \int_0^t V(Z_s,\omega)ds\Big\}\, P_0\,, \tag{2}$$

with $\omega$ typical, $Z_.$ the canonical $d$-dimensional Brownian motion, $P_0$ the Wiener measure, $S_{t,\omega}$ the normalizing constant:

$$S_{t,\omega} = E_0\Big[\exp\Big\{ - \int_0^t V(Z_s,\omega)ds\Big\}\Big]\,, \tag{3}$$

and of

$$Q_t = \frac{1}{S_t} \exp\Big\{ - \int_0^t V(Z_s,\omega)ds\Big\}\, P_0 \otimes \mathbb{P}\,, \tag{4}$$

with $\mathbb{P}$ the law of the Poisson cloud and $S_t$ the normalizing constant:

$$S_t = \mathbb{E} \otimes E_0\Big[\exp\Big\{ - \int_0^t V(Z_s,\omega)ds\Big\}\Big]\,, \tag{5}$$

where $\mathbb{E}$ stands for the expectation with respect to the Poisson cloud.

The path measures $Q_{t,\omega}$ and $Q_t$ respectively describe 'quenched' and 'annealed' Brownian motion in a Poissonian potential. The terminology 'quenched' and 'annealed' is inherited from metallurgy; the distinction between quenched and annealed point of views is a recurrent feature of models of random media. To obtain a better grasp of what is at stake, it is useful to provide the respective interpretations of these measures in terms of random trappings.

The measure $Q_{t,\omega}$ describes the behavior of the trajectory of a particle diffusing from the origin in a typical realization of the partially absorbing medium, conditioned on the atypical event that it has not been absorbed up to a (long) time $t$. On the other hand the $\mathbb{P}$-integration in (4), (5) should be viewed as resulting from a spatial ergodic theorem. Thus $Q_t$ describes the relative displacement and environment, measured from its starting point, for a particle, launched at a location uniformly distributed over a large box centered at the origin, which diffuses in a typical realization of the partially absorbing medium, conditioned on the atypical event that it is not absorbed up to a (long) time $t$.

In contrast to what would be the case for a periodic function $V$, the path $Z_.$ under the respective quenched and annealed path measures displays substantially different behaviors, when $t$ is large. Intuitively the fluctuations present in $V$ create certain pockets where long time survival to trapping is less difficult. As a result, the annealed measure will tend to favour a starting point in a 'good neighborhood', on the other hand the starting point will be 'by force' the origin under the quenched measure. Other physical interpretations of the measures in (2) and (4) can be given. For instance, they can be viewed as descriptions of directed polymers in the presence of 'columnar disorder' (cf. Chapter 6 §1), in this case $t$ represents the transversal thickness of a material rather than time.

The measures $Q_{t,\omega}$ and $Q_t$ are not the only path measures of interest in the subject. For instance,

$$\hat{P}^{\lambda}_{x,\omega} = \frac{1\{H(x) < \infty\}}{e_{\lambda}(0,x,\omega)} \exp\Big\{ - \int_0^{H(x)} (\lambda + V(Z_s,\omega))ds \Big\} \, P_0 \,, \tag{6}$$

with $\lambda \geq 0$, $x \in \mathbb{R}^d$, $H(x)$ the entrance time of $Z_.$ in the ball of radius 1 around $x$, $e_{\lambda}(0,x,\omega)$ the normalizing constant and:

$$\hat{P}^{\lambda}_{x} = \frac{1\{H(x) < \infty\}}{\overline{e}_{\lambda}(x)} \exp\Big\{ - \int_0^{H(x)} (\lambda + V(Z_s,\omega))ds \Big\} \, P_0 \otimes \mathbb{P} \,, \tag{7}$$

with $\overline{e}_{\lambda}(x)$ the normalizing constant, define the respective quenched and annealed path measures for Brownian crossings in a Poissonian potential. They describe the behavior of a particle conditioned to perform a (long) crossing without being trapped, respectively in the quenched and annealed picture. These measures are naturally related to 'directed polymers in random environments', cf. Chap. 7 §3 and Krug-Spohn [KS91]. Their overall mathematical understanding for large $x$ is at this time rather primitive.

Compared to the investigation of path measures, questions related to the principal asymptotic behavior of the normalizing constants coming in the definition of the path measures (2), (4), (6), (7), are simpler. They also provide further connections to other domains of disordered media. For instance the constants $S_{t,\omega}$ and $S_t$ in (3) and (5) can be represented, thanks to the Feynman-Kac formula as:

$$(8) \qquad S_{t,\omega} = u_\omega(t,0) \ \text{ and } \ S_t = \mathbb{E}[u_\omega(t,0)] \, ,$$

where $u_\omega$ is the nonnegative bounded solution of

$$(9) \qquad \begin{cases} \partial_t \, u_\omega &= \frac{1}{2} \, \Delta u_\omega - V u_\omega \, , \\ u_\omega(0,x) &= 1 \, . \end{cases}$$

The study of respective asymptotics for $S_{t,\omega}$ and $S_t$ thus can be viewed as a prototypical problem of intermittency theory, cf. Molchanov [Mol94], Zeldovich et al. [ZMRS88]. The principal asymptotic behavior of $S_t$ was first derived in the celebrated paper [DV75c] of Donsker-Varadhan. It is also closely linked to that of

$$(10) \qquad L(t) = \mathbb{E}[r_\omega(t,0,0)] \, ,$$

where $r_\omega(s,x,y)$ stands for the fundamental solution of (9). This last asymptotics naturally appears in the context of the study of Lifshitz tail effects for the density of state of random Schrödinger operators. Indeed, one has the identity

$$(11) \qquad \int_0^\infty e^{-\lambda t} d\ell(\lambda) = L(t), \ t > 0 \, ,$$

where the deterministic measure $\ell$ stands for the density of states, i.e. the $\mathbb{P}$-a.s. vague limit of

$$(12) \qquad \ell^N = \frac{1}{|B(0,N)|} \sum_{i \geq 1} \delta_{\lambda_\omega^{i,N}} \, ,$$

with $\lambda_\omega^{i,N}$, $i \geq 1$, the Dirichlet eigenvalues, counted with multiplicity, of $-\frac{1}{2}\,\Delta + V$ in $B(0,N)$, the ball of radius $N$ centered at 0. The unusual smallness of $\ell([0,\lambda])$ for $\lambda \to 0$, indicates the rarefaction of low eigenvalues, a phenomenon already discussed at a physical level by I.M. Lifshitz in 1964, see [Lif65] Section 3 and [Lif68]. It can be derived from the large $t$ asymptotics of $L(t)$, via a Tauberian result, (see Chapter 4 §5).

One interest of the investigation of the large $t$ behavior of quantities like (2), (3), (4), (5) is that it displays one of the recurrent paradigms of random media. Specifically these asymptotics are dominated by the occurrence of atypical pockets in the medium. The size of these pockets is on the one hand small when compared to the relevant portion of the medium under consideration (in our case roughly a box of size $t$ centered at the origin), but on the other hand large compared to the unit scale, where the medium displays independence properties. This feature occurs in several situations where evolutions in random media are investigated, for instance random walks in random environments, cf. [DPZ96], stochastic dynamics of spin systems with random interaction, cf. [Mar98], models of intermittency, cf. [Mol94]. The nature of what these 'pockets' are, varies from case to case, but they usually are linked to abnormally low eigenvalues locally describing the system.

In the case of Brownian motion in a soft Poissonian potential, these pockets, in the most naive view correspond to 'large areas where $V = 0$'. This is only a naive description since the location where $V$ vanishes may very well be substantial, and not at all 'pocket-like'. As for the local description of Brownian motion in a Poissonian potential, it is closely related to expressions such as:

$$R^U_{s,\omega} f(x) = E_x\Big[f(Z_s)\ \exp\Big\{-\int_0^s V(Z_u,\omega)du\Big\},\ T_U > s\Big]\,, \tag{13}$$

where $U$ is some open subset of $\mathbb{R}^d$ and $T_U$ the associated exit time of the process $Z_.$ from $U$. These Feynman-Kac type expressions naturally give rise to self-adjoint semigroups on $L^2(U, dx)$. Their generators can be identified with the operators $-\frac{1}{2}\,\Delta + V(\cdot,\omega)$ with Dirichlet boundary condition on $U$. In particular this brings spectral theory to disposal when trying to control expressions like (13). One is then naturally led to consider the 'principal eigenvalues'

$$\lambda_\omega(U) = \begin{array}{l}\text{bottom of the spectrum of } -\frac{1}{2}\,\Delta + V(\cdot,\omega)\\ \text{with Dirichlet boundary conditions on } U\,.\end{array} \tag{14}$$

Such numbers usually govern $L^2$-decay of semigroups, but in our case encode in a very quantitative way the decay properties of the $R^U_{s,\omega}$. Indeed, it is shown in Chapter 3 §1 that

$$\|R^U_{s,\omega}\|_{\infty,\infty} \le c(1 + (\lambda_\omega(U)s)^{d/2})\ e^{-\lambda_\omega(U)s},\ s,\omega \text{ arbitrary}\,, \tag{15}$$

with a constant $c$ which is merely dimension-dependent.

To illustrate the relation of the loose concept of 'pockets' to the notion of 'low local eigenvalue', one can for instance consider hard obstacles given by balls of radius $a$ centered at the points of a Poisson cloud of intensity $\nu$. The intersection of the vacancy set (i.e. the complement of the obstacles) with a box $(-\frac{L}{2}, \frac{L}{2})^d$ determines a finite number of connected components. Its principal Dirichlet eigenvalue $\lambda_L(\omega)$ for $-\frac{1}{2}\,\Delta$, is the smallest of the principal eigenvalues attached to these various components. When $a$ is large, the vacancy set has $\mathbb{P}$-a.s. only bounded components, and the maximal volume of the components meeting the box $(-\frac{L}{2}, \frac{L}{2})^d$ grows logarithmically with $L$, cf. Chapter 5 of [Kes82]. Since balls have lowest principal Dirichlet eigenvalue among open sets of given volume, and empty balls of logarithmic volume typically occur within $(-\frac{L}{2}, \frac{L}{2})^d$, for large $L$, one easily infers that for typical $\omega$:

$$\frac{c}{(\log L)^{2/d}} \le \lambda_L(\omega) \le \frac{c'}{(\log L)^{2/d}} \quad \text{as } L \to \infty \,. \tag{16}$$

On the other hand, when $a$ is sufficiently small, it is known that percolation takes place for the vacancy set: for typical $\omega$ there exists a component of the vacancy set with infinite volume, and it occupies a nonvanishing fraction of $(-\frac{L}{2}, \frac{L}{2})^d$ when $L$ is large. It is a remarkable fact that in spite of this qualitative change of structure of the vacancy set, i.e. the $V = 0$ region, the asymptotic behavior (16) persists. In fact, one can show that (see Chapter 4 §4)

$$\lambda_L(\omega) \sim \frac{c(d,\nu)}{(\log L)^{2/d}}\,, \quad \text{as } L \to \infty, \ \mathbb{P}\text{–a.s}\,, \tag{17}$$

regardless of the size of $a$. So the large $L$ behavior of the principal eigenvalue $\lambda_L$ is still governed by 'pockets of logarithmic volume'. However the definition of what these 'pockets' are, is a more subtle undertaking. Loosely speaking, it can be compared to the elaboration of the notion of clearings within a forest.

Such questions are handled in this book by a 'surgery technique', which in particular enables one to discard the forest part and focus on the clearings. This is the method of enlargement of obstacles of Chapter 4. It gives a very efficient way of deriving uniform controls on the numbers $\lambda_\omega(U)$. The essence of the method is to remodel the regions $V > 0$ and $V = 0$. One builds a coarse-grained picture with much lower combinatorial complexity than the original cloud configurations, easier to handle probabilistically, but with corresponding principal eigenvalues close to the original objects. The remodelling of the regions $V > 0$ and $V = 0$, involves the construction of a trichotomy of $\mathbb{R}^d$. In a first region, true obstacles are 'quickly felt' by Brownian motion, and one imposes extra Dirichlet conditions on this region with little impact on eigenvalues. A second region, where obstacles are insufficiently present and where modification of obstacles potentially influences eigenvalues, is shown to have little volume and hence little importance in probabilistic estimates. The third and last region receives no point of the cloud. Potential theory enters in an essential way in devising this coarse-graining method. The above trichotomy in some sense parallels the one associated to any compact set $K$, by considering the set of regular points of $K$, the set of irregular points of $K$, and the complement of $K$.

One of the aims of the present book is the description in Chapter 6 of the 'pinning effect' which governs the typical large $t$ behavior of the quenched path measure $Q_{t,\omega}$. Most of the techniques which have emerged in the analysis of Brownian motion among Poissonian obstacles enter the study of this effect. Finer asymptotics of the normalizing constant $S_{t,\omega}$ also come as a by-product: for typical $\omega$ and large $t$, $-\log S_{t,\omega}$ is close to the value of the intermittent random variational problem:

$$\inf_{x \in \mathbb{R}^d} \left(\alpha_0(x) + t\lambda_\omega(B(x, R_t))\right), \tag{18}$$

where $\alpha_0(\cdot)$ is a deterministic norm on $\mathbb{R}^d$, and $R_t = \exp\{(\log t)^{1-\chi}\}$, with $\chi > 0$ small enough, is a 'small scale', growing slower than any positive power of $t$. Loosely speaking, the variational problem (18) displays a competition between distance to the origin and occurrence of low local eigenvalues. The norm $\alpha_0(\cdot)$ is one of the (quenched) Lyapunov coefficients constructed in Chapter 5. For typical $\omega$, it governs the exponential decay in the $x$-variable of the normalizing constant $e_0(0, x, \omega)$ of (6) or of the Green function $g_0(0, x, \omega) = \int_0^\infty r_\omega(s, 0, x)ds$:

$$\begin{aligned} \mathbb{P}\text{-a.s.,} \quad e_0(0,x,\omega) &= \exp\{-\alpha_0(x)(1+o(1))\}, \\ g_0(0,x,\omega) &= \exp\{-\alpha_0(x)(1+o(1))\}, \quad \text{as } x \to \infty. \end{aligned} \tag{19}$$

The Lyapunov coefficients of Chapter 5 also enter the formulation of several asymptotic results related to the off-diagonal behavior of Brownian motion in a Poissonian potential, as well as the description of the transition between the small and large drift situation, when a constant drift is added to Brownian motion, cf. Chapter 5 §4, and Chapter 7 §3.

Coming back to the pinning effect, the random variational problem (18) lies at its heart. Under $Q_{t,\omega}$, the path $Z_.$ tends to be attracted to its near minima. In a sense it plays a role similar to the variational problems entering the so far nonrigorous predictions for the behavior of the directed polymer in random environment of the physics literature around the KPZ equation, cf. Krug-Spohn [KS91].

Let us now turn to the organization of this book. The first three chapters make the presentation of the material reasonably self-contained. They can be skipped by the more experienced reader, with occasional glances at specific topics when necessary. On the other hand the last Chapter 7 contains an overview of results on Brownian motion and Poissonian obstacles, as well as connections with neighboring topics. It might well be the first chapter to look at, after reading this introduction!

Chapter 1 develops the link between the probabilistic description (13) of the Feynman-Kac semigroups and their functional analytic construction by means of quadratic forms. This is a remarkably general bridge between probabilistic and functional analytic point of views, which is used throughout the book.

Chapter 2 develops several notions of potential theory. The emphasis is given to capacity, the equilibrium problem, and last exit distribution. The first two of these notions respectively enter the method of enlargement of obstacles in Chapter 4 and the analysis of Lyapunov coefficients in Chapter 5, (the normalizing coefficients $e_\lambda$ in (6) can be viewed as equilibrium potentials).

Chapter 3 presents a number of probabilistic methods for the derivation of lower bounds on principal Dirichlet eigenvalues of $-\frac{1}{2}\,\Delta+V$ in an open set $U$. The interplay between principal eigenvalue and capacity is also investigated.

Chapter 4 is devoted to the method of enlargement of obstacles. As explained above, this turns out to be a powerful tool to control the numbers $\lambda_\omega(U)$. The notion of forest and clearings is developed and later plays an important role in Chapter 6. Applications to the study of the principal asymptotic behavior of $S_{t,\omega}$, $S_t$, and of principal Dirichlet eigenvalues of $-\frac{1}{2}\,\Delta+V$ in large boxes are given.

Chapter 5 discusses Lyapunov coefficients. They are constructed with the help of the subadditive ergodic theorem, in a fashion reminiscent of the proofs of the shape theorems of first passage percolation. They describe off-diagonal properties of Brownian motion in a Poissonian potential, and enter large deviation principles governing the position of $Z_.$ at time $t$ under the respective measures $Q_{t,\omega}$ and $Q_t$. These large deviation principles also have natural applications to the study of Brownian motion with a constant drift among Poissonian obstacles.

Chapter 6 describes the pinning effect for quenched Brownian motion in a Poissonian potential. The analysis involves at some point a coarse-graining procedure of the path $Z_.$, which keeps a reduced information on a system of excursions of the path 'in and out of the clearings'. As a by-product, information on the finer asymptotic behavior of $S_{t,\omega}$ is obtained.

Chapter 7 presents an overview of results on Brownian motion among Poissonian obstacles, which are not necessarily discussed in detail in this book (for instance the confinement property governing the measure $Q_t$). It also discusses connections with a number of other related topics of the literature, and some open problems.

Several techniques and results discussed in this book extend beyond the setting of Brownian motion and Poissonian obstacles. This framework however simplifies the exposition and hopefully still carries the flavour of the generality of the arguments.

# Part I
# SOME TOOLS

In this first part we discuss three main topics which will be helpful in our subsequent investigation of Brownian motion and Poissonian obstacles. We begin in Chapter 1 with the Feynman-Kac formula and provide a functional analytic description of the Dirichlet-Schrödinger semigroups it naturally defines. This will be of constant use in the sequel. We then turn in Chapter 2 to potential theory and more specifically to capacity and the equilibrium problem. This will later come into play in the method of enlargement of obstacles in Chapter 4 and in the construction of Lyapunov exponents in Chapter 5. Finally we discuss in Chapter 3 various estimates on the bottom of the spectrum for the generators of Dirichlet-Schrödinger semigroups. Some of these controls will naturally appear in Chapter 4 and in the study of the pinning effect in Chapter 6.

# 1. The Feynman-Kac Formula and Semigroups

In this chapter we introduce a class of self-adjoint semigroups naturally attached to the Feynman-Kac formula. Section 1 presents in a regular setting the probabilistic functionals we shall study, and begins the discussion of their functional analytic description. Section 2 introduces the class of potential we shall consider in the sequel. Section 3 studies some properties of the semigroups, which are defined in terms of expectations of Brownian motion functionals already encountered in Section 1. In Section 4 we provide by means of quadratic forms, a functional analytic characterization of the semigroups defined in Section 3. We shall amply use throughout the remaining chapters the bridge between functional analytic and probabilistic point of views developed in this chapter.

## 1.1 Feynman-Kac Formula in a Classical Context

The goal of this section is mainly to motivate the study of various functionals we shall recurrently encounter in the sequel. The Feynman-Kac formula can be viewed in several different ways. For the time being, we introduce it as a probabilistic representation of certain perturbations of the Feller semigroup associated to Brownian motion. On the other hand, it can also be viewed as a probabilistic representation for the solutions of certain parabolic partial differential equations, as is explained in the exercise at the end of this section.

We begin with some notations relative to Brownian motion and Wiener measure. For a thorough treatment of these objects we refer for instance to Durrett [Dur84] or Revuz-Yor [RY98]. Throughout the sequel $Z_.$ denotes the canonical $d$-dimensional Brownian motion defined on the space $C(\mathbb{R}_+, \mathbb{R}^d)$ of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}^d$. We let $(\theta_t)_{t\geq 0}$ stand for the canonical shift on the space $C(\mathbb{R}_+, \mathbb{R}^d)$, $P_x$ for the Wiener measure starting from $x \in \mathbb{R}^d$, and $E_x$ for the corresponding expectation. The respective canonical and right continuous canonical filtrations on $C(\mathbb{R}_+, \mathbb{R}^d)$ are denoted by $\mathcal{F}_t, t \geq 0$, and $\mathcal{F}_t^+, t \geq 0$. We shall now recall some classical facts about the Feller semigroup attached to Brownian motion. Additional material on semigroup theory can be found in many references, for instance Reed-Simon, Vol.

2 [RS79] or Rudin [Rud74a]. The Brownian transition density is:

$$p(u,x,y) = (2\pi u)^{-d/2} \exp\Big\{ - \frac{(y-x)^2}{2u}\Big\}, \; u > 0, \; x, y \in \mathbb{R}^d . \tag{1.1}$$

The formula

$$\begin{aligned} R_t\, f(x) &= E_x[f(Z_t)], \;\; \text{for } t \geq 0, x \in \mathbb{R}^d \\ &= \int p(t,x,y)\, f(y)\, dy, \;\; \text{when } t > 0, \end{aligned} \tag{1.2}$$

for $f \in C_0(\mathbb{R}^d)$, the space of continuous functions tending to 0 at infinity, defines a strongly continuous semigroup on $C_0(\mathbb{R}^d)$. That is $R_t\, f \in C_0$ when $f \in C_0$ and $R_t\, f$ converges uniformly to $f$ as $t$ tends to 0. This is the Feller semigroup associated to Brownian motion. The generator $L_0$ of this semigroup has the domain $\mathcal{D}(L_0)$ of functions $f \in C_0(\mathbb{R}^d)$, for which $\frac{1}{t}\,(f - R_t\, f)$ converges in $C_0$ as $t$ tends to 0, and $L_0\, f = \lim_{t\to 0} \frac{1}{t}\,(f - R_t f)$, for $f \in \mathcal{D}(L_0)$, (note the sign convention).

To provide a more explicit description of $L_0$, it is convenient to consider the space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing functions on $\mathbb{R}^d$, that is of smooth functions, which together with all their derivatives decay at infinity faster than any negative power of the distance to the origin, cf. Rudin [Rud74a]. With the help of the Fourier inversion formula and the identity

$$\widehat{R_t\, f}(\xi) = e^{-\frac{1}{2}\, t|\xi|^2}\, \hat{f}(\xi), \;\; f \in \mathcal{S}(\mathbb{R}^d), \; t \geq 0 \,,$$

with $\hat{h}$ the Fourier transform of $h \in \mathcal{S}(\mathbb{R}^d)$, one readily finds that $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{D}(L_0)$ and

$$L_0\, f = -\frac{1}{2}\, \Delta f, \;\; \text{for } f \in \mathcal{S}(\mathbb{R}^d) \,. \tag{1.3}$$

Further by direct inspection, for $t > 0$, $R_t$ maps $C_0(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$. It thus follows from the identity $R_t L_0\, f = L_0 R_t\, f$, $f \in \mathcal{D}(L_0)$, that $\mathcal{S}(\mathbb{R}^d)$ is a core of $\mathcal{D}(L_0)$, i.e. a dense subspace of $\mathcal{D}(L_0)$ endowed with the graph norm $\sup_x |f(x)| + \sup_x |L_0 f(x)|$, which turns it into a Banach space.

We are now ready to introduce the Feynman-Kac formula. We consider a bounded continuous function $V$ on $\mathbb{R}^d$.

**Theorem 1.1:** *The formula*

$$\begin{aligned} &R_t^V\, f(x) = E_x\Big[f(Z_t)\, \exp\Big\{ - \int_0^t V(Z_s) ds\Big\}\Big], \\ &t \geq 0, \; x \in \mathbb{R}^d, \; f \in C_0(\mathbb{R}^d)\,, \end{aligned} \tag{1.4}$$

*defines a strongly continuous semigroup on* $C_0(\mathbb{R}^d)$, *with generator*

$$L_V\, f = L_0\, f + Vf, \ \ \text{for } f \in \mathcal{D}(L_V) = \mathcal{D}(L_0)\,. \tag{1.5}$$

*One also has the perturbation identities*

$$\begin{aligned} R_t^V f(x) &= R_t\, f(x) - \int_0^t R_s(V R_{t-s}^V f)(x) ds\,, \\ &= R_t\, f(x) - \int_0^t R_s^V (V R_{t-s}\, f)(x) ds, \\ & t \geq 0, x \in \mathbb{R}^d,\ f \in C_0(\mathbb{R}^d)\,. \end{aligned} \tag{1.6}$$

**Proof:** We begin with the proof of (1.6). Observe that for $t \geq 0$, we have

$$\begin{aligned} \exp\Big\{-\int_0^t V(Z_s)ds\Big\} &= 1 - \int_0^t V(Z_s)\, \exp\Big\{-\int_s^t V(Z_u)du\Big\} ds \\ &= 1 - \int_0^t V(Z_s)\, \exp\Big\{-\int_0^s V(Z_u)du\Big\} ds\,. \end{aligned} \tag{1.7}$$

Multiplying both members of the first equality of (1.7) by $f(Z_t)$ and integrating we find:

$$\begin{aligned} R_t^V f(x) &= R_t\, f(x) - \int_0^t E_x\Big[V(Z_s)\, \exp\Big\{-\int_s^t V(Z_u)du\Big\}\, f(Z_t)\Big] ds \\ &\overset{\binom{\text{Markov}}{\text{property}}}{=} R_t\, f(x) - \int_0^t R_s(V R_{t-s}^V f)(x) ds\,, \end{aligned}$$

which is the first identity of (1.6). Analogously we find the second identity of (1.6) with the help of the second line of (1.7). This completes the proof of (1.6).

By direct inspection of the first line of (1.6), using dominated convergence, the fact that for $s > 0$, $R_s$ maps bounded measurable functions into continuous functions, and the inequality $|R_t^V f| \leq c(t,V)\, R_t|f|$, it is easy to argue that $R_t^V f \in C_0(\mathbb{R}^d)$, for $f \in C_0(\mathbb{R}^d)$. Moreover for $t, s \geq 0$,

$$\begin{aligned} R_t^V f(x) &= E_x\Big[\exp\Big\{-\int_0^s V(Z_u)du\Big\}\, \exp\Big\{-\int_s^{t+s} V(Z_u)du\Big\} f(Z_t)\Big] \\ &\overset{\binom{\text{Markov}}{\text{property}}}{=} R_s^V (R_t^V\, f)(x)\,. \end{aligned}$$

We thus see that $R_t^V$ defines a semigroup on $C_0(\mathbb{R}^d)$. The strong continuity of this semigroup follows readily by letting $t$ tend to 0 in (1.6).

Let us finally prove (1.5). To this end notice that for $f \in C_0(\mathbb{R}^d)$:

$$(1.8) \qquad \frac{1}{t} \int_0^t R_s(V R_{t-s}^V f)(x) ds \underset{t \to 0}{\longrightarrow} V f(x) \text{ uniformly in } x \,,$$

since $V$ is bounded continuous. Coming back to the first line of (1.6), this proves that the convergence of $\frac{1}{t}(f - R_t^V f)$ or $\frac{1}{t}(f - R_f f)$ as $t$ tends to 0, are equivalent and (1.5) holds. □

Under the light of Theorem 1.1, the right-hand side of the Feynman-Kac formula (1.4) gives a probabilistic representation for the functional analytic object, which is the strongly continuous semigroup with generator (1.5).

For later purpose, it turns out that the type of expressions, which appear in the r.h.s. of (1.4) are not sufficiently general. In particular we wish to be able to constrain the Brownian path to remain inside some open subset $U$ of $\mathbb{R}^d$. In other words, we want to consider

$$(1.9) \qquad R_t^{U,V} f(x) = E_x\Big[f(Z_t) \exp\Big\{ - \int_0^t V(Z_s) ds\Big\},\ T_U > t\Big] \,,$$

in place of (1.4), with

$$(1.10) \qquad T_U = \inf\{t \geq 0,\ Z_t \notin U\} \,.$$

the *exit time* of the Brownian path from the open set $U$, and

$$(1.11) \qquad R_t^U f(x) = E_x[f(Z_t),\ T_U > t] \,,$$

in place of (1.2). However some difficulties begin to appear. Indeed $R_t^U$ need not preserve $C_0(U)$ the space of continuous functions on $U$ tending to 0 at the one point compactification of $U$, when $U$ is not regular; and for the later purpose of 'random obstacles', the relevant $U$ need not be all too regular. In fact even when $R_t^U$ defines a strongly continuous semigroup on $C_0(U)$, its functional analytic description, by means of an explicit core as in (1.3), is not very pleasant, especially when $U$ is not smooth.

All this may give the impression that the hope of providing a functional analytic description of probabilistic expressions, such as the r.h.s. of (1.9), is tied to smoothness assumptions on $U$ and $V$. It is a remarkable fact that such a description can be given in great generality, and in a very tractable fashion, once we abandon the spaces $C_0(U)$ for the spaces $L^2(U, dx)$, and use quadratic forms in place of generators. It will be the principal purpose of the subsequent sections of this chapter.

**Exercise:** Assume $U$ is a bounded smooth domain of $\mathbb{R}^d$, $V(\cdot)$ is smooth on $\overline{U}$, $u(t,x)$ is continuous bounded on $[0,\infty) \times \overline{U}$, has continuous bounded derivatives on $(0,\infty) \times U$, of first order in the $t$ variable, of second order in the $x$ variable, and

$$(1.12) \qquad \begin{cases} \partial_t u &= \frac{1}{2}\,\Delta u - Vu, \text{ on } (0,\infty) \times U, \text{ with} \\ u(0,x) &= f(x) \text{ in } \overline{U},\; u(t,x) = 0,\; t \geq 0, x \in \partial U\,. \end{cases}$$

Show that for $t \geq 0, x \in \overline{U}$,

$$(1.13) \qquad u(t,x) = E_x\Big[f(Z_t)\,\exp\Big\{-\int_0^t V(Z_s)ds\Big\},\; T_U > t\Big]$$

(Hint: apply Ito's formula to $M^n_{s\wedge T_n}$, where

$$M^n_s = u(t-s, Z_s)\,\varphi_n(Z_s)\,\exp\Big\{-\int_0^s V(Z_s)ds\Big\}$$

and $U_n$ is an increasing sequence of relatively compact open subsets of $U$ exhausting $U$, and $\varphi_n \in C_c^\infty(U)$ equal 1 on $U_n$).

Results on the existence of solutions of (1.2) can be found for instance in Section 10.4 of Krylov [Kry96] or Chapter 3 of Friedman [Fri64]. □

## 1.2 The Class $K_d$

The object of this section is to introduce the class of potentials, i.e. of functions $V$, which shall enter the Feynman-Kac formula. We first discuss a natural class of measurable functions $V$ on $\mathbb{R}^d$, for which

$$(2.1) \qquad E_x\Big[\exp\Big\{\int_0^t |V|(Z_s)ds\Big\}\Big] < \infty, \text{ for } t > 0 \text{ and } x \in \mathbb{R}^d\,.$$

Conditions ensuring uniform finiteness of $E_x[\int_0^t |V|(Z_s)ds]$ look at first sight very far from sufficient to yield control over expressions as in (2.1). Khas'minskii's lemma, see [Kha59], may come as a surprise and is instrumental in this section.

**Lemma 2.1:** *Let $f \geq 0$, measurable and $t > 0$, be such that*

$$\alpha = \sup_{x \in \mathbb{R}^d} E_x\Big[\int_0^t f(Z_s)ds\Big] < 1\,, \quad \textit{then} \tag{2.2}$$

$$\sup_{x \in \mathbb{R}^d} E_x\Big[\exp\Big\{\int_0^t f(Z_s)ds\Big\}\Big] \leq \frac{1}{1-\alpha}\,. \tag{2.3}$$

**Proof:** For $x \in \mathbb{R}^d$,

$$E_x\Big[\exp\Big\{\int_0^t f(Z_s)ds\Big\}\Big] = \sum_{n\geq 0} \frac{1}{n!}\, E_x\Big[\Big(\int_0^t f(Z_s)ds\Big)^n\Big] \overset{\text{(symmetry)}}{=}$$

$$\sum_{n\geq 0} E_x\Big[\int_{0<s_1<\ldots<s_n<t} f(Z_{s_1}) \ldots f(Z_{s_n})ds_1 \ldots ds_n\Big] =$$

$$\sum_{n\geq 0} \int_{0<s_1<\ldots<s_n<t} E_x[f(Z_{s_1}) \ldots f(Z_{s_n})ds_1 \ldots ds_n] \overset{\binom{\text{Markov}}{\text{property}}}{=}$$

$$\sum_{n\geq 0} \int_{0<s_1<\ldots<s_{n-1}<t} E_x\Big[f(Z_s) \ldots f(Z_{s_{n-1}})\, E_{Z_{s_{n-1}}}\Big[\int_0^{t-s_{n-1}} f(Z_s)ds\Big]\Big]ds_1 \ldots ds_{n-1} \overset{(2.2)}{\leq}$$

$$\sum_{n\geq 0} \alpha \int_{0<s_1<\ldots<s_{n-1}<t} E_x[f(Z_s) \ldots f(Z_{s_{n-1}})]ds_1 \ldots ds_{n-1} \overset{\text{(induction)}}{\leq}$$

$$\sum_{n\geq 0} \alpha^n = \frac{1}{1-\alpha}\,.$$

□

In view of the above lemma, we introduce the following functional vector space, which in terms of the alternative characterization (2.21) can be found in Kato [Kat73]:

$$K_d = \Big\{f \text{ Borel measurable on } \mathbb{R}^d,\ \lim_{t\downarrow 0}\ \sup_x\ E_x\Big[\int_0^t |f|(Z_s)ds\Big] = 0\Big\}\,. \tag{2.4}$$

**Theorem 2.2:** *If $f \in K_d$, there exist $A, B_f > 0$ such that for $t \geq 0$:*

$$\sup_z\ E_z\Big[\exp\Big\{\int_0^t |f|(Z_s)ds\Big\}\Big] \leq A\ \exp\{B_f t\}\,, \tag{2.5}$$

*(A can be chosen independent of f).*

**Proof:** With no loss of generality we assume $f \geq 0$, and pick $t_0$ such that:

$$\sup_x E_x\Big[\int_0^{t_0} f(Z_s)ds\Big] = \alpha < 1 . \tag{2.6}$$

Then for $t = nt_0 + r$, $0 \leq r < t_0$, $n \in \mathbb{N}$:

$$E_x\Big[\exp\Big\{\int_0^t f(Z_s)ds\Big\}\Big] \leq E_x\Big[\exp\Big\{\int_0^{(n+1)t_0} f(Z_s)ds\Big\}\Big] \overset{\binom{\text{Markov}}{\text{property}}}{=}$$
$$E_x\Big[\exp\Big\{\int_0^{nt_0} f(Z_s)ds\Big\}\, E_{Z_{nt_0}}\Big[\exp\Big\{\int_0^{t_0} f(Z_u)du\Big\}\Big]\Big] \overset{(2.3)}{\leq}$$
$$\frac{1}{1-\alpha}\; E_x\Big[\exp\Big\{\int_0^{nt_0} f(Z_s)ds\Big\}\Big] \overset{\text{(induction)}}{\leq} \Big(\frac{1}{1-\alpha}\Big)^{n+1} \leq$$
$$\frac{1}{1-\alpha}\; \exp\Big\{\tfrac{t}{t_0}\,\log\Big(\frac{1}{1-\alpha}\Big)\Big\},\ \text{since } n \leq \frac{t}{t_0}.$$

This proves our claim. □

It is instructive to discuss an example of an $f$ for which things go wrong.

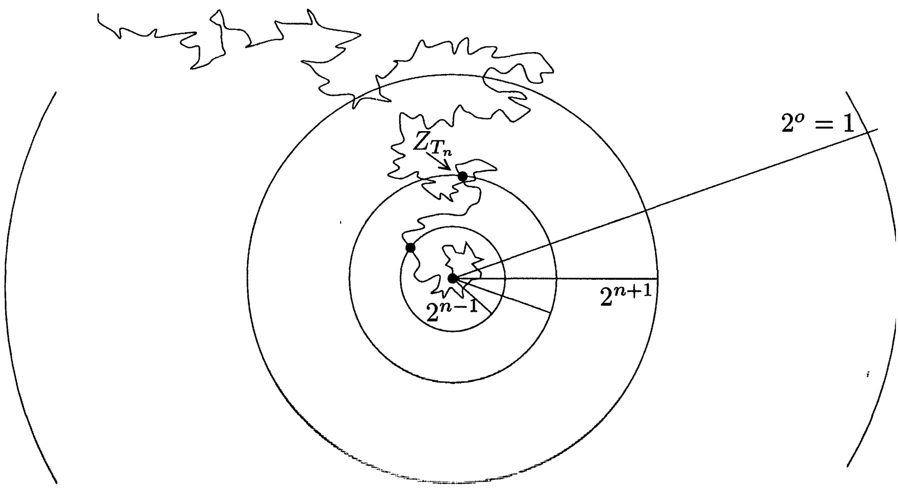
**Example 2.3:** We choose $f(x) = \frac{1}{|x|^2} \cdot 1(|x| \leq 1)$, and assume $d \geq 2$.

Observe that $f \in L^p(\mathbb{R}^d)$ if $1 \leq p < \frac{d}{2}$, indeed:

$$\|f\|_p^p = \int_{\mathbb{R}^d} |f(x)|^p\, dx = \mathrm{vol}(S^{d-1}) \int_0^1 \frac{r^{d-1}}{r^{2p}}\, dr < \infty, \ \text{ if } \ d > 2p . \tag{2.7}$$

On the other hand we are now going to show that:

$$P_0 - \text{a.s., for } \ t > 0,\ \int_0^t f(Z_s)ds = \infty . \tag{2.8}$$



We define for $n \in \mathbb{Z}$

$$T_n = \inf\{s \geq 0,\ |Z_s| \geq 2^n\}\ . \tag{2.9}$$

The key observation is:

**Lemma 2.4:**

(2.10) under $P_0$, for $n \in \mathbb{Z}$, $\displaystyle\int_0^{T_n} \frac{ds}{|Z_s|^2}$ has same law as $\displaystyle\int_0^{T_{n+1}} \frac{ds}{|Z_s|^2}$ .

**Proof:** Observe that under $P_0$,

$$\overline{Z}_. = 2Z_{./4}\ ,$$

is again a Brownian motion. Moreover,

$$\begin{aligned} \frac{1}{4}\, T_{n+1}(\overline{Z}) &= \frac{1}{4}\ \inf\{s \geq 0,\ 2|Z_{s/4}| \geq 2^{n+1}\} \\ &= \inf\left\{\frac{s}{4} \geq 0,\ |Z_{s/4}| \geq 2^n\right\} = T_n(Z)\ . \end{aligned} \tag{2.11}$$

As a result, we see that:

$$\int_0^{T_{n+1}(\overline{Z})} \frac{ds}{|\overline{Z}_s|^2} = \int_0^{T_{n+1}(\overline{Z})} \frac{ds}{4|Z_{s/4}|^2} = \int_0^{\frac{1}{4}T_{n+1}(\overline{Z})} \frac{ds}{|Z_s|^2} = \int_0^{T_n(Z)} \frac{ds}{|Z_s|^2}\ ,$$

which proves our claim. □

It thus follows from (2.10) that for $n \in \mathbb{Z}$,

$$\mathbb{P}_0\text{-a.s.},\ \int_0^{T_n} \frac{ds}{|Z_s|^2} = \infty\ , \tag{2.12}$$

indeed otherwise we would have

$$E_0\Big[\exp\Big\{-\int_0^{T_n} \frac{ds}{|Z_s|^2}\Big\}\Big] > E_0\Big[\exp\Big\{-\int_0^{T_{n+1}} \frac{ds}{|Z_s|^2}\Big\}\Big] =$$
$$E_0\Big[\exp\Big\{-\int_0^{T_n} \frac{ds}{|Z_s|^2}\Big\}\Big]\ .$$

The claim (2.8) now easily follows. □

The above example shows that even in the case of $V \geq 0$, we need some assumptions on $V$, if we want the expression $E_x[\exp\{-\int_0^t V(Z_s)ds\}]$ to be 'classical', i.e. nonzero.

**Exercise:** When $d = 1$, show that

$$(2.13) \qquad P_1\text{-a.s.} \int_0^{H_{\{0\}}} \frac{ds}{|Z_s|^2} = \infty, \quad \text{where}$$

$H_{\{0\}} = \inf\{s \geq 0,\ Z_s = 0\}$.

(Hint: consider $\varphi(x) = E_x[\exp\{-\int_0^{H_{\{0\}}} \frac{ds}{|Z_s|^2}\}]$ .) □

We now give an alternative characterization of $K_d$:

**Proposition 2.5:**

$$(2.14) \qquad K_d = \Big\{f \ \textit{measurable};\ \lim_{r \downarrow 0} \sup_x E_x\Big[\int_0^{H_r} |f|(Z_s)ds\Big] = 0\Big\},$$

$$(2.15) \qquad \textit{where } H_r = \inf\{s \geq 0,\ |Z_s - Z_0| \geq r\}.$$

**Proof:** We first show that the r.h.s. of (2.14) is included in $K_d$. To this end we pick $t > 0$, and $r = \sqrt{t}$, and define $H_r^n$, $n \geq 0$, to be the successive times of displacement of $Z_.$ at distance $r$:

$$(2.16) \qquad H_r^0 = 0,\ H_r^{n+1} = H_r^n + H_r \circ \theta_{H_r^n}\ ,\ n \geq 0\,.$$

Then we have

$$(2.17) \qquad \begin{aligned} E_x\Big[\int_0^t |f|(Z_s)ds\Big] &\leq \sum_{n\geq 0} E_x\Big[H_r^n \leq t,\ \int_{H_r^n}^{H_r^{n+1}} |f|(Z_s)ds\Big] \\ &\overset{\binom{\text{strong Markov}}{\text{property}}}{=} \sum_{n\geq 0} E_x\Big[H_r^n \leq t,\ E_{Z_{H_r^n}}\Big[\int_0^{H_r} |f|(Z_s)ds\Big]\Big] \\ &\leq \sup_x E_x\Big[\int_0^{H_r} |f|(Z_s)ds\Big] \sum_{n\geq 0} E_x[H_r^n \leq t] \\ &\overset{\binom{\text{scaling, translation}}{\text{invariance}}}{=} C \sup_x E_x\Big[\int_0^{H_r} |f|(Z_s)ds\Big], \quad \text{where} \end{aligned}$$

$$C = E_0\Big[\sum_{n\geq 0} 1(H_1^n \leq 1)\Big] < \infty, \text{ since } P_0[H_1^n \leq 1] \leq P_0[H_1 \leq 1]^n\,.$$

From (2.17) we easily conclude that the r.h.s. of (2.14) is included in $K_d$. The converse inclusion is proven analogously, we now simply write, with $t$ and $r$ above:

$$E_x\Big[\int_0^{H_r} |f|(Z_s)ds\Big] \leq \sum_{n\geq 0} E_x\Big[nt < H_r,\ \int_{nt}^{(n+1)t} |f|(Z_s)ds\Big]\,.$$

□

We now want to 'localize' the space $K_d$ and to this end introduce

$$K_d^{\mathrm{loc}} = \Big\{f \text{ measurable; } \forall N \geq 1, \lim_{r \downarrow 0} \sup_{|x| \leq N} E_x\Big[\int_0^{H_r} |f|(Z_s)ds\Big] = 0\Big\}. \tag{2.18}$$

From (2.14), we immediately see that

$$f \in K_d^{\mathrm{loc}} \Longleftrightarrow \forall N \geq 1, \; f \cdot 1\{|\cdot| \leq N\} \in K_d . \tag{2.19}$$

In the sequel when considering the functional $\exp\{-\int_0^t V(Z_s)ds\}$ our usual requirement will be that $V_- = \max(-V, 0) \in K_d$ and $V_+ = \max(V, 0) \in K_d^{\mathrm{loc}}$. In particular the behavior described in Example 2.3 does not occur when $f$ belongs to $K_d^{\mathrm{loc}}$. Indeed,

$$\text{when } f \in K_d^{\mathrm{loc}}, \text{ for } x \in \mathbb{R}^d,\ P_x\text{-a.s.}, \forall t > 0, \int_0^t |f|(Z_s)ds < \infty ; \tag{2.20}$$

this statement simply follows from (2.19) together with the fact that

$$P_x\text{-a.s. for large } N, \int_0^t |f|(Z_s)ds = \int_0^t |f|(Z_s)\, 1\{|Z_s| \leq N\}ds .$$

**Exercise:**

**1)** Alternative characterization of $K_d$:

$$\text{Show that } f \in K_d \iff \begin{cases} \sup_x \int_{|x-y|\leq 1} |f|(y)dy < \infty, \text{ when } d = 1, \\ \lim_{\alpha\downarrow 0} \sup_x \int_{|x-y|\leq\alpha} \log\left(\frac{1}{|x-y|}\right) |f(y)|dy = 0, \\ \quad \text{when } d = 2, \\ \lim_{\alpha\downarrow 0} \sup_x \int_{|x-y|\leq\alpha} \frac{1}{|x-y|^{d-2}} |f(y)|dy = 0, \\ \quad \text{when } d \geq 3 . \end{cases} \tag{2.21}$$

(Hint: use (2.14) and $E_x[\int_0^t |f|(Z_s)ds] = \int Q_t(y-x)\,|f|(y)dy$, where

$$\begin{aligned} Q_t(z) &= \int_0^t \frac{1}{(2\pi s)^{d/2}} \exp\Big\{-\frac{z^2}{2s}\Big\} ds \text{ and using for } d \geq 2: \\ Q_t(z) &= |z|^{2-d} \int_0^{t/z^2} \frac{1}{(2\pi u)^{d/2}} \exp\Big\{-\frac{1}{2u}\Big\} du = \\ &\frac{|z|^{2-d}}{(2\pi)^{d/2}} \int_{z^2/t}^{\infty} e^{-\frac{1}{2}v}\, v^{\frac{d-4}{2}} dv . \end{aligned} \tag{2.22}$$

**2)** When $d \geq 2$, show that $f(x) = \frac{1}{|x|^{2-\epsilon}} \in K_d$, when $\epsilon > 0$. □

## 1.3 Semigroups, Kernels of Semigroups

We return to the Feynman-Kac formula and begin the investigation of the semigroups it naturally induces. Our standing assumption throughout this section is that

$$V_- = \max(-V,0) \in K_d \text{ and } V_+ = \max(V,0) \in K_d^{\mathrm{loc}} . \tag{3.1}$$

For $U \neq \emptyset$ a nonvoid open subset of $\mathbb{R}^d$ and $V$ satisfying (3.1) we want to study expressions like

$$R_t^{U,V} f(x) = E_x\Big[f(Z_t) \exp\Big\{-\int_0^t V(Z_s)ds\Big\},\ T_U > t\Big],\ t > 0\,, \tag{3.2}$$

which we already met in (1.9). Observe that when $f$ is for instance bounded measurable, formula (3.2) makes sense thanks to Theorem 2.2. As it will turn out $R_t^{U,V}$ naturally defines strongly continuous semigroups on $L^p(U,dx)$, $1 \le p < \infty$. Keeping for the time being $f$ bounded measurable, and using the Brownian bridge measure as a regular conditional probability of Wiener measure given $Z_t$ (see Corollary A.2 of the Appendix), we have by (A.11):

$$\begin{aligned} R_t^{U,V} f(x) &= E_x\Big[f(Z_t)\, E^t_{x,Z_t}\Big[\exp\Big\{-\int_0^t V(Z_s)ds\Big\},\ T_U > t\Big]\Big] \\ &= \int f(y)\, r_{U,V}(t,x,y)dy\,, \end{aligned} \tag{3.3}$$

provided we set for $t > 0$, $x, y \in \mathbb{R}^d$:

$$r_{U,V}(t,x,y) = p(t,x,y)\, E^t_{x,y}\Big[\exp\Big\{-\int_0^t V(Z_s)ds\Big\},\ T_U > t\Big]\,, \tag{3.4}$$

where as in (1.1), $p(t,x,y)$ denotes the Brownian transition density. Note by the way that when $x$ or $y$ is not in $U$, $r_{U,V}(t,x,y) = 0$. When $V = 0$, we shall simply write $r_U$ for $r_{U,V=0}$.

As suggested by this calculation, the $r_{U,V}(\cdot,\cdot,\cdot)$ will be the kernels of the semigroups $R_t^{U,V}$. One of our first objects here will be the derivation of estimates on these kernels.

**Proposition 3.1:**

$$\begin{aligned} &r_{U,V}(t,x,y) \le t^{-d/2}\, C_1(V_-)\, \exp\Big\{C_2(V_-)\,t - \frac{(y-x)^2}{4t}\Big\}, \\ &t > 0,\ x, y \in \mathbb{R}^d\,, \end{aligned} \tag{3.5}$$

(3.6) $$r_{U,V}(t,x,y) = r_{U,V}(t,y,x),\ t>0,\ x,y \in \mathbb{R}^d\,,$$

(3.7) $$r_{U,V}(t+s,x,y) = \int_U r_{U,V}(t,x,z)\, r_{U,V}(s,z,y)dz,$$
$$t,s>0,\ x,y\in\mathbb{R}^d,\ \textit{(Chapman-Kolmogorov equations)}\,.$$

**Proof:** We shall drop the subscripts $U,V$, when this causes no confusion.

*Proof of (3.5):* Using the inequality $ab \le \frac{1}{2}\,(a^2+b^2)$, we have

$$\begin{aligned} r(t,x,y) &\le \tfrac{1}{2}\,p(t,x,y)\,E^t_{x,y}\Big[\exp\Big\{-\int_0^{t/2} 2V(Z_s)ds\Big\} \\ &+ \exp\Big\{-\int_{t/2}^t 2V(Z_s)ds\Big\}\Big] \\ &\overset{\binom{\text{symmetry and}}{\text{time reversal (A.12)}}}{=} \tfrac{1}{2}\,p(t,x,y)\,E^t_{x,y}\Big[\exp\Big\{-\int_0^{t/2} 2V(Z_s)ds\Big\}\Big] \\ &+ \tfrac{1}{2}\,p(t,y,x)\,E^t_{y,x}\Big[\exp\Big\{-\int_0^{t/2} 2V(Z_s)ds\Big\}\Big] \\ &= \tfrac{1}{2}\,\Big(E_x\Big[\exp\Big\{-\int_0^{t/2} 2V(Z_s)ds\Big\}\,p(\tfrac{t}{2},\,Z_{t/2},y)\Big] \\ &+ E_y\Big[\exp\Big\{-\int_0^{t/2} 2V(Z_s)ds\Big\}\,p(\tfrac{t}{2},\,Z_{t/2},x)\Big]\Big) \\ &\overset{(2.5)}{\le} (\pi t)^{-d/2}\,A_{2V_-}\,\exp\Big\{B_{2V_-}\cdot\tfrac{t}{2}\Big\}\,. \end{aligned} \tag{3.8}$$

Now applying Cauchy-Schwarz inequality to (3.4) we find:

$$r_{U,V}(t,x,y) \le p(t,x,y)^{1/2}\, r_{U,2V}(t,x,y)^{1/2}\,.$$

Together with (3.8) this shows (3.5).

*Proof of (3.6):* This claim readily follows from symmetry, time reversal and the equivalence $T_U(Z_\cdot) > t \iff T_U(Z_{t-\cdot}) > t$.

*Proof of (3.7):* Using the Markov property for the bridge measure (A.13)

$$E^{t+s}_{x,y}\Big[T_U > t+s,\ \exp\Big\{-\int_0^{t+s} V(Z_u)du\Big\}\Big]$$

$$= E^{t+s}_{x,y}\Big[T_U > t, \exp\Big\{-\int_0^t V(Z_u)du\Big\} E^s_{Z_t,y}\Big[\exp\Big\{-\int_0^s V(Z_u)du\Big\}, T_U > s\Big]\Big]$$

$$= E^{t+s}_{x,y}\Big[T_U > t,\ \exp\Big\{-\int_0^t V(Z_u)du\Big\}\ r(s,Z_t,y)/p(s,Z_t,y)\Big]$$

$$= E_x\Big[T_U > t,\ \exp\Big\{-\int_0^t V(Z_u)du\Big\}\ r(s,Z_t,y)\Big]/p(t+s,x,y)\ .$$

Using our calculation in (3.3), we now obtain (3.7). □

We are now going to apply the above proposition to study expressions like $R^{U,V}_t f(x)$, in (3.2).

**Proposition 3.2:** *When* $f \in L^p(U,dx)$, $1 \le p \le \infty$,

(3.9) $$R^{U,V}_t\,|f|(x) < \infty,\ x \in \mathbb{R}^d,\ t > 0\ ,$$

(3.10) $$R^{U,V}_t f(x) = \int r_{U,V}(t,x,y)\, f(y)dy,\ x \in \mathbb{R}^d,\ t > 0\ ,$$

(3.11) $$\|R^{U,V}_t f\|_p \le A_p(V_-)\ \exp\{B_p(V_-)t\}\ \|f\|_p,\ t \ge 0\ .$$

(3.12) $$R^{U,V}_{t+s}\ f(x) = R^{U,V}_t(R^{U,V}_s\ f)(x),\ s,t > 0,\ x \in \mathbb{R}^d\ .$$

*Proof of (3.9):* When $p \in (1,\infty)$ it follows from Hölder's inequality that:

(3.13) $$R_t\,|f|(x) \le E_x\Big[\exp\Big\{-q\int_0^t V(Z_s)ds\Big\}\Big]^{1/q}\ E_x[|f|^p(Z_t)]^{1/p} < \infty\ ,$$

thanks to Theorem 2.2, provided $q$ stands for the conjugate exponent of $p$. When $p = \infty$, (3.9) follows directly from Theorem 2.2.
When $p = 1$, and $N \ge 0$,

$$R_t\,|f| \wedge N(x) \overset{(3.3)}{=} \int r(t,x,y)(|f| \wedge N)(y)dy$$
$$\overset{(3.5)}{\le} \mathrm{const}(t,V_-)\cdot \|f\|_1\ ,$$

from which (3.9) follows.

*Proof of (3.10):* By (3.9) we have

$$\infty > R_t|f|(x) \geq R_t(|f| \wedge N)(x) \overset{(3.3)}{=} \int r(t,x,y)(|f| \wedge N)(y)dy\,.$$

By monotone convergence we have

$$\int r(t,x,y)\,|f|(y)dy < \infty\,,$$

and (3.10) then follows from (3.3) and dominated convergence.

*Proof of (3.11):* When $p = \infty$ this is an immediate consequence of Theorem 2.2. When $1 < p < \infty$:

$$\begin{aligned} \|R_t\, f\|^p_{L^p} &\overset{(3.13)}{\leq} \int\int p(t,x,y)|f|^p(y)\; E_x\Big[\exp\Big\{-\int_0^t qV(Z_s)ds\Big\}\Big]^{p/q} dxdy \\ &\overset{(2.5)}{\leq} \{A(qV_-)\,\exp\{B(qV_-)t\}\}^{p/q} \times \|f\|^p_p \end{aligned}$$

and (3.11) follows.

When $p = 1$, using (3.10)

$$\|R_t f\|_1 = \int\int r(t,x,y)\,|f|(y)\,dxdy \overset{(3.5)}{\leq} A(V_-)\;\exp\{B(V_-)t\}\;\|f\|_1\,.$$

*Proof of (3.12):* This last claim easily follows from (3.7) and (3.10). □

We now come to

**Proposition 3.3:** *For* $1 \leq p < \infty$,

(3.14) *$f \to R^{U,V}_t f(x)$, generates a strongly continuous* semigroup on $L^p(U,dx)$ .

(3.15) *When $p = 2$, this semigroup is self-adjoint. It is trace* class if $|U| < \infty$ .

(3.16) *When $V \geq 0$, or when there exists $A > 0$ such that* $\|R^{U,V}_t f\|_2 \leq A\|f\|_2$ for any $t > 0$, this semigroup, *is also a contraction semigroup on $L^2(U,dx)$* .

*Proof of (3.14):* Thanks to (3.11) it suffices by a density argument to show that when $f \in C_c(U)$ (i.e. $f$ continuous with compact support in $U$),

(3.17) $$\lim_{t\to 0} \|R^{U,V}_t f - f\|_{L^p} = 0\,.$$

Now if $K = \mathrm{supp}(f)$ and $B(0,N) \supset K$,

$$\|R_t^{U,V} f - f\|_{L^p(U)}^p \le$$

$$(3.18)\quad \|f\|_{L^\infty}^p \int_{B(0,N)^c} E_x\Big[Z_t \in K,\ \exp\Big\{-\int_0^t V(Z_s)ds\Big\}\Big]^p dx +$$
$$\int_{U\cap B(0,N)} \Big|E_x\Big[f(Z_t)\ \exp\Big\{-\int_0^t V(Z_s)ds\Big\}\ 1\{T_U > t\} - f(Z_0)\Big]\Big|^p dx$$

To control the first term in the r.h.s. of (3.18), observe that when $x \notin B(0,N)$:

$$E_x\Big[Z_t \in K,\ \exp\Big\{-\int_0^t V(Z_s)ds\Big\}\Big]^p$$
$$\le P_x[Z_t \in K]^{1/2}\, E_x\Big[\exp\Big\{-\int_0^t 2pV(Z_s)ds\Big\}\Big]^{1/2}$$
$$\le (2\pi t)^{-d/4}\, \exp\Big\{-\frac{1}{4t}\,\mathrm{dist}(x,K)^2\Big\}\cdot|K|\cdot$$
$$A(2pV_-)^{1/2}\, \exp\Big\{\frac{1}{2}\, B(2pV_-)t\Big\},$$

using Theorem 2.2 in the last step. It now follows that the first term in the r.h.s. of (3.18) tends to 0 as $t$ tends to 0. As for the second term observe first that by a very similar argument, when $x \in U$:

$$R_t^{U,V} f(x) - R_t^{\mathbb{R}^d,V} f(x) \to 0, \quad \text{as } t \to 0\,,$$

and that by dominated convergence

$$R_t^{\mathbb{R}^d,V} f(x) \to f(x), \quad \text{as } t \to 0\,.$$

Therefore the second term in the r.h.s. of (3.18) tends to zero. Our claim (3.14) now follows.

*Proof of (3.15):* When $p = 2$, $R_t^{U,V}$ defines a self-adjoint semigroup thanks to the symmetry of the kernel $r_{U,V}(t,\cdot,\cdot)$, see (3.6). When $|U| < \infty$, $r_{U,V}(t,\cdot,\cdot) \in L^2(U\times U)$ and therefore $R_t^{U,V}$ is a Hilbert-Schmidt operator on $L^2(U,dx)$. The trace class property follows from the identity $R_t^{U,V} = R_{\frac{t}{2}}^{U,V} \circ R_{\frac{t}{2}}^{U,V}$.

*Proof of (3.16):* When $V \ge 0$ the claim follows from the case $V = 0$. On the other hand when $\|R_t^{U,V}\|_{L^2\to L^2} \le A$, for $t \ge 0$, by the spectral theorem applied to $R_t$, we have

$$R_t = \int \lambda dG_\lambda\,,$$

for $G_\lambda$ a suitable resolution of the identity on $L^2(U, dx)$. If $f \in L^2(U, dx)$ has unit $L^2$ norm and $\mu_f(d\lambda) = d(G_\lambda f, f)$ denotes its spectral probability, we have for $n \geq 1$:

$$\Big(\int \lambda^{2n} d\mu_f(\lambda)\Big)^{1/n} = \|R_t^n f\|_2^{2/n} \overset{(3.12)}{=} \|R_{tn} f\|_2^{2/n} \leq A^{2/n} \underset{n \to \infty}{\longrightarrow} 1 .$$

This implies that supp $\mu_f(d\lambda) \subset [-1, 1]$. Therefore,

$$\|R_t f\|_2^2 = \int \lambda^2 d\mu_f(\lambda) \leq \mu_f(\mathbb{R}) = 1 ,$$

and $R_t$ is a contraction on $L^2(U, dx)$. □

**Remark 3.4:** The semigroup $R_t^{U,V+c}$, where $c$ is larger than the constant $B_2(V_-)$ from (3.11), is contractive on $L^2(U, dx)$. Indeed we have the inequality

$$\|R_t^{U,V+c}\|_{L^2 \to L^2} \leq A_2(V_-) \ \exp\{(B_2(V_-) - c)t\} ,$$

and (3.16) applies. □

We shall sometimes refer to the semigroup $R_t^{U,V}$ acting on $L^2(U, dx)$, as the *Dirichlet-Schrödinger semigroup* (associated to $U$ and $V$). We shall also use the notation of *Schrödinger semigroup* and *Dirichlet semigroup* when $U = \mathbb{R}^d$, respectively $V = 0$. We close this section with the proof of the continuity of the kernels of the Dirichlet-Schrödinger semigroups.

**Proposition 3.5:**

(3.19) $\qquad (t, x, y) \in (0, \infty) \times U^2 \longrightarrow r_{U,V}(t, x, y)$ *is continuous* .

**Proof:** We drop the dependence on $U, V$ in the notation. From the Chapman-Kolmogorov equations (3.7), and from (3.10) we have

$$r(t, x, y) = (r(\epsilon, x, \cdot),\ R_{t-2\epsilon}\, r(\epsilon, y, \cdot))_{L^2(U,dx)} ,$$

for $t > 2\epsilon$, $x, y \in U$.

From this, (3.11) and the strong continuity of the semigroup (3.14) our claim (3.19) follows once we show that for fixed $t > 0$, $x \in U \to r(t, x, \cdot) \in L^2(U, dx)$ is continuous. Now in view of (3.5), it therefore suffices to show that for $y \in U$, $t > 0$,

(3.20) $\qquad x \in U \to r(t, x, y)$ is continuous .

This in turn will follow from

(3.21) $\ \lim_{\epsilon \to 0} \ \sup_K |r^\epsilon(t, \cdot, y) - r(t, \cdot, y)| = 0,$ for $t > 0$ and $K \subset U$ compact, provided we define,

$$
\begin{aligned}
r^\epsilon(t,x,y) &= \int_U p(\epsilon,x,z)\,p(t-\epsilon,z,y)\,E^{t-\epsilon}_{z,y}\Big[\exp\Big\{-\int_0^{t-\epsilon} V(Z_s)ds\Big\}, \\
&\qquad T_U > t-\epsilon\Big]dz \qquad (3.22)\\
&= p(t,x,y)\,E^t_{x,y}\Big[\exp\Big\{-\int_\epsilon^t V(Z_s)ds\Big\},\ T_U\circ\theta_\epsilon > t-\epsilon\Big]\,.
\end{aligned}
$$

Observe now that

$$
\begin{aligned}
&|r(t,x,y) - r^\epsilon(t,x,y)| \le \\
&p(t,x,y)\Big(E^t_{x,y}\Big[\exp\Big\{-\int_\epsilon^t V(Z_s)ds\Big\}\Big|\exp\Big\{-\int_0^\epsilon V(Z_s)dx\Big\}-1\Big|\Big] \qquad (3.23)\\
&+ E^t_{x,y}\Big[\exp\Big\{\int_0^t V_-(Z_s)ds\Big\},\ T_U\le\epsilon\Big]\Big)\,.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality and (2.3), the last term is smaller than const $\cdot E_x[T_U \le \epsilon]^{1/2} \underset{\epsilon\to 0}{\longrightarrow} 0$, uniformly in $x \in K$. Our claim will therefore follow once we control the first term in the r.h.s. of (3.23). Using once more the Cauchy-Schwarz inequality, it suffices to show:

$$
\lim_{\epsilon\to 0}\ \sup_{x\in K} E_x\Big[\Big(\exp\Big\{-\int_0^\epsilon V(Z_s)ds\Big\}-1\Big)^2\Big] = 0\,. \qquad (3.24)
$$

Observe that for $B(0,N)\supset K$,

$$
\lim_{\epsilon\to 0}\ \sup_{x\in K} E_x\Big[\Big(\exp\Big\{-\int_0^\epsilon V(Z_s)ds\Big\}-1\Big)^2,\ T_{B(0,N)}\le\epsilon\Big] = 0\,.
$$

We can consequently confine ourselves to proving (3.24) to the case $V \in K_d$. Now since for $a \in \mathbb{R}$

$$
(e^a-1)^2 \le (e^{|a|}-1)^2 \le e^{2|a|}-1\,,
$$

we have:

$$
\begin{aligned}
&E_x\Big[\Big(\exp\Big\{-\int_0^\epsilon V(Z_s)ds\Big\}-1\Big)^2\Big] \le \\
&E_x\Big[\exp\Big\{2\int_0^\epsilon |V|(Z_s)ds\Big\}\Big]-1 \overset{(2.3)}{\le} \frac{1}{1-\alpha(\epsilon)}-1\,,
\end{aligned}
$$

provided $\alpha(\epsilon) = \sup_x E_x[\int_0^\epsilon 2|V|(Z_s)ds]$. Our claim (3.24) now follows from the fact that $V \in K_d$ and $\lim_{\epsilon\to 0}\alpha(\epsilon) = 0$. □

## 1.4 Quadratic Forms, Dirichlet Forms

Our goal is to develop in this section an alternative functional analytic characterization of the Dirichlet-Schrödinger semigroups. Our main interest will go to the case $V \geq 0$, the 'obstacle case'.

We begin with some generalities. When $H$ is a real Hilbert space and $(Q_t)_{t \geq 0}$ a strongly continuous semigroup of self-adjoint contractions on $H$, we know from the spectral theorem that we can find a resolution of the identity $E_\lambda$, $\lambda \geq 0$ on, $H$ (see Rudin [Rud74a]) such that $Q_t = \int_0^\infty e^{-t\lambda} dE_\lambda$.

Denoting by $(\cdot, \cdot)$ the scalar product on $H$, for $f \in H$ and $t > 0$,

$$\frac{1}{t}\Big((I - Q_t)f,\, f\Big) = \int_0^\infty \frac{(1 - e^{-\lambda t})}{t}\, d\mu_f(\lambda)\,, \tag{4.1}$$

if $d\mu_f(\lambda) = d(E_\lambda f, f)$ is the spectral measure of $f$ (with mass $\|f\|^2$). From this equality it follows that the quantity on the left-hand side of (4.1) increases as $t$ decreases to 0. We then define the quadratic form $\mathcal{E}(f, f)$ associated to $Q_t$ acting on $H$ via

$$\mathcal{E}(f, f) = \lim_{t \downarrow 0} \frac{1}{t}\Big((I - Q_t)f, f\Big) = \int_0^\infty \lambda\, d\mu_f(\lambda) \in [0, \infty]\,, \tag{4.2}$$

and the domain of the form is

$$\mathcal{D}(\mathcal{E}) = \Big\{f \in H,\ \lim_{t \downarrow 0} \frac{1}{t}\Big((I - Q_t)f, f\Big) < \infty\Big\}\,. \tag{4.3}$$

$\mathcal{D}(\mathcal{E})$ is clearly dense since $E_\lambda(f) \in \mathcal{D}(\mathcal{E})$ for any $\lambda \geq 0$, and $f \in H$.

**Example 4.1:** $U = \mathbb{R}^d$, $V = 0$, $R_t$ the Brownian semigroup on $L^2(\mathbb{R}^d, dx)$. Then:

$$\frac{1}{t}\,((I - R_t)f, f)_{L^2(\mathbb{R}^d)} \overset{\text{(Parseval)}}{=} \frac{1}{t(2\pi)^d} \int \Big(\hat{f}(\xi) - e^{-\frac{1}{2}t\xi^2}\hat{f}(\xi)\Big)\,\overline{\hat{f}(\xi)}\, d\xi\,,$$

if $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{i\xi x} f(x)dx$ is the $(L^2)$ Fourier transform

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{(1 - e^{-\frac{1}{2}t\xi^2})}{t}\, |\hat{f}(\xi)|^2\, d\xi \underset{t \to 0}{\longrightarrow} \frac{1}{(2\pi)^d} \int \frac{1}{2}\,\xi^2\, |\hat{f}(\xi)|^2 d\xi\,.$$

It follows that

$$\begin{aligned}(4.4) \qquad \mathcal{D}(\mathcal{E}) &= \left\{ f \in L^2(\mathbb{R}^d),\ \int_{\mathbb{R}^d} \xi^2 \, |\hat{f}(\xi)|^2 \, d\xi < \infty \right\} \\ &= \{ f \in L^2(\mathbb{R}^d),\ \partial_i f \in L^2(\mathbb{R}^d),\ i = 1, \ldots, d\} = H^1(\mathbb{R}^d)\,,\end{aligned}$$

and

$$(4.5) \qquad \mathcal{E}(f,f) = \frac{1}{2} \int_{\mathbb{R}^d} \sum_{i=1}^{d} (\partial_i f)^2(x)\, dx = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla f|^2\,(x) dx\,,$$

for $f \in \mathcal{D}(\mathcal{E}) = H^1(\mathbb{R}^d)$.

□

An important property of the above densely defined quadratic form associated to $Q_t$ and $H$ is that it is *closed*. That is $\Big(\mathcal{D}(\mathcal{E}),\ \mathcal{E}(\cdot,\cdot) + (\cdot,\cdot)\Big)$ is an Hilbert space, provided $\mathcal{E}(\cdot,\cdot)$ is extended to a symmetric bilinear form via the formula

$$(4.6) \qquad \mathcal{E}(f,g) = \tfrac{1}{4} \Big( \mathcal{E}(f+g, f+g) - \mathcal{E}(f-g, f-g) \Big)\,.$$

**Lemma 4.2:** $\Big(\mathcal{D}(\mathcal{E}),\ \mathcal{E}(\cdot,\cdot) + (\cdot,\cdot)\Big)$ *is a Hilbert space.*

**Proof:** Let $f_n$ be a Cauchy sequence in $\mathcal{E}_1(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + (\cdot,\cdot)$. Then $f_n \underset{H}{\longrightarrow} f$. Looking at the Laplace transform of $\mu_{f_n}$ and $\mu_f$ we find:

$$\begin{aligned}(4.7) \qquad L_f(t) &\overset{\text{def}}{=} \int_0^\infty e^{-\lambda t} d\mu_f(\lambda) = (Q_t f, f) = \lim_{n\to\infty} (Q_t f_n, f_n) \\ &= \lim_n L_{f_n}(t), \ \text{ for } t \geq 0\,.\end{aligned}$$

It follows that $\mu_{f_n}(d\lambda)$ converges weakly to $\mu_f(d\lambda)$ on $[0,\infty)$ and

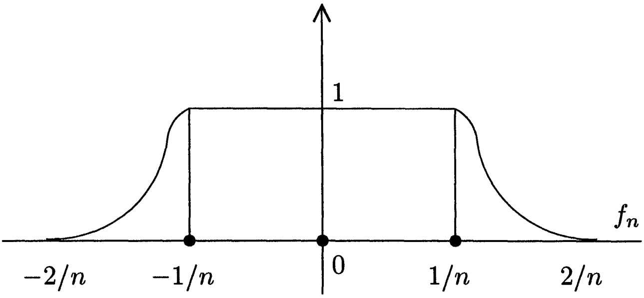
$$\int_0^\infty \lambda\, d\mu_f(\lambda) \leq \underline{\lim}_n \int_0^\infty \lambda\, d\mu_{f_n}(\lambda) = \underline{\lim}_n\ \mathcal{E}(f_n, f_n) < \infty\,.$$

So $f \in \mathcal{D}(\mathcal{E})$ and for given $n$,

$$\mathcal{E}\Big((f - f_n), (f - f_n)\Big) \leq \underline{\lim}_{\substack{p \geq n \\ p \to \infty}}\ \mathcal{E}\Big((f_p - f_n), (f_p - f_n)\Big)\,.$$

This can be made arbitrarily small by picking $n$ large (the sequence $f_p$ is Cauchy). Therefore $f_n \to f$ in $\mathcal{E}_1$ norm. □

**Example 4.3:** We let $C_c^\infty(\mathbb{R})$ denote the space of smooth functions on $\mathbb{R}$ with compact support. The densely defined symmetric bilinear form $\mathcal{E}(f,g) = f(0)\,g(0)$, $f, g \in C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$, is not closable, i.e. has no closed extension:




If $f_n$ is chosen as suggested in the picture, then $f_n$ is Cauchy for $\mathcal{E}_1(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + (\cdot,\cdot)_{L^2}$, $f_n \underset{L^2}{\rightarrow} 0$, but $\mathcal{E}_1(f_n, f_n) \rightarrow 1$. $\square$

**Exercise:** What about the form $\mathcal{E}(f,g) = \frac{1}{2} \int_{\mathbb{R}^d} \nabla f(x) \cdot \nabla g(x) dx + f(0)\, g(0)$, for $f, g \in C_c^\infty(\mathbb{R}^d)$, for $d = 1$ and $d \geq 2$?

(Hint: when $d \geq 3$ use a radial function $g$ with compact support and $g(x) = |x|^{2-d}$ for $|x| \leq 1$. Use a smoothing $f_n$ of the sequence $\frac{1}{n}(g(x) \wedge n) = \widetilde{f}_n(x)$, with $f_n(0) = 1$ and $||f_n||_{H^1} \rightarrow 0$.) $\square$

It is in fact a crucial property that conversely if $\mathcal{E}(\cdot,\cdot)$ is a nonnegative densely defined symmetric bilinear form on $H$, which is closed, then $\mathcal{E}(\cdot,\cdot)$ is the form of a unique (as we shall now see) strongly continuous self-adjoint contraction semigroup $Q_t$ on $H$, (see Reed-Simon [RS79], Vol. 1, Theorem VIII, 15, p. 278).

We are now going to see that the quadratic form $\mathcal{E}$ associated to a strongly continuous self-adjoint contraction semigroup $Q_t$ on $H$ determines $Q_t$.

**Lemma 4.4:** *Let $G_\mu f = \int_0^\infty e^{-\mu s} Q_s f\, ds$, for $\mu > 0$, and $f \in H$, be the $\mu$-resolvant of $f$. Then*

(4.8) *$G_\mu f$ is the unique minimum of the function*
$$g \in \mathcal{D}(\mathcal{E}) \rightarrow \Phi_f(g) = \mathcal{E}_\mu(g,g) - 2(f,g)\,,$$
*(with $\mathcal{E}_\mu(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + \mu(\cdot,\cdot)$), and*

(4.9) *the minimum value of $\Phi_f(\cdot)$ equals: - $\mathcal{E}_\mu(G_\mu f, G_\mu f)$ .*

**Proof:** Observe that for $f, g$ in $H$:

$$\begin{aligned}(G_\mu f, g) &= \int_0^\infty e^{-\mu s}(Q_s f, g) ds = \int_0^\infty e^{-\mu s} \int_0^\infty e^{-\lambda s} d(E_\lambda f, g) ds \\ &= \int_0^\infty \left(\int_0^\infty e^{-(\lambda+\mu)s} ds\right) d(E_\lambda f, g) = \int_0^\infty \frac{1}{\mu + \lambda}\, d(E_\lambda f, g)\,.\end{aligned}$$

In other words:

$$(4.10)\qquad G_\mu f = \int_0^\infty \frac{1}{\mu+\lambda}\, dE_\lambda f \;\; (\text{and } G_\mu f \in \mathcal{D}(\mathcal{E}))\,.$$

It follows that for $g \in \mathcal{D}(\mathcal{E})$

$$\begin{aligned}&\mathcal{E}_\mu(G_\mu f, g) = \mu(G_\mu f, g) + \mathcal{E}(G_\mu f, g) = \int_0^\infty \frac{\mu}{\mu+\lambda}\, d(E_\lambda f, g)\\ &+ \int_0^\infty \frac{\lambda}{\lambda+\mu}\, d(E_\lambda f, g) = \int_0^\infty d(E_\lambda f, g) = (f, g)\,.\end{aligned}$$

Therefore for $g \in \mathcal{D}(\mathcal{E})$

$$(4.11)\qquad \begin{aligned}&\Phi_f(g) = \mathcal{E}_\mu(g,g) - 2(f,g) = \mathcal{E}_\mu(g,g) - 2\mathcal{E}_\mu(G_\mu f, g)\\ &= \mathcal{E}_\mu(g - G_\mu f, g - G_\mu f) - \mathcal{E}_\mu(G_\mu f, G_\mu f)\,,\end{aligned}$$

This proves our claims (4.8), (4.9). □

**Remark 4.5:** Let $L$ denote the generator of the semigroup, that is $Q_t = e^{-tL}$, and

$$(4.12)\quad L\,f = \int_0^\infty \lambda\, dE_\lambda f,\ \text{for } f \in \mathcal{D}(L) = \Big\{f \in H, \int_0^\infty \lambda^2\, d(E_\lambda f, f)\Big\} < \infty\,.$$

It follows from (4.10) that $G_\mu f$ is the solution of the equation

$$(4.13)\qquad \mu\, h + L\, h = f,\quad h \in H\,.$$

One can view (4.8) in a number of cases as a variational method to solve a P.D.E. problem.

For instance in the case of the Brownian semigroup (see Example (4.1)): $G_\mu f$, for $f \in L^2(\mathbb{R}^d)$ solves

$$\begin{aligned}&\mu\, h - \frac{1}{2}\,\Delta h &&= f\\ &\text{for } h \in \mathcal{D}\Big(-\frac{1}{2}\,\Delta\Big) &&= \Big\{f \in L^2, \int_{\mathbb{R}^d} (1+|\xi|^2)^2\, |\hat f(\xi)|^2 d\xi < \infty\Big\}\\ & &&= \Big\{f \in L^2, \Delta f \text{ (distribution sense)} \in L^2\Big\}\,,\end{aligned}$$

and (4.8) gives a variational characterization of $G_\mu f$, as the unique minimum of the functional:

$$g \in H^1(\mathbb{R}^d) \longrightarrow \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g|^2(x)dx + \mu \int_{\mathbb{R}^d} g^2(x)dx - 2\int_{\mathbb{R}^d} f(x)g(x)dx\,.$$

□

As a consequence of the above lemma we have

**Proposition 4.6:** *The quadratic form $(\mathcal{E}(\cdot,\cdot),\mathcal{D}(\mathcal{E}))$ uniquely determines $Q_t$.*

**Proof:** If $Q_t^1$ and $Q_t^2$ possess the same form $\big(\mathcal{E}(\cdot,\cdot),\mathcal{D}(\mathcal{E})\big)$, then $G_\mu^1 f = G_\mu^2 f$ for any $f \in H$ and $\mu > 0$, as follows from (4.8). Since $s \to Q_s^1 f$ and $s \to Q_s^2 f$ are continuous and bounded, uniqueness of the Laplace transform yields $Q_s^1 f = Q_s^2 f$, for $s \geq 0$. □

When the Hilbert space $H$ has the form $H = L^2(X,\mathcal{A},\nu)$, with $\nu$ a $\sigma$-finite measure, one can read the fact that $Q_t$ is positivity-preserving (i.e. $Q_t f \geq 0$, for $f \geq 0$) directly on the associated form $\mathcal{E}$.

**Proposition 4.7** *(first Beurling-Deny criterion)*
*$Q_t$ is positivity-preserving if and only if:*

$$\text{(4.14)}\qquad \textit{for } f \in \mathcal{D}(\mathcal{E}),\ |f| \in \mathcal{D}(\mathcal{E}) \textit{ and } \mathcal{E}(|f|,\,|f|) \leq \mathcal{E}(f,f)\,.$$

**Proof:** If $Q_t$ is positivity-preserving

$$\begin{aligned}\big((I-Q_t)|f|,|f|\big) &= (f,f) - (Q_t|f|,|f|)\\ &\leq (f,f) - (|Q_t f|,|f|)\\ &\leq (f,f) - (Q_t f,f) = \big((I-Q_t)f,f\big)\end{aligned}$$

and from (4.2), we see that whenever $f \in \mathcal{D}(\mathcal{E})$, $|f| \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(|f|,|f|) \leq \mathcal{E}(f,f)$.

Conversely if this last property holds, whenever $f \geq 0$, $\Phi_f(|g|) \leq \Phi_f(g)$, when $g \in \mathcal{D}(\mathcal{E})$, with the notations of (4.8). It now follows from (4.8) that $G_\mu f \geq 0$, when $\mu > 0$ and $f \geq 0$. If one now writes

$$Q_t f = \lim_{n\to\infty}\Big(1+\frac{tL}{n}\Big)^{-n} f = \lim_{n\to\infty}\Big(\frac{t}{n}\Big)^n G^n_{\frac{n}{t}} f$$

one derives that $Q_t$ is positivity-preserving. □

**Exercise:** (second Beurling-Deny criterion)
In the same vein show that $Q_t$ is sub-Markovian (i.e. $0 \leq Q_t f \leq 1$, for $0 \leq f \leq 1$), if and only if:

$$\text{(4.15)}\qquad \begin{aligned}&\text{for } f \in \mathcal{D}(\mathcal{E}),\ (0 \vee f)\wedge 1 \in \mathcal{D}(\mathcal{E}) \text{ and}\\ &\mathcal{E}\big((0\vee f)\wedge 1,(0\vee f)\wedge 1\big) \leq \mathcal{E}(f,f)\,.\end{aligned}$$

(Hint: for the sufficiency of (4.15) use the identity, with the notations of (4.8):

$$\Phi_f(g) + \frac{1}{\mu}\,(f,f) = \mathcal{E}(g,g) + \mu\Big(g - \frac{f}{\mu},\ g - \frac{f}{\mu}\Big)$$

and show that when $0 \le f \le 1$, $0 \le \mu\, G_\mu f \le 1$.

For the necessity of (4.15) consider $f = \sum_i^n \alpha_i\, 1_{A_i}$ where $\nu(A_i) < \infty$, $A_i$ pairwise disjoint, $\alpha_i \in \mathbb{R}$, and use the identity

$$\begin{aligned}((I - Q_t)\, f, f) \quad &= \quad \frac{1}{2} \sum_{i,j} (\alpha_i - \alpha_j)^2 (1_{A_i}, Q_t\, 1_{A_j}) \\ &+ \quad \sum_i \alpha_i^2 (1_{A_i},\ 1 - Q_t\, 1_{\cup A_j}) \quad )\end{aligned}$$

□

If the form $\mathcal{E}$ associated to $Q_t$ satisfies (4.15) it is usually called a *Dirichlet form.*

We are now finished with generalities on forms and are ready to start the description of forms associated to the self-adjoint contraction semigroups $R_t^{U,V}$ acting on $L^2(U, dx)$ in the obstacle case (i.e. $V \ge 0$). We start with

**Lemma 4.8:** *Assume $V_- \in K_d$, $V_+ \in K_d^{\mathrm{loc}}$, for $f \in L^2(U, dx)$ and $t > 0$,*

$$\begin{aligned}&\mathcal{E}^t(f,f) \overset{\text{def}}{=} \frac{1}{t}\,(f - R_t^{U,V} f, f)_{L^2} \\ (4.16) \qquad &= \frac{1}{2t} \int\!\!\int \Big(f(y) - f(x)\Big)^2 r_{U,V}(t,x,y)\,dx dy \\ &+ \frac{1}{t} \int\!\!\int f^2(x)\Big(p(t,x,y) - r_{U,V}(t,x,y)\Big)dx dy\ .\end{aligned}$$

**Proof:** Dropping the subscripts $U, V$ from the notation, we have:

$$\begin{aligned}\mathcal{E}^t(f,f) \;&=\; \frac{1}{t} \int\!\!\int \Big(f(x) - r(t,x,y) f(y)\Big) f(x)\,dx dy \\ &=\; \frac{1}{t} \int\!\!\int \Big(f(x) - f(y)\Big)\, r(t,x,y) f(x)\,dx dy + \\ &\quad\ \frac{1}{t} \int f^2(x)(1 - \int r(t,x,y)dy)dx\ .\end{aligned}$$

Using the symmetry of $r(t,\cdot,\cdot)$ and the equality $1 = \int p(t,x,y)dy$, the above expression equals

$$\frac{1}{2t} \int\!\!\int \Big(f(y) - f(x)\Big)^2 r(t,x,y)\,dx dy + \frac{1}{t} \int\!\!\int f^2(x)\Big(p(t,x,y) - r(t,x,y)\Big)dx dy$$

□

Our next objective is the functional analytic description of the Schrödinger semigroups.

**Theorem 4.9:** *Assume $V = V_+ \in K_d^{\text{loc}}$, $U = \mathbb{R}^d$. If $\mathcal{E}(\cdot,\cdot)$ is the form associated to $R_t^{U,V}$*

$$\mathcal{D}(\mathcal{E}) = \left\{ f \in H^1(\mathbb{R}^d),\ \int f^2(x)\, V(x) dx < \infty \right\} \quad \textit{and} \tag{4.17}$$

$$\textit{for } f \in \mathcal{D}(\mathcal{E}),\ \mathcal{E}(f,f) = \frac{1}{2} \int |\nabla f|^2(x) dx + \int V f^2(x) dx\,. \tag{4.18}$$

*Moreover*

$$C_c^\infty(\mathbb{R}^d) \textit{ is dense in } (\mathcal{D}(\mathcal{E}),\ \mathcal{E}_1(\cdot,\cdot))\,, \tag{4.19}$$

*(recall $\mathcal{E}_1(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + (\cdot,\cdot)_{L^2(\mathbb{R}^d)}$, and $C_c^\infty(\mathbb{R}^d)$ denotes the space of smooth functions on $\mathbb{R}^d$ with compact support).*

**Proof:** Let us briefly explain the strategy of the proof. We denote for convenience by $\widetilde{\mathcal{E}}$ and $\mathcal{D}(\widetilde{\mathcal{E}})$ the form and the domain which appear in the r.h.s. of (4.18) and (4.17). We shall first show that when $V$ is bounded $(\mathcal{E}, \mathcal{D}(\mathcal{E})) = (\widetilde{\mathcal{E}}, \mathcal{D}(\widetilde{\mathcal{E}}))$ holds. We shall then extend this equality to the general case and finally we shall prove (4.19).

*First step:* Case $V \geq 0$ bounded.

Then $\mathcal{D}(\widetilde{\mathcal{E}}) = H^1(\mathbb{R}^d)$. Observe that the two terms in the r.h.s. of (4.16) are nonnegative and that

$$\begin{aligned} &p(t,x,y)(f(y) - f(x))^2 \leq \\ &\qquad r(t,x,y)(f(y) - f(x))^2 + 2(p(t,x,y) - r(t,x,y))(f^2(x) + f^2(y)). \end{aligned}$$

It now follows that

$$\mathcal{D}(\mathcal{E}) = \{ f \in L^2(\mathbb{R}^d),\ \lim_{t \downarrow 0} \mathcal{E}^t(f,f) < \infty \} \subset \mathcal{D}(\mathcal{E}_{\mathbb{R}^d, V=0}) = H^1(\mathbb{R}^d)\,.$$

Observe also that for $f \in H^1(\mathbb{R}^d)$,

$$\begin{aligned} &\frac{1}{2t} \int\int (f(y) - f(x))^2 (p(t,x,y) - r(t,x,y)) dx dy \\ &= \frac{1}{2t} \int\int (f(y) - f(x))^2\, p(t,x,y)\, E_{x,y}^t \Big[ 1 - \exp\Big\{ - \int_0^t V(Z_s) ds \Big\} \Big] dx dy \\ &\leq \text{const} \cdot \|V\|_\infty\, t \to 0, \text{ as } t \to 0\,. \end{aligned}$$

It follows that for $f \in H^1(\mathbb{R}^d)$ the first term in the r.h.s. of (4.16) converges to $\frac{1}{2} \int |\nabla f|^2(x)dx$. Therefore the equality $(\mathcal{E}, \mathcal{D}(\mathcal{E})) = (\widetilde{\mathcal{E}}, \mathcal{D}(\widetilde{\mathcal{E}}))$ will follow once we show that for $f \in L^2(\mathbb{R}^d)$:

$$\text{(4.20)} \quad \lim_{t \downarrow 0} \int f^2(x) \, \frac{1}{t} \, E_x\Big[1 - \exp\Big\{ - \int_0^t V(Z_s)ds\Big\}\Big] dx = \int V(x) f^2(x) dx \, .$$

Notice that

$$\text{(4.21)} \quad \begin{aligned} &\frac{1}{t} \, E_x\Big[1 - \exp\Big\{ - \int_0^t V(Z_s)ds\Big\}\Big] = \\ &\frac{1}{t} \int_0^t ds \; E_x\Big[V(Z_s) \; \exp\Big\{ - \int_0^s V(Z_u)du\Big\}\Big] = \\ &V(x) \, \frac{1}{t} \int_0^t ds \int r(s,x,y)dy + \frac{1}{t} \int_0^t ds \int (V(y) \\ &-V(x)) \, r(s,x,y)dy \, . \end{aligned}$$

Now $\int r(s,x,y)dy = R_t^{\mathbb{R}^d, V} 1(x) \to 1$ as $t \to 0$, and the first term in the last member of (4.21) tends to $V(x)$. As for the second term observe that

$$\text{(4.22)} \quad \begin{aligned} &\int |V(y) - V(x)| \, r(s,x,y)dy \\ &\overset{(3.5)}{\leq} \; c_1 \int |V(y) - V(x)| \; \exp\Big\{ - \frac{(y-x)^2}{4s}\Big\} \, \frac{dy}{s^{d/2}} \\ &\leq \; \frac{c_1}{s^{d/2}} \int_{B(x, M s^{1/2})} |V(y) - V(x)| \, dy \\ &\quad + 2c_1 \, \|V\|_\infty \int_{|z|>M} \exp\Big\{ - \frac{|z|^2}{4}\Big\} dz, \quad \text{for } M > 0 \, . \end{aligned}$$

Now almost every $x$ in $\mathbb{R}^d$ is a differentiability point of $V(y)$ (see Theorem 8.8 of Rudin [Rud74b]), and for fixed $M$ and such $x$, the first term in the last member of (4.22) tends to zero, as $s \to 0$. It now follows that for almost every $x$ the l.h.s. of (4.22) tends to 0 as $s \to 0$, and therefore the l.h.s. of (4.21) tends to $V(x)$ as $t \to 0$. Our claim (4.20) follows.

*Second step:* General case.
Consider the forms $(\mathcal{E}_N, \mathcal{D}(\mathcal{E}_N) = H^1(\mathbb{R}^d))$ associated to $V_N = V \wedge N$, where $N \geq 0$. Then for $f \in H^1(\mathbb{R}^d)$, $\mathcal{E}_N(f,f)$ increases with $N$. One has in fact

$$\text{(4.23)} \quad \begin{aligned} \mathcal{D}(\widetilde{\mathcal{E}}) \;&=\; \{f \in L^2(\mathbb{R}^d), \; \sup_N \, \mathcal{E}_N(f,f) < \infty\}, \;\text{ and} \\ \widetilde{\mathcal{E}}(f,f) \;&=\; \sup_N \, \mathcal{E}_N(f,f), \;\text{ for } \; f \in L^2(\mathbb{R}^d) \, , \end{aligned}$$

(with the convention $\widetilde{\mathcal{E}}(f,f) = \infty$ if $f \notin \mathcal{D}(\widetilde{\mathcal{E}})$).

Now from the first line in (4.16), we have $\mathcal{E}^t(f,f) \geq \mathcal{E}_N^t(f,f)$ for $f \geq 0$, $t > 0$, and $N \geq 0$. Letting $t$ tend to 0 and then $N$ to infinity, we find

$$(4.24) \qquad \widetilde{\mathcal{E}}(f,f) \leq \mathcal{E}(f,f),\ f \in L^2_+(\mathbb{R}^d)\,,$$

($L^2_+(\mathbb{R}^d)$ denotes the set of nonnegative functions in $L^2(\mathbb{R}^d)$). On the other hand for $f \in L^2(\mathbb{R}^d)$ and $t > 0$,

$$(4.25) \qquad \begin{aligned} \mathcal{E}^t(f,f) &= \frac{1}{t}\,(f - R_t^{\mathbb{R}^d,V} f, f) \overset{\binom{\text{dominated}}{\text{convergence}}}{=} \lim_{N\to\infty} \mathcal{E}_N^t(f,f) \\ &\leq \lim_{N\to\infty} \mathcal{E}_N(f,f) = \widetilde{\mathcal{E}}(f,f)\,, \end{aligned}$$

and therefore using (4.24): $\mathcal{E}(f,f) = \widetilde{\mathcal{E}}(f,f)$ for $f \in L^2_+(\mathbb{R}^d)$.

This last equality immediately extends to $L^2(\mathbb{R}^d)$ and yields (4.17), (4.18).

*Third step:*

$$(4.26) \qquad C_c^\infty(\mathbb{R}^d) \text{ is } \widetilde{\mathcal{E}}_1 \text{ dense in } \mathcal{D}(\widetilde{\mathcal{E}})\,.$$

Pick $f \in \mathcal{D}(\widetilde{\mathcal{E}})$. Observe that $f$ can be approximated in $\widetilde{\mathcal{E}}_1$ norm by bounded functions of $\mathcal{D}(\widetilde{\mathcal{E}})$. Indeed, by (3.5), $R_t^{\mathbb{R}^d,V} f$ is bounded for $t > 0$, it belongs to $\mathcal{D}(\mathcal{E}) = \mathcal{D}(\widetilde{\mathcal{E}})$ and converges in $\mathcal{E}_1 = \widetilde{\mathcal{E}}_1$ norm to $f$ as $t \to 0$. We are thus reduced to the case of $f$ bounded in $\mathcal{D}(\widetilde{\mathcal{E}})$. We now pick a smooth function $\psi : \mathbb{R}^d \to [0,1]$, with $\psi = 1$ on $\bar{B}(0,1)$ and compact support. If we set $\psi_n(\cdot) = \psi(\frac{\cdot}{n})$, the sequence $f_n = f \cdot \psi_n$ belongs to $\mathcal{D}(\widetilde{\mathcal{E}})$ and tends to $f$ in $\widetilde{\mathcal{E}}_1$ norm. Indeed

$$\nabla f_n \overset{L^2}{=} \nabla f\, \psi_n + f\ \nabla \psi_n$$

as can be seen by approximation of $f$, the result follows since $\psi_n \to 1$ boundedly and $\nabla \psi_n(\cdot) = \frac{1}{n}\ \nabla\, \psi(\frac{\cdot}{n}) \to 0$ uniformly.

We are therefore reduced to the case of a bounded compactly supported $f \in \mathcal{D}(\widetilde{\mathcal{E}})$. We can regularize $f$ by convolution:

$$f_n(x) = \int \rho_n(x-y) f(y) dz, \text{with } \rho_n = n^d\, \rho(n\cdot), \text{for} n \geq 1, \text{provided} \rho \in C_c^\infty(\mathbb{R}^d)$$

is nonnegative and $\int \rho\, dx = 1$.

Our claim (4.26) will follow from the fact that $f_n \in C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(\widetilde{\mathcal{E}})$, and tends to $f$ in $\widetilde{\mathcal{E}}_1$ norm.

The fact that $f_n$ tends to $f$ in $H^1(\mathbb{R}^d)$ is standard. Let $B$ denote some compact subset of $\mathbb{R}^d$ outside which, $f_n$ and $f$ vanish. Since $V \in K_d^{\text{loc}} \subseteq L^1_{\text{loc}}$, for $M > 0$ we have

$$\begin{aligned}&\overline{\lim_n} \int |f - f_n|^2\, V(x)dx \le M \;\overline{\lim_n} \int |f - f_n|^2\, dx + 4\|f\|^2_\infty \int_{B\cap\{V>M\}} V(x)dx \\ &= 4\|f\|^2_\infty \int_{B\cap\{V>M\}} V(x)dx \overset{M\to\infty}{\longrightarrow} 0\,,\end{aligned}$$

which finishes the proof of (4.26) and of Theorem 4.9. □

**Remark 4.10:** It is clear that in the above theorem the assumption $V = V_+ \in K_d^{\text{loc}}$ is not the most general possible when studying the obstacle case situation. In fact it is not too difficult to see that the arguments we gave can be extended to the case $V = V_+ \in L_1^{\text{loc}}$. Of course in this case one only has $\int_0^t V(Z_s)ds < \infty$, $P_x$-a.s. for a.e. $x$, and one has to slightly modify the proof of (3.14). For more on this we refer to Simon [Sim79], [Sim82]. □

**Exercise:** Show that when $f \in H^1(\mathbb{R}^d)$, for any $a \in \mathbb{R}$,

(4.27) $$\nabla f(x)\; 1\{f(x) = a\} = 0 \;\text{ a.e.}$$

(Hint: approximate $\varphi(x) = x \wedge a$ from above and below by $\varphi_n$, $\widetilde{\varphi}_n$ converging respectively to $1(x \le a)$ and $1(x < a)$, so that $\varphi_n \circ f$ and $\widetilde{\varphi}_n \circ f$ are $H^1$-Cauchy). Show that

(4.28) $$\nabla(f \wedge a) = \nabla f \cdot 1(f \le a) = \nabla f\; 1(f < a) \;\text{ in }\; L^2\,.$$

□

We now shall turn to the functional analytic description of Dirichlet-Schrödinger semigroups, and determine the Dirichlet form $\mathcal{E}_{U,V}$ associated to the semigroup $R_t^{U,V}$ acting on $L^2(U, dx)$, when $U$ is a nonempty open subset of $\mathbb{R}^d$ and $V \ge 0$ belongs to $K_d^{\text{loc}}$.

Let us give a word of comment on the type of difficulty which is involved. It turns out that it is not too difficult to show that $\mathcal{D}(\mathcal{E}_{\mathbb{R}^d,V}) \supset \mathcal{D}(\mathcal{E}_{U,V}) \supset C_c^\infty(U)$, and that on $C_c^\infty(U)$, $\mathcal{E}_{U,V}$ coincides with the expression given in (4.18). However it is more delicate to show that $C_c^\infty(U)$ is $\mathcal{E}_{U,V}(\cdot,\cdot) + (\cdot,\cdot)_{L^2}$ dense in $\mathcal{D}(\mathcal{E}_{U,V})$, in the absence of smoothness assumptions on $U$. To appreciate the type of problem involved, here is the following

**Exercise:** $U = \{x_1 > 0\} \subseteq \mathbb{R}^d$. Let $f \in H^1(\mathbb{R}^d)$ with $f = 0$ a.s. on $U^c$. Show that $f \in H_0^1(U)$ (the closure of $C_c^\infty(U)$ in $H^1(\mathbb{R}^d)$).

(Hint: approximate $f$ by $f(\cdot - \frac{e_1}{n})$ , if $e_1$ denotes the first vector of the canonical basis of $\mathbb{R}^d$, then regularize).

Give a counterexample in the case of a general open set $U$.

(Hint: see Chapter 2, Example 5.2). □

**Theorem 4.11:** *Let $U$ be a nonvoid open subset of $\mathbb{R}^d$ and $V = V_+ \in K_d^{\mathrm{loc}}$. If $\mathcal{E} = \mathcal{E}_{\mathbb{R}^d,V}$ and $\mathcal{E}_{U,V}$ denote respectively the Dirichlet form of (4.18) and the form associated to $R_t^{U,V}$ acting on $L^2(U, dx)$,*

(4.29) $\mathcal{E}$ *extends* $\mathcal{E}_{U,V}$ *and* $\mathcal{D}(\mathcal{E}_{U,V})$ *is the* $\mathcal{E}_1(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + (\cdot,\cdot)_{L^2}$ closure of $C_c^\infty(U)$.

**Proof:** For simplicity of notations we shall drop the subscript $V$. So for instance $r, r_U$ will be the respective kernels of the semigroups $R_t^{\mathbb{R}^d,V}$ and $R_t^{U,V}$ and $\mathcal{E}$, $\mathcal{E}_U$ the associated Dirichlet forms. As follows from (4.16) for $f \in L^2(U, dx)$, and $t > 0$:

$$
\begin{aligned}
\mathcal{E}_U^t(f,f) &= \frac{1}{2t} \int\!\!\int \Big(f(y) - f(x)\Big)^2 r_U(t,x,y)\,dx\,dy \\
&\quad + \frac{1}{t} \int\!\!\int f^2(x)\Big(p(t,x,y) - r_U(t,x,y)\Big)dx\,dy \\
&= \frac{1}{2t} \int\!\!\int \Big(f(y) - f(x)\Big)^2 r(t,x,y)\,dx\,dy \\
&\quad + \frac{1}{t} \int\!\!\int f^2(x)\Big(p(t,x,y) - r(t,x,y)\Big)dx\,dy \\
&\quad - \frac{1}{2t} \int\!\!\int \Big(f(y) - f(x)\Big)^2 (r - r_U)(t,x,y)\,dx\,dy \\
&\quad + \frac{1}{2t} \int\!\!\int \Big(f^2(x) + f^2(y)\Big)(r - r_U)(t,x,y)\,dx\,dy
\end{aligned}
\tag{4.30}
$$

(using symmetry in the last expression)

$$
= \mathcal{E}^t(f,f) + \frac{1}{t} \int\!\!\int f(y)f(x)(r - r_U)(t,x,y)\,dx\,dy\ .
$$

It follows from (4.16) that when $f \in \mathcal{D}(\mathcal{E}_U)$:

$$
\overline{\lim_{t\to 0}}\ \frac{1}{2t} \int\!\!\int \Big(f(y) - f(x)\Big)^2 r_U(t,x,y)\,dx\,dy < \infty, \quad \text{and} \tag{4.31}
$$

$$
\overline{\lim_{t\to 0}}\ \frac{1}{2t} \int\!\!\int \Big(f^2(y) + f^2(x)\Big)\Big(p(t,x,y) - r_U(t,x,y)\Big)dx\,dy < \infty\ . \tag{4.32}
$$

Observe now that $p - r_U = p - r + r - r_U$ with $p - r \geq 0$, $r - r_U \geq 0$, and that $(f(y) - f(x))^2 \leq 2(f^2(y) + f^2(x))$. As now follows from (4.31), (4.32): $\overline{\lim_{t\to 0}}\ \mathcal{E}^t(f,f) < \infty$. We have shown that:

$$
\mathcal{D}(\mathcal{E}_U) \subset \mathcal{D}(\mathcal{E})\ . \tag{4.33}
$$

Moreover as follows from the last line of (4.30), letting $t$ tend to 0:

(4.34) $\quad$ when $f \geq 0$ belongs to $\mathcal{D}(\mathcal{E}_U)$, $\mathcal{E}_U(f,f) \geq \mathcal{E}(f,f)$ .

Our next step is to show

(4.35) $$\begin{aligned}&\text{if } f \in \mathcal{D}(\mathcal{E}) \text{ and } f = 0 \text{ a.s. outside a compact } K \subset U,\\ &\text{then } f \in \mathcal{D}(\mathcal{E}_U) \text{ and } \mathcal{E}_U(f,f) = \mathcal{E}(f,f)\ .\end{aligned}$$

To this end we estimate the rightmost term in the last member of (4.30), using Cauchy-Schwarz inequality, for $f$ as in (4.35) and $t > 0$:

$$\begin{aligned}&\Big|\frac{1}{t}\int\int f(x)f(y)(r-r_U)(t,x,y)dxdy\Big| \overset{\binom{\text{Cauchy-Schwarz}}{\text{and symmetry}}}{\leq}\\ &= \frac{1}{t}\int\int f^2(x)(r-r_U)(t,x,y)dxdy \leq \frac{1}{t}\int_K f^2(x)\ P_x[T_U \leq t]dx\ .\end{aligned}$$

Now $\lim\limits_{t\to 0}\ \sup\limits_{x\in K}\ \frac{1}{t}\,P_x[T_U \leq t] \leq \lim\limits_{t\to 0}\ \frac{1}{t}\ P_0\big[\sup\limits_{u\leq 1}|Z_u| \geq \frac{\mathrm{dist}(K,U^c)}{\sqrt{t}}\,\big] = 0.$

Our claim (4.35) now follows from the last line of (4.30).

We now let $G_1$ and $G_1^U$ stand for the 1-resolvant operators of the semigroups $R_t^{\mathbb{R}^d,V}$, $R_t^{U,V}$, (see Lemma 4.4). Our next objective in order to establish Theorem 4.11 is the derivation of the following a priori bounds:

**Lemma 4.12:** *When $\varphi \geq 0$ belongs to $L^2(U,dx)$:*

(4.36) $$\mathcal{E}_1(G_1^U\varphi, G_1^U\varphi) \leq \mathcal{E}_{U,1}(G_1^U\varphi, G_1^U\varphi) \leq \mathcal{E}_1(G_1\varphi, G_1\varphi) < \infty\ .$$

**Proof:** The leftmost inequality is a consequence of (4.34). Let us now prove the rightmost inequality. Denote by $\mathcal{D}_+(\mathcal{E}_U)$ the set $\mathcal{D}(\mathcal{E}_U)\cap L^2_+(U,dx)$. From (4.9) we have:

$$-\mathcal{E}_{U,1}(G_1^U\varphi, G_1^U\varphi) = \min_{f\in\mathcal{D}(\mathcal{E}_U)} \mathcal{E}_{U,1}(f,f) - 2(f,\varphi)_{L^2}$$

and by (4.14) together with $\varphi \geq 0$,

$$\begin{aligned}&= \min_{f\in\mathcal{D}_+(\mathcal{E}_U)} \mathcal{E}_{U,1}(f,f) - 2(f,\varphi)_{L^2}\\ &\overset{(4.34)}{\geq} \min_{f\in\mathcal{D}_+(\mathcal{E}_U)} \mathcal{E}_1(f,f) - 2(f,\varphi)_{L^2} \overset{(4.33)}{\geq} \min_{f\in\mathcal{D}(\mathcal{E})} \mathcal{E}_1(f,f) - 2(f,\varphi)_{L^2}\\ &\overset{(4.9)}{=} -\mathcal{E}_1(G_1\varphi, G_1\varphi)\ .\end{aligned}$$

Now the rightmost inequality of (4.36) follows. $\square$

We now denote for convenience the set of functions which appears in (4.35) by $\mathcal{D}_U^{\text{comp}}(\mathcal{E})$, that is

$$\mathcal{D}_U^{\text{comp}}(\mathcal{E}) = \{f \in \mathcal{D}(\mathcal{E}),\ \exists K \subset U,\ K \text{ compact and } f \overset{\text{a.s.}}{=} 0 \text{ on } K^c\}\,. \tag{4.37}$$

Our next objective is to show the following

**Lemma 4.13:** *When $K$ is a compact subset of $U$,*

(4.38) *for $\varphi \in L^2(U, dx)$, with $\varphi = 0$ on $K^c$, $G_1^U \varphi$ lies in the* $\mathcal{E}_1$ closure of $\mathcal{D}_U^{\text{comp}}(\mathcal{E})$ (which is also the $\mathcal{E}_{U,1}$ closure of $\mathcal{D}_U^{\text{comp}}(\mathcal{E})$ by (4.35)) .

**Proof:** Observe that we can restrict ourselves without loss of generality to the special case $\varphi \geq 0$. Then introduce an increasing sequence of relatively compact subsets $U_n$ of $U$ with $K \subset U_n$ and $U = \bigcup_{n \geq 1} U_n$.

As follows from (4.36) applied with $U_n$, the sequence $G_1^{U_n}\varphi$ has bounded $\mathcal{E}_1$ norm. However for $x \in U$

$$G_1^U \varphi(x) \geq G_1^{U_n}\varphi(x) = E_x\Big[\int_0^{T_{U_n}} \varphi(Z_s)\ \exp\Big\{-\int_0^s (1+V)(Z_u)ds\Big\}\Big]$$

and since $T_{U_n} \uparrow T_U$ $P_x$-a.s., the last sequence increases to

$$E_x\Big[\int_0^{T_U} \varphi(Z_s)\ \exp\Big\{-\int_0^s (1+V)(Z_u)du\Big\}\Big] = G_1^U\varphi(x), \quad \text{as } n \to \infty\,.$$

Therefore by dominated convergence:

$$G_1^{U_n}\varphi \overset{L^2(U,dx)}{\longrightarrow} G_1^U \varphi\,. \tag{4.39}$$

Now since $G_1^{U_n}\varphi$ is $\mathcal{E}_1$ bounded, there is $\psi \in \mathcal{D}(\mathcal{E})$ and a subsequence $n_k$ such that: $G_1^{U_{n_k}}\varphi \underset{k\to\infty}{\overset{\mathcal{E}_1}{\rightharpoonup}} \psi$ (i.e. $\mathcal{E}_1$-weakly). Now from (4.39) we have necessarily $\psi = G_1^U\varphi$.

This implies that $G_1^U\varphi$ belongs to the $\mathcal{E}_1$ closure of the space of the $G_1^{U_n}\varphi$, (indeed $\mathcal{E}_1(G_1^U\varphi, f) = 0$ whenever $f \in \mathcal{D}(\mathcal{E})$ is orthogonal to the space spanned by the $G_1^{U_n}\varphi$).

Since the $G_1^{U_n}\varphi$ belong to $\mathcal{D}_U^{\text{comp}}(\mathcal{E})$, we have shown (4.38). □

At this point we know from (4.35) that $\mathcal{E}$ and $\mathcal{E}_U$ coincide on $\mathcal{D}_U^{\text{comp}}(\mathcal{E})$. The proof of Theorem 4.12 will be completed once we show

$$\text{(4.40)} \qquad \mathcal{D}_U^{\text{comp}}(\mathcal{E}) \text{ is } \mathcal{E}_{U,1} \text{ dense in } \mathcal{D}(\mathcal{E}_U)$$

$$\text{(4.41)} \qquad C_c^\infty(U) \text{ is } \mathcal{E}_1 \text{ dense in } \mathcal{D}_U^{\text{comp}}(\mathcal{E}) \,.$$

Let us begin with (4.40). In view of (4.38) it suffices to show that any $f \in \mathcal{D}(\mathcal{E}_U)$ is $\mathcal{E}_1$ approximable by functions of the form $G_1^U \varphi$ with $\varphi = 0$ a.s. outside a compact subset of $U$. Now observe that

$$\mathcal{E}_{U,1}(f - \alpha G_\alpha^U f, f - \alpha G_\alpha^U f) = \int_0^\infty (\lambda+1)\Big(1 - \frac{\alpha}{\alpha+1}\Big)^2 d(E_\lambda^U f, f) \underset{\alpha \to +\infty}{\longrightarrow} 0 \,,$$

if $E_\lambda^U$ is a resolution of the identity associated to the semigroup $R_t^{U,V}$. Moreover from the resolvant identity we have

$$G_1^U = G_\alpha^U + (\alpha - 1)\, G_1^U\, G_\alpha^U \,.$$

Indeed for $f, g \in L^2(U, dx)$

$$(G_\alpha^U f, g)_{L^2} + (\alpha-1)(G_1^U\, G_\alpha^U f, g)_{L^2} = \int_0^\infty \frac{1}{\alpha+\lambda}\, d(E_\lambda^U f, g) +$$
$$\int_0^\infty \frac{(\alpha-1)}{(1+\lambda)(\alpha+\lambda)}\, d(E_\lambda^U f, g) = \int_0^\infty \frac{1}{(1+\lambda)}\, d(E_\lambda^U f, g) = (G_1^U f, g)_{L^2} \,.$$

Therefore $G_\alpha^U(L^2(U)) \subseteq G_1^U(L^2(U))$ (in fact one has equality). Our claim (4.40) now simply follows from the fact that functions $\varphi \in L^2(U)$ with compact support are dense in $L^2(U)$.

As for the proof of (4.41) it is a routine truncation argument since by (4.19) of Theorem 4.10, any $f \in \mathcal{D}_U^{\text{comp}}(\mathcal{E})$ is $\mathcal{E}_1$ approximated by a sequence $f_n$ in $C_c^\infty(\mathbb{R}^d)$. □

**Remark 4.14:** Assume that $V$ belongs to $L^2_{\text{loc}} \cap K_d^{\text{loc}}$ is nonnegative and consider the $L^2(U, dx)$-valued densely defined operator

$$\text{(4.42)} \qquad L f = -\frac{1}{2} \Delta f + V f, \; f \in C_c^\infty(U) = \mathcal{D}(L) \,.$$

Then $L$ is symmetric, that is $(Lf, g) = (f, Lg)$ for $f, g \in C_c^\infty(U)$. By a theorem of Friedrichs, since $(Lf, f)_{L^2} \geq 0$, the form $(Lf, f)_{L^2}$ is closable, see Example 4.3 and Reed-Simon [RS79], Vol. 2, Theorem X.23. Its closure is given as

$$q(f, f) = (\sqrt{\hat{L}}\, f,\, \sqrt{\hat{L}}\, f),\; f \in \mathcal{D}(\sqrt{\hat{L}}) \,,$$

where $\hat{L}$ is a nonnegative self-adjoint extension of $L$: the so-called *Friedrichs extension.* It is also the form associated in (4.2), (4.3) to the strongly continuous semigroup $Q_t = e^{-t\hat{L}}$ on $L^2(U, dx)$. By (4.29) the closure $q$ coincides with $\mathcal{E}_{U,V}$, and consequently $\hat{L}$ is the generator of the Dirichlet-Schrödinger semigroup $R_t^{U,V}$. □

**Exercise:** (Generators are more complicated than forms)

Consider $d = 1$, $U = (0,1)$, $V = 0$, and show that $C_c^\infty(0,1)$ is not dense in the domain of the generator $\hat{L}$ of the semigroup $R_t^{(0,1),0}$ on $L^2(0,1)$ for the graph norm:

$$|||f|||^2 = (f,f) + (\hat{L}f, \hat{L}f), \; f \in \mathcal{D}(\hat{L}) .$$

□

We now give an application of Theorem 4.11 concerning the bottom of the spectrum of the generator of the semigroup $R_t^{U,V}$ given by:

$$\begin{aligned} \lambda_V(U) &= \inf\Big\{ \int_0^\infty \lambda d(E_\lambda f, f), \; \|f\|^2_{L^2(U)} = 1 \Big\} \\ &= \inf\{\mathcal{E}_{U,V}(f,f), \; f \in \mathcal{D}(\mathcal{E}_{U,V}), \; \|f\|_{L^2(U)} = 1\} , \end{aligned} \tag{4.43}$$

provided $E_\lambda$ stands for a resolution of the identity associated to the semigroup $R_t^{U,V}$.

**Corollary 4.15:** *Let $U$ be a nonvoid open subset of $\mathbb{R}^d$ and $V \geq 0$ belong to $K_d^{\text{loc}}$, then*

$$\lambda_V(U) = \inf\Big\{ \int_U \frac{1}{2} \mid \nabla f|^2 + V f^2 \, dx, \; f \in C_c^\infty(U), \; \int f^2 dx = 1 \Big\} . \tag{4.44}$$

**Proof:** Immediate from Theorem 4.11. □

Some extensions of the above results to the case of possibly nonpositive potentials can be found in the following exercises, together with some applications to the Cauchy problem.

**Exercise:**

1) Consider $W \geq 0$ in $K_d$. Show that for any $\epsilon > 0$, there exists $M > 0$ such that for $f \in H^1(\mathbb{R}^d)$:

$$\int W \, |f|^2 dx \leq \frac{\epsilon}{2} \int |\nabla f|^2 dx + M \int |f|^2 dx . \tag{4.45}$$

(Hint: choose $M > 0$ so that $R_t^{\mathbb{R}^d, V}$ is a contraction semigroup, where $V = M - \frac{1}{\epsilon} W$, and then apply Theorem 4.9 to find the form associated to the contraction semigroup $R_t^{\mathbb{R}^d, V_N}$, where $V_N = M - (\frac{1}{\epsilon} W) \wedge N$.)

**2)** (The form associated to the self-adjoint contraction semigroup $R_t^{U,V}$).

Assume $V = V_+ - V_-$, where $V_+ \in K_{\text{loc}}^d$, $V_- \in K_d$, is such that $R_t^{U,V}$ is a self-adjoint contraction semigroup on $L^2(U, dx)$, (see Remark 3.4).

**a)** Use 1) to show that the form $q(\cdot,\cdot)$ with domain $\mathcal{D}_{U,V_+}$

$$q(f,f) = \mathcal{E}_{U,V_+}(f,f) - \int V_- \, |f|^2 dx, \; f \in \mathcal{D}_{U,V_+} \,, \tag{4.46}$$

is well defined, nonnegative, closed and that in (4.46) $\mathcal{D}_{U,V_+}$ is the $q_1(\cdot,\cdot) = q(\cdot,\cdot) + (\cdot,\cdot)_{L^2}$ closure of $C_c^\infty(U)$.

**b)** If $\hat{L}$ denotes the nonnegative self-adjoint operator on $L^2(U, dx)$ associated to $q$, and $Q_s = e^{-s\hat{L}}$, $s \geq 0$, show that for $\mu > 0$, $f \in L^2(U, dx)$,

$$\int_0^\infty e^{-\mu s} R_s^{U,V_N} f \, ds \underset{N\to\infty}{\longrightarrow} \int_0^\infty e^{-\mu s} Q_s \, f \, ds \,, \tag{4.47}$$

if $V_N = V_+ - (N \wedge V_-)$.

(Hint: use (4.8) together with the fact that the forms $q_N$ associated to $R_s^{U,V_N}$ have domain $\mathcal{D}_{U,V_+}$ and decrease to $q$, as follows from Theorem 4.11).

**c)** Show that for $\mu > 0$, $f \in L^2(U, dx)$

$$\int_0^\infty e^{-\mu s} R_s^{U,V} f \, ds = \int_0^\infty e^{-\mu s} Q_s \, f \, ds \,,$$

and deduce the fact that

$$(q(\cdot,\cdot), \mathcal{D}_{U,V_+}) \text{ is the form associated to } R_t^{U,V} \,. \tag{4.48}$$

**3)** (Existence of a bounded solution to a Cauchy problem)

Assume $V \in L^2_{\text{loc}} \cap K_d^{\text{loc}}$ is nonnegative and $v$ is a bounded measurable function on $\mathbb{R}^d$. A bounded measurable function $u(s,x)$, $x \geq 0$, $s \in \mathbb{R}^d$, is called a *weak solution* of the Cauchy problem:

$$\begin{cases} \partial_s u & = \; \frac{1}{2} \Delta u - V u \,, \\ u(0,\cdot) & = \; v(\cdot) \end{cases} \tag{4.49}$$

if

$$\text{a)} \quad s \geq 0 \to \int u(s,x)\,\psi(x)dx \text{ is continuous for each } \psi \in C_c^\infty(\mathbb{R}^d),$$

$$\text{(4.50)} \quad \text{b)} \quad \int u(T,x)\,\varphi(T,x)dx - \int v(x)\,\varphi(0,x)dx = \int_{[0,T]\times\mathbb{R}^d} u(s,x)\,(\partial_s + \frac{1}{2}\,\Delta - V)\,\varphi(s,x)dsdx, \text{ for } \varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^d) \text{ and } T > 0\,.$$

Show that

$$\text{(4.51)} \qquad u(t,x) = R_t^{\mathbb{R}^d,V} v(x) \text{ is a weak solution of (4.49)}\,.$$

(Hint: for (4.50) a), use Propositions 3.1 and 3.5. For (4.50) b) define $u_N$ by replacing $v$ with $v 1_{B(0,N)}$ in (4.51), and differentiate the function $s \to (u_N(s,\cdot), \varphi(s,\cdot))_{L^2}$, to recover (4.50) b) with $u$ replaced by $u_N$ and $v$ by $v 1_{B(0,N)}$).

**4)** (Uniqueness of bounded solutions to the Cauchy problem).
Assume $V$ is as in 3). Consider $0 < t_0 < \tau_0$, $\epsilon \in (0, |\tau_0 - t_0|)$, $\varphi_\epsilon(\cdot) = \frac{1}{\epsilon}\,\varphi(\frac{\cdot}{\epsilon})$ a smooth nonnegative approximation of the Dirac mass with support in $(-\epsilon,\epsilon)$, $w_N(\cdot)$, $N \geq 1$, a sequence of $[0,1]$-valued smooth functions with support in $B(0,N+1)$ equal to 1 on $\overline{B}(0,N)$, such that

$$\text{(4.52)} \qquad \sup_{N,i,j} \|\partial_i w_N\|_\infty + \|\partial^2_{ij} w_N\|_\infty < \infty\,.$$

**a)** Assume $u$ is a weak solution of (4.49) with $v = 0$, in the sense of 3). Show that for $x_0 \in \mathbb{R}^d$, $t < t_0 + \epsilon$, $N \geq 1$,

$$\text{(4.53)} \quad \begin{aligned} &\int_{\mathbb{R}^d} u(t_0,y)\,\varphi_\epsilon(t_0 - t)\,w_N(y)\,r(\tau_0 - t, y, x_0)dy = \\ &\int_0^{t_0} ds \int_{\mathbb{R}^d} dy\, u(s,y)\{\varphi'_\epsilon(s-t)\,r(\tau_0 - t, y, x_0)\,w_n(y) \\ &\qquad + \varphi_\epsilon(s-t)(-\hat{L}\,r(\tau_0 - t, y, x_0)\,w_N(y) \\ &\qquad + \nabla r(\tau_0 - t, y, x_0)\,\nabla w_N(y) \\ &\qquad + r(\tau_0 - t, y, x_0)\,\frac{1}{2}\,\Delta w_N(y))\}\,, \end{aligned}$$

with $\hat{L}$ the Friedrichs extension of $L$ in (4.42), and $r \overset{\text{def}}{=} r_{\mathbb{R}^d,V}$.

(Hint: use the fact one can find $\psi_n \in C_c^\infty(\mathbb{R}^d)$ such that $\psi_n \xrightarrow{L^2} r(\tau_0 - t, \cdot, x_0)$ and $L\psi_n \xrightarrow{L^2} \hat{L}r(\tau_0 - t, \cdot, x_0)$, cf. Reed-Simon [RS79], Vol. 2, Theorem X.28, and apply (4.50) b) with $\varphi(s,x) = \varphi_\epsilon(s-t)\,\psi_n(x)\,w_N(x)$).

**b)** Show that with $u$ as in a), for $x_0 \in \mathbb{R}^d$:

$$(4.54) \qquad \int_{\mathbb{R}\times\mathbb{R}^d} dtdy\, u(t_0,y)\,\varphi_\epsilon(t_0-t)\,r(\tau_0-t,y,x_0)=0\,.$$

(Hint: integrate (4.53) over $t$, and then let $N$ tend to infinity, using the fact that
$\partial_t r(\tau_0-t,\cdot,x_0) \overset{L^2}{=} -\hat{L}r(\tau_0-t,\cdot,x_0)$, for $t<t_0+\epsilon$, and
$\int_{[-\epsilon,t_0+\epsilon]\times\mathbb{R}^d}(r^2+|\nabla r|^2)(\tau_0-t,y,x_0)dydt<\infty$, cf. Theorem 4.9.

**c)** Deduce that $u=0$, $dsdx$-a.e.

**d)** Show that when $u$ is a weak solution of (4.49), then

$$(4.55) \qquad u(s,x)=R_s^{\mathbb{R}^d,V}v(x),\ dsdx\text{-a.s.}$$

□

## 1.5 Notes and References

The review article by Simon [Sim82] and the book by Chung and Zhao [CZ95] contain a wealth of information and references about Schrödinger semigroups. In contrast to [CZ95], our treatment privileges quadratic forms over generators. The Feynman-Kac formula goes back to Kac [Kac51], see also [Kac59]. In [Kea77], it is used by McKean to define and study the Schrödinger semigroups, see also Ma [Ma91]. The probabilistic characterization of the space $K_d$ introduced by Kato in [Kat73] can be found in Aizenman-Simon [AS82]. The definition of the transition kernels in (3.4) differs from the traditional definition due to Hunt [Hun56], where the event $\{T_U>t\}$ is replaced by $\{\widetilde{T}_U\geq t\}$, with $\widetilde{T}_U$ the hitting time of $U^c$ (see Chapter 2 §2). The kernels we define here are easier to manipulate, but lead to a less pretty formula (2.2.11), (compare for instance with Bass [Bas95], formula (5.1) of Chapter II).

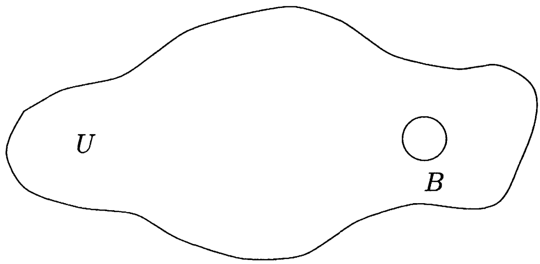
Further material and references on quadratic forms associated to semigroups can be found in volume II of Reed-Simon [RS79] or in Kato [Kat84]. The notions of Dirichlet forms in a probabilistic context are extensively developed and discussed in Fukushima [Fuk80], Fukushima-Oshima-Takeda [FOT94], Ma-Röckner [MR92]. The general theory of Dirichlet forms gives a way to characterize the form associated to the operation of killing when exiting an open set (cf. Theorem 4.4.2 of [Fuk80]). Here we have used a rather 'down to earth' approach to the proof of Theorem 4.11. Connections between Dirichlet forms and kernel estimates can for instance be found in Carlen-Kusuoka-Stroock [CKS87] and in the book of Davies [Dav89]. Further results on the Cauchy problem in Hölder classes appear in the books of Friedman [Fri64] and Krylov [Kry96]. For the case of weak solutions, we also refer to Chapter I and III in the book of Ladyzhenskaya-Solonnikov-Ural'tseva [LSU68].

# 2. Some Potential Theory

The aim of this chapter is to develop in the context of obstacles (i.e. $V \geq 0$), the notions of potential theory, we shall later use. In Section 2 we collect some results on Green functions. In Section 3 we introduce, à la Chung, equilibrium potentials, equilibrium distributions and capacities by means of last exit distributions. Section 4 develops the maximum principle, the variational principle as well as a number of examples. In Section 5 we link capacity and Dirichlet forms, through another variational formula.

## 2.1 A Warm-up Calculation

We shall now perform a slightly formal calculation which will somehow point out some of the objects of interest for the rest of this chapter.




We consider $U$ a bounded smooth open subset of $\mathbb{R}^d$, $V \geq 0$ a smooth function on $\mathbb{R}^d$ and $B \subseteq U$ a closed ball. We are interested in a certain 'equilibrium problem' which we now describe. To this end, we consider the 'equilibrium potential' $v(\cdot)$ of $B$ when $\partial U$ is grounded. It is the solution of

$$(1.1) \qquad \begin{cases} \frac{1}{2}\,\Delta v - V\,v = 0 \text{ on } U \backslash B \\ v|_{\partial U} = 0, \; v|_B = 1 \,. \end{cases}$$

We also introduce the Green function relative to $U, V$:

$$(1.2) \qquad g(x,y) = \int_0^\infty r_{U,V}(s,x,y)ds \,.$$

The '*equilibrium problem*' is then to find a measure $\mu$ on $B$ such that

$$v(x) = \int_B g(x,y)\, \mu(dy),\ x \in U\,. \tag{1.3}$$

We now provide a formal derivation of the formula expressing $\mu$ in terms of $v$, and $V$.

Observe that for $x \in U$

$$\frac{1}{2}\,\Delta g^x - V g^x = -\delta_x, \text{ in the distribution sense,} \tag{1.4}$$

if $g^x$ stands for the function $g(x,\cdot)$.

Indeed by a computation analogous to the one following (1.4.10), for $\varphi, \psi \in C_c^\infty(U)$:

$$\begin{aligned} \int \varphi\,\psi\, dx \quad &= \quad \mathcal{E}(G\varphi,\psi) = \int \frac{1}{2}\,\nabla(G\varphi)\,\nabla\psi + V\,G\varphi\,\psi\, dx \\ &\underset{\text{by parts}}{\overset{\text{integration}}{=}} \quad \int G\varphi\,(-\frac{1}{2}\,\Delta + V)\psi\, dx\,, \end{aligned}$$

where $G\,\varphi(x) = \int g(x,y)\,\varphi(y)dy$.

If we now let $\varphi$ tend to $\delta_y$, $y \in U$, we find

$$\psi(y) = \int g(x,y)\Big(-\frac{1}{2}\,\Delta + V\Big)\,\psi(x)dx\,, \tag{1.5}$$

from which (1.4) follows by symmetry of $g(\cdot,\cdot)$.

Moreover for a smooth bounded domain $O$ of $\mathbb{R}^d$, and smooth functions

$$F : \bar{O} \to \mathbb{R}^d \text{ and } \varphi,\psi : \bar{O} \to \mathbb{R},$$

Green's identity, cf. §26 of [DFN93], implies that:

$$\int_O F\cdot\nabla\varphi\, dx = \int_{\partial O} (F\cdot n)\,\varphi\, dS - \int_O \operatorname{div} F\,\varphi\, dx\,, \tag{1.6}$$

where $n$ stands for the exterior normal to $\partial O$ and $dS$ for the surface measure on $\partial O$. An application of (1.6) with $F = \nabla\psi$ and then with the role of $\varphi$ and $\psi$ exchanged yields the second Green's identity:

$$\int_O \Delta\psi\,\varphi\, dx = \int_O \psi\,\Delta\varphi\, dx + \int_{\partial O} \frac{\partial\psi}{\partial n}\,\varphi - \psi\,\frac{\partial\varphi}{\partial n}\, dS\,. \tag{1.7}$$

We now apply (1.5) and (1.7) to $O_\epsilon = U \backslash \bar{B}(y, \epsilon)$ with a small $\epsilon > 0$, and $y \in U$, we find

$$\begin{aligned} \psi(y) &= \lim_{\epsilon \to 0} \int_{O_\epsilon} g(x,y)\Big(-\tfrac{1}{2}\,\Delta + V\Big)\,\psi(x)dx \\ &= \lim_{\epsilon \to 0} \frac{1}{2} \int_{\partial B(y,\epsilon)} \frac{\partial g^y}{\partial n}\,\psi - g^y\,\frac{\partial \psi}{\partial n}\,dS\,, \end{aligned} \tag{1.8}$$

and $n$ is the exterior normal vector to $O_\epsilon$.

If we now apply (1.8) with $\psi$ replaced by the equilibrium potential $v$, and $y \in U \backslash B$, we obtain

$$v(y) = \lim_{\epsilon \to 0} A_\epsilon\,, \tag{1.9}$$

where $A_\epsilon$ is the analogous expression to the last member of (1.8) with $\psi$ replaced by $v$. We now define for small $\epsilon$, $D_\epsilon = U \backslash (\bar{B}(y,\epsilon) \cup B)$. We see that adding and subtracting the term corresponding to $\partial D_\epsilon \cap B$, we find:

$$\begin{aligned} A_\epsilon = &\ \frac{1}{2} \int_{\partial D_\epsilon} \frac{\partial g^y}{\partial n}\,v - g^y\,\frac{\partial v}{\partial n}\,dS \\ &-\frac{1}{2} \int_{\partial D_\epsilon \cap B} \frac{\partial g^y}{\partial n}\,v - g^y\,\frac{\partial v}{\partial n}\,dS\,. \end{aligned} \tag{1.10}$$

Now using (1.7) we see that the first term in the r.h.s. of (1.10) equals:

$$\int_{D_\epsilon} \frac{1}{2}\,\Delta g^y\,v - \frac{1}{2}\,g^y\,\Delta v\,dx = 0\,,$$

since $\frac{1}{2}\,\Delta g^y - V g^y = 0 = \frac{1}{2}\,\Delta v - V v$ on $D_\epsilon$.

On the other hand, if we apply (1.7) with $O = \mathring{B}$, and observe that $v = 1$ on $B$ so that $\frac{\partial v}{\partial n} = 0$ on $\partial B$, we see that:

$$-\frac{1}{2} \int_{\partial B} \frac{\partial g^y}{\partial n}\,dS = \int_B \frac{1}{2}\,\Delta g^y\,dx = \int_B V g^y\,dx \tag{1.11}$$

where we used the fact that $\frac{1}{2}\,\Delta g^y - V g^y = 0$ on $B$ and $n$ stands for the interior normal vector to $B$.

Inserting in (1.10) and keeping track of the orientation of the normal vectors involved we obtain for small $\epsilon > 0$:

$$A_\epsilon = \frac{1}{2} \int_{\partial B} g^y\,\frac{\partial v}{\partial n}\,dS + \int_B V g^y\,dx\,.$$

And now from (1.9) and using the symmetry of $g$ we obtain the representation formula:

$$(1.12)\quad v(x) = \frac{1}{2} \int_{\partial B} g(x,y)\, \frac{\partial v}{\partial n}\,(y)\; dS(y) + \int_B g(x,y)\; V(y)\; dy,\; x \in U \backslash B\,.$$

On the other hand when $y \in \overset{\circ}{B}$, the analogue of (1.9) is now:

$$1 = \lim_{\epsilon \to 0} \frac{1}{2} \int_{\partial B(y,\epsilon)} \frac{\partial g^y}{\partial n}\; dS\,,$$

where $n$ stands for the interior normal vector to $B(y,\epsilon)$. Using (1.7) on $O_\epsilon = B \backslash \bar{B}(y,\epsilon)$, we obtain

$$1 = \frac{1}{2} \int_{\partial B} \frac{\partial g^y}{\partial n}\; dS + \int_B V\; g^y dx$$

where $n$ is oriented as in (1.11). Applying (1.7) again to $v$ and $g^y$ on $U \backslash B$, we find

$$1 = \frac{1}{2} \int_{\partial B} g^y\, \frac{\partial v}{\partial n}\; dS + \int_B V\; g^y dx\,.$$

Therefore (1.12) holds for $x \in \overset{\circ}{B}$ as well. In fact it is not difficult to see that (1.12) extends by continuity to $\partial B$, so that it holds for $x \in U$.

We now find the *equilibrium measure:*

$$(1.13)\qquad \mu(dy) = \frac{1}{2}\, \frac{\partial v}{\partial n}\,(y)\; dS(y) + V(y)\; 1_B(y) dy\,,$$

where $n$ stands for the interior normal vector to $B$ and $\frac{\partial v}{\partial n}$ is calculated 'from the outside' of $B$.

The total mass of $\mu$ is the *capacity* of $B$ relative to $U$ and $V$ and is given by

$$(1.14)\quad \begin{aligned} \mathrm{cap}_{U,V}(B) &= \int_{\partial B} \frac{1}{2}\, \frac{\partial v}{\partial n}\, dS + \int_B V\, dy = \int_{\partial B} \frac{1}{2}\, \frac{\partial v}{\partial n}\, v\, dS + \\ &\int_B V\, dy \overset{((1.6)\text{ on } U\backslash B)}{=} \int_{U\backslash B} \frac{1}{2}\, |\nabla v|^2\, dy + \\ &\frac{1}{2} \int_{U\backslash B} \Delta v\, v\, dy + \int_B V\, dy = \int_U \frac{1}{2}\; |\nabla v|^2\, dy + \\ &\int_U V\, v^2\, dy \end{aligned}$$

where we made use of (1.1). That is in view of Theorem 4.11 of Chapter I,

$$(1.15)\qquad \mathrm{cap}_{U,V}(B) = \mathcal{E}_{U,V}(v,v)\,.$$

Of course the above calculations are a bit formal, however they give the flavour of the problem. We shall in the subsequent sections provide a probabilistic approach to the equilibrium problem valid in a substantially more general context.

## 2.2 Some Facts about Green Functions

We want to collect in this section several results on Green functions which we shall later use. Throughout this section we assume that

$$U \neq \emptyset \text{ is a connected open subset of } \mathbb{R}^d \, , \tag{2.1}$$

$$V \in K_d^{\text{loc}} \text{ is a nonnegative function .} \tag{2.2}$$

Keeping the notations of Chapter 1 §3, we define the *Green function* relative to $U$ and $V$ as

$$g_{U,V}(z,z') = \int_0^\infty r_{U,V}(s,z,z')ds, \;\; z,z' \in \mathbb{R}^d \, , \tag{2.3}$$

(of course $g_{U,V}(z,z') = 0$ if $z$ or $z'$ is not in $U$). When $V = 0$, we shall simply write $g_U$ for $g_{U,V=0}$. The *hitting time* of a closed subset $B$ of $\mathbb{R}^d$ is defined as

$$\tilde{H}_B = \inf\{s > 0, \; Z_s \in B\} \, . \tag{2.4}$$

In contrast to the *entrance time*

$$\begin{aligned} H_B &= \inf\{s \geq 0, \; Z_s \in B\} \\ &= T_{B^c} \;\; (\text{with the notations of } (1.1.8)) \, , \end{aligned} \tag{2.5}$$

$\tilde{H}_B$ is not necessarily equal to 0 when $Z_0 \in B$. In fact a point $y$ (necessarily in $B$) is called *regular* for $B$ if $P_y[\tilde{H}_B = 0] = 1$. If $y$ in $B$ is such that $P_y[\tilde{H}_B = 0] < 1$, $y$ is called *irregular* for $B$; and as a result of Blumenthal's $0-1$ law, cf. Durrett [Dur84], p. 14, $P_y[\tilde{H}_B = 0] = 0$. When $B$ is not too wild (see (4.23) about Wiener's test below) all points of $B$ are regular.

**Exercise**: Show that when $B \supset C_y \cap B(y,\rho)$, where $C_y$ is an open cone with vertex $y$ and $\rho > 0$, then $y$ is regular for $B$.

(Hint: $P_y[\limsup\{Z_{\frac{1}{n}} \in B\}] \geq \limsup_n \; P_y[Z_{\frac{1}{n}} \in B] > 0$). □

**Lemma 2.1:** *The functions $r_{U,V}$ on $(0,\infty) \times U^2$ and $g_{U,V}$ on $U^2$ are strictly positive.*

**Proof**: From (2.3), it suffices to prove the claim for $r_{U,V}$. Since $V \in K_d^{\text{loc},+}$, $\exp\{-\int_0^t V(Z_s)ds\}$ is strictly positive $P^t_{z,z'}$-a.s. for $z, z' \in \mathbb{R}^d$ and from the definition (1.3.4) of $r_{U,V}$, it suffices to consider the case $V = 0$. Finally by a scaling argument it is enough to show that $r_U(1,\cdot,\cdot) > 0$ on $U^2$. We begin with the observation that when $U = B(0,\rho)$, $\rho > 0$, then for $0 < s < (\frac{\rho}{8})^2$ and $|x| \le 4\sqrt{s}$ (and therefore $|x| < \frac{\rho}{2}$):

$$\begin{aligned} r_U(s,0,x) &= p(s,0,x)(1 - P^s_{0,x}[T_U \le s]) \\ &\ge p(s,0,x) - p\Big(\frac{s}{2},0,0\Big)\Big(E_0\Big[T_U \le \frac{s}{2}\Big] + E_x\Big[T_U \le \frac{s}{2}\Big]\Big) \\ &\ge \pi^{-d/2}\, s^{-d/2}\Big\{2^{-d/2}\, e^{-8} - 2P_0\Big[\sup_{0 \le u \le s/2} |Z_u| > \frac{\rho}{2}\Big]\Big\} \end{aligned}$$

It follows that if $U = B(0,\rho)$, we can find $s_0(d,\rho) \le (\rho/8)^2$ such that

$$\text{(2.6)} \qquad \begin{array}{l} \text{when } 0 < s \le s_0(d,\rho), \text{ and } |x| \le 4\sqrt{s}, \text{ then} \\ r_U(s,0,x) \ge \text{const}\,(d) s^{-d/2} . \end{array}$$

Consider now a general $U$ and $z, z' \in U$. We can choose a polygonal path $\varphi : [0,1] \to U$ with $\varphi(0) = z$, $\varphi(1) = z'$, staying at distance $2\rho > 0$ from $U^c$. We now pick $n$ large enough so that $n^{-2} \le s_0(d,\rho)$ and there exist points $z_0 = z, z_1, \dots, z_{n^2} = z'$ along the path $\varphi$ with $|z_{i+1} - z_i| \le n^{-1}$. Letting $B_j$ stand for the ball $B(z_j, \frac{1}{n}) \subset U$, we find from the Chapman-Kolmogorov equations (1.3.7):

$$\text{(2.7)} \qquad \begin{aligned} r_U(1,z,z') \ge \int_{B_0\times\ldots\times B_{n^2-1}} & r_U\Big(\frac{1}{n^2}, z, x_1\Big) \\ & r_U\Big(\frac{1}{n^2}, x_1, x_2\Big) \ldots r_U\Big(\frac{1}{n^2}, x_{n^2-1}, z\Big) dx_1, \dots, dx_{n^2-1} . \end{aligned}$$

Observe that since $x_j \in B_j$, we have $|x_{j+1} - x_j| \le \frac{3}{n}$, and $B(x_j,\rho) \subset U$ in the above integral. The positivity of $r_U(1,z,z')$ now follows from (2.6). □

We shall also need

**Lemma 2.2**: *Let $U$ be an open ball of $\mathbb{R}^d$, and $V \in K_d^+$. The functions*

$$\text{(2.8)} \qquad v_f(x) = E_x\Big[\exp\Big\{-\int_0^{T_U} V(Z_s)ds\Big\}\, f(Z_{T_U})\Big],\ x \in U ,$$

*for bounded measurable $f$ on $\partial U$, are continuous on $U$. For uniformly bounded $f$'s the continuity is uniform at points of $U$.*

**Proof**: From the Markov property we have for $\epsilon > 0$ and $x \in U$:

$$\text{(2.9)} \qquad \Big|v_f(x) - \int_U r_{U,V}(\epsilon,x,y)\, v_f(y) dy\Big| \le \|f\|_\infty\, P_x[T_U \le \epsilon] .$$

If we now use the continuity of $r_{U,V}$ (see (1.3.19)) our claim now follows by letting $\epsilon \to 0$. □

We now collect some useful identities of the Green function. We drop the subscript $U, V$ from the notation for simplicity.

**Proposition 2.3**: *Let $C$ be a closed subset of $\mathbb{R}^d$. Then for $s > 0$, $x, y \in U$ with $y \in C^c$ or $y$ regular for $C$*

$$\begin{aligned} r_{U,V}(s,x,y) &= r_{U\cap C^c,V}(s,x,y) + E_x\Big[H_C < s \wedge T_U\,, \\ &\qquad \exp\Big\{-\int_0^{H_C} V(Z_u)du\Big\}\, r_{U,V}(s-H_C, Z_{H_C}, y)\Big]\,, \end{aligned} \tag{2.10}$$

*moreover*

$$\begin{aligned} g_{U,V}(x,y) &= g_{U\cap C^c,V}(x,y) + \\ &\qquad E_x\Big[H_C < T_U,\ \exp\Big\{-\int_0^{H_C} V(Z_u)du\Big\}\, g(Z_{H_C}, y)\Big]\,. \end{aligned} \tag{2.11}$$

**Proof**: Observe that (2.11) follows from (2.10) by integration over $s$. Let us now prove (2.10). We have

$$r_{U,V}(s,x,y) = p(s,x,y)\, E^s_{x,y}\Big[s < T_U,\ \exp\Big\{-\int_0^s V(Z_u)du\Big\}\Big]\,.$$

Notice that when $y \in C^c$, $P^s_{x,y}[H_C = s] = 0$. Similarly when $y$ is regular for $C$, $P_y[\tilde{H}_C = 0] = 1$ and therefore $P^s_{y,x}[\tilde{H}_C = 0] = 1$ so that $P^s_{x,y}[H_C < s] = 1$.

It follows that:

$$\begin{aligned} r_{U,V}(s,x,y) &= r_{U\cap C^c,V}(s,x,y) + p(s,x,y)\, E^s_{x,y}\Big[H_C < s < T_U\,, \\ &\qquad \exp\Big\{-\int_0^s V(Z_u)du\Big\}\Big] \\ &\overset{\binom{\text{strong Markov}}{\text{property (A.15)}}}{=} r_{U\cap C^c,V}(s,x,y) + p(s,x,y)\, E^s_{x,y}\Big[H_C < s \wedge T_U, \\ &\qquad \exp\Big\{-\int_0^{H_C} V(Z_u)du\Big\}\, E^{s-H_C}_{Z_{H_C},y}\Big[\widetilde{T}_U > s - H_C, \\ &\qquad \exp\Big\{-\int_0^{s-H_C} V(\widetilde{Z}_u)du\Big\}\Big]\Big] \quad \text{(with obvious notations)} \\ &= r_{U\cap C^c,V}(s,x,y) + E_x\Big[H_C < s \wedge T_U, \\ &\qquad \exp\Big\{-\int_0^{H_C} V(Z_u)du\Big\}\, r_{U,V}(s-H_C, Z_{H_C}, y)\Big]\,. \end{aligned} \tag{2.12}$$

This proves our claim. □

We shall often make the following assumption in the sequel:

(2.13) there exist $z_0, z_0' \in U$ such that: $g_{U,V}(z_0, z_0') < \infty$ .

The next result shows that if $g$ is not identically infinite on $U^2$, it is finite and continuous outside the diagonal of $U^2$.

**Theorem 2.4:** *The function* $1/g_{U,V} : U^2 \to [0,\infty)$ *is continuous.*

*When* $d = 1$, *it is strictly positive and when* $d \geq 2$, *it only vanishes on the diagonal of* $U^2$.

**Proof**: With no loss of generality, we assume that (2.13) holds. Observe that we know from (1.3.19) that $r_{U,V}$ is continuous on $(0,\infty) \times U^2$. When $d \geq 3$, we have

$$r_{U,V}(s,z,z') \leq (2\pi s)^{-d/2} \exp\Big\{ - \frac{(z-z')^2}{2s} \Big\} .$$

The continuity and finiteness of $g$ on $U^2 \backslash \Delta$, if $\Delta$ denotes the diagonal of $U^2$, follows by dominated convergence. Moreover when $d \geq 2$,

(2.14) $$g(z,z') \to +\infty, \ \text{ as } \ (z,z') \to (x,x) \in \Delta .$$

Indeed we can assume $U \supseteq B(x, 2\rho)$ and $V \in K_d^+$. Now for $0 < s \leq s_0(d,\rho) (\leq (\frac{\rho}{8})^2)$ with the notations of (2.6) and $z, z'$ in $B(x, 2\sqrt{s})$ we have by the Cauchy-Schwarz inequality

$$\begin{aligned} r_U(s,z,z')^2 &\leq r_{U,V}(s,z,z')\, r_{U,-V}(s,z,z'), \text{ so that} \\ r_{U,V}(s,z,z') &\geq r_U(s,z,z')^2 \,/\, r_{U,-V}(s,z,z') \\ &\overset{(1.3.5)}{\geq} \text{const } s^{d/2}\, r_U(s,z,z')^2 \\ &\overset{(2.6)}{\geq} \text{const}'\ s^{-d/2} . \end{aligned}$$

Therefore when $(z,z') \in B(x,a)$ with $a^2 \leq 4 s_0(d,\rho)$, $g_{U,V}(z,z') \geq \text{const}'$ $\int_{\frac{a^2}{4}}^{s_0(d,\rho)} s^{-d/2}\, ds$. Our claim (2.14) follows.

Theorem 2.4 is now proved when $d \geq 3$ and there simply remains to treat the case $d = 1, 2$. Dropping the subscript $U, V$ for simplicity, we have

$$\begin{aligned} &\int_{U^2} r(1,z_0,z)\, r(1,z_0',z')\, g(z,z')\, dz dz' \\ &= \int_0^\infty ds \int_{U^2} r(1,z_0,z)\, r(1,z_0',z')\, r(s,z,z') dz dz' \\ &\overset{(1.3.7)}{=} \int_0^\infty ds\, r(2+s, z_0, z_0') \overset{(2.13)}{<} \infty . \end{aligned}$$

Therefore $g(\cdot,\cdot)$ belongs to $L^1_{\mathrm{loc}}(U^2)$. Now consider $z_1 \neq z_1'$ with $g(z_1,z_1') < \infty$, and define $R$ as the minimum of $|z_1 - z_1'|$ and the distances of $z_1$ or $z_1'$ to $U^c$. Then we can find disjoint balls $B = B(z_1,\rho_1)$, $B' = B(z_1',\rho_1')$ with $\rho_1, \rho_1' \in (\frac{R}{4}, \frac{R}{2})$ and

$$\int_{\partial B \times \partial B'} g(y,y')\, ds(y)\, ds'(y') < \infty \,, \tag{2.15}$$

provided $ds$ and $ds'$ stand for the respective surface measures. Now from (2.11) applied to both variables with $C = B^c$ and $B'^c$ respectively we obtain that for $z \in B$, $z' \in B'$:

$$\begin{aligned} &g(z,z') = \\ &E_z \times E_{z'}\Big[\exp\Big\{-\int_0^{T_B} V(Z_s)ds - \int_0^{T_{B'}} V(Z_s')ds\Big\}\, g(Z_{T_B}, Z'_{T_{B'}})\Big] \,, \end{aligned} \tag{2.16}$$

with obvious notation. From this and (2.15) it follows that $g$ is locally bounded on $B \times B'$. If we now apply Lemma 2.2 with $B(z_1, \frac{R}{4})$ and $B(z_1', \frac{R}{4})$ to the above representation (2.16), we see that $g(\cdot,\cdot)$ is continuous on $B(z_1, \frac{R}{4}) \times B(z_1', \frac{R}{4})$. This shows the continuity of $g$ on $U^2 \backslash \Delta$, and proves our claim when $d = 2$.

The last point to show is the finiteness and continuity of $g$ on $\Delta$ when $d = 1$. If $U \neq \mathbb{R}$, the claim is obvious because $r$ is dominated by the kernel of some half line and

$$r_{(0,\infty)}(t,z,z') = (2\pi t)^{-1/2}\Big(\exp\Big\{-\frac{(z-z')^2}{2t}\Big\} - \exp\Big\{-\frac{(z+z')^2}{2t}\Big\}\Big), \; z, z' > 0$$

is smaller than const $t^{-3/2}$ when $t$ is large and $z, z'$ remain bounded. On the other hand when $U = \mathbb{R}$, for $z, z' > a$ we have by (2.11)

$$g_{\mathbb{R},V}(z,z') = g_{I,V}(z,z') + E_z\Big[\exp\Big\{-\int_0^{H_{\{a\}}} V(Z_s)ds\Big\}\Big]\, g_{\mathbb{R},V}(a,z') \tag{2.17}$$

with $I = (a,\infty)$.

The finiteness and continuity of $g_{\mathbb{R}^d,V}(\cdot,\cdot)$ on the diagonal now follows using once more Lemma 2.2 and our previous remarks. □

As Theorem 2.4 shows, the mere finiteness of $g_{U,V}(z_0,z_0')$ for some $z_0, z_0'$ in $U$ has strong consequences. When $d = 1$ or 2, $U = \mathbb{R}^d$, $V = 0$, of course this assumption does not hold, on the other hand we have already seen during the course of the proof of Theorem 2.4 that when $d \geq 3$ or $d = 1$ and $U \neq \mathbb{R}$, (2.13) is automatically fulfilled. Let us still give another instance where (2.13) holds.

**Example 2.5:**

$$U = \mathbb{R}^d,\ d = 1,2 \text{ and } V \geq 0 \text{ is not a.s. equal to } 0\ . \tag{2.18}$$

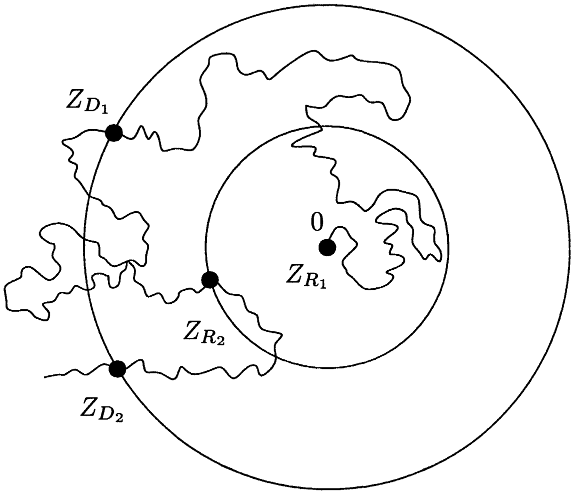
As we shall now see (2.13) holds in this case.

With no loss of generality we assume that $V = 0$ outside $B(0,1)$, and $V \leq 1$. We define $K = \bar{B}(0,1)$, $O = B(0,2)$ and introduce the successive excursions of $Z.$ in $K$ and out of $O$ via:

$$\begin{aligned} R_1 &= \inf\{s \geq 0,\ Z_s \in K\} \\ D_1 &= \inf\{s \geq R_1,\ Z_s \notin O\}, \quad \text{and by induction for } n \geq 1\ , \\ R_{n+1} &= R_1 \circ \theta_{D_n} + D_n\ , \\ D_{n+1} &= D_1 \circ \theta_{D_n} + D_n\ . \end{aligned} \tag{2.19}$$

So we have for any $x \in \mathbb{R}^d$ ($d = 1,2$) $P_x$-a.s.

$$0 \leq R_1 < D_1 < R_2 < D_2 < \ldots < R_n < D_n \ldots \text{ and } R_n, D_n \uparrow \infty\ .$$




Observe that for any $x \in O$, by Lemma 2.1 and since $V$ is not degenerate:

$$\int g_O(x,y)\, V(y)dy = E_x\Big[\int_0^{T_O} V(Z_s)ds\Big] > 0 \tag{2.20}$$

(we recall that $g_O$ stands for $g_{O,V=0}$), and therefore

$$E_x\Big[\exp\Big\{-\int_0^{T_O} V(Z_s)ds\Big\}\Big] < 1, \text{ for } x \in O\,.$$

From Lemma 2.2 we deduce that

$$\sup_{x\in K} E_x\Big[\exp\Big\{-\int_0^{T_O} V(Z_s)ds\Big\}\Big] = c \in (0,1)\,. \tag{2.22}$$

We now have

$$\begin{aligned}
\int_K g_{\mathbb{R}^d,V}(0,x)dx &= E_0\Big[\int_0^\infty 1_K(Z_s)\,\exp\Big\{-\int_0^s V(Z_u)du\Big\}ds\Big]\\
&= E_0\Big[\sum_{i\geq 1}\int_{R_i}^{D_1\circ\theta_{R_i}+R_i} 1_K(Z_s)\,\exp\Big\{-\int_0^s V(Z_u)du\Big\}ds\Big]\\
&\leq \sum_{i=1}^\infty E_0\Big[\exp\Big\{-\int_0^{R_i} V(Z_u)du\Big\}\int_{R_i}^{D_1\circ\theta_{R_i}+R_i} 1_K(Z_s)ds\Big]\\
&\overset{\binom{\text{strong Markov}}{\text{property}}}{=} \sum_{i=1}^\infty E_0\Big[\exp\Big\{-\int_0^{R_i} V(Z_u)du\Big\}\,E_{Z_{R_i}}\Big[\int_0^{D_1} 1_K(Z_s)ds\Big]\Big]\\
&\leq A\cdot\sum_{i=1}^\infty E_0\Big[\exp\Big\{-\int_0^{R_i} V(Z_u)du\Big\}\Big]\,,
\end{aligned}$$

provided $A = E_x[T_O]$ for $|x| = 1$. Observe also that for $i \geq 1$,

$$\begin{aligned}
&E_0\Big[\exp\Big\{-\int_0^{R_{i+1}} V(Z_u)du\Big\}\Big] \overset{\binom{\text{strong Markov}}{\text{property}}}{=}\\
&E_0\Big[\exp\Big\{-\int_0^{R_i} V(Z_u)du\Big\}\,E_{Z_{R_i}}\Big[\exp\Big\{-\int_0^{T_O} V(Z_u)du\Big\}\Big]\Big]\\
&\overset{(2.22)}{\leq} E_0\Big[\exp\Big\{-\int_0^{R_i} V(Z_u)du\Big\}\Big]\cdot c \leq c^i
\end{aligned}$$

by induction in the last step. Therefore

$$\int_K g_{\mathbb{R}^d,V}(0,x)dx \leq A\cdot\sum_{i\geq 1} c^{i-1} = \frac{A}{1-c} < \infty\,. \tag{2.23}$$

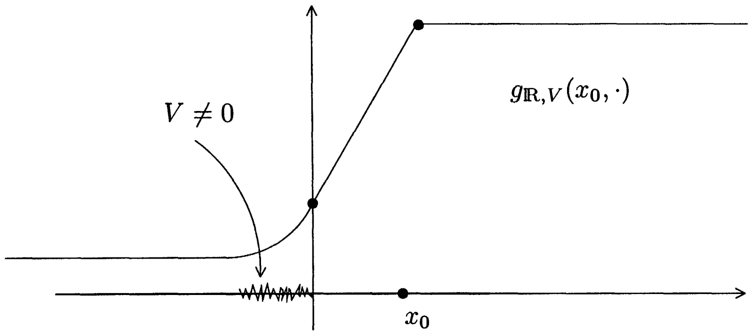
We now see that (2.13) holds.

It should be observed that in contrast to the case of dimension $d \geq 3$ the Green function, even when (2.13) holds, does not necessarily decay at infinity: If for instance $U = \mathbb{R}$ and $V = \frac{1}{2}\, 1_{[-1,0]}$, then for $x_0 \in \mathbb{R}$, $g^{x_0}(\cdot) = g(\cdot, x_0)$ is a solution in the distribution sense of

$$\frac{1}{2}\, g^{x_0 \prime\prime} - V\, g^{x_0} = -\delta_{x_0} \tag{2.24}$$

(the argument used after (1.4) can be made precise: one simply works with $\mathcal{E}_\mu$, with $\mu > 0$ which one lets tend to 0 afterwards). Moreover from (2.11) one sees that $g^{x_0}$ is constant on $[\max(x_0, 0), +\infty)$ and $(-\infty, \min(x_0, -1)]$. Choosing $x_0 > 0$, solving (2.24) with these boundary conditions, we find:

$$\begin{aligned} (e^2 - 1)\, g^{x_0}(\cdot) \;&=\; 2e, && \text{for } x \leq -1\,, \\ &\phantom{=}\; 2(e^2\, e^x + e^{-x}), && \text{for } -1 \leq x \leq 0\,, \\ &\phantom{=}\; 2(e^2 + 1) + 2(e^2 - 1)x, && \text{for } 0 \leq x \leq x_0\,, \\ &\phantom{=}\; 2(e^2 + 1) + 2(e^2 - 1)x_0, && \text{for } x \geq x_0\,. \end{aligned} \tag{2.25}$$




□

We shall close this section with the resolvent identity. We are again in the general situation. We write simply $g_\lambda(\cdot,\cdot)$ for $g_{U,\lambda+V}(\cdot,\cdot)$, when $\lambda$ is a non-negative number.

**Proposition 2.6:** *When $0 \leq \lambda \leq \mu$, for $x, y \in U$:*

$$\begin{aligned} g_\lambda(x,y) &= g_\mu(x,y) + (\mu - \lambda) \int g_\lambda(x,z)\, g_\mu(z,y)\, dz \\ &= g_\mu(x,y) + (\mu - \lambda) \int g_\mu(x,z)\, g_\lambda(z,y) dz\,. \end{aligned} \tag{2.26}$$

**Proof:** With no loss of generality, we assume $\lambda = 0$. We only prove the first identity. The second is proved by analogous methods. The r.h.s. of the first identity equals:

$$\begin{aligned}
&\int_0^\infty e^{-\mu s}\, r(s,x,y)ds + \mu \int_0^\infty \int_0^\infty dsds' \int_U r(s,x,z)e^{-\mu s'} r(s',z,y)dz \\
= &\int_0^\infty e^{-\mu s}\, r(s,x,y)ds + \mu \int_0^\infty \int_0^\infty dsds'\; e^{-\mu s'} r(s+s',x,y) \\
= &\int_0^\infty e^{-\mu s}\, r(s,x,y)ds + \int_0^\infty \Big(\int_0^t e^{-\mu s'} ds'\Big)\, r(t,x,y)dt \\
= &\int_0^\infty e^{-\mu s}\, r(s,x,y)ds + \int_0^\infty (1-e^{-\mu t})\, r(t,x,y)dt = g(x,y)\;.
\end{aligned}$$

□

## 2.3 Last Visit and Equilibrium Problem

In this section we shall first define Brownian motion 'killed at rate $V$ and when exiting $U$". Our main objective is to define a suitable equilibrium problem for a given compact subset $K$ of $U$ and relate the associated equilibrium measure to the last visit of $K$ by our killed Brownian motion. Our standing assumption in this section is that $U \neq \emptyset$ is connected, $V \geq 0$ is in $K_d^{\mathrm{loc}}$ and there are two points $z_0, z_0'$ in $U$ such that $g_{U,V}(z_0, z_0') < \infty$. In other words we assume that (2.1), (2.2), (2.13) hold.

We begin with the definition of *Brownian motion 'killed at rate V and when exiting U '*. We adjoin to $U$ an isolated point $\Delta$ ('cemetery state'), and consider the path space of our process, namely the collection of continuous trajectories in $U$ with possibly finite lifetime:

$$(3.1)\qquad \begin{aligned} C_\Delta(\mathbb{R}_+, U) \;=\; &\{w : \mathbb{R}_+ \to U \cup \{\Delta\},\ \exists \tau \in [0,\infty] \text{ with} \\ &w|_{[0,\tau)} \text{ is continuous with values in } U \text{ and} \\ &w(t) = \Delta \text{ for } t \geq \tau\}\;. \end{aligned}$$

(the case $\tau = 0$ corresponds to $w \equiv \Delta$).

The canonical process is then as usual defined via

$$(3.2)\qquad Z_t(w) = w(t) \in U \cup \{\Delta\},\ \text{for } t \geq 0 \text{ and } w \in C_\Delta(\mathbb{R}_+, U)\;,$$

its *death time* is

$$(3.3)\qquad \zeta(w) = \inf\{s \geq 0, Z_s(w) = \Delta\} \in [0,\infty]\;.$$

The canonical and right continuous canonical filtrations are respectively defined as

$$(3.4) \qquad \mathcal{G}_t = \sigma(Z_s, 0 \le s \le t),\ t \ge 0, \text{ and } \mathcal{G}_t^+ = \bigcap_{\epsilon>0} \mathcal{G}_{t+\epsilon},\ t \ge 0\,.$$

(We recall that the canonical and right continuous canonical filtrations on $C(\mathbb{R}_+, \mathbb{R}^d)$ are denoted by $\mathcal{F}_t$, $t \ge 0$, and $\mathcal{F}_t^+$, $t \ge 0$, respectively, see Chapter 1 §1).

We shall now introduce probability measures $Q_z$, on $C_\Delta(\mathbb{R}_+, U)$ which turn out to have the strong Markov property and define Brownian motion killed at rate $V$ and when exiting $U$.

To this end we consider the product space

$$C(\mathbb{R}_+, \mathbb{R}^d) \times \mathbb{R}_+ = \{(w,\xi) \in C(\mathbb{R}_+, \mathbb{R}^d) \times [0,\infty)\}$$

and for $z \in \mathbb{R}^d$ we define on $C(\mathbb{R}_+, \mathbb{R}^d) \times \mathbb{R}_+$ endowed with its natural product $\sigma$-field the measure:

$$\hat{P}_z = P_z \otimes e^{-\xi} d\xi\,.$$

In other words under $\hat{P}_z$ the canonical components of $C(\mathbb{R}_+, \mathbb{R}^d) \times \mathbb{R}_+$ are independent and respectively distributed as Brownian motion starting from $z$ and as an exponential variable with parameter 1. The 'death time' of the trajectory is then defined as:

$$(3.6) \qquad \tau(w,\xi) = T_U(w) \wedge \inf\left\{s \ge 0, \int_0^s V(Z_u(w))du \ge \xi\right\}.$$

Observe that since $V \in K_d^{\text{loc}}$, $\tau > 0$, $\hat{P}_z$-a.s., when $z \in U$. We now map our construction on the canonical space $C_\Delta(\mathbb{R}_+, U)$ and to this end define for $(w,\xi) \in C(\mathbb{R}_+, \mathbb{R}^d) \times [0,\infty)$

$$(3.7) \qquad \begin{aligned} Y_t(w,\xi) &= w(t), \quad \text{when } 0 \le t < \tau(w,\xi)\,, \\ &= \Delta, \quad \text{when } t \ge \tau(w,\xi)\,. \end{aligned}$$

The map $Y : (w,\xi) \in C(\mathbb{R}_+, \mathbb{R}^d) \times \mathbb{R}_+ \longrightarrow Y_.(w,\xi) \in C_\Delta(\mathbb{R}_+, U)$ is measurable, and we simply define on $C_\Delta(\mathbb{R}_+, U)$

$$(3.8) \qquad Q_z(dw) = \begin{cases} Y \circ \hat{P}_z\,, & \text{when } z \in U\,, \\ \delta_{\text{constant trajectory} = \Delta}\,, & \text{when } z = \Delta\,. \end{cases}$$

Observe that when $A \in B(\mathbb{R}^d)$, $t \geq 0$, $z \in U$:

$$
\begin{aligned}
Q_z(Z_t \in A) &= \hat{P}_z[Z_t \in A, \tau > t] \\
&= \hat{P}_z\Big[Z_t \in A, T_U > t, \int_0^t V(Z_s)ds < \xi\Big] \\
&= E_z\Big[Z_t \in A, T_U > t, \exp\Big\{-\int_0^t V(Z_s)ds\Big\}\Big] \\
&= \int_A r_{U,V}(t,z,z')dz' \,.
\end{aligned} \tag{3.9}
$$

From this one easily sees that for $z \in U \cup \{\Delta\}$, $t \geq 0$:

$$
\widetilde{R}_t(z,dz') \stackrel{\text{def}}{=} Q_z(Z_t \in dz') \tag{3.10}
$$

is equal to

$$
\begin{aligned}
\widetilde{R}_t(z,dz') &= r_{U,V}(t,z,z')dz' + (1 - s_t(z))\,\delta_\Delta, \text{ for } z \in U \,, \\
&= \delta_\Delta \,, \text{ when } z = \Delta \,,
\end{aligned} \tag{3.11}
$$

provided $s_t(z)$ stands for the survival probability at time $t$ when starting from $z \in U$:

$$
s_t(z) = E_z\Big[T_U > t, \exp\Big\{-\int_0^t V(Z_s)ds\Big\}\Big] = \int_U r_{U,V}(t,z,z')dz' \,. \tag{3.12}
$$

The interest of Brownian motion killed at rate $V$ and when exiting $U$ is that on the one hand (3.9) relates the measures $Q_z$ to the semigroups we introduced in Chapter 1, and on the other hand the canonical process $Z_t$, $t \geq 0$, on $C_\Delta(\mathbb{R}_+, U)$ endowed with the measures $Q_z$, $z \in U \cup \{\Delta\}$, and the filtration $(\mathcal{G}_t^+)_{t \geq 0}$, has the strong Markov property, (see exercise below). The main advantage of this construction is that it enables us to handle killing when exiting an open set ('hard obstacle case') or killing at rate $V$ ('soft obstacle case') almost on an equal footing.

**Exercise:**

**1)** Show that $\widetilde{R}_t, t \geq 0$, is a semigroup of probability kernels on $U \cup \{\Delta\}$.

**2)** (simple Markov property)

**a)** Show that when $0 \leq t_1 < t_2 < \ldots < t_n$, $z \in U \cup \{\Delta\}$, and $f_1, \ldots, f_n$ are bounded measurable functions on $U \cup \{\Delta\}$, then for $z \in U \cup \{\Delta\}$

$$
E^{Q_z}[f_1(Z_{t_1}) \ldots f_n(Z_{t_n})] = \widetilde{R}_{t_1}(f_1 \widetilde{R}_{t_2 - t_1}(f_2 \ldots (f_{n-1}\widetilde{R}_{t_n - t_{n-1}} f_n) \ldots))(z)
$$

(Hint: use induction and write

$$E^{\hat{P}_z}[f_1(Y_{t_1})\dots f_n(Y_{t_n})] = E^{\hat{P}_z}[f_1(Y_{t_1})\dots f(Y_{t_n})(1_{\{\tau\le t_{n-1}\}} + 1_{\{t_{n-1}<\tau\le t_n\}}$$
$$+ 1_{\{\tau>t_n\}})] \stackrel{\text{def}}{=} A_1 + A_2 + A_3, \text{ with}$$
$$A_1 = E^{\hat{P}_z}[f_1(Y_{t_1})\dots f_{n-1}(Y_{t_n})\, 1_{\{\tau\le t_{n-1}\}}]\, f_n(\Delta)\,,$$
$$A_2 = E_z\Big[f_1(Z_{t_1})\dots f_{n-1}(Z_{t_{n-1}})\, \exp\Big\{-\int_0^{t_{n-1}} V(Z_u)du\Big\}\, 1_{\{T_U>t_{n-1}\}}$$
$$(1 - s_{t_n-t_{n-1}}(Z_{t_{n-1}}))\Big]\, f_n(\Delta)$$
$$A_3 = E_z\Big[f_1(Z_{t_1})\dots f_{n-1}(Z_{t_{n-1}})\, \exp\Big\{-\int_0^{t_{n-1}} V(Z_u)du\Big\}\, 1_{\{T_U>t_{n-1}\}}$$
$$\int r_{U,V}(t_n - t_{n-1}, Z_{t_{n-1}}, z)\, f_n(z)dz\Big]\ ).$$

**b)** Show the simple Markov property, namely for $z \in U \cup \{\Delta\}$, $t \ge 0$, $f$ bounded $\mathcal{G}_t$-measurable, $g$ bounded $\mathcal{G}_\infty$-measurable and $(\theta_t)_{t\ge 0}$ the canonical shift on $C_\Delta(\mathbb{R}_+, \mathbb{R}^d)$:

$$E^{Q_z}[f \cdot g \circ \theta_t] = E^{Q_z}[f\, E^{Q_{Z_t}}[g]]\,.$$

**3)** (strong Markov property)

Let $T$ be a $(\mathcal{G}_t^+)$-stopping time, and consider the discrete skeleton of $T$:

$$T_n = \sum_{k\ge 0} \frac{k+1}{2^n}\; 1_{\{\frac{k}{2^n}\le T<\frac{k+1}{2^n}\}} + \infty\, 1_{\{T=\infty\}},\quad n \ge 0\,.$$

**a)** Show that when $f$ is bounded $(\mathcal{G}_T^+)$-measurable, and $g$ is bounded $(\mathcal{G}_\infty)$-measurable, then for $z \in U \cup \{\Delta\}$:

$$E^{Q_z}[T_n < \infty,\, f \cdot g \circ \theta_{T_n}] = E^{Q_z}[T_n < \infty, f\, E^{Q_{Z_{T_n}}}[g]]\,.$$

(Hint: Use b) of Exercise 2).

**b)** Show that for $f, g$ as above

$$E^{Q_z}[T < \infty, f \cdot g \circ \theta_T] = E^{Q_z}[T < \infty, f\, E^{Q_{Z_T}}[g]]\,.$$

(Hint: prove first the special case of $g = f_1(Z_{t_1})\dots f_m(Z_{t_m})$, where $f_1, \dots, f_m$ are continuous bounded on $U\cup\{\Delta\}$, using a), the fact that $t \to Z_t(w)$ is right continuous, a) of Exercise 2, and the fact that $\widetilde{R}_t$ sends continuous bounded functions on $U \cup \{\Delta\}$ into continuous bounded functions on $U \cup \{\Delta\}$). □

We shall keep the same notations in the case of killed Brownian motion, for the exit time $T_O$ of an open subset $O$ of $\mathbb{R}^d$, the entrance time $H_B$ or the hitting time $\widetilde{H}_B$ of a closed subset $B$ of $\mathbb{R}^d$. As before these are $\mathcal{G}_t^+$-stopping times.

As a consequence of assumption (2.13), we have the *transience of compact subsets* of $U$:

**Proposition 3.1:** *Let $K$ be a compact subset of $U$ and $z \in U$, then*

(3.13) $$\{t \geq 0,\ Z_t \in K\} \text{ is bounded } Q_z\text{-a.s.}\,.$$

**Proof:** The proof is a variation on the argument used in Example 2.5. With no loss of generality we assume that $K \subset U$ is a closed ball. We pick a concentric open ball $O$ such that $K \subset O \subset \bar{O} \subset U$, and define the successive excursions of our killed Brownian motion in $K$ and out of $O$ essentially as in (2.19):

(3.14) $$\begin{aligned} R_1 &= \inf\{s \geq 0,\ Z_s \in K\} \leq \infty\,, \\ D_1 &= \inf\{s \geq R_1,\ Z_s \notin O\} \leq \infty \end{aligned}$$

(with the convention that $D_1 = \infty$ if $R_1 = \infty$ or if the above set is empty), and by induction for $n \geq 1$:

$$\begin{aligned} R_{n+1} &= R_1 \circ \theta_{D_n} + D_n \leq \infty \\ D_{n+1} &= D_1 \circ \theta_{D_n} + D_n \leq \infty\,, \end{aligned}$$

provided $(\theta_t)_{t\geq 0}$ stands for the canonical shift on $C_\Delta(\mathbb{R}_+, U)$. The $R_i, D_i$ define a sequence of $(\mathcal{G}_t^+)$-shopping times and we have:

(3.15) $$0 \leq R_1 \leq D_1 \leq \ldots \leq R_n \leq D_n \leq \ldots\,,$$

where these inequalities except maybe for the first are strict if the l.h.s. of the inequality is finite. Observe that $R_n,\ D_n \uparrow +\infty$.

As a consequence of the first resolvent identity (2.26), applied with $\lambda = 0$ and $\mu = 1$, and of Theorem 2.4, we know that

(3.16) $$\int_K g(x,z)dz < \infty, \text{ for } x \in U$$

(*the dependence on $U$ and $V$ is dropped from the notation*). Then we obtain:

$$\int_K g(x,z)dz \overset{(3.9)}{=}$$

$$E^{Q_x}\Big[\int_0^\infty 1_K(Z_s)ds\Big] = E^{Q_x}\Big[\sum_{i\geq 1}\int_{R_i}^{D_i} 1_K(Z_s)ds\Big]$$

(3.17) $$= \sum_{i\geq 1} E^{Q_x}\Big[R_i<\infty,\ \int_{R_i}^{D_1\circ\theta_{R_i}+R_i} 1_K(Z_s)ds\Big]$$

$$\overset{\binom{\text{strong Markov}}{\text{property}}}{=} \sum_{i\geq 1} E^{Q_x}\Big[R_i<\infty,\ E^{Q_{Z_{R_i}}}\Big[\int_0^{D_1} 1_K(Z_s)ds\Big]\Big].$$

Observe now that for $z\in K$:

$$E^{Q_z}\Big[\int_0^{D_1} 1_K(Z_s)ds\Big] = E^{Q_z}\Big[\int_0^{T_O} 1_K(Z_s)ds\Big] = \int_K g_{O,V}(z,z')dz',$$

and by Lemma 2.1 and lower semicontinuity,

(3.18) $$\mathcal{A} \overset{\text{def}}{=} \min_{z\in K}\int_K g_{O,V}(z,z')dz' > 0.$$

As a consequence of (3.17) we have

(3.19) $$\mathcal{A}\sum_{i\geq 1} Q_x[R_i<\infty] \leq \int_K g(x,z)dz < \infty.$$

Moreover, by strong Markov property, for $i\geq 1$:

(3.20) $$Q_x[R_i<\infty,\ D_i=\infty] = Q_x[R_i<\infty,\ Q_{Z_{R_i}}[T_O=\infty]] = 0,$$

because for $z\in K$, by the same argument as in (3.16):

(3.21) $$E^{Q_z}[T_O] = \int_O g_{O,V}(z,z')dz' \leq \int_{\bar O} g(z,z')dz' < \infty.$$

Therefore combining (3.19) and (3.20), we see that:

(3.22) $$Q_x\Big[\{R_1=\infty\}\bigcup_{i\geq 2}\{D_{i-1}<\infty, R_i=\infty\}\Big] = 1,$$

and our claim (3.13) now follows. □

For $K$, a relatively closed subset of $U$, we introduce the time of *last visit of killed Brownian motion to* $K$ as:

$$\begin{aligned} &L_K(w) = \sup\{t \in (0,\infty),\ Z_t(w) \in K\}, \text{ for } w \in C_\Delta(\mathbb{R}_+, U) \\ &(\text{with the convention } \sup\emptyset = 0)\,, \end{aligned} \tag{3.23}$$

we then have the equivalence:

$$L_K > 0 \Longleftrightarrow \widetilde{H}_K < \infty\,, \tag{3.24}$$

which relates the last visit to $K$ with the hitting time of $K$.

From now on we assume that $K$ is a nonempty compact subset of $U$. We know from Proposition 3.1 that $K$ is transient (see (3.13)).

We now define the *equilibrium potential* of $K$ relative to $U$ and $V$ as the function:

$$v(x) = Q_x[L_K > 0] = Q_x[\widetilde{H}_K < \infty],\ x \in U\,. \tag{3.25}$$

For the time being, it is unclear that this object has anything to do with the equilibrium potential discussed in Section 2.1. The connection will become more transparent below, see in particular Proposition 3.8. Our goal for the time being is to discuss the 'equilibrium problem' that is to represent $v(\cdot)$ as a suitable mixture of $g(\cdot, y)$, $y \in K$. Before we begin with the study of this question, notice that $v(\cdot)$ can be re-expressed in terms of usual Brownian motion as:

$$\begin{aligned} v(x) &= Q_x[\widetilde{H}_K < \infty] = \hat{P}_x[\widetilde{H}_K(Y) < \infty] \\ &= \hat{P}_x[\{(w,\xi) : \widetilde{H}_K(w) < \tau(w,\xi)\}] \\ &= \hat{P}_x\Big[\Big\{(w,\xi) : \widetilde{H}_K(w) < T_U(w),\ \int_0^{\widetilde{H}_K(w)} V(w(s))ds < \xi\Big\}\Big] \\ &= E_x\Big[\widetilde{H}_K < T_U,\ \exp\Big\{-\int_0^{\widetilde{H}_K} V(Z_s)ds\Big\}\Big],\ x \in U\,. \end{aligned} \tag{3.26}$$

We now introduce for $\epsilon > 0$ the measures on $U$:

$$\psi_\epsilon(z)dz = \frac{1}{\epsilon}\, Q_z[0 < L_K \le \epsilon]dz\,. \tag{3.27}$$

We shall see that as $\epsilon \to 0$, these measures approximate the 'equilibrium measure' of $K$ relative to $U, V$. We also define for $x \in U$ the *last exit distribution* of $K$:

$$L(x, dy) = Q_x[L_K > 0,\ Z_{L_K^-} \in dy]\,. \tag{3.28}$$

Here $Z_{L_K^-}$ denotes the position of $Z$, 'just before time $L$', and the above formula defines a sub-Markovian kernel. Observe that $L(x,dy)$ is concentrated on $K$. Indeed $Q_x$-a.s. on $\{L_K > 0\}$, either $Z_{L_K^-} = Z_{L_K}$ and $L_K < \zeta$ ($\zeta$ defined in (3.3) is the death time) or $Z_{L_K^-} \neq Z_{L_K}$, $L_K = \zeta < \infty$ and $Z_.$ meets $K$ on all intervals $(\zeta - \frac{1}{n}, \zeta)$, $n \geq 1$. The next lemma is simple but crucial:

**Lemma 3.2:**

(3.29) *For* $f \in C_b(U)$, $\lim\limits_{\epsilon \to 0} \int g(x,y)\,\psi_\epsilon(y)\,f(y)dy = \int_K L(x,dy)\,f(y)$ .

**Proof:** We first assume $f \geq 0$. Then we have

$$\int_U g(x,y)\,\psi_\epsilon(y)\,f(y)dy \overset{(3.9)}{=} E^{Q_x}\Big[\int_0^\infty f\,\psi_\epsilon(Z_s)ds\Big]$$

$$= \int_0^\infty E^{Q_x}[f(Z_s)\,Q_{Z_s}[0 < L_K \leq \epsilon]]\,\frac{ds}{\epsilon} \overset{\binom{\text{Markov}}{\text{property}}}{=} \int_0^\infty E^{Q_x}[f(Z_s)$$

$$1\{0 < L_K \leq \epsilon\} \circ \theta_s]\,\frac{ds}{\epsilon} = \int_0^\infty E^{Q_x}[f(Z_s),\ s < L_K \leq s + \epsilon]\,\frac{ds}{\epsilon} \tag{3.30}$$

$$= E^{Q_x}\Big[L_K > \epsilon,\ \frac{1}{\epsilon}\int_{L_K-\epsilon}^{L_K} f(Z_s)ds\Big]$$

$$+ E^{Q_x}\Big[0 < L_K \leq \epsilon,\ \frac{1}{\epsilon}\int_0^{L_K} f(Z_s)ds\Big]\,.$$

As a result of this calculation we see that the expression under the limit sign in the l.h.s. of (3.29) is automatically finite. Moreover the last term of the rightmost member of (3.30) tends to 0 whereas the first term tends to

$$E^{Q_x}[L_K > 0,\ f(Z_{L_K^-})]\,.$$

Our claim (3.29) easily follows for $f \in C_b(U)$. □

Our next step is:

**Proposition 3.3:**

(3.31) *As* $\epsilon \to 0$, $\psi_\epsilon(y)dy$ *converges vaguely to a finite measure* $e(dy)$ *supported on* $K$ .

*Moreover the following identity holds:*

(3.32) $g(x,y)\,e(dy) = L(x,dy)$ *for* $x \in U$ .

*When* $d \geq 2$, $e(dy)$ *has no atom.*

**Proof:** We already know that $g(x,y)\,\psi_\epsilon(y)\,1_U(y)dy$ converges weakly to $L(x,dy)$ as $\epsilon \to 0$. Now when $f \in C_c(U)$, $x \in U$, $y \in U \to \frac{1}{g}(x,y)\,f(y) \in$

$C_c(U)$, provided we use the same convention as in Theorem 2.4: $\frac{1}{g}(x,y)=0$ when $g(x,y)=\infty$, that is when $x=y$ and $d\geq 2$. Therefore we have

$$\int f(y)\,\psi_\epsilon(y)dy = \int g(x,y)\,\psi_\epsilon(y)\,\frac{1}{g}\,(x,y)\,f(y)dy \underset{\epsilon\to 0}{\overset{(3.29)}{\longrightarrow}} \int_K \frac{1}{g}(x,y)f(y)L(x,dy)$$

As a result

$$\text{(3.33)} \qquad \psi_\epsilon(y)dy \text{ converges vaguely on } U \text{ to } \frac{1}{g}\,(x,y)\,L(x,dy), \text{ as } \epsilon\to 0\,.$$

Observe that the measure $\frac{1}{g}\,(x,y)\,L(x,dy)$ which is finite and concentrated on $K$ does not depend on $x$! As a result we can define

$$\text{(3.34)} \qquad e(dy) = \tfrac{1}{g}\,(x,y)\,L(x,dy), \text{ for } x\in U \text{ arbitrary}\,,$$

$e$ is a finite measure on $K$ and (3.31) holds.

When $d=1$, $g$ and $\frac{1}{g}$ are positive continuous by Theorem 2.4 and (3.32) immediately follows. On the other hand when $d\geq 2$, observe that $e(dy)$ has no atom. Indeed from (3.34) for $y_0\in K$

$$e(\{y_0\}) = \frac{1}{g}\,(y_0,y_0)\,L(y_0,\{y_0\}) \overset{\text{(Theorem 2.4)}}{=} 0\cdot L(y_0,\{y_0\}) = 0\,.$$

Let us then prove (3.32). First assume that $x\notin K$. From (3.34):

$$\text{(3.35)} \qquad g(x,y)\,e(dy) = g(x,y)\,\frac{1}{g}\,(x,y)\,L(x,dy) = L(x,dy)$$

since $g(x,\cdot)\times\frac{1}{g}\,(x,\cdot)$ equals 1 on $K$. Now in the case of $x\in K$, the same argument leads to:

$$\text{(3.36)} \qquad g(x,y)\,1_{\{y\neq x\}}e(dy) = 1_{\{y\neq x\}}\,L(x,dy)\,.$$

Since $e$ has no atom, (3.32) will be proved once we know that

$$\text{(3.37)} \qquad L(x,\{x\})=0, \quad \text{for } x\in K\,.$$

Let us check this last point. Observe that $Q_x$-a.s. $Z_{\frac{1}{n}}\neq x$, for $n\geq 1$. Observe now that

$$Q_x[L_K>\frac{1}{n},\, Z_{L_K^-}=x] \underset{n\to\infty}{\longrightarrow} L(x,\{x\}), \text{ and}$$

$$Q_x[L_K>\frac{1}{n},\, Z_{L_K^-}=x] \overset{\binom{\text{Markov}}{\text{property}}}{=} E^{Q_x}[Q_{Z_{\frac{1}{n}}}[L_K>0, Z_{L_K^-}=x],\ Z_{\frac{1}{n}}\in U]$$
$$= E^{Q_x}[L(Z_{\frac{1}{n}},\{x\}),\ Z_{\frac{1}{n}}\in U] = 0$$

because $Z_{\frac{1}{n}} \neq x$ $Q_x$-a.s. and by (3.36) $L(x', \{x\}) = 0$, for $x' \neq x$. This finishes the proof of Proposition 3.3. □

As we shall now see the measure $e(dy)$ introduced in Proposition 3.3 is the unique measure concentrated on $K$ for which (3.32) holds.

**Theorem 3.4:**

(3.38) *For* $x \in U$, $Q_x[\widetilde{H}_K < \infty] = Q_x[L_K > 0] = \int g(x,y)\, e(dy)$ ,

*and* $e(dy)$ *is the only finite measure satisfying (3.38). Moreover for* $x \in U$,

(3.39) $$Q_x[L_K > 0, L_K \in ds,\ Z_{L_K^-} \in dy] = r(s,x,y)\, e(dy) ds \ .$$

**Proof:** Observe that (3.38) follows by integration of (3.32). Consider now another Radon measure $\widetilde{e}$ satisfying:

(3.40) $$\int g(x,y)\, e(dy) = \int g(x,y)\, \widetilde{e}(dy) \ .$$

Then from the second resolvent identity (2.26), applied with $\lambda = 0$ and $\mu > 0$, we find for $x \in U$:

$$\int g_\mu(x,y)\, e(dy) + \mu \int g_\mu(x,z)\, Q_z[\widetilde{H}_K < \infty] dz =$$
$$\int g_\mu(x,y)\, \widetilde{e}(dy) + \mu \int g_\mu(x,z)\, Q_z[\widetilde{H}_K < \infty] dz \ .$$

It now follows by integration in the $x$ variable that for $\mu > 0$ and bounded continuous $f$

(3.41) $$\int \mu\, G_\mu\, f(y)\, e(dy) = \int \mu\, G_\mu\, f(y)\, \widetilde{e}(dy) \ ,$$

provided

$$G_\mu\, f(z) = \int g_\mu(z,z')\, f(z') dz' = E_z\Big[\int_0^{T_U} f(Z_s)\, \exp\Big\{-\mu s - \int_0^s V(Z_u) du\Big\} ds\Big]$$

Now when $\mu \to \infty$, $\mu\, G_\mu\, f(z) \to f(z)$ boundedly (we used here (1.2.20)) and therefore

(3.42) $$\int f(y)\, e(dy) = \int f(y)\, \widetilde{e}(dy) \ .$$

This show the uniqueness part of the statement.

Let us finally prove (3.39). For $t > 0$ and $f$ bounded measurable on $U$:

$$
\begin{aligned}
E^{Q_x}[L_K > t,\ f(Z_{L_K^-})] &= E^{Q_x}[(f(Z_{L_K^-})\, 1_{\{L_K>0\}}) \circ \theta_t] \\
&\overset{\binom{\text{Markov}}{\text{property}}}{=} E^{Q_x}\left[E^{Q_{Z_t}}[L_K > 0,\ f(Z_{L_K^-})]\right] \\
(3.43) \qquad &\overset{(3.32)}{=} E^{Q_x}\left[\int g(Z_t, y)\, f(y)\, e(dy)\right] \\
&\overset{(3.9)}{=} \int r(t,x,z) \int g(z,y)\, f(y)\, e(dy) dz \\
&= \int_t^\infty \int_K r(u,x,y)\, f(y)\, fe(dy) du\ .
\end{aligned}
$$

Our claim now follows by the monotone class theorem. □

The above constructed measure $e$ is therefore the solution of the equilibrium problem for $K$ relative to $U$ and $V$. It is called the *equilibrium measure or distribution* of $K$ and its total mass

(3.44) $$\mathrm{cap}_{U,V}(K) = e(K)$$

is called the *capacity* of $K$ (relative to $U$ and $V$). When $K = \emptyset$, we set $e = 0$, $\mathrm{cap}_{U,V}(\emptyset) = 0$, so that trivially (3.38), (3.39), (3.44) hold.

**Example 3.5:** We consider the case $d = 1$, $U$ is a possibly unbounded interval, (2.13) holds, and $K = \{x_0\} \in U$. In this case (3.38) now reads:

(3.45) $$Q_x[\widetilde{H}_{\{x_0\}} < \infty] = g(x, x_0)\, e(\{x_0\}),\ x \in U\ .$$

Moreover from (3.38) we also have:

(3.46) $$Q_{x_0}[\widetilde{H}_{\{x_0\}} < \infty] \geq P_{x_0}[\widetilde{H}_{\{x_0\}} = 0] = 1\ .$$

We thus see from (3.45) that $e(\{x_0\}) = g(x_0, x_0)^{-1}$. Inserting in (3.45) and (3.39) we find:

(3.47)
$$
\begin{aligned}
&Q_x[\widetilde{H}_{\{x_0\}} < \infty] = Q_x[H_{\{x_0\}} < \infty] = \frac{g(x, x_0)}{g(x_0, x_0)}\ , \text{ and} \\
&Q_x[L_{\{x_0\}} > 0,\ L_{\{x_0\}} \in ds] = \frac{r(s, x, x_0)}{g(x_0, x_0)}\, ds, \quad \text{for } x \in U\ .
\end{aligned}
$$

Observe that in the case discussed at the end of Example 2.5: $U = \mathbb{R}$, $V = \frac{1}{2}\, 1_{[-1,0]}$, the equilibrium potential stays bounded away from 0 on $\mathbb{R}$.

In the case $U = (0, \infty)$, $V = 0$, an analogous calculation to (2.24), (2.25) shows that:

$$g(x,y) = 2(x \wedge y)\,, \quad x, y \in U\,. \tag{3.48}$$

Moreover in this case

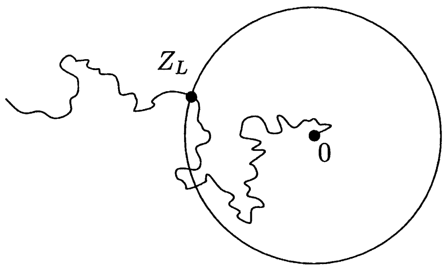
$$r(s,x,y) = (2\pi s)^{-1/2}\Big(\exp\Big\{-\frac{(x-y)^2}{2s}\Big\} - \exp\Big\{-\frac{(x+y)^2}{2s}\Big\}\Big)\,,$$

so that the distribution of the time of last visit to $x_0 > 0$ before hitting 0 when starting from $x > 0$ is given by:

$$P_x[H_{\{x_0\}} < T_{(0,\infty)},\ L_{\{x_0\}} \in ds] = \frac{1}{2x_0}\, r(s,x,x_0)ds\,. \tag{3.49}$$

□

**Example 3.6:** $U = \mathbb{R}^d$, $d \geq 3$, $V = 0$, and $K = \bar{B}(0,R)$ ,




From (2.11) applied to $C = \bar{B}(0,R)$, $x \notin \bar{B}(0,R)$, $y \in B(0,R)$ we find:

$$\begin{aligned} g(x,y) &= E_x[g(Z_{H_K}, y),\ H_K < \infty]\,, \\ &= c(d)\ R^{2-d} P_x[H_K < \infty] \text{ in the case } y = 0, \text{ with} \end{aligned} \tag{3.50}$$

$$c(d) = \frac{1}{2\pi^{d/2}}\, \Gamma\Big(\frac{d}{2} - 1\Big)\,, \tag{3.51}$$

(recall from (2.22) of Chapter 1, that $g(z,z') = c(d)\,|z - z'|^{2-d}$).

Since all points of $K$ are regular for $K$ (see the exercise before Lemma 2.1), the *equilibrium potential* of $K$ equals 1 on $K$. It coincides with $P_.[H_K < \infty]$ outside $K$ and therefore is equal to:

$$v(x) = \Big(\frac{R}{|x|}\Big)^{d-2} \wedge 1\,,\ x \in \mathbb{R}^d\,. \tag{3.52}$$

We shall now find the equilibrium distribution of $K$. To this end we integrate the first equality of (3.50) over $x$ with respect to the normalized surface measure $\bar{d}S_{R'}$ of a sphere of radius $R' > R$ centered at the origin. Conditioned on entering $K$, $Z_{H_K}$ is uniformly distributed on $\partial K = S_R$, if the starting point of $Z.$ is uniformly distributed on $S_{R'}$. Using (3.52), we find that for $y \in B(0,R)$,

$$\int_{\partial B(0,R')} g(x,y)\bar{d}S_{R'}(x) = \Big(\frac{R}{R'}\Big)^{d-2} \int_{\partial B(0,R)} g(x,y)\bar{d}S_R(x)$$

and therefore

$$\int_{\partial B(0,R')} g(x,y)\, \bar{d}S_{R'}(x)/c(d)\, R'^{2-d}$$

does not depend on $R' \geq R$. Letting $R'$ tend to infinity, we find

$$\int_{\partial B(0,R)} g(x,y)\, \frac{\bar{d}S_R(x)}{c(d)R^{2-d}} = 1,\ y \in B(0,R)\ . \tag{3.53}$$

An other application of (2.11) with $x = 0$, $y \notin B(0,R)$, $C = B(0,R)^c$ shows that

$$\begin{aligned} g(0,y) &= \int_{\partial B(0,R)} g(x,y)\ \bar{d}S_R(x) \\ &= c(d)\, |y|^{2-d}\ . \end{aligned} \tag{3.54}$$

Collecting (3.52) - (3.54) we have:

$$\begin{aligned} v(x) &= \int g(x,y)\, e(dy), \ \text{ where} \\ e(dy) &= \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2}-1)}\ R^{d-2}\bar{d}S_R(y) \ \text{ is the } \textit{equilibrium} \\ &\qquad\qquad\qquad\qquad\qquad \textit{measure} \text{ of } \bar{B}(0,R)\ , \\ \mathrm{cap}(\bar{B}(0,R)) &= 2\pi^{d/2}/\Gamma(\tfrac{d}{2}-1)\, R^{d-2}\ . \end{aligned} \tag{3.55}$$

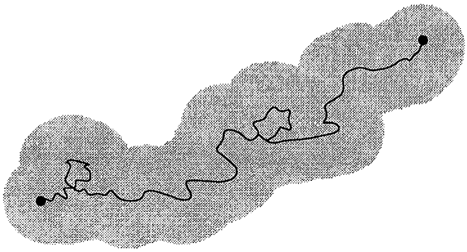
As a result the distribution of the time of last visit of the ball of radius $R$ for Brownian motion starting from 0 is:

$$\begin{aligned} P_0[L_{\bar{B}(0,R)} \in ds] &= \int_K p(s,0,y)\, e(dy) ds \\ &= \frac{R^{d-2}}{2^{d/2-1}\Gamma(\frac{d}{2}-1)s^{d/2}}\ \exp\Big\{-\frac{R^2}{2s}\Big\} ds\ . \end{aligned} \tag{3.56}$$

In the case $d = 3$, since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, this distribution coincides with the distribution of $H_{\{R\}}$ for Brownian motion starting from 0 (see also Revuz-Yor [RY98], p. 94). □

**Example 3.7** The *Wiener sausage* of radius $a > 0$ of Brownian motion up to time $t$ is defined as:

$$W_t^a = \bigcup_{0 \le s \le t} \bar{B}(Z_s, a) \tag{3.57}$$



We shall now discuss the behavior of the volume $|W_t^a|$ of $W_t^a$ when $a$ becomes small, in the case of dimension $d \ge 3$. Observe that for $t \ge 0$,

$$W_t^a = \{x \in \mathbb{R}^d,\ H_{\bar{B}(x,a)} \le t\} .$$

Let us write $H_{x,a}$ for $H_{\bar{B}(x,a)}$ and $L_{x,a}$ for $L_{\bar{B}(x,a)}$ to simplify notations. Then for $t > 0$,

$$\begin{aligned} E_0[|W_t^a|] &= \int dx\, P_0[H_{x,a} \le t] = \int dx\, P_0[0 < L_{x,a} \le t] \\ &\quad + \int dx\, P_0[H_{x,a} \le t,\ L_{x,a} > t \ \text{ or } \ L_{x,a} = 0] . \end{aligned} \tag{3.58}$$

Now by (3.39) and (3.55) the first term in the r.h.s. of the above expression equals:

$$\frac{a^{d-2}}{c(d)} \int_0^t ds \int dx \int_{\partial B(x,a)} p_s(0,y)\, dS_{x,a}(y) \overset{\text{(Fubini)}}{=} t\, a^{d-2}/c(d) , \tag{3.59}$$

whereas the second term equals

$$\begin{aligned} &\int dx\ P_0[H_{x,a} \le t,\ L_{x,a} > t] \le \int dx\ P_0[t < L_{x,a} \le t + \eta] \\ &\qquad + \int dx\ P_0[H_{x,a} \le t,\ L_{x,a} > t + \eta], \ \text{ for } \ \eta > 0 , \\ &\overset{(3.59)}{\le} \eta\, a^{d-2}/c(d) + \int dx\ P_0[H_{x,a} \le t,\ L_{x,a} \circ \theta_{H_{x,a}} > \eta] \\ &\overset{\binom{\text{strong Markov}}{\text{property}}}{=} \eta\, a^{d-2}/c(d) + \int dx\ E_0[H_{x,a} \le t,\ E_{Z_{H_{x,a}}}[L_{x,a} > \eta]] \\ &\overset{\substack{(3.39)\\(3.55)}}{\le} \eta\, a^{d-2}/c(d) + \sup_{\bar{B}(0,a)} \frac{a^{d-2}}{c(d)} \int_\eta^{+\infty} ds \int_{\partial B(0,a)} p(s,\cdot,y) dS_{0,a}(y) \\ &\quad \cdot E_0[|W_t^a|] \le a^{d-2}/c(d)(\eta + \int_\eta^{+\infty} (2\pi s)^{-d/2} ds\ E_0[|W_t^a|]) . \end{aligned} \tag{3.60}$$

Observe that since $W_t^a \subset \bar{B}(0, \sup_{0\le s\le t}|Z_s| + a)$ $P_0$-a.s., $|W_t^a|$ has finite expectation. From (3.58), (3.59) we have:

$$\liminf_{a\to 0} a^{2-d}\, E[|W_t^a|] \geq t/c(d) \; ,$$

and from (3.58) - (3.60), for $\eta > 0$:

$$E[|W_t^a|](1 - a^{d-2}\,\mathrm{const}(d,\eta)) \leq a^{d-2}/c(d)(t+\eta) \; ,$$

so that:

$$\limsup_{a\to 0} a^{2-d} E[|W_t^a|] \leq (t+\eta)/c(d) \; .$$

Since $\eta > 0$ is arbitrary, we have shown that for $t \geq 0$:

$$(3.61) \qquad \begin{aligned} \lim_{a\to 0}\; a^{2-d}\, E[|W_t^a|] &= \mathrm{cap}(\bar{B}(0,1))\, t \\ &= 2\, \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}-1)}\, t \; . \end{aligned}$$

In fact (3.61) can be right away strengthened. Picking $t = 1$, and setting $a^{-2} = \frac{s}{\rho^2}$, using Brownian scaling it shows that:

$$(3.62) \qquad \lim_{s\to\infty} \frac{1}{s}\, E[|W_s^\rho|] = \mathrm{cap}(\bar{B}(0,\rho)), \text{ for } \rho > 0 \; .$$

Now if one defines for $s \leq t$, $W_{s,t}$ as $W_{t-s}\circ\theta_s$, we have an obvious subadditive property:

$$|W_{s,u}| \leq |W_{s,t}| + |W_{t,u}|, \;\text{ for } 0 \leq s \leq t \leq u \; ,$$

and from the subadditive ergodic theorem (see Liggett [Lig85], p. 277) we can conclude that in fact:

$$\frac{|W_s^\rho|}{s} \underset{L^1}{\overset{\text{a.s.}}{\longrightarrow}} \mathrm{cap}(\bar{B}(0,\rho)) \; , \text{ as } s\to\infty \; .$$

Using scaling once more, we see that in fact

$$(3.65) \qquad a^{2-d}|W_t^a| \xrightarrow{L^1} \mathrm{cap}(\bar{B}(0,1))\, t, \text{ as } a\to 0 \; .$$

For more on this see for instance Le Gall [Gal86], [Gal92], and also [Szn87].

□

We shall close this section with a result which provides some connection with the discussion of Section 1.

**Proposition 3.8:** *Let $K$ be a compact subset of $U$, then the equilibrium potential $v$ introduced in (3.25) satisfies*

$$
\text{(3.66)} \qquad
\begin{aligned}
&\text{a)} \quad v \textit{ is continuous on } U \backslash K \ , \\
&\text{b)} \quad v \textit{ is continuous and equals } 1 \textit{ at regular points of } K \\
&\text{c)} \quad \frac{1}{2}\,\Delta v - V v = 0 \textit{ in the distribution sense on } U \backslash K \ .
\end{aligned}
$$

**Proof:**

- Let us begin with the proof of a). If $x_0 \in U \backslash K$ and $O = B(x_0, r)$ is such that $\bar{O} \subset U \backslash K$, then from the strong Markov property for $x \in O$,

$$
\text{(3.67)} \qquad
\begin{aligned}
v(x) &\overset{(3.26)}{=} E_x\Big[H_K < T_U,\ \exp\Big\{-\int_0^{H_K} V(Z_u)ds\Big\}\Big] \\
&= E_x\Big[\exp\Big\{-\int_0^{T_O} V(Z_s)ds\Big\}\, v(Z_{T_O})\Big] \ ,
\end{aligned}
$$

an our claim now follows from Lemma 2.2.

- As for b) observe from (3.26) that $v(x) = 1$, if $x$ is a regular point of $K$. Moreover from (3.38)

$$
\text{(3.68)} \qquad v(\cdot) = \lim_N \uparrow \int g(\cdot, y) \wedge N\, e_K(dy) \ ,
$$

is a lower semicontinuous function of $U$, and the inequality $v \le 1$, forces the continuity of $v$ at points where it equals 1.

- Let us prove c). As $\alpha \downarrow 0$, for $x \in U \backslash K$,

$$
v_\alpha(x) \overset{\text{def}}{=} \int g_\alpha(x,y)\, e_K(dy) \uparrow \int g(x,y)\, e_K(dy) = v(x) \ .
$$

For $\alpha, M > 0$, denote by $g_{\alpha,M}$ the $\alpha$-Green function relative to $U$ and $V \wedge M$. It follows from the discussion in Remark 1.4.14, that for $\varphi \in C_c^\infty(U)$:

$$
\int \Big(-\frac{1}{2}\,\Delta + V \wedge M + \alpha\Big)\, \varphi(x)\, g_{\alpha,M}(x,y)dx = \varphi(y), \ y \in U \ .
$$

From (1.2.4) and (1.2.19), we know that $V(\cdot)\, g_{\mathbb{R}^d,\alpha}(\cdot, y)$ belongs to $L^1_{\text{loc}}(\mathbb{R}^d)$, and by dominated convergence and the fact that $g_{\alpha,M} \downarrow g_\alpha$, as $M \to \infty$,

$$
\text{(3.69)} \qquad \int \Big(-\frac{1}{2}\,\Delta + V + \alpha\Big)\, \varphi(x)\, g_\alpha(x,y)dx = \varphi(y), \ y \in U \ .
$$

Integrating the above equality with respect to $e_K(dy)$, and assuming $\varphi \in C_c^\infty(U \backslash K)$, we obtain

$$
\text{(3.70)} \qquad \int \Big(-\frac{1}{2}\,\Delta + V + \alpha\Big)\, \varphi(x)\, v_\alpha(x)dx = \int \varphi(y)\, e_K(dy) = 0 \ .
$$

Therefore

$$\int \Big( -\frac{1}{2}\,\Delta + V \Big) \varphi\, v_\alpha(x) dx = -\alpha \int \varphi\, v_\alpha(x) dx$$

and letting $\alpha$ tend to 0, we find

$$\int \Big( -\frac{1}{2}\,\Delta + V \Big) \varphi\, v(x) dx = 0\,, \text{ for } \varphi \in C_c^\infty(U \backslash K)\,.$$

This proves c). □

## 2.4 Maximum Principle, Exceptional Sets, Variational Principle

The object of this section is to introduce a Maximum Principle and a Variational Principle which provide ways of concretely comparing and estimating the capacities of compact sets which we introduced in the previous section. We shall also discuss to some extent the 'smallness' of certain exceptional sets. Although this last point may not look like a thrilling perspective, it turns out to be technically very convenient as we shall see.

As before we assume that (2.1), (2.2), (2.13) hold. If $K$ is a compact subset of $U$, then in view of (3.38) either

(4.1) $$\forall x \in U,\ \ Q_x[\widetilde{H}_K < \infty] = 0 \ \text{ or}$$

(4.2) $$\forall x \in U,\ \ Q_x[\widetilde{H}_K < \infty] > 0$$

is true. When (4.1) holds, $K$ is said to be *polar.* Of course one has then

(4.3) $$K \text{ is polar} \iff \mathrm{cap}_{U,V}(K) = 0\,.$$

**Exercise:** Show that the notion of being 'polar' does not depend on $U, V$ and that $K$ is polar if and only if

(4.4) $$\forall x \in \mathbb{R}^d,\ P_x[\widetilde{H}_K < \infty] = 0\,.$$

□

We now begin with the following *maximum principle.*

**Theorem 4.1:** *Let $\mu$ be a finite measure supported on a compact subset $K$ of $U$, then*

(4.5) $$\sup_U \int g(\cdot, y)\, \mu(dy) = \sup_K \int g(\cdot, y)\, \mu(dy)\,.$$

**Proof:** With no loss of generality, we assume that

$$M \overset{\text{def}}{=} \sup_K \int g(\cdot, y)\,\mu(dy) < \infty\,. \tag{4.6}$$

The function $h$ from $U$ to $[0,\infty]$:

$$h(x) = \int g(x,y)\,\mu(dy) = \lim_N \uparrow \int g(x,y) \wedge N\ \mu(dy)\,, \tag{4.7}$$

is lower semi-continuous. For $\epsilon > 0$, we define:

$$B = \{x \in U,\ h(x) \le M + \epsilon\}\,,$$

$B$ is thus a relatively closed subset of $U$. If we also define

$$H_r = \inf\{s \ge 0,\ |Z_s - Z_0| \ge r\}, \quad \text{for } r > 0\,, \tag{4.8}$$

(with the convention that $|\Delta - x| = \infty$, for $x \in U$), then for $x \in U$ and $r$ small enough so that $\bar{B}(x,r) \subset U$:

$$\begin{aligned}
h(x) &= \int_0^\infty \int_K r(s,x,y)\,ds\,d\mu(y) \\
&\overset{\binom{\text{strong Markov}}{\text{property}}}{\ge} \int_0^\infty \int_K E_x\Big[H_r < s, \\
&\qquad \exp\Big\{-\int_0^{H_r} V(Z_u)du\Big\}\, r(s - H_r, Z_{H_r}, y)\Big]\,ds\,d\mu(y) \\
&= \int_K E_x\Big[\exp\Big\{-\int_0^{H_r} V(Z_s)ds\Big\}\, g(Z_{H_r}, y)\Big]\,d\mu(y) \\
&= E_x\Big[\exp\Big\{-\int_0^{H_r} V(Z_s)ds\Big\}\, h(Z_{H_r})\Big] \\
&= E^{Q_x}[h(Z_{H_r})]\,.
\end{aligned} \tag{4.9}$$

Now when $y \in K$ and $\bar{B}(y,r) \subset U$, from the above inequality implies

$$M \ge h(y) \ge E^{Q_y}[h(Z_{H_r})] \ge (M+\epsilon)\,Q_y[Z_{H_r} \in U\backslash B]\,,$$

from which it follows that

$$\begin{aligned}
Q_y[Z_{H_r} \in B] &= Q_y[Z_{H_r} \in U] - Q_y[Z_{H_r} \in U\backslash B] \\
&\ge E_y\Big[\exp\Big\{-\int_0^{H_r} V(Z_s)ds\Big\}\Big] - \frac{M}{M+\epsilon} \ge \alpha > 0\,,
\end{aligned}$$

provided $r \le r_0$ picked small enough. As a result of the $0-1$ law of Blumenthal, $y$ is then a regular point for $B$.

If we now pick $x \notin K$ and apply (2.11) with $C$ a closed subset of $\mathbb{R}^d$ such that $U \cap C = B$, we find for $x \notin K$ and $y \in K$

$$g(x,y) = E_x\Big[H_B < T_U,\, \exp\Big\{-\int_0^{H_B} V(Z_s)ds\Big\}\, g(Z_{H_B}, y)\Big]\,,$$

and integrating with respect to $\mu(dy)$ the above equality we obtain

$$h(x) = E^{Q_x}[h(Z_{H_B})] \le M + \epsilon\,,$$

which shows that $B = U$. Since $\epsilon > 0$ is arbitrary this proves (4.5). □

As a consequence of the maximum principle, we have:

**Proposition 4.2:** *If $\mu$ is a finite measure supported on a compact subset $K$ of $U$*

$$\text{(4.10)} \qquad \sup_{x \in K} \int g(x,y)\,\mu(dy) \le 1 \textit{ implies that } \mu(K) \le \mathrm{cap}(K)\,.$$

**Proof:** Pick $B_n$ as the $\epsilon_n$ closed neighborhood of $K$, where $\epsilon_n \downarrow 0$ and $\epsilon_1$ is small enough so that $B_1 \subset U$. Since $B_n = \bigcup_{x\in K} \bar{B}(x,\epsilon_n)$, every point of $B_n$ is regular for $B_n$. Therefore from (3.38) we have

$$\text{(4.11)} \qquad \begin{aligned} Q_x[\widetilde{H}_{B_n} < \infty] &= \int g(x,y)\, e_{B_n}(dy),\ x \in U\,, \\ &= 1 \text{ when } x \in B_n\,. \end{aligned}$$

Furthermore

$$\begin{aligned} \mu(K) &= \int_K Q_x[\widetilde{H}_{B_n} < \infty]\, \mu(dx) \overset{(3.38)}{=} \int\int g(x,y)\,\mu(dx)\, e_{B_n}(dy) \\ &= \int \Big(\int g(x,y)\,\mu(dx)\Big)\, e_{B_n}(dy)\,. \end{aligned}$$

Observe that $\int g(\cdot,y)\,\mu(dx) \le 1$ on $K$ and therefore on $U$ by (4.5). From the above inequality we deduce

$$\text{(4.12)} \qquad \mu(K) \le \mathrm{cap}(B_n)\,.$$

Moreover when $x_0 \in U \backslash B_1$, $\bigcap_{n\ge 1}\{L_{B_n} > 0\} = \{L_K > 0\}$ $Q_{x_0}$-a.s., and on the set $\{L_K > 0\}$, $L_{B_n} \downarrow L_K$ with in fact $L_{B_n} = L_K$ for $n \ge 1$, on the subset $\{0 < L_K = \zeta\}$. It therefore follows that

$$
(4.13) \qquad \begin{aligned} L_{B_n}(x_0, dy) &= Q_{x_0}[0 < L_{B_n},\ Z_{L^-_{B_n}} \in dy] \underset{n\to\infty}{\overset{\text{weakly}}{\longrightarrow}} \\ L_K(x_0, dy) &= Q_{x_0}[0 < L_K,\ Z_{L^-_{L_K}} \in dy]\,. \end{aligned}
$$

Thus we see that

$$
(4.14) \qquad \begin{aligned} e_{B_n}(dy) &= g(x_0,y)^{-1}\, L_{B_n}(x_0,dy) \underset{n\to\infty}{\overset{\text{weakly}}{\longrightarrow}} e_K(dy) \\ &= g(x_0,y)^{-1} L_K(x_0,dy)\,, \end{aligned}
$$

and therefore $\text{cap}(B_n) \underset{n\to\infty}{\longrightarrow} \text{cap}(K)$, which together with (4.12) implies (4.10). □

As a consequence of the above Proposition, we shall now deduce the monotonicity property of the capacity:

**Corollary 4.3.:**

(4.15) *If $K_1 \subset K_2$ are compact subsets of $U$, then $cap(K_1) \le \text{cap}(K_2)$.*

**Proof:** If $e_{K_1}$ stands for the equilibrium measure of $K_1$,

$$
\int g(x,y)\, e_{K_1}(dy) = Q_x[\widetilde{H}_{K_1} < \infty] \le 1, \text{ for } x \in U\,.
$$

If we now apply (4.10) with $\mu = e_{K_1}$, we see that $\text{cap}(K_1) \le \text{cap}(K_2)$. □

A second consequence of Proposition 4.2 is:

**Corollary 4.4.:** *If $\nu$ is some finite measure on $U$ such that $\int g(x,y)\,\nu(dy)$ is finite $\nu$-a.s., then*

(4.16) $\qquad \nu(K) = 0$ *for any polar compact subset $K$ of $U$.*

**Proof:** By assumption, we have

$$
\begin{aligned} &\nu(K\backslash(\textstyle\bigcup_n K_n)) = 0, \ \text{ where} \\ &K_n = K \cap \{x \in U,\ \int g(x,y)\,\nu(dy) \le n\}, \text{ for } n \ge 1\,. \end{aligned}
$$

Since $x \to \int g(x,y)\,\nu(dy)$ is lower semi-continuous, $K_n$ are compact sets. Moreover,

$$
\int g(x,y)\ 1_{K_n}(y)\ \frac{\nu(dy)}{n} \le 1, \text{ for } x \in K_n
$$

and therefore

$$
\frac{\nu(K_n)}{n} \overset{(4.10)}{\le} \text{cap}(K_n) \overset{(4.15)}{\le} \text{cap}(K) = 0\,.
$$

It thus follows that

$$\nu(K) \le \nu(\cup K_n) + \nu(K \backslash (\cup K_n)) = 0\,.$$

□

**Exercise**: (subadditivity)
Show that when $K_1$, $K_2$ are two compact subsets of $U$

$$\text{cap}(K_1 \cup K_2) \le \text{cap}(K_1) + \text{cap}(K_2)\,. \tag{4.17}$$

□

The next proposition can be traced back to the work of Kellogg [Kel29] and Evans [Eva33]. It shows that the set of irregular points of a compact set is 'small'.

**Proposition 4.5:** *Let $K$ be a compact subset of $U$, then the set of irregular points of $K$ is a countable union of polar compact sets.*

**Proof:** With no loss of generality, we assume $K \neq \emptyset$. We pick $U$ to be some ball containing $K$, $V = 1$, and define

$$K_n = K \cap \left\{x : Q_x[\widetilde{H}_K < \infty] \le 1 - \frac{1}{n}\right\}.$$

It follows from (3.38) and lower semi-continuity that $K_n$ is compact, moreover from (3.26) the set of irregular points of $K$ equals $\bigcup_n K_n$. It therefore suffices to show that each $K_n$ is polar. Observe that

$$Q_x[\widetilde{H}_{K_n} < \infty] \le 1 - \tfrac{1}{n} \text{ for } x \in K_n \text{ and in fact for } x \in U \text{ by (4.5)}\,. \tag{4.18}$$

For a given fixed $n$ and $m \ge m_0$ introduce $B_{n,m}$ as the $\frac{1}{m}$ closed neighborhood of $K_n$, where $m_0$ is large enough so that $B_{n,m}$ is a compact subset of $U$.

Just as in the proof of Proposition 4.2, the compact set $B_{n,m}$ are regular, and when $x_0 \in U \backslash B_{n,m_0}$, then

$$Q_{x_0}\text{-a.s.}, \widetilde{H}_{B_{n,m}} < \widetilde{H}_{K_n} \text{ for } m \ge m_0, \text{ and } \widetilde{H}_{B_{n,m}} \uparrow \widetilde{H}_{K_n}\,. \tag{4.19}$$

Now from the strong Markov property since $\{\widetilde{H}_{K_n} < \infty\} \overset{(4.19)}{=} \{\widetilde{H}_{B_{n,m}} < \infty\} \cap \theta^{-1}_{\widetilde{H}_{B_{n,m}}}(\{\widetilde{H}_{K_n} < \infty\})$ $Q_{x_0}$-a.s., for $m \ge m_0$, we have

$$\begin{aligned} Q_{x_0}[\widetilde{H}_{K_n} < \infty \,|\, \mathcal{G}^+_{\widetilde{H}_{B_{n,m}}}] &= \\ Q_{Z_{\widetilde{H}_{B_{n,m}}}}[\widetilde{H}_{K_n} < \infty] &\overset{(4.18)}{\le} 1 - \frac{1}{n},\ Q_{x_0}\text{-a.s.} \end{aligned} \tag{4.20}$$

on $\{H_{\widetilde{B}_{n,m}} < \infty\}$.

On the other hand by the martingale convergence theorem, (using the notations of (3.4)).

$$\begin{aligned}&Q_{x_0}\text{-a.s. on } \{H_{K_n} < \infty\},\ Q_{x_0}[\widetilde{H}_{K_n} < \infty | \mathcal{G}^+_{\widetilde{H}_{B_{n,m}}}] \\ &\underset{m\to\infty}{\longrightarrow}\ Q_{x_0}[\widetilde{H}_{K_n} < \infty | \vee_{m\geq m_0} \mathcal{G}^+_{\widetilde{H}_{B_{n,m}}}] = 1\,,\end{aligned} \tag{4.21}$$

because $\{\widetilde{H}_{K_n} < \infty\} = \bigcap_{m\geq m_0}\{\widetilde{H}_{B_{n,m}} < \infty\}$ $Q_{x_0}$-a.s. by (3.26) and is therefore $\vee_{m\geq m_0} \mathcal{G}^+_{\widetilde{H}_{B_{n,m}}}$ measurable.

Now (4.20) and (4.21) show that $\{\widetilde{H}_{K_n} < \infty\}$ is $Q_{x_0}$-negligible and therefore $K_n$ is polar by (4.1), (4.2). □

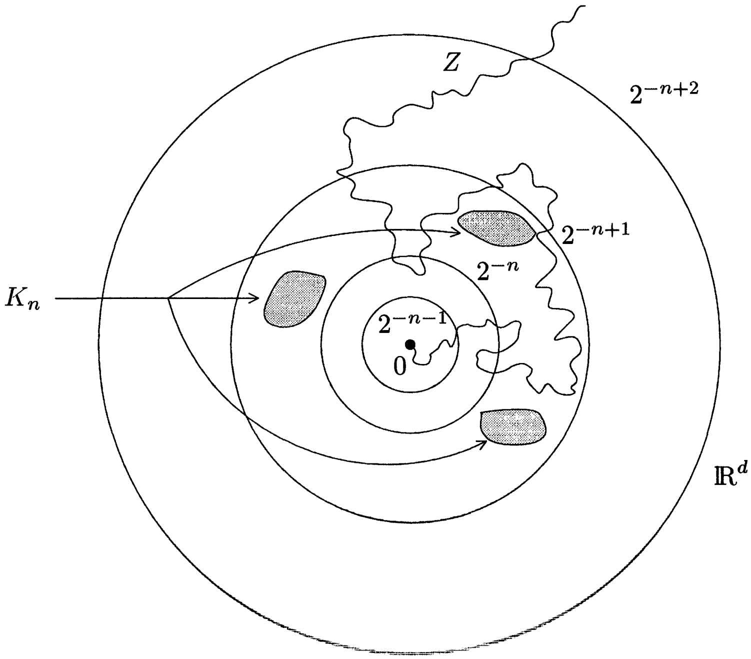
**Example 4.6:** Wiener test

We consider $\mathbb{R}^d$, $d \geq 3$, and $K$ a compact subset of $\mathbb{R}^d$ which contains 0. We want to derive a criterion to test whether 0 is a regular point of $K$ or not.

We define for $n \geq 1$,

$$K_n = K \cap \{x :\ 2^{-n} \leq |x| \leq 2^{-n+1}\}\,, \tag{4.22}$$

and denote by 'cap' the capacity relative to $U = \mathbb{R}^d$ and $V = 0$.



**Theorem 4.7:**

$$(4.23) \qquad 0 \textit{ is regular for } K \textit{ if and only if } \sum_n 2^{n(d-2)}\operatorname{cap}(K_n) = +\infty\,.$$

**Proof:** Consider the sequence of events and of random variables:

$$\begin{aligned} A_n &= \{H_{K_n} < \infty\},\ n \geq 1\,,\\ N_k &= \sum_{n=1}^{k} 1_{A_n}\,,\ k \geq 1\,. \end{aligned}$$

The regularity of 0 for $K$ is equivalent to the nonvanishing of $P_0[N_\infty = \infty]$, (this probability is either 0 or 1, since $\{N_\infty = \infty\}$ belongs to the $P_0$-completion of $\mathcal{F}_0^+$). We shall now prove that there exists dimension dependent constants $c > 1$, $\kappa > 1$, such that

$$(4.24) \qquad \frac{1}{c}\ P_0[A_n] \leq 2^{n(d-2)}\ \operatorname{cap}(K_n) \leq c\ P_0[A_n],\ n \geq 1\,,$$

$$(4.25) \qquad E_0[N_k^2] \leq \kappa\ (E_0[N_k]^2 + 1),\ k \geq 1\,.$$

Our claim (4.23) will then follow. Indeed, when $\sum 2^{n(d-2)}\ \operatorname{cap}(K_n) < \infty$, the left inequality of (4.24) implies that $N_\infty$ is finite $P_0$-a.s., and therefore 0 is not regular for $K$. On the other hand, when $\sum 2^{n(d-2)}\ \operatorname{cap}(K_n) = \infty$, the right inequality of (4.24) implies that $E_0[N_\infty] = \infty$, moreover

$$\begin{aligned} P_0[N_k \geq \tfrac{1}{2}\ E_0[N_k]]\ E_0[N_k^2] &\overset{\binom{\text{Cauchy-Schwarz}}{\text{inequality}}}{\geq} E_0[N_k,\ N_k \geq \tfrac{1}{2}\ E_0[N_k]]^2 \\ (4.26) \qquad &= \Big(E_0[N_k] - E_0\Big[N_k,\ N_k < \tfrac{1}{2}\ E_0[N_k]\Big]\Big)^2 \\ &\geq \frac{1}{4}\ E_0[N_k]^2 \overset{(4.25)}{\geq} \frac{1}{4\kappa}\ E_0[N_k^2] - \frac{1}{4}\,. \end{aligned}$$

It now follows that:

$$P_0[N_\infty = \infty] = \lim_k\ P_0[N_\infty \geq \tfrac{1}{2}\ E_0[N_k]] \geq \frac{1}{4\kappa} > 0\,,$$

and this proves that 0 is regular for $K$.

To prove (4.24), observe that

$$P_0[H_{K_n} < \infty] \overset{(3.38)}{=} \int_{K_n} g(0,y)\, e_{K_n}(dy)\,,$$

where $g(z,y) = \frac{c(d)}{|z-y|^{d-2}}$, with $c(d)$ as in (3.51).

Since $\text{cap}(K_n) = e_{K_n}(K_n)$, and $2^{-n} \le |y| \le 2^{-n+1}$, for $y \in K_n$, the claim (4.24) follows.

One can break $N_k$ into two series of respective even and odd terms. Thus to prove (4.25), it suffices to show the existence of $\kappa' > 1$, such that for $n, n' \ge 1$, with $|n - n'| \ge 2$:

$$P_0[A_n \cap A_{n'}] \le \kappa' \, P_0[A_n] \, P_0[A_{n'}] \,. \tag{4.27}$$

To this end we write:

$$P_0[A_n \cap A_{n'}] = P_0[H_{K_n} < H_{K_{n'}} < \infty] + P_0[H_{K_{n'}} < H_{K_n} < \infty] \,. \tag{4.28}$$

With the help of the strong Markov property and (3.38):

$$\begin{aligned} &P_0[H_{K_n} < H_{K_{n'}} < \infty] = P_0[H_{K_n} < \infty, \int g(Z_{H_{K_{n'}}}, y) \, e_{K_{n'}}(dy)] \\ &\le \kappa_1 \; P_0[H_{K_n} < \infty] \; P_0[H_{K_{n'}} < \infty] \,, \end{aligned}$$

for a suitable dimension dependent $\kappa_1 > 1$, since $\text{dist}(K_n, K_{n'}) \ge 2^{-1-\min(n,n')}$, thanks to the requirement $|n - n'| \ge 2$.

The rightmost term of (4.28) is handled in the same fashion. This completes the proof of (4.27) and of (4.25). $\square$

**Exercise:** Let $K$ be a compact subset of $B(0,1) \subseteq \mathbb{R}^2$, containing 0. Show that 0 is regular for $K$ if and only if

$$\sum n \, \text{cap}_{B(0,1)}(K_n) = \infty \,. \tag{4.29}$$

$\square$

We now want to discuss the other main topic of this section, namely the *variational principle.* For $U, V$ satisfying the assumptions (2.1), (2.2), (2.13) from the beginning of this section, we introduce the set of *measures of finite energy*:

$$\begin{aligned} \mathcal{M} = \{\nu : \nu \text{ is a finite signed measure on } U \text{ and} \\ \int_{U \times U} g(x,y) |\nu|(dx) |\nu|(dy) < \infty\} \,. \end{aligned} \tag{4.30}$$

From Corollary 4.4 and Proposition 4.5, we have:

(4.31) for $\nu \in \mathcal{M}$ and $K$ a compact subset of $U$, the set of irregular points of $K$ is negligible for $|\nu|$.

We begin with the following

**Proposition 4.8:** *$\mathcal{M}$ is a vector space, and when $\mu, \nu \in \mathcal{M}$, $g$ is $|\mu| \otimes |\nu|$ integrable and*

$$(4.32)\quad \begin{aligned} &< \mu, \nu > \stackrel{\text{def}}{=} \int g(x,y)\, \mu(dx)\, \nu(dy) = \\ &2\int_0^\infty ds \int_U \Big(\int_U r(s,x,z)\mu(dx) \int_U r(s,y,z)\nu(dy)\Big) dz \end{aligned}$$

*defines a scalar product on $\mathcal{M}$. For $\mu \in \mathcal{M}$, the quantity $< \mu, \mu >$ is called energy of $\mu$.*

**Proof:** Assume $\mu, \nu$ are positive measures in $\mathcal{M}$. Then

$$(4.33)\quad \begin{aligned} \int g(x,y)\, \mu(dx)\, \nu(dy) &= \int_0^\infty ds \int r(s,x,y)\, \mu(dx)\, \nu(dy) \\ &\stackrel{(I.3.7)}{=} \int_0^\infty ds \int_U \Big(\int_U \mu(dx) r(\tfrac{s}{2},x,z)\Big)\Big(\int_U \nu(dy) r(\tfrac{s}{2},y,z)\Big) dz \\ &\le \frac{1}{2} \int_0^\infty ds \int_U \Big(\int_U \mu(dx)\, r(\tfrac{s}{2},x,z)\Big)^2 + \Big(\int_U \nu(dx)\, r(\tfrac{s}{2},x,z)\Big)^2 dz \\ &\stackrel{\binom{\text{same}}{\text{calculation}}}{=} \frac{1}{2} \int g(x,y)\, \mu(dx)\mu(dy) + \frac{1}{2}\int g(x,y)\, \nu(dx)\, \nu(dy)\,. \end{aligned}$$

The above calculation shows that when $\mu, \nu \in \mathcal{M}$, $g$ is $|\mu| \otimes |\nu|$ integrable and (4.32) holds. When $\mu, \nu$ are positive measures in $\mathcal{M}$, proceeding as above

$$\begin{aligned} &\int g(x,y)(\mu+\nu)(dx)(\mu+\nu)dy) = 2\int_0^\infty \int_U \Big(\int_U (\mu+\nu)(dx)\, r(s,x,z)\Big)^2 dz \\ &\le 2 \int g(x,y)\, \mu(dx)\, \mu(dy) + 2\int g(x,y)\, \nu(dx)\, \nu(dy) < \infty\,. \end{aligned}$$

From this follows that $\mathcal{M}$ is a vector space. Now $< \cdot, \cdot >$ defined in (4.32) is clearly bilinear symmetric, moreover for $\mu \in \mathcal{M}$, $< \mu, \mu > \ \geq 0$.

Observe that when $< \mu, \mu >= 0$, (4.32) implies that

$$\int r(s,x,z)\, \mu(dx) = 0, \ ds\, dz \text{ a.e.}\,.$$

Therefore for $\varphi \in C_c(U)$,

$$\int R_s^{U,V} \varphi(x)\, \mu(dx) = 0, \text{ for a.e. } s\,,$$

and letting $s$ tend to zero in the set of full measure

$$\int \varphi(x)\, \mu(dx) = 0, \text{ for any } \varphi \in C_c(U)\,.$$

As a result $\mu = 0$, and $< \cdot, \cdot >$ defines a scalar product on $\mathcal{M}$. □

We now come to our main object namely the variational principle.

**Theorem 4.9:** *Let $K$ be a compact subset of $U$. The restriction of the scalar product $< \cdot, \cdot >$ to the set $\mathcal{M}_K^+$ of positive measures in $\mathcal{M}$ concentrated on $K$ with mass cap$(K)$ has a unique minimum at $e_K$ the equilibrium measure of $K$. Moreover*

$$\text{cap}(K) = \Big(\inf\Big\{\int g(x,y)\,\mu(dx)\,\mu(dy),\ \mu \in M_1(K)\Big\}\Big)^{-1} \tag{4.34}$$

*($M_1(K)$ stands for the set of probability measures supported on $K$).*

**Proof:** From (4.31) and the fact that the equilibrium potential $v_K(x)$ equals 1 when $x$ is a regular point of $K$, we see that the set $K \cap \{x : v_K(x) < 1\}$ is negligible for any $\nu \in \mathcal{M}$. In particular:

$$\begin{aligned} < e_K, e_K > &= \int\int g(x,y)\,e_K(dy)\,e_K(dx) = \int v_K(x)\,e_K(dx) \\ &= e_K(K) = \text{cap}(K)\,. \end{aligned} \tag{4.35}$$

Therefore $e_K \in \mathcal{M}_K^+$. Now if $\nu \in \mathcal{M}_K^+$, then

$$< \nu, \nu > = < \nu - e_K,\ \nu - e_K > +2 < \nu - e_K,\ e_K > + < e_K,\ e_K >$$

and

$$\begin{aligned} < \nu - e_K, e_K > &= \int (\nu - e_K)(dx) \int g(x,y)\,e_K(dy) \\ &= \int (\nu - e_K)(dx)\,v_K(x) = (\nu - e_K)(K) = 0 \end{aligned}$$

where we used the fact that $v_K(x) = 1$, $(\nu - e_K)$ a.e. . Therefore

$$< \nu, \nu > = < \nu - e_K, \nu - e_K > + < e_K, e_K > \ . \tag{4.36}$$

In other words, '$e_K$ is the orthogonal projection of 0 on $\mathcal{M}_K^+$'. This proves the first part of our statement. Let us finally prove (4.34).

When cap$(K) = 0$, any positive measure in $M$ satisfies $\nu(K) = 0$ by Corollary 4.4 and therefore $\mathcal{M} \cap M_1(K) = \emptyset$ so that (4.34) holds.

When cap$(K) > 0$, then $\frac{1}{\text{cap}(K)}\ e_K \in \mathcal{M} \cap M_1(K)$ and with no loss of generality we can replace $M_1(K)$ by $\mathcal{M} \cap M_1(K)$ in (4.34). Now for $\mu \in \mathcal{M} \cap M_1(K)$, $\nu = \text{cap}(K)\,\mu$ belongs to $M_K^+$ and

$$(\text{cap}\,K)^2 < \mu, \mu > = < \nu, \nu > \overset{(4.35)-(4.36)}{\geq} < e_K, e_K > = \text{cap}(K)$$

therefore for $\mu \in \mathcal{M} \cap M_1(K)$

$$< \mu, \mu > \geq \operatorname{cap}(K)^{-1} ,$$

with equality when $\mu = \operatorname{cap}(K)^{-1} e_K$. This proves (4.34). □

The above theorem gives a powerful way to estimate capacities. As an application we have this alternate characterization of the equilibrium measure.

**Corollary 4.10:** *Let $K$ be a compact subset of $U$. Then $e_K$ is the only positive finite measure supported on the regular points of $K$ which satisfies:*

$$\int g(x,y)\, e_K(dy) = 1 , \tag{4.37}$$

*for all $x$ regular point of $K$.*

**Proof:** We simply have to prove uniqueness. Let $\nu$ be a positive finite measure supported on the regular points of $K$ for which (4.37) holds with $e_K$ replaced by $\nu$. Then $\nu \in \mathcal{M}$ and

$$\begin{aligned} \nu(K) &= \int v_K(x)\, d\nu(x) \overset{(3.38)}{=} \int\int \nu(dx) e_K(dy) g(x,y) \overset{(4.31)(4.37)}{=} \int e_K(dx) \\ &= \operatorname{cap}(K) . \end{aligned}$$

Therefore $\nu \in \mathcal{M}_K^+$ and:

$$< \nu, \nu >= \int \nu(dx) \int g(x,y)\, \nu(dy) \overset{(4.37)}{=} \nu(K) = \operatorname{cap}(K) =< e_K, e_K > .$$

The equality $\nu = e_K$ now follows from Theorem 4.9. □

**Example 4.11:** ($\lambda$-equilibrium measure and $\lambda$-capacity)

We consider the case $U = \mathbb{R}^d$, $d \geq 1$, $V = \lambda > 0$. We shall now see that for $K$ a compact subset of $\mathbb{R}^d$:

$$e_K(dz) = \int_{\mathbb{R}^d} \lambda E_y[\exp\{-\lambda H_K\},\, Z_{H_K} \in dz]\, dy . \tag{4.38}$$

When $x$ is a regular point of $K$:

$$\begin{aligned} &\int_{\mathbb{R}^d} \lambda\, E_y[\exp\{-\lambda H_K\}\, g_\lambda(Z_{H_K}, x)]\, dy \overset{(2.11)}{=} \int_{\mathbb{R}^d} \lambda\, g_\lambda(y,x)\, dy \\ &= \int_0^\infty \lambda e^{-\lambda s} \int_{\mathbb{R}^d} p(s,y,x)\, dy ds = 1 . \end{aligned}$$

We thus see that the r.h.s. of (4.38) defines a finite measure, which is concentrated on $K$. With the help of Proposition 4.5, it puts no mass on the irregular points of $K$. The claim (4.38) now follows from Corollary 4.10. If we denote by $\mathrm{cap}_\lambda$ the capacity relative to $U = \mathbb{R}^d$, $V = \lambda > 0$, we find as a direct consequence of (4.38)

$$\mathrm{cap}_\lambda(K) = \int_{\mathbb{R}^d} \lambda E_y[\exp\{-\lambda H_K\}]\, dy\,. \tag{4.39}$$

We now specialize the situation to the case $d \geq 2$, $K = \overline{B}(0,\epsilon)$. An application of (2.11) shows that in this case, for $y \in \mathbb{R}^d$:

$$g(y,0) = E_y[\exp\{-\lambda H_{\overline{B}(0,\epsilon)}\}\, g(Z_{H_{\overline{B}(0,\epsilon)}}, 0)]\,,$$

so that

$$E_y[\exp\{-\lambda H_{\overline{B}(0,\epsilon)}\}] = \left(\frac{g_\lambda(y,0)}{r}\right) \wedge 1, \quad \text{with } r = g_\lambda(0, \epsilon e_1)\,, \tag{4.40}$$

with $e_1$ the first vector of the canonical basis of $\mathbb{R}^d$.

Coming back to (4.39), we now find

$$\begin{aligned} \mathrm{cap}_\lambda(\overline{B}(0,\epsilon)) &= \lambda\,|\overline{B}(0,\epsilon)| + \frac{\lambda}{r}\int_{\overline{B}(0,\epsilon)^c} g_\lambda(y,0)\, dy \\ &= \lambda\,|\overline{B}(0,\epsilon)| + \frac{1}{r}\left(1 - \int_{\overline{B}(0,\epsilon)} \lambda g_\lambda(y,0)\, dy\right). \end{aligned} \tag{4.41}$$

The above formula is in particular interesting when $\epsilon$ tends to 0. In this case both $r\,|\overline{B}(0,\epsilon)|$ and $\int_{\overline{B}(0,\epsilon)} g_\lambda(y,0)dy$ tend to 0, and we thus find

$$\mathrm{cap}_\lambda(\overline{B}(0,\epsilon)) \sim \frac{1}{r}\,, \quad \text{as } \epsilon \to 0\,.$$

Therefore as $\epsilon \to 0$,

$$\begin{aligned} \mathrm{cap}_\lambda(\overline{B}(0,\epsilon)) &\sim \frac{\epsilon^{d-2}}{c(d)}\,, \quad \text{when } d \geq 3\,, \\ &\sim \frac{\pi}{\log\frac{1}{\epsilon}}\,, \quad \text{when } d = 2\,, \end{aligned} \tag{4.42}$$

with $c(d) = 2\pi^{d/2}/\Gamma(\frac{d}{2} - 1)$, for $d \geq 3$, (compare with (3.55). □

**Exercise:**

**1)** Consider $U \subseteq \mathbb{R}^d$, $d \geq 3$, a nonempty connected open set and $x \in U$. Show that when $\epsilon \to 0$

$$\mathrm{cap}_{U,V=0}(\bar{B}(x,\epsilon)) \sim \epsilon^{d-2}\mathrm{cap}(\bar{B}(0,1)) \tag{4.43}$$

(Hint: use scaling, translation invariance, with

$$\epsilon^{-(d-2)}\text{cap}_U(\bar{B}(x,\epsilon)) = \text{cap}_{\epsilon^{-1}(U-x)}(\bar{B}(0,1)) \le \text{cap}_{B(0,r/\epsilon)}(\bar{B}(0,1))$$

if $B(x,r) \subseteq U$).

**2)** Let $W_t^a$ stand for the Wiener sausage of radius $a > 0$ in time $t$ for Brownian motion in $\mathbb{R}^d$, $d \ge 2$. Define the measure $\nu_a$ on $\mathbb{R}_+$ via:

$$\nu_a[0,t] = E_0[|W_t^a|] \,.$$

Show that for $\lambda > 0$:

$$\int_0^\infty e^{-\lambda s}\, d\nu_a(s) = \int_{\mathbb{R}^d} E_x[\exp\{-\lambda H_{\bar{B}(0,a)}\}]\, dx \,.$$

Deduce from this that when $a \to 0$ ,

$$(4.44) \qquad K_d(a) \int_0^\infty e^{-\lambda s} d\nu_a(s) \longrightarrow \begin{array}{ll} (c(d)\lambda)^{-1}\,, & \text{when } d \ge 3\,, \\ (\pi\lambda)^{-1}\,, & \text{when } d = 2\,. \end{array}$$

$$\text{provided } K_d(a) \begin{array}{lll} = & a^{2-d}\,, & \text{when } d \ge 3\,, \\ = & \log \dfrac{1}{a}\,, & \text{when } d = 2\,, \end{array}$$

and $c(d) = \text{cap}(\bar{B}(0,1))$ is as in (4.42).

Recover from (4.44) that when $a \to 0$, for $t \ge 0$,

$$(4.45) \qquad \begin{array}{l} a^{2-d}\, E_0[|W_t^a|] \longrightarrow c(d)\, t \;\text{ when } d \ge 3, \text{ (see also (3.61))}\,, \\ \left(\log \dfrac{1}{a}\right) E_0[|W_t^a|] \longrightarrow \pi\, t, \;\text{ when } d = 2\,. \end{array}$$

**3)** (Level sets of Green function)

Let $U, V$ fulfill (2.1), (2.2), (2.13), and consider $x_0 \in U$, $r > 0$, and the level set of $g(\cdot, x_0)$:

$$K = \{x \in U,\; g(x,x_0) \ge r\}\,,$$

where we assume that $x_0 \in K$ and $K$ is compact. Show that the equilibrium potential of $K$ is

$$(4.46) \qquad v_K(x) = \left(\frac{g}{r}\,(x,x_0)\right) \wedge 1\,,$$

in particular all points of $K$ are regular for $K$.

(Hint: outside $K$, argue as in (4.40). To handle the possibility of an irregular $x \in K$, introduce a decreasing sequence $O_n \supset K$ of relatively compact open subsets of $U$, with $\bigcup_n \overline{O}_n = K$. Show that

$$v_K(x) \geq E_x\Big[T_{O_n} < T_U,\, \exp\Big\{-\int_0^{T_{O_n}} V(Z_s)ds\Big\}\, v_K(Z_{T_{O_n}})\Big]\,,$$

and $T_{O_n} \downarrow 0$, $P_x$-a.s. .

4) Specialize the situation of 3) to the case $V = 0$. Show that

$$e_K(dz) = \frac{1}{r}\, P_{x_0}[Z_{H_{\partial K}} \in dz] \tag{4.47}$$

is the equilibrium measure of $K$, and consequently $\mathrm{cap}_{U,V=0}(K) = \frac{1}{r}$. (Hint: use Corollary 4.10 and (2.11)). □

**Example 4.12:** (Capacity and entropy)
We consider $K$ a compact subset of $[0,1]^2 \subseteq \mathbb{R}^2$ and $\nu \in M_1(K)$ a probability measure on $K$. If $\lambda$ denotes the restriction of Lebesgue measure to $[0,1]^2$, the relative entropy $H(\nu|\lambda)$ is defined as:

$$H(\nu|\lambda) = \begin{array}{l} \displaystyle\int \log f(x)\, f(x)dx, \;\; \text{if } \nu << \lambda \text{ and } \nu = f\cdot\lambda\,, \\ +\,\infty, \;\; \text{otherwise}\,. \end{array} \tag{4.48}$$

As we shall now see one can choose $c > 0$, so that

$$\frac{1}{c + H(\nu|\lambda)} \leq \mathrm{cap}_{\mathbb{R}^2,V=1}(K), \;\; \text{for any } \nu \in M_1(K)\,. \tag{4.49}$$

With no loss of generality, we can assume that $H(\nu|\lambda) < \infty$. If $g(\cdot,\cdot)$ stands for $g_{\mathbb{R}^2,V=1}(\cdot,\cdot)$, then:

$$\langle \nu, \nu\rangle = \int \nu(dx)\Big(\int g(x,y)\,\nu(dy)\Big)\,,$$

and by the entropy inequality (see Deuschel-Stroock [DS89], p. 69), for $x \in [0,1]^2$,

$$\begin{aligned}\int g(x,y)\,\nu(dy) &\leq \log\Big(\int_{[0,1]^2} e^{g(x,y)}dy\Big) + H(\nu|\lambda) \\ &\leq c + H(\nu|\lambda)\,,\end{aligned}$$

where $c$ is a positive finite constant since $g(x,y) = g(0,y-x) \sim -\frac{1}{\pi}\log|y-x|$ as $y \to x$.

We now see in view of (4.34) that

$$\frac{1}{\mathrm{cap}_{\mathbb{R}^2,V=1}(K)} \leq \langle \nu,\nu\rangle \leq c + H(\nu|\lambda)\,.$$

This proves our claim (4.49). As a special case, observe that when $|K| > 0$, picking $\nu$ to be the normalized Lebesgue measure on $K$, we find:

$$\frac{1}{c + \log \frac{1}{|K|}} \le \mathrm{cap}_{\mathbb{R}^2, V=1}(K) \,. \tag{4.50}$$

□

**Example 4.13**: We consider $U = \mathbb{R}^d$, and $V = 0$, when $d \ge 3$, $V = \lambda > 0$, when $d = 1, 2$. In this case we have

$$g(x, y) = G(|y - x|) \,,$$

where $G(\cdot)$ is the continuous decreasing function $\mathbb{R}_+ \to (0, +\infty]$:

$$\begin{aligned} G(u) &= c(d)\, u^{2-d}, \quad \text{when } d \ge 3, \text{ see } (3.51)\,, \\ &\frac{1}{(2\pi)^{d/2}} \int_0^\infty \exp\Big\{ - \lambda s - \frac{u^2}{2s} \Big\} \frac{ds}{s^{d/2}}\,, \quad \text{when } d = 1, 2\,, \\ &\Big( \text{when } d = 2, \ G(u) \underset{u \to 0}{\overset{(1.2.22)}{\sim}} \frac{1}{\pi} \log \frac{1}{u} \Big)\,. \end{aligned} \tag{4.51}$$

We have a lower bound on the capacity relative to $U = \mathbb{R}^d$ and $V = 0$, of compact sets of positive Lebesgue measure:

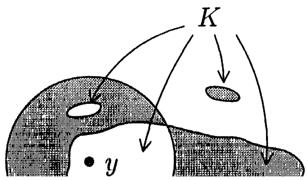
**Proposition 4.14**: *If $K$ is a compact subset of positive Lebesgue measure in $\mathbb{R}^d$, $d \ge 3$, and $|K| = |B(0, R)|$, then*

$$\mathrm{cap}(K) \ge |K| \Big( \int_{B(0,R)} g(0, x) dx \Big)^{-1} \,. \tag{4.52}$$

**Proof**:

$$\frac{1}{\mathrm{cap}(K)} \Big( \int_K g(x, y) dx \Big) \, dy \,. \tag{4.53}$$

Consider $y \in K$ and $B = \overline{B}(y, R)$



We have $|B| = |K|$ and

$$\begin{aligned} |B| &= |B \cap K| + |B \backslash K| \,, \\ |K| &= |B \cap K| + |K \backslash B| \,, \quad \text{therefore} \end{aligned}$$

$$|B \backslash K| = |K \backslash B| \,.$$

Now since $G(\cdot)$ is decreasing:

$$\begin{aligned} \int_K g(x,y)dx &= \int_{B\cap K} g(x,y)dx + \int_{K\backslash B} G(|x-y|)dx \\ &\le \int_{B\cap K} g(x,y)dx + \int_{B\backslash K} G(|x-y|)dx = \int_B g(x,y)dx \\ &= \int_{B(0,R)} g(0,x)dx \,. \end{aligned}$$

Inserting in (4.53), we find our claim (4.52). □

In fact, using symmetric decreasing rearrangements and the Dirichlet form characterization of capacity, see Theorem 5.3 of the next section, one can show that

$$\mathrm{cap}(K) \ge \mathrm{cap}(\overline{B}(0,R)) \,,$$

in the situation of Proposition 4.16. On this we refer the reader to Bandle-Flucher [BF96].

Observe by the way that

$$\begin{aligned} |B(0,R)| \left( \int_{B(0,R)} g(0,x)dx \right)^{-1} &= \frac{2}{d\,c(d)}\, R^{d-2} \quad d \ge 3 \,, \\ &\underset{R\to 0}{\sim} \frac{\pi}{\log \frac{1}{R}} \,, \quad d = 2 \,, \\ &\underset{R\to 0}{\longrightarrow} \mathrm{cap}(\{0\}), \quad d = 1 \,. \end{aligned}$$

□

**Example 4.15**: (Energy and dyadic decomposition)
$U = \mathbb{R}^d$, $d \ge 2$, $V = 0$, when $d \ge 3$, $V = 1$, when $d = 2$.

We consider a dyadic decomposition of $[0,1]^d$ into cubes

(4.54) $$D_m = y_m + 2^{-k}[0,1]^d \,,$$

where the label $m$ runs over the set $I = \bigcup_{k\ge 0} I_k$ of finite sequences of arbitrary length $k$ with values in $\{0,1\}^d$, (the sequence of length 0 is denoted as $\phi$), and

(4.55) $$y_m = \frac{i_1}{2} + \ldots + \frac{i_k}{2^k} \,, \quad \text{for } m = (i_1, \ldots, i_k) \in I_k, \; k \ge 1, \; (\text{and } y_\phi = 0) \,.$$

Just as in (4.51), we can write $g(x,y) = G(|y-x|)$, where $G$ is a continuous decreasing bijection from $[0,\infty)$ onto $(0,\infty]$.

**Proposition 4.16**: *There exists a dimension dependent $c > 1$, such that for any nonnegative finite measure $\nu$ on $[0,1]^d$:*

$$\frac{1}{c}\, I(\nu) \le \langle \nu, \nu \rangle \le c\, I(\nu), \quad \textit{with} \tag{4.56}$$

$$I(\nu) \stackrel{\text{def}}{=} \sum_{k\ge 1} 2^{k(d-2)} \sum_{m\in I_k} \nu(D_m)^2 \,. \tag{4.57}$$

**Proof**: Lower bound:

$$\langle \nu, \nu \rangle = \int\!\!\int G(|y-x|)\, \nu(dx)\, \nu(dy) \ge \sum_{k\ge 1} (G(2^{-k}) - G(2^{1-k}))\, \nu\otimes\nu(|x-y| \le 2^{-k})$$

Observe that when $x, y \in D_m$, with $m \in I_{k+k_0(d)}$, then $|x-y| \le 2^{-k}$. Therefore

$$\langle \nu, \nu \rangle \ge \sum_{k\ge 1} \Big(G(2^{-k}) - G(2^{1-k}\Big) \sum_{m\in I_{k+k_0}} \nu(D_m)^2 \,. \tag{4.58}$$

Moreover if $m' \in I_k$, and $m \succ m'$ means that $m$ 'extends' $m'$,

$$\begin{aligned} \nu(D_{m'})^2 \;&\le\; \Big(\sum_{m\in I_{k+k_0}, m\succ m'} \nu(D_m)\Big)^2 \\ &\overset{\text{(Cauchy-Schwarz)}}{\le} \Big(\sum_{m\in I_{k+k_0}, m\succ m'} \nu(D_m)^2\Big)\, 2^{dk_0} \,. \end{aligned}$$

Therefore

$$\langle \nu, \nu \rangle \ge 2^{-dk_0} \sum_{k\ge 1} (G(2^{-k}) - G(2^{1-k})) \sum_{m\in I_k} \nu(D_m)^2 \,. \tag{4.59}$$

Upper bound:

$$\begin{aligned} \langle \nu, \nu \rangle \le &\sum_{k\ge 1} (G(2^{-k}) - G(2^{1-k}))\, \nu\otimes\nu(|x-y| \le 2^{1-k}) \\ &+ G(1)\, \nu([0,1]^d)^2 \,. \end{aligned} \tag{4.60}$$

If $J_k$ denotes the symmetric set of couples $m, m' \in I_k$ with dist $(D_m, D_{m'}) \le 2^{1-k}$, we have

$$\nu\otimes\nu(|x-y| \le 2^{1-k}) \le \sum_{J_k} \nu(D_m)\, \nu(D_{m'}) \le \frac{1}{2} \sum_{J_k} \nu(D_m)^2 + \nu(D_{m'})^2 \,.$$

We can find $N(d) \geq 1$, such that for any $m \in I_k$

$$\begin{aligned} \#\{m', (m,m') \in J_k\} &\leq N(d), \text{ and therefore} \\ \nu \otimes \nu(|x-y| \leq 2^{1-k}) &\leq N(d) \sum_{I_k} \nu(D_m)^2 . \end{aligned}$$

Inserting in (4.60), we find

$$\begin{aligned} \langle \nu, \nu \rangle \leq N(d) \sum_{k \geq 1} (G(2^{-k}) - G(2^{1-k})) \sum_{I_k} \nu(D_m)^2 \\ + G(1)\, 2^k \sum_{I_1} \nu(D_m)^2 . \end{aligned} \tag{4.61}$$

Observe that for a suitable dimension dependent $\gamma > 1$

$$\gamma^{-1}\, 2^{k(d-2)} \leq G(2^{-k}) - G(2^{1-k}) \leq \gamma 2^{k(d-2)}, \ k \geq 1 ,$$

this is indeed immediate when $d \geq 3$, and otherwise follows from the calculation in Chapter 4 after (4.3.7), when $d = 2$. The claim (4.56) is now a consequence of (4.59), (4.61). □

Incidentally observe that when $\nu$ is a probability concentrated on a compact set $K \subset [0,1]^d$,

$$N_k \sum_{m \in I_k} \nu(D_m)^2 \geq \nu(K)^2 = 1 ,$$

if $N_k$ denotes the number of dyadic cubes of generation $k$ which intersect $K$. As a result one has an upper bound on $\mathrm{cap}(K)$:

$$\mathrm{cap}(K) \leq c \Big( \sum_{k \geq 1} \frac{2^{k(d-2)}}{N_k} \Big)^{-1} . \tag{4.62}$$

For more on this we refer to Chapter 4 of Carleson [Car67], Benjamini-Peres [BP92], Peres [Per96].

**Exercise:**

1) Show that a line segment in $\mathbb{R}^2$ is nonpolar, but that it is polar in $\mathbb{R}^3$.

2) Show that for $Z_.$ a Brownian motion in $\mathbb{R}^3$

$$E_0\Big[\int_0^1 ds \int_0^1 ds' \frac{1}{|Z_s - Z_{s'}|}\Big] \leq 2E_0\Big[\int_0^1 \frac{ds}{|Z_s|}\Big] < \infty .$$

(use Markov property).

Deduce that

(4.63) $P_0$-a.s. $Z_{[\alpha,\beta]}$ for $0 \leq \alpha < \beta < \infty$ is a nonpolar compact set in $\mathbb{R}^3$.

**3)** With the assumptions of the beginning of this section show that when $K_n, n \geq 1$, is a decreasing sequence of compact subsets of $U$, and $K = \bigcap_n K_n$:

$$\text{cap}(K_n) \downarrow \text{cap}(K), \quad \text{as } n \to \infty . \tag{4.64}$$

On the other hand if $U_n$, $n \geq 1$, are connected nonempty open sets which increase to $U$, and $K \subseteq U_1$ is a compact subset, show that

$$\text{cap}_{U_n,V}(K) \downarrow \text{cap}_{U,V}(K) . \tag{4.65}$$

**4)** (Subadditivity and mutual energy).

$\nu_A, \nu_B$ are probabilities respectively supported by the nonpolar compact subsets $A, B$ of $U$, $\alpha \geq 0$, $\beta \geq 0$, with $\alpha + \beta = 1$, $\nu = \alpha\, \nu_A + \beta \nu_B$. Show that

$$\langle \nu, \nu \rangle \geq \alpha^2 \langle \nu_A, \ \ \nu_B \rangle + \beta^2 \langle \nu_B, \ \ \nu_B \rangle \geq \frac{1}{\text{cap}(A) + \text{cap}(B)} .$$

Deduce (4.17): $\text{cap}(A \cup B) \leq \text{cap}(A) + \text{cap}(B)$.

If $\nu_A, \nu_B$ are the normalized equilibrium measures of $A$ and $B$ what is the minimum value of $\alpha^2 \langle \nu_A, \nu_A \rangle + \beta^2 \langle \nu_B, \nu_B \rangle$ under the above constraints $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta = 1$? □

## 2.5 Some Connections with Dirichlet Forms

We shall now briefly discuss the connection of the energy of a measure, as defined in the last section, with the Dirichlet form associated to $U, V$. Throughout this section, we assume (2.1), (2.2), (2.13). For the coming proposition we shall use the following *coercivity condition* on the Dirichlet form $\mathcal{E}_{U,V}$:

$$\mathcal{E}_{U,V}(f, f) \geq c(f, f), \quad \text{for } f \in \mathcal{D}(\mathcal{E}_{U,V}) , \tag{5.1}$$

$c$ a positive constant. This condition is fulfilled for instance when $V \geq \text{const} > 0$, or $U$ is bounded, so that $\lambda_V(U) > 0$. We shall drop the subscript $U, V$ in the sequel for convenience. We consider a (positive) measure $\mu$ of finite energy concentrated on a compact subset $K$ of $U$, and define:

$$\psi(x) = \int g(x, y)\, d\mu(y), \ x \in U . \tag{5.2}$$

**Proposition 5.1** *Assume (5.1), then:*

$$\psi \in \mathcal{D}(\mathcal{E}) \ \textit{and} \ \mathcal{E}(\psi, \psi) = \langle \mu, \mu \rangle . \tag{5.3}$$

**Proof:** For $\epsilon > 0$, we define the function

$$f_\epsilon(x) = \int r(\epsilon, x, y)\, \mu(dy), \ x \in U . \tag{5.4}$$

Then $f_\epsilon \in \mathcal{D}(\mathcal{E})$, and for $x \in U$,

(5.5)
$$\begin{aligned}\psi_\epsilon(x) &\stackrel{\text{def}}{=} \int_U g(x,y)\, f_\epsilon(y)dy = \int_\epsilon^\infty \int_U r(s,x,z)\,\mu(dz)ds \\ &\uparrow \psi(x), \text{ as } \epsilon \to 0\,.\end{aligned}$$

From the spectral theorem and (5.1) we have:

(5.6)
$$\begin{aligned}\mathcal{E}(\psi_\epsilon,\psi_\epsilon) &= \int_c^\infty \lambda\, d(E_\lambda\, \psi_\epsilon, \psi_\epsilon) = \int_c^\infty \frac{1}{\lambda}\, d(E_\lambda\, f_\epsilon, f_\epsilon) \\ &= (\psi_\epsilon, f_\epsilon)_{L^2} = \int\int f_\epsilon(x)\, g(x,y)\, f_\epsilon(y)dxdy \stackrel{(4.47)}{=} \\ &2\int_\epsilon^\infty ds \int_U \Big(\int_U r(s,x,y)\, d\mu(x)\Big)^2 dy \uparrow < \mu, \mu > < \infty\,.\end{aligned}$$

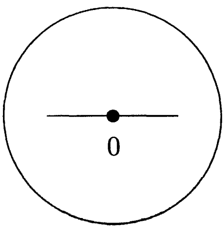
From this, (5.1) and (5.5), we see that $\mathcal{E}(\psi,\psi) + (\psi,\psi)_{L^2} < \infty$ so that $\psi \in \mathcal{D}(\mathcal{E})$.

Finally for $\epsilon, \epsilon' > 0$ we have by the above calculation:

$$\begin{aligned}&\mathcal{E}(\psi_\epsilon - \psi_{\epsilon'}, \psi_\epsilon - \psi_{\epsilon'}) = 2\int_\epsilon^\infty ds \int_U \Big(\int_U r(s,x,z)\, \mu(dx)\Big)^2 dz \\ &+ 2\int_{\epsilon'}^\infty ds \int_U \Big(\int_U r(s,x,z)\, \mu(dx)\Big)^2 dz \\ &- 4\int_0^\infty ds \int_U \Big(\int_U r(s+\epsilon,x,z)\, \mu(dx) \int_U r(s+\epsilon',y,z)\, \mu(dy)\Big) dz \\ &\longrightarrow 0, \text{ as } \epsilon,\epsilon' \to 0, \Big(\text{using } h(\epsilon + \cdot, \cdot) \xrightarrow[\epsilon\to 0]{L^2} h(\cdot,\cdot), \text{ for } h \in L^2(\mathbb{R}_+ \times U)\Big)\end{aligned}$$

This and (5.5) shows that $\psi_\epsilon \to \psi$, as $\epsilon \to 0$; together with (5.6) this finishes our proof of (5.3). □

**Example 5.2:**




Consider $U = B(0,2) \subseteq \mathbb{R}^2$, $K$ a line segment $\subseteq U$. The equilibrium potential

$v_K$ of $K$ relative to $U$, has the form

$$(5.7) \qquad v_K(x) = \int g(x,y)\, e_K(dy),\ x \in U\,,$$

where $e_K$ is the equilibrium measure of $K$ (relative to $U$). All points of $K$ are regular (see Exercise 1) at the end of last section). Therefore by Proposition 3.8, $v_K$ is continuous on $U$, harmonic on $U \backslash K$, equal to 1 on $K$. Moreover from (5.3), $v_K \in H_0^1(U)$. Observe that $v_K \in H^1(\mathbb{R}^2)$, $v_K = 0$, a.e. on the complement of $U \backslash K$, but $v_K \notin H_0^1\ (U \backslash K)$ (see the exercise preceding Theorem 4.11 in Chapter 1). □

We shall now discuss the characterization of the capacity in terms of the *Dirichlet principle.* We are now back in the 'general situation' and only assume (2.1), (2.2), (2.13).

**Theorem 5.3**: *If $K$ is a compact subset of $U$,*

$$(5.8) \quad cap(K) = \inf\{\mathcal{E}(\psi,\psi);\ \psi \in \mathcal{D}(\mathcal{E}),\ \psi \geq 1, \text{ on a neighborhood of } K\}\,.$$

**Proof:** We begin with the observation that it suffices to prove (5.8) in the case of a bounded connected $U \neq \emptyset$.

Indeed the l.h.s. of (5.8) $\text{cap}_{U,V}(K)$ is decreasingly approximated by $\text{cap}_{U_n,V}(K)$, for $U_n \supset K$, $n \geq 1$, an increasing sequence of bounded connected nonempty open sets, with $\cup U_n = U$, see (4.65).

As for the r.h.s., using the second Beurling-Deny criterion (1.4.15), we can assume that $\psi = 1$ on a neighborhood of $K$. Using a smooth function with support in this neighborhood and equal to 1 on a neighborhood of $K$, we can approximate $\psi$ in Dirichlet norm by functions in $C_c^\infty(U)$ equal to 1 on a neighborhood of $K$. In other words we can replace the conditions $\psi \in \mathcal{D}(\mathcal{E})$, and $\psi \geq 1$ on a neighborhood of $K$, by $\psi \in C_c^\infty(U)$, and $\psi = 1$ on a neighborhood of $K$.

It is now plain that the r.h.s. of (5.8) is the decreasing limit of the corresponding expressions for $U_n$ instead of $U$. As a result we now assume that $U$ is bounded.

If we approximate $K$ from above by decreasing compact neighborhoods $K_n \subset U$ of $K$, we have $\text{cap}(K_n) \downarrow \text{cap}(K)$, see (4.64). The equilibrium potential $v_{K_n}$ of $K_n$ relative to $U, V$ equals 1 on a neighborhood of $K$. If $e_{K_n}$ stands for the equilibrium measure, we have

$$\mathcal{E}(v_{K_n}, v_{K_n}) \overset{(5.3)}{=} < e_{K_n}, e_{K_n} >= \text{cap}(K_n) \downarrow \text{cap}(K)\,.$$

This shows we only need to prove that the r.h.s. of (5.8) is bigger than the l.h.s.. By the approximation argument used above it suffices to show that:

$$\text{(5.9)} \qquad \begin{array}{l} \mathrm{cap}(K) \le \mathcal{E}(\psi,\psi), \text{ for } \psi \in C_c^\infty(U), \text{ with } \psi = 1 \\ \text{on a neighborhood of } K\ . \end{array}$$

Now if $v$ and $e$ denote respectively the equilibrium potential and equilibrium measure of $K$, we have:

$$\begin{aligned} \mathcal{E}(\psi,\psi) &= \mathcal{E}(v,v) + 2\,\mathcal{E}(\psi - v, v) + \mathcal{E}(\psi - v, \psi - v) \\ &\overset{(5.3)}{\ge} \langle e, e\rangle + 2\mathcal{E}(\psi - v, v) \\ &\overset{(5.3)}{=} \mathrm{cap}(K) + 2\Big(\mathcal{E}(\psi, v) - \int_K v(x)\, e(dx)\Big)\ . \end{aligned}$$

Our claim (5.9) will follow from the identity

$$\text{(5.10)} \qquad \mathcal{E}(\psi, v) = \int_K \psi(x)\, e(dx)\ .$$

To prove (5.10) we simply approximate $v$ in Dirichlet norm, as in (5.4), (5.5) and use the spectral theorem. We then find:

$$\mathcal{E}(\psi, v_\epsilon) = \int_U \psi(x) \int_U r(\epsilon, x, y)\, e(dy) dx\ .$$

Letting $\epsilon \to 0$, this proves (5.10). □

In general it may very well happen that the equilibrium potential $v$ does not belong to $\mathcal{D}(\mathcal{E})$, (see the end of §1 of this chapter). However, even when $v \in \mathcal{D}(\mathcal{E})$ one drawback of the above theorem is that the class of functions which appears in the r.h.s. of (5.8) need not contain $v$, which is the natural minimizer.

The formulation used in Theorem 5.3 is due to the fact that the condition $\psi \ge 1$ on a neighborhood of $K$ is meaningful for an a.e. defined function $\psi \in \mathcal{D}(\mathcal{E})$, whereas the condition $\psi \ge 1$ on $K$ is problematic, since $K$ can be negligible (see Example 5.2). This can be resolved by showing that $\psi \in \mathcal{D}(\mathcal{E})$ implies some regularity on $\psi$ which enables to construct better versions of $\psi$, the so-called 'quasi continuous versions'. On this we refer the reader to Fukushima [Fuk80].

## 2.6 Notes and References

The present notes do not even attempt to give an account of the gigantic literature around potential theory. Calculations qualitatively similar to §1 can be found in Chapter IV of Courant-Hilbert [CH62], Vol. II. Chapter II of Dautray-Lions [DL90], Vol. 1 offers further material in this direction.

The books of Landkof [Lan72] and Tsuji [Tsu75] contain a wealth of information on nonprobabilistic potential theory. We also mention the recent review article by Bandle and Flucher [BF96] about harmonic radius. Classical probabilistic approaches to potential theory are developed in Chapter VI of Blumenthal and Getoor [BG68], and also in Port and Stone [PS78]. The sections 3 and 4 of this chapter are close to Chung [Chu73] and [Chu82]. The law of large number for the volume of the Wiener sausage, as in Example 3.7, goes back to Kesten-Spitzer-Whitman, see Spitzer [Spi64b], p. 40, more detailed asymptotics on the expected volume of the sausage appear in Spitzer [Spi64a]. Further developments on the subject as well as references can be found in the monograph [Gal92] of Le Gall. The Wiener test for regular points of a compact set, see Theorem 4.7, and the Kellogg-Evans theorem on the smallness of the set of irregular points of a compact set, cf. Proposition 4.5, underlie the method of enlargement of obstacles in Chapter 4. The Wiener test goes back to Wiener [Wie24], who introduced it in the context of the Dirichlet problem. Wiener also gives credit to G. Bouligand [Bou24] for independently obtaining conditions on regularity comparable in generality. The connection with Brownian motion appears in Kakutani [Kak44]. The Kellogg-Evans theorem goes back to Evans [Eva33], who proved the general case of a conjecture of Kellogg in [Kel29], p. 337. Section 5 provides some connections between the Dirichlet form point of view on capacity, as in Fukushima [Fuk80] or Fukushima-Oshima-Takeda [FOT94], and the point of view developed in Section 3 and 4.

# 3. Some Principal Eigenvalue Estimates

The main purpose of this chapter is to develop in the context of obstacles, a collection of methods and quantitative bounds, relating the Dirichlet-Schrödinger semigroups of Chapter 1 and the bottom of their spectrum. General methods and some first examples are discussed in Section 1. Section 2 presents two instances of the profound link between capacity and bottom of the spectrum. In Section 3 we develop certain one-dimensional bounds on principal Dirichlet eigenvalues of the Laplacian in the presence of obstacles. We apply these bounds to study the asymptotic behavior of a certain random variational problem for Poissonian obstacles. This is a caricature of the 'pinning effect' we shall discuss in Chapter 6.

## 3.1 General Results

We are now interested in the case where $V \geq 0$ (obstacle case). Our tacit assumption throughout this chapter is that:

(1.1) $$U \neq \emptyset \text{ is an open subset of } \mathbb{R}^d,\ d \geq 1, \text{ and } 0 \leq V \in K_d^{\mathrm{loc}}\,.$$

We know from Chapter 1 that the quadratic form of the self-adjoint contraction semigroup $R_t^{U,V}$ acting on $L^2(U, dx)$ is the closure of $f \in C_c^\infty(U) \to \int \frac{1}{2}\, |\nabla f|^2 + V f^2 dx$, see Theorem 1.4.11. Moreover, if $\lambda_V(U)$ denotes the bottom of the spectrum of the generator of this semigroup, it follows from (1.4.43), (1.4.44) and the spectral theorem that:

(1.2)
$$\begin{aligned}
\lambda_V(U) &= \inf\{\mathcal{E}_{U,V}(f,f),\ f \in \mathcal{D}(\mathcal{E}_{U,V}),\ \|f\|_{L^2} = 1\} \\
&= \inf\Big\{ \int_U \frac{1}{2}\, |\nabla f|^2 + V f^2\, dx,\ f \in C_c^\infty(U),\ \int f^2 dx = 1\Big\} \\
&= \inf\Big\{ \int_0^\infty \lambda d(E_\lambda f, f),\ f \in L^2(U,dx),\ \int f^2\, dx = 1\Big\} \\
&= -\frac{1}{t}\, \log \|R_t^{U,V}\|_{L^2 \to L^2},\ \text{ for } t > 0\,.
\end{aligned}$$

We shall sometimes with an abuse of terminology refer to $\lambda_V(U)$, as the principal Dirichlet eigenvalue of $-\frac{1}{2}\,\Delta + V$ in $U$. In the case $V = 0$, i.e. for Dirichlet semigroups, we shall simply write

$$\lambda(U) \stackrel{\text{def}}{=} \lambda_{V=0}(U) \,. \tag{1.3}$$

The second line of (1.2) can easily be used to derive upper bounds on $\lambda_V(U)$. It is often more difficult to obtain lower bounds on $\lambda_V(U)$. The principal object of this section is to discuss several techniques which provide ways to derive such lower estimates on $\lambda_V(U)$. We begin with the simple but useful

**Lemma 1.1:** *When $U \supseteq \widetilde{U}$ and $\widetilde{V} \geq V \geq 0$ satisfy (1.1)*

$$\lambda_V(U) \leq \lambda_{\widetilde{V}}(\widetilde{U}) \,. \tag{1.4}$$

*If $(U_i)_{i \in I}$ denote the collection of connected components of $U$, then:*

$$\lambda_V(U) = \inf_i \lambda_V(U_i) \,. \tag{1.5}$$

*For $\alpha > 0$ and $V_\alpha(\cdot) = \alpha^{-2}\, V(\frac{\cdot}{\alpha})$, we have*

$$\lambda_{V_\alpha}(\alpha U) = \alpha^{-2} \lambda_V(U) \,. \tag{1.6}$$

**Proof:** Observe that (1.4) is an immediate consequence of the second line of (1.2). The claim (1.5) in view of (1.4) will follow from

$$\lambda_V(U) \geq \inf_i \lambda_V(U_i) \,. \tag{1.7}$$

Now for $f \in C_c(U)$ and $f_i = f \cdot 1_{U_i} \in C_c^\infty(U_i)$, the second line of (1.2) implies that

$$\begin{aligned} \int \frac{1}{2}\, |\nabla f|^2 + V f^2 dx &= \sum_i \int \frac{1}{2}\, |\nabla f_i|^2 + V f_i^2 dx \\ &\geq \sum_i \lambda_V(U_i) \int f_i^2 dx \geq \inf_i \lambda_V(U_i) \int f^2 dx \,, \end{aligned}$$

which proves (1.7) and therefore (1.5).

Let us finally show (1.6). Note that the collection of $g \in C_c^\infty(\alpha U)$ with $\|g\|_{L^2} = 1$ coincides with the collection of $f_\alpha(\cdot) = \alpha^{-d/2} f(\frac{\cdot}{\alpha})$ with $f \in C_c^\infty(U)$ and $\|f\|_2 = 1$. Therefore the second line of (1.2) shows that

$$\lambda_{V_\alpha}(\alpha U) = \inf \Big\{ \int \frac{1}{2}\, |\nabla f_\alpha|^2 + V_\alpha\, f_\alpha^2 dx,\ f \in C_c^\infty(U),\ \|f\|_{L^2} = 1 \Big\} \,.$$

On the other hand:

$$\begin{aligned}\int \frac{1}{2}\,|\nabla f_\alpha|^2 + V_\alpha f_\alpha^2 dx &= \alpha^{-d}\int \frac{1}{\alpha^2}\Big(\frac{1}{2}\,|\nabla f|^2(\tfrac{x}{\alpha}) + V(\tfrac{x}{\alpha})\, f^2(\tfrac{x}{\alpha})\Big)dx \\ &= \alpha^{-2}\int \frac{1}{2}\,|\nabla f|^2 + V f^2 dx\,.\end{aligned}$$

Our claim (1.6) easily follows. □

It will be very useful for the sequel to relate $\lambda_V(U)$ to the decay properties, as $t \to \infty$, of the Dirichlet-Schrödinger semigroup, measured in terms of $\|R_t^{U,V}\|_{L^\infty\to L^\infty} = \sup_x R_t^{U,V} 1(x)$. This is precisely the object of

**Theorem 1.2:**

(1.8) *For* $t \ge 0$, $\exp\{-t\,\lambda_V(U)\} = \|R_t^{U,V}\|_{L^2\to L^2} \le \|R_t^{U,V}\|_{L^\infty\to L^\infty}$ .

*There exists a constant* $c(d) \in (1,\infty)$ *such that for any* $U, V$ *as in (1.1),* $t \ge 0$*:*

$$(1.9)\qquad \begin{aligned}\|R_t^{U,V}\|_{L^\infty\to L^\infty} &= \sup_x E_x\Big[T_U > t,\ \exp\Big\{-\int_0^t V(Z_s)ds\Big\}\Big] \\ &\le c(d)((\lambda_V(U)t)^{d/2}+1)\ \exp\{-\lambda_V(U)t\}\,.\end{aligned}$$

*When* $U$ *is connected, for each* $x \in U$*:*

$$(1.10)\qquad \lambda_V(U) = -\lim_{t\to\infty}\ \frac{1}{t}\ \log R_t^{U,V} 1(x)\,.$$

**Proof:**

*Proof of (1.8):* We only have to prove the rightmost inequality. Let us denote by $\mu_f$ the spectral probability measure of $f \in C_c^\infty(U)$ with $\|f\|_{L^2} = 1$. Then for $n \ge 0$, $t \ge 0$,

$$(1.11)\qquad \begin{aligned}\exp\{-nt\mathcal{E}(f,f)\} &= \exp\Big\{-nt\int_0^\infty \lambda\, d\mu_f(\lambda)\Big\} \overset{\text{(Jensen)}}{\le} \int_0^\infty e^{-nt\lambda} d\mu_f(\lambda) \\ &= (R_t^n f, f)_{L^2} \le \|f\|_1\ \|R_t\|^n_{L^\infty\to L^\infty} \|f\|_\infty\,.\end{aligned}$$

Taking the $n^{\text{th}}$ root and letting $n$ tend to $\infty$, we find:

$$(1.12)\qquad \exp\{-t\mathcal{E}(f,f)\} \le \|R_t\|_{L^\infty\to L^\infty}\ ,\ t \ge 0\,.$$

Taking the supremum of the l.h.s. over $f \in C_c^\infty(U)$ with $\|f\|_{L^2} = 1$, we find (1.8).

*Proof of (1.9):* We first assume $\lambda_V(U) \le 1$. Then for $x \in U$, $t \ge 1$ and $L > 0$:

$$(1.13)\quad \begin{aligned} R_t^{U,V} 1(x) &= R_t^{U,V}(1_{B(x,L)})(x) + R_t^{U,V}(1_{B(x,L)^c})(x) \\ &\overset{(1.3.7)-(1.3.10)}{=} (r_{U,V}(1,x,\cdot),\ R_{t-1}^{U,V}(1_{B(x,L)}))_{L^2} + R_t^{U,V}(1_{B(x,L)^c})(x) \end{aligned}$$

Therefore

$$\begin{aligned} \|R_t^{U,V}\|_{L^\infty \to L^\infty} &\le \exp\{-(t-1)\,\lambda_V(U)\}\|r_{U,V}(1,x,\cdot)\|_2\,\|1_{B(x,L)\cap U}\|_2 \\ &\quad + P_0[Z_t \notin B(0,L)] \\ &\le c_1(d)\,L^{d/2}\exp\{-(t-1)\,\lambda_V(U)\} + P_0\Big[Z_1 \notin B\Big(0, \frac{L}{\sqrt{t}}\Big)\Big] \end{aligned}$$

Now for a suitable $c_2(d) > 0$, we have

$$P_0[Z_1 \notin B(0,2a)] \le c_2(d)\ \exp\{-a^2\},\ \text{for } a \ge 0\,.$$

Picking $L = 2t$ and using the inequality $\exp\{-t\} \le \exp\{-\lambda_V(U)t\}$, we obtain for $t \ge 1$:

$$(1.14)\qquad \|R_t^{U,V}\|_{L^\infty \to L^\infty} \le c_3(d)(t^{d/2}+1)\ \exp\{-\lambda_V(U)t\}\,.$$

Using once more the fact that $\lambda_V(U) \le 1$, $c_3(d)$ can be increased if necessary so that (1.14) holds for all $t \ge 0$, and $U, V$ with $\lambda_V(U) \le 1$.

In the case of $U, V$ with $\lambda_V(U) \ne 0$, then by (1.6):

$$(1.15)\qquad \lambda_{V_\alpha}(\alpha U) = \alpha^{-2}\,\lambda_V(U) = 1,\ \text{ provided } \alpha = \lambda_V(U)^{1/2}\,.$$

Now using Brownian scaling, for $x \in U$ and $t \ge 0$:

$$(1.16)\qquad \begin{aligned} R_{\alpha^2 t}^{\alpha U, V_\alpha} 1(\alpha x) &= E_{\alpha x}\Big[T_{\alpha U} > \alpha^2 t,\ \exp\Big\{-\int_0^{\alpha^2 t} V_\alpha(Z_s)ds\Big\}\Big] \\ &= R_t^{U,V} 1(x)\,. \end{aligned}$$

Using (1.14) we find:

$$\begin{aligned} \|R_t^{U,V}\|_{L^\infty \to L^\infty} &\le c_3(d)((\alpha^2 t)^{d/2}+1)\ \exp\{-\alpha^2 t\} \\ &= c_3(d)((\lambda_V(U)t)^{d/2}+1)\ \exp\{-\lambda_V(U)t\}\,. \end{aligned}$$

This shows (1.9) when $\lambda_V(U) \ne 0$. The case where $\lambda_V(U) = 0$ is immediate since $R_t^{U,V}\,1 \le R_t^{\mathbb{R}^d, V=0}\,1 = 1$. This finishes the proof of (1.9).

*Proof of (1.10):* For any $\eta > 0$, we can find $f \in \mathcal{D}(\mathcal{E}_{U,V}) \cap C_c(U)$ with $\|f\|_2 = 1$ and

$$\lambda_V(U) \le \mathcal{E}_{U,V}(f,f) \le \lambda_V(U) + \eta\,.$$

Using (1.4.14) we can assume that $f \geq 0$. As in (1.12) we have:

$$\exp\{-t(\lambda_V(U)+\eta)\} \leq (f, R_t^{U,V} f) = \int_U f(x) \int_U r_{U,V}(t,x,y)\, f(y) dy\, dx\,.$$

If $x_0 \in U$ we know from Lemma 2.1 of Chapter 2 and (1.3.19) that $r(1, x_0, \cdot)$ is a strictly positive continuous function on $U$. From the above inequality we deduce that for $x_0 \in U$ and $t \geq 0$:

$$\begin{aligned} &\exp\{-t(\lambda_V(U)+\eta)\} \leq \|f\|_\infty / \{\inf_{\operatorname{supp} f} r(1, x_0, \cdot)\} \\ (1.17)\qquad &\int_U r(1,x_0,x) \int r(t,x,y)\, f(y) dy\, dx \leq \frac{\|f\|_\infty^2}{\inf_{\operatorname{supp} f} r(1,x_0 .\cdot)}\, R_{1+t}^{U,V}\, 1(x_0)\,. \end{aligned}$$

We then see that

$$\varliminf_{t\to\infty}\ t^{-1} \log\{R_t^{U,V} 1(x)\} \geq -\lambda_V(U) - \eta\,.$$

Our claim (1.10) follows by letting $\eta$ tend to 0. □

Beyond the obstacle case, when $V = V_+ - V_-$, with $V_+ \in K_d^{\text{loc}}$, $V_- \in K_d$, a less uniform bound than (1.9) is known to hold, see Simon [Sim82], and the Exercise 2) after Corollary 1.3. However there are examples where $\lambda_V(U) = 0$ and $\lim_t \|\mathbb{R}_t^{\mathbb{R}^d, V}\|_{\infty,\infty} = \infty$, see [Sim81], Section 3. Thus (1.9) breaks down in this case.

**Exercise:** Show that for $d \geq 1$, $M > 0$, $L > 0$, there is a constant $C(d, M, L) \in (1, \infty)$ such that when $\operatorname{diam}(U) \leq L$ and $0 \leq V \in K_d^{\text{loc}}$, then for $x \in \mathbb{R}^d$:

$$(1.18)\qquad R_t^{U,V} 1(x) \leq c(d,M,L)\ \exp\{-(\lambda_V(U) \wedge M)t\}\,.$$

(Hint: use (1.13)). □

Let us recall that in the notations of Chapter 2 §3, $Q_z$, for $z \in \mathbb{R}^d$, denotes the law of Brownian motion starting from $z$, killed at rate $V$ and when exiting from $U$, and $\zeta$ stands for its death time. Then for $\lambda \geq 0$, $x \in \mathbb{R}^d$, the $\lambda$-exponential moment under $Q_x$ of the death time can be expressed as:

$$\begin{aligned} E^{Q_x}[\exp\{\lambda\zeta\}] &= 1 + \int_0^\infty \lambda e^{\lambda u} Q_x[\zeta > u] du \\ (1.19)\qquad &= 1 + \int_0^\infty \lambda e^{\lambda u} E_x\Big[T_U > u,\ \exp\Big\{-\int_0^u V(Z_s) ds\Big\}\Big] du \end{aligned}$$

**Corollary 1.3** *There exists a constant $K(d) \in (1,\infty)$ such that for $\rho \in (0,1)$, and $U, V$ as in (1.1):*

$$\sup_{x\in\mathbb{R}^d} E^{Q_x}[\exp\{\lambda_V(U)(1-\rho)\zeta\}] \le \frac{K(d)}{\rho^{\frac{d}{2}+1}} \,. \tag{1.20}$$

**Proof:** In view of (1.9), (1.19), the l.h.s. of (1.20) is smaller than:

$$\begin{aligned} 1 \;&+\; \int_0^\infty c(d)\,\lambda_V(U)(1-\rho)(1+(\lambda_V(U)t)^{d/2})\,\exp\{\lambda_V(U)(1-\rho)t \\ &-\; \lambda_V(U)t\}dt \\ &=\; 1 + c(d)(1-\rho)\int_0^\infty (1+u^{d/2})\,\exp\{-\rho u\}du \\ &=\; 1 + c(d)(1-\rho)\Big[\frac{1}{\rho} + \frac{\Gamma(\frac{d}{2}+1)}{\rho^{\frac{d}{2}+1}}\Big]\,, \text{ if } \lambda_V(U) \neq 0\,, \\ &=\; 1, \text{ if } \lambda_V(U) = 0\,. \end{aligned}$$

Our claim follows. □

**Exercise:**

**1)** Give an example of a $U$ such that for $x \in U$:

$$E_x[\exp\{\lambda(U)\,T_U\}] < \infty\,. \tag{1.21}$$

**2)** Show that when $V = V_+ - V_-$, with $V_+ \in K_d^{\text{loc}}$, $V_- \in K_d$,

$$\begin{aligned} &\|R_t^{U,V}\|_{L^\infty\to L^\infty} \le c(d,V)(t^{d/2}+1)\,\exp\{-\lambda_V(U)t\}, \text{ for } t\ge 0, \text{ and} \\ &\sup_x \int_0^\infty \lambda e^{\lambda t}\, R_t^{U,V}\,1(x)\,dt < \infty\,, \text{ for } \lambda < \lambda_V(U)\,, \end{aligned}$$

here $c(d,V)$ is a suitable positive constant and $\lambda_V(U)$ is the bottom of the generator of $R_t^{U,V}$ acting on $L^2(U,dx)$, (see Proposition 3.3 of Chapter 1). □

We now provide two simple propositions which illustrate how lower bounds on $\lambda_V(U)$ can be derived by probabilistic arguments. These methods rely on the derivation of pointwise 'absorption' estimates rather than on the sole control of the quadratic form $\mathcal{E}_{U,V}$.

**Proposition 1.4:** *If $T > 0$ and $c \in (0,1)$ are such that for $x \in U$:*

$$E_x\Big[T_U > T,\, \exp\Big\{-\int_0^T V(Z_s)ds\Big\}\Big] \le 1-c\,, \tag{1.22}$$

*then $\lambda_V(U) \ge \frac{1}{T}\log\frac{1}{1-c} \ge \frac{c}{T}$.*

**Proof:** From (1.8) we find:

$$\exp\{-\lambda_V(U)T\} \leq \sup_x R_T^{U,V} 1(x) \leq 1 - c\,,$$

and our claim follows. $\square$

**Exercise:** Assume (1.22), denote by $b$ a bounded measurable $\mathbb{R}^d$-valued vector function, and by $Q_x$ the law on $C(\mathbb{R}_+, \mathbb{R}^d)$ of the weak solution of the stochastic differential equation:

$$\begin{cases} dZ_s &= d\beta_s + b(Z_s)\, ds \\ Z_0 &= x\,, \end{cases}$$

With $\beta_.$ a $d$-dimensional Brownian motion, (see Durrett [Dur84]). Show that when $||b||_\infty$ is small enough,

$$\sup_{x \in U} E^{Q_x}\Big[T_U > T,\ \exp\Big\{-\int_0^T V(Z_s)ds\Big\}\Big] < 1\,.$$

(Hint: use Girsanov's formula, as well as Hölder or Cauchy-Schwarz inequality). $\square$

Another argument uses the finiteness of exponential moments of the death time:

**Proposition 1.5:** *If $\mu \geq 0$ is such that:*

$$\textit{for } x \in U,\ E^{Q_x}[\exp\{\mu\zeta\}] < \infty,\ \textit{then}$$

(1.23) $$\mu \leq \lambda_V(U)\,.$$

**Proof:** Assume $\mu > \lambda_V(U)$, then by (1.5) $\mu > \lambda_V(U_i)$ for some connected component $U_i$ of $U$. On the other hand by (1.10) and (1.19), for $x \in U_i$:

$$\begin{aligned} E^{Q_x}[\exp\{\mu\zeta\}] &= 1 + \int_0^\infty dv\, \mu e^{\mu v}\, R_v^{U_i,V} 1(x) dv \\ &= \infty, \text{ a contradiction}\,. \end{aligned}$$

$\square$

**Exercise:** Show that when $V = V_+ - V_-$, with $V_+ \in K_d^{\mathrm{loc}}$, $V_- \in K_d$, the condition:

$$\int_0^\infty e^{\lambda t}\, R_t^{U,V}\, 1(x)\, dt < \infty, \text{ for all } x \in U,$$

implies that

$$\lambda \leq \lambda_V(U)\,,$$

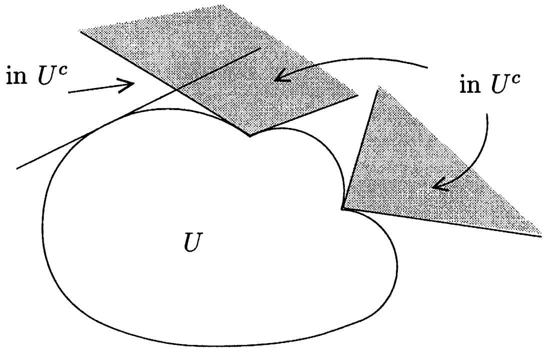
here the notations are as in the Exercise 2) after Corollary 1.3.

(Hint: use Exercise 2) at the end of Chapter 1 to extend (1.5), (1.10) to this context). □

**Example 1.6:** We consider $\mathbb{R}^d$, $d \geq 3$. A nonvoid connected open set $U$ is called a *uniformly $\Delta$-regular domain* if there exist $c > 0$ such that:

$$\text{for } x \in \partial U \text{ and } r > 0, \ \operatorname{cap}(U^c \cap B(x,r)) \geq c\, r^{d-2} \,, \tag{1.24}$$

($\operatorname{cap}(\cdot)$ denotes the capacity relative to $U = \mathbb{R}^d$ and $V = 0$).




an example of a uniformly $\Delta$-regular domain

**Proposition 1.7:** *There exists a positive constant $A(d,c)$ such that for any domain $U$ in $\mathbb{R}^d$, $d \geq 3$, satisfying (1.24):*

$$\frac{A(d,c)}{R^2} \leq \lambda(U) \leq \frac{\lambda_d}{R^2}\,, \quad \textit{where} \tag{1.25}$$

$$R = \sup\{r > 0, \ \exists\, x \in U, \ B(x,r) \subseteq U\}$$

*is the inner radius of $U$ and*

$$\lambda_d \overset{\text{def}}{=} \lambda(B(0,1)) \,.$$

**Proof:** With no loss of generality we assume $R < \infty$. If we use the scaling property of the capacity:

$$\operatorname{cap}(\alpha K) = \alpha^{d-2} \operatorname{cap}(K) \,,$$

$\alpha > 0, K$ compact subset of $\mathbb{R}^d$, we see that $\alpha U$ satisfies (1.24) with the same $c > 0$ as $U$. Moreover $\lambda(\alpha U) \overset{(1.6)}{=} \alpha^{-2} \lambda(U)$. Picking $\alpha = R^{-1}$, we see it suffices to prove (1.25) in the special case of $U$ with inner radius $R = 1$.

Now if $R = 1$, then for any $x_0 \in U$, $\bar{B}(x_0,1) \cap \partial U \neq \emptyset$. Pick $y_0 \in \bar{B}(x_0,1) \cap \partial U$, then by (1.24) $\operatorname{cap}(K) \geq c$, provided $K = \bar{B}(y_0,1) \cap U^c$. Letting $e(dz)$ stand for the equilibrium measure of $K$, we find:

$$\begin{aligned} P_{x_0}[T_U \le 1] &\ge P_{x_0}\Big[\frac{1}{2} < L_K \le 1\Big] \\ &\overset{(2.3.39)}{=} \int_{\frac{1}{2}}^{1} \int_K p(s, x_0, z)\, e(dz) ds \\ &\ge \int_{\frac{1}{2}}^{1} (2\pi)^{-d/2} \exp\Big\{ -\sup_K \frac{|x_0 - \cdot|^2}{2s} \Big\} \operatorname{cap}(K) ds \\ &\ge \frac{1}{2} (2\pi)^{-d/2} e^{-4} c\,. \end{aligned}$$

In view of Proposition 1.4, this gives the lower bound of (1.25). The upper bound readily follows from (1.2). □

We refer to Ancona [Anc86], for further results about uniformly $\Delta$-regular domains.

**Exercise:** Show that when $d = 2$, there exists an $A > 0$ such that for any Jordan domain $U$ of $\mathbb{R}^2$:

(1.26) $$\frac{A}{R^2} \le \lambda(U) \le \frac{\lambda_2}{R^2}\,,$$

provided $R$ is the inner radius of $U$. □

For the next result we consider a Brownian stopping time $S_1$, and introduce the sequence of iterates of $S_1$:

(1.27) $$0 = S_0 \le S_1 \le \infty, \text{ and by induction, for } k \ge 1:$$

$$S_{k+1} = S_1 \circ \theta_{S_k} + S_k \le \infty\,.$$

We thus have:

$$0 = S_0 \le S_1 \le S_2 \le \ldots \le S_k \le \ldots \le \infty\,.$$

The following proposition provides an abstract scheme for the derivation of lower bounds on principal Dirichlet eigenvalues.

**Proposition 1.8:** *If $U, V$ satisfy (1.1), $\lambda$ is a positive number, $S_1$ is a Brownian stopping time and*

(1.28) *for all $x \in \mathbb{R}^d$, $\lim_k \uparrow S_k \ge T_U$, $P_x$-a.s.,*

(1.29) $$\alpha \overset{\text{def}}{=} \sup_x E_x\Big[S_1 < T_U,\ \exp\Big\{\lambda S_1 - \int_0^{S_1} V(Z_s) ds\Big\}\Big] < 1\,,$$

(1.30) $$\beta \overset{\text{def}}{=} \sup_x \int_0^{\infty} \lambda e^{\lambda u}\, E_x\Big[S_1 \wedge T_U > u,\ \exp\Big\{-\int_0^u V(Z_s) ds\Big\}\Big] du < \infty\,,$$

then

$$\lambda \le \lambda_V(U) \ \ \textit{and} \ \ \sup_x \int_0^\infty \lambda e^{\lambda u} R_u^{U,V} 1(x) du \le \frac{\beta}{1-\alpha} \,. \tag{1.31}$$

**Proof:** In view of Proposition 1.5 and (1.19) it is enough to show that

$$\text{for } x \in \mathbb{R}^d, \ E_x\Big[\int_0^{T_U} \lambda \exp\Big\{\lambda u - \int_0^u V(Z_s)ds\Big\}du\Big] \le \frac{\beta}{1-\alpha} \,. \tag{1.32}$$

Observe that $\widetilde{S}_k \stackrel{\text{def}}{=} S_k \wedge T_U$ is the sequence of iterates of $\widetilde{S}_1 = S_1 \wedge T_U$. (i.e. $\widetilde{S}_{k+1} = \widetilde{S_k} + \widetilde{S}_1 \circ \theta_{\widetilde{S}_k}$), and (1.28), (1.29), (1.30) hold relative to $\widetilde{S}_k$, with the same $\alpha, \beta$. In other words we can assume in addition to (1.28) that $S_k \le T_U$, for all $k$.

Then the expression in (1.32) equals:

$$\begin{aligned}
&E_x\Big[\sum_{k\ge 0} 1\{S_k < T_U\} \int_{S_k}^{S_{k+1}} \lambda \exp\Big\{\lambda u - \int_0^u V(Z_s)ds\Big\}du\Big] \\
&= \sum_{k\ge 0} E_x\Big[S_k < T_U, \ \exp\Big\{\lambda S_k - \int_0^{S_k} V(Z_s)ds\Big\}\Big(\int_0^{S_1} \lambda \exp\Big\{\lambda u \\
&\qquad - \int_0^u V(Z_s)ds\Big\}du\Big) \circ \theta_{S_k}\Big] \\
&\overset{\binom{\text{strong}}{\text{Markov}}}{\le} \ \sum_{k\ge 0} E_x\Big[S_k < T_U, \ \exp\Big\{\lambda S_k - \int_0^{S_k} V(Z_s)ds\Big\}\Big]\,\beta \,.
\end{aligned}$$

If we now define for $k \ge 0$:

$$a_k = E_x\Big[S_k < T_U, \ \exp\Big\{\lambda S_k - \int_0^{S_k} V(Z_s)ds\Big\}\Big] \,,$$

then

$$\begin{aligned}
a_{k+1} &\overset{\binom{\text{strong}}{\text{Markov}}}{=} E_x\Big[S_k < T_U, \exp\Big\{\lambda S_k - \int_0^{S_k} V(Z_s)ds\Big\} \\
&\qquad E_{Z_{S_k}}\Big[S_1 < T_U, \ \exp\Big\{\lambda S_1 - \int_0^{S_1} V(Z_s)ds\Big\}\Big]\Big] \\
&\le \ \alpha\, a_k \,.
\end{aligned}$$

It follows by induction that $a_k \le \alpha^k$, and therefore

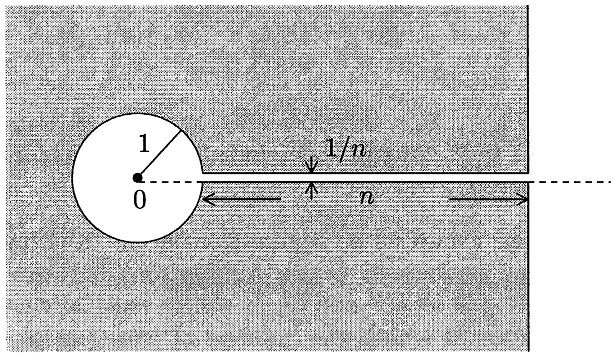
$$\sup_x E_x\Big[\int_0^{T_U} \lambda \exp\Big\{\lambda u - \int_0^u V(Z_s)ds\Big\}\Big] \le \beta \cdot \sum_{k=1}^{\infty} \alpha^k = \frac{\beta}{1-\alpha} < \infty \,.$$

This proves (1.32). □

**Example 1.9:** Consider $\mathbb{R}^d$, $d \geq 1$. It is an easy consequence of the second line of (1.2) that when $U_n$, $n \geq 1$, is an increasing sequence of nonempty open subsets of $\mathbb{R}^d$, and $U = \bigcup_{n \geq 1} U_n$, then

$$\lambda(U_n) \downarrow \lambda(U), \quad \text{as } n \to \infty . \tag{1.33}$$

On the other hand, when $U_n$, $n \geq 1$ is a decreasing sequence and $U = \bigcap_{n \geq 1} U_n$ is a nonempty open subset, $\lambda(U_n)$ is an increasing sequence (see (1.4)), however $\lambda(U_n)$ need not converge to $\lambda(U)$. For instance if $U_n \subseteq \mathbb{R}^2$ is the complement of the shaded region in the picture:




then $\bigcap_{n \geq 1} U_n = B(0,1)$, however $\lambda(U_n) = 0$ for all $n$, and $\lambda(U_n)$ does not converge to $\lambda(U) = \lambda(B(0,1))$. As we shall now see the trouble comes from the unbounded character of the opens sets $U_n$.

**Proposition 1.10:** *Assume that $U_n$, $n \geq 1$, is a decreasing sequence of open sets in $\mathbb{R}^d$, $d \geq 1$, $U_1$ is bounded, and $U = \bigcap_{n \geq 1} U_n$ is a nonvoid open set, then*

$$\lambda(U_n) \uparrow \lambda(U) \quad \textit{as } n \to \infty . \tag{1.34}$$

**Proof:** It suffices to show that for any $\rho \in (0,1)$,

$$\text{for large } n, \ \lambda(U_n) \geq \lambda \stackrel{\text{def}}{=} (1-\rho)\,\lambda(U) . \tag{1.35}$$

Pick a fixed $U_n$ which will play the role of $U$ in Proposition 1.8; $V = 0$ here. Now define

$$S_1 = 1 + T_U \circ \theta_1 .$$

Condition (1.28) (relative to $T_{U_n}$!) is clearly fulfilled. Moreover, we have

$$\begin{aligned} \beta \le \sup_x E_x[e^{\lambda S_1}] & \overset{\binom{\text{Markov}}{\text{property}}}{\le} e^{\lambda} \sup_x E_x[e^{\lambda T_U}] \\ & \overset{(1.20)}{\le} e^{\lambda}\, \frac{K(d)}{\rho^{\frac{d}{2}+1}} < \infty\,, \end{aligned}$$

and condition (1.30) is satisfied. Finally

$$\begin{aligned} \alpha = \ & \sup_x E_x\Big[\exp\{\lambda S_1\},\ S_1 < T_{U_n}\Big] \overset{\binom{\text{Markov}}{\text{property}}}{\le} \\ & e^{\lambda} \sup_x E_x[T_{U_n} > 1,\ E_{Z_1}[e^{\lambda T_U},\ T_U < T_{U_n}]] \\ & \le e^{\lambda} \int_{U_1} (2\pi)^{-d/2} E_z[e^{\lambda T_U},\ Z_{T_U} \in U_n] dz\,. \end{aligned}$$

Now $Z_{T_U} \in U^c = \bigcup_{m \ge 1} U_m^c$, therefore $\{Z_{T_U} \in U_n\} \downarrow \emptyset$. Using dominated convergence (thanks to (1.20) and $U_1$ bounded) we see that for large enough $n$, $\alpha < 1$. This proves (1.35). □

The coming theorem comes as an application of the 'abstract scheme' of Proposition 1.8, and enables to derive comparisons between $\lambda_V(U_1)$ and $\lambda_V(U_2)$, for suitable open sets $U_1$ and $U_2$. It will be especially useful in Chapter 4, when we discuss at length the 'method of enlargement of obstacles'. We first need some notations.

We consider two open subsets $U_1$, $U_2$ of $\mathbb{R}^d$, $d \ge 1$, $U_2 \ne \emptyset$, a Brownian stopping time $\tau \ge 0$, and a positive number $\lambda > 0$. We introduce the three quantities:

$$(1.36)\qquad \begin{aligned} A &= \sup_x 1 + \int_0^{\infty} \lambda e^{\lambda u} R_u^{U_1,V} 1(x)\, du\,, \\ B &= \sup_{x \notin U_1} \int_0^{\infty} \lambda e^{\lambda u} E_x\Big[\tau \wedge T_{U_2} > u,\ \exp\Big\{-\int_0^u V(Z_s)ds\Big\}\Big] du\,, \\ C &= \sup_{x \notin U_1} E_x\Big[\tau < T_{U_2},\ \exp\Big\{\lambda\tau - \int_0^{\tau} V(Z_s)ds\Big\}\Big]\,, \\ & \text{(with the convention that } A = 1 \text{ when } U_1 = \emptyset)\,. \end{aligned}$$

Typically $\lambda$ will be chosen 'close to $\lambda_V(U_1)$', and $\tau$ will be a stopping time such that when the process $Z_.$ starts outside $U_1$, it has a 'good chance' of either being killed by the soft potential $V$ or exiting $U_2$. Finally we introduce

$$(1.37)\qquad S_1 = \tau \circ \theta_{T_{U_1}} + T_{U_1}\,,$$

and denote by $S_k$, $k \ge 0$ the iterates of $S_1$ (see (1.27)).

**Theorem 1.11:** *Assume*

(1.38) $$\text{for } x \in \mathbb{R}^d,\ \lim_k \uparrow S_k \geq T_{U_2},\ P_x\text{-a.s.},$$

(1.39) $$A < \infty,\ B < \infty,\ \text{and}$$

(1.40) $$A \cdot C < 1\,,$$

*it then follows that*

(1.41) $$\lambda \leq \lambda_V(U_2)\,.$$

**Proof:** With the notations of (1.29) and (1.30) ($U_2$ playing the role of $U$), it suffices to prove

(1.42) $$\alpha < 1 \text{ and } \beta < \infty\,.$$

To this end note the identity:

(1.43) $$\begin{aligned} 1 &+ \int_0^t \lambda e^{\lambda u} \exp\Big\{-\int_0^u V(Z_s)ds\Big\}du \\ &= \int_0^t V(Z_u) \exp\Big\{\lambda u - \int_0^u V(Z_s)ds\Big\}du \\ &+ \exp\Big\{\lambda t - \int_0^t V(Z_s)ds\Big\}, \text{ for } t \geq 0\,. \end{aligned}$$

Then:

$$\begin{aligned} \beta &= \sup_x E_x\Big[\int_0^{S_1 \wedge T_{U_2}} \lambda \exp\Big\{\lambda u - \int_0^u V(Z_s)ds\Big\}du\Big] \\ &\overset{\binom{\text{strong Markov}}{\text{property}}}{\leq} \sup_x E_x\Big[\int_0^{T_{U_1} \wedge T_{U_2}} \lambda \exp\Big\{\lambda u - \int_0^u V(Z_s)ds\Big\}du\Big] \\ &\quad + \sup_x E_x\Big[T_{U_1} \wedge T_{U_2} < \infty, \\ &\quad \exp\Big\{\lambda T_{U_1} \wedge T_{U_2} - \int_0^{T_{U_1} \wedge T_{U_2}} V(Z_s)ds\Big\} \\ &\quad E_{Z_{T_{U_1} \wedge T_{U_2}}}\Big[\int_0^{\tau \wedge T_{U_2}} \lambda \exp\Big\{\lambda u - \int_0^u V(Z_s)ds\Big\}du\Big]\Big]\,. \end{aligned}$$

Observe that the inner expectation in the last term vanishes when $T_{U_2} < T_{U_1}$. Thus one only needs to consider the case $T_{U_1} \wedge T_{U_2} = T_{U_1} < \infty$ so that $Z_{T_{U_1} \wedge T_{U_2}} \notin U_1$. From this and (1.43) we have

$$\beta \leq A + AB \overset{(1.30)}{<} \infty\,.$$

On the other hand we have

$$
\begin{aligned}
\alpha &= \sup_x E_x\Big[S_1 < T_{U_2}, \exp\Big\{\lambda S_1 - \int_0^{S_1} V(Z_s)ds\Big\}\Big] \\
&\overset{\substack{(\text{strong Markov} \\ \text{property})}}{\le} \quad \sup_x E_x\Big[T_{U_1} < T_{U_2}, \exp\Big\{\lambda T_{U_1} - \int_0^{T_{U_1}} V(Z_s)ds\Big\}\Big] \\
&\qquad \cdot \sup_{x \notin U_1} E_x\Big[\tau < T_{U_2}, \exp\Big\{\lambda \tau - \int_0^{\tau} V(Z_s)ds\Big\}\Big] \\
&\overset{(1.43)}{\le} \quad A \cdot C \overset{(1.40)}{<} 1\,.
\end{aligned}
$$

This finishes the proof of (1.42). □

**Exercise:**

**1)** Consider $U \neq \emptyset$ a connected open subset from $\mathbb{R}^d$, $d \ge 1$, such that $T_U$ is $P_x$-a.s. finite for any $x \in \mathbb{R}^d$. Further assume that $V \ge 0$ belongs to $K^d_{\text{loc}}$ and that $\lambda > 0$. Define for $x \in \mathbb{R}^d$:

$$
g(x) = E_x\Big[\exp\Big\{\lambda T_U - \int_0^{T_U} V(Z_s)ds\Big\}\Big] \in [0,\infty]\,. \tag{1.44}
$$

**a)** Show that $g$ is bounded away from 0 on compact subsets of $U$.

(Hint: use Lemma 2.2 from Chapter 2).

**b)** Show that

$$
g \ge e^{\lambda s}\, R_s^{U,V} g\,, \quad \text{for } s \ge 0\,.
$$

**c)** Show that when $g(x_0) < \infty$, for some $x_0 \in U$, then

$$
\lambda \le \lambda_V(U)\,. \tag{1.45}
$$

(Hint: mimick the proof of (1.10)).

Note: the function $g$ is sometimes called a *gauge* function. For more on this see for instance Chung-Zhao [CZ95].

**2)** Consider $U \neq \emptyset$ an open subset of $\mathbb{R}^d$, $V = V_+ - V_-$ with $V_+ \in K_d^{\text{loc}}$, $V_- \in K_d$, and $f \in C^2(\overline{U})$ a strictly positive function. Define

$$
\lambda = \inf_U \frac{-\frac{1}{2}\,\Delta f + Vf}{f} \in \mathbb{R} \cup \{-\infty\}\,, \tag{1.46}
$$

and show that

$$
\lambda \le \lambda_V(U)\,.
$$

(Hint: when $\lambda \in \mathbb{R}$, apply Ito's formula to

$$\exp\{\lambda t \wedge T_U - \int_0^{t\wedge T_U} V(Z_s)ds\}\ f(Z_{t\wedge T_U}),$$

to show that $e^{\lambda t}\,R_t^{U,V} f \leq f$, when $t \geq 0$. Then proceed as in the exercise after Proposition 1.5.) □

We shall close this section with a lower bound, which will refine (1.10). We shall use it later in Chapter 4, 5 and 6.

In what follows, $U$ is a nonempty bounded domain of $\mathbb{R}^d$ and $V$ a nonnegative function in $K_d^{\text{loc}}$. Since $U$ is a bounded domain, $R_t^{U,V}$ defines a self-adjoint trace class operator on $L^2(U,dx)$, see Proposition 3.3 of Chapter 1. The Dirichlet form in (1.4.2) can in fact be written as:

$$\mathcal{E}_{U,V}(f,f) = \sum_{i\geq 1} \lambda_i (f,\varphi_i)^2_{L^2},\ f \in L^2(U,dx), \tag{1.47}$$

where $\lambda_i$, $i \geq 1$, denote the increasing sequence of nonnegative eigenvalues of the generator of the semigroup, counted with multiplicity, and $\varphi_i$, $i \geq 1$, are associated normalized eigenfunctions, forming an orthonormal basis of $L^2(U,dx)$.

In this situation $\lambda_1 = \lambda_V(U)$ is a simple eigenvalue. Indeed from the Beurling-Deny criterion (1.4.14), for any normalized eigenfunction $\varphi$ associated to $\lambda_V(U)$, $|\varphi|$ and therefore $\varphi_+ = \max(\varphi,0)$ and $\varphi_- = \max(-\varphi,0)$ are eigenfunctions as well. The functions $\varphi_+$ and $\varphi_-$ are clearly orthogonal in $L^2$, and the positivity of $r_{U,V}$ (Lemma 2.1 of Chapter 2) implies that either $\varphi_+$ or $\varphi_-$ vanishes. In other words $\varphi$ has a constant sign. Using once again the positivity of $r_{U,V}$ we see that $\varphi$ is unique up to multiplication by $\pm 1$. We assume that we choose $\varphi = \varphi_1$ positive. The eigenfunction $\varphi$ is also continuous bounded on $U$, in view of (1.3.19) and the boundedness of $r_{U,V}(1,\cdot,\cdot)$.

We also consider a possibly unbounded domain $U'$ containing the bounded domain $U$, and a closed subset $A$ of $U'$ intersecting $U$.

**Proposition 1.12**: *If $A, U, U'$ are as above, then for $t \geq 2$ and $x \in U'$,*

$$\begin{aligned} R_t^{U',V}\,1(x) \ \geq\ & \inf_{A\times A} r_{U',V}(2,\cdot,\cdot) \times \int_A \varphi^2(x)dx \times \exp\{-\lambda_V(U)\,t\} \\ & E_x\Big[\exp\Big\{-\int_0^{H_A} V(Z_s)ds\Big\},\ H_A < T_{U'}\Big]\,. \end{aligned} \tag{1.48}$$

**Proof**: The l.h.s. side of (1.48) is bigger than

$$
\begin{aligned}
&E_x\Big[\exp\Big\{-\int_0^{H_A+t} V(Z_s)ds\Big\},\ H_A+t<T_{U'}\Big] \overset{\text{(strong Markov property)}}{=} \\
&E_x\Big[\exp\Big\{-\int_0^{H_A} V(Z_s)ds\Big\},\ H_A<T_{U'},\ R_t^{U',V}\,1(Z_{H_A})\Big]\,,
\end{aligned}
\tag{1.49}
$$

moreover

$$
\begin{aligned}
&R_t^{U',V}\,1(Z_{H_A}) \ge \\
&E_{Z_{H_A}}\Big[T_{U'}>2, Z_2\in A,\ \exp\Big\{-\int_0^2 V(Z_s)ds\Big\}\ R_{t-2}^{U,V}\,1(Z_2)\Big]\,.
\end{aligned}
\tag{1.50}
$$

Observe that for any $z$

$$
\begin{aligned}
0\le\varphi(z) &= \exp\{\lambda_V(U)\}\int r_{U,V}(1,z,z')\,\varphi(z')dz' \\
&\le \exp\{\lambda_V(U)\}
\end{aligned}
\tag{1.51}
$$

using Cauchy-Schwarz inequality and $r_{U,V}(1,\cdot,\cdot)\le 1$, in the last step. Thus the r.h.s. of (1.50) is bigger than

$$
\begin{aligned}
&\exp\{-\lambda_V(U)\}\,E_{Z_{H_A}}\Big[T_{U'}>2, Z_2\in A, \\
&\qquad \exp\Big\{-\int_0^2 V(Z_s)ds\Big\}\ R_{t-2}^{U,V}\,\varphi(Z_2)\Big] \\
&= \exp\{-(t-1)\,\lambda_V(U)\}\ R_2^{U',V}(1_A\,\varphi)(Z_{H_A}) \\
&\ge \exp\{-(t-1)\,\lambda_V(U)\}\ \inf_{A\times A} r_{U',V}(2,\cdot,\cdot)\times\int_A\varphi\,dx \\
&\overset{(1.51)}{\ge} \exp\{-t\,\lambda_V(U)\}\ \inf_{A\times A} r_{U',V}(2,\cdot,\cdot)\int_A\varphi^2\,dx\,.
\end{aligned}
\tag{1.52}
$$

Inserting this lower bound on $R_t^{U',V}\,1(Z_{H_A})$ in (1.49), we find our claim (1.48). □

**Exercise:**

**1)** Show the existence of $c(d)>0$, such that for $t\ge 0$, $R>0$,

$$
P_0[T_{B(0,R)}>t]\ge c(d)\ \exp\Big\{-\frac{\lambda_d}{R^2}\,t\Big\}\,.
\tag{1.53}
$$

(Hint: use scaling to assume $R=1$ and (1.51)).

**2)** $U$ is a nonempty bounded domain of $\mathbb{R}^d$, $d \ge 1$, $V = V_+ \in K_d^{\text{loc}}$, $\varphi$ in $L^2(U, dx)$ is an eigenfunction of the semigroup $R_t^{U,V}$ attached to the eigenvalue $\lambda > 0$, (we implicitly choose a version of $\varphi$, which is continuous in $U$ and equals 0 on $U^c$, so that $\varphi(x) = e^{-\lambda s}\, R_s^{U,V}\varphi(x)$ for $s \ge 0$ and arbitrary $x$).

**a)** $U' \subseteq U$ is open and we assume $\lambda_V(U') > \lambda$. Show that for $y \in U'$,

$$\varphi(y) = E_y\Big[\varphi(Z_{T_{U'}})\,\exp\Big\{\int_0^{T_{U'}} (\lambda - V(Z_s))ds\Big\},\ Z_{T_{U'}} \in U \backslash U'\Big]\,. \tag{1.54}$$

(Hint: show that for $s > 0$, $y \in U'$:

$$\varphi(y) = e^{\lambda s}\, R_s^{U',V}\,\varphi(y) + E_y\Big[T_{U'} \le s,\ \exp\Big\{\int_0^{T_{U'}} (\lambda - V)(Z_s)ds\Big\}\,\varphi(Z_{T_{U'}})\Big]\,,$$

and let $s$ tend to $\infty$ with the help of (1.43) and (1.20) ).

**b)** Show that for a suitable $c(d) > 0$,

$$\|\varphi\|_\infty \le c(d)\,\lambda^{\frac{d}{4}}\,\|\varphi\|_2\,. \tag{1.55}$$

(Hint: proceed as in (1.51) and write for $z \in U$, $\varphi(z) = e^{\lambda s}(R_s^{U,V}\varphi)(z)$, with $s = \lambda^{-1}$. Then use Cauchy-Schwarz inequality). □

## 3.2 Capacity and Principal Eigenvalues

There is a profound interplay between capacity and principal eigenvalues. This is not too surprising in view of the Dirichlet principle of Chapter 2 §5 and the variational formula (1.2) for $\lambda_V(U)$. In fact we have already seen an instance of this connection in Example 1.6 of last section, for uniformly $\Delta$-regular domains. We shall further illustrate this connection in the present section. This theme will come again in Chapter 4, in the context of the method of enlargement of obstacles.

### A) An Inequality of Thirring and Variations

In what follows $U$ in a nonempty bounded domain of $\mathbb{R}^d$, $V \equiv 0$, and we consider the corresponding $\lambda(U)$ and $\varphi$ (see after (1.47)). We denote by $\mu(U)$ the second eigenvalue (i.e. the second Dirichlet eigenvalue of $-\frac{1}{2}\Delta$ in $U$, see (1.4.29)):

$$\mu(U) = \inf\Big\{\frac{1}{2}\int_U |\nabla f|^2 dx,\ \langle f, \varphi\rangle_{L^2} = 0,\ f \in H_0^1(U),\ \int_U f^2 dx = 1\Big\}\,. \tag{2.1}$$

We suppose we are also given a $V \in K_d^{\text{loc}}$, such that

$$V > 0, \text{ on } U . \tag{2.2}$$

We shall now give a lower bound on the shift $\lambda_V(U) - \lambda(U)$:

**Theorem 2.1:** *(Thirring's inequality)*

$$\lambda(U) + \int_U V\,\varphi^2 dx \geq \lambda_V(U) \geq \min\Big(\mu(U),\ \lambda(U) + \Big(\int_U \frac{\varphi^2}{V}\,dx\Big)^{-1}\Big) . \tag{2.3}$$

**Proof:** The left inequality is immediate. Indeed $\varphi \in \mathcal{D}(\mathcal{E}_{U,V})$ (see Theorem 4.9, 4.11 of Chapter 1) and

$$\mathcal{E}_{U,V}(\varphi,\varphi) = \frac{1}{2} \int_U |\nabla\varphi|^2 dx + \int_U V\varphi^2\, dx . \tag{2.4}$$

As for the right inequality, we assume $\int \frac{\varphi^2}{V}\, dx < \infty$ (otherwise it is trivial). Then for $f \in C_c^\infty(U)$, application of Cauchy-Schwarz inequality yields:

$$\int V f^2\, dx \int \frac{\varphi^2}{V}\, dx \geq \Big(\int f\,\varphi\, dx\Big)^2 . \tag{2.5}$$

Therefore,

$$\int_U \frac{1}{2}\, |\nabla f|^2 + V f^2\, dx \geq \int \frac{1}{2}\, |\nabla f|^2 dx + \Big[\int \frac{\varphi^2}{V}\, dx\Big]^{-1} (\varphi, f)^2_{L^2} .$$

If we now write $f = (\varphi, f)_{L^2}\, \varphi + \psi$, with $(\varphi,\psi)_{L^2} = 0 = \mathcal{E}_{U,V=0}\ (\varphi,\psi)$, see (2.1), we find that for $f \in C_c^\infty(U)$:

$$\begin{aligned}
&\int \frac{1}{2}\, |\nabla f|^2 + V f^2\, dx \\
&\geq \mathcal{E}_{U,V=0}(\psi,\psi) + \Big[\lambda(U) + \Big[\int \frac{\varphi^2}{V}\, dx\Big]^{-1}\Big] (\varphi, f)^2 \qquad (2.6)\\
&\geq \mu(U)(\|f\|^2_{L^2} - (\varphi, f)^2_{L^2}) + \Big[\lambda(U) + \Big[\int \frac{\varphi^2}{V}\, du\Big]^{-1}\Big] (\varphi, f)^2 .
\end{aligned}$$

Our claim now follows. $\square$

**Example 2.2:** We shall now apply Thirring's inequality to the case of a potential $V = \infty \cdot 1_K$, $K$ a compact subset of $U$. Of course this potential does not fulfill the requirements of Theorem 2.1, and we apply Theorem 2.1 with $V = \alpha + \beta\, 1_K$, where $\alpha, \beta > 0$. We find:

$$\lambda(U\backslash K) \geq \lambda_V(U) - \alpha \overset{(2.3)}{\geq} \min\Big(\mu(U), \lambda(U) + \Big(\int \frac{\varphi^2}{V}\, dx\Big)^{-1}\Big) - \alpha .$$

Letting $\beta \uparrow \infty$, we obtain:

$$\lambda(U\backslash K) \geq \min\Big(\mu(U), \lambda(U) + \alpha\Big[\int_{U\backslash K} \varphi^2 dx\Big]^{-1}\Big) - \alpha\,.$$

Optimizing over $\alpha$, we choose

$$\alpha = (\mu(U) - \lambda(U)) \int_{U\backslash K} \varphi^2\, dx\,,$$

and obtain the following lowerbound on $\lambda(U\backslash K)$:

$$\lambda(U\backslash K) \geq \lambda(U) \int_{U\backslash K} \varphi^2 dx + \mu(U) \int_K \varphi^2 dx\,. \tag{2.7}$$

One might be tempted to deduce from this inequality that the 'shift' $\lambda(U\backslash K) - \lambda(U)$ is governed by the volume $|K|$. As we shall now discuss, this does not capture the right order of magnitude, and indicates the limitations of the method employed. □

We shall now present a variation of Thirring's inequality. It provides a lower bound on the shift $\lambda(U\backslash K) - \lambda(U)$, which underlines the role of capacity.

**Theorem 2.3:** *If $U$ is a nonempty bounded domain of $\mathbb{R}^d$, and $K$ a compact subset of $U$, then*

$$\lambda(U\backslash K) \geq \lambda(U) + \frac{\mu(U) - \lambda(U)}{\mu(U)} \cdot c, \quad \text{where } c = \inf_K \varphi^2 \cdot \mathrm{cap}_U(K)\,, \tag{2.8}$$

*(a reinforcement of this inequality is given in (2.10) below).*

**Proof:** Approximating $K$ from above, we can restrict to the case where $K$ is a smooth compact set, (see (1.33) and after (2.4.64)). Denote by $Q$ and $Q_\perp$ the orthogonal projections relative to $\mathcal{E}(f,g) = \frac{1}{2}\int \nabla f\, \nabla g\, dx$, on the closed subspace $H_0^1(U\backslash K)$ of $H_0^1(U)$ and its orthogonal. Then for $\alpha > 0$, $\beta > 0$,

$$\begin{aligned}
&(1+\alpha)\,\lambda(U\backslash K) \\
&= \inf_{f\in H_0^1(U\backslash K), \|f\|_2=1} \{\mathcal{E}(f,f) + \alpha\mathcal{E}(f,Qf) + (\alpha+\beta)\,\mathcal{E}(f,Q_\perp f)\} \\
&\geq \inf_{f\in H_0^1(U), \|f\|_2=1} \{\mathcal{E}(f,f) + \alpha\mathcal{E}(f,Qf) + (\alpha+\beta)\,\mathcal{E}(f,Q_\perp f)\}\,.
\end{aligned} \tag{2.9}$$

Observe that for $f \in H_0^1(U)$:

$$\begin{aligned}
\mathcal{E}(\varphi,f)^2 &= \mathcal{E}((\alpha^{-\frac{1}{2}}\, Q + (\alpha+\beta)^{-\frac{1}{2}}\, Q_\perp)\varphi, (\alpha^{\frac{1}{2}} Q + (a+\beta)^{\frac{1}{2}}\, Q_\perp) f)^2 \\
&\overset{(\text{Cauchy-Schwarz})}{\leq} \mathcal{E}((\alpha^{-1}\, Q + (\alpha+\beta)^{-1}\, Q_\perp)\varphi, \varphi) \\
&\qquad \cdot \mathcal{E}((\alpha Q + (\alpha+\beta)\, Q_\perp) f, f)\,.
\end{aligned}$$

Inserting in (2.9), we find:

$$\begin{aligned}
&(1+\alpha)\lambda(U\backslash K) \\
&\geq \inf_{f \in H_0^1(U), \|f\|_2=1} \{\mathcal{E}(f,f) + \mathcal{E}(\alpha^{-1}\, Q + (\alpha+\beta)^{-1}\, Q_\perp\, \varphi, \varphi)^{-1} \mathcal{E}(\varphi, f)^2\} \\
&\overset{(2.1)}{=} \min\{\mu(U), \lambda(U) + \mathcal{E}((\alpha^{-1}\, Q + (\alpha+\beta)^{-1}\, Q_\perp)\varphi, \varphi)^{-1} \lambda(U)^2\}\,.
\end{aligned}$$

Letting $\beta$ tend to infinity, we find:

$$(1+\alpha)\,\lambda(U\backslash K) \geq \min\left\{\mu(U), \lambda(U) + \alpha\, \frac{\lambda(U)^2}{\mathcal{E}(Q\varphi,\varphi)}\right\}.$$

Using $\mathcal{E}(Q\varphi,\varphi) = \lambda(U) - \kappa$, where $\kappa = \mathcal{E}(Q_\perp\,\varphi, Q_\perp\,\varphi)$, and optimizing over $\alpha$, we find

$$\begin{aligned}
\lambda(U\backslash K) \;&\geq\; \frac{\mu(U)\,\lambda(U)}{\lambda(U) + (\mu(U)-\lambda(U))\, \dfrac{(\lambda(U)-\kappa)}{\lambda(U)}} \\
&= \frac{\lambda(U)}{1-\kappa\, \dfrac{(\mu(U)-\lambda(U))}{\lambda(U)\,\mu(U)}} \geq \lambda(U) + \frac{\mu(U)-\lambda(U)}{\mu(U)}\;\kappa\,. 
\end{aligned} \tag{2.10}$$

The claim (2.8) thus follows from the lemma below, (a quicker proof can be given when knowing a bit more about Dirichlet spaces, cf. [Szn91b], p. 1168).

**Lemma 2.4:**

$$\kappa \geq c\,. \tag{2.11}$$

**Proof:** Integrating over $s$ the identity:

$$\exp\{-\lambda(U)\,s\}\,\varphi(x) = \int r(s,x,y)\,\varphi(y)dy,\; x \in U\,,$$

we find for $x \in U$:

$$\varphi(x) = \int_U g(x,y)\,\lambda(U)\varphi(y)dy\,. \tag{2.12}$$

Using (2.2.11) (with the role of $x$ and $y$ exchanged), together with the fact that all points of $K$ are regular we deduce that:

$$\varphi(x) = \psi(x), \text{ when } x \in K\,, \tag{2.13}$$

provided we define for $x \in U$

$$\psi(x) = \int g(x,y)\,\mu(dy), \quad \text{and} \tag{2.14}$$

$$\mu(dz) = \int_U \lambda(U)\varphi(y)\, E_y[H_K < T_U, Z_{H_K} \in dz]dy\,. \tag{2.15}$$

Observe that $\mu$ is supported on $K$ and that $\psi$ is uniformly bounded by the maximum principle (2.4.5) and (2.13). Therefore $\mu$ has finite energy and from (2.5.3), $\psi \in H_0^1(U)$. Moreover as in (2.5.10), we have

$$\mathcal{E}(f,\psi) = \int_K f\,d\mu = 0\,,$$

for any $f \in C_c^\infty(U\backslash K)$, so that $\psi \perp H_0^1(U\backslash K)$. On the other hand $\varphi - \psi = 0$ on $K$, and since $K$ is a smooth compact set, $\varphi - \psi \in H_0^1(U\backslash K)$, using at this point an argument similar to the one used in the exercise before Theorem 4.11 of Chapter 1. We have thus shown:

$$\psi = Q_\perp \varphi\,, \tag{2.16}$$

$\psi$ is usually called the reduced function of $\varphi$ over $K$.

Finally observe that $\psi \geq \inf_K \varphi$ on $K$, so that denoting by $v$ the equilibrium potential and by $e$ the equilibrium measure of $K$:

$$\mu(K) = \int v\,d\mu = \int\int g(x,y)\,\mu(dx)\,e(dy) = \int \psi\,de \geq \inf_K \varphi\,\mathrm{cap}_U(K)\,.$$

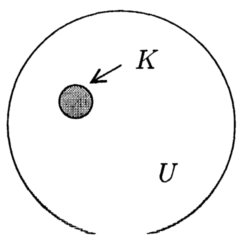
Our claim now follows from:

$$\mathcal{E}(Q_\perp\varphi, Q_\perp\varphi) \overset{(2.5.3)}{=} \langle\mu,\mu\rangle = \int \psi\,d\mu \geq \inf_K \varphi^2\,\mathrm{cap}_U(K)\,. \tag{2.17}$$

□

**Example 2.5:** (deletion of a ball of small radius)

$d \geq 3$, $U = B(0,1)$, $K = \bar{B}(x,\epsilon)$, where $x$ is a point of $U$ and $\epsilon$ is small so that $K \subset U$. We also write $U_\epsilon = U\backslash K$.



Keeping the notations of Lemma 2.4, $\varphi - \psi \in H_0^1(U\backslash K)$ and using (2.17):

$$
\begin{aligned}
&\mathcal{E}(\varphi-\psi,\varphi-\psi) = \mathcal{E}(\varphi,\varphi) - \mathcal{E}(\psi,\psi) = \lambda(U) - \int \psi\, d\mu \\
&\le \lambda(U) - \inf_K \varphi^2 \operatorname{cap}_U(K) \le \lambda(U) - \inf_{\bar{B}(x,\epsilon)} \varphi^2 \cdot \operatorname{cap}(\bar{B}(0,1)) \cdot \epsilon^{d-2}\,,
\end{aligned} \tag{2.18}
$$

since $\operatorname{cap}(K) \le \operatorname{cap}_U(K)$.

On the other hand:

$$
\begin{aligned}
\|\varphi-\psi\|_2^2 &\ge 1 - 2\int \varphi\psi\, dx \overset{(2.14)}{=} 1 - \frac{2}{\lambda(U)} \int\int g(x,y)\, \lambda(U)\varphi(x) dx d\mu(y) \\
&\overset{(2.12)}{=} 1 - \frac{2}{\lambda(U)} \int \varphi\, d\mu\,.
\end{aligned}
$$

Since $\mu$ is supported on $K$ and $\psi = \int g(\cdot,y)\, d\mu(y) \le \sup_K \varphi$, it follows from (2.4.10) that $\mu(K) \le \sup_K \varphi \operatorname{cap}_U(K)$. We thus find:

$$
\|\varphi-\psi\|_2^2 \ge 1 - \frac{2}{\lambda(U)} (\sup_K \varphi)^2 \operatorname{cap}_U(K)\,. \tag{2.19}
$$

Now as $\epsilon \to 0$, $\operatorname{cap}_U(K) \sim \operatorname{cap}(K) = \operatorname{cap}(\bar{B}(0,1))\, \epsilon^{d-2}$ (see (2.4.43)).

Therefore we see from (2.18), (2.19), that

$$
\overline{\lim_{\epsilon\to 0}}\ \epsilon^{-(d-2)}(\lambda(U_\epsilon) - \lambda(U)) \le \varphi^2(x) \cdot \operatorname{cap}(\bar{B}(0,1))\,. \tag{2.20}
$$

Furthermore (2.8) implies that

$$
\underline{\lim_{\epsilon\to 0}}\ \epsilon^{-(d-2)}(\lambda(U_\epsilon) - \lambda(U)) \ge \frac{\mu(U) - \lambda(U)}{\mu(U)} \cdot \varphi^2(x) \operatorname{cap}(\bar{B}(0,1))
$$

and we see that in this case of a small obstacle, (2.8) in contrast to (2.7), captures the right order of magnitude of the shift of the principle eigenvalue. In fact more can be found in the following

**Exercise:**

**1)** Generalize Theorem 2.3 to the situation of $U$ a nonempty bounded domain in $\mathbb{R}^d$, $V \ge 0$ in $K_d$, and $K$ a compact subset of $U$, to derive a lower bound on the shift $\lambda_V(U\backslash K) - \lambda_V(U)$.

(Hint: repeat the proof of Theorem 2.3, note by the way that $\mathcal{D}(\mathcal{E}_{U,V}) = H_0^1(U)$, $\mathcal{D}(\mathcal{E}_{U\backslash K,V}) = H_0^1(U\backslash K)$, and $\mathcal{E}_{U,V}$ is equivalent to $\mathcal{E}$, (cf. (1.4.45), (2.5.1)).

**2)** In the case of Example (2.5), denote by $\varphi_\epsilon$ the nonnegative normalized eigenfunction of $U_\epsilon$.

**a)** Show that for small $\epsilon$,

$$\|\varphi - \varphi_\epsilon\|_2^2 \le \text{const}(d)\, \epsilon^{d-2} \tag{2.21}$$

**b)** Show the identity:

$$(\lambda(U_\epsilon) - \lambda(U)) \int \varphi\varphi_\epsilon\, dx = \lambda(U_\epsilon) \int \psi\, \varphi_\epsilon\, dx\, .$$

(Hint: show $\lambda(U) \int \varphi\varphi_\epsilon\, dx = \mathcal{E}(\varphi, \varphi_\epsilon)$ and $\lambda(U_\epsilon) \int (\varphi - \psi)\, \varphi_\epsilon\, dx = \mathcal{E}(\varphi - \psi, \varphi_\epsilon)$).

**c)** Show that:

$$\int \psi(\varphi - \varphi_\epsilon) dx = o(\epsilon^{d-2}), \quad \text{as } \epsilon \to 0\, .$$

(Hint: break the integration on $\bar{B}(x, \epsilon^\alpha)$ and its complement, for $0 < \alpha < \frac{1}{2}$). Deduce from (2.21) and (2.20) that:

$$\lim_{\epsilon \to 0} \epsilon^{-(d-2)}(\lambda(U_\epsilon) - \lambda(U)) = \varphi^2(x)\, \text{cap}(\bar{B}(0,1))\, . \tag{2.22}$$

(For more on this see Flucher [Flu95] and references therein, as well as Swanson [Swa63]). □

We shall now give an example of application of (2.8) which illustrates the interest of lowerbounds on the shift which are uniform in the 'obstacle' $K$.

**Example 2.6:** We consider $\mathbb{R}^d$, $d \ge 3$, $U = B(0,1)$, and the Wiener sausage of small radius:

$$W_t^{\epsilon(t)} = \bigcup_{0 \le s \le t} \bar{B}(Z_s, \epsilon(t)), \quad \text{with } \epsilon(t) = t^{-\alpha},\ \alpha > 0\, . \tag{2.23}$$

**Proposition 2.7:** *Assume that $\alpha \in (0, \frac{1}{d-2})$, then for $\rho \in (0,1)$ and $y \in U$:*

$$P_y[W_t^{\epsilon(t)} \supset \bar{B}(0,\rho) \,|\, T_{B(0,1)} > t] \underset{t \to \infty}{\longrightarrow} 1\, . \tag{2.24}$$

*('Given that Brownian motion stays a long time in $U$, it comes close to each point of $U$').*

**Proof:** For large $t$, we can find a collection of $N(t) \le \text{const}(d)\, \epsilon(t)^{-d}$ closed balls $\bar{B}(x_i, \frac{\epsilon(t)}{2})$ with centers $x_i \in B(0, \frac{1+\rho}{2})$, which cover $\bar{B}(0,\rho)$. Then, for $y \in B(0,1)$:

$$P_y[W_t^{\epsilon(t)} \not\supset \bar{B}(0,\rho) \,|\, T_{B(0,1)} > t] \leq$$
$$P_y\Big[\bigcup_{i=1}^{N(t)} \{H_{\bar{B}(x_i,\frac{\epsilon(t)}{2})} > t\} \,|\, T_{B(0,1)} > t\Big] \leq$$
$$N(t) \sup_{x_i} P_y[H_{\bar{B}(x_i,\frac{\epsilon(t)}{2})} > t \,|\, T_{B(0,1)} > t] =$$
$$N(t) \sup_{x_i} P_y[T_{B(0,1)\backslash \bar{B}(x_i,\frac{\epsilon(t)}{2})} > t]/P_y[T_{B(0,1)} > t] \,.$$

Observe that:

$$\begin{aligned} P_y[T_{B(0,1)} > t] &\geq \int_{\bar{B}(0,\frac{1}{2})} r(1,y,z) \int_{\bar{B}(0,\frac{1}{2})} r(t-1,z,z')dzdz' \\ &\geq \inf_{\bar{B}(0,\frac{1}{2})} r(1,y,\cdot) \exp\{-\lambda(B(0,1))(t-1)\} \, (\varphi, 1_{\bar{B}(0,\frac{1}{2})})^2 \,. \end{aligned} \tag{2.25}$$

Moreover, when $t$ is large $\sup_{x_i} \lambda(B(0,1)\backslash \bar{B}(x_i, \frac{\epsilon(t)}{2})) \leq c(d)$, so that using (1.18) and the above inequalities we find

$$P_y[W_t^{\epsilon(t)} \not\supset \bar{B}(0,\rho) \,|\, T_{B(0,1)} > t] \leq \text{const}(d,y)\, \epsilon(t)^{-d} \cdot$$
$$\exp\Big\{ -\inf_{x_i} \Big[\lambda\Big(B(0,1)\backslash \bar{B}\Big(x_i, \frac{\epsilon(t)}{2}\Big)\Big) - \lambda(B(0,1))\Big]\, t\Big\}$$
$$\overset{(2.8)}{\leq} \text{const}(d,y)\, \epsilon(t)^{-d} \exp\{-c(d,\rho)\, \epsilon(t)^{d-2}\, t\} \to 0 \,.$$

This proves our claim. □

The condition $\alpha < \frac{1}{d-2}$, in Proposition 2.7 is important, as is shown in the following

**Exercise:** Show that when $x \neq 0$, and $\alpha > \frac{1}{d-2}$,

$$\varliminf_{t\to\infty} P_0[H_{\bar{B}(x,\epsilon(t))} > t \,|\, T_{B(0,1)} > t] > 0 \,. \tag{2.26}$$

(Hint: use the analogue of (2.25) for $B(0,1)\backslash B(x,\epsilon(t))$, together with (2.20), (2.21)). □

**Example 2.8:** (intersection of Wiener sausages)

We consider two independent Brownian motions $Z^1_\cdot$, $Z^2_\cdot$ in $\mathbb{R}^d$, $d \geq 5$. We denote by $W^1$ and $W^2$ the Wiener sausages of radius 1 generated by the full paths $Z^1_\cdot$ and $Z^2_\cdot$:

$$W^i = \bigcup_{s\geq 0} \bar{B}(Z^i_s, 1),\ i = 1,2, \,. \tag{2.27}$$

As a consequence of the high dimension ($d \geq 5$), $W^1 \cap W^2$ has finite volume. Indeed:

$$E_0^1 \otimes E_0^2[|W^1 \cap W^2|] = \int_{\mathbb{R}^d} dx\, P_0^1[x \in W^1]\, P_0^2[x \in W^2]$$

$$= \int_{\mathbb{R}^d} dx\, P_0[H_{\bar{B}(x,1)} < \infty]^2 \overset{(2.3.52)}{=} \int_{\mathbb{R}^d} \widetilde{g}(0,x)^2\, dx < \infty\,,$$

where $\widetilde{g}(z,z') = 2^{d-2}\{(\frac{1}{|z-z'|^{d-2}}) \wedge 1\}$.

We also have:

$$E_0^1 \otimes E_0^2[|W^1 \cap W^2|^2] = \int dxdy\, P_0^1[x \in W^1, y \in W^1]\, P_0^2[x \in W^2, y \in W^2]$$

$$= \int dxdy (P_0[H_{\bar{B}(x,1)} < H_{\bar{B}(y,1)} < \infty] + P_0[H_{\bar{B}(y,1)} < H_{\bar{B}(x,1)} < \infty])^2$$

$$\leq \int dxdy (\widetilde{g}(0,x)\, \widetilde{g}(x,y) + \widetilde{g}(0,y)\, \widetilde{g}(y,x))^2$$

$$\leq 2 \int dxdy (\widetilde{g}(0,x)^2\, \widetilde{g}(x,y)^2 + \widetilde{g}(0,y)^2\, \widetilde{g}(y,x)^2) < \infty\,.$$

In the same fashion, one can see that $|W^1 \cap W^2|$ has finite moments of any order. On the other hand, as we now see it has no finite exponential moment:

**Proposition 2.9:** *For $\eta > 0$,*

(2.28) $$\varliminf_{a \to \infty} a^{-(\frac{d-2}{d}+\eta)} \log P_0^1 \otimes P_0^2[|W^1 \cap W^2| \geq a] > -\infty\,.$$

**Proof:** Define $r$ and $t$ via:

$$a = |B\Big(0, \frac{1}{2}\Big)|\, r^d, \quad t = a^{1+\eta}\,,$$

we then have:

$$P_0^1 \otimes P_0^2[|W^1 \cap W^2| \geq a] \geq P_0\Big[W_t^1 \supset \bar{B}\Big(0, \frac{r}{2}\Big),\ T_{B(0,r)} > t\Big]^2$$

(2.29) (with the notation of (2.3.57))

$$\overset{(\text{scaling})}{=} P_0\Big[W_{t/r^2}^{1/r} \supset \bar{B}\Big(0, \frac{1}{2}\Big),\ T_{B(0,1)} > \frac{t}{r^2}\Big]^2\,.$$

Observe that:

$$\frac{1}{r} = \text{const}\; a^{-1/d} = \text{const}\; t^{-1/d(1+\eta)}, \quad \text{so that}$$

$$\frac{1}{r} = \text{const}\, \Big(\frac{t}{r^2}\Big)^{-\frac{1}{d(1+\eta)-2}}\,.$$

From (2.24), we find that for large $a$,

$$P_0^1 \otimes P_0^2[|W^1 \cap W^2| \geq a] \geq \text{const } P_0\Big[T_{B(0,1)} > \frac{t}{r^2}\Big]^2 \overset{(2.25)}{\geq}$$
$$c_1 \exp\Big\{ - c_2 \,\frac{t}{r^2}\Big\} = c_1 \exp\{-c_3\, a^{1+\eta-2/d}\}\,,$$

which proves our claim (2.28). □

## B) The Constant Capacity Regime

We shall now discuss the case of a periodic configuration of small holes in $\mathbb{R}^d$, $d \geq 3$, such that the capacity of each hole is comparable to the volume of the fundamental domain of the periodic configuration. The holes are assumed spherical for simplicity, however our arguments naturally cover the case of a 'general shape', (see Exercise 3) at the end of the section). The proof of Theorem 2.10 below further illustrates the use of probabilistic arguments for the purpose of deriving eigenvalue estimates. In what follows, we consider

(2.31) $$U_\epsilon = \mathbb{R}^d \setminus \bigcup_{q \in \mathbb{Z}^d} \bar{B}(2\epsilon q, r), \quad \text{where } \epsilon > 0 \text{ and}$$

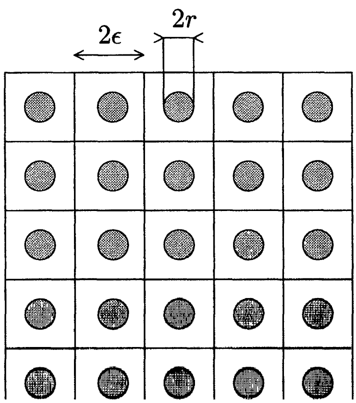
(2.32) $$r(\epsilon) = a\,\epsilon^{\frac{d}{d-2}} \quad (<< \epsilon \text{ when } \epsilon \text{ is small})\,.$$

Indeed the capacity of 'each hole' $\bar{B}(2\epsilon q, r)$ is

(2.33) $$\text{cap}(\bar{B}(0,1))\, r^{d-2} \overset{(2.3.55)}{=} 2\, \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}-1)}\, a^{d-2}\, \epsilon^d\,,$$

whereas the volume of the fundamental cell is: $2^d\,\epsilon^d$.

Thus pretending that capacity is additive, 'each unit volume of the material' contains a roughly 'constant capacity of holes'. This accounts for the terminology of the constant capacity regime.



We denote by $v_q(\cdot)$ and $e_q(dy)$ the respective equilibrium potential and equilibrium measure of $\overline{B}(2\epsilon q, r)$, and define the 'profile of equilibrium measures of the holes':

$$m_\epsilon(dx) = \sum_{q \in \mathbb{Z}^d} e_q(dx) \,. \tag{2.34}$$

Observe that in view of (2.33):

$$m_\epsilon(dx) \underset{\epsilon \to 0}{\overset{\text{vaguely}}{\longrightarrow}} m \stackrel{\text{def}}{=} 2^{-d} \operatorname{cap}(\bar{B}(0,a)) \cdot dx \,. \tag{2.35}$$

**Theorem 2.10:**

$$\lim_{\epsilon \to 0} \lambda(U_\epsilon) = 2^{-d} \operatorname{cap}(\bar{B}(0,a)) \,. \tag{2.36}$$

**Proof:** *Upperbound:*

Consider for $N \geq 1$, the functions

$$\psi_N = N^{-d/2}\, 1_{[0,N]^d} \,.$$

It follows from the spectral theorem that for $t > 0$, $\epsilon$ small enough,

$$\begin{aligned} \exp\{-\lambda(U_\epsilon)t\} \;&\geq\; (\psi_N,\, R_t^{U_\epsilon, V=0} \psi_N) \\ &=\; \frac{1}{N^d} \int_{[0,N]^d} dx\, E_x[T_{U_\epsilon} > t,\, Z_t \in [0,N]^d], \\ &\quad \text{for } N \geq 1 \,. \end{aligned} \tag{2.37}$$

Letting $N$ tend to infinity, and using periodicity we find

$$\exp\{-\lambda(U_\epsilon)t\} \geq (2\epsilon)^{-d} \int_{\|x\| \leq \epsilon} dx\, P_x[T_{U_\epsilon} > t] \,, \tag{2.38}$$

with $\|z\| \stackrel{\text{def}}{=} \sup |z^i|$, for $z \in \mathbb{R}^d$.

Summing (2.38) over closed periodic cells contained in $[0,1]^d$ and denoting by $C_\epsilon$ their union, we find that for $t > 0$, $\epsilon$ small enough:

$$\begin{aligned} \exp\{-\lambda(U_\epsilon)\, t\} \;&\geq\; \int_{C_\epsilon} dx\, P_x[T_{U_\epsilon} > t] \\ &\geq\; |C_\epsilon| - \int_{[0,1]^d} P_x[T_{U_\epsilon} \leq t]\, dx \,. \end{aligned} \tag{2.39}$$

Denote by $A_q$ the event:

$$A_q = \{H_q \leq t\} \,, \tag{2.40}$$

where $H_q$ (resp. $L_q$) are the entrance time (resp. the time of last visit) in $\overline{B}(2\epsilon q, r)$. Of course $\{T_{U_\epsilon} \le t\} = \bigcup_{q\in\mathbb{Z}^d} A_q$, and therefore:

$$\int_{[0,1]^d} P_x[T_{U_\epsilon} \le t] \le \int_{[0,1]^d} dx \sum_q P_x[A_q] \overset{\text{def}}{=} A \,. \tag{2.41}$$

Now for $\eta > 0$, $A_q \subseteq \{0 < L_q \le t+\eta\} \cup (A_q \cap \theta^{-1}_{H_q}\{L_q \in (\eta,\infty)\})$, and using a similar argument as in (2.3.60), essentially revolving around (2.3.39), we find:

$$\begin{aligned} A &\le B + A\, r^{d-2}\, c(d,\eta), \quad \text{where} \\ B &= \langle m_\epsilon, \int_{[0,t+\eta]\times[0,1]^d} ds dx\, p(s,x,\cdot)\rangle \,, \end{aligned}$$

in the notations of (2.34), and $c(d,\eta)$ is a positive constant.

Since $y \to \int_{[0,t+\eta]\times[0,1]^d} ds\, dx\, p(s,x,y)$ is a continuous function which decays rapidly at infinity and $\overline{\lim_{\epsilon\to 0}} \langle m_\epsilon, (1+|y|)^{-K}\rangle < \infty$, when $K$ is large enough, it follows from (2.35) that for $t,\eta > 0$, $B$ converges to

$$2^{-d}\text{cap}(\overline{B}(0,a)) \int_{\mathbb{R}^d\times[0,t+\eta]\times[0,1]^d} dy\, ds\, dx\, p(s,x,y) = 2^{-d}\text{cap}(\overline{B}(0,a))\,(t+\eta) \,,$$

when $\epsilon$ tends to 0. Letting $\epsilon$ and then $\eta$ tend to 0, we thus find that

$$\overline{\lim_{\epsilon\to 0}}\; A \le 2^{-d}\, \text{cap}(B(0,a))\, t \,.$$

Since $|C_\epsilon| \to 1$, as $\epsilon \to 0$, we see by letting $\epsilon$ tend to 0 in (3.39) that for $t > 0$:

$$\exp\{-\overline{\lim_{\epsilon\to 0}}\; \lambda(U_\epsilon)\, t\} \ge 1 - 2^{-d}\, \text{cap}(\overline{B}(0,a))\, t \,, \tag{2.42}$$

from which it follows that

$$\overline{\lim_{\epsilon\to 0}}\; \lambda(U_\epsilon) \le 2^{-d}\, \text{cap}(\overline{B}(0,a)) \,. \tag{2.43}$$

*Lower bound:*

We shall use Proposition 1.4 (with $U = \mathbb{R}^d \backslash \bigcup_q \bar{B}(2\epsilon q, r)$ and $V = 0$). To this end we wish to derive for small $t > 0$ and small $\epsilon > 0$, uniform lower bounds on:

$$P_x[\bigcup_{q\in\mathbb{Z}^d} A_q] \,.$$

Using periodicity, we need only consider the case of $x$ with $\|x\| \overset{\text{def}}{=} \sup_i |x^i| \le \epsilon$. We introduce $M > 0$, and define:

$$I = \{q \in \mathbb{Z}^d \backslash \{0\},\ |2\epsilon q| \le M\sqrt{t}\} \,.$$

We now see that for $t > 0$, and $\epsilon > 0$ small enough:

$$
\begin{aligned}
& P_x\big[\bigcup_{q\in\mathbb{Z}^d} A_q\big] \geq P_x\big[\bigcup_{q\in I} A_q\big] \geq \sum_{q\in I} P_x[A_q] \\
& - \sum_{\{q,q'\}\subseteq I} P_x[A_q \cap A_{q'}] \overset{(2.3.39)}{\geq} \sum_{q\in I} \int_0^t \int p(s,x,y)\, e_q(dy) ds \\
& - \sum_{\{q,q'\}\subseteq I} (P_x[H_q < H_{q'} < \infty] + \\
& P_x[H_{q'} < H_q < \infty]) \geq \sum_{q\in I} \int_0^t \int p(s,x,y)\, e_q(dy) ds \\
& - \sum_{q\in I} \sum_{q'\in I\setminus\{q\}} P_x[H_q < \infty,\ v_{q'}(Z_{H_q})]\,.
\end{aligned}
\tag{2.44}
$$

Observe that for $q \neq q'$ in $I$, $2\epsilon|q'-q| \leq 2M\sqrt{t}$. We thus have:

$$
\begin{aligned}
& \sum_{q\in I} \sum_{q'\in I\setminus\{q\}} P_x[H_q < \infty,\ v_{q'}(Z_{H_q})] \\
& \leq \sum_{q\in I} v_q(x) \cdot \sum_{0<2\epsilon|q''|\leq 2M\sqrt{t}} \Big(\frac{r}{2\epsilon|q''|-r}\Big)^{d-2} \\
& \leq \sum_{0<2\epsilon|q|\leq M\sqrt{t}} \sup_{\|x\|\leq\epsilon} \frac{r^{d-2}}{|2\epsilon q - x|^{d-2}} \cdot \sum_{0<2\epsilon|q''|\leq 2M\sqrt{t}} \Big(\frac{r}{\epsilon|q''|}\Big)^{d-2} \\
& \leq \sum_{0<2\epsilon|q|\leq M\sqrt{t}} \Big(\frac{r}{\epsilon|q|}\Big)^{d-2} \cdot \sum_{0<2\epsilon|q|\leq 2M\sqrt{t}} \Big(\frac{r}{\epsilon|q|}\Big)^{d-2}.
\end{aligned}
\tag{2.45}
$$

Recall that $r^{d-2} = a^{d-2}\epsilon^d$. We thus have for small $\epsilon > 0$:

$$
\sum_{0<\epsilon|q|\leq M\sqrt{t}} \Big(\frac{r}{\epsilon|q|}\Big)^{d-2} \leq a^{d-2} \int_{|y|\leq 2M\sqrt{t}} \Big(\frac{c(d)}{|y|}\Big)^{d-2} dy \leq C(d,a,M)t\,.
$$

This shows that for small $\epsilon$ the expression in (2.45) is uniformly bounded by $C'(d,a,M)t^2$. On the other hand,

$$
\nu^\epsilon(dy) = \sum_{q\in I} e_q(dy) \underset{\epsilon\to 0}{\overset{\text{vaguely}}{\longrightarrow}} \kappa\, 1_{B(0,M\sqrt{t})}\, dy\,,
$$

so that for $t' < t$:

$$\varliminf_{\epsilon \to 0} \inf_{|x| \le \epsilon} \int_{t'}^{t} \int p(s,x,y)\, \nu^\epsilon(dy) ds \ge (t-t')\, \kappa\, P_0[|Z_1| \le M] .$$

We now have:

$$\begin{aligned}\varliminf_{\epsilon \to 0} \; \lambda(U_\epsilon) &\overset{(1.22)}{\ge} \varliminf_{\epsilon \to 0} \; \frac{1}{t} \inf_x P_x[\bigcup_{q \in \mathbb{Z}^d} A_q] \\ &\overset{(2.44)}{\ge} \frac{1}{t}\,(\kappa\, t\, P_0\,[|Z_1| \le M] - c'(d,a,M)t^2)\,, \text{ (letting } t' \to 0) .\end{aligned}$$

Now letting $t$ tend to 0 and then $M$ to $+\infty$, we find

$$\varliminf_{\epsilon \to 0} \; \lambda(U_\epsilon) \ge \kappa = 2^{-d}\, \mathrm{cap}(\bar{B}(0,a)) \,,$$

which finishes the proof of (2.36). □

Theorem 2.10 can be viewed as an instance of the rough heuristic principle governing the 'constant capacity regime'. Namely 'a profile of equilibrium measures $m_\epsilon$ of hard obstacles close to $V(x)dx$ tends to behave like the soft obstacle $V$'. In this light $2^{-d}\, \mathrm{cap}(\overline{B}(0,a))$ should be viewed as

$$\lambda_V(\mathbb{R}^d), \text{ where } V = 2^{-d}\, \mathrm{cap}(\overline{B}(0,a)) \,.$$

For more on this see Baxter-Jain [BJ87], Baxter-Chacon-Jain [BCJ86], Cioranescu-Murat [CM82], Kac [Kac74], Rauch-Taylor [RT75], Simon [Sim79].

**Exercise:**

**1)** Show that when $r(\epsilon)$ is chosen so that

$$\lim_{\epsilon \to 0} \frac{\mathrm{cap}(B(0,r(\epsilon)))}{|B(0,\epsilon)|} = 0\,, \tag{2.46}$$

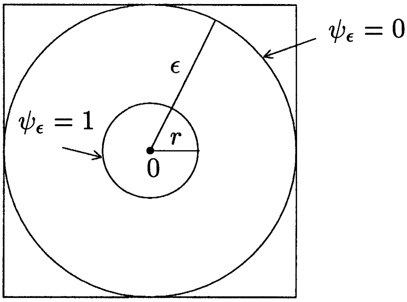
and $U_\epsilon$ is as in (2.31),

$$\lim_{\epsilon \to 0} \lambda(U_\epsilon) = 0\,. \tag{2.47}$$

Show that when the limit in (2.46) is $+\infty$, instead of 0, then (2.47) holds with $+\infty$ replacing 0. (The obstacles 'solidify' in this case).

**2)** Define the functions $\psi_\epsilon \in H^1(\mathbb{R}^d)$:

$$\begin{aligned}\psi_\epsilon(x) &= c_\epsilon \left(\frac{r^{d-2}}{|x|^{d-2}} \wedge 1 - \frac{r^{d-2}}{\epsilon^{d-2}}\right)_+, \text{ with} \\ c_\epsilon &= \left(1 - \frac{r^{d-2}}{\epsilon^{d-2}}\right)^{-1} \sim 1, \text{ as } \epsilon \to 0\,.\end{aligned}$$




If $\psi_{\epsilon,q} = \psi_\epsilon(0 - 2\epsilon q)$, for $q \in \mathbb{Z}^d$, show that

$$\sum_q \psi^2_{\epsilon,q}(x)dx \underset{\epsilon\to 0}{\overset{\text{vaguely}}{\longrightarrow}} 0, \sum_{q\in\mathbb{Z}^d} |\nabla\psi_{\epsilon,q}(x)|dx \underset{\epsilon\to 0}{\overset{\text{vaguely}}{\longrightarrow}} 0 \text{ and}$$

$$\sum_{q\in\mathbb{Z}^d} \frac{1}{2}\, |\nabla\psi_{\epsilon,q}(x)|^2\, dx \underset{\epsilon\to 0}{\overset{\text{vaguely}}{\longrightarrow}} m(dx)\,.$$

Deduce that for $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $\varphi_\epsilon \overset{\text{def}}{=} \varphi(1 - \sum_{q\in\mathbb{Z}^d} \psi_{\epsilon,q})$,

$$\varphi_\epsilon \in H_0^1(U_\epsilon),\ ||\varphi - \varphi_\epsilon||_2 \underset{\epsilon\to 0}{\longrightarrow} 0 \text{ and}$$

$$\int \frac{1}{2}\, |\nabla\varphi_\epsilon|^2\, dx \underset{\epsilon\to 0}{\longrightarrow} \int \frac{1}{2}\, |\nabla\varphi_\epsilon|^2\, dx + 2^{-d}\, \mathrm{cap}(\overline{B}(0,a))\,.$$

**3)** Consider the case of

$$U_\epsilon = \mathbb{R}^d \backslash \bigcup_{q\in\mathbb{Z}^d} (2\epsilon q + \epsilon^{\frac{d}{d-2}} K)\,,$$

where $K$ is some fixed compact set. Generalize Theorem 2.10 and show

$$\lambda(U_\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 2^{-d}\,\mathrm{cap}(K)\,.$$

□

## 3.3 Some One-dimensional Estimates

This section is in some sense a warm up for the random media aspects we shall develop in the next chapters. We introduce certain 'soft obstacles' obtained by translating a given nonnegative measurable bounded function with compact support, at the points of a locally finite sequence on $\mathbb{R}$. Our main objective will be the derivation of bounds relating the corresponding principal eigenvalues with those obtained by imposing instead Dirichlet conditions at the points of the sequence.

In the sequel $\omega$ will stand for a generic locally finite simple pure point measure on $\mathbb{R}$, in other words $\omega = \sum_i \delta_{x_i}$ is such that:

$$(3.1) \qquad \begin{aligned} &\omega(I) < \infty, && \text{for } I \subset \mathbb{R} \text{ compact}\,, \\ &\omega(\{x\}) = 0 \text{ or } 1\,, && \text{for } x \in \mathbb{R}\,. \end{aligned}$$

The 'cloud $\omega$' will only become random in Example 3.3 below. We also consider a given nonnegative bounded measurable function $W(\cdot)$ on $\mathbb{R}$, with compact support such that

$$(3.2) \qquad \lim_{x \to 0_+} \frac{1}{x} \int_0^x W(y)dy > 0 \text{ and } \lim_{x \to 0_+} \frac{1}{x} \int_{-x}^0 W(y)dy > 0\,,$$

(0 is a density point of $W$).

We shall write

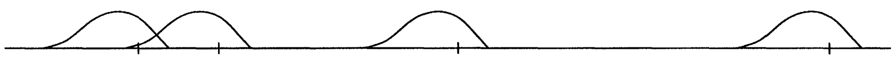
$$(3.3) \qquad \begin{array}{ll} a & \text{for the smallest number in } [0,\infty) \text{ with} \\ W = 0 & \text{on the complement of } [-a,+a], \text{ and} \end{array}$$

$$(3.4) \qquad c_+ = \int_0^a W(y)dy,\ c_- = \int_{-a}^0 W(y)dy,\ c = \int_{-a}^a W(y)dy\,.$$

These are positive numbers in view of (3.2).

The *soft obstacle* attached to the cloud of points $\omega$ and the shape function $W(\cdot)$ is then defined as

$$(3.5) \qquad V(x,\omega) = \sum_i W(x - x_i) = \int_{\mathbb{R}} W(x-y)\,\omega(dy),\ x \in \mathbb{R}\,.$$



When $U$ is an open subset of $\mathbb{R}$, we shall write

$$(3.6) \qquad \lambda_\omega(U) = \lambda_{V(\cdot,\omega)}(U)\,,$$

and denote by $I_k$ the connected component(s) of

$$(3.7) \qquad U \backslash \operatorname{supp} \omega = \bigcup_k I_k\,.$$

Recall that $\lambda((a,b)) = \frac{\pi^2}{2(b-a)^2}$, when $a < b$. Our main object is the following one-dimensional eigenvalue estimate:

**Theorem 3.1:** *There exist $c_1(W)$, $c_2(W) > 0$, such that for any $\omega$ and $U$:*

$$\inf_k \frac{\pi^2}{2(|I_k| - c_2)_+^2} \geq \lambda_\omega(U) \geq c_1 \wedge \inf_k \frac{\pi^2}{2(|I_k| + c_2)^2} \, . \tag{3.8}$$

**Proof:** The leftmost inequality of (3.8) is obvious in view of (1.4) - (1.5), and holds if $c_2 > 2a$. Our main concern is therefore the proof of the rightmost inequality.

With no loss of generality we shall assume that $U$ is an interval, see (1.5), that supp $\omega \cap U \neq \phi$, and that each $I_k$ is bounded (otherwise both members of the inequality to be proved simply equal 0).

Consider $u \in C_c^\infty(U)$. If both endpoints of $I_k = (a_k, b_k)$ are in supp $\omega$:

$$\int_{I_k} V(x,\omega)\, u^2(x) dx \geq \int_{a_k}^{b_k} (W(x - a_k) + W(x - b_k))\, u^2(x) dx \, . \tag{3.9}$$

Otherwise some endpoint of $I_k (\subseteq U)$, say $a_k$, is an endpoint of $U$, and since $u = 0$ outside $U$ and supp $W \subseteq [-a, a]$:

$$\int_{I_k} V(x,\omega)\, u^2(x) dx \geq \int_{a_k - a}^{b_k} (W(x - (a_k - a)) + W(x - b_k))\, u^2(x) dx \, . \tag{3.10}$$

It then follows that (this argument is part of the so-called 'Dirichlet-Neumann' bracketing principle, see Reed-Simon [RS79], Vol. 4):

$$\begin{aligned} &\int_U \frac{1}{2}\, u'^2(x) + V(x,\omega)\, u^2(x) dx = \sum_k \int_{I_k} \frac{1}{2}\, u'^2(x) + V(x,\omega)\, u^2(x) dx \\ &\geq \inf_k \, \nu(\ell_k) \cdot \sum_k \int_{I_k} u^2(x) dx = \inf_k \, \nu(\ell_k) \int_U u^2(x) dx \, , \end{aligned}$$

where for $\ell > 0$,

$$\begin{aligned} \nu(\ell) = \; &\inf \Big\{ \int_0^\ell \frac{1}{2}\, v'^2(x) + (W(x) + W(x - \ell))\, v^2(x) dx \, , \\ &v \in C^1([0,\ell]), \int_0^\ell v^2(x) dx = 1 \Big\}, \text{ and} \end{aligned} \tag{3.12}$$

$$\ell_k = \begin{cases} |I_k|, \text{ when } \partial I_k \subseteq \operatorname{supp} \omega \, , \\ |I_k| + a, \text{ otherwise} \, . \end{cases} \tag{3.13}$$

We have thus shown that

$$\lambda_\omega(U) \geq \inf_k \nu(\ell_k) \,, \tag{3.14}$$

and the proof of Theorem 3.1 will be completed when we show

**Lemma 3.2:**

$$\forall L > 0, \ \inf_{(0,L]} \nu(\ell) > 0 \,. \tag{3.15}$$

$$\begin{aligned} &\textit{There exist } \kappa(W),\ \kappa'(W) > 0 \textit{ such that} \\ &\nu(\ell) \geq \frac{\pi^2}{2(\ell+\kappa)^2} \,, \quad \textit{for } \ell \geq \kappa' \,. \end{aligned} \tag{3.16}$$

**Proof:**

*Proof of (3.15):* Assume (3.15) does not hold and chose a sequence $\ell_n \to \ell \in [0, L]$ and a sequence $u_n \in C^1([0, \ell_n])$, with $\|u_n\|_2 = 1$ such that

$$\int_0^{\ell_n} \frac{1}{2}\, u_n'^2(x) + (W(x) + W(x - \ell_n))\, u_n^2(x)dx \underset{n\to\infty}{\longrightarrow} 0 \,, \tag{3.17}$$

Writing $u_n = v_n + d_n$, where

$$d_n = \frac{1}{\ell_n} \int_0^{\ell_n} u_n(x)dx \ \text{ and } \ \frac{1}{\ell_n} \int_0^{\ell_n} v_n(x)dx = 0 \,, \tag{3.18}$$

it follows from Poincaré's inequality that

$$\int_0^{\ell_n} u_n'^2(x)dx = \int_0^{\ell_n} v_n'^2(x)dx \geq \frac{\pi^2}{L^2}\, \|v_n\|_2^2 \,,$$

so that by (3.17):

$$\|v_n\|_2 \to 0 \quad \text{and} \tag{3.19}$$

$$d_n^2\, \ell_n \to 1 \ \text{ as } n \to \infty \,. \tag{3.20}$$

Since $W(\cdot)$ is bounded, it follows from (3.17), (3.19), (3.20) that:

$$\begin{aligned} 0 &= \lim_n \int_0^{\ell_n} (W(x) + W(x - \ell_n))(v_n^2 + 2v_n d_n + d_n^2)dx \\ &= \lim_n d_n^2\, \ell_n \times \frac{1}{\ell_n} \int_0^{\ell_n} W(x) + W(x - \ell_n)dx \end{aligned}$$

however this last quantity due to (3.20), (3.2) is $> 0$. This is a contradiction

*Proof of (3.16):* Using scaling (i.e. $v(\cdot) = \sqrt{\ell}\, u(\ell \cdot))$, we have:

$$\begin{aligned} \ell^2 \nu(\ell) &= \inf \Big\{ \int_0^1 \frac{1}{2}\, v'^2(x) + (\ell^2\, W(x\ell) + \ell^2\, W(\ell(x-1))\, v^2(x) dx\,, \\ & \qquad v \in C^1([0,1]),\ \int_0^1 v^2(x) dx = 1 \Big\}\,. \end{aligned}$$

By considering a minimizing sequence, we can find a $v_\ell \in C([0,1])$ with $v'_\ell \in L^2((0,1))$ and $\|v_\ell\|_2 = 1$, such that:

$$\ell^2\, \nu(\ell) \geq \int_0^1 \frac{1}{2}\, v'^2_\ell(x) + \ell^2 \Big( W(\ell x) + W(\ell(x-1)) \Big)\, v^2_\ell(x) dx\,. \tag{3.21}$$

(It is not difficult to see that both members of (3.21) are in fact equal). With no loss of generality since $|v_\ell|' \overset{L^2}{=} \mathrm{sign}(v_\ell)\, v'_\ell$, we can assume $v_\ell \geq 0$. From now on, we suppose that

$$\ell > 4a \tag{3.22}$$

and define

$$\begin{aligned} \eta_+ &= \inf \Big\{ v_\ell(x),\ x \in \Big[0,\ \frac{a}{\ell}\Big] \Big\} \\ &= v_\ell(\alpha_\ell), \text{ for some } \alpha_\ell \in \Big[0,\ \frac{a}{\ell}\Big]\,, \\ \eta_- &= \inf \Big\{ v_\ell(x),\ x \in \Big[1 - \frac{a}{\ell}\,, 1\Big] \Big\} \\ &= v_\ell(\beta_\ell), \text{ for some } \beta_\ell \in \Big[1 - \frac{a}{\ell}\,, 1\Big]\,, \\ \eta &= \eta_+ + \eta_-\,, \end{aligned} \tag{3.23}$$

$$u_\ell(x) = v_\ell(x) - \frac{(\beta_\ell - x)}{\beta_\ell - \alpha_\ell}\, v_\ell(\alpha_\ell) - \frac{(x - \alpha_\ell)}{\beta_\ell - \alpha_\ell}\, v_\ell(\beta_\ell),\ x \in [0,1]\,, \tag{3.24}$$

so that $u_\ell(\alpha_\ell) = u_\ell(\beta_\ell) = 0$. In view of (3.22), $\beta_\ell - \alpha_\ell > \frac{1}{2}$, so that:

$$\|u_\ell - v_\ell\|_\infty \leq 2 \times \max(\eta_+, \eta_-) \leq 2\eta \tag{3.25}$$

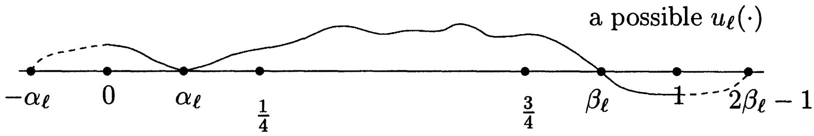
$$\|u'_\ell - v'_\ell\|_\infty \leq 2 \times \max(\eta_+, \eta_-) \leq 2\eta \tag{3.26}$$

It now follows that

$$\begin{aligned} \ell^2\, \nu(\ell) &\overset{(3.21)-(3.22)}{\geq} \frac{1}{2}\, \|v'_\ell\|_2^2 + (c_+\, \eta_+^2 + c_-\, \eta_-^2)\, \ell \\ &\overset{(3.26)}{\geq} \frac{1}{2}\, \max(\|u'_\ell\|_2 - 2\eta,\, 0)^2 + (c_+\, \eta_+^2 + c_-\, \eta_-^2)\, \ell\,. \end{aligned}$$

Since, $u_\ell(\alpha_\ell) = u_\ell(\beta_\ell) = 0$, extending $u$ by symmetry across 0 and 1, it is easy to argue that:

$$\int_0^1 u_\ell'^2\, dx \geq \pi^2 \int_0^1 u_\ell^2\, dx, \text{ and}$$




$$\begin{aligned} \ell^2\, \nu(\ell) &\geq \frac{1}{2} \max(\pi\, \|u_\ell\|_2 - 2\eta,\, 0)^2 + (c_+\, \eta_+^2 + c_-\, \eta_-^2)\, \ell \\ &\overset{(3.25)}{\geq} \frac{1}{2} \max(\pi - 2(1+\pi)\, \eta,\, 0)^2 + c\eta^2\, \ell\,, \,. \end{aligned}$$

provided $c = \frac{1}{4} \min(c_+, c_-)(> 0)$.

The minimum of rightmost expression in the above inequality over $\eta \in [0,\infty) = [0, \frac{\pi}{2(1+\pi)}] \cup [\frac{\pi}{2(1+\pi)}, \infty)$ is bigger than:

$$\begin{aligned} &\min\Big(c\, \frac{\pi^2}{4(1+\pi)^2}\, \ell,\ \frac{\pi^2}{2} + \min_{\eta \geq 0} \{c\ell\, \eta^2 - 2\pi(1+\pi)\, \eta\}\Big) \\ &= \min\Big(\text{const}(W)\, \ell,\ \frac{\pi^2}{2} - \frac{\text{const}(W)}{\ell}\Big) \geq \frac{\pi^2}{2} - \frac{\text{const}(W)}{\ell}\,, \end{aligned}$$

when $\ell$ is large enough.

This last inequality easily implies our claim (3.16). □

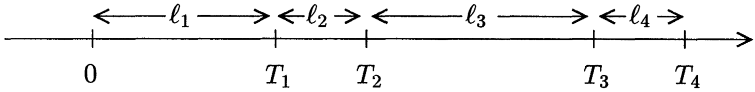
**Example 3.3:** (Poissonian obstacles on $(0,\infty)$).
We consider the situation where the cloud $\omega$ is Poisson with constant intensity $\nu > 0$ on $(0,\infty)$. The space $\Omega$ of single pure point locally finite measures is naturally endowed with the $\sigma$-algebra generated by the applications $A \to \omega(A)$, $A \in \mathcal{B}(\mathbb{R})$, and $\mathbb{P}$ is the Poisson law with intensity measure $\nu 1_{(0,\infty)} dx$, see Neveu [Nev77]. We can then write on a set of full $\mathbb{P}$ measure,

$$\omega = \sum_{i \geq 1} \delta_{T_i}\,,$$

where the random variables $T_i$ are strictly positive, increasing tend to infinity and

$$\ell_i \overset{\text{def}}{=} T_i - T_{i-1}, i \geq 1\,, \tag{3.27}$$

(with the convention $T_0 \equiv 0$) are *iid* exponential variables with parameter $\nu > 0$.




We then also introduce

(3.28) $$M_n = \max(\ell_i, 1 \le i \le n) ,$$

The asymptotic behavior of $M_n$ will be useful. We have

**Lemma 3.4:**

(3.29) $$\mathbb{P}\text{-}a.s. \ \frac{M_n}{\log n} \to \frac{1}{\nu}\ , \ \textit{as } n \to \infty\ , \ \textit{and:}$$

(3.30) *$\nu\, M_n - \log n$ converges in distribution towards the double exponential distribution, i.e.*
$$\mathbb{P}\Big[M_n - \frac{1}{\nu}\ \log n \le a\Big] \underset{n\to\infty}{\longrightarrow} \exp\{-e^{-\nu a}\},\ a \in \mathbb{R}\ .$$

**Proof:**

*Proof of (3.29)*: Pick $\alpha > 1$, then $\{\overline{\lim_n} \frac{M_n}{\log n} > \frac{\alpha}{\nu}\} \subseteq \overline{\lim_i}\ \{\ell_i > \frac{\alpha}{\nu}\ \log i\}$, but $\sum_{i\ge 1} \mathbb{P}[\ell_i > \frac{\alpha}{\nu}\ \log i] = \sum_{i\ge 1} \frac{1}{i^\alpha} < \infty$, since $\alpha > 1$.

Using Borel-Cantelli's lemma, we find that:

(3.31) $$\mathbb{P}\text{-a.s. } \overline{\lim_n}\ \frac{M_n}{\log n} \le \frac{\alpha}{\nu}$$

($\alpha > 1$ arbitrary).
As for the lowerbound, pick $\alpha < 1$. Then

$$\sum_k \mathbb{P}[M_{2^k} < \frac{\alpha}{\nu}\ \log 2^k] = \sum_k \mathbb{P}\Big[\bigcap_1^{2^k} \Big\{\ell_i < \frac{\alpha}{\nu}\ \log 2^k\Big\}\Big]$$
$$= \sum_k \Big(1 - \frac{1}{2^{\alpha k}}\Big)^{2^k} \le \sum_k e^{-2^k \cdot 2^{-\alpha k}} < \infty\ ,$$

using the inequality $0 \le 1 - x \le e^{-x}$, for $x \in [0,1]$ in the last step. It then follows from Borel-Cantelli's lemma that:

$$\mathbb{P}\text{-a.s. for large } k,\ M_{2^k} \ge \frac{\alpha}{\nu}\ \log 2^k\ ,$$

which in turns, writing $2^k \le n < 2^{k+1}$, implies that:

$$\mathbb{P}\text{-a.s. for large } n,\ M_n \ge \frac{\alpha}{\nu}\ \log \frac{n}{2}\ ,$$

and therefore we find:

$$\text{(3.32)} \qquad \mathbb{P}\text{-a.s. } \underline{\lim}_n \ \frac{M_n}{\log n} \geq \frac{\alpha}{\nu}$$

($\alpha < 1$ arbitrary).

Combining (3.31) and (3.32), we obtain (3.29).

*Proof of (3.30)*: (see also Resnick [Res87]).

$$\text{For } a \in \mathbb{R}\text{: } \mathbb{P}[M_n - \tfrac{1}{\nu} \log n \leq a] =$$
$$\mathbb{P}\Big[\bigcap_1^n \{\ell_i \leq \frac{1}{\nu} \log n + a\}\Big] = \Big(1 - \frac{e^{-\nu a}}{n}\Big)^n \to e^{-e^{-\nu a}} .$$

□

We now want to describe the large $L$ behavior of

$$\text{(3.33)} \qquad \lambda_L(\omega) \overset{\text{def}}{=} \lambda_\omega((0,L)),\ L > 0 .$$

**Proposition 3.5:**

(3.34) $\lambda_L(\omega)$ *is measurable in* $\omega$ *and continuous decreasing in* $L$ .

$$\text{(3.35)} \qquad \mathbb{P}\text{-}a.s.\ \lambda_L \sim \frac{(\pi\nu)^2}{2(\log L)^2},\ \text{as } L \to \infty .$$

*For* $M > 0$,

$$\text{(3.36)} \qquad \underline{\lim}_L \ \mathbb{P}\Big[\lambda_L \leq \frac{(\pi\nu)^2}{2(\log L + M)^2}\Big] > 0 ,$$

$$\text{(3.37)} \qquad \underline{\lim}_L \ \mathbb{P}\Big[\lambda_L \geq \frac{(\pi\nu)^2}{2(\log L - M)^2}\Big] > 0 ,\ \text{and}$$

$$\text{(3.38)} \qquad \lim_{M\to\infty} \ \underline{\lim}_{L\to\infty} \ \mathbb{P}\Big[\lambda_L \in \Big[\frac{(\pi\nu)^2}{2(\log L - M)^2},\ \frac{(\pi\nu)^2}{2(\log L + M)^2}\Big]\Big] = 1 .$$

**Proof:**

*Proof of (3.34)*: The second line of (1.2) immediately shows the measurability of $\lambda_L$ for fixed $L$. Moreover for fixed $\omega$, $L > 0 \to \lambda_L(\omega)$ is clearly left continuous decreasing (see (1.33)). On the other hand, using a similar argument as in (3.21), we see that when $L_n \downarrow L > 0$,

$$\underline{\lim}_n \ \lambda_{L_n} \geq \frac{1}{2} \int_0^L v'^2 dx + \int_0^L V(x,\omega)\, v^2\, dx ,$$

for some $v \in C([0,L])$, $v(0) = v(L) = 0$, $\int_0^L v^2\,dx = 1$, and $v' \in L^2((0,L))$. This implies that $v \in H_0^1((0,L))$ and shows that $\underline{\lim}_n\ \lambda_{L_n} \geq \lambda_L$.

This finishes the proof of (3.34).

*Proof of (3.35)*: By the law of large numbers:

$$\frac{T_n}{n} \xrightarrow[n\to\infty]{} \frac{1}{\nu} \quad \mathbb{P}\text{-a.s}\,. \tag{3.39}$$

On a set of full measure where this holds, when $\alpha > 1$, for large $L$,

$$(0,\ T_{[\frac{\nu L}{\alpha}]}) \subseteq (0,L) \subset (0, T_{[\alpha\nu L]})\,,$$

so that by (3.8), for large $L$:

$$\frac{\pi^2}{2(M_{[\frac{\nu}{\alpha}\,L]} - c_2)^2} \geq \lambda_L \geq c_1 \wedge \frac{\pi^2}{2(M_{[\alpha\nu L]} + c_2)^2}\,, \tag{3.40}$$

and (3.35) follows from (3.29).

*Proof of (3.36):* Using (3.40), it is enough to show that for $M > 0$,

$$\underline{\lim}_{L}\ \mathbb{P}\Big[\frac{\pi^2}{2(M_{[\frac{\nu L}{\alpha}]} - c_2)^2} \leq \frac{(\pi\nu)^2}{2(\log L + M)^2}\Big] > 0\,.$$

Now the quantity on the l.h.s. of the above inequality is bigger than

$$\underline{\lim}_{L}\ \mathbb{P}\Big[M_{[\frac{\nu L}{\alpha}]} - c_2 > \frac{\log L + M}{\nu}\Big]$$

which is $> 0$, in view of (3.30) and the fact that $\log L < \log\,[\frac{\nu L}{\alpha}] + \log\frac{2\alpha}{\nu}$ for large $L$. The proof of (3.37) is analogous.

*Proof of (3.38)*: It suffices to prove that:

$$\lim_{M\to\infty}\ \overline{\lim_{L\to\infty}}\ \mathbb{P}\Big[\lambda_L \leq \frac{(\pi\nu)^2}{2(\log L + M)^2}\Big] = 0\,, \tag{3.41}$$

$$\lim_{M\to\infty}\ \overline{\lim_{L\to\infty}}\ \mathbb{P}\Big[\lambda_L \geq \frac{(\pi\nu)^2}{2(\log L - M)^2}\Big] = 0\,. \tag{3.42}$$

We shall only prove (3.41), the proof of (3.42) being quite analogous. Using (3.40)

$$\begin{aligned}
&\overline{\lim_{L\to\infty}}\ \mathbb{P}\Big[\lambda_L \le \frac{(\pi\nu)^2}{2(\log L + M)^2}\Big] \le \overline{\lim_L}\ \mathbb{P}\Big[c_1 \wedge \frac{\pi^2}{2(M_{[\alpha\nu L]}+c_2)^2} \\
\le\ &\frac{(\pi\nu)^2}{2(\log L + M)^2}\Big] = \overline{\lim_L}\ \mathbb{P}\Big[\frac{\log L + M}{\nu} \le M_{[\alpha\nu L]} + c_2\Big] \\
\le\ &\overline{\lim_L}\ \mathbb{P}\Big[\frac{1}{\nu}\,(\log[\alpha\nu L] + M - \log\alpha\nu) - c_2 \le M_{[\alpha\nu L]}\Big] \\
\overset{(3.30)}{=}\ &1 - \exp\{-e^{-M+\log(\alpha\nu)+\nu c_2}\} \underset{M\to\infty}{\longrightarrow} 0\,.
\end{aligned}$$

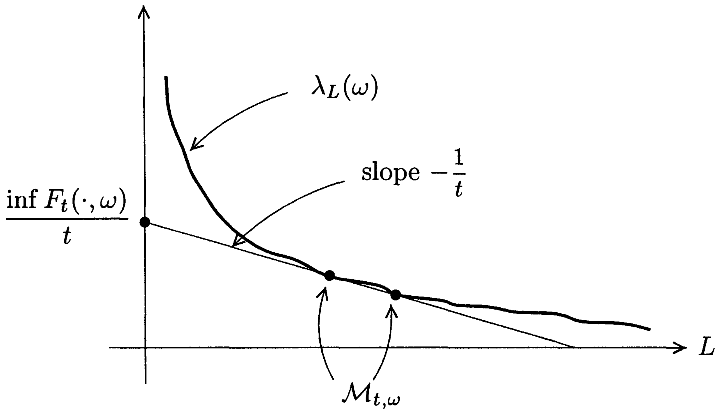
This shows (3.41). The proof of (3.42) is analogous. This finishes the proof of (3.38). □

As a result of the above corollary, we see that for large $L$, $\frac{(\pi\nu)^2}{2(\log L)^2}$ is a 'centering' for the distribution of $\lambda_L$, and that the size of fluctuations of $\lambda_L$ around this centering is of order const $\cdot\,(\log L)^{-3}$.

We now apply this to the study of the location of minima of the function

$$F_t(L,\omega) = L + t\,\lambda_L(\omega), \quad L > 0\,. \tag{3.43}$$

This kind of random variational problem shows up naturally in the study of the 'pinning effect' for Brownian motion in a Poissonian potential, which we shall discuss in Chapter 6.




The locus of minima of $F_t(\cdot,\omega)$ is a compact subset of $(0,\infty)$ defined as

$$\mathcal{M}_{t,\omega} = \{L \in (0,\infty),\ F_t(L,\omega) = \inf_{(0,\infty)} F_t(\cdot,\omega)\}\,. \tag{3.44}$$

We are interested in the asymptotic behavior of $\mathcal{M}_{t,\omega}$. It turns out that (3.35) does not determine the principal asymptotic behavior of $\mathcal{M}_{t,\omega}$, as can be seen from the following

**Exercise:**

**1)** Show that the location $m_t$ of the minimum of

$$L > 0 \longrightarrow L + t(\log L)^{-2}$$

has principal asymptotic behavior governed by:

$$m_t \sim t(\log t)^{-3}, \quad \text{as } t \to \infty \, .$$

**2)** (effect of a bump):

Pick $0 < \ell_0 < \ell_1 < \infty$, and define

$$\begin{aligned} \mu_L = \quad & (\log L)^{-2}, \quad 0 < L < \ell_0 \, , \\ & (\log \ell_1)^{-2}, \quad \ell_0 \le L \le \ell_1 \, , \\ & (\log L)^{-2}, \quad \ell_1 \le L \, . \end{aligned}$$

Find the location of minima of the function: $L > 0 \to L + t\mu_L$.

**3)** Construct two functions $\lambda_L$, $L > 0$, and $L_t$, $t > 0$, such that:

- $\lambda_L$ is continuous decreasing and $\lambda_L \sim (\log L)^{-2}$, as $L \to \infty$ ,
- $L_t$ is a minimum of $L > 0 \to L + t\lambda_L$ ,
- $\varliminf_{t\to\infty} L_t/(t(\log t)^{-3}) = 0$ ,

  $\varlimsup_{t\to\infty} L_t/(t(\log t)^{-3}) = +\infty$. □

We shall now see that $\mathcal{M}_{t,\omega}$ typically lies at distance of order $t(\log t)^{-3}$ from the origin.

**Theorem 3.6:**

$$\lim_{\rho\to 0} \varliminf_{t\to\infty} \mathbb{P}\Big[\mathcal{M}_{t,\omega} \subseteq \Big[\rho t(\log t)^{-3}, \; \frac{1}{\rho}\, t(\log t)^{-3}\Big]\Big] = 1 \, . \tag{3.45}$$

$$\begin{aligned} &\lim_{\rho\to 0} \varliminf_{t\to\infty} \mathbb{P}\Big[\,\Big|\inf F_t(\cdot,\omega) - \frac{(\pi\nu)^2}{2}\, \frac{t}{(\log)^2}\, \Big(1 + \frac{2\log\log t}{\log t}\Big)\Big| \\ &\le \frac{1}{\rho}\, \frac{t}{(\log t)^3}\Big] = 1 \, . \end{aligned} \tag{3.46}$$

**Proof:** For convenience we define for $t > 1$:

$$L_t = t(\log t)^{-3} \, .$$

Our claim (3.45) will follow from:

$$\lim_{k_0\to\infty} \varliminf_{t\to\infty} \mathbb{P}[\mathcal{M}_{t,\omega} \subset [2^{-k_0} L_t, \infty)]] = 1\,, \tag{3.47}$$

$$\lim_{k_0\to\infty} \varliminf_{t\to\infty} \mathbb{P}[\mathcal{M}_{t,\omega} \subset [0, 2^{k_0} L_t)]] = 1\,. \tag{3.48}$$

We begin with the proof of (3.47). Observe that the inequality

$$\lambda_{2^{-k_0} L_t} > \lambda_{L_t} + \frac{L_t}{t} = \lambda_{L_t} + (\log t)^{-3}\,,$$

implies that $\mathcal{M}_{t,\omega} \subset [2^{-k_0} L_t, \infty)$. Now for $\epsilon > 0$, we can choose $M(\epsilon) > 0$, in view of (3.38), so that for any $k_0 \geq 1$:

$$\varliminf_{t\to\infty} \mathbb{P}\Big[\lambda_{2^{-k_0} L_t} > \frac{(\pi\nu)^2}{2(\log(2^{-k_0} L_t))^2} - \frac{M}{(\log(2^{-k_0} L_t))^3}\Big] > 1 - \frac{\epsilon}{2}$$

and

$$\varliminf_{t\to\infty} \mathbb{P}\Big[\lambda_{L_t} < \frac{(\pi\nu)^2}{2(\log L_t)^2} + \frac{M}{(\log L_t)^3}\Big] > 1 - \frac{\epsilon}{2}\,.$$

Then, given any $k_0 \geq 1$, with $\mathbb{P}$ probability bigger than $1 - \epsilon$ when $t$ is large

$$\begin{aligned} \lambda_{2^{-k_0} L_t} - \lambda_{L_t} - \frac{L_t}{t} &> \frac{(\pi\nu)^2}{2(\log L_t - k_0 \log 2)^2} - \frac{(\pi\nu)^2}{2(\log L_t)^2} \\ &- \frac{M}{(\log L_t - k_0 \log 2)^3} - \frac{M}{(\log L_t)^3} - (\log t)^{-3} \\ &\sim [(\pi\nu)^2\, k_0 \log 2 - 2M - 1](\log t)^{-3}, \ \text{ as } t \to \infty\,, \end{aligned}$$

provided $k_0$ is large enough so that $(\pi\nu)^2\, k_0 \log 2 > 2M + 1$. This shows that for any $\epsilon > 0$:

$$\varliminf_{t\to\infty} \mathbb{P}[\mathcal{M}_{t,\omega} \subset [2^{-k_0} L_t, \infty)] > 1 - \epsilon\,,$$

and proves (3.47).

Let us now prove (3.48). We introduce for $t \geq 2(\log 2)^3$, $n(t)$ defined by:

$$2^n (\log 2^n)^3 \leq t < 2^{n+1} (\log 2^{n+1})^3\,. \tag{3.49}$$

It then follows that $2^{n(t)}$ and $L_t$ are comparable in the sense that for large $t$:

$$0 < C < \frac{2^{n(t)}}{L_t} < C' < \infty\,.$$

Now when $k_0 \geq 1$ is such that:

$$\forall k \geq k_0, \ \ \lambda_\omega((0, 2^{n(t)})) + \frac{2^{n(t)}}{t} < \lambda_\omega((0, 2^{n(t)+k+1})) + \frac{2^{n(t)+k}}{t},$$

then $\mathcal{M}_{t,\omega} \subset [0, 2^{n(t)+k_0}]$.

It thus suffices to show that

$$\lim_{k_0 \to \infty} \varliminf_{t \to \infty} \mathbb{P}\Big[ \bigcap_{k \geq k_0} \Big\{ \lambda_{2^n} < \lambda_{2^{n+k+1}} + \frac{2^{n+k}}{t} \Big\} \Big] = 1 .$$

Now for $\epsilon > 0$, using (3.38), we can choose $M(\epsilon) > 0$ such that:

$$\varliminf_{t \to \infty} \mathbb{P}\Big[ \lambda_{2^{n(t)}} < \frac{(\pi\nu)^2}{2(\log 2^{n(t)} - M)^2} \Big] > 1 - \epsilon .$$

It is therefore enough to prove that for $M > 0$,

$$\lim_{k_0 \to \infty} \varliminf_{t \to \infty} \mathbb{P}\Big[ \bigcap_{k \geq k_0} \Big\{ \frac{(\pi\nu)^2}{2(\log 2^n - M)^2} < \lambda_{2^{n+k+1}} + \frac{2^{k-1}}{(\log 2^{n+1})^3} \Big\} \Big] = 1 .$$

Picking $\alpha > 1$, it is enough to prove the above equality with $\lambda_{2^{n+k+1}}$ replaced by

$$\lambda_\omega((0, T_{[\alpha\nu 2^{n+k+1}]})) \overset{(3.8)}{\geq} c_1 \wedge \frac{\pi^2}{2(M_{[\alpha\nu 2^{n+k+1}]} + c_2)^2} .$$

This will now follow from:

$$(3.50) \qquad \begin{aligned} &\lim_{k_0 \to \infty} \varliminf_{t \to \infty} \mathbb{P}\Big[ \bigcap_{k \geq k_0} \Big\{ \frac{(\pi\nu)^2}{2(\log 2^n - M)^2} - \frac{2^{k-1}}{(\log 2^{n+1})^3} \\ &< \frac{\pi^2}{2(M_{[\alpha\nu 2^{n+k+1}]} + c_2)^2} \Big\} \Big] = 1 . \end{aligned}$$

Using the fact that $(1+x)^{-2} \geq 1 - 2x$, for $x \geq 0$,

$$\begin{aligned} &\frac{(\pi\nu)^2}{2(\log 2^n - M)^2} - \frac{2^{k-1}}{(\log 2^{n+1})^3} \\ \leq\ & \frac{(\pi\nu)^2}{2} \Big( \log 2^n - M + \frac{2^{k-1}}{\pi^2\nu^2} \Big( \frac{\log 2^n - M}{\log 2^{n+1}} \Big)^3 \Big)^{-2} \\ <\ & \frac{(\pi\nu)^2}{2} \Big( \log 2^n - M + \frac{2^{k-2}}{\pi^2\nu^2} \Big)^{-2} \end{aligned}$$

for large $t$.

If we denote by $A$ the event:

$$A = \bigcap_{k \geq k_0} \{M_{[\alpha\nu 2^{n+k+1}]} < a_{n,k}\} \tag{3.51}$$

with

$$a_{n,k} = \frac{1}{\nu}\,(\log 2^n - M + (4\pi^2\nu^2)^{-1}\,2^k) - c_2\,, \tag{3.52}$$

our claim (3.50) is a consequence of:

$$\lim_{k_0 \to \infty}\ \varliminf_{t \to \infty}\ \mathbb{P}[A] = 1\,. \tag{3.53}$$

To show this last point, we have

$$\begin{aligned} \mathbb{P}[A^c] &\leq \sum_{k \geq k_0} 1 - \mathbb{P}[M_{[\alpha\nu 2^{n+k+1}]} \leq a_{n,k}] \\ &= \sum_{k \geq k_0} 1 - (1 - \exp\{-\nu\, a_{n,k}\})^{[\alpha\nu 2^{n+k+1}]} \end{aligned}$$

and using the inequality $1 - e^{-x} \leq x$, for $x \geq 0$,

$$\leq \sum_{k \geq k_0} (\alpha\nu 2^{n+k+1}) \log\{(1 - e^{-\nu a_{n,k}})^{-1}\}$$

and for large enough $n(t)$,

$$\begin{aligned} &\leq \sum_{k \geq k_0} 2(\alpha\nu 2^{n+k+1})\, e^{-\nu a_{n,k}} = \sum_{k \geq k_0} \alpha\nu 2^{n+k+2}\ e^{-\log 2^n + M - c 2^k + \nu c_2} \\ &= 4\alpha\nu \sum_{k \geq k_0} 2^k\, e^{M - c2^k + \nu c_2} \underset{k_0 \to \infty}{\longrightarrow} 0\,. \end{aligned}$$

This proves (3.53) and finishes the proof of (3.45).

As for (3.46), observe that on the event

$$\Big\{\mathcal{M}_{t,\omega} \subseteq \frac{t}{(\log t)^3}\,\Big[\rho, \frac{1}{\rho}\Big]\Big\}\,,$$

we have

$$t\,\lambda_{\frac{t}{\rho(\log t)^3}}(\omega) \leq \inf F_t(\cdot,\omega) \leq \frac{t}{\rho(\log t)^3} + t\,\lambda_{\frac{\rho t}{(\log t)^3}}(\omega)\,.$$

As a result, our claim (3.46) follows from (3.38) and (3.45). $\square$

Let us close this section with some comments on the variational problem (3.45). If $s(t)$ denotes the 'typical scale' in which the minima of (3.45) occur and $\sigma(L)$ the 'size of fluctuations' of $\lambda_L(\omega)$ around a median, the heuristic principle connecting both quantities is:

$$s(t) \approx t\,\sigma(s(t)), \quad \text{for large } t\,. \tag{3.54}$$

This roughly corresponds to balancing out the extra cost of moving at distances of order $s(t)$ and the associated gain induced by the decrease of the principal eigenvalue. In view of Proposition 3.5, $\sigma(L) \approx (\log L)^{-3}$, for large $L$, and as a result we recover from (3.54),

$$s(t) \approx t(\log t)^{-3}\,.$$

This is of course the same order one finds, when one naively replaces $\lambda_L(\omega)$ by $(\log L)^{-2}$ (in view of (3.34)) and looks at the minimum of

$$L \to L + t(\log L)^{-2}\,.$$

However the exercises preceding Theorem 3.6 indicate that this naive replacement is not a heuristic justification of the occurrence of the scale $t(\log t)^{-3}$, in Theorem 3.6. For more on this we refer to [Szn97b].

## 3.4 Notes and References

The connection between the large time asymptotics of the Feynman-Kac formula and principal eigenvalues is already present in Kac [Kac51]. Principal eigenvalues in a non-self-adjoint probabilistic context are extensively discussed in the book of Pinsky [Pin95]. Some of the material of §1 is close to the Appendix of [Szn97a]. More results and references around the gauge functions, cf. (1.44), can be found in Chung-Zhao [CZ95], see also Cranston-Fabes-Zhao [CFZ88], and Khasminski [Kha59].

Saying that the interplay between eigenvalues and capacity has been extensively studied is an understatement, and the present notes only provide a limited account of the literature. The effect of the presence of a small obstacle already appears in Samarskii [Sam48], and Swanson [Swa63]. Further results can be found in a series of articles by Ozawa, see for instance [Oza81], [Oza83a], as well as in the book by Maz'ja-Nazarov-Plamenevskii [MNP91]. Similar questions in a geometric context can be found in Courtois [Cou95] and in the book of Chavel [Cha84]. The article of Flucher [Flu95] gives many references on the whole subject. The probabilistic point of view and the connection with the Wiener sausage in the case of many small random obstacles goes back to the stimulating article [Kac74] of Kac. There is a large literature on the constant capacity regime and 'the mathematics of crushed ice', for instance the references [BCJ86], [BJ87], [CM82], [RT75], [Sim79] are mentioned in B) of §2, we also refer to Ozawa [Oza83b].

Theorem 2.1 is a special case of inequalities of Thirring [Thi79], Vol. 3, p. 156-157; a variant of this result is used in [KM83]. Theorem 2.3 originates from Lemma 4.6 of [Szn91b]. Further results can be found in [Nol98]. Bounds with some similar flavor to Theorem 2.3 appear in Mc Gillivray [Gil96]. Example 2.8 about intersection of Wiener sausages in the slightly different setting of simple random walks, is investigated in [KMSS94] by Khanin et al., who prove an upper bound (with exponent $\frac{d-2}{d} - \eta$, $\eta > 0$ arbitrary), qualitatively complementing the lower bound of Proposition 2.9. The results of §3 are close to Section 5 of [Szn96a], I, and offer a toy model for the random scales of Chapter 6 §3, where a similar Ansatz as (3.54) is expected to hold, see also [Szn97b].

# Appendix: Brownian Bridge Measures

We collect here some useful results about the Brownian bridge measures, which are used in the previous chapters. We first need some notations. For $t > 0$, we consider the space $C([0,t], \mathbb{R}^d)$ of continuous functions on $[0,t]$ with values in $\mathbb{R}^d$, endowed with its canonical $\sigma$-field and the topology of uniform convergence. With a slight abuse of notations we still denote by $Z_.$ the canonical process on $C([0,t], \mathbb{R}^d)$, and by $\mathcal{F}_s$, $0 \le s \le t$, the canonical filtration. We write $P_x^t$ for the Wiener measure in time $t$, starting in $x$, and $E_x^t$ for the corresponding expectation. As we shall see the Brownian bridge measures $P_{x,y}^t$, $y \in \mathbb{R}^d$, constitute a natural choice of regular conditional probabilities of $P_x^t$ given $Z_t = y$. We begin with a proposition where the measures $P_{x,y}^t$ are constructed.

**Proposition A.1:** *For $x, y \in \mathbb{R}^d$, $t > 0$, there exists a unique probability measure $P_{x,y}^t$ on $C([0,t], \mathbb{R}^d)$ such that for $s \in [0,t)$, $A \in \mathcal{F}_s$:*

$$(A.1) \qquad P_{x,y}^t[A] = \frac{1}{p(t,x,y)}\, E_x^t[A, p(t-s, Z_s, y)]\,.$$

*Moreover,*

$$(A.2) \qquad P_{x,y}^t[Z_t = y] = 1, \quad \textit{and}$$

*when $S$ is an $(\mathcal{F}_s)$-stopping time and $A \in \mathcal{F}_S$, then:*

$$(A.3) \qquad P_{x,y}^t[A \cap \{S < t\}] = \frac{1}{p(t,x,y)}\, E_x^t[A \cap \{S < t\}, p(t-S, Z_S, y)]\,.$$

**Proof:** We begin with the construction of the Brownian bridge measures and the proof of the characteristic property (A.1). Observe that (A.1) uniquely determines $P_{x,y}^t$ since $\mathcal{F}_t = \bigvee_{0 \le s < t} \mathcal{F}_s$. We thus only need to show the existence of $P_{x,y}^t$ for which (A.1) holds. To this end note that

$$(A.4) \qquad p(t-s, Z_s, y),\ 0 \le s < t, \ \text{ is an } (\mathcal{F}_s)\text{-martingale}\,.$$

Indeed, when $0 \le s < s' < t$ and $A \in \mathcal{F}_s$:

$$E_x^t[A, p(t-s', Z_{s'}, y)] \overset{\binom{\text{Markov}}{\text{property}}}{=} E_x^t[A, E_{Z_s}[p(t-s', \widetilde{Z}_{s'-s}, y)]]$$
$$= E_x^t\Big[A, \int_{\mathbb{R}^d} p(t-s', z, y)\, p(s'-s, Z_s, z) dz\Big] = E_x^t[A, p(t-s, Z_s, y)]\,.$$

Thus $A \in \mathcal{F}_s \to \frac{1}{p(t,x,y)} E_x^t[A, p(t-s, Z_s, y)] \in [0,1]$, defines a consistant family of probabilities and using Kolmogorov's consistency theorem, cf. [CT88], one sees that there exists a unique probability $\widetilde{P}_{x,y}^t$ on $C([0,t), \mathbb{R}^d)$ such that (with a slight abuse of notations):

$$(A.5) \quad \widetilde{P}_{x,y}^t[A] = \frac{1}{p(t,x,y)}\, E_x^t[A, p(t-s, Z_s, y)], \quad \text{for } s \in [0,t) \text{ and } A \in \mathcal{F}_s\,.$$

We shall now prove that:

$$(A.6) \qquad \widetilde{P}_{x,y}^t \text{ is concentrated on } C([0,t], \mathbb{R}^d)\,,$$

(which is a measurable subspace of $C([0,t), \mathbb{R}^d)$). The construction of $P_{x,y}^t$ will then be completed by defining $P_{x,y}^t$ as the restriction of $\widetilde{P}_{x,y}^t$ to $C([0,t]$, $\mathbb{R}^d)$. The crucial consequence of (A.5) is that:

$$(A.7) \quad 1/p(t-s, Z_s, y), 0 \le s < t, \text{ is a nonnegative martingale under } P_{x,y}^t\,.$$

It then follows from the martingale convergence theorem that $\widetilde{P}_{x,y}^t$-a.s., $1/p(t-s, Z_s, y) = [2\pi(t-s)]^{d/2} \exp\{\frac{1}{2}\, \frac{(Z_s-y)^2}{t-s}\}$ converges to a finite limit as $s$ tends to $t$. This implies that

$$(A.8) \qquad \widetilde{P}_{x,y}^t\text{-a.s., } Z_s \text{ converges to } y \text{ as } s \text{ tends to } t\,,$$

and proves both (A.6) and (A.2).

We now turn to the proof of (A.3). Observe that for $0 < \epsilon < t$, $A \cap \{S \le t - \epsilon\}$ belongs to $\mathcal{F}_{S \wedge (t-\epsilon)}$. With the help of (A.1) and the stopping theorem applied to the martingale in (A.4):

$$(A.9) \quad \begin{aligned} &P_{x,y}^t[A \cap \{S \le t-\epsilon\}] \\ &= \frac{1}{p(t,x,y)}\, E_x^t[A \cap \{S \le t-\epsilon\},\, p(\epsilon, Z_{t-\epsilon}, y)] \\ &= \frac{1}{p(t,x,y)}\, E_x^t[A \cap \{S \le t-\epsilon\}, p(t - S \wedge (t-\epsilon),\, Z_{S\wedge(t-\epsilon)}, y)] \\ &= \frac{1}{p(t,x,y)}\, E_x^t[A \cap \{S \le t-\epsilon\},\, p(t-S, Z_S, y)]\,, \end{aligned}$$

since $S = S \wedge (t-\epsilon)$ on the event $\{S \le t-\epsilon\}$. Letting $\epsilon$ converge to 0, (A.3) follows by monotone convergence. $\square$

Here are some useful consequences of the above proposition:

**Corollary A.2:** *For* $0 < s, < \ldots < s_n < t$ *and* $x, y \in \mathbb{R}^d$

$$(A.10)\qquad \begin{aligned} &P^t_{x,y}[Z_{s_1} \in dz_1, \ldots, Z_{s_n} \in dz_n] = \\ &\frac{1}{p(t,x,y)}\, p(s_1,x,z_1)\, p(s_2-s_1,z_1,z_2)\ldots p(t-s_n,z_n,y)dz_1\ldots dz_n\,. \end{aligned}$$

$(A.11)$ $y \to P^t_{x,y}$ *is a regular conditional probability of* $P^t_x$ *given* $\{Z_t = y\}$ ,

*(i.e.* $y \to P^t_{x,y}$ *is a probability kernel,* $P^t_{x,y}[Z_t = y] = 1$, *for* $A$ *measurable and* $B \in \mathcal{B}(\mathbb{R})$, $E^t_x[A, Z_t \in B] = E^t_x[Z_t \in B,\ P^t_{x,Z_t}[A]]$*).*

$(A.12)$ *The image of* $P^t_{x,y}$ *under the map* $w(\cdot) \to w(t-\cdot)$ *is* $P^t_{y,x}$ .

**Proof:** The claim (A.10) immediately follows from (A.1). As for (A.11), the fact that $y \to P^t_{x,y}$ is a probability kernel immediately follows from (A.10) and a monotone class argument. Further we know from (A.2) that $P^t_{x,y}[Z_t = y] = 1$, and finally when $A \in \mathcal{F}_s$, $0 \le s < t$ and $B \in \mathcal{B}(\mathbb{R})$:

$$\begin{aligned} E^t_x[A, Z_t \in B] &\overset{\binom{\text{Markov}}{\text{property}}}{=} \int_B E^t_x[A, p(t-s, Z_s, y)]dy \\ &\overset{(A.1)}{=} \int_B p(t,x,y)\ P^t_{x,y}[A]dy = E^t_x[Z_t \in B,\ E^t_{x,Z_t}[A]]\,. \end{aligned}$$

The case of a general $A \in \mathcal{F}_t$ then follows by a monotone class argument. This proves (A.11). Finally (A.12) is an immediate consequence of (A.10) and the symmetry of $p(u,\cdot,\cdot)$ for $u > 0$. $\square$

We now come to the statement of the strong Markov property. When $S$ is an $(\mathcal{F}_s)$-stopping time, and $t' \in [0,t]$, it will be convenient in what follows to view $Z_{(S+u)\wedge t'}$, $0 \le u \le t$, as a random element of $C([0,t], \mathbb{R}^d)$.

**Proposition A.3:** *(strong Markov property)*
*For* $x, y \in \mathbb{R}^d$, $t > 0$, $S$ *an* $(\mathcal{F}_s)$*-stopping time,* $A \in \mathcal{F}_S$ *and* $B \in \mathcal{F}_t$, *one has:*

$$(A.13)\qquad \begin{aligned} &P^t_{x,y}[A \cap \{S < t\} \cap \{Z_{(S+\cdot)\wedge t} \in B\}] = \\ &E^t_{x,y}[A \cap \{S < t\},\ P^{t-S}_{Z_S,y}[\widetilde{Z}_{\cdot\wedge(t-S)} \in B]]\,. \end{aligned}$$

**Proof:** We consider $0 < \epsilon' < \epsilon < t$, $A \in \mathcal{F}_S$ and $f$ a bounded continuous function on $C([0,t], \mathbb{R}^d)$. Then we have

$$\begin{aligned}
&p(t,x,y)\, E^t_{x,y}\Big[A \cap \{S < t-\epsilon\}, E^{t-S}_{Z_S,y}\,[f(\widetilde{Z}_{\cdot \wedge (t-\epsilon'-S)})]\Big] \\
\overset{(A.3)}{=}\quad & E^t_x\Big[A \cap \{S < t-\epsilon\}, p(t-S, Z_S, y)\, E^{t-S}_{Z_S,y}[f(\widetilde{Z}_{\cdot\wedge(t-\epsilon'-S)})]\Big] \\
\overset{(A.3)}{=}\quad & E^t_x\Big[A \cap \{S < t-\epsilon\},\ E^{t-S}_{Z_S}[f(\widetilde{Z}_{\cdot\wedge(t-\epsilon'-S)})\, p(\epsilon', \widetilde{Z}_{t-\epsilon'-S}, y)]\Big] \\
\overset{\binom{\text{strong Markov}}{\text{property}}}{=}\quad & E^t_x[A \cap \{S < t-\epsilon\},\ f(Z_{(S+\cdot)\wedge(t-\epsilon')})\, p(\epsilon', Z_{t-\epsilon'}, y)] \\
\overset{(A.1)}{=}\quad & p(t,x,y) E^t_{x,y}[A \cap \{S < t-\epsilon\},\ f(Z_{(S+\cdot)\wedge(t-\epsilon')})]\,.
\end{aligned}$$

Letting $\epsilon'$ first tend to zero and then letting $\epsilon$ tend to zero, we find

$$E^t_{x,y}[A \cap \{S < t\},\ E^{t-S}_{Z_S,y}\,[f(\widetilde{Z}_{\cdot\wedge(t-S)})]] = E^t_{x,y}[A \cap \{S < t\},\ f(Z_{(S+\cdot)\wedge t})]\,.$$

The claim (A.13) now follows. □

**Exercise:**

**1)** (scaling property)

Show that $P^t_{x,y}$ is the image of $P^1_{0,0}$ under the map:

$$(A.14) \qquad w(\cdot) \to x + \frac{\cdot}{t}\,(y-x) + \sqrt{t}\, w\big(\tfrac{\cdot}{t}\big)\,.$$

(Hint: use (A.10)).

**2)** Consider $x, y \in \mathbb{R}^d$, $t > 0$, and $S$ an $(\mathcal{F}_t)$-stopping time. Show that when $f(w, \widetilde{w})$ is bounded $\mathcal{F}_S \otimes \mathcal{F}_t$-measurable,

$$(A.15) \qquad E^t_{x,y}[S < t, f(w, Z_{(S+\cdot)\wedge t})] = E^t_{x,y}[S < t, E^{t-S}_{Z_S,y}\,[f(w, \widetilde{Z}_{\cdot\wedge(t-S)})]]$$

(Hint: use (A.13) and the monotone class theorem).

**3)** Show that under $P^1_{0,0}$, $Z_s, 0 \le s \le t$, is a centered Gaussian process with covariance

$$E^1_{0,0}[Z^i_s\, Z^j_t] = (s \wedge t - st)\delta_{ij},\ 0 \le s,t \le 1,\ 1 \le i,j \le d\,.$$

□

# Part II
# BROWNIAN MOTION AND RANDOM OBSTACLES

We now begin the study of Brownian motion and Poissonian obstacles. We first develop two helpful techniques in Chapter 4 and 5, namely the method of enlargement of obstacles and the Lyapunov exponents, we also discuss some first applications. We subsequently provide a more substantial application in Chapter 6 with the study of the pinning effect. The last Chapter 7 provides an overview of known results on Brownian motion and Poissonian obstacles. It further presents a number of connections with other topics of random media as well as some currently open problems.

# 4. The Method of Enlargement of Obstacles

The object of this chapter is to present a coarse graining method, which provides lower bounds on the bottom of the Dirichlet spectrum of the Laplacian in regions receiving many small and possibly random obstacles. This technique will for instance turn out to be very helpful in the study of various problems related to Poissonian obstacles. The method is outlined in Section 1. The results are then developed in Section 2 (eigenvalue estimates) and Section 3 (capacity and volume estimates). Section 4 gives several applications of the method to the study of principal Dirichlet eigenvalues of the Laplacian with Poissonian obstacles, in arbitrary dimension. In Section 5 we apply the method to investigate the long time behavior of certain quenched and annealed Wiener expectations.

## 4.1 Orientation

In this section we shall somewhat informally outline the principle of this 'method of enlargement of obstacles', and motivate the developments of the subsequent sections. We begin with some notations.

We let $\Omega$ stand for the set of locally finite simple pure point measures on $\mathbb{R}^d$, $d \geq 1$. An element $\omega$ of $\Omega$ has the form

$$\begin{aligned} \omega &= \sum_i \delta_{x_i}, && \text{and} \\ \omega(K) &< \infty, && \text{for } K \subset \mathbb{R}^d \text{ compact}\,, \\ \omega(\{x\}) &= 0 \text{ or } 1, && \text{for } x \in \mathbb{R}^d\,. \end{aligned} \tag{1.1}$$

Keeping in mind that these cloud configurations will typically be random in our applications, we shall endow $\Omega$ with the canonical $\sigma$-algebra, which is generated by the applications:

$$\omega \in \Omega \to \omega(A) \in \mathbb{N} \cup \{\infty\}, \quad A \in \mathcal{B}(\mathbb{R}^d)\,.$$

Very much in the spirit of Chapter 3 §3, we attach to each point of supp $\omega$ an obstacle. We shall either be interested in the case of soft or hard obstacles, which will be suitably scaled.

In the case of *soft obstacles*, we choose a fixed function $W(\cdot)$, nonnegative bounded measurable, compactly supported and not a.e. equal to 0. It is the (unscaled) model for the obstacle attached to each point of supp $\omega$. We then define for $\epsilon \in (0,1)$, $\omega \in \Omega$, $x \in \mathbb{R}^d$:

$$(1.2) \qquad V_\epsilon(x,\omega) = \sum_i \epsilon^{-2}\, W\Big(\frac{x-x_i}{\epsilon}\Big) = \int \epsilon^{-2}\, W\Big(\frac{x-y}{\epsilon}\Big)\, \omega(dy)\,.$$

The nonnegative potential $V_\epsilon(\cdot,\omega)$ is the soft obstacle attached to $\omega$ and $\epsilon$.

The model in the case of *hard obstacles* will be a given fixed nonpolar subset $C$ of $\mathbb{R}^d$. The hard obstacle (or trap configuration) attached to $\omega = \sum_i \delta_{x_i} \in \Omega$ and $\epsilon \in (0,1)$ is the closed set:

$$(1.3) \qquad S_\epsilon = \bigcup_i x_i + \epsilon\, C\,.$$

We shall impose Dirichlet conditions on $S_\epsilon$. Informally, the hard obstacle case corresponds to the singular potential $W(\cdot) = \infty \cdot 1_C(\cdot)$.

We also denote by $a$ the positive number

$$(1.4) \qquad a(W) = \inf\{s > 0;\ W = 0 \ \text{ on } \ \bar{B}(0,s)^c\}\,,$$

in the soft obstacle case, and

$$(1.5) \qquad a(C) = \inf\{s > 0,\ \ C \subseteq \bar{B}(0,s)\}\,,$$

in the hard obstacle case.

The basic objects of interest are the 'principal eigenvalues' (although they need not be eigenvalues)

$$(1.6) \qquad \lambda_\omega^\epsilon(U) \overset{\text{def}}{=} \lambda_{V_\epsilon(\cdot,\omega)}(U),\ \ \epsilon \in (0,1),\ \ \omega \in \Omega,\ U \subseteq \mathbb{R}^d \ \text{ open}\,,$$

in the soft obstacle case and:

$$(1.7) \qquad \lambda_\omega^\epsilon(U) \overset{\text{def}}{=} \lambda(U \backslash S_\epsilon)\,,$$

in the hard obstacle case, using the notation (3.1.3) to denote the bottom of the Dirichlet spectrum of $-\frac{1}{2}\,\Delta$ in a given open set.

The motivation for the technique we develop here is as follows. In several applications one is directly interested in finding good lower bounds for the numbers $\lambda_\omega^\epsilon(U)$ (or some of them). One serious difficulty comes from the fact that the obstacle configurations inside $U$ may have a high combinatorial complexity reflecting the 'high variety of possibilities for $\omega$'. The rough idea to circumvent this fact is to replace true obstacles, with size of order $a\epsilon$, by much larger obstacles of size $\epsilon^\gamma$, where $0 < \gamma < 1$, without inducing a noticeable upward shift of the eigenvalue.

The point of working with large obstacles is that it produces a reduction of the combinatorial complexity or 'entropy' of the problem. On the other hand, the situation of the 'constant capacity regime' described in Theorem 2.10 of Chapter 3, indicates that we cannot in general hope to replace each obstacle by a bigger obstacle without inferring a substantial increase of eigenvalues.

Thus, in order to develop the above program, we shall only enlarge obstacles at certain 'good locations', where this operation creates almost no increase of principal eigenvalues.

This will involve certain pointwise bounds from which eigenvalue estimates will follow. In parallel we shall also derive enough controls so that these modified configurations can still be analysed in a probabilistic context. This will correspond to certain volume estimates.

Let us now somewhat more precisely describe how this works. To $\omega \in \Omega$ and $\epsilon \in (0,1)$, we shall associate two Borel subsets of $\mathbb{R}^d$: $\mathcal{D}_\epsilon(\omega)$, and $\mathcal{B}_\epsilon(\omega)$, which measurably depend on $\omega$. At the heart of the definition of $\mathcal{D}_\epsilon(\omega)$ and $\mathcal{B}_\epsilon(\omega)$ lies a certain quantitative Wiener criterion for the obstacles. It will be precisely stated in Section 2.

The set $\mathcal{D}_\epsilon(\omega)$ is the 'density set' and this is the location where we shall 'enlarge obstacles': we shall impose Dirichlet conditions on $\overline{\mathcal{D}}_\epsilon(\omega)$. The set $\mathcal{B}_\epsilon(\omega)$ is a 'bad set', where obstacles are left untouched. We shall derive volume estimates on $\mathcal{B}_\epsilon(\omega)$. The sets $\mathcal{D}_\epsilon(\omega)$, $\mathcal{B}_\epsilon(\omega)$, $\mathbb{R}^d \backslash (\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega))$ define a trichotomy of $\mathbb{R}^d$, which in some sense parallels that induced by the regular points, the irregular points, and the complement of a compact set $K$. The location where certain series are divergent (see (2.4.23)) describes the set of regular points of a compact $K$. Starting from such points, Brownian motion quickly feels $K$, which is a 'natural boundary condition'. On the other hand, the intersection of $K$ with the location where these series converge describes the set of irregular points of $K$. This is a 'small set' (see Proposition 2.4.5).

The sets $\mathcal{D}_\epsilon(\omega)$ and $\mathcal{B}_\epsilon(\omega)$ are constructed in such a fashion that for $\epsilon \in (0,1)$, $\omega \in \Omega$:

(1.8) $$\mathcal{D}_\epsilon(\omega) \cap \mathcal{B}_\epsilon(\omega) = \emptyset\,,$$

(i.e. $\mathbb{R}^d$ is partitioned into $\mathcal{D}_\epsilon(\omega)$, $\mathcal{B}_\epsilon(\omega)$, $(\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega))^c$),

(1.9) $$\omega(\mathbb{R}^d \backslash (\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega))) = 0\,,$$

(i.e. no point of $\omega$ falls in the complement of $\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega)$),

(1.10) for each box $C_q = q + [0,1)^d$, $q \in \mathbb{Z}^d$, the ranges of the applications $\omega \in \Omega \to C_q \cap \mathcal{D}_\epsilon(\omega) \in \mathcal{P}(C_q)$(subsets of $C_q$), and $\omega \in \Omega \to C_q \cap \mathcal{B}_\epsilon(\omega) \in \mathcal{P}(C_q)$, are finite and have cardinality smaller than $2^{\epsilon^{-d\beta}}$ where $\beta \in (0,1)$ is a fixed number .

This last condition ensures the promised reduction of combinatorial complexity. Very loosely speaking, the description of obstacles of size $\sim \epsilon$ in a box $C_q$ of size 1 has combinatorial complexity $2^{\epsilon^{-d}}$ (thinking of a grid of mesh $\epsilon$ in order to keep track of those subboxes which receive a point of supp $\omega$). The reduction of complexity now corresponds to the replacement of $2^{\epsilon^{-d}}$ by $2^{\epsilon^{-d\beta}}$.

Furthermore, we shall construct $\mathcal{D}_\epsilon(\omega)$ and $\mathcal{B}_\epsilon(\omega)$ in such a way that the following two estimates hold:

**Theorem A**: *(eigenvalue estimate)*

$\exists \rho > 0$, $\forall M > 0$,

$$\lim_{\epsilon \to 0} \sup_{\omega, U} \epsilon^{-\rho}(\lambda_\omega^\epsilon(U \backslash \overline{\mathcal{D}}_\epsilon(\omega)) \wedge M - \lambda_\omega^\epsilon(U) \wedge M) = 0 \,, \tag{1.11}$$

*where $\omega$ runs over $\Omega$ and $U$ over all open subsets of $\mathbb{R}^d$.*

**Theorem B**: *(volume estimate)*

$$\exists \kappa > 0, \ \overline{\lim_{\epsilon \to 0}} \sup_{q \in \mathbb{Z}^d, \omega} \epsilon^{-\kappa} \, |\mathcal{B}_\epsilon(\omega) \cap C_q| < 1 \,, \tag{1.12}$$

*where we recall $C_q = q + [0,1)^d$.*

We shall now discuss one example which displays how one can use (1.8) - (1.12). It also provides us with the opportunity to describe somewhat more explicitly the meaning of the expression 'coarse graining scheme'.

**Example 1.1**: $\mathbb{P}_\epsilon$ denotes the Poisson law on $\mathbb{R}^d$, $d \geq 1$, with constant intensity $\nu\epsilon^{-d}$, (see Neveu [Nev77]). Here $\nu > 0$, is some fixed number. We assume that we are in the hard obstacle situation, where obstacles are modelled on closed balls of radius $a\epsilon$. We denote by $U$ the open set $(-1,1)^d$, and we pick some fixed constant:

$$c > \lambda_d \ \left(\text{we recall that } \lambda_d \overset{\text{def}}{=} \lambda(B(0,1))\right) , \tag{1.13}$$

and we want to discuss the small $\epsilon$ behavior of

$$\mathbb{P}_\epsilon[A_\epsilon], \ \text{with} \ A_\epsilon = \{\lambda^\epsilon_\omega(U) \le c\} \,. \tag{1.14}$$

Here $\lambda^\epsilon_\omega(U)$ is the principal Dirichlet eigenvalue of $-\frac{1}{2}\,\Delta$ in the vacant set $U\backslash S_\epsilon$. The asymptotic regime we consider corresponds to a 'roughly constant volume' of the vacant set. Indeed, denoting by $\mathbb{E}_\epsilon$ the $\mathbb{P}_\epsilon$ expectation:

$$\begin{aligned} \mathbb{E}_\epsilon[|U\backslash S_\epsilon|] &= \int_U dx\ \mathbb{P}_\epsilon[\omega(\overline{B}(x,a\epsilon)) = 0] \\ &= |U|\ e^{-\nu/\epsilon^d\,|\overline{B}(0,a\epsilon)|} = 2^d\ e^{-\nu\omega_d a^d}\,. \end{aligned} \tag{1.15}$$

Moreover:

$$\mathbb{E}_\epsilon[|U\backslash S_\epsilon|^2] = \int_{U^2} dxdy\ \mathbb{P}_\epsilon[\{\omega(B(x,a\epsilon)) = 0\} \cap \{\omega(B(y,a\epsilon)) = 0\}]$$

and the two events inside the $\mathbb{P}_\epsilon$ expectation are independent whenever $|x - y| > 2a\epsilon$. It then easily follows that

$$\mathrm{Var}_{\mathbb{P}_\epsilon}(|U\backslash S_\epsilon|) \to 0, \ \text{ as } \epsilon \to 0\,, \tag{1.16}$$

which together with (1.15) shows that the vacant set $U\backslash S_\epsilon$ tends to have constant volume $|U|\,e^{-\nu|B(0,a)|}$, in the asymptotic regime we consider. We shall however see as a by product of the estimates below (see (1.28)) that $\lambda^\epsilon_\omega(U)$ tends to blow up as $\epsilon \to 0$.

We shall now study the small $\epsilon$ behavior of $\mathbb{P}_\epsilon[A_\epsilon]$ by a *coarse graining procedure.* This first step involves a

*Lower bound:* We pick $R > 0$, such that

$$\frac{\lambda_d}{R^2} = c, \ \text{ so that } \ R \in (0,1) \text{ in view of (1.13)}\,. \tag{1.17}$$

One possible strategy for 'producing' $A_\epsilon$ is that no point of $\omega$ falls in $B(0,R+a\epsilon)$. In other words

$$A_\epsilon \supseteq \widetilde{A}_\epsilon, \ \text{ provided } \ \widetilde{A}_\epsilon \stackrel{\text{def}}{=} \{\omega(B(0,R+a\epsilon)) = 0\}\,, \tag{1.18}$$

Therefore:

$$\mathbb{P}_\epsilon[A_\epsilon] \ge \mathbb{P}_\epsilon[\widetilde{A}_\epsilon] = \exp\Big\{ -\frac{\nu}{\epsilon^d}\,\omega_d\Big(\sqrt{\frac{\lambda_d}{c}} + a\epsilon\Big)^d\Big\}\,. \tag{1.19}$$

The upper bound we shall now derive on $\mathbb{P}_\epsilon[A_\epsilon]$ shows in particular that this lower bound captures the correct principal logarithmic order of the quantity.

The coarse graining proof in a certain way makes sense of the idea that 'there is no essentially better strategy than $\widetilde{A}_\epsilon$ to produce the occurrence of $A_\epsilon$'.

*Upper bound:*
In view of (1.10), the sets $U \cap \mathcal{D}_\epsilon(\omega)$ and $U \cap \mathcal{B}_\epsilon(\omega)$ each range among at most $2^{2^d \epsilon^{-d\beta}}$ possible different sets as $\omega$ varies over $\Omega$. We denote respectively by $\mathcal{C}^\epsilon_{\mathcal{D}}$ and $\mathcal{C}^\epsilon_{\mathcal{B}}$ these respective collections of possible sets. We now can cover $\Omega$, the space of 'all cloud configurations' by a collection of events

$$\mathcal{G}_\epsilon = \{G_{D,B};\ D \in \mathcal{C}^\epsilon_{\mathcal{D}},\ B \in \mathcal{C}^\epsilon_{\mathcal{B}}\}\,, \tag{1.20}$$

where

$$G_{D,B} = \{\omega:\ U \cap \mathcal{D}_\epsilon(\omega) = D,\ U \cap \mathcal{B}_\epsilon(\omega) = B\}\,, \tag{1.21}$$

for $D \in \mathcal{C}^\epsilon_{\mathcal{D}},\ B \in \mathcal{C}^\epsilon_{\mathcal{B}}$. We thus have

$$|\mathcal{G}_\epsilon| \le 2^{2^{d+1}\epsilon^{-d\beta}}\,. \tag{1.22}$$

We can now write

$$\mathbb{P}_\epsilon[A_\epsilon] \le \sum_{G_{D,B} \cap A_\epsilon \neq \emptyset} \mathbb{P}_\epsilon[A_\epsilon \cap G_{D,B}]\,. \tag{1.23}$$

*Use of Theorem A:*
When $\epsilon$ is small, choosing $M = 2c$ in (1.11), we find that for $\omega \in \Omega$:

$$\begin{aligned} \epsilon^\rho + \lambda^\epsilon_\omega(U) \wedge 2c &\ge \lambda^\epsilon_\omega(U \backslash \overline{\mathcal{D}}_\epsilon(\omega)) \wedge 2c \\ &\ge \lambda(U \backslash \overline{\mathcal{D}}_\epsilon(\omega)) \wedge 2c\,. \end{aligned}$$

Thus when $\omega \in G_{D,B} \cap A_\epsilon \neq \emptyset$, assuming $\epsilon^\rho < c$, we find that

$$\begin{aligned} c + \epsilon^\rho &\ge \epsilon^\rho + \lambda^\epsilon_\omega(U) \wedge 2c \ge \lambda(U \backslash \overline{D}) \wedge 2c \\ &= \lambda(U \backslash \overline{D}) \ge \lambda_d \Big(\frac{|B(0,1)|}{|U \backslash \overline{D}|}\Big)^{2/d}\,, \end{aligned}$$

using Faber-Krahn's inequality in the last step, see Chavel [Cha84], p. 87-92 Therefore when $\epsilon$ is small the condition $A_\epsilon \cap G_{D,B} \neq \emptyset$ implies:

$$|U \backslash D| \ge |U \backslash \overline{D}| \ge \omega_d \cdot \Big(\frac{\lambda_d}{c + \epsilon^\rho}\Big)^{d/2}\,. \tag{1.24}$$

*Use of Theorem B:*
On the other hand, in view of (1.9)

$$A_\epsilon \cap G_{D,B} \subseteq \{\omega(U \backslash (D \cup B)) = 0\}\,, \tag{1.25}$$

so that for any term entering the sum on the r.h.s. of (1.23):

$$
\begin{aligned}
\mathbb{P}_\epsilon[A_\epsilon \cap G_{D,B}] &\le \exp\Big\{-\tfrac{\nu}{\epsilon^d}\,|U\backslash(D\cup B)|\Big\}\\
&= \exp\Big\{-\tfrac{\nu}{\epsilon^d}\,(|U\backslash D| - |B|)\Big\} \qquad (1.26)\\
&\overset{(1.24)}{\le} \exp\Big\{-\tfrac{\nu}{\epsilon^d}\Big(\omega_d\Big(\frac{\lambda_d}{c+\epsilon^\rho}\Big)^{d/2} - |B|\Big)\Big\}
\end{aligned}
$$

and using now Theorem B, we see that for small $\epsilon$ for any $B \in \mathcal{C}_B^\epsilon$:

$$|B| \le 2^d\,\epsilon^\kappa\,.$$

Collecting this fact and (1.22), (1.23), (1.26), we find:

$$
\begin{aligned}
\mathbb{P}_\epsilon[A_\epsilon] &\le |\mathcal{G}_\epsilon| \sup_{G_{D,B}\cap A_\epsilon \neq \emptyset} \mathbb{P}_\epsilon[A_\epsilon \cap G_{D,B}] \qquad (1.27)\\
&\le 2^{2^{d+1}\epsilon^{-d\beta}} \exp\Big\{-\tfrac{\nu}{\epsilon^d}\Big(\omega_d\Big(\frac{\lambda_d}{c+\epsilon^\rho}\Big)^{d/2} - 2^d\,\epsilon^\kappa\Big)\Big\}\,,
\end{aligned}
$$

when $\epsilon$ is small enough.

This should be compared with the lower bound (1.19), and implies in particular that

$$
\begin{aligned}
\lim_{\epsilon\to 0} \epsilon^d \log \mathbb{P}_\epsilon[A_\epsilon] &= -\nu\,\omega_d\Big(\frac{\lambda_d}{c}\Big)^{d/2} \qquad (1.28)\\
&= -\nu\,|B(0,R)|\,.
\end{aligned}
$$

□

We shall encounter in this chapter and the next chapters several instances of coarse graining schemes to study certain asymptotic expectations. Very much like the example we have just discussed, the line of proof will involve two steps: a lower bound step and an upper bound step. Typically they will look as follows.

The lower bound step will correspond to devising a 'specific strategy' to derive a good asymptotic lower bound on the expectation under consideration.

The upper bound step on the other hand, will possibly first involve a partitioning of the space over which the integration is performed, into an essential part $E$ and an inessential part $E^c$. The contribution of the inessential part $E^c$ to the total expectation will be negligible with respect to the lower bound derived in the first step; in the above example we did not really need this step (i.e. $E^c = \emptyset$, $\Omega = E$). On the other hand, the essential part $E$ will be covered by a family $\mathcal{G}$ of events $G$, in such a fashion that the cardinality of $\mathcal{G}$ is not too high and we have in the asymptotic regime, simultaneously good upper bounds for the contributions on the various $G$ of the expectation under study.

## 4.2 Eigenvalue Estimates

The object of this section is to first define the 'density set' $\mathcal{D}_\epsilon(\omega)$ and then show how it can be used to prove lower bounds on principal eigenvalues, in the spirit of Theorem A of the previous section. We shall develop the results in the context of soft obstacles. This case is technically more involved, and we shall indicate how the results have to be adapted in the simpler situation of hard obstacles. It should also be pointed out that throughout this section the environment $\omega$ will be nonrandom, and the estimates will be performed in a uniform fashion over $\omega$.

The definition of $\mathcal{D}_\epsilon(\omega)$ hinges on a certain quantitative Wiener criterion for a 'skeleton of the obstacles', see (2.12) below. To construct $\mathcal{D}_\epsilon(\omega)$, we shall need four parameters: $L$, $\alpha$, $\gamma$, $\delta$ where

$$\begin{aligned} &L \geq 2\,, \quad \text{is an integer}\,, \\ &0 < \alpha < \gamma < 1\,, \\ &\delta > 0\,. \end{aligned} \tag{2.1}$$

The integer $L$ will be used to determine an $L$-adic decomposition of $\mathbb{R}^d$, $\alpha$, $\gamma$ will determine 'two distinct asymptotic scales':

$$\epsilon << \epsilon^\gamma << \epsilon^\alpha << 1, \ \text{ when } \epsilon \to 0\,,$$

and finally $\delta$ will enter the definition of the quantitative Wiener criterion. The unit scale will essentially describe the size of the 'clearings' we are interested in, whereas $\epsilon$ will correspond to the size of the obstacles.

*L-adic decomposition of* $\mathbb{R}^d$*:*
To $m = (i_0, i_1, \dots, i_k)$, where $k \geq 0$, $i_0 \in \mathbb{Z}^d$ and $i_1, \dots, i_k \in \{0, \dots, L-1\}^d$, we associate a box of generation $k$, with size $L^{-k}$:

$$\begin{aligned} C_m &= y_m + \frac{1}{L^k}\,[0,1)^d, \ \text{provided} \\ y_m &= i_0 + \frac{i_1}{L} + \dots + \frac{i_k}{L^k}\,. \end{aligned} \tag{2.2}$$

When $k = 0$, with a slight abuse of notation, we shall write $C_q$, where $q \in \mathbb{Z}^d$, to denote the box $q + [0,1)^d$. For $k \geq 0$, $\mathcal{I}_k$ will stand for the set of labels $m$ as above of generation $k$, whereas $\mathcal{I}$ will stand for the set of all possible labels. The notation

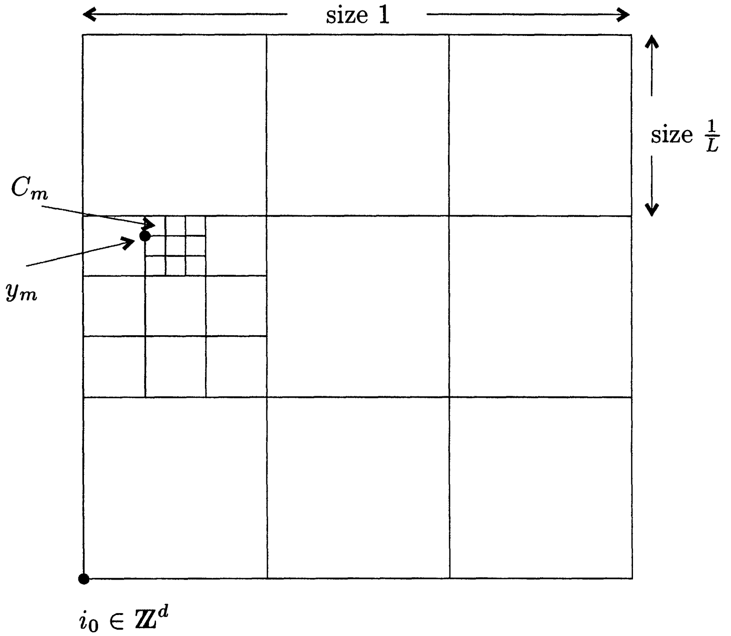
$$m \preceq m' \tag{2.3}$$

will mean that '$m'$ extends $m$', that is:

$$m = (i_0, \dots, i_k),\ m' = (i'_0, \dots, i'_{k'}) \ \text{with} \ k \leq k' \ \text{and} \ i_0 = i'_0, \dots, i_k = i'_k\,.$$

When $m = (i_0, \dots, i_k) \in \mathcal{I}_k$ and $j \in \{0, \dots, L-1\}^d$ we shall define the concatenation of $m$ and $j$:

(2.4) $$m \cdot j = (i_0, \dots, i_k, j) \in \mathcal{I}_{k+1} \;.$$




The truncation of $m = (i_0, \dots, i_k) \in \mathcal{I}_k$ at generation $k'$, where $0 \le k' \le k$, will be denoted by

(2.5) $$[m]_{k'} = (i_0, \dots, i_{k'}) \in \mathcal{I}_{k'} \;.$$

*Asymptotic scales:*

n For $\epsilon \in (0,1)$, we introduce

(2.6) $$n_\alpha(\epsilon) = \left[\alpha \, \frac{\log \frac{1}{\epsilon}}{\log L}\right], \quad n_\gamma(\epsilon) = \left[\gamma \, \frac{\log \frac{1}{\epsilon}}{\log L}\right], \quad \text{so that :}$$

(2.7) $$L^{-n_\alpha(\epsilon)} \ge \epsilon^\alpha > L^{-n_\alpha(\epsilon)-1}, \; L^{-n_\gamma(\epsilon)} \ge \epsilon^\gamma > L^{-n_\gamma(\epsilon)-1} \;.$$

In the context of the above $L$-adic decomposition it will be convenient to let $L^{-n_\alpha(\epsilon)}$ and $L^{-n_\gamma(\epsilon)}$ play the role of the scales $\epsilon^\alpha$ and $\epsilon^\gamma$. We shall also use the notation

(2.8) $$\ell(\epsilon) = n_\gamma(\epsilon) - n_\alpha(\epsilon) \;,$$

for the number of generations separating the scales $L^{-n_\alpha(\epsilon)}$ and $L^{-n_\gamma(\epsilon)}$.

*Quantitative Wiener criterion:*
For $m \in \mathcal{I}_k$, $k \geq 0$, $\omega = \sum_i \delta_{x_i} \in \Omega$, $\epsilon \in (0,1)$, we define in the soft obstacle case, the compact set

$$K_m = L^k\Big(\bigcup_{x_i \in C_m} \overline{B}(x_i, a\epsilon)\Big)\,, \tag{2.9}$$

where it should be observed that $L^k\, C_m$ is a box of size 1. In the hard obstacle case (see previous section), we simply define

$$K_m = L^k\Big(\bigcup_{x_i \in C_m} x_i + \epsilon C\Big)\,. \tag{2.9'}$$

We then introduce

$$\mathrm{cap}_m = \mathrm{cap}(K_m)\,, \tag{2.10}$$

where $\mathrm{cap}(\cdot)$ denotes the capacity relative to

$$\begin{aligned} g(x,y) &= g_{\mathbb{R}^d, V=0}(x,y)\,, \quad \text{when } d \geq 3\,, \\ &\phantom{=} g_{\mathbb{R}^d, V=1}(x,y)\,, \quad \text{when } d = 1 \text{ or } 2\,. \end{aligned} \tag{2.11}$$

With this choice, $\mathrm{cap}(\cdot)$ is translation invariant:

$$\mathrm{cap}(K+x) = \mathrm{cap}(K), \quad K \subseteq \mathbb{R}^d \text{ compact}, \ x \in \mathbb{R}^d\,.$$

We shall say that $m \in \mathcal{I}_{n_\gamma(\epsilon)}$ is a *density index* (or $C_m$ a *density box*) if

$$\sum_{n_\alpha(\epsilon) < k \leq n_\gamma(\epsilon)} \mathrm{cap}_{[m]_k} \geq \delta(n_\gamma(\epsilon) - n_\alpha(\epsilon))\,. \tag{2.12}$$

If (2.12) does not hold we shall say that $m \in \mathcal{I}_{n_\gamma(\epsilon)}$ is a *rarefaction index* (or that $C_m$ is a *rarefaction box*).

This is the promised *quantitative Wiener criterion.* It should be observed that when $\epsilon$ is small, the fact that $C_m$ is a density box or a rarefaction box has little to do with the actual obstacles which fall in $C_m$.

For $\epsilon \in (0,1)$ and $\omega \in \Omega$, we can now define

$$\begin{aligned} \mathcal{D}_\epsilon(\omega) &= \bigcup_{m \in \mathcal{I}_{n_\gamma(\epsilon)},\, m \text{ density index}} C_m\,, \quad \text{`the density set'}\,, \\ \mathcal{R}_\epsilon(\omega) &= \bigcup_{m \in \mathcal{I}_{n_\gamma(\epsilon)},\, m \text{ rarefaction index}} C_m\,, \quad \text{`the rarefaction set'}\,. \end{aligned} \tag{2.13}$$

**Exercise:**

**1)** Show that when $d = 1$ and $\delta < \text{cap}(\{0\})$, $\mathcal{R}_\epsilon(\omega) = \emptyset$.

**2)** Give an example of

- a density box which receives no point of $\omega$
- a rarefaction box $C_m$, for which $K_m \supseteq L^{n_\gamma(\epsilon)} C_m$.

**3)** Consider the hard obstacle periodic configuration corresponding to 'the constant capacity regime' of Chapter 3 §2 B). That is $d \geq 3$, and for $\epsilon \in (0,1)$:

$$\begin{aligned} \omega_\epsilon &= \sum_{q \in \mathbb{Z}^d} \delta_{2\epsilon' q}\,, \quad \text{where } \epsilon' = \epsilon^{1-\frac{2}{d}}, \text{ and} \\ S_\epsilon &= \bigcup_{q \in \mathbb{Z}^d} \overline{B}(2\epsilon' q,\, a\epsilon)\,. \end{aligned}$$

Show that with any choice of constants in (2.1), for small $\epsilon$:

(2.14) $$\mathcal{D}_\epsilon(\omega_\epsilon) = \emptyset\,.$$

**4)** Show that for $\epsilon \in (0,1)$

(2.15) $$(x,\omega) \to 1_{\mathcal{D}_\epsilon(\omega)}(x) \text{ is jointly measurable on } \mathbb{R}^d \times \Omega\,.$$

(Hint: show that for $m \in \mathcal{I}_k$, $\omega \to \text{cap}_m$ is measurable. To this end, use for instance Theorem 5.3 of Chapter 2 to show that one can find a countable collection of $\varphi_\ell \in C_c^\infty(\mathbb{R}^d)$ and $A_\ell \in B(\mathbb{R}^d)$ so that:

$$\text{cap}_m(\omega) = \min_\ell F_\ell(\omega)\,, \quad \text{where}$$

$$\begin{aligned} F_\ell(\omega) = \mathcal{E}(\varphi_\ell, \varphi_\ell)\,, &\quad \text{on } \{\omega(C_m \cap A_\ell) = 0\} \\ \infty\,, &\quad \text{on } \{\omega(C_m \cap A_\ell) > 0\}\,.\ ) \end{aligned}$$

□

The interest of introducing the set $\mathcal{D}_\epsilon(\omega)$ lies in the following lemma, which somehow is the root of the subsequent principal eigenvalue lower bounds of this section. It provides pointwise estimates which show that in the asymptotic regime, when starting on $\mathcal{D}_\epsilon(\omega)$, Brownian motion has a high probability of being killed by the obstacles before moving to a distance of order $\epsilon^\alpha$.

For $k \geq 0$, we introduce the stopping time

(2.16) $$H_k = \inf\{s \geq 0,\ \|Z_s - Z_0\| \geq L^{-k}\}, \quad \text{where}$$

(2.17) $$\|x\| = \sup_{i=1,\dots,d} |x_i|, \quad \text{for } x \in \mathbb{R}^d\,.$$

**Lemma 2.1**: *There exists $c_1(d,W) > 0$, such that when*

$$4a\epsilon < L^{-n_\gamma(\epsilon)}\,, \tag{2.18}$$

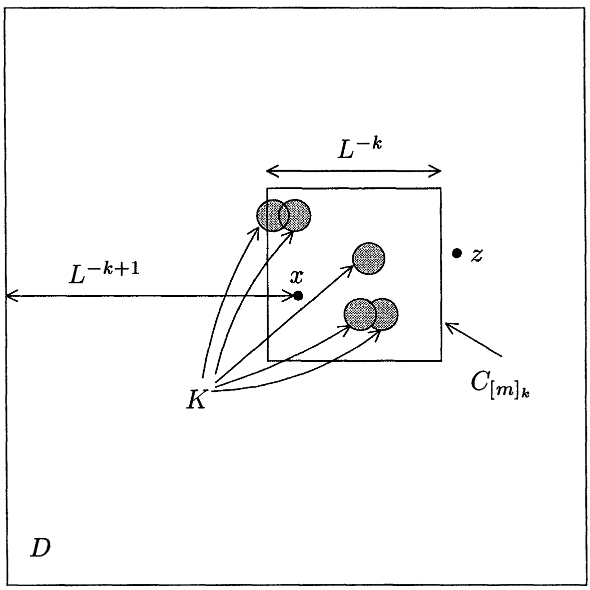
*for $\omega \in \Omega$, $m \in \mathcal{I}_{n_\gamma(\epsilon)}$ and $x \in \overline{C}_m$,*

$$\begin{aligned} &E_x\Big[\exp\Big\{-\int_0^{H_{n_\alpha(\epsilon)}} V_\epsilon(Z_s,\omega)ds\Big\}\Big] \le \\ &\exp\Big\{-c_1 \sum_{n_\alpha(\epsilon)<k\le n_\gamma(\epsilon)} \mathrm{cap}_{[m]_k}\Big\}\,. \end{aligned} \tag{2.19}$$

**Proof**: We shall analyse the process at the successive times $H_k$, $n_\gamma \ge k > n_\alpha$.

Assume that $4a\epsilon < L^{-n_\gamma(\epsilon)}$, and consider $m \in \mathcal{I}_{n_\gamma(\epsilon)}$, $x \in \overline{C}_m$, $\omega \in \Omega$, $k$ with $n_\alpha(\epsilon) < k \le n_\gamma(\epsilon)$, and $z$ some point with $\|z - x\| \le L^{-k}$, playing the role of $Z_{H_k}$. Let $K$ stand for the compact set:

$$K = L^{-k}\, K_{[m]_k} = \bigcup_{x_i \in C_{[m]_k}} \overline{B}(x_i, a\epsilon)\,. \tag{2.20}$$




Observe that

(2.21) the $3a\epsilon$ closed $\|\cdot\|$-neighborhood of $K$ is contained in $D \overset{\text{def}}{=} B_{\|\cdot\|}(x, L^{-k+1})$.

Indeed if $\mathrm{dist}_{\|\cdot\|}(y,K) \le 3a\epsilon$, then $\mathrm{dist}_{\|\cdot\|}(y,\overline{C}_{[m]_k}) \le 4a\epsilon$ so that:

$$\|y-x\| \le L^{-k} + 4a\epsilon < L^{-k} + L^{-n_\gamma} \le 2L^{-k} \le L^{-k+1}\,.$$

We now have:

$$\begin{aligned}
&E_z\Big[\exp\Big\{-\int_0^{T_D} V_\epsilon(Z_s,\omega)ds\Big\}\Big] \le P_z[T_D < H_K] + \\
(2.22)\quad &E_z\Big[H_K < T_D,\, E_{Z_{H_K}}\Big[\exp\Big\{-\int_0^{T_D} V_\epsilon(Z_s,\omega)ds\Big\}\Big]\Big] \le \\
&1 - P_z[H_K < T_D]\Big(1 - \sup_{z'\in K} E_{z'}\Big[\exp\Big\{-\int_0^{T_D} V_\epsilon(Z_s,\omega)ds\Big\}\Big]\Big)\,.
\end{aligned}$$

Consider now $z' \in K$. By the very definition of $K$ we can find $x_i \in \mathrm{supp}\,\omega$, such that $|x_i - z'| \le a\epsilon$. It follows from (2.21) that

$$\begin{aligned}
&E_{z'}\Big[\exp\Big\{-\int_0^{T_D} V_\epsilon(Z_s,\omega)ds\Big\}\Big] \le \\
&E_{z'}\Big[\exp\Big\{-\int_0^{T_{B(x_i,3a\epsilon)}} V_\epsilon(Z_s,\omega)ds\Big\}\Big] \le \\
(2.23)\quad &E_{z'}\Big[\exp\Big\{-\int_0^{T_{B(x_i,3a\epsilon)}} \frac{1}{\epsilon^2}\, W\Big(\frac{Z_s - x_i}{\epsilon}\Big)ds\Big\}\Big] \overset{\text{(scaling)}}{\le} \\
&\sup_{x\in\overline{B}(0,a)} E_x\Big[\exp\Big\{-\int_0^{T_{B(0,3a)}} W(Z_s)ds\Big\}\Big] \overset{\text{def}}{=} K(d,W)\,.
\end{aligned}$$

Observe that

$$(2.24)\qquad K(d,W) \in (0,1)\,.$$

Indeed the expectation under the supremum in the rightmost member of (2.23) is a continuous function of $x \in \overline{B}(0,a)$ in view of Lemma 2.2 of Chapter 2, and it takes values in $(0,1)$, since:

$$E_x\Big[\int_0^{T_{B(0,3a)}} W(Z_s)ds\Big] = \int_{B(0,3a)} g(x,y)\,W(y)\,dy > 0\,,$$

by Lemma 2.1 of Chapter 2. We thus find that

$$(2.25)\quad E_z\Big[\exp\Big\{-\int_0^{T_D} V_\epsilon(Z_s,\omega)ds\Big\}\Big] \le 1 - P_z[H_K < T_D](1 - K(d,W))\,.$$

Using scaling and translation invariance:

$$P_z[H_K < T_D] = P_{L^k z - L^k x}[H_{K_{[m]_k} - L^k x} < T_{B_{\|\cdot\|}(0,L)}]\,,$$

where

$$K_{[m]_k} - L^k x \subset \overline{B}_{\|\cdot\|}\Big(0, \frac{5}{4}\Big), \;\text{ and } \|L^k z - L^k x\| \le 1\,. \tag{2.26}$$

Using Theorem 3.4 of Chapter 2

$$P_z[H_K < T_D] \ge \int \widetilde{g}\,(L^k z - L^k x, y)\, \widetilde{e}(dy)\,, \tag{2.27}$$

where $\widetilde{g}$ and $\widetilde{e}$ stand for the Green function and for equilibrium measure of $K_{[m]_k} - L^k x$, relative to $U = B_{\|\cdot\|}(0, L)$, $V = 0$ when $d \ge 3$, and $V = 1$ when $d = 1, 2$. Of course $\widetilde{g} \le g$ so that $\int \widetilde{e}(dy) \ge \mathrm{cap}(K_{[m]_k} - L^k x) = \mathrm{cap}_{[m]_k}$. We thus find:

$$P_z[H_K < T_D] \ge \inf_{\overline{B}_{\|\cdot\|}(0,1)\times\overline{B}_{\|\cdot\|}(0,\frac{5}{4})} \widetilde{g}(\cdot,\cdot) \times \mathrm{cap}_{[m]_k}\,,$$

so that

$$E_z\Big[\exp\Big\{ - \int_0^{T_D} V_\epsilon(Z_s,\omega)ds\Big\}\Big] \le 1 - c_1(d,W)\,\mathrm{cap}_{[m]_k}\,, \tag{2.28}$$

where

$$c_1(d,W) \stackrel{\text{def}}{=} \inf_{\overline{B}_{\|\cdot\|}(0,1)\times\overline{B}_{\|\cdot\|}(0,\frac{5}{4})} \widetilde{g}(\cdot,\cdot) \times (1 - K(d,W))\,. \tag{2.29}$$

Coming back to the expression in (2.19), we find:

$$\begin{aligned}
& E_x\Big[\exp\Big\{ - \int_0^{H_{n_\alpha}} V_\epsilon(Z_s,\omega)ds\Big\}\Big] \stackrel{\text{(strong Markov property)}}{=} \\
& E_x\Big[\exp\Big\{ - \int_0^{H_{n_\alpha+1}} V_\epsilon(Z_s,\omega)ds\Big\} \\
& E_{Z_{H_{n_\alpha+1}}}\Big[\exp\Big\{ - \int_0^{T_D} V_\epsilon(Z_s,\omega)ds\Big\}\Big]\Big]
\end{aligned} \tag{2.30}$$

(here $k = n_\alpha + 1$, in the notations of the beginning of the proof),

$$\begin{aligned}
& \stackrel{(2.28)}{\le} E_x\Big[\exp\Big\{ - \int_0^{H_{n_\alpha+1}} V_\epsilon(Z_s,\omega)ds\Big\}\Big](1 - c_1\,\mathrm{cap}_{[m]_{n_\alpha+1}}) \\
& \stackrel{\text{induction}}{\le} \prod_{n_\alpha<k\le n_\gamma} (1 - c_1\,\mathrm{cap}_{[m]_k}) \le \exp\Big\{ - c_1 \sum_{n_\alpha<k\le n_\gamma} \mathrm{cap}_{[m]_k}\Big\}\,,
\end{aligned}$$

since $0 \le 1 - x \le \exp\{-x\}$, for $x \in [0,1]$. This proves our claim. □

Here are some further complements to the pointwise estimates of Lemma 2.1.

**Remark 2.2**:

**1)** In the case of hard obstacles the claim (2.19) is replaced by the simpler

$$P_x[H_{n_\alpha(\epsilon)} < T] \le \exp\Big\{ - c_1(d) \sum_{n_\alpha(\epsilon)<k\le n_\gamma(\epsilon)} \mathrm{cap}_{[m]_k} \Big\} , \tag{2.19$'$}$$

where $T = H_{\cup_i x_i + \epsilon C}$ stands for the entrance time in the obstacle set, and

$$c_1(d) = \inf_{\overline{B}_{\|\cdot\|}(0,1)\times \overline{B}_{\|\cdot\|}(0,\frac{5}{4})} \widetilde{g}(\cdot,\cdot) \, . \tag{2.29}$$

**2)** In the case of soft obstacles, the constant $c_1(d,W)$ has the following invariance property, which is inherited from $K(d,W)$ in (2.23):

$$c_1(d, W(\cdot)) = c_1\Big(d, \frac{1}{\lambda^2}\, W\Big(\frac{\cdot}{\lambda}\Big)\Big) \quad \text{for } \lambda > 0 \, . \tag{2.31}$$

**3)** We did not really require that $V_\epsilon(Z_s,\omega)$ be given as in (1.2) to prove Lemma 2.1, but merely that

$$V_\epsilon(x,\omega) \ge \sup_{x_i \in \operatorname{supp}\omega} \frac{1}{\epsilon^2}\, W\Big(\frac{x-x_i}{\epsilon}\Big) , \tag{2.32}$$

see (2.23).

In fact all the results of this chapter can be routinely adapted to the case of *truncated potentials*:

$$V_\epsilon^K(x,\omega) \stackrel{\text{def}}{=} \Big(\int \epsilon^{-2}\, W\Big(\frac{x-y}{\epsilon}\Big)\, \omega(dy)\Big) \wedge (K\epsilon^{-2}) , \tag{2.33}$$

where $K > 0$ is some fixed constant. □

The estimates of Lemma 2.1 can in fact accommodate the presence of a drift.

**Exercise:** (pointwise bounds in the presence of a drift)

Consider a uniformly bounded measurable function $b(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$. Denote by $Q_x$, for $x \in \mathbb{R}^d$, the law on $C(\mathbb{R}_+, \mathbb{R}^d)$ of the weak solution of the stochastic differential equation:

$$\begin{cases} dZ_s &= d\beta_s + b(Z_s)\, ds \, , \\ Z_0 &= x \end{cases}$$

with $\beta_.$ a $d$-dimensional Brownian motion, (see Durrett [Dur84]). Show that there exists $c(d) > 0$, such that when $4a\epsilon < L^{-n_\gamma(\epsilon)}$ and $\|b\|_\infty L^{-n_\alpha(\epsilon)} < c(d)$, for $\omega \in \Omega$ and $x \in \mathcal{D}_\epsilon(\omega)$:

$$E^{Q_x}\Big[\exp\Big\{-\int_0^{H_{n_\alpha(\epsilon)}} V_\epsilon(Z_s,\omega)ds\Big\}\Big] \le 2\exp\Big\{-\tfrac{c_1}{2}\,\delta(n_\gamma(\epsilon)-n_\alpha(\epsilon))\Big\}\,.$$

(Hint: use Girsanov's formula, Cauchy-Schwarz inequality, and Lemma 2.1).

□

The Lemma we just proved is instrumental in the eigenvalue lower bounds we shall derive in this section. Our first application is an 'eigenvalue estimate', which is a precise version of the theorem A of the previous section:

**Theorem 2.3**: *If* $\rho \in (0, \delta c_1(d,W)\frac{(\gamma-\alpha)}{(d+2)\log L})$, *then for* $M > 0$,

(2.34) $$\lim_{\epsilon\to 0}\ \sup_{\omega,U}\ \epsilon^{-\rho}(\lambda^\epsilon_\omega(U\backslash\overline{\mathcal{D}}_\epsilon(\omega))\wedge M - \lambda^\epsilon_\omega(U)\wedge M) = 0\,,$$

*where the supremum is taken over* $\omega \in \Omega$ *and over all open subsets* $U$ *of* $\mathbb{R}^d$.

**Proof**: The proof is an application of Theorem 1.11 of Chapter 3. Pick $M > 0$, $0 < \rho < \rho' < \delta c_1\,\frac{(\gamma-\alpha)}{(d+2)\log L}$, and for $\epsilon \in (0,1)$, $\omega \in \Omega$, $U$ an open subset of $\mathbb{R}^d$, which we may assume nonempty, define:

(2.35) $$\lambda = (\lambda^\epsilon_\omega(U_1)\wedge M - \epsilon^{\rho'})_+\,,$$

where $U_1 = U\backslash\overline{\mathcal{D}}_\epsilon(\omega)$, $U_2 = U$ (in the notations of Theorem 1.11 of Chapter 3).

- Either $\lambda = 0$, so that

(2.36) $$\lambda^\epsilon_\omega(U_1)\wedge M - \lambda^\epsilon_\omega(U_2)\wedge M \le \epsilon^{\rho'}\,,$$

- or $\lambda > 0$ and then $M > \epsilon^{\rho'}$ so that:

(2.37) $$\lambda \le \lambda^\epsilon_\omega(U_1)\Big(1-\frac{\epsilon^{\rho'}}{M}\Big)\,.$$

Choose $\tau = H_{n_\alpha(\epsilon)}$ in Theorem 1.11 of Chapter 3. Obviously the iterates of $S_1 \overset{\text{def}}{=} \tau\circ\theta_{T_{U_1}} + T_{U_1}$ tend to $\infty$, so that (3.1.38) holds. Moreover from (2.37) and (3.1.20), we have

(2.38) $$A \overset{\text{def}}{=} 1 + \sup_x \int_0^\infty \lambda e^{\lambda u}\, R_u^{U_1,V_\epsilon}1(x)du \le K(d)\Big(\frac{M}{\epsilon^{\rho'}}\Big)^{\frac{d}{2}+1}\,.$$

We now choose $\epsilon$ small enough so that:

$$4a\epsilon < L^{-n_\gamma(\epsilon)}, \text{ and} \tag{2.39}$$

$$E_0[\exp\{2M\tau\}] \le K'(d) < \infty\,, \tag{2.40}$$

using for instance (3.1.20), for (2.40). It follows that in the notations of (3.1.36)

$$B \le E_0[\exp\{M\tau\}] \le K'(d) < \infty\,. \tag{2.41}$$

On the other hand

$$\begin{aligned} C^2 &\stackrel{\text{def}}{=} \sup_{x\in U_1^c} E_x\Big[\tau < T_{U_2},\\ &\qquad\qquad \exp\Big\{\lambda\tau - \int_0^\tau V_\epsilon(Z_s,\omega)ds\Big\}\Big]^2\\ &\overset{\text{(Cauchy Schwarz)}}{\le} \sup_{x\in\mathbb{R}^d} E_x[e^{2M\tau}]\ \sup_{x\in U_1^c} E_x\Big[\tau < T_{U_2},\\ &\qquad\qquad \exp\Big\{-2\int_0^\tau V_\epsilon(Z_s,\omega)ds\Big\}\Big]\,. \end{aligned} \tag{2.42}$$

In the above rightmost expression, the expectation vanishes whenever $x \notin U_2$. But $U_2\backslash U_1 \subseteq \overline{\mathcal{D}}_\epsilon(\omega)$, and for $x \in \overline{\mathcal{D}}_\epsilon(\omega)$, it follows from Lemma 2.1, since (2.39) holds that:

$$E_x\Big[\exp\Big\{-\int_0^\tau V_\epsilon(Z_s,\omega)ds\Big\}\Big] \le \exp\{-c_1\,\delta\ell(\epsilon)\}\,. \tag{2.43}$$

Therefore:

$$A\cdot C \le K(d,M)\ \exp\Big\{\rho'\Big(\tfrac{d}{2}+1\Big)\log\frac{1}{\epsilon} - \frac{c_1}{2}\,\delta\,\ell(\epsilon)\Big\}\,. \tag{2.44}$$

However

$$\ell(\epsilon) \sim \frac{\gamma-\alpha}{\log L}\log\frac{1}{\epsilon}\,, \quad \text{as } \epsilon\to 0\,, \tag{2.45}$$

and in view of our choice of $\rho'$, we see that for $\epsilon < \epsilon_0(d,\alpha,\gamma,\delta,\rho',L,W,M)$,

$$A\cdot C < 1\,.$$

It follows from Theorem 1.11, that

$$\lambda = \lambda_\omega^\epsilon(U_1)\wedge M - \epsilon^{\rho'} \le \lambda_\omega^\epsilon(U_2) \text{ and of course } \lambda \le M\,,$$

therefore

$$\lambda^\epsilon_\omega(U_1) \wedge M - \lambda^\epsilon_\omega(U_2) \wedge M \le \epsilon^{\rho'} . \tag{2.46}$$

Combining (2.36) and (2.46), our claim (2.34) easily follows. □

In fact Lemma 2.1 can be used to control the smallness of Dirichlet eigenfunctions of $-\frac{1}{2}\,\Delta + V_\epsilon$ on the set $\mathcal{D}_\epsilon$.

**Exercise:** (smallness of eigenfunctions on $\mathcal{D}_\epsilon$)

Show that there exist positive constants $c(d)$ and $c'(d)$ such that when $4a\epsilon < L^{-n_\gamma(\epsilon)}$ and $\lambda L^{-2n_\alpha(\epsilon)} < c(d)$, then for $\omega \in \Omega$, $U$ a bounded open subset and $\varphi$ a normalized eigenfunction of the semigroup $R_t^{U,V_\epsilon}$, attached to the eigenvalue $\lambda$, and $x \in \mathcal{D}_\epsilon(\omega)$:

$$|\varphi(x)| \le c'(d)\;\lambda^{\frac{d}{4}}\;\exp\Big\{-\tfrac{c_1}{2}\;\delta(n_\gamma(\epsilon) - n_\alpha(\epsilon))\Big\}\,.$$

(Hint: for $x \in U$, apply (3.1.54) to the open set $U' = U_x \cap B_{\|\cdot\|}(x, L^{-n_\alpha})$, with $U_x$ the connected component of $U$ containing $x$, and show that:

$$\varphi(x) = E_x\Big[\varphi(Z_{H_{n_\alpha}})\;\exp\Big\{\lambda H_{n_\alpha} - \int_0^{H_{n_\alpha}} V_\epsilon(Z_s,\omega)ds\Big\},\;H_{n_\alpha} < T_U\Big]\,.$$

Then use (3.1.55), Cauchy-Schwarz inequality and lemma 2.1). Incidentally the smallness of eigenfunctions on $\mathcal{D}_\epsilon(\omega)$ can also be used to derive eigenvalue estimates. □

The above Theorem 2.3 will be used to provide lower bounds for $\lambda^\epsilon_\omega(U)$ in terms of $\lambda^\epsilon_\omega(U\backslash\overline{\mathcal{D}}_\epsilon(\omega))$ or even $\lambda(U\backslash\overline{\mathcal{D}}_\epsilon(\omega))$, for open subsets $U$ which have a size which does not increase too rapidly when $\epsilon \to 0$, (in order to keep control of the combinatorial complexity of $U\backslash\overline{\mathcal{D}}_\epsilon(\omega)$).

We are now going to develop some further eigenvalue controls which enable to discard whole portions of space where $U \cap \mathcal{R}_\epsilon(\omega) = U \cap \mathcal{D}_\epsilon(\omega)^c$ is locally thin (see (2.49)), as being 'unimportant' for the value of $\lambda^\epsilon_\omega(U) \wedge M$, when 'the asymptotic regime takes over'.

**Proposition 2.4**: *There exists a constant $c_2(d) \in (0,\infty)$, such that when $\epsilon \in (0,1)$ and $r \in (0,\frac{1}{4})$ satisfy:*

$$4a\epsilon < L^{-n_\gamma(\epsilon)} < L^{-n_\alpha(\epsilon)} < r \;\text{ and} \tag{2.47}$$

$$\delta\,c_1(n_\gamma(\epsilon) - n_\alpha(\epsilon)) > \log 2\,, \tag{2.48}$$

*then for any $\omega \in \Omega$ and open set $U$ in $\mathbb{R}^d$ such that*

$$\sup_{q \in \mathbb{Z}^d} |(U \backslash \overline{\mathcal{D}}_\epsilon(\omega)) \cap C_q| < r^d \,, \tag{2.49}$$

*one has*

$$\lambda_\omega^\epsilon(U) > c_2(d)/r^2 \,. \tag{2.50}$$

**Proof**: We shall use Proposition 1.8 of Chapter 3. We define

$$S_1 = \inf\{s \geq 0, \; \|Z_s - Z_0\| \geq 4r\} \,.$$

Then for $x \in \mathbb{R}^d$,

$$E_x\Big[S_1 < T_U, \; \exp\Big\{-\int_0^{S_1} V_\epsilon(Z_s,\omega)ds\Big\}\Big]$$

$$\leq \; P_x[T_{B_{\|\cdot\|}(x,3r)} \leq H_{\overline{\mathcal{D}} \cup U^c}] + P_x\Big[H_{\overline{\mathcal{D}} \cup U^c} < T_{B_{\|\cdot\|}(x,3r)} < S_1 < T_U \,,$$
$$\exp\Big\{-\int_{H_{\overline{\mathcal{D}} \cup U^c}}^{H_{\overline{\mathcal{D}} \cup U^c} + H_{n_\alpha} \circ \theta_{H_{\overline{\mathcal{D}} \cup U^c}}} V_\epsilon(Z_s,\omega)ds\Big\}\Big]$$

$$\leq \; 1 - P_x[H_{\overline{\mathcal{D}} \cup U^c} < T_{B_{\|\cdot\|}(x,3r)}] + E_x\Big[H_{\overline{\mathcal{D}} \cup U^c} < T_{B_{\|\cdot\|}(x,3r)} \wedge T_U \,,$$
$$E_{Z_{H_{\overline{\mathcal{D}} \cup U^c}}}\Big[\exp\Big\{-\int_0^{H_{n_\alpha}} V_\epsilon(Z_s,\omega)ds\Big\}\Big]\Big] \,.$$

Observe that on the event $\{H_{\overline{\mathcal{D}} \cup U^c} < T_{B_{\|\cdot\|}(x,3r)} \wedge T_U\}$, $Z_{H_{\overline{\mathcal{D}} \cup U^c}} \in \overline{\mathcal{D}}$, so that by Lemma 2.1:

$$E_{Z_{H_{\overline{\mathcal{D}} \cup U^c}}}\Big[\exp\Big\{-\int_0^{H_{n_\alpha}} V_\epsilon(Z_s,\omega)ds\Big\}\Big] \leq \exp\{-c_1\,\delta\ell\} \overset{(2.48)}{\leq} \frac{1}{2} \,.$$

We thus find that for $x \in \mathbb{R}^d$

$$\begin{aligned} &E_x\Big[S_1 < T_U, \; \exp\Big\{-\int_0^{S_1} V_\epsilon(Z_s,\omega)ds\Big\}\Big] \leq \\ &1 - \frac{1}{2}\, P_x[H_{\overline{\mathcal{D}} \cup U^c} < T_{B_{\|\cdot\|}(x,3r)}] \,. \end{aligned} \tag{2.51}$$

Observe now that $\overline{\mathcal{D}} \cup U^c$ occupies a nonvanishing fraction of the volume of $B_{\|\cdot\|}(x,2r)$ for any $x \in \mathbb{R}^d$. Indeed if $x \in C_q$, since $r < \frac{1}{4}$:

$$\begin{aligned} &|B_{\|\cdot\|}(x,2r) \cap (\overline{\mathcal{D}} \cup U^c) \cap C_q| \geq |B_{\|\cdot\|}(x,2r) \cap C_q| \\ &-|(U \backslash \overline{\mathcal{D}}) \cap C_q| \overset{(2.49)}{\geq} 2^d r^d - r^d \geq r^d \,. \end{aligned} \tag{2.52}$$

On the other hand it is clear from Theorem 3.4 and 4.9 of Chapter 2, that for any compact subset $K$ of $\overline{B}_{\|\cdot\|}(0,2)$ with $|K| = 1$,

$$P_0[H_K < T_{B_{\|\cdot\|}(0,3)}] \geq c(d) > 0\,.$$

Using translation and scaling invariance, (2.51), (2.52) now imply that:

$$\sup_{x \in \mathbb{R}^d} E_x\Big[S_1 < T_U,\ \exp\Big\{-\int_0^{S_1} V_\epsilon(Z_s,\omega)ds\Big\}\Big] \leq \gamma(d) < 1\,. \tag{2.53}$$

If we now pick $c_2(d)$ small enough, then

$$E_0[\exp\{2c_2\, T_{B_{\|\cdot\|}(0,4)}\}] = \overline{c}_2 < \frac{1}{\gamma(d)}\,. \tag{2.54}$$

Choosing $\lambda = c_2/r^2$, we see that

$$\begin{aligned}\beta &\overset{\text{def}}{=} \sup_x \int_0^\infty \lambda e^{\lambda u}\, E_x\Big[S_1 \wedge T_U > u, \exp\Big\{-\int_0^u V_\epsilon(Z_s,\omega)ds\Big\}\Big]\, du \\ &\leq E_0[\exp\{\lambda S_1\}] \overset{\text{(scaling)}}{<} \overline{c}_2 < \infty\,, \quad \text{and}\end{aligned} \tag{2.55}$$

$$\begin{aligned}\alpha &\overset{\text{def}}{=} \sup_x E_x\Big[S_1 < T_U, \exp\Big\{\lambda S_1 - \int_0^{S_1} V_\epsilon(Z_s,\omega)ds\Big\}\Big] \\ &\leq \sup_x E_x[\exp\{2\lambda S_1\}]^{1/2}\, E_x\Big[S_1 < T_U, \\ &\qquad \exp\Big\{-2\int_0^{S_1} V_\epsilon(Z_s,\omega)ds\Big\}\Big]^{1/2} \\ &\overset{(2.53)-(2.54)}{\leq} \sqrt{\overline{c}_2\,\gamma} < 1\,.\end{aligned} \tag{2.56}$$

Our claim (2.50) now follows from Proposition 1.8 of Chapter 3. □

One possible case where the above applies, corresponds to defining for $\epsilon \in (0,1)$, $\omega \in \Omega$, $r \in (0,\frac{1}{4})$ the *clearing boxes* as the $C_q$, $q \in \mathbb{Z}^d$, for which

$$|C_q \backslash \mathcal{D}_\epsilon(\omega)| \geq r^d \tag{2.57}$$

and the *forest boxes* as the $C_q$, $q \in \mathbb{Z}^d$, for which (2.57) fails. If one defines

$$\mathcal{A}_\epsilon(\omega) = \bigcup_{C_q:\ \text{clearing box}} \overline{C}_q\,, \tag{2.58}$$

then $U = \mathcal{A}_\epsilon(\omega)^c$ by definition satisfies (2.49). Thus in 'the asymptotic regime', i.e. when (2.47) and (2.48) hold, one has

$$\lambda_\omega^\epsilon(\mathcal{A}_\epsilon(\omega)^c) \geq c_2(d)/r^2\,. \tag{2.59}$$

We are now going to use Proposition 2.4 to derive upper bounds on the shift

(2.60) $$\lambda^{\epsilon}_{\omega}(U_1) \wedge M - \lambda^{\epsilon}_{\omega}(U_2) \wedge M ,$$

in the 'asymptotic regime', when $U_1, U_2$ are open sets such that

(2.61) $$U_2 \supset U_1 ,$$

where $\mathcal{A}$ is a closed subset such that

(2.62) $$\sup_{q \in \mathbb{Z}^d} \left|\Big(U_2 \backslash (\mathcal{A} \cup \overline{\mathcal{D}}_\epsilon(\omega))\Big) \cap C_q\right| < r^d, \text{ and}$$

(2.63) $$\mathrm{dist}_{\|\cdot\|}(U_2 \backslash U_1, \mathcal{A} \cap U_2) \geq R ,$$

here $R$ is some suitable $> 0$ number, (in other words $U_1$ contains the trace on $U_2$ of an $R$-neighborhood of the trace of $\mathcal{A}$ on $U_2$). It is helpful to keep in mind the following:

**Example 2.5**:

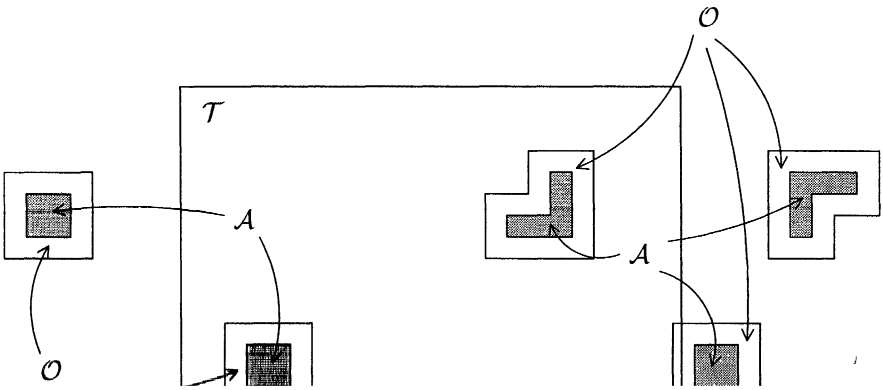
**1)** We consider $\mathcal{A}$ given as in (2.58) (i.e. the union of closure of clearing boxes) and

(2.64) $$\mathcal{O} = \{x \in \mathbb{R}^d, \ \mathrm{dist}_{\|\cdot\|}(x, \mathcal{A}) < R\} .$$

Then if $\mathcal{T}$ is some open subset of $\mathbb{R}^d$ (possibly depending on $\epsilon$)

(2.65) $$U_2 = \mathcal{T}, \ U_1 = \mathcal{T} \cap \mathcal{O}$$

satisfies (2.61) - (2.63).

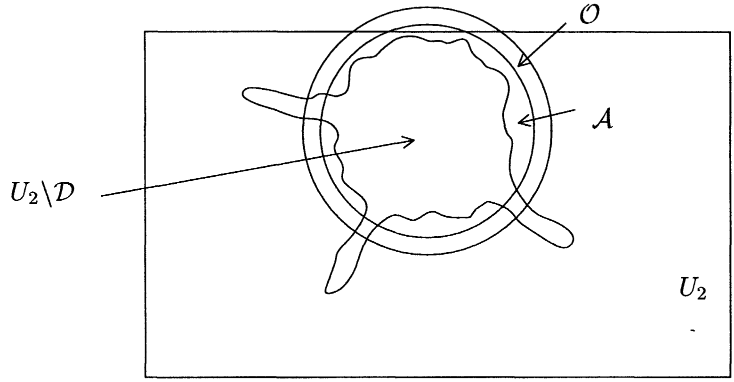


**2)** Another type of instance where (2.61) - (2.63) occur is when $\mathcal{A}$ is some closed ball, $\mathcal{O}$ a slightly larger concentric ball, with $d_{\|\cdot\|}(\mathcal{O}^c, \mathcal{A}) \geq R$, $U_2$ is some open set so that:

$$(2.66) \qquad |U_2 \backslash (\mathcal{A} \cup \mathcal{D}_\epsilon(\omega))| < r^d ,$$

that is $\mathcal{R}$ is 'thin' on $U_2 \backslash \mathcal{A}$, and

$$U_1 = U_2 \cap \mathcal{O} .$$




□

The next theorem could be nicknamed Theorem A' in the terminology of §1. It will in particular enable to discard in practise most of the forest part when estimating principal eigenvalues of large boxes. This will bring the main focus on (not too large) neighborhoods of clearings.

**Theorem 2.6**: *There exists $c_3(d) \in (0,\infty)$, $c_4(d,M) \in (1,\infty)$, $r_0(d,M) \in (0,\frac{1}{4})$, such that for $M > 0$,*

$$(2.67) \qquad \overline{\lim_{\epsilon \to 0}} \ \widetilde{\sup} \ \exp\left\{c_3 \left[\frac{R}{4r}\right]\right\} (\lambda_\omega^\epsilon(U_1) \wedge M - \lambda_\omega^\epsilon(U_2) \wedge M) \leq 1 ,$$

*where $[\ ]$ denotes the integer part and $\widetilde{\sup}$ the supremum over all $\omega \in \Omega$, $U_2 \supset U_1$, $\mathcal{A}$, $R > 0$, $r > 0$, so that (2.61) - (2.63) hold together with*

$$(2.68) \qquad L^{-n_\alpha(\epsilon)} < r < r_0 , \quad and$$

$$(2.69) \qquad \frac{R}{4r} > c_4 .$$

**Proof**: We consider $M > 0$, and $r_0(d,M) \in (0,\frac{1}{4})$ such that

$$(2.70) \qquad c_2/r_0^2 > 4M ,$$

(see Proposition 2.4).

We assume that $\epsilon$ from now on is small enough so that:

$$4a\epsilon < L^{-n_\gamma(\epsilon)} < L^{-n_\alpha(\epsilon)} < r_0 \,,$$
$$\delta c_1 \, \ell(\epsilon) > \log 2 \,,$$

and consider $r \in (L^{-n_\alpha(\epsilon)}, r_0)$, $R > 4r$, $U_1, U_2, \mathcal{A}$ for which (2.61) - (2.63) hold. We assume with no loss of generality that $U_2 \neq \emptyset$.

We shall use Theorem 1.11 of Chapter 3. We define

$$\lambda = \Big(\lambda_\omega^\epsilon(U_1) \wedge M - \exp\Big\{ - c_3 \, \Big[\frac{R}{4r}\Big]\Big\}\Big)_+ \,, \tag{2.71}$$

where $c_3 > 0$ is to be determined further below (see (2.82)). Either $\lambda = 0$, in which case

$$\lambda_\omega^\epsilon(U_1) \wedge M - \lambda_\omega^\epsilon(U_2) \wedge M \le \exp\Big\{ - c_3 \, \Big[\frac{R}{4r}\Big]\Big\} \,, \tag{2.72}$$

or $\lambda > 0$, so that

$$0 < \lambda \le \lambda_\omega^\epsilon(U_1)\Big(1 - \frac{\exp\{-c_3[\frac{R}{4r}]\}}{M}\Big) \,. \tag{2.73}$$

We choose $\tau = H_{\mathcal{A}}$, so that in the notations of Theorem 1.11 of Chapter 3,

$$S_1 = T_{U_1} + \tau \circ \theta_{T_{U_1}} \,,$$

and $S_k$, $k \ge 0$, denote the iterates of $S_1$.

Observe that when $S_{k+1} < T_{U_2}$, $Z.$ travels at least at distance $R$ on $[S_k, S_{k+1}]$, thanks to (2.63). It follows that for any $x \in \mathbb{R}^d$,

$$P_x\text{-a.s.} \;\; \lim_k \uparrow S_k \ge T_{U_2} \,,$$

that is (3.1.38) holds.

If $A, B, C$ are defined as in (3.1.36), (2.73) and (3.1.20) imply that

$$\begin{aligned} A = 1 + \sup_x \int_0^\infty & \lambda e^{\lambda u} \, R_u^{U_1, V_\epsilon} \, 1(x) du \le \\ & K(d)\Big(M \, \exp\Big\{c_3\Big[\frac{R}{4r}\Big]\Big\}\Big)^{(\frac{d}{2}+1)} \,. \end{aligned} \tag{2.74}$$

As for the quantity

$$B = \sup_{x \notin U_1} \int_0^\infty \lambda e^{\lambda u} \, E_x\Big[\tau \wedge T_{U_2} > u, \; \exp\Big\{ - \int_0^u V_\epsilon(Z_s, \omega) ds\Big\}\Big] du \,,$$

observe that $\tau \wedge T_{U_2} = T_U$, where $U = U_2 \backslash \mathcal{A}$ is such that

$$\lambda_\omega^\epsilon(U) \ge 4M \,, \tag{2.75}$$

in view of Proposition 2.4 and our choice of $r$. Therefore $B$ is finite and (3.1.39) holds.

As for the quantity $C$, we have

$$C = \sup_{x \notin U_1} E_x\Big[\tau < T_{U_2},\ \exp\Big\{\lambda\tau - \int_0^\tau V_\epsilon(Z_s,\omega)ds\Big\}\Big]\,. \tag{2.76}$$

The expectation in the r.h.s. vanishes when $x \notin U_2$, and applying Cauchy-Schwarz inequality we find

$$\begin{aligned} C^2 \le \;& \sup_{x \in U_2\setminus U_1} E_x\Big[\tau < T_{U_2},\ \exp\Big\{2M\,T_U - \int_0^{T_U} V_\epsilon(Z_s,\omega)ds\Big\}\Big] \\ & \sup_{x \in U_2\setminus U_1} E_x\Big[\tau < T_{U_2},\ \exp\Big\{-\int_0^\tau V_\epsilon(Z_s,\omega)ds\Big\}\Big]\,, \end{aligned} \tag{2.77}$$

since $\tau = \tau \wedge T_{U_2} = T_U$, on $\{\tau < T_{U_2}\}$. Now with the help of the identity (3.1.43), we see that the first term in the r.h.s. of (2.77) is smaller than:

$$\sup_x \Big(1 + \int_0^\infty 2M\,e^{2Mu}\,R_u^{U,V_\epsilon}\,1(x)du\Big) \overset{(2.75)-(3.1.20)}{\le} K(d)\;2^{(\frac{d}{2}+1)}\,. \tag{2.78}$$

As for the last term in the r.h.s. of (2.77), note that when $x \in U_2\setminus U_1$, it takes $P_x$-a.s. on $\{\tau < T_{U_2}\}$, at least $[\frac{R}{4r}]$ successive displacements at $\|\cdot\|$-distance $4r$ to enter $\mathcal{A}$. In other words

$$\{\tau < T_{U_2}\} \subseteq \Big\{T_{[\frac{R}{4r}]} \le T_U\Big\}\,,\quad P_x - \text{a.s.}\,, \tag{2.79}$$

if $T_k$, $k \ge 0$, denote the iterates of the stopping time

$$T = \inf\{s \ge 0,\ \|Z_s - Z_0\| \ge 4r\}\,.$$

If we now use (2.53) (where $S_1$ stands for $T$), and the strong Markov property, we see that for $x \in U_2\setminus U_1$:

$$E_x\Big[\tau < T_{U_2},\ \exp\Big\{-\int_0^\tau V_\epsilon(Z_s,\omega)ds\Big\}\Big] \le \gamma(d)^{[\frac{R}{4r}]-1}\,. \tag{2.80}$$

We thus find

$$\begin{aligned} AC \;&\le\; K(d,M)\,\exp\Big\{\Big(\frac{d}{2}+1\Big)\,c_3\Big[\frac{R}{4r}\Big] - \frac{1}{2}\,\log\Big(\frac{1}{\gamma(d)}\Big)\,\Big[\frac{R}{4r}\Big]\Big\} \\ &\le\; K(d,M)\,\exp\Big\{-\Big(\frac{d}{2}+1\Big)\,c_3\Big[\frac{R}{4r}\Big]\Big\} < 1\,, \end{aligned} \tag{2.81}$$

provided we choose $c_3(d, M)$ via:

$$\left(\frac{d}{2} + 1\right) c_3 = \frac{1}{4} \log \frac{1}{\gamma(d)}\,, \tag{2.82}$$

as well as $\frac{R}{4r} > c_4(d, M) > 1$, where

$$K(d, M) \exp\left\{ - \left(\frac{d}{2} + 1\right) c_3\, c_4 \right\} < 1\,. \tag{2.83}$$

In view of Theorem 1.11, (2.81) now implies that $\lambda \le \lambda^\epsilon_\omega(U_2)$.

Since we also have $\lambda \le M$, we find

$$\lambda^\epsilon_\omega(U_1) \wedge M - \lambda^\epsilon_\omega(U_2) \wedge M \le \exp\left\{ - c_3 \left[\frac{R}{4r}\right] \right\}. \tag{2.84}$$

Collecting (2.72) and (2.84), we see this proves our claim (2.67). □

## 4.3 Capacity and Volume Estimates

In this section we shall construct the 'bad set' $\mathcal{B}_\epsilon(\omega)$, and derive volume estimates in the spirit of Theorem $B$ of Section 4 §1. Roughly speaking the set $\mathcal{B}_\epsilon(\omega)$ will be a union of boxes of size $\approx \epsilon^\beta$, where $\epsilon << \epsilon^\beta << \epsilon^\gamma$, which contain some point of the cloud $\omega$ and which are included in the rarefaction set $\mathcal{R}_\epsilon(\omega)$.

In order to obtain volume control on the set $\mathcal{B}_\epsilon(\omega)$, which will be precisely defined below, see (3.47), we shall first derive certain capacity estimates on the set $\mathcal{R}_\epsilon(\omega)$. We are now ready to begin.

Observe first that when $\epsilon \in (0,1)$ is small enough so that:

$$4a\epsilon < L^{-n_\gamma(\epsilon)}\,, \tag{3.1}$$

then in view of (2.9),

(3.2) for $m \in \mathcal{I}_k$, with $n_\alpha < k \le n_\gamma$, the closed $\frac{1}{4}$-neighborhood of $L^k\, C_m$ relative to $\|\cdot\|$ contains $K_m$ .

Moreover the $K_m$, $m \in \cup_{n_\alpha < k \le n_\gamma} \mathcal{I}_k$ , satisfy the consistency relation:

$$K_m = \bigcup_{j \in \{0,\dots,L-1\}^d} \frac{1}{L}\, K_{m \cdot j}, \quad \text{for any } m \in \mathcal{I}_k,\ n_\alpha < k < n_\gamma\,. \tag{3.3}$$

We are going to derive a certain exponential estimate in Theorem 3.1 below, which is quite general. It applies to an arbitrary collection of possibly empty compact sets $K_m$, $m \in \bigcup_{n_\alpha < k \le n_\gamma} \mathcal{I}_k$, (here $0 \le n_\alpha < n_\gamma$ are some given integers), which satisfies (3.2) and (3.3).

It is not required that the $K_m$ are given by (2.9), and $\epsilon$ plays no role here. For instance the result applies to the situation where $K$ is some given closed subset of $\mathbb{R}^d$ and:

$$K_m = L^k(K \cap \overline{C}_m),\ m \in \bigcup_{n_\alpha < k \le n_\gamma} \mathcal{I}_k\ ,$$

(3.2) and (3.3) obviously hold in this case. Of course as before $\text{cap}_m = \text{cap}(K_m)$ as in (2.10) - (2.11), and the rarefaction boxes are the boxes $C_m$, $m \in \mathcal{I}_{n_\gamma}$, where

$$\sum_{n_\alpha < k \le n_\gamma} \text{cap}_{[m]_k} < \delta(n_\gamma - n_\alpha)\ ,$$

i.e. the quantitative Wiener criterion breaks down for $C_m$.

In the case of dimension 1, as soon as $\delta < \text{cap}\{0\}$, then $K_m = \emptyset$, whenever $C_m$ is a rarefaction box. This will later have the consequence that the (not yet defined) bad set is empty when $d = 1$, provided we choose $\delta < \text{cap}\{0\}$. For this reason we restrict ourselves to the situation $d \ge 2$. We have the following capacity estimate:

**Theorem 3.1**: *Assume $d \ge 2$, and $L$ large enough so that (3.12) holds. Then there exists positive constants $\delta_0(d, L)$, $c_4(d, L)$, $c_5(d, L)$ such that for any collection of compact sets $K_m$, $m \in \bigcup_{n_\alpha < k \le n_\gamma} \mathcal{I}_k$, satisfying (3.2), (3.3), and $\delta < \delta_0(d, L)$ we have for any $m_0 \in \mathcal{I}_{n_\alpha}$:*

$$(3.4) \quad \frac{1}{L^{d(n_\gamma - n_\alpha)}} \sum_{\substack{m \succ m_0 \\ m \in \mathcal{I}_{n_\gamma},\ \text{rarefaction index}}} \text{cap}_m \le c_4 \exp\Big\{ - c_5 \Big(1 - \frac{\delta}{\delta_0}\Big)(n_\gamma - n_\alpha)\Big\}\ .$$

**Proof**: With no loss of generality, we assume that $n_\alpha = 0$, $n_\gamma = \ell > 0$, $m_0 = 0 \in \mathbb{Z}^d$, (this simply amounts to working with the new collection of compact sets

$$\begin{aligned} \widetilde{K}_m = \quad & K_{m_0\cdot(i_1,\dots,i_k)} - L^{n_\alpha + k}\, y_{m_0} \text{ if } m = (0, i_1, \dots, i_k),\ 0 < k \le \ell\ , \\ & \text{(see (2.2) for the notation } y_{m_0}) \\ & \emptyset\ , \text{ for the other } m \text{ in } \bigcup_{0 < k \le \ell} \mathcal{I}_k\ . \end{aligned}$$

This new collection satisfies (2.2), (2.3).

We shall establish three relations which govern the behavior of the numbers $\mathrm{cap}_m$ 'from generations to generations'. The first relation is immediate from (2.2). There exists $c_6(d) > 0$ such that

$$\mathrm{cap}_m \leq c_6(d), \quad m \in \bigcup_{0<k\leq \ell} \mathcal{I}_k \,. \tag{3.5}$$

For the second relation observe that when $d \geq 3$, $g(\cdot,\cdot)$ (defined in (2.11)) has the exact scaling:

$$g\Big(\frac{x}{L}\,,\frac{y}{L}\Big) = L^{(d-2)}\, g(x,y), \quad x,y \in \mathbb{R}^d\,, \tag{3.6}$$

whereas in dimension $d = 2$, for $\|x - y\| \leq 4$,

$$g\Big(\frac{x}{L}\,,\frac{y}{L}\Big) \leq c'(d=2)\, \log L + g(x,y) \leq c(d=2)\, \log L \cdot g(x,y)\,. \tag{3.7}$$

Indeed, if $z = y - x$

$$\begin{aligned} g\Big(\frac{x}{L}, \frac{y}{L}\Big) - g(x,y) &= \frac{1}{2\pi} \int_0^\infty \Big(\exp\Big\{-\frac{s}{L^2}\Big\} - \exp\{-s\}\Big)\, \exp\Big\{-\frac{z^2}{2s}\Big\}\, \frac{ds}{s} \\ &= \frac{1}{2\pi} \int_0^\infty \Big(\int_{\frac{s}{L^2}}^{s} e^{-u}\, du\Big)\, \exp\Big\{-\frac{z^2}{2s}\Big\}\, \frac{ds}{s} \\ &= \frac{1}{2\pi} \int_0^\infty e^{-u} \Big(\int_u^{uL^2} \exp\Big\{-\frac{z^2}{2s}\Big\}\frac{ds}{s}\Big) du \end{aligned}$$

which is a decreasing function of $|z|$ tending to

$$\frac{1}{2\pi} \int_0^\infty e^{-u} \int_u^{uL^2} \frac{ds}{s}\, du = \frac{1}{\pi}\, \log L \ \ \text{, as } \ |z| \to 0\,.$$

Observe that in view of (3.3), when $0 < k < \ell$, $m \in \mathcal{I}_k$ and $j \in \{0,\ldots,L-1\}^d$,

$$L^{-1}\, K_{m\cdot j} \subseteq K_m\,,$$

combining this and (3.6), (3.7) we obtain our second relation:

$$\begin{aligned} &\mathrm{cap}_{m\cdot j} \leq c_7\, \mathrm{cap}\Big(\frac{1}{L}\, K_{m\cdot j}\Big) \leq c_7\, \mathrm{cap}_m, \\ &m \in \mathcal{I}_k,\ 0 < k < \ell,\ j \in \{0,\ldots,L-1\}^d\,, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} c_7(d,L) &= L^{d-2}, \quad \text{when } d \geq 3\,, \\ &= c(d=2)\, \log L, \quad \text{when } d = 2,\ (c_7 > 1 \text{ by construction})\,. \end{aligned} \tag{3.9}$$

We now come to the third and main relation. If as in (2.4.51), $G : \mathbb{R}_+ \to (0, +\infty]$ denotes the continuous decreasing function such that:

$$g(x,y) = G(|y-x|) \,, \tag{3.10}$$

we define the constants $\widetilde{\delta}_1(d,L)$, $\delta_1(d,L)$, $c_8(d,L)$ via:

$$\begin{aligned} \widetilde{\delta}_1 \cdot L^d \cdot G\Big(\frac{1}{2L}\Big) &= 1\,, \;\; \delta_1 = c_7\,\widetilde{\delta}_1 \\ c_8 = \frac{L^d}{(3^d+1)c_7} &= \frac{L^2}{3^d+1}\,, \;\; \text{when } d \geq 3\,, \\ & \phantom{=} \frac{1}{10\,c(d=2)}\;\frac{L^2}{\log L}\,, \;\; \text{when } d = 2\,. \end{aligned} \tag{3.11}$$

An important point is that when $L \geq L_0(d)$

$$c_8(d,L) > 1\,. \tag{3.12}$$

We shall from now on assume that $L \geq L_0(d)$. The third relation is the object of

**Lemma 3.2**: *For $m \in \mathcal{I}_k$, $0 < k < \ell$:*

$$\text{cap}_m \geq c_8\,\frac{1}{L^d} \sum_{j\in\{0,\dots,L-1\}^d} \text{cap}_{m\cdot j} \wedge \delta_1\,. \tag{3.13}$$

**Proof**: Consider $m$ as above. With no loss of generality, we can assume that $\text{cap}_{m\cdot j} \neq 0$ at least for one $j \in \{0,\dots,L-1\}^d$. We then choose compact subsets $\widetilde{K}_j$ of $\frac{1}{L}\,K_{m\cdot j}$ for $j \in \{0,\dots,L-1\}^d$ such that

$$\begin{aligned} \widetilde{K}_j &= \frac{1}{L}\,K_{m\cdot j}\,, \;\text{ if } \text{cap}\Big(\frac{1}{L}\,K_{m\cdot j}\Big) \leq \widetilde{\delta}_1\,, \;\text{ and} \\ \text{cap}(\widetilde{K}_j) &= \widetilde{\delta}_1\,, \;\text{ if } \text{cap}\Big(\frac{1}{L}\,K_{m\cdot j}\Big) > \widetilde{\delta}_1\,. \end{aligned} \tag{3.14}$$

This can for instance be done with the help of the $L$-adic decomposition into boxes. For a given $j$ such that $\text{cap}(\frac{1}{L}\,K_{m\cdot j}) > \widetilde{\delta}_1$, one constructs two sequences of compact sets, $K_\ell \subset K'_\ell \subset \frac{1}{L}\,K_{m\cdot j}$, $\ell \geq 0$, with $K_\ell \uparrow$, $K'_\ell \downarrow$,

$$\begin{aligned} \text{cap}(K_\ell) &\leq \widetilde{\delta}_1\,, \;\; \text{cap}(K'_\ell) \geq \widetilde{\delta}_1\,, \text{ and} \\ \text{cap}(K'_\ell) &\leq \text{cap}(K_\ell) + \text{cap}\Big(\frac{1}{L^\ell}\,[0,1]^d\Big)\,. \end{aligned}$$

If we now define:

$$\begin{aligned} &\widetilde{K}_j = \bigcap_\ell K'_\ell, \;\text{ then} \\ &\text{cap}(\widetilde{K}_j) \overset{(2.4.64)}{=} \lim_\ell \downarrow \text{cap}(K'_\ell) = \widetilde{\delta}_1\,, \end{aligned}$$

which proves that the choice of $\widetilde{K}_j$ in (3.14) is possible.

Denote by $\nu_j$ the equilibrium measure of $\widetilde{K}_j$ for $j \in \{0, \dots, L-1\}^d$, so that:

$$\nu_j(\widetilde{K}_j) = \mathrm{cap}(\widetilde{K}_j) = \mathrm{cap}\Big(\frac{1}{L}\, K_{m\cdot j}\Big) \wedge \widetilde{\delta}_1 \,, \tag{3.15}$$

and not all $\nu_j$ vanish. Consider then

$$\nu = \frac{1}{\sum_j \mathrm{cap}(\widetilde{K}_j)} \sum_j \nu_j \in M_1\Big(\bigcup_j \frac{1}{L}\, K_{m\cdot j} \overset{(3.3)}{=} K_m\Big) \,. \tag{3.16}$$

In the notations of Theorem 2.4.11, we have

$$\begin{aligned} \frac{1}{\mathrm{cap}_m} &\overset{(2.4.34)}{\le} \langle \nu, \nu \rangle = \frac{1}{(\sum_j \mathrm{cap}(\widetilde{K}_j))^2} \Big(\sum_j \langle \nu_j, \nu_j \rangle + \sum_{j \neq j'} \langle \nu_j, \nu_{j'} \rangle\Big) \\ &= \frac{1}{\sum_j \mathrm{cap}(\widetilde{K}_j)} \Big(1 + \sum_j \sum_{j' \neq j} \langle \frac{\nu_j}{\sum_{j''} \mathrm{cap}(\widetilde{K}_{j''})}, \nu_{j'} \rangle\Big) \,, \end{aligned} \tag{3.17}$$

where we used $\langle \nu_j, \nu_j \rangle = \mathrm{cap}(\widetilde{K}_j)$ in the last step. Now for fixed $j$:

$$\sum_{j' \neq j} \int g(x,y)\, \nu_{j'}(dy) \le 3^d - 1 + \sum_{j'}{}' \int g(x,y)\, \nu_{j'}(dy) \,, \tag{3.18}$$

where $\sum'_{j'}$ denotes the sum over indices $j'' \neq j$ such that $C_{m\cdot j'}$ is not a neighbor of $C_{m\cdot j}$, (i.e. $\overline{C}_{m\cdot j'} \cap \overline{C}_{m\cdot j} = \emptyset$). It now follows from (3.2) that for such $j'$:

$$x \in \mathrm{supp}\, \nu_j \quad y \in \mathrm{supp}\, \nu_{j'} \Longrightarrow |x - y| \ge \frac{1}{2L} \,,$$

and therefore $g(x,y) \le G(\frac{1}{2L})$. Inserting in the r.h.s. of (3.18), we thus find for $x \in \mathrm{supp}\, \nu_j$:

$$\sum_{j' \neq j} \int g(x,y)\, \nu_{j'}(dy) \le 3^d - 1 + L^d\, G\Big(\frac{1}{2L}\Big)\, \widetilde{\delta}_1 \overset{(3.11)}{=} 3^d \,. \tag{3.19}$$

Coming back to (3.17) we find:

$$\begin{aligned} \mathrm{cap}_m &\ge \frac{1}{(1+3^d)} \sum_j \mathrm{cap}(\widetilde{K}_j) \\ &\overset{(3.15)}{=} \frac{1}{(1+3^d)} \sum_j \mathrm{cap}\Big(\frac{1}{L}\, K_{m\cdot j}\Big) \wedge \widetilde{\delta}_1 \\ &\overset{(3.8)}{\ge} \frac{1}{(1+3^d)c_7} \sum_j \mathrm{cap}_{m\cdot j} \wedge c_7\, \widetilde{\delta}_1 \\ &\overset{(3.11)}{=} \frac{c_8}{L^d} \sum_j \mathrm{cap}_{m\cdot j} \wedge \delta_1 \,. \end{aligned} \tag{3.20}$$

This proves our claim. □

**Remark 3.3**: The inequality (3.13) reflects both a growth ($c_8(d,L) > 1$) and saturation (truncation by $\delta_1$) effect for the capacity.

The saturation effect for instance is easily displayed when

$$\begin{aligned} K_{m\cdot j} &= L^{k+1}\,\overline{C}_{m\cdot j} \text{ for all } j, \text{ so that by (3.3)} \\ K_m &= L^k\,\overline{C}_m\,. \end{aligned}$$

In this case we have for all $j$

$$\mathrm{cap}_{m\cdot j} = \mathrm{cap}([0,1]^d) = \mathrm{cap}_m\,.$$

Since we are specifically interested in $c_8(d,L) > 1$, this 'explains' the 'need' for the presence of $\delta_1$ in (3.13). On the other hand, when interaction between the various components $\frac{1}{L}\,K_{m\cdot j}$ can be neglected, capacity is almost additive, (see Exercise 6) before Example 2.4.13). This corresponds to the first inequality in (3.20). Then the discrepancy between the number of terms ($L^d$) and the scaling effect (say $\frac{1}{L^{d-2}}$, when $d \ge 3$), brings the 'growth effect', where $c_8 > 1$, when $L$ is large enough. □

We shall now exploit the three basic relations (3.5), (3.8), (3.13). To this end, we introduce the probability space:

$$\sum = (\{0,\dots,L-1\}^d)^\ell\,, \quad (\ell = n_\gamma - n_\alpha) \tag{3.21}$$

endowed with the uniform probability $Q$. We denote by $X_1,\dots,X_\ell$, the canonical $\{0,\dots,L-1\}^d$ valued coordinates on this space and by $\mathcal{G}_k$, $k \ge 0$, the filtration on $\sum$ defined via:

$$\mathcal{G}_0 = \{\emptyset, \textstyle\sum\} \quad \mathcal{G}_k = \sigma(X_1,\dots,X_{k\wedge\ell}),\quad k \ge 1\,. \tag{3.22}$$

We now view $(0,X_1,\dots,X_k)$, $1 \le k \le \ell$, as a random index in $\mathcal{I}_k$ and define the stochastic process

$$Y_k \stackrel{\text{def}}{=} \mathrm{cap}_{(0,X_1,\dots,X_k)}\; 1 \le k \le \ell\,. \tag{3.23}$$

We can now reexpress (3.5), (3.8), (3.13) as:

$$0 \le Y_k \le c_6,\quad 1 \le k \le \ell\,, \tag{3.24}$$

$$Y_{k+1} \le c_7\,Y_k,\quad 1 \le k < \ell\,, \tag{3.25}$$

$$Y_k \ge c_8\,E[Y_{k+1} \wedge \delta_1 \,|\, \mathcal{G}_k]\,,\quad 1 \le k < \ell\,. \tag{3.26}$$

We now define

$$\delta_0 = \frac{1}{2}\,\delta_1\,c_7^{-1}(<\delta_1\,c_7^{-1} < \delta_1)\,. \tag{3.27}$$

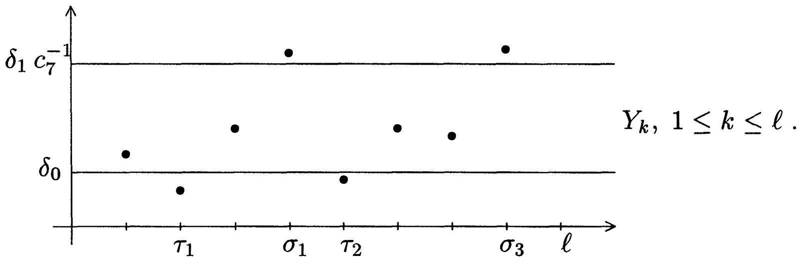
We shall construct a supermartingale based on the $Y_k$, $1 \le k \le \ell$, which will produce our claim (3.4). We introduce the excursions of $Y_.$ below level $\delta_0$ and above level $\delta_1\,c_7^{-1}$ as follows:

$$\begin{aligned} \tau_1 &= \inf\{k \ge 1,\ Y_k \le \delta_0\} \wedge \ell \\ \sigma_1 &= \inf\{k \ge \tau_1,\ Y_k \ge \delta_1\,c_7^{-1}\} \wedge \ell \ \text{ and for } i \ge 2: \\ \tau_i &= \inf\{k \ge \sigma_{i-1},\ Y_k \le \delta_0\} \wedge \ell \\ \sigma_i & \quad \inf\{k \ge \tau_i,\ Y_k \ge \delta_1\,c_7^{-1}\} \wedge \ell\,. \end{aligned} \tag{3.28}$$

Then $\tau_i, \sigma_i$ are $(\mathcal{G}_k)$-stopping times and

$$1 \le \tau_1 \le \sigma_1 \le \ldots \le \sigma_i \le \tau_i \le \ldots \le \ell\,,$$

and these inequalities except may be for the first one are strict provided the l.h.s. is strictly smaller than $\ell$.




**Lemma 3.4**:

(3.29) *For each $i \ge 1$, $U_n^i \overset{\text{def}}{=} c_8^{n\wedge(\sigma_i-\tau_i)}\,Y_{(\tau_i+n)\wedge\sigma_i}$, $n \ge 0$, is a $(\mathcal{G}_{\tau_i+n})_{n\ge 0}$ supermartingale*.

**Proof**: Observe that $U_n^i$, $n \ge 0$, is $\mathcal{G}_{\tau_i+n}$ adapted since $n \wedge (\sigma_i - \tau_i) = (n+\tau_i) \wedge \sigma_i - \tau_i$ is $\mathcal{G}_{\tau_i+n}$ measurable and $Y_{(\tau_i+n)\wedge\sigma_i}$ is $\mathcal{G}_{(\tau_i+n)\wedge\sigma_i} \subset \mathcal{G}_{\tau_i+n}$ measurable.

Moreover for $n \ge 0$:

$$\begin{aligned} E\,[U_{n+1}^i \,|\, \mathcal{G}_{\tau_i+n}] &= E\,[c_8^{n+1}\,Y_{\tau_i+n+1}\,,\ \tau_i + n < \sigma_i \,|\, \mathcal{G}_{\tau_i+n}] \\ &+ E\,[c_8^{n\wedge(\sigma_i-\tau_i)}\,Y_{(\tau_i+n)\wedge\sigma_i}\,,\ \tau_i + n \ge \sigma_i \,|\, \mathcal{G}_{\tau_i+n}] \end{aligned}$$

and the last term equals $1\{\sigma_i \le \tau_i + n\}\, U_n^i$. On the other hand on $\{\tau_i + n < \sigma_i\}$:

$$Y_{\tau_i+n+1} \overset{(3.25)}{\le} c_7\, Y_{\tau_i+n} \le c_7\, \frac{\delta_1}{c_7} = \delta_1\,.$$

Consequently:

$$\begin{aligned}
&E\,[c_8^{n+1}\, Y_{\tau_i+n+1},\ \tau_i + n < \sigma_i \,|\, \mathcal{G}_{\tau_i+n}] = \\
&c_8^n\, E\,[c_8(Y_{\tau_i+n+1} \wedge \delta_1),\ \tau_i + n < \sigma_i \,|\, \mathcal{G}_{\tau_i+n}] \\
&\overset{(3.26)}{\le}\ c_8^n\, Y_{\tau_i+n} \cdot 1\,\{\tau_i + n < \sigma_i\} = U_n^i\, 1\{\sigma_i > \tau_i + n\}\,.
\end{aligned}$$

Collecting, we now see that:

$$E[U_{n+1}^i \,|\, \mathcal{G}_{\tau_i+n}] \le U_n^i,\ n \ge 0\,.$$

□

Picking $n = \ell$, we see that:

$$E\,[c_8^{\sigma_i-\tau_i}\, Y_{\sigma_i} \,|\, \mathcal{G}_{\tau_i}] \le Y_{\tau_i}\,,\quad \text{for } i \ge 1\,. \tag{3.30}$$

Using the convention $\frac{Y_{\sigma_i}}{Y_{\tau_i}} = 0$ on $\{Y_{\tau_i} = 0\}$, we see by repeated use of (3.30) that

$$E\Big[\prod_{i=1}^{\ell} c_8^{\sigma_i-\tau_i}\, \frac{Y_{\sigma_i}}{Y_{\tau_i}}\ \Big|\ \mathcal{G}_{\tau_1}\Big] \le 1\,. \tag{3.31}$$

We shall now derive a lower bound for $H$ on the set $\bigcap_{i\ge1} \{Y_{\tau_i} > 0\} = \{Y_\ell > 0\}$, with

$$H \overset{\text{def}}{=} \prod_{i\ge1} \frac{Y_{\sigma_i}}{Y_{\tau_i}} = \frac{1}{Y_{\tau_1}}\, \frac{Y_{\sigma_1}}{Y_{\tau_2}} \cdots \frac{Y_{\sigma_{M-1}}}{Y_{\tau_M}}\, Y_{\sigma_M}\,, \tag{3.32}$$

$M = \inf\{i \ge 1,\ \tau_i \ge \ell\}$, and $\sigma_M = \ell = \tau_M$. If $M = 1$, then $H = \frac{Y_\ell}{Y_{\tau_1}}$ $(= 1)$.

Otherwise when $M \ge 2$, either

- $\sigma_{M-1} < \ell \le \tau_M$ (in fact $\tau_M = \ell$), so that:

$$Y_{\sigma_{M-1}} \ge \frac{\delta_1}{c_7}\,,\quad Y_\ell = Y_{\tau_M} \le c_6\,,$$

and $Y_{\sigma_i} \ge Y_{\tau_{i+1}} \ge 1$ for $1 \le i < M - 1$.

As a consequence:

$$H \ge \frac{Y_\ell}{Y_{\tau_1}}\, \frac{\delta_1}{c_7\, c_6}\,.$$

- Or $\tau_{M-1} < \ell \le \sigma_{M-1}$, so that

$$H \ge \frac{Y_\ell}{Y_{\tau_1}} \,.$$

If we now define

$$c_9 = \Big(\frac{\delta_1}{c_6 \cdot c_7}\Big) \wedge 1 \,, \tag{3.33}$$

inserting the above lower bounds on $H$ in (3.31), we find

$$c_9 \, E\Big[c_8^{\Sigma_i \sigma_i - \tau_i} \, \frac{Y_\ell}{Y_{\tau_1}} \,, \; Y_\ell > 0 \Big| \mathcal{G}_{\tau_1}\Big] \le 1 \,.$$

Now since $Y_\ell = 0$ on $\{Y_{\tau_1} = 0\}$, see (3.25), multiplying the above by $Y_{\tau_1}$ and integrating we obtain:

$$E\Big[c_8^{\Sigma_i \sigma_i - \tau_i} \; Y_\ell\Big] \le \frac{1}{c_9} \; E[Y_{\tau_1}] \overset{(3.24)}{\le} \frac{c_6}{c_9} \,. \tag{3.34}$$

Assume now that $\delta < \delta_0(d, L)$.

For $k \in [1, \tau_1 - 1] \cup \bigcup_{i \ge 2} [\sigma_{i-1}, \tau_i - 1]$, we have $Y_k > \delta_0$. Moreover

$$\tau_1 - 1 + \sum_{i \ge 2} (\tau_i - \sigma_{i-1}) + 1 + \sum_{i \ge 1} \sigma_i - \tau_i = \ell \,. \tag{3.35}$$

Therefore on the event

$$A \overset{\text{def}}{=} \{(0, X_1, \dots, X_\ell) \;\text{ is a rarefaction index}\} \,, \tag{3.36}$$

we find

$$\delta \ell \ge \sum_1^\ell Y_k > \delta_0 \Big(\tau_1 - 1 + \sum_{i \ge 2} \tau_i - \sigma_{i-1}\Big) \,,$$

and thus from (3.35):

$$1 + \sum_{i \ge 1} \sigma_i - \tau_i \ge \Big(1 - \frac{\delta}{\delta_0}\Big) \, \ell, \;\text{ on } A \,. \tag{3.37}$$

Coming back to (3.34) we see that:

$$\frac{1}{L^{d \cdot \ell}} \sum_{\substack{m \succ 0 \\ m:\, \text{rarefaction index}}} \text{cap}_m = E[Y_\ell \, 1_A] \le \frac{c_6}{c_9} \, c_8^{1 - (1 - \delta/\delta_0)\ell} \,. \tag{3.38}$$

This precisely proves our claim (3.4). □

It should be observed that the pair made of Lemma 2.1 and Theorem 3.1 can be viewed as a type of quantitative uniform Wiener test and Kellogg-Evans theorem. In Theorem 2.4.7 we have characterized the regular points of a compact set as the points where the Wiener test holds, and we have seen in Proposition 2.4.5 that the irregular points, where the Wiener test fails, are exceptional, (Kellogg-Evans theorem). Here similar roles are played by Lemma 2.1 (see also 1) of Remark 2.2) of the previous section and the above Theorem 3.1.

Let us now come back to our specific situation, where $K_m$ are defined as in (2.9). It would be possible at this point to derive volume controls on the set of rarefaction boxes which receive a point of $\omega$. Let us introduce for $d \geq 2$

$$(3.39) \qquad \begin{aligned} c_{10}(d) = \ & \sup \Big\{ \frac{|K|}{\mathrm{cap}(K)} \,, \ K \text{ compact}, \\ & K \subset \Big[ -\frac{1}{4}, \frac{5}{4} \Big]^d, \ |K| > 0 \Big\} \overset{(2.4.61)}{<} \infty \,. \end{aligned}$$

We see that under the conditions of Theorem 3.1, summing the inequalities (3.4) for all $m_0 \in \mathcal{I}_{n_\alpha}$ with $m_0 \succ q$, where $q \in \mathbb{Z}^d$ is fixed:

$$(3.40) \qquad \begin{aligned} & c_{10}\, c_4 \, \exp\Big\{ -c_5 \Big(1 - \frac{\delta}{\delta_0}\Big) \, \ell(\epsilon) \Big\} \geq \\ & \frac{1}{L^{d \cdot n_\gamma}} \sum_{\substack{m \succ q \\ m:\ \text{rarefaction index}}} |K_m| \,, \ \text{and} \\ & \frac{1}{L^{d \cdot n_\gamma}} \, |K_m| \overset{(2.9)}{=} \Big| \bigcup_{x_i \in C_m} \overline{B}(x_i, a\epsilon) \Big| \geq |\overline{B}(0, a\epsilon)| \end{aligned}$$

if $C_m$ receives at least a point of $\omega$. Moreover

$$|\overline{B}(0, a\epsilon)| \overset{(2.7)}{\geq} |C_m| \times \epsilon^{d(1-\gamma)} \, \frac{|B(0,a)|}{L^d} \,, \quad \text{and } \ \ell(\epsilon) \underset{\epsilon \to 0}{\sim} \frac{\gamma - \alpha}{\log L} \, \log \frac{1}{\epsilon} \,.$$

We thus see that (3.40) shows the 'smallness' of

$$\Big| \bigcup_{\substack{m \succ q \\ C_m:\ \text{rarefaction box}, \omega(C_m) \neq 0}} C_m \Big| \,, \quad \text{as } \epsilon \to 0 \,,$$

provided:

$$(3.41) \qquad c_5 \Big(1 - \frac{\delta}{\delta_0}\Big) \, \frac{(\gamma - \alpha)}{\log L} > d(1 - \gamma) \,.$$

It is then tempting to define bad boxes as the rarefaction boxes which receive a point of $\omega$. However using a better definition of 'bad boxes', it is possible to derive 'Theorem B' (see Section 4 §1)) under conditions where $d$ is essentially replaced by $(d-2)$ in (3.41).

To this end, we consider $\beta$ such that:

$$0 < \alpha < \gamma < \beta < 1\,, \tag{3.42}$$

and introduce for $\epsilon \in (0,1)$, as in (2.6),

$$n_\beta(\epsilon) = \left[\beta\,\frac{\log 1/\epsilon}{\log L}\right], \quad \text{so that} \tag{3.43}$$

$$L^{-n_\beta(\epsilon)-1} < \epsilon^\beta \le L^{-n_\beta(\epsilon)}\,. \tag{3.44}$$

A box $C_m$, with $m \in \mathcal{I}_{n_\beta(\epsilon)}$ will be called a *bad box* if:

$$C_m \text{ is contained in a rarefaction box (i.e. } C_{[m]_{n_\gamma}}) \tag{3.45}$$

$$\omega(C_m) \neq 0\,. \tag{3.46}$$

Accordingly for $\epsilon \in (0,1)$ and $\omega \in \Omega$, we define the *bad set*

$$\mathcal{B}_\epsilon(\omega) = \bigcup_{m \in \mathcal{I}_{n_\beta(\epsilon)},\, C_m \text{is a bad box}} C_m\,. \tag{3.47}$$

In the case of dimension 1, as soon as:

$$\delta < \mathrm{cap}\{0\}\,, \tag{3.48}$$

any rarefaction box receives no point of $\omega$ and therefore $\mathcal{B}_\epsilon(\omega) = \emptyset$.

In the case of dimension $d \ge 2$, the next theorem is a type of 'solidification estimate'. It shows for instance that when $d = 2$, for small $\epsilon$ replacing $K_m = L^{n_\gamma}(\bigcup_{x_i \in C_m} \overline{B}(x_i, a\epsilon))$ by

$$L^{n_\gamma}\Big(\overline{\bigcup_{\substack{m' \succ m \\ C_{m'}:\ \text{bad boxes}}} C_{m'}}\Big)$$

when $m \in \mathcal{I}_{n_\gamma}$, essentially does not increase capacity.

**Theorem 3.5**: $(d \ge 2)$

$$\overline{\lim_{\epsilon \to 0}}\ \epsilon^{(d-2)(1-\beta)} \sup_{\omega \in \Omega, m \in \mathcal{I}_{n_\gamma(\epsilon)}} \frac{\mathrm{cap}(L^{n_\gamma}(\overline{\mathcal{B}_\epsilon(\omega) \cap C_m}))}{\mathrm{cap}_m} < \infty \tag{3.49}$$

*(when* $\mathrm{cap}_m = 0$ *and therefore* $\mathcal{B}_\epsilon(\omega) \cap C_m = \emptyset$*, the above fraction is understood as equal to* 0*).*

**Proof**: We assume that $\epsilon$ is small enough so that

$$4a\epsilon < L^{-n_\beta(\epsilon)} \,. \tag{3.50}$$

We consider $\omega \in \Omega$ and $m_0 \in \mathcal{I}_{n_\gamma(\epsilon)}$ such that $\mathcal{B}_\epsilon(\omega) \cap C_{m_0} \neq \emptyset$. Denote by $\mathcal{P}_{m_0}$ some maximal collection of indices $m \in \mathcal{I}_{n_\beta(\epsilon)}$, $m \succ m_0$, of bad subboxes of $C_{m_0}$, such that any two distinct subboxes with labels in $\mathcal{P}_{m_0}$ are not neighbors. Define

$$\mathcal{B}_{m_0} = \overline{\mathcal{B}_\epsilon(\omega) \cap C_{m_0}} \text{ and } \mathcal{B}'_{m_0} = \bigcup_{m \in \mathcal{P}_{m_0}} \overline{C}_m \,. \tag{3.51}$$

Since $\mathcal{P}_{m_0}$ is maximal, $\mathcal{B}_{m_0}$ is included in the union of $3^d$ translates of $\mathcal{B}'_{m_0}$. It follows from (2.4.17) that:

$$\mathrm{cap}(L^{n_\gamma}\, \mathcal{B}_{m_0}) \le 3^d \,\mathrm{cap}(L^{n_\gamma}\, \mathcal{B}'_{m_0}) \,. \tag{3.52}$$

Let $\mu$ stand for the equilibrium measure of $L^{n_\gamma}\, \mathcal{B}'_{m_0}$, and define for $m \in \mathcal{P}_{m_0}$:

$$\begin{aligned} \mu_m &= 1_{L^{n_\gamma}\overline{C}_m} \cdot \mu, \text{ so that :} \\ \mu &= \sum_{m \in \mathcal{P}_{m_0}} \mu_m \,. \end{aligned} \tag{3.53}$$

When $m \in \mathcal{P}_{m_0}$ and $x \in L^{n_\gamma}\, \overline{C}_m$

$$\int g(x,y)\, \mu_m(dy) \le 1 \,,$$

and by Proposition 2.4.2:

$$\begin{aligned} \mu_m(L^{n_\gamma}\, \overline{C}_m) &\le \mathrm{cap}(L^{n_\gamma}\, \overline{C}_m) = \mathrm{cap}(L^{n_\gamma - n_\beta}[0,1]^d) \\ &= L^{-(d-2)(n_\gamma - n_\beta)}\, \mathrm{cap}([0,1]^d), \text{ when } d \ge 3 \,, \\ &\overset{(2.4.41)}{\le} \frac{c(L)}{n_\beta(\epsilon) - n_\gamma(\epsilon)} \,, \text{ when } d = 2, \text{ and } \epsilon \text{ is small} \,. \end{aligned} \tag{3.54}$$

Choose for each $m \in \mathcal{P}_{m_0}$ a point $x_m \in \mathrm{supp}\, \omega \cap C_m$, and define:

$$\nu = \sum_{m \in \mathcal{P}_{m_0}} \mu_m(1)\, \overline{e}_m \,, \tag{3.55}$$

where $\overline{e}_m$ stands for the normalized equilibrium measure of $L^{n_\gamma}\, \overline{B}(x_m, a\epsilon)$. Observe that

$$\mathrm{supp}\, \nu \subseteq L^{n_\gamma}\Big( \bigcup_{m \in \mathcal{P}_{m_0}} \overline{B}(x_m, a\epsilon)\Big) \subseteq K_{m_0} \,. \tag{3.56}$$

Moreover, when $x \in L^{n_\gamma}\,\overline{B}(x_m, a\epsilon)$, $m \in \mathcal{P}_{m_0}$,

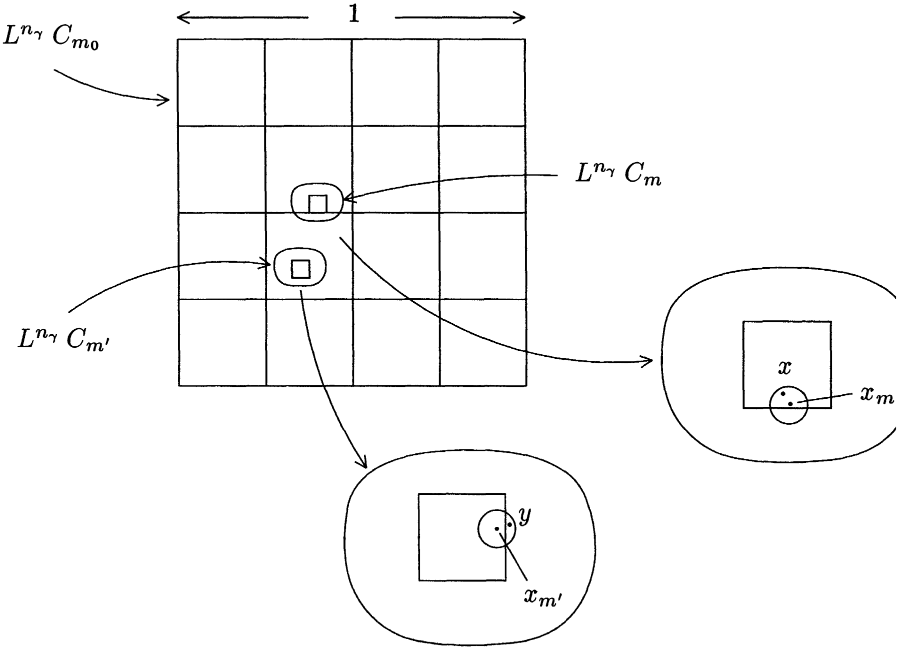
$$\begin{aligned}\int g(x,y)\,\nu(dy) = & \frac{\mu_m(1)}{\mathrm{cap}(L^{n_\gamma}\,\overline{B}(0,a\epsilon))} \\ & + \sum_{\substack{m' \neq m \\ m' \in \mathcal{P}_{m_0}}} \mu_{m'}(1) \int g(x,y)\,\bar{e}_m(dy)\,. \end{aligned} \tag{3.57}$$

Now when $x \in L^{n_\gamma}\,\overline{B}(x_m, a\epsilon)$, $y \in L^{n_\gamma}\,\overline{B}(x_{m'}, a\epsilon)$, with $m \neq m'$ in $\mathcal{P}_{m_0}$, then $x_m \in C_m$ and $x_{m'} \in C_{m'}$, where $C_m$ and $C_{m'}$ are nonneighboring boxes of size $L^{-n_\beta}$. If $y'$ is some point in $L^{n_\gamma}\,\overline{C}_{m'}$, it follows from (3.50) that

$$|x - y'| \leq |x-y| + |y-y'| \leq |x-y| + \left(\sqrt{d} + \frac{1}{4}\right) L^{n_\gamma - n_\beta}$$

Moreover $|x-y| \overset{(3.50)}{\geq} \frac{1}{2}\, L^{n_\gamma - n_\beta}$, and therefore

$$|x-y'| \leq |x-y| \Big(1 + 2\Big(\sqrt{d} + \frac{1}{4}\Big)\Big)\,. \tag{3.58}$$




It now follows (as in (3.6), (3.7)), that for a suitable constant $\gamma(d) > 0$,

$$g(x,y) \leq \gamma(d)\; g(x,y')\,. \tag{3.59}$$

We thus find that for $x \in L^{n_\gamma}\,\overline{B}(x_m, a\epsilon)$

$$\begin{aligned}
\int g(x,y)\,\nu(dy) &\le \frac{\mu_m(1)}{\mathrm{cap}(L^{n_\gamma}\,\overline{B}(0,a\epsilon))} + \\
&\sum_{\substack{m'\neq m \\ m'\in\mathcal{P}_{m_0}}} \mu_{m'}(1) \int \gamma(d)\, g(x,y')\,\frac{\mu_{m'}(dy')}{\mu_{m'}(1)} \\
&\le \frac{\mu_m(1)}{\mathrm{cap}(L^{n_\gamma}\,\overline{B}(0,a\epsilon))} + \gamma(d)\int g(x,y')\,\mu(dy') \\
&\le \frac{\mu_m(1)}{\mathrm{cap}(L^{n_\gamma}\,\overline{B}(0,a\epsilon))} + \gamma(d) \overset{(3.54)}{\le} \frac{\mathrm{cap}(L^{n_\gamma - n_\beta}[0,1]^d)}{\mathrm{cap}(L^{n_\gamma}\,\overline{B}(0,a\epsilon))} + \gamma(d) \overset{\text{def}}{=} A\,.
\end{aligned} \tag{3.60}$$

In view of (3.56) and Proposition 2.4.2:

$$\frac{\nu(1)}{A} = \frac{\mu(1)}{A} \le \mathrm{cap}\,K_{m_0} = \mathrm{cap}_{m_0}\,. \tag{3.61}$$

We thus see that:

$$\frac{1}{3^d}\,\mathrm{cap}(L^{n_\gamma}\mathcal{B}_{m_0}) \overset{(3.52)}{\le} \mathrm{cap}(L^{n_\gamma}\,\mathcal{B}'_{m_0}) = \mu(1) \le \mathrm{cap}_{m_0}\cdot A\,. \tag{3.62}$$

Finally from the definition of the constant $A$, we have:

$$\begin{aligned}
A &\le \mathrm{const}(d,a,L)\epsilon^{-(d-2)(1-\beta)} + \gamma(d), \text{ when } d \ge 3, \\
&\quad 2\,\frac{1-\gamma}{\beta-\gamma} + \gamma(d), \quad \text{when } d=2 \text{ and } \epsilon \text{ is small enough}\,.
\end{aligned} \tag{3.63}$$

Our claim (3.49) now follows. □

It is now convenient to define the following constant when $d \ge 2$:

$$\begin{aligned}
\kappa_0 &= \Big(1-\frac{\delta}{\delta_0}\Big)\,(\gamma-\alpha)\Big(2-\frac{\log(3^d+1)}{\log L}\Big) - (d-2)(1-\beta), \\
&\quad \text{when } d \ge 3, \\
&\quad \Big(1-\frac{\delta}{\delta_0}\Big)\,(\gamma-\alpha)\Big(2-\frac{\log[10\,c(d=2)\log L]}{\log L}\Big), \\
&\quad \text{when } d = 2\,.
\end{aligned} \tag{3.64}$$

By convention we set $\kappa_0 = \infty$, when $d = 1$.

The role of $\kappa_0$ comes from the identity ($d \ge 2$):

$$\kappa_0 = \lim_{\epsilon\to 0}\Big\{c_5\Big(1-\frac{\delta}{\delta_0}\Big)\,\ell(\epsilon) - (d-2)(1-\beta)\,\log\frac{1}{\epsilon}\Big\}\Big/\log\frac{1}{\epsilon}\,.$$

The main object of this section the following precise version of Theorem B of §1:

**Theorem 3.6**: *Assume $d \geq 2$, $L$ large enough so that $c_8(d,L) > 1$, and $\delta < \delta_0(d,L)$, then*

$$\varlimsup_{\epsilon \to 0} \sup_{q \in \mathbb{Z}^d, \omega \in \Omega} \epsilon^{-\kappa_0} \, |\mathcal{B}_\epsilon(\omega) \cap C_q| < \infty \,. \tag{3.65}$$

**Proof:**

$$|\mathcal{B}_\epsilon(\omega) \cap C_q| = \frac{1}{L^{dn_\gamma}} \sum_{\substack{m \succ q, m \in \mathcal{I}_{n_\gamma} \\ m:\ \text{rarefaction index}}} |L^{n_\gamma}(\mathcal{B}_\epsilon(\omega) \cap C_m)|$$

$$\overset{(3.39)}{\leq} c_{10}(d) \sup_{\omega \in \Omega, m \in \mathcal{I}_{n_\gamma}} \frac{\operatorname{cap}(L^{n_\gamma}(\mathcal{B}_\epsilon(\omega) \cap C_m))}{\operatorname{cap}_m} \cdot \frac{1}{L^{dn_\gamma}} \sum_{\substack{m \succ q, m \in \mathcal{I}_{n_\gamma} \\ m:\ \text{rarefaction index}}} \operatorname{cap}_m$$

$$\overset{(3.4)(3.49)}{\leq} \text{const}\ \epsilon^{-(d-2)(1-\beta)} \exp\Big\{ - c_5 \Big(1 - \frac{\delta}{\delta_0}\Big)\, \ell(\epsilon) \Big\} \,.$$

Our claim (3.65) now comes from the definition of $\kappa_0$ together with the fact that

$$\Big| \ell(\epsilon) - \frac{(\gamma - \alpha)}{\log L} \log \frac{1}{\epsilon} \Big| \leq 2 \,.$$

□

If one picks $\beta \in (\gamma, 1)$ close to $\gamma$, one sees that in the formula for $\kappa_0$, the term $-(d-2)(1-\beta)$, close to $-(d-2)(1-\gamma)$, improves on the term $-d(1-\gamma)$ which would show up if one directly performed volume estimates on the 'naive definition of the bad set' as explained in (3.40), (3.41). This motivates our introduction of the new scale $L^{-n_\beta(\epsilon)} \approx \epsilon^\beta$ to define the bad set.

In applications of the method an *admissible collection of parameters* will refer to a collection $\alpha, \gamma, \beta, L, \delta, \rho, \kappa$, such that

$$\begin{aligned}
&0 < \alpha < \gamma < \beta < 1 \,, \\
&L \geq 2, \text{with } L \text{ such that } c_8(d,L) > 1, \text{ when } d \geq 2, \\
&\delta > 0, \text{ with } \delta < \delta_0(d,L), \text{ if } d \geq 2, \text{ and } \delta < \operatorname{cap}\{0\}, \text{ if } d = 1 \,, \\
&\rho \in \Big(0,\ \rho_0 = \delta\, c_1 \frac{(\gamma - \alpha)}{(d+2)\log L}\Big) \\
&\kappa_0 > 0, \ \text{ and } \ \kappa \in (0, \kappa_0) \,.
\end{aligned} \tag{3.66}$$

With such a collection of parameters, our requirements (1.8), (1.9), (1.10) hold and we can apply our results of Section 2 and 3.

It may be useful to provide here some comments on the nature of (3.66). The asymptotic scale $\epsilon^\beta$ is that of finest details of the coarse grained picture, $\epsilon$ is the asymptotic scale of the true obstacles, and the unit scale corresponds to the size of 'clearings' in which we are interested. When $d \geq 3$, as soon as $1 > \beta > \frac{d-2}{d}$, it is possible to imbed $\beta$ into an admissible collection of parameters, as can be seen from (3.64), (3.66). The intuitive meaning of this is that as $\epsilon \to 0$,

$$\epsilon^{d\beta} << \epsilon^{d-2} ,$$

i.e. 'when measured in the scale of clearings under consideration, the ratio of the capacity of the true obstacles to the volume of details of the coarse grained picture tends to infinity'. In agreement with this heuristics principle when $d = 1, 2$, any $\beta \in (0,1)$ can be imbedded into an admissible collection of parameters. This principle will play a role at the end of Section 5 in the context of slightly rarefied traps, see also [Szn90b].

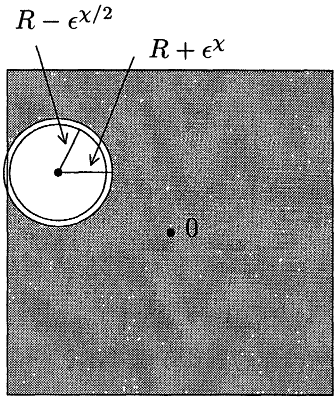
Let us finally mention that in the case of hard obstacles, in view of Remark 2.2, one can choose collection of parameters which are simultaneously admissible for any hard obstacle, (the constant $c_1$ solely depends on $d$).

## 4.4 Principal Eigenvalues and Poissonian Obstacles

We shall discuss in this section several applications of the method of enlargement of obstacles to the study of principal eigenvalues of the Laplacian in the presence of Poissonian obstacles.

### A) Geometric Controls

We consider the situation of Example 1.1 in the $d = 2$ dimensional case. As we already observed, the occurrence of a disc of radius $R$ within $(-1,1)^2$ receiving no obstacle ensures that $\lambda_\omega^\epsilon((-1,1)^2) \leq \lambda_2/R^2$. We now want to show that when $\epsilon$ is small, conditional on the 'deviant event' $A_\epsilon \overset{(1.14)}{=} \{\lambda_\omega^\epsilon((-1,1)^2) \leq c\}$ an empty disc of radius $\approx R$ where $c \overset{(1.17)}{=} \lambda_2/R^2$ occurs with high probability. The intuitive idea behind this, is that for a given fundamental Dirichlet tone, discs achieve minimal area among open sets, (Faber-Krahn's inequality, when $d = 2$). Thus the occurrence of an empty disc within $(-1,1)^2$ should be the 'minimal cost solution' to 'produce the event $A_\epsilon$'. As we shall now see, the method of enlargement of obstacles as well as quantitative two dimensional isoperimetric controls known as Bonnesen inequalities, see Burago-Zalgaller [BZ88] or Osserman [Oss79], give us a way to make this heuristics precise.




**Theorem 4.1**: *Consider $c > \lambda_2$, $R \in (0,1)$ with $c = \lambda_2 \,/\, R^2$. There exists a numerical constant $\chi \in (0,1)$ such that for any $\mu > 0$,*

$$\lim_{\epsilon \to 0} \mathbb{P}_\epsilon[E^\mu / \lambda^\epsilon_\omega((-1,1)^2) \le c] = 1\,, \quad \textit{where} \tag{4.1}$$

$$\begin{aligned} E^\mu \;&=\; \bigcup_{q \in \mathbb{Z}^2} \{\lambda^\epsilon_\omega((-1,1)^2 \cap B(q\epsilon^\chi, R + \epsilon^\chi)) \\ &\le\; c + \epsilon^\mu,\ \omega(B(q\epsilon^\chi,\ R - \epsilon^{\chi/2})) = 0\} \end{aligned} \tag{4.2}$$

**Proof**: Consider as in (3.66) an admissible choice of parameters with the additional condition $\frac{1}{2} < \beta$, (this choice does not depend on $a$ or $\nu$). From (1.19) we know that for small $\epsilon$:

$$\mathbb{P}_\epsilon[A_\epsilon] \ge \exp\Big\{ - \frac{\nu}{\epsilon^2}\, \pi R^2 - \frac{\nu}{\epsilon}\, 2\pi\, Ra - \nu\pi a^2 \Big\}\,. \tag{4.3}$$

Consider $\beta' \in (\beta, 1)$, and define the event:

$$G = \{\omega \in \Omega,\ |(-1,1)^2 \backslash (\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega))| \le \pi R^2 + \epsilon^{2(1-\beta')}\}\,. \tag{4.4}$$

For small $\epsilon$,

$$\begin{aligned} \mathbb{P}_\epsilon[G^c] \;&\overset{(1.9)-(1.10)}{\le}\; 2^{4 \times 2\epsilon^{-2\beta}} \exp\Big\{ - \frac{\nu}{\epsilon^2}\, (\pi R^2 + \epsilon^{2(1-\beta')}) \Big\} \\ &\le\; \exp\Big\{ - \frac{\nu}{\epsilon^2}\, \pi R^2 - \frac{1}{2}\, \epsilon^{-2\beta'} \Big\}\,, \end{aligned} \tag{4.5}$$

and since $\frac{1}{2} < \beta < \beta' < 1$, it follows from (4.3) that for small $\epsilon$:

$$\mathbb{P}_\epsilon\,[G | A_\epsilon] \ge 1 - \exp\{-\epsilon^{-2\beta}\}\,. \tag{4.6}$$

Our claim (4.1) will thus be shown once we prove that we can choose $\chi > 0$, (independent of $a$ and $\nu$), such that

$$\text{for any } \mu > 1, \text{ for small } \epsilon, \ G \cap A_\epsilon \subseteq E^\mu \ . \tag{4.7}$$

For small $\epsilon$, and any $\omega \in G \cap A_\epsilon$, it follows from Theorem 3.6 and (4.4) that:

$$|(-1,1)^2 \backslash \overline{\mathcal{D}}_\epsilon(\omega)| \le \pi R^2 + \epsilon^{2(1-\beta')} + \epsilon^\kappa \ , \tag{4.8}$$

and choosing $M = 2c$ in Theorem 2.3 that

$$c + \epsilon^\rho \ge \lambda_\omega^\epsilon((-1,1)^2) + \epsilon^\rho \ge \lambda((-1,1)^2 \backslash \overline{\mathcal{D}}_\epsilon(\omega)) \ . \tag{4.9}$$

Faber-Krahn's inequality says that $\lambda(U) \ge \lambda(B)$, if $B$ is a disc such that $|U| = |B|$. In this light (4.8), (4.9) show that $\lambda((-1,1)^2 \backslash \overline{\mathcal{D}}_\epsilon(\omega))$ is close to the fundamental tone of a disc with area $|(-1,1)^2 \backslash \mathcal{D}_\epsilon(\omega)|$. Using now a refinement of Faber-Krahn's inequality, which is based on Bonnesen's isoperimetric inequality (see Theorem 4.1 of Melas [Mel92] or the appendix of [Szn97b]), we can find $0 < \chi'(\alpha, \beta', \kappa, \rho) < \alpha$, and a closed disc $D'$ with radius

$$R'_\epsilon = R + \epsilon^{\chi'}, \quad \text{such that} \tag{4.10}$$

$$|(-1,1)^2 \backslash (\overline{\mathcal{D}}_\epsilon(\omega) \cup D')| \le \epsilon^{2\chi'} \ . \tag{4.11}$$

We can now apply Theorem 2.6, in the same setting as Example 2.5 2). That is we choose $\mathcal{A} = D'$, $r = 2\epsilon^{\chi'}$, $U_2 = (-1,1)^2$, $\mathcal{O}$ the open disc concentric to $D'$ with radius:

$$R_\epsilon = R + \epsilon^{\chi''}, \quad \text{where } \ 0 < \chi'' < \chi' \ . \tag{4.12}$$

It now follows from (2.67) that for $\mu > 1$, when $\epsilon$ is small for any $\omega \in G \cap A_\epsilon$:

$$\lambda_\omega^\epsilon((-1,1)^2 \cap \mathcal{O}) \le \lambda_\omega^\epsilon((-1,1)^2) + \epsilon^\mu \le \frac{\lambda_2}{R^2} + \epsilon^\mu \ . \tag{4.13}$$

On the other hand, we can apply the lower bound on the shift of principal eigenvalue (3.2.8), as well as a lower bound on the capacity of a disc $B(x, a\epsilon) \subset \mathcal{O} \subset B(x, 2R)$, see (2.4.47). It follows that when $\chi''' < \chi''$, and small $\epsilon$:

$$\lambda_\omega^\epsilon((-1,1)^2 \cap \mathcal{O}) \ge \lambda_\omega^\epsilon(\mathcal{O}) > \frac{\lambda_2}{R^2} + \epsilon^\mu \tag{4.14}$$

whenever a point of $\omega$ falls within distance $R - \epsilon^{\chi'''/2}$ from the center of $\mathcal{O}$. Combining this and (4.13), we see that when $\epsilon$ is small for any $\omega \in G \cap A_\epsilon$ the concentric disc to $\mathcal{O}$ with radius $R - \epsilon^{\chi'''/2}$ receives no point of $\omega$.

Finally if we choose $\chi < \chi'''$, and for small $\epsilon$ and $\omega \in G \cap A_\epsilon$ some point of the form $q\epsilon^\chi$, $q \in \mathbb{Z}^2$, within distance $\frac{1}{\sqrt{2}}\,\epsilon^\chi$ from the center of $\mathcal{O}$, we see that

$$B(q\epsilon^\chi,\, R+\epsilon^\chi) \supset \mathcal{O} \text{ and } \omega(B(q\epsilon^\chi,\, R-\epsilon^{\chi/2})) = 0\,,$$

so that combining with (4.13), (4.7) follows. This finishes our proof. □

## B) Confidence Intervals

The setting is similar to Example 3.3 of Chapter 3, although not restricted to dimension 1. We consider a Poisson point process of constant intensity $\nu > 0$, in $\mathbb{R}^d$, $d \geq 1$, and denote by $\mathbb{P}$ its law on $\Omega$. We want to derive confidence intervals on the random variable

$$(4.15) \qquad \lambda_\ell(\omega) \overset{\text{def}}{=} \lambda_{V(\cdot,\omega)}((-\ell,\ell)^d)\,,$$

for large $\ell$, where for $\omega = \sum_i \delta_{x_i} \in \Omega$, $x \in \mathbb{R}^d$

$$(4.16) \qquad V(x,\omega) = \sum_i W(x-x_i)\,,$$

and $W(\cdot)$, the model for the obstacles satisfies our standing assumptions of Section 1. The function $V$ defines the *soft Poissonian obstacles.*

We shall now use scaling to put ourselves in the setting of the previous sections. We choose as unit length $(\log \ell)^{1/d}$, which is the typical order of magnitude of the radius of the largest spherical hole of the Poisson configuration in $(-\ell,\ell)^d$, for large $\ell$. We thus define

$$(4.17) \qquad \epsilon = (\log \ell)^{-1/d}\,,$$

when $\ell$ is large enough so that $\epsilon \in (0,1)$, and

(4.18) $\lambda_\ell(\omega)$ has same distribution under $\mathbb{P}$, as $\epsilon^2 \lambda^\epsilon_\omega(\mathcal{T})$ under $\mathbb{P}_\epsilon$,

where $\mathbb{P}_\epsilon$ is the Poisson law with constant intensity $\nu\epsilon^{-d}$, and

$$(4.19) \qquad \mathcal{T} = \Big(-\frac{\ell}{(\log \ell)^{1/d}}\,,\ \frac{\ell}{(\log \ell)^{1/d}}\Big)^d = (-\epsilon\, e^{\epsilon^{-d}},\, \epsilon\, e^{\epsilon^{-d}})^d\,.$$

The derivation of confidence intervals on $\lambda_\epsilon(\omega)$ will involve two main steps.

In the first step, we shall see that $\lambda^\epsilon_\omega(\mathcal{T})$ is close to the minimum of the not too dependent variable $\lambda^\epsilon_\omega(\mathcal{T} \cap B)$, where $B$ runs over a collection of blocks of size much smaller than $\mathcal{T}$. This step involves the use of Theorem 2.6 in the context of Example 2.5. 1) and an estimate on the size of connected components of the set $\mathcal{O} \cap \mathcal{T}$. Estimates on the probability of occurrence of holes in the cloud configuration will provide the upper bounds we need.

In the second step, we shall derive lower bounds on the variables $\lambda_\omega^\epsilon(\mathcal{T} \cap B)$ by using 'Theorem A and B'.

We shall need the following constants:

$$c(d,\nu) = \lambda_d \left(\frac{d}{\nu\omega_d}\right)^{-2/d}, \quad R_0(d,\nu) = \left(\frac{d}{\nu\omega_d}\right)^{1/d}, \tag{4.20}$$

$$M = 2c(d,\nu) . \tag{4.21}$$

We specify the clearing boxes (see (2.57)) by a fixed choice of $r \in (0, r_0)$, such that (see (2.50))

$$\frac{c_2(d)}{r^2} > 4M . \tag{4.22}$$

The size of the neighborhood $\mathcal{O}$ of $\mathcal{A}$, the closure of the union of clearing boxes (see (2.64)), is chosen as $R(\ell, d, \nu) \geq 1$ the smallest positive integer for which (see Theorem 2.6):

$$\frac{R}{4r} > c_4(d, M) \text{ and } c_3(d) \left[\frac{R}{4r}\right] \geq 3 \log\log \ell . \tag{4.23}$$

Finally in order to apply the method of enlargement of obstacles we consider some admissible collection of parameters as in (3.66).

We first derive some estimates on the probability of occurrence of clearing boxes and on the size of connected components of $\mathcal{O}$ which intersect $\mathcal{T}$.

**Proposition 4.2**: *For small $\epsilon$, when $q \in \mathbb{Z}^d$,*

$$\mathbb{P}_\epsilon[C_q \textit{ is a clearing box}] \leq \exp\left\{ -\frac{\nu}{2}\, \epsilon^{-d} r^d \right\} . \tag{4.24}$$

*There exist $\gamma_1(d,\nu) > 0$, such that for small $\epsilon$,*

$$\mathbb{P}_\epsilon[C] \geq 1 - \exp\{-d\epsilon^{-d}\} = 1 - \ell^{-d} , \quad \textit{if} \tag{4.25}$$

$$\begin{aligned} C = \{\omega \in \Omega,\ &\textit{all connected components of } \mathcal{O} \textit{ intersecting } \mathcal{T} \\ &\textit{are contained in some } q + (0, [\gamma_1 \log \epsilon^{-d}])^d,\ q \in \mathbb{Z}^d\} . \end{aligned} \tag{4.26}$$

**Proof**:

- *Proof of (4.24)*: For small $\epsilon$ and arbitrary $q \in \mathbb{Z}^d$

$$\mathbb{P}_\epsilon[C_q \subseteq \mathcal{A}_\epsilon] = \mathbb{P}_\epsilon[|C_q \backslash \mathcal{D}_\epsilon(\omega)| \geq r^d] \overset{(3.65)-(3.66)}{\leq}$$
$$\mathbb{P}_\epsilon[|C_q \backslash (\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega))| \geq r^d - \epsilon^\kappa] \overset{(1.9)-(1.10)}{\leq}$$
$$\exp\{2\epsilon^{-d\beta} \log 2 - \nu\epsilon^{-d}(r^d - \epsilon^\kappa)\} \leq \exp\Big\{-\frac{\nu}{2}\,\epsilon^{-d}\,r^d\Big\}\,.$$

- *Proof of (4.25)*: Define

$$M_0 = 2\Big(\Big[\frac{4d}{\nu r^d}\Big] + 1\Big), \quad \mu = \Big(\frac{\nu r^d}{4d}\Big) \wedge \frac{1}{2}\,, \tag{4.27}$$

$$\mathcal{K} = \text{collection of blocks } q + [0,[\ell^\mu]]^d,\ q \in \mathbb{Z}^d, \text{ which intersect } \mathcal{T}\,. \tag{4.28}$$

We let $\widetilde{C}$ stand for the event:

$$\widetilde{C} = \{\omega \in \Omega, \text{all blocks in } \mathcal{K} \text{ contain at most } M_0 \text{ clearing boxes}\}\,. \tag{4.29}$$

For small $\epsilon$:

$$\begin{aligned} &\mathbb{P}_\epsilon[\widetilde{C}^c] \\ &\leq \quad \ell^d\, \mathbb{P}_\epsilon[\,[0,[\ell^\mu]]^d \text{ contains at least } M_0 \text{ clearing boxes}] \\ &\overset{\text{(independence)}}{\leq} \quad \ell^{d+d\mu M_0}\, \mathbb{P}_\epsilon[[0,1)^d \subseteq \mathcal{A}_\epsilon]^{M_0} \\ &\overset{(4.24)}{\leq} \quad \exp\Big\{\epsilon^{-d}\Big(d - M_0\Big(\frac{\nu}{2}\,r^d - d\mu\Big)\Big)\Big\} \\ &\overset{(4.27)}{\leq} \quad \exp\Big\{\epsilon^{-d}\Big(d - M_0\,\frac{\nu r^d}{4}\Big)\Big\} \overset{(4.27)}{\leq} \exp\{-d\epsilon^{-d}\} = \ell^{-d}\,. \end{aligned} \tag{4.30}$$

Our claim (4.25) will thus follow once we show that for large $\ell$:

$$\widetilde{C} \subseteq C\,. \tag{4.31}$$

Indeed consider some connected component $\mathcal{V}$ of $\mathcal{O}$ intersecting $\mathcal{T}$. Then $\mathcal{V}$ contains a clearing box $C_q$ within $\|\cdot\|$-distance $R$ from $\mathcal{T}$. Let $B_q$ be a block of $\mathcal{K}$ with middle point in $C_q$, and denote by $W_q$ the union of open $R$-neighborhoods in $\|\cdot\|$-distance of clearing boxes in $B_q$. When $\ell$ is large,

$$\mathcal{O} \cap \Big(q + \Big(-\frac{\ell^\mu}{4}, \frac{\ell^\mu}{4}\Big)^d\Big) \subseteq W_q\,. \tag{4.32}$$

However, on the event $\widetilde{C}$, the projection of $W_q$ on each coordinate axis has measure smaller than $(2R+1)\,M_0$.

Observe that $q \in \mathcal{V} \cap (q + (-\frac{\ell^\mu}{4}, \frac{\ell^\mu}{4})^d)$. Thus the connected component of $\mathcal{V} \cap (q + (-\frac{\ell^\mu}{4}, \frac{\ell^\mu}{4})^d)$ containing $q$ has $\|\cdot\|$-diameter smaller than $(2R+1)\, M_0 \le \frac{\ell^\mu}{4}$, ($\ell$ is large).

Thus $\mathcal{V} \subseteq q + (-\frac{\ell^\mu}{4}, \frac{\ell^\mu}{4})^d)$ and has $\|\cdot\|$-diameter smaller than $(2R+1)\, M_0 \le \gamma_1(d,\nu) \log\log \ell$. Our claim (4.25) follows. □

We denote by

$$\text{(4.33)} \qquad \begin{aligned} &\mathcal{C}_\ell = \text{the collection of blocks } q + (0, [\gamma_1 \log\log \ell])^d, \\ &q \in \mathbb{Z}^d, \text{ which intersect } \mathcal{T}, \end{aligned}$$

and recall that $\log \ell = \epsilon^{-d}$. We shall now see that $\lambda^\epsilon_\omega(\mathcal{T})$ is 'close' to the minimum of the not too dependent variables $\lambda^\epsilon_\omega(B \cap \mathcal{T})$, $B \in \mathcal{C}_\ell$.

**Proposition 4.3**: *One can choose $\gamma_2(d, \nu, a(W))$ such that for large $\ell$*

$$\text{(4.34)} \qquad \mathbb{P}_\epsilon[F] \ge 1 - 2\ell^{-d},$$

*if $F$ denotes the event, when $d \ge 2$,*

$$\text{(4.35)} \qquad \begin{aligned} F = \Big\{ c(d,\nu) + \gamma_2\, \epsilon \;&\ge \inf_{B \in \mathcal{C}_\ell} \lambda^\epsilon_\omega(B \cap \mathcal{T}) \ge \lambda^\epsilon_\omega(\mathcal{T}) \\ &\ge \inf_{B \in \mathcal{C}_\ell} \lambda^\epsilon_\omega(B \cap \mathcal{T}) - \epsilon^{2d} \Big\}, \end{aligned}$$

*and when $d = 1$, $\gamma_2\, \epsilon$ is replaced by $\gamma_2 \log \frac{1}{\epsilon}$, where $\gamma_2$ solely depends on $\nu$.*

**Proof**: We have of course

$$\lambda^\epsilon_\omega(\mathcal{T}) \le \inf_{B \in \mathcal{C}_\ell} \lambda^\epsilon_\omega(B \cap \mathcal{T}).$$

On the other hand, as a result of Theorem 2.6, for large $\ell$ and any $\omega$:

$$\text{(4.36)} \qquad \begin{aligned} \lambda^\epsilon_\omega(\mathcal{T}) \wedge M \;&\ge\; \lambda^\epsilon_\omega(\mathcal{T} \cap \mathcal{O}) \wedge M - 2 \exp\Big\{ - c_3 \Big[\frac{R}{4r}\Big]\Big\} \\ &\overset{(4.23)}{\ge}\; \lambda^\epsilon_\omega(\mathcal{T} \cap \mathcal{O}) \wedge M - 2\epsilon^{3d}. \end{aligned}$$

Observe that on the event $C$ defined in (4.26)

$$\text{(4.37)} \qquad \begin{aligned} &\lambda^\epsilon_\omega(\mathcal{T} \cap \mathcal{O}) \wedge M \ge \inf_{B \in \mathcal{C}_\ell} \lambda^\epsilon_\omega(B \cap \mathcal{T}) \wedge M, \\ &\text{and } \mathbb{P}_\epsilon[C] \ge 1 - \ell^{-d}, \ \text{ for large } \ell. \end{aligned}$$

Finally, for large $\ell$, as soon as $\mathcal{T}$ contains a ball of radius

$$\text{(4.38)} \qquad \begin{aligned} R \overset{\text{def}}{=} \;& R_0 - a\epsilon, \ d \ge 2, \\ & R_0 - c(\nu)\, \epsilon \log \frac{1}{\epsilon} \ \ d = 1, \text{ where } c(\nu) \text{ is chosen after (4.40)}, \end{aligned}$$

which receives no point of $\omega$,

$$\inf_{B \in \mathcal{C}_\ell} \lambda^\epsilon_\omega(B \cap \mathcal{T}) \le c(d,\nu) + \gamma_2(d,\nu,a(W))\,\epsilon \le 2c(d,\nu) = M \,, \tag{4.39}$$

when $d \ge 2$, and when $d = 1$, $\gamma_2\,\epsilon$ is replaced by $\gamma_2(\nu)\,\epsilon \log \frac{1}{\epsilon}$. If we chop $\mathcal{T}$ in boxes of size $2R_0$, we see that the probability of nonoccurrence of such a hole within $\mathcal{T}$ is smaller than:

$$\begin{aligned} &(1 - \exp\{-\nu\epsilon^{-d}\omega_d\,R^d\})^{\gamma_3(d,\nu)\,\ell^d/\log \ell} \\ &\le \exp\Big\{ -\gamma_3(d,\nu)\,\frac{\ell^d}{\log \ell}\;\exp\{-\nu\epsilon^{-d}\omega_d\,R^d\}\Big\}, \\ &\text{since } 0 \le 1-x \le e^{-x}, \text{ for } x \in [0,1], \\ &\le \exp\{-\gamma_3(d,\nu)(\log \ell)^{-1}\;\exp\{c(\epsilon)\}\}\,, \end{aligned} \tag{4.40}$$

where

$$\begin{aligned} c(\epsilon) &= \text{const}(d,\nu)\,\epsilon^{-(d-1)}, \quad \text{when } d \ge 2\,, \\ &= c(\nu)\,\log \tfrac{1}{\epsilon}\,, \quad \text{when } d = 1, \text{ and } \gamma_3(1,\nu)\,c(\nu) = 2\,. \end{aligned}$$

This is smaller than $\ell^{-d}$, when $\ell$ is large and our claim (4.34) follows. $\square$

We shall now begin with the second step of the proof where our main task is now to derive lower bounds for the variables $\inf_{B \in \mathcal{C}_\ell} \lambda^\epsilon_\omega(B \cap \mathcal{T})$. We choose some

$$\beta' \in (\beta, 1)\,, \tag{4.41}$$

and define

$$G = \Big\{\omega \in \Omega,\; \sup_{B \in \mathcal{C}_\ell} |B \backslash (\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega))| \le \frac{d}{\nu} + \epsilon^{d(1-\beta')}\Big\}\,. \tag{4.42}$$

**Lemma 4.4**: *For small* $\epsilon$

$$\mathbb{P}_\epsilon[G] \ge 1 - \exp\Big\{ -\frac{\nu}{2}\,(\log \ell)^{\beta'}\Big\}\,. \tag{4.43}$$

**Proof**: This is very similar to (4.5). For small $\epsilon$ and $B \in \mathcal{C}_\ell$

$$\begin{aligned} &\mathbb{P}_\epsilon\Big[|B \backslash (\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega))| \ge \frac{d}{\nu} + \epsilon^{d(1-\beta')}\Big] \overset{(1.9)-(1.10)}{\le} \\ &2^{2\gamma_1^d(\log\log \ell)^d\,\epsilon^{-d\beta}}\,\exp\{-d\epsilon^{-d} - \nu\epsilon^{-d\beta'}\}\,, \end{aligned} \tag{4.44}$$

and therefore when $\ell$ is large:

$$\begin{aligned} \mathbb{P}_\epsilon[G^c] &\le (4\ell\epsilon)^d\, \mathbb{P}_\epsilon\Big[|(0,\gamma_1 \log\log \ell)^d \backslash (\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega))| \\ &\qquad \ge \frac{d}{\nu} + \epsilon^{d(1-\beta')}\Big] \\ &\le 4^d\, \epsilon^d\, \exp\{2\gamma_1^d (\log \epsilon^{-d})^d\, \epsilon^{-d\beta} - \nu\epsilon^{-d\beta'}\} \\ &\le \exp\Big\{-\frac{\nu}{2}\, \epsilon^{-d\beta'}\Big\}\,. \end{aligned} \tag{4.45}$$

□

The promised confidence intervals for $\lambda_\ell(\omega)$ come in

**Theorem 4.5**: *There exist $b, \gamma_2 > 0$, $\chi \in (0,d)$, such that for large $\ell$*

$$\begin{aligned} &\mathbb{P}\Big[\frac{c(d,\nu)}{(\log \ell)^{2/d}} - \frac{1}{(\log \ell)^{\frac{2+\chi}{d}}} \le \lambda_\ell(\omega) \le \frac{c(d,\nu)}{(\log \ell)^{2/d}} + \frac{\gamma_2}{(\log \ell)^{3/d}}\Big] \\ &\ge 1 - \exp\{-(\log \ell)^b\}\,, \end{aligned} \tag{4.46}$$

*when $d \ge 2$, and in the case $d = 1$, $\gamma_2 (\log \ell)^{-3/d}$ is replaced by $\gamma_2 (\log \ell)^{-3} \log\log \ell$.*

**Proof**: We know that for large $\ell$:

$$\mathbb{P}_\epsilon[F \cap G] \ge 1 - \exp\{-(\log \ell)^\beta\}\,, \tag{4.47}$$

and we pick $b = \beta$. Denote by $\mathcal{C}'_\ell$ the subcollection of boxes $B$ in $\mathcal{C}_\ell$ such that:

$$c(d,\nu) + \gamma_2\, c(\epsilon) \ge \lambda_\omega^\epsilon\, (B \cap \mathcal{T})\,, \tag{4.48}$$

with $c(\epsilon) = \epsilon$, when $d \ge 2$, and $c(\epsilon) = \epsilon \log \frac{1}{\epsilon}$, when $d = 1$. For large $\ell$, when $\omega \in F \cap G$, $\mathcal{C}'_\ell$ is not empty and $c(d,\nu) + \gamma_2\, c(\epsilon) + \epsilon^\rho < M$.

It now follows from Theorem 2.3 ('Theorem A') that for small $\epsilon$ and any $B \in \mathcal{C}'_\ell$:

$$\begin{aligned} \lambda_\omega^\epsilon(B \cap \mathcal{T}) &\ge \lambda_\omega^\epsilon\, (B \cap \mathcal{T} \backslash \overline{\mathcal{D}}_\epsilon(\omega)) - \epsilon^\rho \\ &\overset{\text{(Faber–Krahn)}}{\ge} \lambda_d \Big(\frac{\omega_d}{|B \cap \mathcal{T} \backslash \overline{\mathcal{D}}_\epsilon(\omega)|}\Big)^{2/d} - \epsilon^\rho\,. \end{aligned} \tag{4.49}$$

On the other hand by Theorem 3.6 ('Theorem B') and (4.42), for small $\epsilon$:

$$\begin{aligned} |B \cap \mathcal{T} \backslash \mathcal{D}_\epsilon(\omega)| &\le |B \cap \mathcal{T} \backslash (\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega))| + |B \cap \mathcal{B}_\epsilon(\omega)| \\ &\le \frac{d}{\nu} + \epsilon^{d(1-\beta')} + \epsilon^\kappa\, \gamma_1^d (\log\log \ell)^d\,. \end{aligned} \tag{4.50}$$

Combining (4.49) and (4.50), we thus see that for small $\epsilon$, and $\omega \in F \cap G$:

$$\lambda_\omega^\epsilon(\mathcal{T}) \overset{(4.35)-(4.38)}{\geq} \inf_{B \in \mathcal{C}'_\ell} \lambda_\omega^\epsilon(B \cap \mathcal{T}) - \epsilon^{2d} \overset{(4.49)-(4.50)}{\geq} c(d,\nu) - \epsilon^\chi , \tag{4.51}$$

provided $0 < \chi < d(1-\beta') \wedge \rho \wedge \kappa$. This proves our claim. □

In the one-dimensional case, with minor modifications due to the special role of 0 in Chapter 3 §2, the proof of Proposition 3.3.5 shows that for large $\ell$, the fluctuations of $\lambda_\ell$ around a median have typical size $(\log \ell)^{-3}$, see also section V of [Szn97a]. Incidentally this coincides with the order of fluctuations for large $\ell$ of $\widetilde{\lambda}_\ell$ the principal Dirichlet eigenvalue of $-\frac{1}{2}\,\Delta$ in the largest ball in $(-\ell,\ell)^d$ receiving no point of $\omega$. This feature essentially follows from the fact that the volume $v_\ell$ of the largest ball in $(-\ell,\ell)^d$ receiving no point of the cloud, for large $\ell$ has fluctuations of order 1 around a median which is equivalent to $\frac{d}{\nu}\, \log \ell$, see Hall [Hal85], Theorem 2, or Janson [Jan84].

### C) Almost sure Behavior

We shall now discuss two results related to the typical behavior of $\lambda_\omega(U)$ where $U$ is a large open set. The first result, which is an application of Theorem 4.5, describes the typical behavior of $\lambda_\ell(\omega)$, for large $\ell$.

**Theorem 4.6**: *There exists $\gamma > 0$ and $\chi \in (0,d)$, such that on a set of full $\mathbb{P}$-measure, for large $\ell$*

$$\frac{c(d,\nu)}{(\log \ell)^{2/d}} - \frac{1}{(\log \ell)^{\frac{2+\chi}{d}}} \leq \lambda_\ell(\omega) \leq \frac{c(d,\nu)}{(\log \ell)^{2/d}} + \frac{\gamma}{(\log \ell)^{3/d}} , \tag{4.52}$$

*when $d \geq 2$, and when $d = 1$, $\gamma(\log \ell)^{-3/d}$ is replaced by $\gamma(\log \ell)^{-3} \log\log \ell$.*

**Proof**: Observe that $\lambda_\ell(\omega)$ is a decreasing function of $\ell$. Apply (4.46) to the blocks $(-2^k, 2^k)^d$, $k \geq 1$, and Borel-Cantelli's lemma. □

The next theorem shows that for typical $\omega$ and large $\ell$, when $U, U' \subseteq (-\ell,\ell)^d$ are disjoint and $\lambda_\omega(U)$, $\lambda_\omega(U')$ are close to $c(d,\nu)(\log \ell)^{-2/d}$, then the diameter of $U \cup U'$ has to be large.

**Theorem 4.7:** *Given $\zeta \in (0,1)$, on a set of full $\mathbb{P}$-measure,*

$$\begin{aligned} &\varlimsup_{\ell\to\infty} \sup_{U,U'} \left(\frac{c(d,\nu)(\log \ell)^{-2/d}}{\lambda_\omega(U)}\right)^{d/2} + \left(\frac{c(d,\nu)(\log \ell)^{-2/d}}{\lambda_\omega(U')}\right)^{d/2} \\ &\leq (1+\zeta) \vee \left(1 + \left(\frac{2}{3}\right)^{d/2}\right) < 2 , \end{aligned} \tag{4.53}$$

*where the supremum runs over $U, U'$ disjoint open subsets of $(-\ell,\ell)^d$ with $\mathrm{diam}(U \cup U') \leq \ell^\zeta$.*

**Proof:** We first come back to the setting of §4 B). If $C_{(\epsilon)}$ denotes the event in (4.26), a similar argument to (4.36) - (4.37) shows that for large $\ell$, for any $\omega \in C_{(\epsilon)}$ and any open subset $U$ of $\mathcal{T}$ (see (4.19) - (4.26) for the notation),

$$\lambda_\omega^\epsilon(U) \geq \inf_{B \in \mathcal{C}_\ell : B \cap U \neq \emptyset} \lambda_\omega^\epsilon(B \cap U) \wedge (2c(d,\nu)) - \epsilon^{2d} \,. \tag{4.54}$$

Furthermore, using Theorem 2.3, 3.6 and Faber-Krahn's inequality as in (4.49), (4.50), we find that for large $\ell$, $\omega \in C_{(\epsilon)}$ and any $U \subseteq \mathcal{T}$ open with $\lambda_\omega^\epsilon(U) \leq \frac{3}{2}\, c(d,\nu)$

$$\begin{aligned} \lambda_\omega^\epsilon(U) \geq & \inf_{B \in \mathcal{C}_\ell : B \cap U \neq \emptyset} \lambda_d \omega_d^{2/d} (|B \cap U \backslash (\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega))| \\ & + \epsilon^\kappa\, \gamma_1^d (\log\log \ell)^d)^{-2/d} - \epsilon^\rho - \epsilon^{2d} \,. \end{aligned} \tag{4.55}$$

We define

$\widetilde{\mathcal{C}}_\ell =$ the collection of blocks $q + (0, 2[\gamma_1 \log\log \ell])^d$, $q \in \mathbb{Z}^d$, which intersect $\mathcal{T}$

We then consider $\beta' \in (\beta, 1)$ as in (4.42) and introduce the event

$$\begin{aligned} H_{(\epsilon)} = \; & \Big\{\omega \in \Omega;\ \text{for } B, B' \in \mathcal{C}_\ell \text{ with } B \cap B' = \emptyset \text{ and} \\ & \text{diam}\,(B \cup B') \leq \ell^\zeta,\ |B \backslash (\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega))| \\ & + |B' \backslash (\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega))| \leq \frac{d}{\nu}\,(1+\zeta) + \epsilon^{d(1-\beta')}, \\ & \text{and for } \widetilde{B} \in \widetilde{\mathcal{C}}_\ell,\ |\widetilde{B} \backslash (\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega))| \leq \frac{d}{\nu} + \epsilon^{d(1-\beta')}\Big\} \,. \end{aligned} \tag{4.56}$$

Analogously to Lemma 4.4, we have

**Lemma 4.8:** *For large $\ell$,*

$$\mathbb{P}_\epsilon[H_{(\epsilon)}] \geq 1 - \exp\Big\{-\frac{\nu}{2}\,(\log \ell)^{\beta'}\Big\} \,. \tag{4.57}$$

**Proof:** Just as in (4.44), when $B$, $B'$ are two disjoint blocks in $\mathcal{C}_\ell$, and $\ell$ is large:

$$\begin{aligned} & \mathbb{P}_\epsilon[|B \backslash (\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega))| + |B' \backslash (\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega))| \geq \frac{d}{\nu}\,(1+\zeta) + \epsilon^{d(1-\beta')}] \\ & \leq 2^{4\gamma_1^d (\log\log \ell)^d (\log \ell)^\beta} \exp\{-d(1+\zeta)\log \ell - \nu(\log \ell)^{\beta'}\} \,, \end{aligned}$$

and when $\widetilde{B} \in \widetilde{\mathcal{C}}_\ell$:

$$\begin{aligned} & \mathbb{P}_\epsilon[|\widetilde{B} \backslash (\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega))| \geq \frac{d}{\nu} + \epsilon^{d(1-\beta')}] \\ & \leq 2^{2^d \gamma_1^d (\log\log \ell)^d (\log \ell)^\beta} \exp\{-d \log \ell - \nu(\log \ell)^{\beta'}\} \,. \end{aligned}$$

On the other hand, when $\ell$ is large, $\mathbb{P}_\epsilon[H]$ is no bigger than the sum of $\ell^{d(1+\zeta)}$ times the first expression and $\ell^d$ times the second expression. Our claim (4.57) follows. □

Applying Borel-Cantelli's lemma together with (4.25), (4.58), we find that for $\omega = \sum_i \delta_{x_i}$ in a set of full $\mathbb{P}$-probability for large $n$, $\omega_{\epsilon_n} \in C_{(\epsilon_n)} \cap H_{(\epsilon_n)}$, provided $\epsilon_n = (\log 2^n)^{-1/d}$ and $\omega_{\epsilon_n} = \sum_i \delta_{\epsilon_n x_i}$. We choose from now on $\omega$ in the intersection of the above set of full $\mathbb{P}$-probability with the other set of full $\mathbb{P}$-probability determined by (4.52). We also view $n$ as a function of $\ell$ via the requirement:

$$2^{n-1} \le \ell < 2^n \stackrel{\text{def}}{=} \ell_0 \,. \tag{4.58}$$

Consider $U, U'$ fulfilling the requirement after (4.53). If $\lambda_\omega(U)$ or $\lambda_\omega(U') \ge \frac{3}{2}\, c(d,\nu)(\log \ell_0)^{-2/d}$, then the expression under the supremum in (4.53) is smaller than

$$\Big(\frac{c(d,\nu)(\log \ell)^{-2/d}}{\lambda_\ell(\omega)}\Big)^{d/2} + \Big(\frac{2}{3}\Big)^{d/2} \frac{\log \ell_0}{\log \ell} \underset{\ell\to\infty}{\xrightarrow{(4.52)}} 1 + \Big(\frac{2}{3}\Big)^{d/2} . \tag{4.59}$$

On the other hand scaling together with the fact that $\omega_{\epsilon_n} \in C_{(\epsilon_n)} \cap H_{(\epsilon_n)}$ for large $n$ imply that for large $\ell$, when both $\lambda_\omega(U)$ and $\lambda_\omega(U')$ are smaller than $\frac{3}{2}\, c(d,\nu)(\log \ell_0)^{-2/d}$, we can find $B$ and $B'$ in $\mathcal{C}_\ell$ respectively intersecting $\epsilon_n U$ and $\epsilon_n U'$ for which (4.54) and (4.55) hold. If $B \cap B' \neq \emptyset$, we can find $\widetilde{B} \in \widetilde{\mathcal{C}}_\ell$ with $B \cup B' \subseteq \widetilde{B}$, and

$$\begin{aligned}
&\Big(\frac{c(d,\nu)(\log \ell_0)^{-2/d}}{\lambda_\omega(U) + (\epsilon_n^\rho + \epsilon_n^{2d})(\log \ell_0)^{-2/d}}\Big)^{d/2} + \\
&\Big(\frac{c(d,\nu)(\log \ell_0)^{-2/d}}{\lambda_\omega(U') + \ (\epsilon_n^\rho + \epsilon_n^{2d})(\log \ell_0)^{-2/d}}\Big)^{d/2} \\
&\stackrel{(4.55)}{\le} \frac{\nu}{d}\ (|B \cap U \backslash (\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega))| \\
&\quad + |B' \cap U' \backslash (\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega))|) + \frac{2\nu}{d}\ \epsilon_n^\kappa\ \gamma_1^d (\log\log \ell_0)^d
\end{aligned} \tag{4.60}$$

and since $U \cap U' = \emptyset$ and $B \cup B' \subseteq \widetilde{B}$,

$$\begin{aligned}
&\le \frac{\nu}{d}\ |\widetilde{B} \backslash (\mathcal{D}_\epsilon(\omega) \cup \mathcal{B}_\epsilon(\omega))| + \frac{2\nu}{d}\ \epsilon_n^\kappa\ \gamma_1^d (\log\log \ell_0)^d \\
&\stackrel{(4.56)}{\le}\ 1 + \frac{\nu}{d}\ \epsilon^{d(1-\beta')} + \frac{2\nu}{d}\ \epsilon_n^\kappa\ \gamma_1^d (\log\log \ell_0)^d \,.
\end{aligned}$$

On the other hand if $B \cap B' = \emptyset$, by an analogous reasoning we find

$$(4.61)\qquad \begin{aligned} &\Big(\frac{c(d,\nu)(\log \ell_0)^{-2/d}}{\lambda_\omega(U) + (\epsilon_n^\rho + \epsilon_n^{2d})(\log \ell_0)^{-2/d}}\Big)^{d/2} \\ + &\Big(\frac{c(d,\nu)(\log \ell_0)^{-2/d}}{\lambda_\omega(U') + (\epsilon_n^\rho + \epsilon_n^{2d})(\log \ell_0)^{-2/d}}\Big)^{d/2} \\ \overset{(4.55)-(4.56)}{\le} &\ (1+\zeta) + \frac{\nu}{d}\,\epsilon_n^{d(1-\beta')} + \frac{2\nu}{d}\,\epsilon_n^\kappa \gamma_1^d (\log\log \ell_0)^d\,. \end{aligned}$$

Furthermore, as follows from $U, U' \subseteq (-\ell,\ell)^d$ and (4.52), when $\ell$ is large $\lambda_\omega(U)$ and $\lambda_\omega(U')$ are both larger than $\frac{1}{2}c(d,\nu)(\log \ell_0)^{-2/d}$. Coming back to (4.60) - (4.61) we thus see that

$$(4.62)\quad \overline{\lim_{\ell\to\infty}}\ \widetilde{\sup_{U,U'}}\ \Big(\frac{c(d,\nu)(\log \ell)^{-2/d}}{\lambda_\omega(U)}\Big)^{d/2} + \Big(\frac{c(d,\nu)(\log \ell)^{-2/d}}{\lambda_\omega(U')}\Big)^{d/2} \le (1+\zeta)\,,$$

where $\widetilde{\sup}$ means that additionally to the conditions after (4.53),

$$\lambda_\omega(U) \vee \lambda_\omega(U') \le \frac{3}{2}\, c(d,\nu)(\log \ell_0)^{-2/d}\,.$$

This and the upper bound (4.59) finishes this proof of (4.53). □

The following corollary will be of use in Chapter 6 §2.

**Corollary 4.9:** *There exist $\kappa(d,\zeta) \in (1,2)$, $\kappa'(d,\zeta) > 1$, for $d \ge 1$, $\zeta \in (0,1)$, such that on a set of full $\mathbb{P}$-measure, for any $\zeta \in (0,1)$, for large $\ell$, whenever $U, U'$ are disjoint open subsets of $(-\ell,\ell)^d$ with*

$$(4.63)\qquad diam(U \cup U') \le \ell^\zeta,\ \textit{then}$$

$$(4.64)\qquad \lambda_\omega(U) \le \kappa\,\frac{c(d,\nu)}{(\log \ell)^{2/d}}\ \textit{implies}\ \lambda_\omega(U') \ge \kappa'\,\lambda_\omega(U)\,.$$

**Proof:** This is an immediate application of (4.53). □

## 4.5 Large-time Wiener Asymptotics

We consider soft Poissonian obstacles on $\mathbb{R}^d$, $d \ge 1$, just as in (4.16). In this section we shall mainly be concerned with the large time behavior of:

$$(5.1)\quad S_{t,\omega} = E_0\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big]\,,\ \text{‘the quenched asymptotics’},$$

for $\mathbb{P}$-typical $\omega$, and

$$S_t = \mathbb{E}[S_{t,\omega}] = \mathbb{E} \otimes E_0\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big], \tag{5.2}$$

'the annealed asymptotics' .

It is also convenient to define, for $\omega \in \Omega$, $t \geq 0$, $x \in \mathbb{R}^d$

$$u_\omega(t,x) = E_x\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big], \tag{5.3}$$

$$h_\omega(t,x) = -\log u_\omega(t,x) . \tag{5.4}$$

As we shall see the quenched and annealed asymptotics have quite different behavior. This reflects a strong inhomogeneity of the Poissonian random medium. The terminology 'quenched' and 'annealed' is inherited from disordered media physics, and originally comes from metallurgy. The next exercise illustrates the difference of status between the two asymptotics and explains the 'correct interpretation' of the $\mathbb{P}$-expectation in (5.2).

**Exercise:**

**1)** Show that for $\mathbb{P}$-a.e. $\omega$, for $t \geq 0$,

$$S_t = \lim_{N\to\infty} \frac{1}{|B(0,N)|}\int_{B(0,N)} u_\omega(t,x)dx , \tag{5.5}$$

whereas

$$S_{t,\omega} = u_\omega(t,0) .$$

(Hint: use the multidimensional ergodic theorem, see Krengel [Kre85], p. 210, or a bound on the variance of the r.h.s. of (5.5) and a Borel-Cantelli argument).

**2)** Consider $V(s,x)$ on $\mathbb{R}_+ \times \mathbb{R}^d$, nonnegative locally bounded and measurable. Show that

$$h(t,x) \overset{\text{def}}{=} -\log E_{t,x}\Big[\exp\Big\{-\int_t^T V(s,Z_s)ds\Big\}\Big],\ t \in [0,T],\ x \in \mathbb{R}^d , \tag{5.6}$$

where $E_{t,x}$ denotes the expectation with respect to a Brownian motion $Z$. starting in $x$ at time $t$, satisfies:

$$h(t,x) = \inf_\gamma E^\gamma\Big[\int_t^T (V + \tfrac{1}{2}\,|\gamma|^2)(s,Z_s)ds\Big],\ t \in [0,T],\ x \in \mathbb{R}^d , \tag{5.7}$$

where $E^\gamma$ denotes the expectation with respect to the (weak) solution of the SDE:

$$(5.8) \qquad \begin{cases} dZ_s &= d\beta_s + \gamma(s, Z_s)ds\,, \quad t \le s \le T\,, \\ Z_t &= x\,, \end{cases}$$

and $\gamma$ varies over the class of bounded measurable drifts.

(Hint: assume first $V \ge 0$ bounded continuous and uniformly Lipschitz in $x$. Show the '$\ge$' part of (5.7) with the help of Cameron-Martin-Girsanov's formula and Jensen's inequality. To prove the '$\le$' part of (5.7) choose

$$(5.9) \qquad \gamma^*(s,x) = -\nabla h(s,x)\,,$$

(which is bounded measurable with our choice of $V$, in (5.6)), and show that (see Durrett [Dur84], p. 226):

$$\begin{aligned} &\partial_s h + \frac{1}{2}\,\Delta h - \frac{1}{2}\,|\nabla h|^2 = -V\,, \quad \text{and} \\ &P^{\gamma^*}\text{-a.s. } h(t,x) = \int_t^T \gamma(s,Z_s)\,d\beta_s + \int_t^T \Big(V + \frac{1}{2}\,|\gamma|^2\Big)(s,Z_s)ds\,. \end{aligned}$$

Show then the case $V \ge 0$ uniformly bounded measurable, and then the general case).

Formula (5.7) interprets $h$ as the value function of a certain control problem. For more on this we refer the reader to Fleming-Soner [FS93], Chapter 6, or Krylov [Kry80]. □

## A) Quenched Asymptotics

We want to study the typical large $t$ behavior of $h_\omega(t,x)$. Our main goal here is

**Theorem 5.1**: *There is a set of full* $\mathbb{P}$*-measure such that*

$$(5.10) \qquad \text{for } x \in \mathbb{R}^d \;\; h_\omega(t,x) \sim c(d,\nu)\,\frac{t}{(\log t)^{2/d}}\,, \quad \text{as } t \to \infty\,,$$

*(see (4.20) for the definition of* $c(d,\nu)$*). In particular*

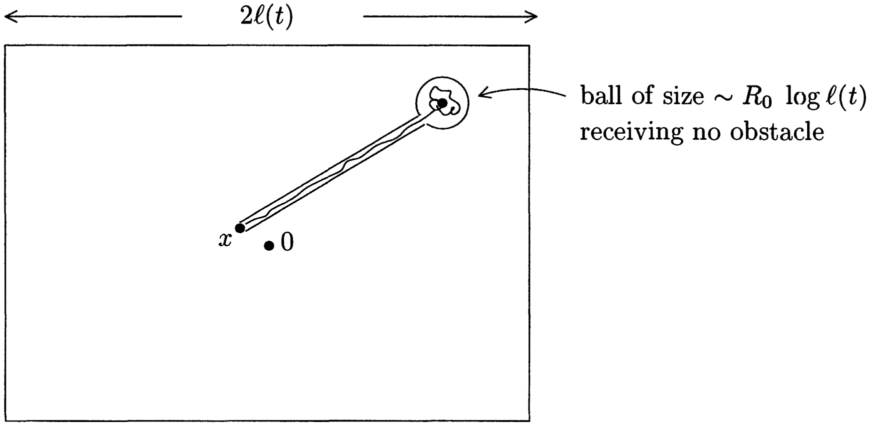
$$S_{t,\omega} = \exp\Big\{ - c(d,\nu)\,\frac{t}{(\log t)^{2/d}}\,(1+o(1))\Big\}\,, \quad \text{as } t \to \infty\,.$$

**Proof**: Upper bound on $h_\omega$:

Heuristically the upper bound corresponds to devising one survival strategy for Brownian motion in the soft Poissonian potential, somewhat the spirit of (5.7). It will correspond to letting Brownian motion go in time of order $\ell(t)$ to a spherical hole of size of order $R_0(\log \ell(t))^{1/d}$ which occurs in the Poisson configuration within distance $\ell(t)$ from the origin. This argument will provide the correct principal order given in (5.10), provided we choose $\ell(t) \to \infty$, as $t \to \infty$, so that

$$(5.11) \qquad t^{\chi} = o(\ell(t)), \text{ when } \chi < 1, \text{ and } \ell(t) = o(t(\log t)^{-\frac{2}{d}-1}) \, .$$

In fact a slightly finer analysis of the argument would show that one can allow $t(\log t)^{-2/d}$ instead of $t(\log t)^{-\frac{2}{d}-1}$ in (5.11), and still obtain the same conclusions.




**Lemma 5.2**: *On a set of full* $\mathbb{P}$*-measure*

$$(5.12) \qquad \sup_{q \in \mathbb{Z}^d} \omega(q + [0,1)^d) \cap (-\ell,\ell)^d) = o(\log \ell), \ \text{as } \ell \to \infty \, ,$$

*(as a result* $\sup_{(-\ell,\ell)^d} V(\cdot,\omega) = o(\log \ell)$, *as* $\ell \to \infty$*), and*

(5.13) $(-\ell,\ell)^d$ *contains a ball of radius* $R_\ell \stackrel{\text{def}}{=} R_0(\log \ell)^{1/d} - (\log\log \ell)^2$ *on which* $V(\cdot,\omega) = 0$ .

**Proof**: Let us begin with (5.12). For $u > 0$, $\lambda > 0$

$$\mathbb{P}[\omega([0,1)^d) > u] \le \exp\{-\lambda u + \nu(e^\lambda - 1)\} \, .$$

Thus for $u = \eta \log |q|$, with $q \in \mathbb{Z}^d \backslash \{0\}$ choosing $\lambda = \frac{2d}{\eta}$,

$$(5.14) \qquad \mathbb{P}[\omega(q + [0,1)^d) > \eta \log |q|] \le \exp\{-2d \log |q| + \nu(e^\lambda - 1)\} \, .$$

The r.h.s. is summable in $q \in \mathbb{Z}^d$, so that by Borel-Cantelli's lemma, $\mathbb{P}$-a.s.

$$\omega(q + [0,1)^d) \le \eta \log |q| \ \text{ for large } q\,.$$

Letting $\eta > 0$ (rational) vary, we see that $\mathbb{P}$-a.s.

$$\omega(q + [0,1)^d) = o(\log |q|)\,, \tag{5.15}$$

which easily implies (5.12).

Finally (5.13) is an easy consequence of the bound (4.40) and Borel-Cantelli's lemma applied to blocks $(-2^k, 2^k)^d$. □

We pick some $\omega$ in the set of full $\mathbb{P}$-measure of Lemma 5.2. For large $\ell$ we denote by $x_\ell$ some point such that

$$V(\cdot,\omega) = 0 \ \text{ on } \ B_\ell \overset{\text{def}}{=} B(x_\ell, R_\ell)\,. \tag{5.16}$$

We let $\varphi_\ell$ stand for the principal Dirichlet eigenfunction $-\frac{1}{2}\,\Delta$ in $B_\ell$. We now specify some deterministic function $\ell(t)$ for which (5.11) holds.

We can apply Proposition 3.1.12, with $U' = \mathbb{R}^d$, $U = B_{\ell(t)}$, $A = \overline{B}(x_{\ell(t)}, 1)$, for large enough $t$. We find

$$\begin{aligned} E_x\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big] &\ge c(d) \int_{\overline{B}(x_{\ell(t)},1)} \varphi^2_{\ell(t)}\,dx \\ &\cdot \exp\Big\{-\frac{\lambda_d}{R^2_{\ell(t)}}\,t\Big\}\, E_x\Big[\exp\Big\{-\int_0^{H_t} V(Z_s,\omega)ds\Big\},\ H_t < \infty\Big]\,, \end{aligned} \tag{5.17}$$

provided $H_t$ stands for the entrance time of $Z_.$ in $\overline{B}(x_{\ell(t)}, 1)$. Observe that because of scaling:

$$\int_{\overline{B}(x_{\ell(t)},1)} \varphi^2_{\ell(t)}\,dx \ge c'(d)\, R^{-d}_{\ell(t)}\,. \tag{5.18}$$

As for the last term of (5.17), for large $t$:

$$\begin{aligned} &E_x\Big[\exp\Big\{-\int_0^{H_t} V(Z_s,\omega)ds\Big\},\ H_t < \infty\Big] \ge \\ &E_x\Big[\exp\Big\{-\int_0^{\ell(t)} V(Z_s,\omega)ds\Big\}, \\ &\sup_{0\le s\le \ell(t)} \Big|Z_s - x - \frac{s}{\ell(t)}\,(x_{\ell(t)} - x)\Big| < 1\Big] \ge \\ &\exp\Big\{-\ell(t) \sup_{(-t,t)^d} V(\cdot,\omega)\Big\}\, P_0\Big[\sup_{0\le s\le \ell(t)} \Big|Z_s - \frac{s}{\ell(t)}\,(x_{\ell(t)} - x)\Big| < 1\Big]\,. \end{aligned} \tag{5.19}$$

Using Cameron-Martin-Girsanov's formula, the last term of the above expression equals

$$
\begin{aligned}
(5.20)\qquad & E_0\Big[\sup_{0\le s\le \ell(t)} |Z_s| < 1, \\
& \exp\Big\{-\frac{1}{\ell(t)}\,(x_{\ell(t)}-x)\cdot Z_{\ell(t)} - \frac{1}{2\ell(t)}\,|x_{\ell(t)}-x|^2\Big\}\Big] \\
& \overset{\text{(Jensen)}}{\geq} P_0[C]\,\exp\Big\{-\frac{1}{\ell(t)}\,(x_{\ell(t)}-x)\,E_0[Z_{\ell(t)}|C] \\
& \qquad -\frac{1}{2\ell(t)}\,|x_{\ell(t)}-x|^2\Big\}\,,
\end{aligned}
$$

provided

$$
C = \{\sup_{0\le s\le \ell(t)} |Z_s| < 1\}\,.
$$

The conditional expectation in the last expression of (5.20) vanishes as can be seen by replacing $Z$ with $-Z$. If we use (3.1.53), to produce a lower bound on $P_0[C]$, coming back to (5.17) we see that for large $t$

$$
\begin{aligned}
(5.21)\qquad & E_x\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big] \\
& \geq \exp\Big\{-\ell(t)(c+\sup_{(-t,t)^d} V(\cdot,\omega)) - \frac{\lambda_d}{R^2_{\ell(t)}}\,t\Big\}\,.
\end{aligned}
$$

In view of (5.11) and (5.12) we find

$$
(5.22)\qquad \overline{\lim_t}\ \frac{h_\omega(x,t)}{t(\log t)^{-2/d}} \leq \frac{\lambda_d}{R_0^2} = c(d,\nu), \quad \text{for } x\in \mathbb{R}^d\,.
$$

Lower bound on $h_\omega$:

Consider a set of full $\mathbb{P}$-measure on which (4.52) holds. For $\omega$ in this set, we have for $x \in \mathbb{R}^d$

$$
\begin{aligned}
& E_x\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big] \leq P_x[T_{(-t,t)^d}\leq t] + R_t^{(-t,t)^d, V(\cdot,\omega)}1(x) \\
(5.23)\qquad & \\
& \overset{(3.1.9)}{\leq}\ P_{\frac{x}{\sqrt t}}\,[T_{(-\sqrt t,\sqrt t)^d}\leq 1] + c(d)((\lambda_t(\omega)t)^{d/2}+1)\,e^{-\lambda_t(\omega)t}\,.
\end{aligned}
$$

The first term in the right-most member of (5.23) has exponential decay in $t$ and it now follows from (4.52) that for $x\in\mathbb{R}^d$

$$
\underline{\lim}_{t\to\infty}\ h_\omega(x,t)/t(\log t)^{-2/d} \geq -c(d,\nu)\,.
$$

Together with (5.22), this finishes our proof. □

Observe that the asymptotic behavior (5.10) does not depend on the function $W(\cdot)$. It is a quite robust asymptotics (see also the exercise below). The same asymptotics hold in the case of sufficiently small hard obstacles, provided the starting point $x$ lies in an infinite trap free cluster (see [Szn93b]). On the other hand when obstacles are balls of a large enough radius the complement of the set of traps is $\mathbb{P}$-a.s. the union of bounded components, (this can for instance be seen with the help of a comparison argument with site percolation, see also Meester-Roy [MR96], p. 936, and Sarkar [Sar97]). In this case (5.10) clearly breaks down.

**Exercise**: Consider $r_\omega(t,x,y) \overset{\text{def}}{=} r_{\mathbb{R}^d, V(\cdot,\omega)}(t,x,y)$, for $t > 0$, $x,y \in \mathbb{R}^d$. Show that in a set of full $\mathbb{P}$-measure

$$(5.24) \qquad r_\omega(t,0,0) = \exp\Big\{ - c(d,\nu)\,\frac{t}{(\log t)^{2/d}}\,(1+o(1))\Big\}, \quad \text{as } t \to \infty.$$

(Hint: show that for $t \geq 1$:

$$r_\omega(t,0,0) \leq (2\pi)^{-d/2}\, S_{t-1,\omega}\,,$$

and for $t > s > 0$,

$$r_\omega(t,0,0) \geq \exp\{-s \sup_{(-t,t)^d} V(\cdot,\omega)\}\,(2\pi s)^{-d/2}\,\exp\Big\{-\frac{dt^2}{s}\Big\}\, R_{t-s}^{(-t,t)^d, V(\cdot,\omega)} 1(0).\ )$$

□

### B) Annealed Asymptotics

We now want to discuss the large time behavior of $S_t$, defined in (5.2). If one performs the $\mathbb{P}$-expectation one finds

$$(5.25) \qquad S_t = E_0\Big[\exp\Big\{ - \nu \int (1 - e^{-\int_0^t W(Z_s - y)ds})dy\Big\}\Big],\ t \geq 0\,.$$

We shall also be interested in the case of hard obstacles, modelled on a given nonpolar compact subset $C$ of $\mathbb{R}^d$, $d \geq 1$, (see Section 1). So for hard obstacles we instead consider:

$$(5.26) \qquad S_t = \mathbb{P} \otimes P_0[T > t] = E_0[\exp\{-\nu\,|W_t^C|\}]\,,$$

with $T$ the entrance time in the random obstacle set:

$$(5.27) \qquad S = \bigcup_i x_i + C\,, \text{ if } \omega = \sum_i \delta_{x_i}\,,$$

and $W_t^C$ the Wiener sausage in time $t$ modelled on $-C$:

$$(5.28) \qquad W_t^C = \bigcup_{0\leq s\leq t} Z_s - C\,.$$

We shall give a proof of Donsker-Varadhan's asymptotics of Wiener sausage [DV75c] governing the principal logarithmic behavior of $S_t$, as an application of the method of enlargement of obstacles.

**Theorem 5.3**: *(soft or hard obstacles)*

$$S_t = \exp\{-\widetilde{c}(d,\nu)\, t^{\frac{d}{d+2}}(1+o(1))\}, \quad \textit{as } t \to \infty \textit{ with} \tag{5.29}$$

$$\begin{aligned} \widetilde{c}(d,\nu) &= \inf_{U \text{ open}} \{\nu\,|U| + \lambda(U)\} \\ &= (\nu\,\omega_d)^{\frac{2}{d+2}} \left(\frac{d+2}{2}\right) \left(\frac{2\lambda_d}{d}\right)^{\frac{d}{d+2}} . \end{aligned} \tag{5.30}$$

It should be observed that the principal behavior of $S_t$ given in (5.29) is the same for soft and hard obstacles. This is a very robust asymptotics even more so than in the quenched case, (see discussion at the end of Section A)).

The value of the variational problem in (5.30) does not change if one further restricts $U$ to be a ball. This simply follows from Faber-Krahn's inequality, and one thus finds

$$\widetilde{c}(d,\nu) = \inf_{R>0} \left\{\nu\,\omega_d\, R^d + \frac{\lambda_d}{R^2}\right\}, \tag{5.31}$$

which then yields the second line of (5.30), for an optimal radius:

$$\widetilde{R}_0(d,\nu) = \left(\frac{2\lambda_d}{d\nu\,\omega_d}\right)^{1/(d+2)} . \tag{5.32}$$

**Proof of Theorem 5.3**: *Lower bound:*

For $R>0$, $t \geq 0$, with $a$ defined in (1.4), (1.5),

$$\begin{aligned} S_t &\geq \mathbb{P}\,[\omega(\overline{B}(0,R+a)) = 0]\; P_0[T_{B(0,R)} > t] \\ &\overset{(3.1.53)}{\geq} c(d)\, \exp\Big\{ -\nu\,\omega_d(R+a)^d - \frac{\lambda_d}{R^2}\, t\Big\} \end{aligned} \tag{5.33}$$

optimizing in $R$ (in the case $a=0$) we choose

$$R = \widetilde{R}_0\, t^{\frac{1}{d+2}}\,, \quad \text{so that} \tag{5.34}$$

$$S_t \geq c(d)\, \exp\{-t^{\frac{d}{d+2}}\, [\nu\,\omega_d(\widetilde{R}_0 + a t^{-\frac{1}{d+2}})^d + \lambda_d\, \widetilde{R}_0^{-2}]\},\ t \geq 0\,, \tag{5.35}$$

this now implies:

$$\underline{\lim}_{t\to\infty}\; t^{-\frac{d}{d+2}} \log S_t \geq -\widetilde{c}(d,\nu)\,. \tag{5.36}$$

*Upper bound (first version):*

We present the case of soft obstacles. The proof is routinely adapted to the hard obstacle case. Motivated by (5.34), we consider $t^{\frac{1}{d+2}}$ as the 'unit length', and introduce for $t > 1$:

$$\epsilon = t^{-\frac{1}{d+2}} \in (0,1) \,. \tag{5.37}$$

Using scaling in (5.25), we find

$$S_t = E_0\Big[\exp\Big\{-\frac{\nu}{\epsilon^d}\int\Big(1 - e^{-\int_0^{t\epsilon^2} \epsilon^{-2}\, W(\frac{Z_s - y}{\epsilon})ds}\Big)dy\Big\}\Big] \,. \tag{5.38}$$

Consider some integer $\ell \geq 1$. It follows from the inequality

$$1 - e^{-a-b} \leq 1 - e^{-a} + 1 - e^{-b},\ a, b \geq 0\,, \tag{5.39}$$

that for $u \geq 0$, and $W_\epsilon(\cdot) = \epsilon^{-2}\, W(\epsilon^{-1}\,\cdot)$:

$$\begin{aligned} &\int_{[0,\ell)^d}\Big(1 - \exp\Big\{-\sum_{p\in\mathbb{Z}^d}\int_0^u W_\epsilon(Z_s - y - p\ell)ds\Big\}dy\Big) \leq \\ &\int_{\mathbb{R}^d}\Big(1 - \exp\Big\{-\int_0^u W_\epsilon(Z_s - y)ds\Big\}\Big)dy \,. \end{aligned} \tag{5.40}$$

Inserting in (5.38), we thus find

$$S_t \leq S_t^\ell \stackrel{\text{def}}{=} \mathbb{E}_\epsilon^\ell \otimes E_0\Big[\exp\Big\{-\int_0^{t\epsilon^2} V_\epsilon(Z_s, \omega)ds\Big\}\Big] \,, \tag{5.41}$$

in the notations of (1.2), with $\mathbb{P}_\epsilon^\ell$ standing for the law of the $\ell$-periodic point process on $\mathbb{R}^d$ with restriction to $[0,\ell)^d$ Poissonian of intensity $\nu\epsilon^{-d}$. Here we simply used the fact that for an $\ell$ periodic $\omega$:

$$\int_0^t V_\epsilon(Z_s, \omega)ds = \int_{[0,\ell)^d}\int_0^t \sum_{p\in\mathbb{Z}^d} W_\epsilon(Z_s - y - p\ell)ds\, \omega(dy) \,.$$

We shall bound $S^\ell(t)$ using a coarse graining procedure as explained in Section 1. We choose an admissible collection of parameters. We denote by $\Omega_\ell \subseteq \Omega$ the set of $\ell$ periodic configurations $\omega$, on which $\mathbb{P}_\epsilon^\ell$ is supported. We introduce a covering $\mathcal{G}_\epsilon^\ell$ of $\Omega_\ell$ by the events:

$$G_{D,B} = \{\omega \in \Omega_\ell,\ [0,\ell)^d \cap \mathcal{D}_\epsilon(\omega) = D,\ [0,\ell)^d \cap \mathcal{B}_\epsilon(\omega) = B\} \,, \tag{5.42}$$

where $D, B$ run over the at most $2^{\ell^d \epsilon^{-d\gamma}}$ and $2^{\ell^d \epsilon^{-d\beta}}$ respective possible shapes of $[0,\ell)^d \cap \mathcal{D}_\epsilon(\omega)$ and $[0,\ell)^d \cap \mathcal{B}_\epsilon(\omega)$. We of course have:

$$|\mathcal{G}_\epsilon^\ell| = \exp\{o(\epsilon^{-d})\}, \text{ as } \epsilon \to 0 . \tag{5.43}$$

It is convenient to introduce

$$\tau = t\epsilon^2 = \epsilon^{-d} = t^{\frac{d}{d+2}}, \text{ and} \tag{5.44}$$

$$M = 2\,\widetilde{c}(d,\nu) . \tag{5.45}$$

Then for small $\epsilon$ and any $G_{D,B}$ in $\mathcal{G}_\epsilon^\ell$, $t \geq 0$, we have

$$\begin{aligned} &\mathbb{E}_\epsilon^\ell \otimes E_0\Big[G_{D,B},\ \exp\Big\{-\int_0^\tau V_\epsilon(Z_s,\omega)ds\Big\}\Big] \overset{(3.1.9)}{\leq} \\ &\mathbb{E}_\epsilon^\ell[G_{D,B},\ c(d)(1+(\lambda_\omega^\epsilon(\mathbb{R}^d)\,\tau)^{d/2})\ e^{-\lambda_\omega^\epsilon(\mathbb{R}^d)\,\tau}] \leq \\ &c(d,\eta)\ \mathbb{E}_\epsilon^\ell\,[G_{D,B},\ \exp\{-(\lambda_\omega^\epsilon(\mathbb{R}^d) \wedge M)(1-\eta)\,\tau\}] , \end{aligned} \tag{5.46}$$

for $\eta \in (0,1)$ some given number. If we now use Theorem 2.3 ('Theorem A'), we see that for small $\epsilon$ and any $G_{D,B} \in \mathcal{G}_\epsilon^\ell$ the l.h.s. of (5.46) is smaller than:

$$c(d,\eta)\ \mathbb{E}_\epsilon^\ell\,[\exp\{-(\lambda(\overline{\mathcal{D}}_\epsilon^c(\omega)) \wedge M - \epsilon^\rho)(1-\eta)\,\tau\},\ G_{D,B}] ,$$

where $\lambda(U)$, see (3.1.3), denotes the bottom of the Dirichlet spectrum of $-\frac{1}{2}\,\Delta$ in $U$. However on the event $G_{D,B}$

$$\mathcal{D}_\epsilon(\omega) = D_{\text{per}} \overset{\text{def}}{=} \bigcup_{p \in \mathbb{Z}^d} D + p\ell, \text{ and} \tag{5.47}$$

$$\omega([0,\ell)^d \setminus (D \cup B)) = 0 . \tag{5.48}$$

Using Theorem 3.6 ('Theorem B') we see that for small $\epsilon$ and any in $G_{D,B}$ the l.h.s. of (5.46) is smaller than:

$$\begin{aligned} &c(d,\eta)\ \exp\Big\{-(\lambda(\overline{D}_{\text{per}}^c) \wedge M)(1-\eta)\,\tau + \epsilon^\rho\,\tau \\ &\qquad -\frac{\nu}{\epsilon^d}\,|[0,\ell)^d \backslash (D \cup B)|\Big\} \\ &\leq c(d,\eta)\ \exp\{-\lambda(\overline{D}_{\text{per}}^c) \wedge M(1-\eta)\,\tau \\ &\qquad -\nu\tau\,|[0,\ell)^d \backslash D| + \epsilon^\rho\,\tau + \nu\,\ell^d\,\epsilon^\kappa\,\tau\} . \end{aligned} \tag{5.49}$$

It now follows from (5.43) that summing the above inequalities:

$$\overline{\lim_{t\to\infty}}\ t^{-\frac{d}{d+2}} \log S_t^\ell \leq -(1-\eta)\ \inf_U\{\lambda(U) + \nu\,|[0,\ell)^d \cap U|\} \wedge M ,$$

where $U$ runs over $\ell$ periodic open sets of $\mathbb{R}^d$. Since $\eta \in (0,1)$ is arbitrary, we obtain:

$$\overline{\lim_{t\to\infty}}\ t^{-\frac{d}{d+2}} \log S_t^\ell \le - \inf_{U\ \ell-\text{periodic}} \{\lambda(U) + \nu\,|[0,\ell)^d \cap U|\} \wedge M\,. \tag{5.50}$$

To conclude we use

**Lemma 5.4**: *For any $\delta > 0$, when $\ell$ is large*

$$\begin{aligned} &\delta + \inf_{U\ \ell-\text{periodic}} \{\lambda(U) + \nu\,|[0,\ell)^d \cap U|\}(1+\delta) \ge \\ &\inf_{\widetilde{U}} \{\lambda(\widetilde{U}) + \nu\,|\widetilde{U}|\} = \widetilde{c}(d,\nu)\,. \end{aligned} \tag{5.51}$$

**Proof**: For $\ell$ sufficiently large, we consider a function $h_\ell$ in $C^1(\mathbb{R}^d)$ such that:

$$\begin{aligned} &0 \le h_\ell \le 1,\ h_\ell = 1 \ \text{ on } \ [\sqrt{\ell},\ \ell - \sqrt{\ell}]^d\,, \\ &h_\ell \text{ has compact support in } (0,\ell)^d,\ |\nabla h_\ell| \le c_\ell \to 0, \text{ as } \ell \to \infty\,. \end{aligned} \tag{5.52}$$

Denote by $S_\ell$ the set:

$$S_\ell = \bigcup_{q\in\mathbb{Z}^d} (\sqrt{\ell}, \ell - \sqrt{\ell})^d + q\,\ell\,. \tag{5.53}$$

For $U$ an $\ell$-periodic open set, pick $\varphi \in C_c^\infty(U)$ with

$$\frac{1}{2}\int |\nabla\varphi|^2\, dx \le \lambda(U) + \frac{\delta}{2} \ \text{ and } \ \int \varphi^2 dx = 1\,. \tag{5.54}$$

Observe that

$$\begin{aligned} &\frac{1}{\ell^d}\int_{[0,\ell)^d} dy \int \varphi^2(x)\, 1_{S_\ell^c}(x-y)dx = \\ &\int dx\, \varphi^2(x)\, \frac{1}{\ell^d}\int_{[0,\ell)^d} 1_{S_\ell^c}(x-y)dy \overset{\text{(periodicity and (5.54))}}{=} \\ &\frac{1}{\ell^d}\,|S_\ell^c \cap [0,\ell)^d| \le \frac{2d}{\sqrt{\ell}}\,. \end{aligned} \tag{5.55}$$

We can thus pick $y \in [0,\ell)^d$ such that:

$$\int \varphi^2(x)\, 1_{S_\ell}(x-y)dx \ge 1 - \frac{2d}{\sqrt{\ell}}\,, \tag{5.56}$$

and define

$$\psi(x) = \varphi(x)\, h_\ell(x-y)\,.$$

Then $\psi$ is $C^1$ and has compact support in the $\ell$-periodic open set

$$U' \overset{\text{def}}{=} U \cap \Big( \bigcup_{q \in \mathbb{Z}^d} q\ell + (0,\ell)^d + y \Big), \quad \text{and} \tag{5.57}$$

$$1 \geq \int \psi^2 \, dx \geq 1 - \frac{2d}{\sqrt{\ell}} \,. \tag{5.58}$$

Using the notation $h^y$ to denote $h_\ell(\cdot \; - y)$ we have:

$$\begin{aligned} \frac{1}{2} \int |\nabla \psi|^2 \, dx &= \frac{1}{2} \int \nabla \varphi^2 \, (h^y)^2 + 2\varphi \, h^y \, \nabla \varphi \, \nabla h^y + \varphi^2 \, |\nabla h^y|^2 \, dx \\ &\overset{(5.52)-(5.54)}{\leq} \lambda(U) + \frac{\delta}{2} + \Big( \int \varphi^2 \, |\nabla h^y|^2 \, dx \Big)^{1/2} \Big( \int |\nabla \varphi|^2 \, dx \Big)^{1/2} + \frac{c_\ell^2}{2} \\ &\leq \lambda(U) + \frac{\delta}{2} + c_\ell \sqrt{2\lambda(U) + \delta} + \frac{c_\ell^2}{2} \\ &\leq \lambda(U)(1 + 2c_\ell) + \frac{\delta}{2} + c_\ell \Big( \delta + \frac{c_\ell}{2} + \frac{1}{4} \Big) \,. \end{aligned} \tag{5.59}$$

Each connected component of $U'$ is included in some $q\ell + (0,\ell)^d + y$, for some $q \in \mathbb{Z}^d$. It follows from (5.58) and (5.59) that when $\ell$ is large for any $\ell$ periodic open set $U$

$$\begin{aligned} (1+\delta) \, \lambda(U) + \delta \geq \lambda(U') &= \inf_{q \in \mathbb{Z}^d} \lambda(U \cap (q\ell + (0,\ell)^d + y)) \\ &\overset{\text{(periodicity)}}{=} \lambda(U \cap ((0,\ell)^d + y)) \,. \end{aligned} \tag{5.60}$$

If we now define

$$\widetilde{U} = U \cap ((0,\ell)^d + y) \,, \tag{5.61}$$

then from periodicity $|U \cap [0,\ell)^d| = |\widetilde{U}|$ and

$$(1+\delta)(\lambda |U| + \nu \, |U \cap [0,\ell)^d|) + \delta \geq \nu \, |\widetilde{U}| + \lambda(\widetilde{U}) \,. \tag{5.62}$$

This proves (5.51). □

With the help of the above lemma and (5.50), we thus see that

$$\overline{\lim} \; t^{-\frac{d}{d+2}} \, \log S_T \leq -\widetilde{c}(d,\nu) \,, \tag{5.63}$$

and conclude our (first) proof of (5.29).

This first version uses the argument of projection on the torus (5.41) in order to dominate the true problem. As a result we do not need to use Theorem 2.6

(so to speak Theorem A'!). However this brings us to the variational problem (5.50) and we need Lemma 5.4 to conclude.

One drawback of the projection argument is that it introduces a certain rigidity in the proof, and is difficult to adapt to non-Poissonian point processes for instance describing nonoverlapping traps, see [Szn93b].

*Upper bound (second version):*
As in (5.38), it follows from scaling that for $t > 1$,

$$S_t = \mathbb{E}_\epsilon \otimes E_0\Big[\exp\Big\{ - \int_0^\tau V_\epsilon(Z_s,\omega)ds\Big\}\Big]\,, \tag{5.64}$$

with $\tau = t^{\frac{d}{d+2}}$ as in (5.44).

We shall now split $\Omega \times C(\mathbb{R}_+, \mathbb{R}^d)$ into an essential part $E$ and an inessential part $E^c$. To this end we choose some collection of admissible parameters (see (3.66)). We define (compare with (4.19) - (4.23)) the domain

$$\mathcal{T} = (-[t],[t])^d = (-[\epsilon^{-(d+2)}],[\epsilon^{-(d+2)}])^d\,, \tag{5.65}$$

the constant

$$M = 2\,\tilde{c}(d,\nu)\,, \tag{5.66}$$

then pick $r \in (0, r_0)$, see Theorem 2.6, so that

$$\frac{c_2(d)}{r^2} > 4M\,, \tag{5.67}$$

and choose $R(t,d,\nu) \geq 1$, to be the smallest integer for which

$$\frac{R}{4r} > c_4(d,M), \quad c_3(d)\Big[\frac{R}{4r}\Big] \geq \log t\,. \tag{5.68}$$

As in Section 4B), $r$ specifies the clearing boxes and $R$ the size of the open $\|\cdot\|$-neighborhood around the clearing boxes. Finally we introduce $n_0(d,\nu) \geq 1$, as the smallest integer for which

$$\frac{\nu}{2}\, n_0\, r^d > 2\,\tilde{c}(d,\nu)\,. \tag{5.69}$$

Our essential set $E$ is now

$$\begin{aligned} E &= \{\lambda_\omega^\epsilon(\mathcal{T}) \leq 2\,\tilde{c}(d,\nu),\ |\mathcal{A}_\epsilon \cap 2\mathcal{T}| \leq n_0\} \cap \{T_\mathcal{T} > t^{\frac{d}{d+2}}\} \\ &\overset{\text{def}}{=} F \cap \{T_\mathcal{T} > t^{\frac{d}{d+2}}\}\,. \end{aligned} \tag{5.70}$$

$E$ is essential in the sense that we know (5.36) and

**Lemma 5.5**:

$$\overline{\lim_{t\to\infty}}\; t^{-\frac{d}{d+2}} \log \mathbb{E}_\epsilon \otimes E_0\Big[\exp\Big\{-\int_0^T V_\epsilon(Z_s,\omega)ds\Big\}, E^c\Big] \le -2\,\widetilde{c}(d,\nu)\ . \tag{5.71}$$

**Proof**: It is immediate that

$$\overline{\lim_{t\to\infty}}\; \tau^{-1}\log P_0[T_{\mathcal{T}} \le \tau] = -\infty\ . \tag{5.72}$$

Moreover for large $t$

$$\begin{aligned}
&\mathbb{E}_\epsilon[R_\tau^{\mathcal{T},V_\epsilon}1(0),\ \lambda_\omega^\epsilon(\mathcal{T}) \ge 2\,\widetilde{c}(d,\nu)] \overset{(3.1.9)}{\le} \\
&c(d)\ \mathbb{E}_\epsilon[(1+(\lambda_\omega^\epsilon(\mathcal{T})\,\tau)^{d/2})\,e^{-\lambda_\omega^\epsilon(\mathcal{T})\tau},\ \lambda_\omega^\epsilon(\mathcal{T}) \ge 2\,\widetilde{c}(d,\nu)] \le \\
&c(d)(1+(2\,\widetilde{c}(d,\nu)\tau)^{d/2})\ \exp\{-2\,\widetilde{c}(d,\nu)\tau\}\ .
\end{aligned} \tag{5.73}$$

Finally using (4.24), we see that for large $t$:

$$\begin{aligned}
&\mathbb{P}_\epsilon[|\mathcal{A}_\epsilon \cap 2\mathcal{T}| \ge n_0] \le (4t)^{dn_0}\ \mathbb{P}_\epsilon[[0,1)^d \subseteq \mathcal{A}]^{n_0} \le \\
&(4t)^{dn_0}\ \exp\Big\{-\frac{\nu}{2}\ n_0\, r^d\tau\Big\}\ .
\end{aligned} \tag{5.74}$$

If we now collect (5.72), (5.73), (5.74), our claim (5.71) follows. □

We recall that in the notations of Example 2.5 1), $\mathcal{O}$ denotes open $R\,\|\cdot\|$-neighborhood of $\mathcal{A}$ and we now introduce

$$\begin{aligned}
U(\omega) &= \mathcal{T}\cap\mathcal{O}\backslash\overline{\mathcal{D}}_\epsilon(\omega) \\
V(\omega) &= \mathcal{T}\cap\mathcal{O}\backslash(\mathcal{D}_\epsilon(\omega)\cup\mathcal{B}_\epsilon(\omega))\ ,
\end{aligned} \tag{5.75}$$

so that for large $t$ and any $\omega \in E$,

$$\omega \text{ has no point in } V(\omega), \text{ (see (1.9))}, \tag{5.76}$$

$$|U| \overset{\text{(Theorem 3.6)}}{\le} |V| + |\mathcal{T}\cap\mathcal{O}|\,\epsilon^\kappa \le |V| + (2R+1)^d\ n_0\,\epsilon^\kappa\ . \tag{5.77}$$

Observe that when $\omega$ varies in $E$, the couple of sets $U(\omega)$, $V(\omega)$ can take no more than

$$A(t) = (n_0+1)\ (4t)^{dn_0}\ 2^{2(2R+1)^d\,n_0\,\epsilon^{-d\beta}} \tag{5.78}$$

possible different values, corresponding to the specification of at most $n_0$ clearing boxes in $2\mathcal{T}$ and of the sets $\mathcal{D}_\epsilon(\omega)$, $\mathcal{B}_\epsilon(\omega)$ in the $R$-neighborhood of these clearing boxes.

We thus introduce a covering $\mathcal{G}_t$ of $F$ (and therefore of $E$ (see (5.70)) made of events

$$G_{U,V} = \{\omega \in \Omega,\, U(\omega) = U,\; V(\omega) = V\}\,, \tag{5.79}$$

where $G_{U,V} \cap F \neq \emptyset$. We thus have

$$|\mathcal{G}_t| \leq A(t)\,. \tag{5.80}$$

Now when $t$ is large, for any $G_{U,V}$ in $\mathcal{G}_t$, we have

$$\begin{aligned}
&\mathbb{E}_\epsilon \otimes E_0\Big[G_{U,V} \cap E,\ \exp\Big\{-\int_0^{\mathcal{T}} V_\epsilon(Z_s,\omega)ds\Big\}\Big] \overset{(3.1.9)}{\leq} \\
&\qquad c(d)(1+(2\,\widetilde{c}\,\tau)^{d/2})\,\mathbb{E}_\epsilon[G_{U,V}\ \exp\{-(\lambda^\epsilon_\omega(\mathcal{T}) \wedge M)\tau\}] \\
&\overset{(\text{Theorem 2.6})}{\leq} c'(d)(1+(2\,\widetilde{c}\,\tau)^{d/2}) \\
&\qquad \mathbb{E}_\epsilon[G_{U,V}\ \exp\{-(\lambda^\epsilon_\omega(\mathcal{T} \cap \mathcal{O}) \wedge M)\tau\}] \\
&\overset{(\text{Theorem 2.3})}{\leq} c'(d)(1+(2\,\widetilde{c}\,\tau)^{d/2}) \\
&\qquad \exp\{-(\lambda(U) \wedge M)\tau + \epsilon^\rho\,\tau\}\,\mathbb{P}_\epsilon[G_{U,V}] \\
&\overset{(5.76)-(5.77)}{\leq} c'(d)(1+(2\,\widetilde{c}\,\tau)^{d/2})\ \exp\{\epsilon^\rho\,\tau + (2R+1)^d\,n_0\,\epsilon^\kappa\,\tau \\
&\qquad -(\lambda(U) \wedge M + \nu\,|U|)\,\tau\}\,.
\end{aligned} \tag{5.81}$$

Notice that

$$\lambda(U) \wedge M + \nu\,|U| \geq (\lambda(U) + \nu\,|U|) \wedge M \geq \widetilde{c} \wedge M = \widetilde{c}\,.$$

Collecting (5.71), (5.80), (5.81), and the lower bound shown in (5.35), we have thus shown

**Theorem 5.6**: *If $\mu > \max(\beta,\, 1 - \frac{\rho_0}{d},\, 1 - \frac{\kappa_0}{d})$ for some admissible collection of parameters, then for large $t$*

$$\exp\{-\widetilde{c}(d,\nu)\,t^{\frac{d}{d+2}} - \gamma(d,\nu)\,t^{\frac{d-1}{d+2}}\} \leq S_t \leq \exp\{-\widetilde{c}(d,\nu)\,t^{\frac{d}{d+2}} + t^{\frac{d\mu}{d+2}}\}\,. \tag{5.82}$$

□

**Exercise:**

**1)** For $t > 0$, define

$$\begin{aligned}
L(t) &= \mathbb{E}[r_{\mathbb{R}^d, V(\cdot,\omega)}(t,0,0)] \\
&= (2\pi t)^{-d/2}\,E^t_{0,0}\Big[\exp\Big\{-\nu\int(1 - e^{-\int_0^t W(Z_s - y)ds})dy\Big\}\Big]
\end{aligned} \tag{5.83}$$

a) Show that for $U$ a bounded open subset of $\mathbb{R}^d$

(5.84)
$$L(t) \geq \frac{1}{|U|} \exp\{-\nu\,|U^a| - t\lambda(U)\}$$

where $U^a$ denotes the $a$-neighborhood of $U$.

(Hint: $L(t)\,|U| = \mathbb{E}\Big[\int_U r_{\mathbb{R}^d, V(\cdot,\omega)}(t,x,x)dx\Big]$ ).

b) Show that for $0 < t' < t$:

(5.85)
$$L(t) \leq (2\pi(t-t'))^{-d/2}\, S_{t'}\ .$$

c) Show that

(5.86)
$$\lim_{t\to\infty} t^{-\frac{d}{d+2}} \log L(t) = -\widetilde{c}(d,\nu)\ .$$

2) (Lifshitz tail of the density of states.)
Consider the normalized counting measure:

(5.87)
$$\ell^N_\omega(d\lambda) = \frac{1}{|B(0,N)|} \sum_{i\geq 1} \delta_{\lambda^{i,N}_\omega}\ ,$$

where $\lambda^{i,N}_\omega$ are the Dirichlet eigenvalues (counted with multiplicity) of $-\frac{1}{2}\,\Delta + V(\cdot,\omega)$ in $B(0,N)$, see (3.1.47).

a) Show that on a set of full $\mathbb{P}$-measure, for $t > 0$,

$$\begin{aligned} L^N(t) &\stackrel{\text{def}}{=} \int e^{-t\lambda}\, \ell^N_\omega(d\lambda) \\ &= \frac{1}{|B(0,N)|} \int_{B(0,N)} r_{B(0,N),V(\cdot,\omega)}(t,x,x)dx \underset{N\to\infty}{\longrightarrow} L(t)\ . \end{aligned}$$

(Hint: see Krengel [Kre85], p. 210, or bound $\mathrm{var}(L^N(t))$ and use Borel-Cantelli's lemma).

b) Deduce that on a set of full $\mathbb{P}$-measure

(5.88) $\ell^N_\omega$ converges vaguely on $[0,\infty)$ to a deterministic measure $\ell$ with Laplace transform $L(t)$ ($\ell$ is the so-called *density of states*) .

c) Show that

(5.89)
$$\ell([0,\lambda]) = \exp\Big\{ -\nu\,\Big|B\Big(0, \sqrt{\frac{\lambda_d}{\lambda}}\Big)\Big|(1+o(1))\Big\},\ \text{ as } \lambda\to 0.$$

(Hint: use the de Bruinj-Minlos-Povzner exponential Tauberian theorem, see Bingham-Goldie-Teugels [BGT87], p. 254).

The asymptotics (5.89) is the *Lifshitz tail* effect. It indicates the rarefaction of small eigenvalues due to the presence of obstacles. It should be contrasted to the case $V = 0$ (no obstacles), where one has the usual Weyl asymptotics:

$$\ell([0,\lambda]) \sim \frac{1}{(2\pi)^{d/2}\,\Gamma(\frac{d}{2}+1)}\,\lambda^{d/2}, \quad \text{as } \lambda \to 0\,. \tag{5.90}$$

For more on *Lifshitz tail* effects and *density of states*, see Carmona-Lacroix [CL91], Pastur-Figotin [PF92].

**3)** For $p \geq 1$, an integer and $t \geq 0$, define

$$\begin{aligned} S_{t,p} &= \mathbb{E}\Big[E_0\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big]^p\Big] \\ &= E_0^1 \otimes \ldots \otimes E_0^p\Big[\exp\Big\{-\nu\int_{\mathbb{R}^d}\Big(1-e^{-\sum_1^p \int_0^t W(Z_s^i-y)ds}\Big)dy\Big\}\Big]\,. \end{aligned} \tag{5.91}$$

Show that

$$\begin{aligned} \lim_{t\to\infty} t^{-\frac{d}{d+2}} \log S_{t,p} &= -\inf_{U \text{ open}} \{\nu\,|U| + p\lambda(U)\} \\ &= -p^{\frac{d}{d+2}}\,\widetilde{c}(d,\nu)\,. \end{aligned} \tag{5.92}$$

Thus $(S_{t,p+1})^{\frac{1}{p+1}}$ is much larger in logarithmic principal order than $(S_{t,p})^{\frac{1}{p}}$ when $t$ becomes large. This feature is sometimes used as definition for *intermittency*, see Molchanov [Mol94], Chapter III. □

*Complement: slightly rarefied traps.*

We shall now close this subsection with a discussion of the large $t$ behavior of

$$S_{t,\mu} = \mathbb{E}^\mu \otimes E_0[T > t] = E_0\Big[\exp\Big\{-\frac{\nu}{t^\mu}\,|W_t^a|\Big\}\Big]\,, \tag{5.93}$$

where $\mu > 0$ is some given not too large number, $\mathbb{P}^\mu$ is the Poisson law with constant (rarefied) intensity $\nu t^{-\mu}$, and the hard obstacles are modelled on the closed ball of radius $a$ centered at 0. We are interested here in the not too large $\mu$ situation, (slightly rarefied traps), for which the intuitive picture of the case $\mu = 0$ persists. We shall see how this relates to the formulas (3.64) for $\kappa_0$.

**Theorem 5.7**: $(d \geq 2)$
*Assume* $\mu < \frac{2}{d}$, *then*

(5.94) $$S_t^\mu = \exp\{-t^{\frac{d-2\mu}{d+2}}\, \widetilde{c}(d,\nu)(1+o(1))\}, \quad \text{as } t \to \infty\,.$$

**Proof**: *Lower bound:*
We proceed as in (5.33), and find

$$S_t^\mu \geq c(d)\ \exp\{-\nu t^{-\mu}\, \omega_d (R+a)^d - \lambda_d\, t\, R^{-2}\}\,.$$

We now 'optimize' on $R$ (setting $a = 0$), and choose

(5.95) $$R = \widetilde{R}_0\ t^\rho$$

with $\widetilde{R}_0$ as in (5.32) and $\rho$ such that:

$$-\mu + d\rho = 1 - 2\rho\,, \quad \left(\text{i.e. } \rho = \frac{1+\mu}{d+2}\right)\,.$$

With this choice, we have

$$1 - 2\rho = \frac{d-2\mu}{d+2} > 0 \ \ (\text{since } d \geq 2),$$

and we find

(5.96) $$\varliminf_{t\to\infty}\ t^{-\frac{d-2\mu}{d+2}}\ \log S_t^\mu \geq -\widetilde{c}(d,\nu)\,.$$

*Upper bound:* We define

(5.97) $$\epsilon = t^{-\rho} = t^{-\frac{1+\mu}{d+2}}, \quad \tau = t\epsilon^2 = t^{\frac{d-2\mu}{d+2}}\,,$$

so that using the projection argument, we find

(5.98) $$S_t^\mu \leq \mathbb{P}^{\mu,\ell} \otimes P_0\,[T > \tau]\,,$$

when $\mathbb{P}^{\mu,\ell}$ denotes the law of the periodic point process with period $\ell \geq 1$ (integer), with restriction to $[0,\ell)^d$ Poissonian with intensity $\nu t^{-\mu}\,\epsilon^{-d} = \nu\tau$, and $T$ is the entrance time in the trap configurations $\cup_i \overline{B}(x_i, a\epsilon)$, for $\omega = \sum_i \delta_{x_i}$.

We can proceed exactly as in the proof of the upper bound (first version) of Theorem 5.3, provided we can choose admissible parameters such that in the notations of (5.42)

(5.99) $$|\mathcal{G}_\epsilon^\ell| = \exp\{o(\tau)\}\,.$$

This is certainly the case if we can choose $\beta$ so that

$$\rho d\beta < \frac{d-2\mu}{d+2}\,, \quad \text{i.e.}\ \ d\beta < \frac{d-2\mu}{1+\mu}\,. \tag{5.100}$$

Coming back to (3.66) and the formula (3.64) for $\kappa_0$, we see that when $d = 2$, we can choose $\beta$ as small as we wish. On the other hand, when $d \geq 3$, the condition $\kappa_0 > 0$, only enables to choose $\beta > \beta^*$, where

$$2\beta^* - (d-2)(1-\beta^*) = 0 \ \text{ i.e. } \ d\beta^* = d-2\,.$$

(This corresponds to choosing $L$ large, $\delta$ small, $\alpha$ close to 0, $\gamma$ close to $\beta$, and imposing $\kappa_0 > 0$ in the first line of (3.64)).

Observe that the condition $\mu < \frac{2}{d}$ precisely imposes that $\frac{d-2\mu}{1+\mu} > d-2$, so that we can choose admissible parameters for which (5.100) and therefore (5.99) hold. It then follows as in the proof of Theorem 5.3 that

$$\overline{\lim_{t\to\infty}}\ \tau^{-1}\log S_t^\mu \leq -\widetilde{c}(d,\nu)\,. \tag{5.101}$$

This concludes our proof. □

The condition $\mu < \frac{2}{d}$ is a natural threshold for the validity of Theorem 5.7. Indeed it follows from Jensen's inequality that

$$\begin{aligned} S_t^\mu \ &\geq\ \exp\Big\{-\frac{\nu}{t^\mu}\,E_0[|W_t^a|]\Big\} \\ &\overset{(2.3.61)}{=} \exp\{-\nu\, t^{1-\mu}\,\mathrm{cap}(\overline{B}(0,a))(1+o(1))\}, \\ &\qquad\quad \text{as } t\to\infty, \text{ when } d\geq 3\,, \\ &\overset{(2.4.44)}{=} \exp\Big\{-\nu\, t^{1-\mu}\,\frac{2\pi}{\log t}\,(1+o(1))\Big\}, \\ &\qquad\quad \text{as } t\to\infty, \text{ when } d=2\,. \end{aligned} \tag{5.102}$$

Moreover when $\mu = \frac{2}{d}$, we have

$$1-\mu = \frac{d-2\mu}{d+2} = \frac{d-2}{d}\,.$$

Thus when $\mu = \frac{2}{d}$, the expression in the right-most member of (5.102) is bigger than the r.h.s. of (5.94), when $d = 2$, or $d \geq 3$, and $a$ is small enough.

**Exercise:**

**1)** (Asymptotics for the shrinking Wiener sausage)

Show that when $d \geq 2$,

$$\lim_{t\to\infty}\ t^{-\frac{d}{d+2}}\ \log E_0[\exp\{-\nu\,|W_t^{a(t)}|\}\ = -\widetilde{c}(d,\nu)\,, \tag{5.103}$$

where

$$a(t) = c\,t^{-\alpha}, \text{ with } c > 0 \text{ and}$$

$$0 \le \alpha < \frac{2}{d^2 - 4}, \text{ when } d \ge 3,\ \alpha \ge 0, \text{ when } d = 2\,.$$

(Hint: use scaling and (5.94)).

For more on this see Bolthausen [Bol90], Bolthausen-den Hollander [BdH94], and [Szn90b].

**2)** (An upper bound in the critical regime)

We keep the notations of Theorem 5.7, and assume $\mu = \frac{2}{d}$, where $d \ge 3$.

**a)** for $t > 1$, consider the collection of subcubes

$$t^{-\frac{1-\mu}{d}}\,(q + [0,1)^d),\quad q \in \mathbb{Z}^d\,,$$

which are included in $[0,1)^d$. Denote by $N_t(\omega)$, for $\omega$ a 1-periodic cloud configuration, the number of subcubes receiving a point of $\omega$. Show that for $\kappa \in (0,1)$

$$\text{(5.104)}\qquad \lim_{t\to\infty}\ \inf_{\omega:N_t(\omega)\ge\kappa\, t^{1-\mu}}\ \mathrm{cap}\Big(\bigcup_{x_i\in[0,1)^d} \overline{B}(x_i,\epsilon a)\Big) > 0\,.$$

(Hint: the capacity of each obstacle has same order as the volume of a subcube. Use an analogous procedure as in (3.17) or (3.60); see also the calculations of Theorem 3.2.10).

**b)** Show that for small $\kappa > 0$, (with the notation of (5.98)):

$$\text{(5.105)}\qquad \overline{\lim_{t\to\infty}}\ t^{-(1-\mu)} \log \mathbb{P}^{\mu,\ell=1}[N_t \le \kappa\, t^{1-\mu}] < 0\,.$$

**c)** Prove that

$$\text{(5.106)}\qquad \overline{\lim_{t\to\infty}}\ t^{-(1-\mu)} \log S_t^{\mu} < 0\,.$$

(Hint: use (5.98), (5.105), and (5.104) together with Proposition 3.1.4).

**3)** Show that (5.106) holds as well for $d \ge 3$ and $\frac{2}{d} < \mu < 1$.

(Hint: use Jensen's inequality and (5.106)). □

One can in fact give a heuristic guess of the value of the constant which ought to replace $\tilde{c}(d,\nu)$ in (5.94), when $\mu = \frac{2}{d}$, with $d \ge 3$. The new constant should be

$$\text{(5.107)}\qquad c = \inf_{W\ge 0,\|\varphi\|_2=1} \int \frac{1}{2}\,|\nabla\varphi|^2 + \alpha W\varphi^2 + W\log\frac{W}{\nu} - W + \nu\,dx\,,$$

with $\alpha = \text{cap}(\overline{B}(0,a))$. Intuitively the term $\int W \log \frac{W}{\nu} - W + \nu \, dx$ describes the large deviation cost as $t$ tends to infinity, of producing a profile $\sum_i \frac{1}{t^{1-\mu}} \delta_{x_i}$ 'close to' $W(x)dx$ under $\mathbb{P}^{\mu,\ell}$, ($\ell$ 'large'). In this situation, the hard obstacles in a constant capacity regime, 'behave' like a potential $V = \alpha W$, (cf. comments after Theorem 3.2.10). This corresponds to the term $\inf_{\|\varphi\|_2=1} \frac{1}{2} |\nabla\varphi|^2 + \alpha W \varphi^2 dx$ in (5.107). Performing the minimization over $W$ first, one thus formally finds

$$
\begin{aligned}
c = & \inf_{\|\varphi\|_2=1} \int \frac{1}{2} \, |\nabla\varphi(x)|^2 \\
& + \inf_{w \geq 0} \, (\alpha\varphi^2(x) \, w + w \log \frac{w}{\nu} - w + \nu) dx \\
= & \inf_{\|\varphi\|_2=1} \int \frac{1}{2} \, |\nabla\varphi|^2 + \nu(1 - e^{-\alpha\varphi^2}) dx \, .
\end{aligned}
\tag{5.108}
$$

For much more on the critical regime, we refer to van den Berg- Bolthausen-den Hollander [BvdBdH98].

## 4.6 Notes and References

Earlier versions of the method of enlargement of obstacles are discussed in [Szn90a], [Szn93b] and in the survey article [Szn94a]. The sections 2 and 3 are close to [Szn97a]. A version of the method of enlargement of obstacles in a discrete setting appears in Antal [Ant95], and in the case of Brownian motion on the Sierpinski gasket in Pietruska-Paluba [PP91]. Further applications are mentioned in Chapter 7 §1 and §3.

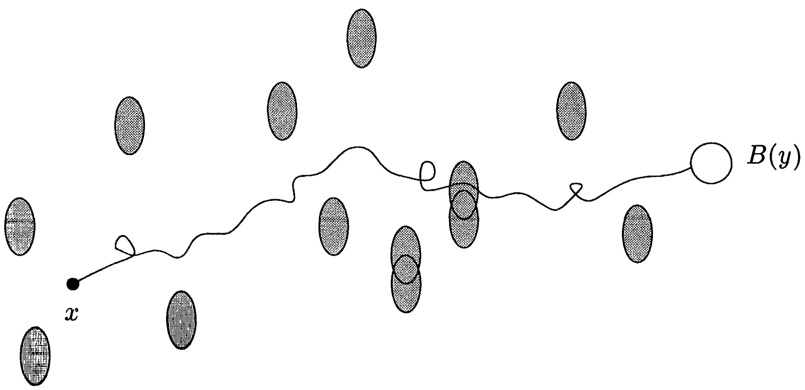
Confidence intervals on the principal Dirichlet eigenvalue of a large block in the presence of Poissonian obstacles, see §4 B), are also derived in Beliaev-Yurinsky [BY95] and Yurinsky [Yur98]. Further results on the size of fluctuations of the principal eigenvalue of a large block appear in section 3 of [Szn97b].

The investigation of the large time Wiener asymptotics of §5 B) has a long history. It appears in rather distinct contexts of the Physics literature, both in the study of spectral properties of the Schrödinger equation with random potentials, and in the investigation of trapping problems of particles diffusing among randomly distributed reaction sites. For instance (5.86) is conjectured, when $d = 3$, in the work of Kac-Luttinger around Bose-Einstein condensation in [KL73], [KL74]. The authors mention the connection with Lifshitz's work [Lif65], [Lif68], and are able to derive lower bounds on the Laplace transform $L(t)$ of the density of states. The asymptotic behavior (5.29) then appears as a conjecture in the mathematical article [Kac74] of Kac. The conjecture is proved in the famous article of Donsker-Varadhan [DV75c]. We also refer to the book of Pastur-Figotin [PF92], which contains a thorough treatment of Lifshitz-tail effects. On the other hand the investigation of trapping problems goes back to von Smoluchowski [vS18]. An account of the development of the subject can be found in the survey article of den Hollander and Weiss [HW94]. The asymptotics (5.29) appear in Grassberger-Procaccia [GP82], see also Kayser-Hubbard [KH84], Berezkhovskii et al. [BMBM93], [BMS86] for examples of further developments in physical chemistry. The end of §5 as well as Chapter 7 provide further references on mathematical works around trapping problems.

# 5. Lyapunov Exponents

We introduce in this chapter two families of norms, the 'quenched' and 'annealed' Lyapunov exponents, which measure various costs attached to performing long crossings among Poissonian obstacles. These norms have various interpretations and in particular measure the respective directional exponential decay of the Green function and its expectation. They naturally appear in the description of several large deviation principles governing long displacements of quenched and annealed Brownian motion among Poissonian obstacles. One of the quenched norms comes in the formulation of the intermittent variational problem controlling the pinning effect in Chapter 6. In Section 1 we give an informal discussion of the objects of interests in this chapter. In Section 2, we construct the quenched Lyapunov exponents, and in Section 3 the annealed Lyapunov exponents. Section 4 presents some large deviation principles where Lyapunov exponents appear. As a natural application of these large deviation principles, we study the transition of behavior for Brownian motion with a constant drift among Poissonian obstacles as the size of the drift varies.

## 5.1 Orientation




We consider throughout this chapter a Poisson point process on $\mathbb{R}^d$, $d \geq 1$ of constant intensity $\nu > 0$, with law $\mathbb{P}$, and soft obstacles defined by:

(1.1) $$V(x,\omega) = \sum_i W(x - x_i), \quad \text{for } x \in \mathbb{R}^d, \ \omega = \sum_i \delta_{x_i} \in \Omega\,.$$

Here notations are similar to Chapter 4 §1, and $W(\cdot)$ is a nonnegative bounded measurable compactly supported function, which is not a.e. equal to 0. We also sometimes consider hard obstacles, which are given as closed balls of radius $a$ centered at the points of supp $\omega$.

For $y \in \mathbb{R}^d$, we denote respectively by

(1.2) $$B(y) \stackrel{\text{def}}{=} \overline{B}(y,1) \ \text{ and } \ H(y) \stackrel{\text{def}}{=} H_{B(y)}\,,$$

the closed unit ball around $y$ and its entrance time for the canonical process $Z_.$ . We are interested in the typical behaviour in 'the large' of quantities such as

(1.3) $$e_\lambda(x,y,\omega) = E_x\Big[\exp\Big\{-\int_0^{H(y)} (\lambda + V)(Z_s,\omega)ds\Big\},\ H(y) < \infty\Big]\,,$$
$$x, y \in \mathbb{R}^d,\ \lambda \geq 0,\ \omega \in \Omega\,,$$

($e_\lambda(\cdot, y,\omega)$ is for typical $\omega$ the $(\lambda + V(\cdot,\omega))$-equilibrium potential of $\overline{B}(y)$, see (2.3.26)),

(1.4) $$a_\lambda(x,y,\omega) = -\inf_{B(x)} \log e_\lambda(\cdot,y,\omega)\,,$$

(1.5) $$d_\lambda(x,y,\omega) = \max\big(a_\lambda(x,y,\omega),\ a_\lambda(y,x,\omega)\big)\,.$$

It turns out that $d_\lambda(\cdot,\cdot,\omega)$ defines for $\mathbb{P}$-a.e. $\omega$ a distance function on $\mathbb{R}^d$, which induces the usual topology. This function somehow measures the cost attached to connecting $x$ and $y$, when Brownian motion moves in the potential $\lambda + V$.

**Example 1.1**: Assume $d = 3$, $\lambda \geq 0$, $\omega = 0$. Then for $x, y \in \mathbb{R}^d$:

(1.6) $$e_\lambda(x,y,\omega) = \frac{1}{|x-y| \vee 1} \exp\Big\{-\sqrt{2\lambda}\,(|x-y| - 1)_+\Big\}$$

(this follows from (2.4.40) and the fact that $g_\lambda(x,y) = \frac{1}{2\pi}\,\frac{e^{-\sqrt{2\lambda}\,|x-y|}}{|x-y|}$, when $d = 3$) and thus:

(1.7) $$d_\lambda(x,y) = \sqrt{2\lambda}\,|x-y| + \log(1 + |x-y|)\,.$$

Observe that due to the term $\log(1 + |x-y|)$, there are no genuine flat triangles for $d_\lambda(\cdot,\cdot)$, that is:

$$d_\lambda(x,z) + d_\lambda(z,y) = d_\lambda(x,y) \Longrightarrow x = z \ \text{ or } \ y = z\,.$$

As a result $d_\lambda$ is not a geodesic distance (for which the distance between any two points is achieved by the length of some geodesic curve joining the two points, cf. Chapter 1 of Gromov [Gro81]). This intuitively corresponds to the fact that Brownian motion has more than one way of going from $x$ to $y$. $\square$

We shall construct in Section 2 a collection of norms $\alpha_\lambda(\cdot)$ on $\mathbb{R}^d$, the *quenched Lyapunov coefficients*, which describe the principal order in the large of the above distances $d_\lambda$, and are such that:

$$\text{(1.8)} \qquad \mathbb{P}\text{-a.s.}, \; d_\lambda(0,y,\omega) \sim \alpha_\lambda(y), \text{ as } y \to \infty, \text{ for } \lambda \geq 0 .$$

It is a special feature of the randomness of the Poissonian obstacles, that even when $\lambda = 0$, (i.e. at the bottom of the spectrum of $-\frac{1}{2}\,\Delta + V(\cdot,\omega)$, for typical $\omega$), $d_\lambda(0,y,\omega)$ has a linear growth when $y \to \infty$. This is for instance in contrast to (1.7), corresponding to the atypical $\omega = 0$.

The quenched Lyapunov coefficients have a certain robustness, and we can replace in (1.7), $d_\lambda$ by $a_\lambda$, $-\log e_\lambda$, or $-\log g_\lambda$, where for $x, y \in \mathbb{R}^d$, $\lambda \geq 0$, $\omega \in \Omega$:

$$\text{(1.9)} \qquad g_\lambda(x,y,\omega) = \int_0^\infty e^{-\lambda s}\, r(s,x,y,\omega)ds \; ,$$

denotes the $(\lambda + V(\cdot,\omega))$-Green function, and for $t > 0$,

$$\text{(1.10)} \qquad \begin{aligned} r(t,x,y,\omega) &= (2\pi t)^{-d/2} \exp\Big\{-\frac{(y-x)^2}{2t}\Big\} \\ &\quad E^t_{x,y}\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big] \\ &= r_{\mathbb{R}^d, V(\cdot,\omega)}(t,x,y), \end{aligned}$$

in the notations of Chapter 1 §3 .

The distance functions $d_\lambda(\cdot,\cdot,\omega)$ have certain similarities (but also differences) to the (geodesic) distances which one considers in first passage percolation theory (see Hammersley-Welsh [HW10], Kesten [Kes86]), and go under the name of point to point passage time. In this light, and in view of the spirit of the proof of (1.8), which relies on the subadditive ergodic theorem, one should view (1.8) and its variants as 'shape theorems'. The terminology 'shape' simply refers to the unit ball of $\alpha_\lambda(\cdot)$.

Presently, one does not know too much in general, when $d \geq 2$, about the norms $\alpha_\lambda(\cdot)$. However, when $W(\cdot)$ is rotationally invariant, the norms $\alpha_\lambda(\cdot)$ turn out to be proportional to the Euclidean norm, although the proportionality factor is unknown.

We shall introduce in Section 3, a second collection of norms on $\mathbb{R}^d$, the *annealed Lyapunov coefficients* $\beta_\lambda(\cdot)$.

The role of $e_\lambda(\cdot,\cdot,\omega)$ and $g_\lambda(\cdot,\cdot,\omega)$ in the above discussion is played in this case by $\mathbb{E}[e_\lambda(\cdot,\cdot,\omega)]$ and $\mathbb{E}[g_\lambda(\cdot,\cdot,\omega)]$. As $y \to \infty$,

$$-\log \mathbb{E}[e_\lambda(0,y,\omega)] \sim \beta_\lambda(y) \, , \tag{1.11}$$

with a similar result with $g_\lambda$ in place of $e_\lambda$.

One can define the annealed norms $\beta_\lambda(\cdot)$ in the case of hard obstacles as well. In this situation $e_\lambda$ stands for:

$$\begin{aligned} &e_\lambda(x,y,\omega) = E_x[H(y) < T, \ \exp\{-\lambda H(y)\}], \\ &\lambda \geq 0, \ x,y \in \mathbb{R}^d, \ \omega \in \Omega \, , \end{aligned} \tag{1.12}$$

with $T$ the entrance time in the obstacle set

$$S = \bigcup_i \overline{B}(x_i,a), \quad \omega = \sum_i \delta_{x_i} \, , \tag{1.13}$$

One defines $g_\lambda$ in a similar fashion. Unlike the case of soft obstacles, $e_\lambda$ and $g_\lambda$ may now take the value 0.

It follows from Jensen's inequality that in the soft obstacles case:

$$\beta_\lambda(\cdot) \leq \alpha_\lambda(\cdot) \, , \tag{1.14}$$

and it is possible to choose $W$ so that $\beta_\lambda(\cdot)$ and $\alpha_\lambda(\cdot)$ are distinct. One suspects, that in sufficiently high dimension and rarefied soft obstacles $\alpha_\lambda(\cdot) = \beta_\lambda(\cdot)$, on grounds of analogies with the case of directed polymers, see Bolthausen [Bol89], Sinai [Sin95].

The norms $\alpha_\lambda(\cdot)$ and $\beta_\lambda(\cdot)$ play an important role in the respective descriptions of the 'off-diagonal' properties of quenched and annealed Brownian motion in a Poissonian potential, cf. Chapter 7 §2. Quenched and annealed *Brownian motion in a Poissonian potential* are respectively governed by the *quenched path measure*:

$$Q_{t,\omega}(dw) \overset{\text{def}}{=} \frac{1}{S_{t,\omega}} \exp\Big\{ - \int_0^t V(Z_s,\omega)ds \Big\} \, P_0(dw) \, , \tag{1.15}$$

and the *annealed path measure*:

$$Q_t(dw,d\omega) \overset{\text{def}}{=} \frac{1}{S_t} \exp\Big\{ - \int_0^t V(Z_s,\omega)ds \Big\} \, P_0(dw) \, \mathbb{P}(d\omega) \, , \tag{1.16}$$

The most direct physical interpretation of these measures, as mentioned in the Introduction, corresponds to the motion of a particle diffusing in a partially absorbing medium, conditioned not to be absorbed up to (a long) time $t$. The particle evolves in a 'typical medium' $\omega$; for the quenched path measure it starts from the origin, whereas for the annealed measure, the starting point is chosen uniformly over a 'large box' by an ergodic mean, and $Z_.$ describes relative displacements.

We shall see in Section 4 that the position $Z_t$ of the path at time $t$ satisfies large deviation principles in scale $t$ relative to these measures. Informally this corresponds to:

(1.17) $Q_{t,\omega}(\text{`}Z_t \sim t \cdot x\text{'}) \approx e^{-tI(x)}$, as $t \to \infty$, for $x \in \mathbb{R}^d$, $\omega$ being typical ,

(1.18) $$Q_t(\text{`}Z_t \sim t \cdot x\text{'}) \approx e^{-tJ(x)}, \ \text{as } t \to \infty,\ x \in \mathbb{R}^d\ .$$

The convex rate functions $I(\cdot)$ and $J(\cdot)$, $\mathbb{R}^d \to [0,\infty)$, can be expressed in terms of the respective Lyapunov norms $\alpha_\lambda(\cdot)$ and $\beta_\lambda(\cdot)$.

A motivation for these large deviation results is their applications to the study of quenched and annealed Brownian motion with a constant drift $h$ in a Poissonian potential (i.e. one replaces Wiener measure $P_0$ in (1.15), (1.16) by Wiener measure with a constant drift $P_0^h$). As we shall see in Section 4, a transition of regime between a 'sub-ballistic regime' for 'small $h$' and a 'ballistic regime' for large $h$ takes place. The critical values of $h$ correspond to the respective unit spheres of the dual norms to $\alpha_0(\cdot)$ and $\beta_0(\cdot)$.

## 5.2 Quenched Lyapunov Exponents

Our objective is to construct the quenched Lyapunov exponents, and study some of their properties. We are in the 'soft obstacle' setting, and we keep the notations of the previous section. We shall begin with some controls on the quantities introduced in Chapter 5 §1.

**Lemma 2.1**:

(2.1) *$e_\lambda(x,y,\omega)$, $a_\lambda(x,y,\omega)$ are measurable in $\omega \in \Omega$, and for fixed $\omega \in \Omega$, are jointly continuous in $(\lambda,x,y) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ .*

(2.2) $$a_\lambda(x,y,\omega) \le a_\lambda(x,z,\omega) + a_\lambda(z,y,\omega),\ x,y,z \in \mathbb{R}^d,\ \omega \in \Omega\ ,$$

*and the same holds for $d_\lambda$.*

(2.3) *For $\mathbb{P}$-a.e. $\omega, d_\lambda(\cdot,\cdot,\omega)$, for $\lambda \ge 0$, defines a distance function on $\mathbb{R}^d$, which induces the usual topology.*

**Proof:**

*Proof of (2.1)*: The measurability of $e_\lambda$ in the $\omega$ variable comes from the joint measurability on $C(\mathbb{R}_+, \mathbb{R}^d) \times \Omega$ of:

$$(w, \omega) \to \int_0^{H(y)} (\lambda + V)(Z_s(w), \omega) ds = \lambda H(y)(w) +$$
$$\int_0^\infty \int_{\mathbb{R}^d} W(Z_s(w) - y)\, 1(s \le H(y)(w))\, ds\, \omega(dy)\,.$$

Let us now show the joint continuity of $e_.(\cdot, \cdot, \omega)$, $\omega \in \Omega$ fixed. If $\lambda \ge 0$, $y \in \mathbb{R}^d$, $\omega \in \Omega$ are fixed, $e_\lambda(\cdot, y, \omega)$ is continuous.

Indeed if $\omega = 0$, $\lambda = 0$, and $d = 1, 2$, it equals 1, and otherwise, this follows from Proposition 2.3.8. As a result of the strong Markov property:

$$(2.6) \quad e_\lambda(x, y, \omega) \ge e_\lambda(x, z, \omega) \inf_{B(z)} e_\lambda(\cdot, y, \omega), \;\; x, y, z \in \mathbb{R}^d, \; \lambda \ge 0, \; \omega \in \Omega\,.$$

Dropping $\omega$ from the notation, we thus see that:

$$(2.7) \quad \begin{aligned} &|e_\lambda(x', y') - e_\lambda(x, y)| \le |e_\lambda(x', y) - e_\lambda(x, y)| \\ &+ |e_\lambda(x', y) - e_\lambda(x', y')| \overset{(2.6)}{\le} |e_\lambda(x', y) - e_\lambda(x, y)| \\ &+ 1 - \exp\{-d_\lambda(y', y)\}\,. \end{aligned}$$

It is also plain that for $\omega \in \Omega$, $\lambda \ge 0$, $R > 0$

$$(2.8) \qquad \lim_{\eta \to 0} \sup_{\substack{|z'-z| \le \eta \\ |z| \le R, |z'| \le R}} a_\lambda(z, z', \omega) = 0\,.$$

Coming back to (2.7), we thus see that $e_\lambda(\cdot, \cdot, \omega)$ is continuous. On the other hand, for fixed $x, y, \omega$, $e_\lambda(x, y, \omega)$ is continuous decreasing in $\lambda$, and the joint continuity now follows by a Dini type argument.

*Proof of (2.2)*: This is immediate from (2.6).

*Proof of (2.3)*: $d_\lambda(\cdot, \cdot, \omega)$ is symmetric and in view of (2.2) satisfies the triangle inequality. It is also continuous in the $(\lambda, x, y)$ variables. Observe that for $\mathbb{P}$-a.e. $\omega$

$$(2.9) \qquad \int_0^{H(y)} \lambda + V(Z_s, \omega) ds \text{ is not } P_x \text{ negligible if } |x - y| > 1\,.$$

As a result for such $\omega$:

$$d_\lambda(x, y, \omega) = 0 \Longrightarrow x = y\,,$$

and $d_\lambda$ defines a (continuous) distance function on $\mathbb{R}^d$, which is such that

$$\underline{\lim}_{x \to \infty} d_\lambda(x, y, \omega) > 0, \;\; \text{for } y \in \mathbb{R}^d\,.$$

It follows that $d_\lambda(\cdot, \cdot, \omega)$ induces the usual topology on $\mathbb{R}^d$. □

We shall now derive controls which are helpful in comparing the various quantities $-\log e_\lambda$, $-\log g_\lambda$, $a_\lambda$, $d_\lambda$. We define for $\lambda \geq 0$, $z \in \mathbb{R}^d$, $\omega \in \Omega$, the nonnegative function:

$$(2.10)\quad F_\lambda(z,\omega) = \lambda + \log^+\Big(\int_{B(z)\times B(z)} g_\lambda(x_1,x_2,\omega)dx_1\,dx_2\Big) + \sup_{\overline{B}(z)} V(\cdot,\omega)\ ,$$

Except for the case of dimension $d = 1,2$, $\lambda = 0$, $\omega = 0$, $F_\lambda$ is locally bounded, (see the proof of Theorem 2.2.4). In fact we shall see that for typical $\omega$, $F_\lambda(\cdot,\omega)$ does not grow too fast at infinity. Our main interest in $F_\lambda$ comes from:

**Proposition 2.2**: *There exists a positive constant $c(d)$ such that for $\omega \in \Omega$, $\lambda > 0$, $x,y \in \mathbb{R}^d$, with $|x-y| > 4$,*

$$(2.11)\quad \begin{aligned} &\max\{|d_\lambda(x,y,\omega) + \log g_\lambda(x,y,\omega)|,\ |d_\lambda(x,y,\omega) + \log e_\lambda(x,y,\omega)|\ ,\\ &|d_\lambda(x,y,\omega) - a_\lambda(x,y,\omega)|\} \leq c(d)(1 + F_\lambda(x,\omega) + F_\lambda(y,\omega))\ . \end{aligned}$$

**Proof**: With no loss of generality, we assume $d \geq 3$, or $\lambda > 0$, or $\omega \neq 0$, otherwise $F_0(\cdot,\omega = 0)$ and there is nothing to prove. The assumptions (2.2.1), (2.2.2), (2.2.13) are then fulfilled and $e_\lambda(\cdot,y,\omega)$ now appears as the $(\lambda + V(\cdot,\omega))$-equilibrium potential of $B(y)$, (see (2.3.26)). As a result of (2.3.38)

$$(2.12)\quad e_\lambda(x,y,\omega) = \int_{B(y)} g_\lambda(x,z,\omega)\, e_y^{\lambda,\omega}(dz)\ ,$$

where $e_y^{\lambda,\omega}$ is the $(\lambda + V(\cdot,\omega))$-equilibrium measure of $B(y)$.

We shall now derive certain quantitative local estimates on $g_\lambda$ (Harnack-type inequalities). When $x' \in B(x)$, $y' \in B(y)$, with $|x-y| > 4$, it follows from (2.2.11) that:

$$(2.13)\quad \begin{aligned} &g_\lambda(x',y',\omega) =\\ &E_{x'}\Big[\exp\Big\{-\int_0^{T_{B(x,2)}} (\lambda + V)(Z_s,\omega)ds\Big\}\, g_\lambda(Z_{T_{B(x,2)}},y')\Big]\ . \end{aligned}$$

Conditioning on $Z_{T_{B(x,2)}}$, and using Jensen's inequality, we thus obtain

$$(2.14)\quad \begin{aligned} &g_\lambda(x',y',\omega) \geq\\ &E_{x'}\,[\exp\{-E_{x'}\,[(\lambda + \sup_{\overline{B}(x,2)} V(\cdot,\omega))\,T_{B(x,2)}|Z_{T_{B(x,2)}}]\}\\ &\qquad g_\lambda(Z_{T_{B(x,2)}},y')]\ . \end{aligned}$$

We now introduce the density $v_x(x',z)$ of the harmonic measure $P_{x'}[Z_{T_{B(x,2)}} \in dz]$ with respect to the normalized surface measure $\overline{d}S$ on $\partial\overline{B}(x,2)$, which equals (see Durrett [Dur84], p. 36):

$$v_x(x',z) = R^{d-2} \,\frac{R^2 - |x'-z|^2}{|x'-z|^d}\,, \quad \text{when } d \geq 2, \text{ with } R = 2\,, \tag{2.15}$$

and

$$\begin{aligned} v_x(x', x+2) &= \frac{1}{2}\,(x' - x + 2),\; v_x(x', x-2) \\ &= \frac{1}{2}\,(2 + x - x'), \;\text{ when } d = 1\,, \end{aligned} \tag{2.16}$$

(the normalized surface measure is understood as $\frac{1}{2}\,(\delta_{x-2} + \delta_{x+2})$ in this case). We have

**Lemma 2.3**:

$$E_{x'}\,[T_{B(x+2)} | Z_{T_{B(x,2)}}] = h_x(x', Z_{T_{B(x,2)}}),\; P_{x'}\text{–a.s.}\,, \tag{2.17}$$

*where for $x' \in B(x,2)$, $z \in \partial B(x,2)$:*

$$h_x(x',z) = \frac{1}{v_x(x',z)} \int_{B(x,2)} g_{B(x,2),V=0}\,(x',y)\, v_x(y,z)\, dy \tag{2.18}$$

**Proof**: Since $v_x(y,z) \sim \text{const } |y-z|^{1-d}$ near $z$, one easily argues that $h(\cdot,\cdot)$ is uniformly bounded on compact subsets of $B(x,2) \times \partial B(x,2)$. Moreover, if $\varphi$ is bounded measurable on $\mathbb{R}^d$, and $T$ stands for $T_{B(x,2)}$:

$$\begin{aligned} E_{x'}\,[h_x(x',Z_T)\,\varphi(Z_T)] &= \int_{\partial B(x,2)} v_x(x',z)\, h_x(x',z)\,\varphi(z)\,\overline{d}S(z) \\ &\overset{(2.18)}{=} \int_0^\infty \int_{\partial B(x,2)} r_{B(x,2),V=0}\,(s,x',y)\, v_x(y,z)\,\varphi(z)\, dy\, ds\, \overline{d}S(z) \\ &= \int_0^\infty ds\, E_{x'}\,[T > s,\, E_{Z_s}\,[\varphi(Z_T)]] \\ &\overset{\left(\substack{\text{strong Markov} \\ \text{property}}\right)}{=} \int_0^\infty ds\, E_{x'}\;\, [T > s,\; \varphi(Z_T)] = E_{x'}\,[T\varphi(Z_T)]\,. \end{aligned}$$

This proves our claim. □

With the help of the above lemma, we find:

$$\begin{aligned} &g_\lambda(x',y',\omega) \geq \\ &E_{x'}\,[\exp\{-(\lambda + \sup_{\overline{B}(x,2)} V(\cdot,\omega))\, h(x', Z_{T_{B(x,2)}})\}\, g(Z_{T_{B(x,2)}}, y')] \end{aligned} \tag{2.19}$$

iterating on the $y'$ variable,

$$\geq E_{x'} \otimes E_{y'} \Big[\exp\{-(\lambda + \sup_{\overline{B}(x,2)} V(\cdot,\omega))\, h(x', Z_{T_{B(x,2)}}) - (\lambda + \sup_{\overline{B}(y,2)} V(\cdot,\omega))\, h(y', Z'_{T_{B(y,2)}})\}\, g(Z_{T_{B(x,2)}},\, Z'_{T_{B(x,2)}})\Big]\,,$$

with obvious notations .

Since we also have

$$g_\lambda(x', y', \omega) \leq E_{x'} \otimes E_{y'} \left[g(Z_{T_{B(x,2)}},\, Z'_{T_{B(x,2)}})\right]\,, \tag{2.20}$$

we obtain the following Harnack-type inequality: for $x'_1, x'_2 \in B(x)$, $y'_1, y'_2 \in B(y)$,

$$\begin{aligned} &\frac{1}{A} \leq g_\lambda(x'_1, y'_1, \omega) g_\lambda(x'_2,\, y'_2, \omega) \leq A\,, \quad \text{where} \\ &A = \left(\frac{\sup v_0}{\inf v_0}\right)^2 \exp\{(\sup h_0)(2\lambda + \sup_{\overline{B}(x,2)} V(\cdot,\omega) + \sup_{\overline{B}(y,2)} V(\cdot,\omega)\}\,, \end{aligned} \tag{2.21}$$

and the above sup and inf run over $\overline{B}(0,1) \times \partial B(0,2)$.

Let us note that by an analogous reasoning for $|x-y| > 3$, $x'_1, x'_2 \in B(x)$, and $\omega \in \Omega$,

$$\begin{aligned} &\frac{1}{\widetilde{A}} \leq e_\lambda(x'_1, y, \omega) \,/\, e_\lambda(x'_2, y, \omega) \leq \widetilde{A}, \quad \text{where} \\ &\widetilde{A} = \frac{\sup v_0}{\inf v_0} \;\exp\{(\sup h_0)(\lambda + \sup_{\overline{B}(x,2)} V(\cdot,\omega))\}\,. \end{aligned} \tag{2.22}$$

Coming back to formula (2.12), we also know that for $z \in \mathbb{R}^d$

$$e_z^{\lambda,\omega}\,(B(z))^{-1} \overset{(2.4.49)}{=} \inf\left\{\int\!\!\int g_\lambda(z_1, z_2, \omega)\, d\mu(z_2),\, \mu \in M_1(B(z))\right\}. \tag{2.23}$$

Since for $z_1, z_2 \in B(z)$,

$$\begin{aligned} g_\lambda(z_1, z_2, \omega) \;&\geq\; \int_0^1 ds\, e^{-\lambda s}\, r_{B(z,2),V}\,(s, z_1, z_2) \\ &\geq\; c(d)\, \exp\{-(\lambda + \sup_{\overline{B}(z,2)} V(\cdot,\omega))\}\,. \end{aligned} \tag{2.24}$$

We thus see that for $z \in \mathbb{R}^d$

$$\begin{aligned} &|B(0)|^2 \left(\int_{B(z)\times B(z)} g_\lambda(z_1, z_2, \omega)\, dz_1\, dz_2 \vee 1\right)^{-1} \\ &\leq e_z^{\lambda,\omega}(B(z)) \leq c^{-1}(d)\, \exp\{\lambda + \sup_{\overline{B}(z,2)} V(\cdot,\omega)\}\,. \end{aligned} \tag{2.25}$$

Our claim (2.11) now immediately follows from the representation formula (2.12),the respective definitions of $\alpha_\lambda$, $d_\lambda$, and the estimates (2.21),(2.25). □

As we shall now see the functions $F_\lambda(\cdot,\omega)$ typically have moderate growth at infinity.

**Lemma 2.4**: *For* $1 \le q < \infty$, $R > 0$,

$$\mathbb{E}\Big[\Big(\int_{B(0,R)\times B(0,R)} g_0(z,z',\omega)dz\,dz'\Big)^q\Big] < \infty\,. \tag{2.26}$$

*On a set of full* $\mathbb{P}$*-measure, for any* $M > 0$,

$$\overline{\lim_{|z|\to\infty}}\,\{\sup_{0\le\lambda\le M} F_\lambda(z,\omega)\}/\log|z| < \infty\,. \tag{2.27}$$

**Proof**: The claim (2.27) easily follows by a Borel-Cantelli argument from (2.26), (4.5.12) and the definition (2.10) of $F_\lambda$.

We shall now prove (2.26).

The claim only requires a proof when $d = 1, 2$, since otherwise $g_0(\cdot,\cdot,\omega) \le g_{\mathbb{R}^d, V=0}(\cdot,\cdot)$. In any case the argument we provide holds for arbitrary dimension. We have for $R > 0$, and $q \ge 1$ an integer:

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\frac{1}{|B(0,R)|}\int_{B(0,R)\times B(0,R)} g_0(z,z',\omega)\,dz\,dz'\Big)^q\Big] \overset{\text{(Jensen)}}{\le} \\
&\mathbb{E}\Big[\frac{1}{|B(0,R)|}\int_{B(0,R)} dz\Big(\int_{B(0,R)} g_0(z,z',\omega)dz'\Big)^q\Big] \overset{\left(\substack{\text{translation}\\ \text{invariance}}\right)}{\le} \\
&\mathbb{E}\Big[\Big(\int_{\mathbb{R}^d} g_0(0,z',\omega)\,dz'\Big)^q\Big] = \\
&\int_0^\infty ds_1 \dots \int_0^\infty ds_q\, \mathbb{E}\Big[E_0\Big[\exp\Big\{-\int_0^{s_1} V(Z_u,\omega)du\Big\}\Big]\dots \\
&E_0\Big[\exp\Big\{-\int_0^{s_q} V(Z_u,\omega)du\Big\}\Big]\Big] \overset{\text{(Hölder)}}{\le} \\
&\int_0^\infty ds_1 \dots \int_0^\infty ds_q\, \mathbb{E}\times E_0\Big[\exp\Big\{-\int_0^{s_1} q\,V(Z_u,\omega)du\Big\}\Big]^{1/q}\dots \\
&\mathbb{E}\otimes E_0\Big[\exp\Big\{-\int_0^{s_q} q\,V(Z_u,\omega)du\Big\}\Big]^{1/q} = \\
&\Big(\int_0^\infty ds\,\mathbb{E}\otimes E_0\Big[\exp\Big\{-\int_0^s q\,V(Z_u,\omega)du\Big]\Big)^q \overset{\text{(Theorem 4.5.3)}}{<} \infty\,.
\end{aligned} \tag{2.28}$$

This proves (2.26). □

**Exercise**: Show that

$$(2.29)\qquad \int_{B(z,1)} g_0(z,z',\omega)dz' \;\leq\; \begin{cases} c(W(\cdot))(1+\log^+\{d(z,\operatorname{supp}\omega)\}), \\ \quad \text{when } d=2\,, \\ c'(W(\cdot))(1+d(z,\operatorname{supp}\omega)), \\ \quad \text{when } d=1\,. \end{cases}$$

(Hint: use scaling and a similar calculation as in example 2.2.5). □

We are now ready to prove the shape theorem. The Lyapunov coefficients $\alpha_\lambda(\cdot)$ for the time being appear as semi-norms (i.e. we do not yet know whether $\alpha_\lambda(x) = 0 \Longrightarrow x = 0$). We shall later see in Proposition 2.8 that they are bona fide norms on $\mathbb{R}^d$.

**Theorem 2.5**: *(shape theorem)*

*There exists a nonnegative function $\alpha_\lambda(x)$ defined on $[0,\infty) \times \mathbb{R}^d$, which is jointly continuous, concave increasing in the $\lambda$ variable, which for fixed $\lambda$ defines a (semi)-norm on $\mathbb{R}^d$, such that on a set of full $\mathbb{P}$-measure, for each $M > 0$,*

$$(2.30)\qquad \lim_{x\to\infty}\ \sup_{0\leq\lambda\leq M}\ \frac{1}{|x|}\,|d_\lambda(0,x,\omega) - \alpha_\lambda(x)| = 0\,.$$

*One can replace $d_\lambda(0,x,\omega)$ by either one of $-\log e_\lambda(0,x,\omega)$, $-\log e_\lambda(x,0,\omega)$, $a_\lambda(0,x,\omega)$, $a_\lambda(x,0,\omega)$, $-\log g_\lambda(0,x,\omega)$ in the above statement. These limits hold in $L^1(\mathbb{P})$ as well. Moreover, for $x \in \mathbb{R}^d$*

$$(2.31)\qquad \alpha_\lambda(x) \leq \sqrt{2(\lambda + \lambda_d + \|W\|_\infty\, \nu\omega_d(a+2)^d)}\ |x|\,.$$

**Proof**: The proof will consist of several steps.

*First step:* lower bounds on $e_\lambda$.

Using similar tubular estimates for Brownian motion as in (4.5.20), we have for $x,y \in \mathbb{R}^d$, $z \in B(x)$, $\lambda \geq 0$, $\omega \in \Omega$, $t > 0$:

$$(2.32)\qquad \begin{aligned} e_\lambda(z,y,\omega) &\geq P_z\Big[\sup_{0\leq s\leq t}\Big|Z_s - \Big(z + \tfrac{s}{t}\,(y-z)\Big)\Big| < 1\Big] \times \\ &\exp\Big\{-\int_0^t \Big[\lambda + \omega(\overline{B}\Big(x + \tfrac{s}{t}\,(y-x), 2+a\Big)\ \|W\|_\infty\Big] ds\Big\} \\ &\geq c(d)\ \exp\Big\{-\lambda_d\, t - \tfrac{1}{2}\,\frac{(y-z)^2}{t} - \lambda t \\ &-\|W\|_\infty \int_0^t \omega\Big(\overline{B}\Big(x + \tfrac{s}{t}\,(y-x), 2+a\Big)\Big)\, ds\Big\}\,. \end{aligned}$$

Exchanging the role of $x$ and $y$ in (2.32), we see that

$$\mathbb{E}[d_\lambda(x,y,\omega)] < \infty, \quad \text{for } x,y \in \mathbb{R}^d, \ \lambda \geq 0 . \tag{2.33}$$

*Second step:* Construction of $\alpha_\lambda(\cdot)$.

We choose a fixed $\lambda \geq 0$, $v \in \mathbb{R}^d \backslash \{0\}$. Then the doubly indexed sequence

$$X_{m,n} \stackrel{\text{def}}{=} d_\lambda(mv, nv, \omega), \ 0 \leq m \leq n ,$$

thanks to (2.3), the translation invariance of $\mathbb{P}$ and (2.33) satisfies:

- $X_{0,0} = 0, \ X_{m,n} \leq X_{m,k} + X_{k,n} \ , 0 \leq m \leq k \leq n,$
- $\{X_{m,n}, \ 0 \leq m \leq n\}$ has same distribution under $\mathbb{P}$ as $\{X_{m+1,n+1}, 0 \leq m \leq n\}$,
- $\mathbb{E}[X_{0,1}] < \infty$.

The subadditive ergodic theorem (see Liggett [Lig85], p. 277) thus applies to the $\{X_{m,n}, 0 \leq m \leq n\}$. As a result, there exists nonnegative coefficients $\alpha_\lambda(v)$ such that:

$$\lim_{n\to\infty} \frac{1}{n} \, d_\lambda(0, nv, \omega) \underset{L^1(\mathbb{P})}{\overset{\mathbb{P}-\text{a.s.}}{=}} \alpha_\lambda(v), \ v \in \mathbb{R}^d , \tag{2.34}$$

(the case $v = 0$, is of course trivial). Moreover

$$\alpha_\lambda(v) = \inf_{n\geq 1} \frac{1}{n} \, \mathbb{E}[d_\lambda(0, nv, \omega)] = \lim_{n} \frac{1}{n} \, \mathbb{E}[d_\lambda(0, nv, \omega)] . \tag{2.35}$$

Observe that for $v, v' \in \mathbb{R}^d$

$$\mathbb{E}[d_\lambda(0, n(v+v'), \omega)] \overset{\left(\substack{\text{(2.3) and trans-} \\ \text{lation invariance}}\right)}{\leq} \mathbb{E}[d_\lambda(0, nv, \omega)] + \mathbb{E}[d_\lambda(0, nv', \omega)]$$

as a consequence

$$\alpha_\lambda(v+v') \leq \alpha_\lambda(v) + \alpha_\lambda(v'), \ v, v' \in \mathbb{R}^d . \tag{2.36}$$

Using similar arguments:

$$\mathbb{E}[d_\lambda(0, nv, \omega)] \overset{\left(\substack{\text{translation} \\ \text{invariance}}\right)}{=} \mathbb{E}[d_\lambda(-nv, 0, \omega)] \overset{(2.3)}{=} \mathbb{E}[d_\lambda(0, -nv, \omega)]$$

so that

$$\alpha_\lambda(v) = \alpha_\lambda(-v), \ v \in \mathbb{R}^d . \tag{2.37}$$

It also follows from (2.32) with $x = 0$, $y = nv$, and $y = 0$, $x = nv$, where $t = n|v|/\sqrt{2(\lambda + \lambda_d + \|W\|_\infty \, \nu\omega_d(a+2)^d)}$, that

$$
\begin{aligned}
\alpha_\lambda(v) &= \lim_{n\to\infty} \frac{1}{n} \, \mathbb{E}[d_\lambda(0, nv, \omega)] \\
&\le \sqrt{2(\lambda + \lambda_d + \|W\|_\infty \, \nu\omega_d(a+2)^d)} \; |v| \, ,
\end{aligned} \tag{2.38}
$$

which proves (2.31).

Then for $p \ge 0$, $q \ge 1$ integers and $v$ in $\mathbb{R}^d$:

$$
\begin{aligned}
\alpha_\lambda\Big(\frac{p}{q}\, v\Big) &= \lim_{n\to\infty} \frac{1}{nq} \, \mathbb{E}\Big[d_\lambda(0, nq\, \frac{p}{q}\, v, \omega)\Big] \\
&= \frac{p}{q} \lim_{n\to\infty} \frac{1}{np} \, \mathbb{E}[d_\lambda(0, npv, \omega)] = \frac{p}{q} \, \alpha_\lambda(v) \, .
\end{aligned}
$$

Using (2.36) - (2.38) we immediately deduce that

$$
\alpha_\lambda(\gamma v) = |\gamma| \, \alpha_\lambda(v), \quad \gamma \in \mathbb{R}, \; v \in \mathbb{R}^d \, . \tag{2.39}
$$

In other words we have proved that for fixed $\lambda$, $\alpha_\lambda(\cdot)$ defines a semi-norm on $\mathbb{R}^d$. We shall now prove that $\alpha_.(\cdot)$ is jointly continuous, concave increasing in the $\lambda$ variable.

As a consequence of Proposition 2.2 and Lemma 2.4, we can replace $d_\lambda$ by $-\log e_\lambda$ in the second equality of (2.35). Notice that using Hölder's inequality in (1.3), $\lambda \ge 0 \to -\log e_\lambda(0, nv, \omega)$ is a concave function.

With the help of the above remark we thus see that

$$
\lambda \ge 0 \to \alpha_\lambda(v) \;\text{ is concave increasing for } v \in \mathbb{R}^d \, . \tag{2.40}
$$

This also shows that the function $\alpha_.(v)$ is lower semicontinuous on $[0, \infty)$, the only possible discontinuity point being $\lambda = 0$.

On the other hand the bounds (2.32) and Lemma 2.1, show that for each $n$

$$
\lambda \ge 0 \to \mathbb{E}[d_\lambda(0, nv, \omega)] \;\text{ is a continuous function} \, .
$$

It now follows from the first equality in (2.35) that $\alpha_.(\nu)$ is upper semicontinuous on $[0, \infty)$. This proves the continuity of $\lambda \ge 0 \to \alpha_\lambda(v)$, for $v \in \mathbb{R}^d$. Using a Dini type argument, the joint continuity of $(\lambda, v) \to \alpha_\lambda(v)$ now follows.

*Third step:* patching limits

We now consider a fixed $\lambda \ge 0$, and want to show that on a set of full $\mathbb{P}$-measure

$$
\lim_{|x|\to\infty} \frac{1}{|x|} \, |d_\lambda(0, x, \omega) - \alpha_\lambda(x)| = 0 \, . \tag{2.41}
$$

In order to prove this we shall rely on a 'maximal lemma', to control the size of $d_\lambda$.

**Lemma 2.6**: *There exists $A(d, \nu, W, \lambda) > 0$, such that for large enough $\rho > 0$, and $\eta > 0$*

$$\mathbb{P}[\sup_{\|x\|<\rho} d_\lambda(0, x, \omega) > \eta] \leq (4\rho + 1)^d \exp\{-\eta + A\rho\} . \tag{2.42}$$

Let us admit Lemma 2.6 for the time being and see how (2.41) follows.

If (2.42) with $\eta = 2A\,\epsilon\,|p|$, $\rho = \epsilon\,|p|$, $p \in \mathbb{Z}^d$, $\epsilon \in (0,1)$ fixed, we have for large $p$:

$$\mathbb{P}[\sup_{\|y-p\|<\epsilon|p|} d_\lambda(p, y, \omega) > 2A\,\epsilon\,|p|] \leq (4\epsilon\,|p| + 1)^d \exp\{-A\epsilon\,|p|\} ,$$

which is summable in $p$. It thus follows that on a set of full $\mathbb{P}$-measure:

$$\begin{aligned} &\text{for any } \epsilon \in \mathbb{Q} \cap (0,1), \text{ for large } p , \\ &\sup_{\|y-p\|<\epsilon|p|} \{d_\lambda(p, y, \omega)\} \leq 2A\epsilon\,|p| . \end{aligned} \tag{2.43}$$

Consider now $\omega$ fixed in a set of full $\mathbb{P}$-measure where both (2.43), and (2.34) for all $v \in \mathbb{Q}^d$, are satisfied. We shall now see that (2.41) holds as well. It suffices to prove that for any sequence $x_k \to \infty$, with

$$\frac{x_k}{|x_k|} \to e \in S^{d-1} , \tag{2.44}$$

we have

$$\lim_k \frac{1}{|x_k|} |d_\lambda(0, x_k, \omega) - \alpha_\lambda(x_k)| = 0 . \tag{2.45}$$

Choose $\epsilon \in \mathbb{Q} \cap (0,1)$, $v \in \mathbb{Q}^d$, and an integer $M \geq 1$, with

$$|v - e| < \frac{\epsilon}{2} \text{ and } Mv \in \mathbb{Z}^d . \tag{2.46}$$

Define

$$y_k = \left[\frac{|x_k|}{M}\right] \cdot Mv ,$$

then for large $k$:

$$\begin{aligned} |y_k - x_k| &\leq \left|\frac{|x_k|}{M}\,\frac{Mx_k}{|x_k|} - \left[\frac{|x_k|}{M}\right]\,\frac{Mx_k}{|x_k|}\right| + \left[\frac{|x_k|}{M}\right] M\left|\frac{x_k - v}{|x_k|}\right| \\ &\leq M + |x_k|\left|\frac{x_k}{|x_k|} - v\right| < \frac{\epsilon}{2}\,|x_k| . \end{aligned} \tag{2.47}$$

Thus for large $k$:

$$|y_k| > \left(1 - \frac{\epsilon}{2}\right) |x_k| \geq \frac{1}{2} |x_k|$$

and inserting back in (2.47), for large $k$

$$|y_k - x_k| < \epsilon \, |y_k| \, . \tag{2.48}$$

Thus for large $k$:

$$\begin{aligned}
&\left|\frac{1}{|x_k|} d_\lambda(0, x_k, \omega) - \alpha_\lambda\left(\frac{x_k}{|x_k|}\right)\right| \leq \frac{1}{|x_k|} |d_\lambda(0, x_k, \omega) - d_\lambda(0, y_k, \omega)| + \\
&\left|\frac{1}{|x_k|} - \frac{1}{|y_k|}\right| d_\lambda(0, y_k, \omega) + \left|\frac{1}{|y_k|} d_\lambda(0, y_k, \omega) - \alpha_\lambda(v)\right| + \\
&\left|\alpha_\lambda(v) - \alpha_\lambda\left(\frac{x_k}{|x_k|}\right)\right| \overset{(2.3)}{\leq} \frac{1}{|x_k|} d_\lambda(x_k, y_k, \omega) + \frac{1}{|y_k|} d_\lambda(0, y_k, \omega) \\
&\left|\frac{|y_k|}{|x_k|} - 1\right| + \left|\frac{1}{|y_k|} d_\lambda(0, y_k, \omega) - \alpha_\lambda(v)\right| + \left|\alpha_\lambda(v) - \alpha_\lambda\left(\frac{x_k}{|x_k|}\right)\right| .
\end{aligned}$$

It now follows from (2.43), (2.47), (2.34), (2.46) that

$$\overline{\lim_k} \left|\frac{1}{|x_k|} d_\lambda(0, x_k, \omega) - \alpha_\lambda\left(\frac{x_k}{|x_k|}\right)\right| \leq 2A\epsilon + \left(\frac{\epsilon}{2} + \frac{\epsilon}{2}\right) \sup_{|y| \leq 1} \alpha_\lambda(y) \, .$$

Letting $\epsilon$ tend to 0, this proves our claim (2.45). □

*Proof of Lemma 2.6:* Choosing $t = 1$ in (2.32), and exchanging the role of $x$ and $y$, we see that there exists $K(d) > 0$ such that

$$d_\lambda(x, y, \omega) \leq B(x, \omega), \text{ whenever } |x - y| \leq \sqrt{d} \, , \tag{2.49}$$

provided we define $B(x, \omega)$ as

$$B(x, \omega) = K(d) + \lambda + ||W||_\infty \, \omega(\overline{B}(x, 2 + \sqrt{d} + a)) \, . \tag{2.50}$$

We can now pick an integer $L(d, a) \geq 1$ (for instance $L = [4 + 2\sqrt{d} + 2a] + 1$) such that for any (finite) collection $p_i$, of points at mutual distance larger than $L$, the $B(p_i, \omega)$ are independent variables.

Observe that for $e$ some unit vector in $\mathbb{R}^d$ and $k \geq 1$,

$$\begin{aligned}
A_k(e) &\overset{\text{def}}{=} \sup_{y : |y - ke| \leq \sqrt{d}} d_\lambda(0, y, \omega) \overset{(2.49)}{\leq} \sum_{j=0}^{k} B(j\,e, \omega) \\
&\leq \sum_{j=0}^{L-1} \sum_{\ell=0}^{[\frac{k}{L}]} B((\ell L + j)\, e, \omega) \, .
\end{aligned} \tag{2.51}$$

Using Hölder's inequality, stationarity and independence, we find

$$\begin{aligned}(2.52)\qquad & \mathbb{E}[\exp\{dA_k\}] \le \mathbb{E}\Big[\exp\Big\{dL\sum_{\ell=0}^{[\frac{k}{L}]} B(\ell\, Le)\Big\}\Big] \\ & = \mathbb{E}[\exp\{dLB(0)\}]^{[\frac{k}{L}]+1}\,.\end{aligned}$$

Now for $\ell \ge 1$, any $q \in \mathbb{Z}^d$, with $\|q\| < \ell L$, can be connected to 0 with a nearest neighbor path in $\mathbb{Z}^d \cap B_{\|\cdot\|}(0, \ell L)$, made of at most $d$ successive segments parallel to one of the axes, each segment with length at most $\ell L - 1$.

Thus for $q \in \mathbb{Z}^d$ with $\|q\| < \ell L$, using Hölders inequality, (2.52) and translation invariance:

$$\begin{aligned}(2.53)\qquad & \mathbb{E}[\exp\{\sup_{|y-q|\le\sqrt{d}} d_\lambda(0,y)\}] \le \exp\{\ell c\}, \quad \text{where} \\ & c = \log\,\mathbb{E}[\exp\{d\,L\,B(0)\}] < \infty\,.\end{aligned}$$

If we now choose $\rho > L$, $\ell \ge 1$, with $\rho \le \ell\, L \le 2\rho$, and $\eta > 0$, we have

$$\begin{aligned}\mathbb{P}[\sup_{\|y\|<\rho} d_\lambda(0,y,\omega) > \eta] &\le \sum_{\substack{q\in\mathbb{Z}^d \\ \|q\|<\ell L}} \mathbb{P}[\sup_{|y-q|\le\sqrt{d}} d_\lambda(0,y,\omega) > \eta] \\ &\overset{(2.53)}{\le} (4\rho+1)^d \exp\Big\{-\eta + \frac{2}{L}\,\rho\, c\Big\}\,.\end{aligned}$$

Picking $A = \frac{2c}{L}$, our claim (2.42) follows. □

*Fourth step:* Conclusion

Consider a set of full $\mathbb{P}$-measure, where(2.41) holds for any nonnegative rational $\lambda$. The map

$$\lambda \ge 0 \to \alpha_\lambda(e) \text{ is uniformly continuous in } e \in S^{d-1}, \text{ and increasing}\,.$$

Since $d_\lambda$ is increasing in $\lambda$, we can conclude by a Dini type argument that on the above set of full measure, for any $M > 0$,

$$\lim_{x\to\infty} \sup_{0\le\lambda\le M} \Big|\frac{1}{|x|}\, d_\lambda(0,x,\omega) - \alpha_\lambda\Big(\frac{x}{|x|}\Big)\Big| = 0\,.$$

This proves (2.30).

Using Proposition 2.2, the fact that $F_\lambda \le F_{[\lambda]+1}$ (see (2.10)), and (2.27), it follows that on a suitable set of full $\mathbb{P}$-measure, (2.30) holds with $d_\lambda$ replaced by either one of $-\log e_\lambda(0,x,\omega)$, $-\log e_\lambda(x,0,\omega)$, $a_\lambda(0,x,\omega)$, $a_\lambda(x,0,\omega)$, $-\log g_\lambda(0,x,\omega)$.

As a result of (2.42), for each $M > 0$, the random variables $\frac{1}{|x|}\, d_M(0,x,\omega)$, $|x| \geq 1$, are uniformly integrable. The various statements about $L^1$ convergence now follow with the help of Proposition 2.2 and Lemma 2.4. This concludes the proof of Theorem 2.5. $\square$

**Example 2.7**: (the one-dimensional case)

When $d = 1$, the shape theorem yields

$$\alpha_\lambda(1) = \lim_{x\to\infty} -\frac{1}{|x|} \log e_\lambda(x,0,\omega), \quad \mathbb{P}\text{-a.s.}\,, \tag{2.54}$$

where $v(x) = e_\lambda(x,0,\omega), x \geq 1$ is continuous and satisfies (see Proposition 2.3.8):

$$\begin{aligned} &\frac{1}{2}\,\Delta v - (\lambda + V)\, v = 0, \text{ in the distribution sense on } (1,\infty), \\ &v(0) = 1, \text{ and } \lim_{x\to\infty} v(x) = 0 \text{ (for typical } \omega, \text{ when } \lambda = 0)\,. \end{aligned} \tag{2.55}$$

This provides a connection with the 'traditional Lyapunov exponents' (see Carmona-Lacroix [CL91], Chapter VII).

Moreover when $n \geq 2$, $\lambda \geq 0$, $\omega \in \Omega$:

$$\begin{aligned} -\frac{1}{n} \log e_\lambda(0,n,\omega) &= -\frac{1}{n} \log\Big(E_0\Big[\exp\Big\{-\int_0^{H_{\{n-1\}}} (\lambda+V)(Z_s,\omega)ds\Big\}\Big]\Big) \\ &\overset{\binom{\text{strong Markov}}{\text{property}}}{=} -\frac{1}{n}\sum_{k=0}^{n-2} \log\Big(E_k\Big[\exp\Big\{-\int_0^{H_{\{k+1\}}} (\lambda+V)(Z_s,\omega)ds\Big\}\Big]\Big)\,. \end{aligned}$$

Applying the ergodic theorem, with the help of the bounds (2.32), we thus obtain:

$$\begin{aligned} \alpha_\lambda(1) &\overset{\mathbb{P}\text{-a.s.}}{=} \lim -\frac{1}{n} \log e_\lambda(0,n,\omega) \\ &= \mathbb{E}\Big[-\log\Big(E_0\Big[\exp\Big\{-\int_0^{H_{\{1\}}} (\lambda+V)(Z_s,\omega)ds\Big\}\Big]\Big)\Big]\,. \end{aligned} \tag{2.56}$$

This formula appears in a slightly different context in Freidlin [Fre85], p. 502. It can also be used to prove the fact that $\alpha_\lambda(1)$ extends to an analytic function in the region $Re\,\lambda > 0$, see [Szn94b], p. 1682. It is a natural question to wonder whether similar analyticity properties hold in higher dimension. $\square$

We shall now prove the nondegeneracy of the Lyapunov exponents. To this end, we begin with a result which will also be helpful in the annealed situation.

**Proposition 2.8**: *There exists positive constants $c_1(d,\nu,W)$, $c_2(d,\nu,W)$ such that for $R \geq 0$,*

$$\mathbb{E} \otimes E_0\Big[\exp\Big\{-\int_0^{T_{B(0,R)}} V(Z_s,\omega)ds\Big\}\Big] \leq c_1 \exp\{-c_2\, R\}\,. \tag{2.57}$$

**Proof**: Performing the $\mathbb{P}$-integration first, the l.h.s. of (2.57) equals:

$$E_0\Big[\mathbb{E}\Big[\exp\Big\{-\int_{\mathbb{R}^d}\int_0^{T_{B(0,R)}} W(Z_s-y)\,ds\,\omega(dy)\Big\}\Big]\Big] = \tag{2.58}$$
$$E_0\Big[\exp\Big\{-\nu\int_{\mathbb{R}^d}\Big(1-\exp\Big\{-\int_0^{T_{B(0,R)}} W(Z_s-y)ds\Big\}\Big)dy\Big\}\Big].$$

Denote for $q \in \mathbb{Z}^d$ by $D(q)$ the cube centered at $4aq$ with side length $4a$:

$$D(q) = \{z \in \mathbb{R}^d,\ -2a \le z^i - 2a\,q^i < 2a, \text{ for } i = 1,\dots,d\}\,, \tag{2.59}$$

and by $\widetilde{D}(q)$ the open cube of sidelength $8a$ with same center as $D(q)$. For each $z \in \mathbb{R}^d$, there is a unique $q$ such that $z \in D(q)$ and we shall write

$$D[z] \stackrel{\text{def}}{=} D(q) \text{ and } \widetilde{D}[z] \stackrel{\text{def}}{=} \widetilde{D}(q)\,. \tag{2.60}$$

Consider the sequence of stopping times $\tau_i, i \ge 0$, defined via:

$$\tau_0 = 0, \text{ and by induction for } i \ge 0\,, \tag{2.61}$$
$$\tau_{i+1} = \inf\Big\{s \ge 0,\ Z_s \notin \bigcup_{k=0}^{i} \widetilde{D}[Z_{\tau_k}]\Big\}\,.$$

Observe that with this definition $D[Z_{\tau_{i+1}}]$ is a neighboring cube of one of the $D[Z_0]$, $D[Z_{\tau_1}],\dots,D[Z_{\tau_i}]$, and thus

$$|Z_s - Z_0| \le a\sqrt{d}(4i+8) \text{ for } s \in [0,\tau_{i+1}],\ i \ge 0\,. \tag{2.62}$$

Moreover, for $i \ge 0$,

$$E_0\Big[\exp\Big\{-\nu\int_{\mathbb{R}^d}\Big(1-\exp\Big\{-\int_0^{\tau_{i+1}} W(Z_s-y)ds\Big\}\Big)dy\Big\}\Big]$$
$$\le E_0\Big[\exp\Big\{-\nu\sum_{k=0}^{i}\int_{D[Z_{\tau_k}]}\Big(1-\exp\Big\{-\int_{\tau_k}^{\tau_{k+1}} W(Z_s-y)ds\Big)dy\Big\}\Big]$$
$$\le E_0\Big[\exp\Big\{-\nu\sum_{k=0}^{i-1}\int_{D[Z_{\tau_k}]}\Big(1-\exp\Big\{-\int_{\tau_k}^{\tau_{k+1}} W(Z_s-y)ds\Big\}\Big)dy \tag{2.63}$$
$$+\int_{D[Z_{\tau_i}]}\Big(1-\exp\Big\{-\int_{\tau_i}^{T_{\widetilde{D}[Z_{\tau_i}]}\circ\theta_{\tau_i}+\tau_i} W(Z_s-y)ds\Big\}\Big)dy\Big\}\Big]$$
$$\overset{\substack{\text{(strong Markov pro-}\\\text{perty and induction)}}}{\le} \chi^i\,,$$

where

$$\chi(d,\nu,W) \stackrel{\text{def}}{=} \tag{2.64}$$
$$\sup_{z\in D(0)} E_z\Big[\exp\Big\{-\nu\int\Big(1-\exp\Big\{-\int_0^{T_{\widetilde{D[0]}}} W(Z_s-y)ds\Big\}\Big)dy\Big\}\Big]\,.$$

Since $W(\cdot)$ is not negligible and supported in $\overline{B}(0,a)$, it easily follows that

$$\chi\in(0,1)\,. \tag{2.65}$$

Thus combining (2.62), (2.63), we see that the expression in (2.58) is smaller than $c_1\ \exp\{-c_2\ R\}$ for suitable positive constants $c_1(d,\nu,\omega)$, $c_2(d,\nu,W)$. This proves our claim (2.57). □

We can now prove the nondegeneracy of the Lyapunov exponents $\alpha_\lambda(\cdot)$. Keeping the notations of Proposition 2.8, we have

**Proposition 2.9**:

$$\alpha_\lambda(x)\geq \max(\sqrt{2\lambda},c_2)\ |x|,\quad \textit{for}\ \lambda\geq 0,\ x\in\mathbb{R}^d\,. \tag{2.66}$$

*In particular for $\lambda\geq 0$, $\alpha_\lambda(\cdot)$ is a norm on $\mathbb{R}^d$.*

**Proof**: We have for $|x|\geq 1$:

$$\begin{aligned}\mathbb{E}[\log e_\lambda(0,x)] &\overset{\binom{\text{Jensen's}}{\text{inequality}}}{\leq} \log\mathbb{E}[e_\lambda(0,x)]\\ &\leq \log(\min\{\exp\{-\sqrt{2\lambda}\,(|x|-1)\},\ c_1\ \exp\{-c_2(|x|-1)\}\})\,,\end{aligned} \tag{2.67}$$

where we used (2.57), as well as the fact that for a one-dimensional Brownian motion, $\lambda\geq 0$, $y\in\mathbb{R}$, $E_0[\exp\{-\lambda H_{\{y\}}] = \exp\{-\sqrt{2\lambda}\,|y|\}$.

Our claim now follows, since we know from Theorem 2.5 that

$$\frac{1}{n}\ \mathbb{E}[-\log e_\lambda(0,nx)] \underset{n\to\infty}{\longrightarrow} \alpha_\lambda(x),\ \lambda\geq 0,\ x\in\mathbb{R}^d\,.$$

□

The next result is an immediate consequence of Theorem 2.5 and Proposition 2.9, and explains the terminology 'shape theorem'.

**Corollary 2.10**: *On a set of full $\mathbb{P}$-measure, for $\lambda\geq 0$, $\epsilon\in(0,1)$*

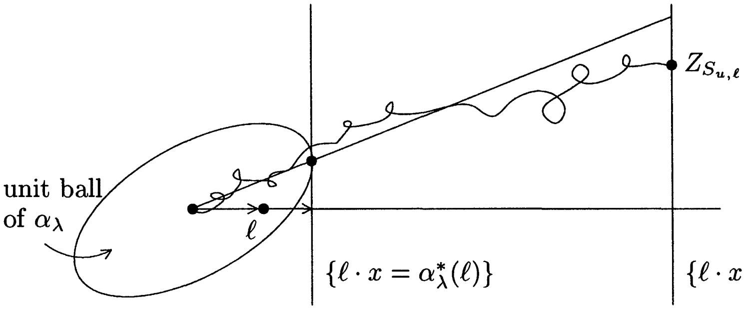
$$\{\alpha_\lambda(\cdot)\leq(1-\epsilon)r\}\subseteq\{d_\lambda(0,\cdot,\omega)\leq r\}\subseteq\{\alpha_\lambda(\cdot)\leq(1+\epsilon)\,r\}\,, \tag{2.68}$$

*for large enough $r$.*

The same holds true with either one of $-\log e_\lambda(0,\cdot,\omega)$, $-\log e_\lambda(\cdot,0,\omega)$, $a_\lambda(0,\cdot,\omega)$, $a_\lambda(\cdot,0,\omega)$, $-\log g_\lambda(0,\cdot,\omega)$ in place of $d_\lambda(0,\cdot,\omega)$.

We can also state an analogue of the so-called 'point to hyperplane' results of first passage percolation (see [Kes86], [Szn94b]). To this end it is convenient to introduce the dual norm to $\alpha_\lambda(\cdot)$:

(2.69) $$\alpha^*_\lambda(\ell) \stackrel{\text{def}}{=} \sup_{x \neq 0} \Big\{\frac{\ell \cdot x}{\alpha_\lambda(x)}\Big\}, \quad \ell \in \mathbb{R}^d,\ \lambda \geq 0\,.$$




For $\ell \neq 0$, and $u \geq 0$, we denote by $S_{u,\ell}$ the stopping time

(2.70) $$S_{u,\ell} = \inf\{s \geq 0,\ Z_s \cdot \ell \geq u\}\,.$$

We now have the 'point to hyperplane' result

**Corollary 2.11**: *On the set of full $\mathbb{P}$-measure introduced in Theorem 2.5, for $\lambda \geq 0$, $\ell \neq 0$,*

(2.71) $$\lim_{u\to\infty} \frac{1}{u} \log E_0\Big[\exp\Big\{-\int_0^{S_{u,\ell}} (\lambda + V)(Z_s,\omega)ds\Big\}\Big] = -\frac{1}{\alpha^*_\lambda(\ell)}\,.$$

**Proof**: Pick a fixed $\omega$ in the above set of full $\mathbb{P}$-measure. For $\lambda \geq 0$, $\ell \neq 0$, choose $x^* \neq 0$, such that

(2.72) $$x^* \cdot \ell = 1, \quad \alpha_\lambda(x^*) = \frac{1}{\alpha^*_\lambda(\ell)}\,.$$

Then the expression under the limit sign in (2.71) is larger than:

$$\frac{1}{u} \log e_\lambda\Big(0, ux^* + \frac{\ell}{|\ell|}, \omega\Big) \underset{u\to\infty}{\longrightarrow} -\alpha_\lambda(x^*) \ \text{(by Theorem 2.5)}\,.$$

This proves the lower bound part of (2.71).

For the upper bound part of (2.71), choose some parallelepiped

$$D = \{z, -M \le z \cdot \ell \le 1,\ |z \cdot x_i| \le M,\ i = 2, \dots, d\}$$

where $\frac{\ell}{|\ell|}$, $x_2, \dots, x_d$ is an orthonormal basis of $\mathbb{R}^d$ and $M$ is large enough so that

$$\inf_{\partial D} \alpha_\lambda(\cdot) = \alpha_\lambda(x^*) .$$

For the homothetic $uD, u > 0$, we can choose an at most polynomially growing number of points $x_j(u)$ on $\partial(uD)$ such that the balls $B(x_j|u|)$ cover $\partial(uD)$. We then have:

$$\begin{aligned} &E_0\Big[\exp\Big\{-\int_0^{S_{u,\ell}} (\lambda + V)(Z_s, \omega) ds\Big\}\Big] \\ &\le \; E_0\Big[\exp\Big\{-\int_0^{H_{\partial(uD)}} (\lambda + V)(Z_s, \omega) ds\Big\}\Big] \\ &\le \; \sum_j e_\lambda(0, x_j(u), \omega) \le \#\{x_j(u)\} \cdot e_\lambda(0, x^*(u), \omega) , \end{aligned}$$

provided $x^*(u)$ is one $x_j(u)$ at which the maximum value of $e_\lambda(0, x_j(u), \omega)$ is attained. Using Theorem 2.5 we see that the l.h.s. of (2.71) is smaller than:

$$-\varliminf_{u\to\infty} \frac{1}{u}\, \alpha_\lambda(x^*(u)) \le -\varliminf_{u\to\infty} \frac{1}{u}\, \inf\{\alpha_\lambda(x), x \cdot \ell \ge u\} = -\frac{1}{\alpha_\lambda^*(\ell)}$$

and this concludes the proof of (2.71). □

We shall now close this section with a result we shall use in Chapter 6.

**Proposition 2.12**: *There exist positive constants $C_1, C_2, C_3$, depending on $d, \nu, W(\cdot)$, such that on a set of full $\mathbb{P}$-measure, for large $\ell$:*

- *for $x \in (-\ell, \ell)^d$, $v \ge C_2 \log \ell$,*

$$(2.73)\qquad \begin{aligned} &E_x\Big[\exp\Big\{-\int_0^{T_v} V(Z_s, \omega) ds\Big\}\Big] \le \exp\{-C_1 v\}, \quad \textit{with} \\ &T_v = \inf\{s \ge 0,\ |Z_s - Z_0| \ge v\} , \end{aligned}$$

- *for $\rho \ge C_2 \log \ell$, $x, y \in (-\ell, \ell)^d$, with $|x - y| \le \rho$,*

$$(2.74)\qquad e_0(x, y, \omega) \ge \exp\{-C_3\, \rho\} .$$

**Proof:**

*Proof of (2.73)*: We write $\mathcal{T}_\ell = (-\ell,\ell)^d \cap (\frac{1}{\sqrt{d}}\,\mathbb{Z}^d)$. It follows from Proposition 2.8 that for $x \in \mathbb{R}^d$ and $v \geq 0$,

$$\begin{aligned}(2.75)\qquad &\mathbb{P}\Big[E_x\Big[\exp\Big\{-\int_0^{T_v} V(Z_s,\omega)ds\Big\}\Big] \geq c_1\,\exp\Big\{-\frac{c_2}{2}\,v\Big\}\Big]\\ &\leq \exp\Big\{-\frac{c_2}{2}\,v\Big\}.\end{aligned}$$

If $\kappa > 0$ is some number and $\ell \geq 1$ is an integer

$$\begin{aligned}&\mathbb{P}\Big[\bigcup_{\substack{x\in\mathcal{T}_\ell\\ v>\kappa\log\ell,\ v\ \text{integer}}} \Big\{E_x\Big[\exp\Big\{-\int_0^{T_v} V(Z_s,\omega)ds\Big\}\Big] \geq c_1\,\exp\Big\{-\frac{c_2}{2}\,v\Big\}\Big\}\Big]\\ &\leq \text{const}\ \ell^d \sum_{\substack{v>\kappa\log\ell\\ v\ \text{integer}}} \exp\Big\{-\frac{c_2}{2}\,v\Big\} \leq \text{const}\ \ell^{d-\kappa\,\frac{c_2}{2}}\,.\end{aligned}$$

This last expression is the general term of a summable series, when $\kappa\,c_2 \geq 2(d+2)$, which we assume from now on. Thus on a set of full $\mathbb{P}$-measure for large integer $\ell$, for $x \in \mathcal{T}_\ell$, and integer $v > \kappa\log\ell$,

$$(2.76)\qquad E_x\Big[\exp\Big\{-\int_0^{T_v} V(Z_s,\omega)ds\Big\}\Big] \leq c_1\exp\Big\{-\frac{c_2}{2}\,v\Big\}\,.$$

Moreover with the help of the Harnack inequality (as in (2.22)), for $v > 4$, $x, x' \in \mathbb{R}^d$, with $|x - x'| \leq 1$,

$$\begin{aligned}(2.77)\qquad &E_{x'}\Big[\exp\Big\{-\int_0^{T_v} V(Z_s,\omega)ds\Big\}\Big] \leq E_x\Big[\exp\Big\{-\int_0^{T_{v-1}} V(Z_s,\omega)ds\Big\}\Big]\\ &\cdot c(d)\,\exp\Big\{c'(d)\sup_{B(x,2)} V(\cdot,\omega)\Big\}\,.\end{aligned}$$

We know from (4.5.12), that $\mathbb{P}$-a.s., $\sup_{(-u,u)^d} V(\cdot,\omega) = o(\log u)$, as $u \to \infty$. It is now easy to conclude from (2.76) and (2.77), that (2.73) holds for suitable positive constants $C_1(d,\nu,W)$, $C_2(d,\nu,W)$.

*Proof of (2.74):* Observe that when $x, x', y \in \mathbb{R}^d$, and $|x - x'| \leq 1$,

$$(2.78)\qquad e_0(x',y,\omega) \geq \exp\{-d_0(x,y,\omega)\}\,.$$

It follows from Lemma 2.6 that for $\ell$ integer and $\kappa > 0$:

$$\mathbb{P}\Big[\bigcup_{\substack{x\in\mathcal{T}_\ell\\ v>\kappa\log\ell,\ v\ \text{integer}}} \Big\{\sup_{\|x-y\|\leq v} d_0(x,y,\omega) > 2Av\Big\}\Big] \leq \text{const}\ \ell^{\mathrm{d}} \sum_{\substack{v>\kappa\log\ell\\ v\ \text{integer}}} (4v+1)^d\,e^{-Av}\,,$$

which is the general term of a convergent series (in $\ell$), when $A\kappa > d+2$, which we assume from now on. As a result, on a set of full $\mathbb{P}$-probability, for large integer $\ell$, for any $x_0 \in \mathcal{T}_\ell$, for integer $v \geq \kappa \log \ell$,

$$\sup_{y:\|x_0-y\|\leq v} d_0(x_0,y,\omega) \leq 2Av\,. \tag{2.79}$$

Together with (2.78), this is easily seen to imply (2.74); the constant $C_2$ being increased if necessary. □

## 5.3 Annealed Lyapunov Exponents

We shall now construct the annealed Lyapunov exponents and study some of their properties. The overall strategy is similar to that of the previous section, however the situation is somewhat simpler. We consider both the soft and hard obstacle case, and keep the notations of Section 5 §1. The annealed Lyapunov exponents will measure the directional exponential decay at infinity of

$$\overline{e}_\lambda(y) \stackrel{\text{def}}{=} \mathbb{E}[e_\lambda(0,y,\omega)],\ \lambda \geq 0,\ y \in \mathbb{R}^d\,. \tag{3.1}$$

Performing the $\mathbb{P}$-integration, we find in the *soft obstacle case*:

$$\begin{aligned}
\overline{e}_\lambda(y) &= E_0\Big[\mathbb{E}\Big[\exp\Big\{-\int_{\mathbb{R}^d}\int_0^{H(y)} W(Z_s-x)ds\,\omega(dx)\Big\}\Big]\\
&\qquad e^{-\lambda H(y)},\ H(y)<\infty\Big]\\
&= E_0\Big[\exp\Big\{-\nu\int_{\mathbb{R}^d}\Big(1-\exp\Big\{-\int_0^{H(y)} W(Z_s-x)ds\Big\}\Big)\,dx\\
&\qquad -\lambda H(y)\Big\},\ H(y)<\infty\Big] \stackrel{\binom{\text{translation}}{\text{invariance}}}{=}\\
&\quad E_{-y}\Big[\exp\Big\{-\nu\int_{\mathbb{R}^d}\Big(1-\exp\Big\{-\int_0^{H(0)} W(Z_s-x)ds\Big\}\Big)\,dx\\
&\qquad -\lambda H(0)\Big\},\ H(0)<\infty\Big]\,,\quad \text{for } \lambda\geq 0, y\in\mathbb{R}^d\,.
\end{aligned}\tag{3.2}$$

Analogously in the *hard obstacle case*:

$$\begin{aligned}
\overline{e}_\lambda(y) &= E_0[\mathbb{P}[H(y)<T]\,e^{-\lambda H(y)},\ H(y)<\infty]\\
&= E_0[\exp\{-\nu|W^a_{H(y)}|-\lambda H(y)\},\ H(y)<\infty]\\
&\stackrel{\binom{\text{translation}}{\text{invariance}}}{=} E_{-y}[\exp\{-\nu|W^a_{H(0)}|-\lambda H(y)\},\ H(0)<\infty],\\
&\qquad \text{for } \lambda\geq 0, y\in\mathbb{R}^d\,,
\end{aligned}\tag{3.3}$$

where we used the notation of (2.3.57)

(3.4) $$W_t^a = \bigcup_{0\le s\le t} \overline{B}(Z_s,a),\ t\ge 0\,,$$

for the Wiener sausage of radius $a$ around $Z_.$ in time $t$.

We shall also consider for $\lambda \ge 0$, $y \in \mathbb{R}^d$, the quantities:

(3.5) $$\overline{a}_\lambda(y) = -\inf_{z\in B(0)} \log \overline{e}_\lambda(y-z)\,,$$

(3.6) $$\overline{d}_\lambda(y) = \max(\overline{a}_\lambda(y),\ \overline{a}_\lambda(-y))\,,$$

and in the case of soft obstacles:

(3.7) $$\overline{g}_\lambda(y) = \mathbb{E}[g_\lambda(0,y,\omega)]\,.$$

We begin with

**Lemma 3.1**:

(3.8) $\overline{e}_\lambda(y)$ *and* $\overline{a}_\lambda(y)$, *are jointly continuous on* $\mathbb{R}_+ \times \mathbb{R}^d$,

(3.9) $$\overline{a}_\lambda(x+y) \le \overline{a}_\lambda(x) + \overline{a}_\lambda(y),\ x,y\in\mathbb{R}^d\,,$$

*and the same holds for* $\overline{d}_\lambda$.

(3.10) $x,y \to \overline{d}_\lambda(y-x)$ *defines a distance function on* $\mathbb{R}^d$ *which induces the usual topology*.

**Proof**:

*Proof of (3.8)*: It is enough to prove the statement for $\overline{e}_.(\cdot)$. In the soft obstacle case it immediately follows from (3.1) and Lemma 2.1.

In the hard obstacle case, the continuity in the $\lambda$ variable is immediate. Moreover, from the last line of (3.3), we see that $\overline{e}_\lambda(\cdot)$ is a rotationally invariant function, which equals 1 on $\overline{B}(y)$ and decreases with $|y|$. Observe that when $0 \le u \le t$, and $\sup_{0\le s\le u} |Z_s - Z_0| < \epsilon$,

(3.11) $$W_t^a \subseteq (W_{t-u}^a \circ \theta_u) \cup (\overline{B}(Z_u, a+\epsilon)\backslash \overline{B}(Z_u,a))\,,$$

and it thus follows by an application of the strong Markov property that for $x, y \in \mathbb{R}^d$, with $|y-x| < \epsilon$,

(3.12) $$\begin{aligned}\overline{e}_\lambda(y) \ge\ & \exp\{-\nu\omega_d((a+\epsilon)^d - a^d)\}\, \overline{e}_\lambda(x)\\ & E_y[\exp\{-\lambda H_{\overline{B}(0,|x|)}\},\ H_{\overline{B}(0,|x|)} < T_{B(x,\epsilon)}]\,,\end{aligned}$$

(where the last quantity vanishes when $x=0$).

We can write a similar inequality exchanging the role of $x$ and $y$. If we let $y$ tend to $x \neq 0$, and then let $\epsilon$ tend to 0, we find that

$$\lim_{y \to x} \overline{e}_\lambda(y) = x, \quad \text{when } x \neq 0, \text{ for } \lambda \geq 0\,.$$

The continuity in $x = 0$, is immediate, and the joint continuity of $\overline{e}_.(\cdot)$ now follows by a Dini type argument.

*Proof of (3.9):* We explain the argument for soft obstacles, the case of hard obstacles is analogous and easier. We use the elementary inequality:

$$\begin{aligned} 1 - e^{-(a+b)} &= 1 - e^{-a} + 1 - e^{-b} - (1-e^{-a})(1-e^{-b}) \\ &\leq 1 - e^{-a} + 1 - e^{-b}, \ a, b \geq 0\,. \end{aligned} \tag{3.13}$$

We then find that for $x, y \in \mathbb{R}^d$, $z \in B(0)$,

$$\begin{aligned} &\overline{e}_\lambda(x+y-z) = \\ &E_z\Big[\exp\Big\{-\nu\int\Big(1-\exp\Big\{-\int_0^{H(x+y)} W(Z_s - z_1)ds\Big\}\Big)dz_1 \\ &-\lambda H(x+y)\Big\},\ H(x+y)<\infty\Big] \\ &\overset{\binom{(3.13)\text{ and strong}}{\text{Markov property}}}{\geq} \\ &E_z\Big[\exp\Big\{-\nu\int\Big(1-\exp\Big\{-\int_0^{H(x)} W(Z_s - z_1)ds\Big\}\Big)dz_1 \\ &-\lambda H(x)\Big\},\ H(x)<\infty\Big] \\ &\inf_{z'\in B(x)} E_{z'}\Big[\exp\Big\{-\nu\int\Big(1-\exp\Big\{-\int_0^{H(x+y)} W(Z_s - z_2)ds\Big\}dz_2 \\ &-\lambda H(x+y)\Big\}\,,\ H(x+z)<\infty\Big]\cdot \\ &\overset{\binom{\text{translation}}{\text{invariance}}}{=} \overline{e}_\lambda(x-z)\ \exp\{-\overline{a}_\lambda(y)\}\,. \end{aligned} \tag{3.14}$$

Taking the infimum over $z \in \overline{B}(0)$, our claim (3.9) immediately follows, both for $\overline{a}_\lambda$ and $\overline{d}_\lambda$.

*Proof of (3.10):* We use similar arguments as for (2.3). □

**Remark 3.2:** If we were working on $\mathbb{Z}^d$, the proof of (3.9) could invoke the FKG inequality (see Liggett [Lig85]) along the following line

$$\mathbb{E}[e_\lambda(0,x+y,\omega)] \overset{\binom{\text{strong Markov}}{\text{property}}}{\geq} \mathbb{E}[e_\lambda(0,x,\omega)\, e_\lambda(x,y,\omega)] \overset{\text{(FKG)}}{\geq} \mathbb{E}[e_\lambda(0,x,\omega)]\,\mathbb{E}[e_\lambda(x,x+y,\omega)] \overset{\binom{\text{translation}}{\text{invariance}}}{=} \mathbb{E}[e_\lambda(0,x,\omega)]\,\mathbb{E}[e_\lambda(0,y,\omega)]\,,$$

since $e_\lambda(z,z',\omega)$ are decreasing functions of $\omega$. The FKG inequality is known to hold in the Poissonian setting as well (see Janson [Jan84]). However the above 'proof of (3.9) in a discrete setting' is not directly transposable in the Poissonian setting, since after the application of the strong Markov property, an infimum over $B(x)$ appears inside the $\mathbb{P}$-expectation. □

**Proposition 3.3**: *In the soft obstacle case, for $M > 0$,*

$$\lim_{y\to\infty} \sup_{0\le\lambda\le M} \frac{1}{|y|} \max\{|\overline{a}_\lambda(y) + \log \overline{g}_\lambda(y)|,\ |-\log \overline{e}_\lambda(y) + \log \overline{g}_\lambda(y)|,\ |\overline{a}_\lambda(-y) + \log \overline{g}_\lambda(y)|,\ |\overline{d}_\lambda(y) + \log \overline{g}_\lambda(y)|\} = 0\,. \tag{3.15}$$

*In the hard obstacle case, for $\lambda \ge 0, y \in \mathbb{R}^d$,*

$$\overline{a}_\lambda(y) = \overline{a}_\lambda(-y) = \overline{d}_\lambda(y) = -\log \overline{e}_\lambda(x)\,, \tag{3.16}$$

*with $x \in \mathbb{R}^d$, such that $|x| = |y| + 1$. Moreover, for $M > 0$,*

$$\lim_{y\to\infty} \sup_{0\le\lambda\le M} \frac{1}{|y|} \max\{|\overline{a}_\lambda(y) + \log \overline{e}_\lambda(y)|,\ |\overline{a}_\lambda(-y) + \log \overline{e}_\lambda(y)|\} = 0\,. \tag{3.17}$$

**Proof:**

*Proof of (3.15)*: Using the symmetry of $g_\lambda(\cdot,\cdot,\omega)$ and translation invariance, we have in the soft obstacle case:

$$\overline{g}_\lambda(y) = \overline{g}_\lambda(-y),\ \lambda \ge 0,\ y \in \mathbb{R}^d\,. \tag{3.18}$$

Furthermore, observe that for $|y| > 3$, $z \in B(0)$, both in the soft and hard obstacle case,

$$\overline{e}_\lambda(y-z) \ge \exp\{-\nu\,|B(0,2+a)|\}\, E_z[\exp\{-\lambda T_{B(0,2)}\}\, \overline{e}_\lambda(x - Z_{T_{B(0,2)}})]\,,$$

using Jensen's inequality and Lemma 2.3:

$$\overline{e}_\lambda(y-z) \ge \exp\{-\nu\,|B(0,2+a)|\} E_z[\exp\{-\lambda h_0(z, Z_{T_{B(0,2)}})\}\, \overline{e}_\lambda(x - Z_{T_{B(0,2)}})]\,. \tag{3.19}$$

On the other hand:

$$\overline{e}_\lambda(y-z) \le E_z[\overline{e}_\lambda(x - Z_{T_{B(0,2)}})] \,. \tag{3.20}$$

As a result for $z, z' \in B(0)$, $|y| > 3$,

$$\begin{aligned} &\overline{e}_\lambda(y-z)/\overline{e}_\lambda(y-z') \le \\ &\exp\{-\nu\,|B(0,2+a)|\} \times \frac{\sup v_0}{\inf v_0}\; e^{\lambda \sup h_0} < \infty\,, \\ &\text{(soft and hard obstacles)}\,, \end{aligned} \tag{3.21}$$

where we use the notation of Lemma 2.3, and the supremum and infimum run over $B(0) \times \partial B(0,2)$.

As a result of (3.18) and (3.21), our claim (3.15) will follow once we prove:

$$\lim_{y\to\infty} \sup_{0\le\lambda\le M} \frac{1}{|y|}\, |-\log \overline{e}_\lambda(y) + \log \overline{g}_\lambda(y)| = 0\,. \tag{3.22}$$

Observe now that as a result of (2.12) and (2.21), for $|y| > 4$:

$$\widetilde{B}_\lambda\, g_\lambda(0,y,\omega) \le e_\lambda(0,y,\omega) \le \widetilde{A}_\lambda\, g_\lambda(0,y,\omega), \quad \text{where} \tag{3.23}$$

$$\widetilde{A}_\lambda = A \times e_y^{\lambda,\omega}(B(y)),\; \widetilde{B}_\lambda = \frac{1}{A}\, e_y^{\lambda,\omega}(B(y))\,, \tag{3.24}$$

and $A$ is defined as in (2.22) with $x = 0$, and $y$ as above. As a result of (2.25) and of Lemma 2.4,

$$\sup_{0\le\lambda\le M} E[\widetilde{A}_\lambda^p + \widetilde{B}_\lambda^p] < \infty \;\text{ for any }\; p < \infty\,. \tag{3.25}$$

If now $p \in (1,\infty)$ and $q$ is the conjugate exponent of $p$ (i.e. $\frac{1}{p} + \frac{1}{q} = 1$), it follows from Hölders inequality that for $|y| > 4$ and $\lambda \ge 0$:

$$\mathbb{E}[e_\lambda^{1/p}(0,y,\omega)] \le \mathbb{E}[e_\lambda(0,y,\omega)/\widetilde{A}_\lambda]^{1/p}\, \mathbb{E}[\widetilde{A}_\lambda^{q/p}]^{1/q}$$

which together with the right-most inequality of (3.23) implies:

$$\begin{aligned} \log \overline{g}_\lambda(y) &\ge p\, \log(\mathbb{E}[e_\lambda^{1/p}(0,y,\omega)]) - \frac{p}{q}\, \log(\mathbb{E}[\widetilde{A}_\lambda^{q/p}]) \\ &\ge p\, \log \overline{e}_\lambda(y) - \frac{p}{q}\, \log(\mathbb{E}[\widetilde{A}_\lambda^{q/p}])\,, \end{aligned} \tag{3.26}$$

where we used the fact that $0 \le e_\lambda \le 1$ in the last step. It thus follows from Jensen's inequality and monotonicity in $\lambda$ that:

$$\log \overline{g}_\lambda(y) - \log \overline{e}_\lambda(y) \ge (p-1)\, \mathbb{E}[\log e_M(0,y,\omega)] - \frac{p}{q}\, \log(\mathbb{E}[\widetilde{A}_\lambda^{q/p}])\,, \tag{3.27}$$

for $|y| > 4$ and $0 \le \lambda \le M$.

Similarly, it follows from the left-most inequality in (3.23) and Hölder's inequality, that for $|y| > 4$ and $\lambda \geq 0$:

$$\begin{aligned}\log \overline{g}_\lambda(y) &\leq \frac{1}{p} \log(\mathbb{E}[e_\lambda(0,y,\omega)^p]) + \frac{1}{q} \log \mathbb{E}[\widetilde{B}_\lambda^{-q}] \\ &\leq \frac{1}{p} \log \overline{e}_\lambda(y) + \frac{1}{q} \log \mathbb{E}[\widetilde{B}_\lambda^{-q}] ,\end{aligned}$$

and by similar arguments as above,

$$\log \overline{g}_\lambda(y) - \log \overline{e}_\lambda(y) \leq -\frac{(p-1)}{p} \mathbb{E}[\log e_M(0,y,\omega)] + \frac{1}{q} \log \mathbb{E}[\widetilde{B}_\lambda^{-q}] , \tag{3.28}$$

for $|y| > 4$ and $0 \leq \lambda \leq M$.

Our claim (3.22) now follows from (3.26) and (3.28), letting first $y$ tend to $\infty$ and then $p$ tend to 1, with the help of (2.30) (or (2.32)) to control $\mathbb{E}[-\log e_M(0,y,\omega)]$ and (3.25).

*Proof of (3.16):* We are now in the hard obstacle case, and the statement follows immediately from the fact that $\overline{e}_\lambda(\cdot)$ is rotationally invariant decreasing with $|y|$.

*Proof of (3.17):* This claim follows immediately from (3.21), which as we mentioned above, holds in the hard obstacle case as well. □

We can now state the shape theorem for the annealed situation.

**Theorem 3.4:** *(shape theorem)*

*There exists a nonnegative function $\beta_\lambda(x)$ defined on $[0,\infty) \times \mathbb{R}^d$, which is jointly continuous, concave increasing in the $\lambda$ variable, which for fixed $\lambda$ defines a norm on $\mathbb{R}^d$, such that for $M > 0$,*

$$\lim_{x\to\infty} \sup_{0\leq\lambda\leq M} \frac{1}{|x|} |\overline{d}_\lambda(x) - \beta_\lambda(x)| = 0 . \tag{3.29}$$

*One can replace $\overline{d}_\lambda(x)$ by either one of $-\log \overline{e}_\lambda(x)$, $\overline{a}_\lambda(x)$, $-\log \overline{g}_\lambda(x)$ (soft obstacle case), in the above statement. Moreover,*

$$\max(\sqrt{2\lambda},\, c_2)\, |x| \leq \beta_\lambda(x) \leq \min(k_\lambda\, |x|, \alpha_\lambda(x)),\ \lambda \geq 0,\ x \in \mathbb{R}^d , \tag{3.30}$$

*where $c_2(d,\nu,W)$ is defined in Proposition 2.8, in the soft obstacle case, and equals $\lim_{u\to\infty} \uparrow c_2(d,\nu,u 1_{\{|\cdot|\leq a\}})$ in the hard obstacle case, and*

$$\begin{aligned} k_\lambda &= \nu + \sqrt{2\lambda}\ (= \beta_\lambda^{\text{hard obstacle}}(1)), \quad \textit{when } d = 1 , \\ &= \min_{r>0} \left(\nu\, \omega_{d-1}(a+r)^{d-1} + \sqrt{2(\lambda_{d-1} + \lambda r^2)}/r\right), \quad \textit{when } d \geq 2 . \end{aligned} \tag{3.31}$$

**Proof:**

*Proof of (3.29):* We shall sketch the argument, which is simpler and essentially follows the scheme of the proof of Theorem 2.5. Using continuity $\overline{d}_\lambda(\cdot)$ is uniformly bounded on compact sets, and using subadditivity one defines

$$(3.32) \qquad \beta_\lambda(x) = \lim_{u\to\infty} \frac{1}{u}\, \overline{d}_\lambda(ux) = \inf_{u\geq 0} \frac{1}{u}\, \overline{d}_\lambda(ux)\,,$$

first as a directional limit. Proceeding as in the second step of Theorem 2.5, with the information of Lemma 3.1, one sees that $\beta_\lambda(\cdot)$ defines a seminorm on $\mathbb{R}^d$, which depends in a continuous concave increasing fashion on $\lambda$. One then 'patches together' the various directional limits in (3.32). This is of course simpler than in Theorem 2.5, since no maximal lemma is now required.

*Proof of (3.30):* The left-most inequality of (3.30) follows from Proposition 2.8, just as in the proof of (2.66). We shall now prove the right-most inequality. As a result of Jensen's inequality, we have

$$\overline{e}_\lambda(x) = \mathbb{E}[e_\lambda(0,x,\omega)] \geq \exp\{\mathbb{E}[\log e_\lambda(0,x,\omega)]\}\,,$$

and as a result of Theorem 2.5 and the previous step,

$$(3.33) \qquad \beta_\lambda(x) \leq \alpha_\lambda(x), \quad \text{for } \lambda \geq 0, x \in \mathbb{R}^d\,.$$

It now suffices to show that in the hard obstacle case:

$$(3.34) \qquad \beta_\lambda(x) \leq k_\lambda\,|x|\,,$$

holds for $\lambda \geq 0$, $x \in \mathbb{R}^d$, with $k_\lambda$ as in (3.31). From rotational invariance, we can assume that $x = e_1$, the first vector of the canonical basis of $\mathbb{R}^d$.

If we denote by $C_{u,r}$ for $u > 0, r > 0$, the cylinder

$$(3.35) \qquad C_{u,r} = (-\sqrt{u}, u + \sqrt{u}) \times B_r\,,$$

where $B_r$ stands for the open ball of radius $r$ in $\mathbb{R}^{d-1}$, when $d \geq 2$, and is omitted when $d = 1$, we have for $u > 0$, $t > 0$, $\lambda \geq 0$:

$$\begin{aligned} \overline{e}_\lambda(ue_1) \;\;\geq\;\; & E_0[\exp\{-\nu\,|W^a_{H(ue_1)}| - \lambda H(ue_1)\}\,, \\ & T_{C_{u,r}} > t,\ Z^1_t \in [u, u+1],\ H(ue_1)\circ\theta_t < \infty]\,. \end{aligned}$$

It follows from the Markov property and the inequality,

$$|W^a_{H(ue_1)}| \leq |W^a_t| + |W^a_{H(ue_1)} \circ \theta_t| \text{ on } \{H(ue_1)\circ\theta_t < \infty\}\,,$$

that:

$$\begin{aligned}&\bar{e}_\lambda(ue_1) \geq \\ &\exp\{-\nu\,|C^a_{u,r}| - \lambda t\}\ \mathbb{P}_0^{d-1}[T_{B_r} > t]\ P_0^1[T_{(-\sqrt{u},u+\sqrt{u})} > t, Z_t \in [u,u+1]] \\ &\inf\{\bar{e}_\lambda(-x),\ x \in [0,1] \times B_r\}\ ,\end{aligned}$$

where $C^a_{u,r}$ denotes the $a$-neighborhood of $C_{u,r}$, $P_0^{d-1}$ and $P_0^1$ the respective $(d-1)$ and 1 dimensional Wiener measure, and the $(d-1)$ dimensional expression is absent when $d=1$. Choosing $t = \rho u$, with $\rho > 0$, we find

$$\text{(3.37)}\quad \begin{aligned}\beta_\lambda(e_1) \;\leq\; &\overline{\lim_{u\to\infty}}\ \frac{\nu}{u}\ |C^a_{u,r}| + \lambda_\rho + \frac{\lambda_{d-1}}{r^2}\ \rho \\ &-\frac{1}{u}\ \log P_0^1[T_{(-\sqrt{u},u+\sqrt{u})} > t,\ Z_t \in [u,u+1]]\ .\end{aligned}$$

We provide an upper bound on the last term in the following

**Lemma 3.5:**

$$\text{(3.38)}\quad \begin{aligned}&P_0^1[T_{(-\sqrt{u},u+\sqrt{u})} > t,\ Z_t \in [u,u+1]] \geq \frac{1}{\sqrt{2\pi t}}\ \Big(\exp\Big\{-\frac{(u+1)^2}{2t}\Big\} \\ &-\exp\Big\{-\frac{(u+2\sqrt{u})^2}{2t}\Big\} - \exp\Big\{-\frac{(u+2\sqrt{u}-1)^2}{2t}\Big\}\Big)\end{aligned}$$

*for $u > 1, t > 0$.*

**Proof:** The claim follows by an application of the method of images. Indeed for $\epsilon > 0$, the function

$$G^\epsilon(s,y) = p(s+\epsilon,0,y) - p(s+\epsilon,-\sqrt{u},y) - p(s+\epsilon,2u+2\sqrt{u},y),\ \ s>0, y \in \mathbb{R}\ ,$$

is negative on the boundary of $(-\sqrt{u}, u+\sqrt{u})$. Applying Ito's formula, we find for $y \in (-\sqrt{u}, u+\sqrt{u})$, $t > 0$:

$$\begin{aligned}&E_y^1[G^\epsilon(0,Z_t)\ 1\{T_{(-\sqrt{u},u+\sqrt{u})} > t\}] = \int r_{(-\sqrt{u},u+\sqrt{u}),V=0}\ (t,y,x)\ G^\epsilon(0,x)dx \\ &\geq E_y^1[G^\epsilon(t - t\wedge T_{(-\sqrt{u},u+\sqrt{u})}),\ Z_{t\wedge T_{(-\sqrt{u},u+\sqrt{u})}})] \\ &\overset{\text{(Ito's formula)}}{=} G^\epsilon(t,y)\ .\end{aligned}$$

Letting $\epsilon$ tend to 0, we find:

$$\text{(3.39)}\quad r_{(-\sqrt{u},u+\sqrt{u})}(t,0,y) \geq p(t,0,y) - p(t,-\sqrt{u},y) - p(r,2u+2\sqrt{u},y)$$

for $u > 0$, $t > 0$, $y \in (-\sqrt{u}, u+\sqrt{u})$. The inequality (3.38) now follows by integration of (3.39) over $y \in [u,u+1]$. □

Coming back to (3.37), we now see that

$$\begin{aligned}\beta_\lambda(e_1) &\le \nu\omega_{d-1}(a+r)^{d-1} + \lambda\rho + \frac{\lambda_{d-1}}{r^2}\,\rho + \frac{1}{2\rho}\,, \quad \text{when } d \ge 2;\,, \\ &= \nu + \lambda\rho + \frac{1}{2\rho}\,, \quad \text{when } d = 1\,.\end{aligned} \tag{3.40}$$

Optimizing over $\rho > 0$, we find (3.34). Finally, observe that in the one-dimensional case:

$$\begin{aligned}\bar{e}_\lambda(ue_1) &\le \exp\{-\nu(u-1)\}\, E_0[\exp\{-\lambda H(ue_1)\}] \\ &= \exp\{-(\nu+\sqrt{2\lambda})(u-1)\}, \quad \text{for } u > 1\,.\end{aligned}$$

This together with the previous bound shows that

$$\beta_\lambda^{\text{hard obstacles}}(1) = \nu + \sqrt{2\lambda}, \quad \text{for } \lambda \ge 0, \text{ when } d = 1\,, \tag{3.41}$$

and concludes the proof of Theorem 3.4. □

As an immediate consequence of Theorem 3.4, we have

**Corollary 3.6:** *For $\lambda \ge 0$, $\epsilon > 0$,*

$$\{\beta_\lambda(\cdot) \le (1-\epsilon)\,r\} \subseteq \{\bar{d}_\lambda(\cdot) \le r\} \subset \{\beta_\lambda(\cdot) \le (1+\epsilon)\,r\}\,, \tag{3.42}$$

*for large $r$.*

Furthermore, if we introduce the dual norm

$$\beta_\lambda^*(\ell) = \sup_{x \ne 0} \left\{\frac{\ell \cdot x}{\beta_\lambda(x)}\right\}, \quad \ell \in \mathbb{R}^d,\ \lambda \ge 0\,, \tag{3.43}$$

and keep the notation (2.70) for $S_{u,\ell}$, we have a point to hyperplane result

**Corollary 3.7:** *For $\lambda \ge 0, \ell \ne 0$,*

$$\lim_{u \to 0} \frac{1}{u} \log \mathbb{E} \otimes E_0\Big[\exp\Big\{-\int_0^{S_{u,\ell}} (\lambda + V)(Z_s,\omega)ds\Big\}\Big] = -\frac{1}{\beta^*(\ell)}\,, \tag{3.44}$$

*where in the hard obstacle case the expression under the logarithm stands for*

$$E_0[\exp\{-\nu|W^a_{S_{u,\ell}}| - \lambda S_{u,\ell}\}]\,.$$

**Proof:** One uses the same argument as in the proof of Corollary 2.10. □

## 5.4 Some Large Deviation Estimates

The Lyapunov exponents have natural applications to the description of large displacements of the particle under the respective quenched path measure $Q_{t,\omega}$ or annealed path measure $Q_t$, for large $t$, (see (1.15), (1.16)). They appear for instance in the description of several large deviation principles for $Z_t$, discussed in Chapter 7, and in the variational problem associated to the pinning effect, which is the principal object of Chapter 6.

Here we shall discuss two large deviation principles governing the occurrence of displacements of $Z_t$ of order $t$, in the quenched and annealed situation. These large deviation principles play an important role in understanding the transition of behavior for Brownian motion with a constant drift among Poissonian obstacles, as the size of the drift increases.

We begin with the study of the *quenched path measures*. We are in the soft obstacle case, and consider for $t > 0$, $\omega \in \Omega$:

$$Q_{t,\omega} = \frac{1}{S_{t,\omega}} \exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\} P_0(dw)\,. \tag{4.1}$$

The rate function governing for 'typical $\omega$' the large deviation principle of $Z_t$ in scale $t$ under $Q_{t,\omega}$ (that is heuristically (1.17)), turns out to be

$$I(x) \overset{\text{def}}{=} \sup_{\lambda\geq 0}\, (\alpha_\lambda(x) - \lambda),\ x \in \mathbb{R}^d\,. \tag{4.2}$$

We begin with

**Lemma 4.1**: *$I(x)$ is a nonnegative convex continuous function, and*

$$c_2\,|x| \vee \frac{x^2}{2} \leq I(x) \leq \begin{cases} c_3\,|x|\,, & \textit{for } |x| \leq c_3\,, \\ \dfrac{x^2}{2} + \dfrac{c_3^2}{2}\,, & \textit{for } |x| > c_3\,, \end{cases} \tag{4.3}$$

*with $c_2$ as in (2.57) and*

$$c_3 = \sqrt{2(\lambda_d + \|W\|_\infty\, \omega_d(a+2)^d)}\,. \tag{4.4}$$

**Proof:** The convexity and continuity follows easily from the convexity in the $x$ variable and joint continuity of $\alpha_\lambda(x)$, together with the upper bound (2.31). The inequalities in (4.3) are immediate consequences of

$$(c_2 \vee \sqrt{2\lambda})\,|x| \overset{(2.66)}{\leq} \alpha_\lambda(x) \overset{(2.31)}{\leq} \sqrt{2\lambda + c_3}\,|x|,\ x \in \mathbb{R}^d\,, \tag{4.5}$$

together with the definition (4.4). $\square$

We are now ready to state and prove

**Theorem 4.2:** *(quenched large deviation principle in scale $t$)*

*There is a set of full $\mathbb{P}$-measure such that when $\omega$ belongs to this set:*

(4.6)
*for any closed subset $A$ of $\mathbb{R}^d$,*
$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log Q_{t,\omega}(Z_t \in tA) \le - \inf_{x\in A} I(x)\,,$$

(4.7)
*for any open subset $O$ of $\mathbb{R}^d$,*
$$\underline{\lim_{t\to\infty}} \frac{1}{t} \log Q_{t,\omega}(Z_t \in tO) \ge - \inf_{x\in O} I(x)\,.$$

**Proof:** Choose a set $\Omega_1$ of full $\mathbb{P}$-measure for which the asymptotic behavior (4.5.10) of $S_{t,\omega}$ as well as the shape Theorem 2.5 hold.

*Proof of (4.6):* 'the upper estimate'

*First step:* the case of $A$ compact

Assume first that $A$ is bounded measurable. We can choose a finite collection of points $\mathbb{L}_{tA}$ in $tA$, whose number grows at most polynomially in $t$, such that:

$$tA \subset C_t \stackrel{\text{def}}{=} \bigcup_{p\in\mathbb{L}_{tA}} B(p), \quad \text{for } t \ge 1\,, \tag{4.8}$$

with the notation (1.2). Then for $\omega \in \Omega_1$, $t \ge 1$, $\lambda \ge 0$,

$$\begin{aligned}
&\exp\{-\lambda t\}\, S_{t,\omega}\, Q_{t,\omega}\,(Z_t \in tA) = \\
&\exp\{-\lambda t\}\, E_0\Big[Z_t \in tA,\ \exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big] \le \\
&E_0\Big[\exp\Big\{-\int_0^{H_{C_t}} (\lambda + V)(Z_s,\omega)ds\Big\},\ H_{tA} < \infty\Big] \le \\
&\sum_{p\in\mathbb{L}_{tA}} e_\lambda(0,p,\omega) \le \#\,\mathbb{L}_{tA}\cdot e_\lambda(0,p_*(t),\omega)\,,
\end{aligned} \tag{4.9}$$

where $p_*(t) \in \mathbb{L}_{tA}$ is a point at which the maximal value of $e_\lambda(0,p,\omega)$, $p \in \mathbb{L}_{tA}$, is attained.

Since $S_{t,\omega}$ has a subexponential decay in $t$, it then follows from the shape Theorem 2.5 that

$$\begin{aligned}
&-\lambda + \overline{\lim_{t\to\infty}} \frac{1}{t} \log Q_{t,\omega}(Z_t \in tA) \le \\
&\overline{\lim_{t\to\infty}} \frac{1}{t} \log e_\lambda(0,p_*(t),\omega) = \overline{\lim_{t\to\infty}} - \frac{1}{t}\,\alpha_\lambda(p_*(t)) \le -\inf_A \alpha_\lambda(x)\,.
\end{aligned}$$

We have thus shown that for $A$ bounded measurable subset of $\mathbb{R}^d$:

$$(4.10) \qquad \overline{\lim_{t\to\infty}} \tfrac{1}{t} \log Q_{t,\omega}(Z_t \in tA) \le -\sup_{\lambda\ge 0} (\inf_A \alpha_\lambda(x) - \lambda) .$$

We now assume that $A$ is compact. To prove (4.6) for $A$, we need to exchange supremum and infimum in (4.10). We use a classical argument (see for instance Donsker-Varadhan [DV75b]). The idea is to localize (4.10) with the help of the special structure of its l.h.s.. As a consequence of (4.10), we have:

$$(4.11) \quad \overline{\lim_{t\to\infty}} \tfrac{1}{t} \log Q_{t,\omega}(Z_t \in tA) \le - \sup_{A_1,\dots,A_\ell} \inf_{1\le i\le \ell} \sup_\lambda \inf_{A_i} (\alpha_\lambda(x) - \lambda) ,$$

where $A_1, \dots, A_\ell$ runs over all possible finite coverings of $A$ by bounded measurable sets. Observe now that for $\epsilon > 0$,

$$A \subset \bigcup_{\lambda\ge 0} \{\alpha_\lambda(\cdot) - \lambda > \inf_A I - \epsilon\} ,$$

and from the compactness of $A$, we can choose $\lambda_1, \dots, \lambda_\ell$ such that:

$$A \subset \bigcup_1^\ell A_i, \text{ with } A_i = \{\alpha_{\lambda_i}(\cdot) - \lambda_i > \inf_A I - \epsilon\} \cap A ,$$

and therefore:

$$\inf_{1\le i\le \ell} \sup_\lambda \inf_{A_i} \{\alpha_\lambda(x) - \lambda\} \ge \inf_A I - \epsilon .$$

Since $\epsilon > 0$, is arbitrary, this shows that the r.h.s. of (4.11) is smaller than $-\inf_A I$. This proves (4.6) for a compact $A$.

*Second step:* general case

The case of a general closed $A$ simply follows from the fact that

$$\lim_{R\to\infty} \overline{\lim_{t\to\infty}} \frac{1}{t} \log Q_{t,\omega}(Z_t \notin t\overline{B}(0,R)) =$$
$$\lim_{R\to\infty} \overline{\lim_{t\to\infty}} \frac{1}{t} \log E_0\Big[\exp\Big\{-\int_0^{T_{B(0,Rt)}} V(Z_s,\omega)ds\Big\}\Big] \overset{\text{(Corollary 2.10)}}{=} -\infty ,$$

so that one can replace $A$ by $A \cap \overline{B}(0,R)$, for a large enough $R$ in the l.h.s. of (4.6). We have thus proved (4.6) in the general situation.

*Proof of (4.7):* 'the lower estimate'

Let us first discuss the heuristics of the proof of the lower estimate.

The rough idea in order to produce a lower estimate on $Q_{t,\omega}$('$Z_t \sim tx$') is to use the measures:

$$\hat{P}^{0,y}_{\lambda,\omega}(dw) \stackrel{\text{def}}{=}$$

(4.12)
$$\frac{1}{e_\lambda(0,y,\omega)} \exp\Big\{ - \int_0^{H(y)} (\lambda + V)(Z_s,\omega)ds \Big\} \, 1\{H(y) < \infty\} \, P_0(dw) ,$$

with $y = tx$ and $\lambda = \lambda(x)$ chosen in such a way that for large $t$, $H(y = tx)$ is typically of order $t$ under $\hat{P}^{\lambda,y}_\omega$. One then 'obtains' a lower bound

$$\begin{aligned} Q_{t,\omega}(\text{`}Z_t \sim tx\text{'}) \quad &\text{`}\geq\text{'} \quad \exp\{\lambda(x)t\} \, e_{\lambda(x)}(0,tx,\omega) \\ &= \quad \exp\{-(\alpha_{\lambda(x)}(x) - \lambda(x))\, t + o(t)\} \, . \end{aligned}$$

On the other hand in 'good cases', (see (4.37)), $H(tx)$ for large $t$, is typically of order $\frac{\partial \alpha_\lambda}{\partial \lambda}\,(x)t$ under $\hat{P}^{0,y=tx}_{\lambda,\omega}$, and this motivates choosing $\lambda(x)$ so that:

(4.13)
$$\frac{\partial \alpha_\lambda}{\partial \lambda}\,(x)\Big|_{\lambda=\lambda(x)} = 1 \, .$$

This is on the other hand an equation for the locus of maxima of the function $\lambda \to \alpha_\lambda(x) - \lambda$, (which is concave and tends to $-\infty$ at $\infty$), and the previous heuristic lower bound on $Q_{t,\omega}(\text{`}Z_t \sim tx\text{'})$ is now linked to the function $I(\cdot)$ in (4.2).

There are several problems with the above strategy of proof. First we do not know that the concave functions $\lambda \to \alpha_\lambda(x)$ are differentiable. Moreover it turns out that for all $x$

(4.14)
$$\alpha'_\lambda(x)_+ \,|_{\lambda=0} < \infty \, ,$$

($\alpha'_\lambda(x)_+$, $\alpha'_\lambda(x)_-$ denote the respective right and left derivative of $\alpha_\lambda(x)$), this result is proved in [Szn95b]. This shows that even if $\lambda \to \alpha_\lambda(x)$ is differentiable, the equation (4.13) need not have a solution for small $x$. Intuitively, (4.14) corresponds to the existence of a 'minimal velocity' $1/\alpha'_\lambda\,(\frac{x}{|x|})_+ \,|_{\lambda=0}$ for performing crossings from 0 to $tx$ under $\hat{P}^{0,tx}_{\lambda,\omega}$ when $t$ is large and $\lambda \geq 0$. It shows that one cannot always tune $\lambda(x)$ so that $H(tx)$ is of order $t$ under $\hat{P}^{0,tx}_{\lambda(x),\omega}$.

Let us now return to the proof of (4.7).

For $0 < \gamma_1 < \gamma_2 < \infty$, $v \in \mathbb{R}^d \backslash \{0\}$, and $0 \leq m \leq n$, we introduce the stopping time

(4.15)
$$S_{m,n,v,\gamma_1} = H(nv) \circ \theta_{(n-m)\gamma_1} + (n-m)\gamma_1 \, ,$$

as well as the event $A_{m,n,v,\gamma_1,\gamma_2}$ 'the trajectory $Z_.$ has touched $B(nv)$ sometimes during $[(n-m)\gamma_1, (n-m)\gamma_2]$':

(4.16)
$$A_{m,n,v,\gamma_1,\gamma_2} = \{S_{m,n,v,\gamma_1} \leq (n-m)\gamma_2\} \, .$$

It is useful to consider the quantities

$$(4.17)\qquad \begin{aligned} &b_\lambda(m,n,v,\gamma_1,\gamma_2,\omega) = \\ &- \inf_{z\in B(mv)} \log E_z\Big[A_{m,n,v,\gamma_1,\gamma_2},\ \exp\Big\{-\int_0^{S_{m,n,v,\gamma_1}} (\lambda+V)(Z_s,\omega)ds\Big\}\Big]\,. \end{aligned}$$

It measures the 'cost' of going from $B(mv)$ to $B(nv)$ sometimes during $[(n-m)\gamma_1, (n-m)\gamma_2]$ when moving in the potential $\lambda+V$. It is clear from Markov property and (1.3.19) that $b_\lambda$ is measurable in $\omega$. Moreover we have

**Lemma 4.3:** *For* $0<\gamma_1<\gamma_2<\infty$, $\lambda\ge 0$, $v\in \mathbb{R}^d\backslash\{0\}$, *as* $n\to\infty$,

$$(4.18)\qquad \frac{1}{n}\, b_\lambda(0,n,v,\gamma_1,\gamma_2,\omega) \underset{L^1(\mathbf{P})}{\overset{\mathbf{P}-\text{a.s.}}{\longrightarrow}} \kappa_\lambda(v,\gamma_1,\gamma_2)\in[0,\infty)\,.$$

*Moreover, if* $\lambda>0$, *and*

$$(4.19)\qquad (\gamma_1,\gamma_2)\cap[\alpha'_\lambda(x)_+,\ \alpha'_\lambda(x)_-]\neq\emptyset,\ \textit{then}$$

$$(4.20)\qquad \kappa_\lambda(v,\gamma_1,\gamma_2)\le\alpha_\lambda(v)\,.$$

Let us admit Lemma 4.3 for the time being and prove (4.7). Since $I(\cdot)$ is continuous, and $S_{t,\omega}$ has subexponential decay in $t$ for typical $\omega$, (4.7) will follow once we find a set $\Omega_2$ of full $\mathbb{P}$-measure such that

$$(4.21)\qquad \begin{aligned} &\text{for } \omega\in\Omega_2,\ v\in\mathbb{Q}^d\backslash\{0\},\ r>0\,, \\ &\varliminf_{t\to\infty}\ \frac{1}{t}\ \log E_0\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\},\ Z_t\in tB(v,r)\Big]\ge -I(v)\,. \end{aligned}$$

We shall now prove (4.21). We begin with the observation that we can find $c(d)>0$, such that on a set of full $\mathbb{P}$-measure:

$$(4.22)\qquad \begin{aligned} &\text{for large } |x|, \text{ there exists } q\in\mathbb{Z}^d \text{ with } |q-x|<\sqrt{|x|} \text{ and} \\ &V(\cdot,\omega)=0\ \text{ on }\ B(q,c(d)\log|x|)\,. \end{aligned}$$

Indeed for large $p\in\mathbb{Z}^d$, the probability that all cubes with center $p'\in p+L\mathbb{Z}^d\cap B(p,\frac{1}{2}\,|p|^{1/2})$ and side length $L=4c(\log|p|)^{1/2}$ receive a point of $\omega$ is smaller than

$$(1-\exp\{-\nu L^d\})^{\text{const}(d,c)|p|^{d/2}/\log|p|}\,,$$

which is the general term of a summable series in $p\in\mathbb{Z}^d$, when $c$ is small enough. The claim (4.22) thus follows from Borel-Cantelli's lemma.

We shall now define $\Omega_2$. To this end we introduce

$$\lambda_\infty(v) = \inf\{\lambda \in (0,\infty) \cap \mathbb{Q},\ \alpha'_\lambda(v)_+ < 1\} \in [0,\infty)\ , \tag{4.23}$$

notice that the set in brackets is never empty since in view of (2.31), (2.66), the decreasing functions $\alpha'_\lambda(v)_- \geq \alpha'_\lambda(v)_+$ tend to 0 as $\lambda$ tends to $\infty$. It is plain that either $\lambda_\infty(v) = 0$, and $\alpha_\lambda(v) - \lambda$, is a decreasing function of $\lambda$, or $\lambda_\infty(v) > 0$, so that $\alpha'_{\lambda_\infty}(v)_+ \leq 1 \leq \alpha'_{\lambda_\infty}(v)_-$. In both cases we have

$$I(v) = \alpha_{\lambda_\infty}(v) - \lambda_\infty\ . \tag{4.24}$$

We define $\Omega_2$ as the intersection of $\Omega_1$ (see after (4.7)) with a set of full $\mathbb{P}$-measure where (4.18) holds for all $v \in \mathbb{Q}^d \backslash \{0\}$, all rationals $0 < \gamma_1 < \gamma_2 < 1$, and all $\lambda \in (0,\infty) \cap (\mathbb{Q} \cup \{\lambda_\infty(v)\})$, where (4.22) holds, as well as the upper bound (4.5.12) on the growth of $V(\cdot,\omega)$.

Keep $\omega \in \Omega_2$, and consider $v \in \mathbb{Q}^d \backslash \{0\}$ together with rationals $0 < \gamma_1 < \gamma_2 < 1$ and $\lambda \in (0,\infty) \cap (\mathbb{Q} \cup \{\lambda_\infty(v)\})$ for which

$$(\gamma_1,\gamma_2) \cap [\alpha'_\lambda(v)_+,\ \alpha'_\lambda(v)_-] \neq \emptyset\ . \tag{4.25}$$

Using (4.22), we now denote for large $t$ by $y(t)$ some point in $\mathbb{Z}^d$ for which:

$$|y(t) - v[t]| < \sqrt{t\,|v|},\ \text{ and }\ V(\cdot,\omega) = 0\ \text{ in }\ B(y(t),\, c\log(\sqrt{|v|[t]}))\ . \tag{4.26}$$

Then for large $t$, and $r > 0$, we find:

$$\begin{aligned}
&E_0\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\},\ Z_t \in B(vt,rt)\Big] \geq \\
&E_0\Big[A_{0,[t],v,\gamma_1,\gamma_2}, \exp\Big\{-\int_0^{S_{0,[t],v,\gamma_1}} V(Z_s,\omega)ds\Big\}\Big]\ \cdot \\
&\inf_{z\in B(v[t])} E_z\Big[\exp\Big\{-\int_0^{H(y(t))} V(Z_s,\omega)ds\Big\},\ H(y(t)) \leq t^{3/4}\Big]\ \cdot \\
&P_0[T_{B(0,\frac{c}{2}\log(\sqrt{|v|[t]}))} > t]\ .
\end{aligned} \tag{4.27}$$

Using tubular estimates on Brownian motion similar to (4.5.19) to control the second term on the r.h.s. of (4.27), and (3.1.53) to control the rightmost term of (4.27), we find

$$\begin{aligned}
&\varliminf_{t\to\infty} \frac{1}{t}\ \log E_0\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\},\ Z_t \in B(vt,rt)\Big] \\
&\geq \lambda\gamma_1 + \varliminf_{t\to\infty} \frac{1}{t}\ \log E_0\Big[A_{0,[t],v,\gamma_1,\gamma_2}, \\
&\exp\Big\{-\int_0^{S_{0,[t],v,\gamma_1}} (\lambda+V)(Z_s,\omega)ds\Big\}\Big] \\
&\overset{(4.18)-(4.19)-(4.25)}{\geq} \lambda\gamma_1 - \alpha_\lambda(v)\ .
\end{aligned} \tag{4.28}$$

If $\lambda_\infty = 0$, we can let $\lambda$ decrease to 0 adjusting $\gamma_1, \gamma_2$ so that (4.25) holds. We find that the l.h.s. of (4.28) is bigger than $-\alpha_0(v) = -I(v)$.

If $\lambda_\infty > 0$, either $\alpha'_{\lambda_\infty}(v)_+ < 1 \le \alpha'_{\lambda_\infty}(v)_-$, in which case we choose $\lambda = \lambda_\infty$ in (4.28) and $\gamma_1$ arbitrarily close to 1, or $\alpha'_{\lambda_\infty}(v)_+ = 1$, in which case we let $\lambda$ decrease to $\lambda_\infty$ and $\gamma_1$ tend to 1. As a result we see that the l.h.s. of (4.28) is bigger than $\alpha_{\lambda_\infty}(v) - \lambda_\infty \overset{(4.24)}{=} -I(v)$. This concludes the proof of (4.21), and therefore of (4.7).

*Proof of Lemma 4.3:*

*Proof of (4.18):* We consider some fixed $v \in \mathbb{R}^d \backslash \{0\}$, $\lambda \ge 0$, $0 < \gamma_1 < \gamma_2 < \infty$, and omit the dependence on $v, \gamma_1, \gamma_2$ in the notations, for simplicity. The doubly indexed sequence $b_\lambda(m, n)$ of (4.17) satisfies:

- $b_\lambda(0,0) = 0$, $b_\lambda(m,n) \le b_\lambda(m,k) + b_\lambda(k,n)$, $0 \le m \le k \le n$, thanks to the strong Markov property, and the fact that on $A_{m,k} \cap \theta^{-1}_{S_{m,k}}\{A_{k,n}\}$,

$$S_{m,n} \le S_{m,k} + S_{k,n} \circ \theta_{S_{m,k}} \in [\gamma_1(n-m),\ \gamma_2(n-m)]\,.$$

- $\{b_\lambda(m,n),\ 0 \le m \le n\}$ has same distribution under $\mathbb{P}$ as $\{b_\lambda(m+1, n+1),\ 0 \le m \le n\}$,
- $\mathbb{E}[b_\lambda(0,1)] < \infty$,

using the r.h.s. of (2.32) with $v = y$, $x = 0$, $t = \frac{\gamma_1+\gamma_2}{2}$ to obtain a lower bound on $b_\lambda(0,1)$. Our claim (4.18) is now a consequence of the subadditive ergodic theorem (Liggett [Lig85], p. 277).

*Proof of (4.20):* With the help of (4.19), we can choose $\rho \in (0,1)$ and $\eta > 0$, with

$$\rho\alpha'_\lambda(v)_+ + (1-\rho)\,\alpha'_\lambda(v)_- + [-\eta, \eta] \subset (\gamma_1, \gamma_2)\,. \tag{4.29}$$

It now follows from strong Markov property that

$$\begin{aligned}
&\exp\{-b_\lambda(0,n)\} \ge \inf_{x\in B(0)} E_x\Big[H(\rho n v) \in \rho n[\alpha'_\lambda(v)_+ \\
&-\eta,\ \alpha'_\lambda(v)_+ + \eta]\,, \\
&\exp\Big\{-\int_0^{H(\rho n v)} (\lambda+V)(Z_s,\omega)ds\Big\}\Big] \\
&\cdot \inf_{z\in B(\rho n v)} E_z\Big[H(nv) \in (1-\rho)n\,[\alpha'_\lambda(v)_- - \eta,\ \alpha'_\lambda(v)_- + \eta], \\
&\exp\Big\{-\int_0^{H(\rho n v)} (\lambda+V)(Z_s,\omega)ds\Big\}\Big]\,.
\end{aligned} \tag{4.30}$$

Thus for $0 < \lambda_2 < \lambda < \lambda_1$, with the notation of (1.4),

$$
\begin{aligned}
&\frac{1}{n}\, b_\lambda(0,n) \le \\
&\frac{1}{n}\, a_{\lambda_1}(0,\rho n v) + \frac{1}{n}\, a_{\lambda_2}(\rho n v, nv) + (\lambda-\lambda_2)(\alpha'_\lambda(v)_- + \eta) \\
&-\frac{1}{n} \inf_{x\in B(0)} \log \hat{P}^{x,\rho n v}_{\lambda_1,\omega}\Big[\frac{H(\rho n v)}{\rho n} \in (\alpha'_\lambda(v)_+ - \eta, \alpha'_\lambda(v)_+ + \eta)\Big] \\
&-\frac{1}{n} \inf_{z\in B(\rho n v)} \log \hat{P}^{z,nv}_{\lambda_2,\omega}\Big[\frac{H(nv)}{n} \in (\alpha'_\lambda(v)_- - \eta, \alpha'_\lambda(v)_- + \eta)\Big]
\end{aligned}
\tag{4.31}
$$

Let us admit for a moment the following claim:

(4.32) if $0 < \lambda_2 < \lambda < \lambda_1$ are such that $\alpha'_{\lambda_1}(v)$ and $\alpha'_{\lambda_2}(v)$ exist and respectively belong to $[\alpha'_\lambda(v)_+,\ \alpha'_\lambda(v)_+ + \eta)$, $(\alpha'_\lambda(v)_- - \eta, \alpha'_\lambda(v)_-]$, then the last two terms of (4.31) converge to 0 in probability .

Choosing $\lambda_1, \lambda_2$ as indicated in (4.32), it follows from the shape theorem, (4.18), using convergence in probability as $n\to\infty$, that

$$
\kappa_\lambda(v,\gamma_1,\gamma_2) \le \rho\alpha_{\lambda_1}(v) + (1-\rho)\,\alpha_{\lambda_2}(v) + (\lambda-\lambda_2)(\alpha'_\lambda(v)_- + \eta)\ . \tag{4.33}
$$

Letting $\lambda_1 \downarrow \lambda$ and $\lambda_2 \uparrow \lambda$, this proves (4.20). As for the proof of (4.32), observe that when $\overline{\lambda} > 0$ and $\alpha'_{\overline{\lambda}}(v)$ exists,

$$
\begin{aligned}
&\hat{P}^{z,\mu v}_{\overline{\lambda},\omega}\,[H(uv) \le u(\alpha'_{\overline{\lambda}}(v) - \epsilon)] \le \exp\{\mu u(\alpha'_{\overline{\lambda}}(v)-\epsilon)\} \\
&e_{\overline{\lambda}+\mu}(z,\mu v,\omega)/e_{\overline{\lambda}}(z,uv,\omega), \quad \text{for } z\in B(0),\ \epsilon>0,\ u>0,\ \mu>0\ .
\end{aligned}
\tag{4.34}
$$

It now follows from the shape theorem 2.4 and the Harnack inequalities (2.22) applied to compare $e_\lambda(z,uv,\omega)$ with $e_\lambda(0,uv,\omega)$ that $\mathbb{P}$-a.s., for $\epsilon>0$, $\mu>0$

$$
\begin{aligned}
&\overline{\lim_{u\to\infty}}\ \frac{1}{u}\, \log \sup_{B(0)} \hat{P}^{z,\mu v}_{\overline{\lambda},\omega}\Big[\frac{H(uv)}{u} \le \alpha'_{\overline{\lambda}}(v) + \epsilon\Big] \\
&\le -\mu\epsilon + (\alpha_{\overline{\lambda}}(v) - \alpha_{\overline{\lambda}+\mu}(v) + \mu\alpha'_{\overline{\lambda}}(v))
\end{aligned}
\tag{4.35}
$$

and this last number is strictly negative when $\mu$ is small. In a similar fashion we have $\mathbb{P}$-a.s. for $\epsilon>0$, $0<\mu<\overline{\lambda}$

$$
\begin{aligned}
&\overline{\lim_{u\to\infty}}\ \frac{1}{u}\, \log \sup_{B(0)} \hat{P}^{z,\mu v}_{\overline{\lambda},\omega}\Big[\frac{H(uv)}{u} \ge \alpha'_{\overline{\lambda}}(v) + \epsilon\Big] \\
&\le -\mu\epsilon + (\alpha_{\overline{\lambda}}(v) - \alpha_{\overline{\lambda}-\mu}(v) - \mu\alpha'_\lambda(v))
\end{aligned}
\tag{4.36}
$$

which is strictly negative for small $\mu$. Thus, $\mathbb{P}$-a.s., for $\epsilon > 0$,

$$(4.37) \qquad \overline{\lim_{\mu\to\infty}}\ \frac{1}{u}\ \log\ \sup_{B(0)}\ \hat{P}^{z,uv}_{\frac{\lambda}{},\omega}\Big[\Big|\frac{H(uv)}{u} \geq \alpha'_{\frac{}{\lambda}}(v)\Big| \geq \epsilon\Big] < 0\,,$$

and our claim (4.32) easily follows. This concludes the proof of Theorem 4.2.

□

**Exercise:** Show that

$$K = \{x \in \mathbb{R}^d,\ \alpha'_0(x) \leq 1\}$$

is a compact subset of $\mathbb{R}^d$, star shaped around 0, and

$$(4.38) \qquad K = \{x \in \mathbb{R}^d,\ I(x) = \alpha_0(x) \leq 1\}\,.$$

(Hint: $\alpha'_0(x) = \sup_{n\geq 1}\ n(\alpha_{\frac{1}{n}}(x) - \alpha_0(x))$, and use the definition (4.2)). As already mentioned, it can be shown that $\alpha'_0(x) < \infty$, for $x \in \mathbb{R}^d$, see [Szn95b], and therefore $K$ has nonempty interior. This has interesting consequences for quenched Brownian motion with a constant drift in the Poissonian potential $V(\cdot, \omega)$, see (4.66) . □

We now turn to the discussion of the large deviation principle for the *annealed path measures*. In the *soft obstacle case*, we consider for $t > 0$,

$$(4.39) \qquad Q_t = \frac{1}{S_t}\ \exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\ P_0 \otimes \mathbb{P}\,,$$

whereas in the *hard obstacle case*, we instead consider

$$(4.40) \qquad \begin{aligned} Q_t &= \frac{1}{S_t}\ 1\{T > t\}\ P_0 \otimes \mathbb{P} \\ &= P_0 \otimes \mathbb{P}[\,\cdot\,|T > t]\,, \end{aligned}$$

with $T$ the entrance time in the obstacle set $S = \bigcup_i \overline{B}(x_i, a)$.

The rate function governing the large deviation principle of $Z_t$ in scale $t$ under $Q_t$, as we shall see, will now be

$$(4.41) \qquad J(x) \stackrel{\text{def}}{=} \sup_{\lambda\geq 0}\ (\beta_\lambda(x) - \lambda),\ x \in \mathbb{R}^d\,,$$

with $\beta_\lambda(\cdot)$ the annealed Lyapunov coefficients introduced in Section 3.

**Lemma 4.4:** *$J(x)$ is a continuous convex function, and in the notation of (2.57), (3.31),*

$$(4.42)\qquad c_2\,|x| \vee \frac{x^2}{2} \le J(x) \le \begin{cases} \nu\,|x| + \dfrac{x^2}{2}\,(= J^{\text{hard obstacle}}(x)), & \textit{when } d=1\,,\\ k_0\,|x|, \textit{ for } |x| \le \sqrt{2\lambda_{d-1}}/r_0, & \textit{when } d \ge 2\,,\\ \min\limits_{r>0}\Big(\nu\omega_{d-1}(a+r)^{d-1}\,|x| + \dfrac{\lambda_{d-1}}{r^2}\Big) + \dfrac{x^2}{2}, & \\ \textit{for } |x| > \sqrt{2\lambda_{d-1}}/r_0, & \end{cases}$$

*provided $r_0 > 0$ denotes the unique value of $r$ for which*

$$\nu\,\omega_{d-1}(a+r)^{d-1} + \sqrt{2\lambda_{d-1}}/r = k_0 (= \textit{minimum value})\,.$$

*In the soft obstacle case,*

$$(4.43)\qquad J(x) \le I(x),\ x \in \mathbb{R}^d\,,$$

**Proof:** The proof is analogous to Lemma 4.1, and follows readily from the estimates of Theorem 3.4. □

We now have

**Theorem 4.5:** *(annealed large deviation principle in scale t)*

*Under $Q_t$, $Z_t/t$ satisfies a large deviation principle at rate $t$ with rate function $J(\cdot)$, as $t$ tends to infinity, i.e.:*

$$(4.44)\qquad \begin{aligned} &\textit{for any closed subset } A \textit{ of } \mathbb{R}^d,\\ &\overline{\lim_{t\to\infty}}\ \frac{1}{t}\log Q_t(Z_t \in tA) \le -\inf_{x\in A} J(x)\,, \end{aligned}$$

$$(4.45)\qquad \begin{aligned} &\textit{for any open subset } A \textit{ of } \mathbb{R}^d,\\ &\underline{\lim_{t\to\infty}}\ \frac{1}{t}\log Q_t(Z_t \in tA) \ge -\inf_{x\in O} J(x)\,. \end{aligned}$$

**Proof:** The proof of the upper estimate (4.44) is analogous to the proof of (4.6). We simply use Proposition 2.8 instead of Corollary 2.10, in the reduction step to the case of a compact subset $A$. As for the subexponential decay of the normalizing constant $S_t$, it is a consequence of (4.5.29) (in fact the lower estimate).

The proof of the lower estimate (4.45), is also analogous to that of (4.7), though simpler. In the notations of (4.15), (4.16), we consider for $v \in \mathbb{R}^d\backslash\{0\}$, $0 < \gamma_1 < \gamma_2 < \infty$, $\lambda \ge 0$, $n \ge 0$,

$$(4.46)\qquad \begin{aligned} c_\lambda(n,v,\gamma_1,\gamma_2) = -\inf_{z\in B(0)} \log E_z\Big[A_{0,n,v,\gamma_1,\gamma_2}\,,\ \exp\Big\{-\lambda S_{n,v,\gamma_1}\\ -\nu\int\Big(1-\exp\Big\{-\int_0^{S_{n,v,\gamma_1}} W(Z_s-y)ds\Big\}\Big)dy\Big\}\Big]\,, \end{aligned}$$

in the soft obstacle case, and with an obvious modification in the hard obstacle case. Analogously to Lemma 4.3, we now have

**Lemma 4.6:** *For* $0 < \gamma_1 < \gamma_2 < \infty$, $\lambda \geq 0$, $v \in \mathbb{R}^d \backslash \{0\}$*:*

$$\frac{1}{n}\, c_\lambda(n, v, \gamma_1, \gamma_2) \underset{n\to\infty}{\longrightarrow} \delta_\lambda(v, \gamma_1, \gamma_2) \in [0, \infty)\,, \tag{4.47}$$

*Moreover, if* $\lambda > 0$, *and*

$$(\gamma_1, \gamma_2) \cap [\beta'_\lambda(v)_+, \beta'_\lambda(v)_-] \neq \emptyset, \quad \textit{then} \tag{4.48}$$

$$\delta_\lambda(v, \gamma_1, \gamma_2) \geq \beta_\lambda(v)\,. \tag{4.49}$$

**Proof:** With the help of (3.13) and strong Markov property, one shows subadditivity of the $c_\lambda(n)$, so that

$$\frac{1}{n}\, c_\lambda(n) \underset{n\to\infty}{\longrightarrow} \delta_\lambda(v, \gamma_1, \gamma_2) = \inf_{n\geq 1} \frac{c_\lambda(n)}{n}\,. \tag{4.50}$$

As for the proof of (4.49), the argument is similar to the proof of (4.20), with the help of (3.13). □

From then on the argument is essentially a repetition, with obvious modifications of the proof of (4.7). □

We shall now discuss some applications of the large deviation principle we have just derived to the analysis of the large $t$ behavior of quenched and annealed Brownian motion with a constant drift moving among Poissonian obstacles.

We denote by $P^h_x$ the Wiener measure with constant drift $h \in \mathbb{R}^d$, starting from $x \in \mathbb{R}^d$. We now want to consider the *quenched path measure with drift*:

$$\begin{aligned} Q^h_{t,\omega} &= \frac{1}{S^h_{t,\omega}} \exp\Big\{-\int_0^t V(Z_s, \omega)ds\Big\}\, P^h_0,\ t > 0,\ \omega \in \Omega\,, \\ &= \frac{1}{\widetilde{S}^h_{t,\omega}} \exp\Big\{h\cdot Z_t - \int_0^t V(Z_s, \omega)ds\Big\}\, P_0\,, \end{aligned} \tag{4.51}$$

where $S^h_{t,\omega}$ is the normalizing constant and

$$\widetilde{S}^h_{t,\omega} = E_0\Big[\exp\Big\{h\cdot Z_t - \int_0^t V(Z_s, \omega)ds\Big\}\Big]\,. \tag{4.52}$$

Analogously, the *annealed path measure with drift* are defined as

$$
\begin{aligned}
Q_t^h &= \frac{1}{S_t^h}\,\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\,P_0^h\otimes \mathbb{P}, \quad \textit{(soft obstacles)}\\
&= \frac{1}{\widetilde{S}_t^h}\,\exp\Big\{h\cdot Z_t-\int_0^t V(Z_s,\omega)ds\Big\}\,P_0\otimes\mathbb{P}\,, \quad \text{with} \qquad (4.53)\\
\widetilde{S}_t^h &= \mathbb{E}\otimes E_0\Big[\exp\Big\{h\cdot Z_t-\int_0^t V(Z_s,\omega)ds\Big\}\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
Q_t^h &= P_0\otimes\mathbb{P}\,[\,\cdot\,|\,T>t], \quad \textit{(hard obstacles)}\\
&= \frac{1}{\widetilde{S}_t^h}\,\exp\{h\cdot Z_t\}\;1\{T>t\}\;P_0\otimes\mathbb{P}, \quad \text{with} \qquad (4.54)\\
\widetilde{S}_t^h &= \mathbb{E}\otimes E_0[e^{h\cdot Z_t},T>t]=E_0[\exp\{h\cdot Z_t-\nu\,|W_t^a|\}]\,,
\end{aligned}
$$

with $T$ the entrance time in the obstacle set (see (1.13)), and $W_t^a$ as in (3.4).

We shall now describe a transition between the small and large $h$ situation. For this purpose, the size of $h$ will be quantified in terms of the dual norm $\alpha_0^*$ (see (2.69)), in the quenched case, and the dual norm $\beta_0^*$ (see (3.43)), in the annealed case.

We begin with the quenched situation.

**Theorem 4.7:** *(soft obstacles)*

*When $\omega$ belongs to the set of full measure appearing in Theorem 4.2, then for any $h\in\mathbb{R}^d$, as $t\to\infty$,*

(4.55) $\quad \dfrac{Z_t}{t}$ *satisfies a large deviation principle under $Q_{t,\omega}^h$, with rate $t$ and rate function*

$$I^h(x)=I(x)-h\cdot x+\sup_y\,(h\cdot y-I(y))\,. \tag{4.56}$$

*Moreover,*

$$
\begin{aligned}
\lim_{t\to\infty}\frac{1}{t}\,\log\widetilde{S}_{t,\omega}^h=\sup_y(h\cdot y-I(y)) \;&=\; 0, \quad \textit{if } \alpha_0^*(h)\le 1\,,\\
&>\; 0, \quad \textit{if } \alpha_0^*(h)>1\,.
\end{aligned} \tag{4.57}
$$

**Proof:** The large deviation principle (4.55), and the first equality of (4.57) follow from Theorem 4.2 and a classical argument of large deviation theory (see Deuschel-Stroock [DS89], p. 43, 51), once we prove the 'exponential tightness estimate':

$$
\begin{aligned}
&\lim_{L\to\infty}\;\overline{\lim_{t\to\infty}}\;\frac{1}{t}\,\log E_0\Big[h\cdot Z_t\ge Lt,\\
&\exp\Big\{h\cdot Z_t-\int_0^t V(Z_s,\omega)ds\Big\}\Big]=-\infty\,.
\end{aligned} \tag{4.58}
$$

However the expression under the logarithm is smaller than:

$$\left(\frac{t}{2\pi h^2}\right)^{1/2} \int_L^\infty \exp\left\{\left(u - \frac{u^2}{2h^2}\right) t\right\} du\,,$$

so that (4.58) immediately follows. To conclude our proof we shall now see that:

$$\sup_y \,(h \cdot y - I(y)) = 0 \Longleftrightarrow \alpha_0^*(h) \le 1\,. \tag{4.59}$$

Assume $\alpha_0^*(h) \le 1$. Since $I \ge \alpha_0$, we find

$$\sup_y \; h \cdot y - I(y) \le \sup_y \; (h \cdot y - \alpha_0(y)) \le 0\,,$$

from the definition of $\alpha_0^*$ and the condition $\alpha_0^*(h) \le 1$. Next observe that

$$\lim_{u \to 0} \frac{I(uy)}{u} = \alpha_0(y) \tag{4.60}$$

(in fact it can be shown that $I(\cdot)$ coincides with $\alpha_0(\cdot)$ on a neighborhood of the origin, see (4.38)). Indeed we know that

$$I(uy) \overset{(4.24)}{=} u\alpha_{\lambda_\infty(uy)}(y) - \lambda_\infty(uy), \quad \text{where} \tag{4.61}$$

$\lambda_\infty(uy) = \inf\{\lambda \in (0,\infty) \cap \mathbb{Q},\ u\alpha'_\lambda(y)_+ < 1\}$ tends to 0 as $u$ tends to 0. Thus

$$u\alpha_0(y) \le I(uy) \le u\alpha_{\lambda_\infty(uy)}(y)\,,$$

and (4.60) follows. Assume now that $\alpha_0^*(h) > 1$, and choose $y$ with

$$\alpha_0(y) = 1 \;\text{ and }\; h \cdot y = \alpha_0^*(h) > 1\,. \tag{4.62}$$

Then the function

$$u \ge 0 \to uh \cdot y - I(uy)$$

has a right derivative in 0 equal to $h \cdot y - \alpha_0(y) > 0$. Therefore $\sup_x (h \cdot x - I(x)) > 0$. This finishes the proof of (4.59). □

As an application of the above result, we find:

**Corollary 4.8:** *On the set of full* $\mathbb{P}$*-measure appearing in Theorem 4.2, when* $\alpha_0^*(h) < 1$,

$$\frac{Z_t}{t} \to 0 \;\textit{ in } Q_{t,\omega}^h \textit{ probability, as } t \to \infty, \tag{4.63}$$

*and when* $\alpha_0^*(h) > 1$,

$$\mathrm{dist}\left(\frac{Z_t}{t}, M\right) \to 0, \;\textit{ in } Q_{t,\omega}^h \textit{ probability, as } t \to \infty\,, \tag{4.64}$$

*where $M$ is the compact subset of* $\mathbb{R}^d \backslash \{0\}$:

$$M = \{x \in \mathbb{R}^d,\ h \cdot x - I(x) = \sup_y \{h - y - I(y)\} \ . \tag{4.65}$$

**Proof:** When $\alpha_0^*(h) < 1$, the rate function of the large deviation principle satisfied by $\frac{Z_t}{t}$ under $Q_{t,\omega}^h$ equals

$$I(x) - h \cdot x \geq \alpha_0(x) - h \cdot x \ .$$

It only vanishes when $x = 0$, and the claim (4.63) now easily follows.

On the other hand, when $\alpha_0^*(h) < 1$, the rate function of the large deviation principle satisfied by $\frac{Z_t}{t}$ under $Q_{t,\omega}^h$ only vanishes on the compact set $M$. From this (4.64) easily follows. Observe also that $M$ does not contain the origin since in view of (4.57),

$$\sup(h \cdot y - I(y)) > 0 \ .$$

□

Corollary 4.8 thus displays a transition between a 'sub-ballistic' behavior of $Z_t$ for 'small $h$' and a 'ballistic' behavior of $Z_t$ for large $h$. The unit sphere of $\alpha_0^*$ corresponds to the set of critical drifts.

**Exercise:** Show that

$$M \cap K^0 = \emptyset \ , \tag{4.66}$$

where $M$ is defined in (4.65) and $K^0$ is the interior of

$$K = \{x \in \mathbb{R}^d,\ \alpha_0'(x) \leq 1\} \ .$$

(Hint: use (4.38)). □

We have analogous large deviation results in the annealed situation:

**Theorem 4.9:** *(soft and hard obstacles)*

*For any $h \in \mathbb{R}^d$, as $t \to \infty$,*

(4.67) $\frac{Z_t}{t}$ *satisfies a large deviation principle under $Q_t^h$, with rate $t$, and rate function*

$$J^h(x) = J(x) - h \cdot x + \sup_y (h \cdot y - J(y)) \ . \tag{4.68}$$

Moreover,

$$\lim_{t\to\infty} \frac{1}{t} \log \tilde{S}_t^h = \sup_y (h \cdot y - I(y)) \begin{array}{ll} = 0, & \text{if } \beta_0^*(h) \leq 1 \ , \\ > 0, & \text{if } \beta_0^*(h) > 1 \ . \end{array} \tag{4.68}$$

As an application one derives:

**Corollary 4.10:** *When* $\beta_0^*(h) < 1$,

$$\frac{Z_t}{t} \to 0 \text{ in } Q_t^h \text{ probability, as } t \to \infty\,, \tag{4.69}$$

*and when* $\beta_0^*(h) > 1$,

$$\text{dist}\left(\frac{Z_t}{t}, N\right) \to 0, \text{ in } Q_t^h \text{ probability, as } t \to \infty\,, \tag{4.70}$$

*where* $N$ *is the compact subset of* $\mathbb{R}^d \backslash \{0\}$:

$$N = \{x \in \mathbb{R}^d,\ h \cdot x - J(x) = \sup_y \{h \cdot y - J(y)\}\}\,. \tag{4.71}$$

In the annealed case, one thus has a similar transition between a sub-ballistic and ballistic behavior, and 'critical drifts' are those belonging to the unit sphere of $\beta_0^*$. Further results are known in the 'small $h$' regime, cf. Chapter 7. On the other hand, the 'large $h$' regime is for the time being rather poorly understood.

Let us also mention that in the case of one-dimensional hard obstacles,

$$\frac{\partial}{\partial\lambda}\, \beta_\lambda^{\text{hard obstacles}}(x)\Big|_{\lambda=0} \overset{(3.31)}{=} \frac{\partial}{\partial\lambda}\left(\nu\,|x| + \sqrt{2\lambda}\right)\Big|_{\lambda=0} = \infty\,.$$

This is unlike the quenched situation (see [Szn94b] Theorem 2.6 when $d = 1$, or [Szn95b] when $d \geq 2$), and perhaps restricted to one-dimension, which plays a special role in the annealed case, see Chapter 7.

## 5.5 Notes and References

The material in this chapter is close to [Szn94b] and [Szn95a], I. As mentioned in §1, shape theorems have first been derived in the context of first passage percolation, see Hammersley-Welsh [HW10], and Kesten [Kes86] for a more recent account. In a one-dimensional setting Freidlin and Gärtner have used analogous quantities to the $e_\lambda$ for the purpose of proving certain large deviation estimates related to the study of wave front propagation in a one-dimensional random medium, see Chapter VII of Freidlin [Fre85], and Chapter 7 §2. In a (multidimensional) discrete setting, Lyapunov exponents have been constructed by Zerner in [Zer98], and used to derive a quenched large deviation principle for the simple random walk among nonnegative i.i.d. potentials, see also Chapter 7 §2. The strategy developed in this chapter has also been applied by Zerner in [Zer97], to obtain a quenched large deviation principle for a rather general random walk in random environment.

The presence of a transition between a sub-ballistic and a ballistic regime for annealed Brownian motion with a constant drift among spherical traps appears in the literature of physical chemistry, see Grassberger-Procaccia [GP88]. Eisele and Lang investigated the problem in [EL87], and in particular derived estimates on the size of the critical drift. The small $h$ situation was further analyzed in [Szn91a], where it was for instance shown that $e^{\frac{1}{2}|h|^2 t} S_t^h$ has the same principal asymptotic behavior (4.5.29) of the case $h = 0$. The analysis was later extended to subcritical drifts in [Szn95a], II. In the one-dimensional case, Povel obtained a description of the limiting macroscopic and microscopic behavior of the path $Z_.$ under $Q_t^h$, for subcritical $h$, see also Chapter 7 §3. Let us also mention that the situation of supercritical drifts, where the motion is ballistic, is so far rather poorly understood.

# 6. Quenched Path Measure and Pinning Effect

In this chapter we present the proof of the pinning effect for the quenched path measure. This effect is related to a certain random variational problem which also governs the finer asymptotic behavior of the normalizing constant $S_{t,\omega}$. In Section 1 the pinning effect is discussed at a heuristic level. Section 2 shows how a certain weak pinning property can be bootstrapped into a strong pinning property. In Section 3 we introduce a simplified random variational problem in the spirit of Chapter 3 §3, and use it to introduce certain random length scales. Section 4 is devoted to the construction of a coarse graining method on the path space, which relies on the forest and clearings picture of Chapter 4. Finally Section 5 applies the coarse graining method of Section 4 to derive finer asymptotics on $S_{t,\omega}$ and establish a weak pinning property, which together with the results of Section 2 proves the full pinning effect.

## 6.1 Heuristics

Throughout this chapter we shall consider Poissonian soft obstacles. We keep the same notations as in the Chapters 4 and 5. Our main object in this chapter is to describe for typical cloud configurations $\omega$ and large $t$, the behavior of the trajectory $Z_.$ under the quenched path measure $Q_{t,\omega}$ of (5.1.15):

$$Q_{t,\omega} = \frac{1}{S_{t,\omega}} \exp\left\{ - \int_0^t V(Z_s, \omega) ds \right\} P_0 \,, \tag{1.1}$$

with $S_{t,\omega}$ the normalizing constant.

The most direct interpretation of the probability $Q_{t,\omega}$ is given in terms of a trapping problem. The function $V(x,\omega)$ is viewed as the absorption rate at location $x$ of the random medium. The probability $Q_{t,\omega}$ thus describes the behavior of a particle launched at the origin and diffusing in a partially absorbing medium conditioned on the atypical event that it has not been absorbed up to a (long) time $t$. In this language, we are interested in the resulting 'survival strategy' of the particle.

Other physical interpretations of the measure $Q_{t,\omega}$ can be given. For instance the couple $(s, Z_s)$, $0 \le s \le t$, under $Q_{t,\omega}$ can be viewed as a 'directed polymer in the presence of columnar disorder', which can model the behavior of flux lines in high-temperature superconductors, see Krug-Halpin Healy [KH93], Nelson-Vinokur [NV92]. In this situation $t$ does not represent time anymore, but instead thickness of the material in the direction of the columnar defects, which are described by $V(x,\omega)$.

Formula (1.1) gives in a sense a 'static description' of the whole path $Z_s$, $0 \le s \le t$, under the quenched path measure. It is also possible to give a 'dynamic description' of the path, which we now discuss informally. The general line is close to the ideas in Exercise 2 at the beginning of Chapter 4 §5. As in (4.5.3), (4.5.4), we consider for $t \ge 0$, $x \in \mathbb{R}^d$, $\omega \in \Omega$:

$$u(t,x) = E_x\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big]\,, \tag{1.2}$$

$$h(t,x) = -\log u(t,x)\,, \tag{1.3}$$

where the subscript $\omega$ has been omitted from the notation for simplicity. Disregarding questions of regularity, the function $h$ satisfies

$$\begin{aligned} \partial_t h &= \tfrac{1}{2}\,\Delta h - \tfrac{1}{2}\,|\nabla h|^2 + V \\ h(0,\cdot) &= 0\,. \end{aligned} \tag{1.4}$$

On the other hand, we can express (1.1) as

$$Q_{t,\omega} = \exp\Big\{h(t,Z_0) - h(0,Z_t) - \int_0^t V(Z_s,\omega)ds\Big\}\,P_0\,. \tag{1.5}$$

Thus the application of Ito's formula yields:

$$\begin{aligned} h(0,Z_t) - h(t,Z_0) &= \int_0^t \nabla h(t-s,Z_s)\,dZ_s \\ &+ \int_0^t \Big(-\partial_s + \tfrac{1}{2}\,\Delta\Big)\,h(t-s,Z_s)\,ds \\ &\overset{(1.4)}{=} \int_0^t \nabla h(t-s,Z_s)\,dZ_s \\ &+ \int_0^t \tfrac{1}{2}\,|\nabla h|^2\,(t-s,Z_s) - V(Z_s,\omega)\,ds\,. \end{aligned} \tag{1.6}$$

Inserting the above identity in (1.5), we find:

$$Q_{t,\omega} = \exp\Big\{\int_0^t -\nabla h(t-s,Z_s)\,dZ_s - \tfrac{1}{2}\int_0^t |\nabla h|^2(t-s,Z_s)\,ds\Big\}\,P_0\,. \tag{1.7}$$

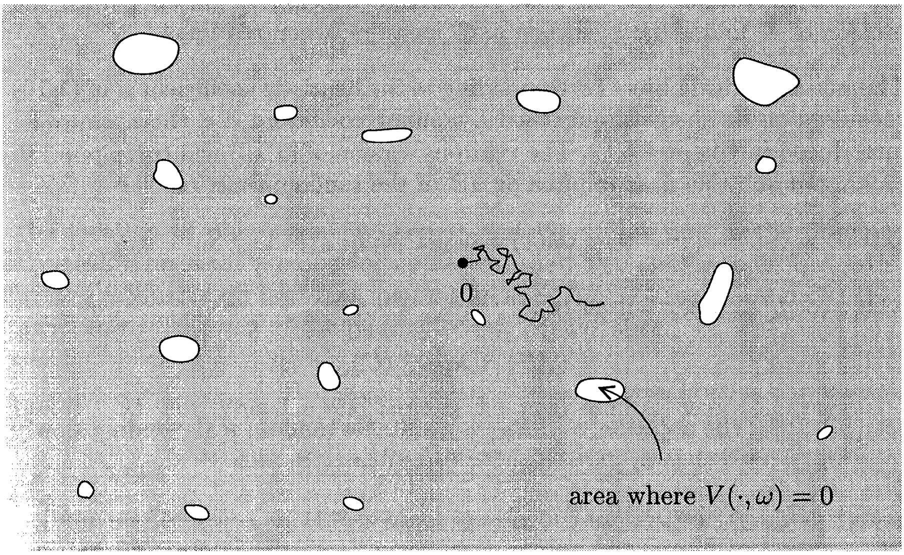
With the help of Girsanov's formula, we thus see that under $Q_{t,\omega}$, $Z_.$ satisfies the stochastic differential equation

$$(1.8)\qquad \begin{aligned} dZ_s &= d\beta_s - \nabla h(t-s, Z_s)\,ds, \quad 0 \le s \le t\,, \\ Z_0 &= 0\,, \end{aligned}$$

where $\beta_.$ is an $\mathbb{R}^d$-valued Brownian motion. In other words, under the quenched path measure, $(Z_s)_{0\le\Delta\le t}$ is a Brownian motion feeling a time inhomogeneous random gradient drift $-\nabla h(t-s,\cdot) = \frac{\nabla u}{u}\,(t-s,\cdot)$.

It turns out that we shall not need to develop a fully rigorous version of (1.8). However both 'static' and 'dynamic' point of views on the path $Z_.$, will be helpful.

We now turn to the heuristic discussion of the behavior of $Z_.$ under the quenched path measure when $\omega$ is typical and $t$ large. From (1.1), it is clear that the path $Z_.$ feels an incentive to go to large areas where $V$ vanishes, in order to avoid the penalty $\int_0^t V(Z_s,\omega)\,ds$.




Nevertheless, it is not clear where $Z_.$ really ought to go. We shall see that $Z_t$ under $Q_{t,\omega}$ tends to be concentrated around near minima of a random variational problem, which we now motivate. Informally, if we use an 'eigenfunction expansion', to express the normalizing factor $S_{t,\omega}$, we find

$$(1.9)\qquad S_{t,\omega} = [R_t^{\mathbb{R}^d, V(\cdot,\omega)}\,1]\,(0)\ \text{`}=\text{'} \sum_i \varphi_i(0) < \varphi_i,\, 1 > e^{-\lambda_i t}\,,$$

where $\lambda_i \geq 0$, are the 'eigenvalues' of $-\frac{1}{2}\,\Delta + V(\cdot,\omega)$ in $\mathbb{R}^d$ and $\varphi_i$ the corresponding normalized 'eigenfunctions'. This 'picture' is motivated by the localization theory for random Schrödinger operators, see Carmona-Lacroix [CL91], Chapter 9, Pastur-Figotin [PF92], Molchanov [Mol94]. Although not really rigorous in our context, this picture should have 'some truth', close to the bottom of the spectrum of $-\frac{1}{2}\,\Delta + V(\cdot,\omega)$.

For large $t$, only $\lambda_i$ close to 0 should matter in (1.9). The corresponding eigenfunction $\varphi_i$ should be localized 'near some large hole in the cloud', and exhibit an exponential decay 'at rate $\alpha$' away from this domain of localization. The term $< \varphi_i, 1 >$ should be viewed as 'almost constant' for our purpose. Thus the dominating terms in the series (1.9) should come from the locus of minima in:

$$\min_{x_i} \{\alpha\,|x_i| + t\,\lambda_i\}\,, \tag{1.10}$$

with $x_i$ a localization point for the eigenfunction corresponding to $\lambda_i$.

This is of course a very nonrigorous heuristics, which nevertheless motivates the fact that a suitable random variational principle ought to govern the motion of $Z_.$ under the quenched path measure, when $t$ is large.

The mathematical object corresponding to the heuristic coefficient $\alpha$ in (1.10) turns out to be the 0-th quenched Lyapunov coefficient, i.e. the norm $\alpha_0(\cdot)$ introduced in Chapter 5 §2. The rigorous version of (1.10) will correspond to looking at suitable near minima on $\mathbb{R}^d$ of the random function:

$$x \to \alpha_0(x) + t\lambda_\omega(B(x, R_{t,\omega}))\,, \tag{1.11}$$

where for $U$ an open subset of $\mathbb{R}^d$,

$$\lambda_\omega(U) = \lambda_{V(\cdot,\omega)}(U)\,, \tag{1.12}$$

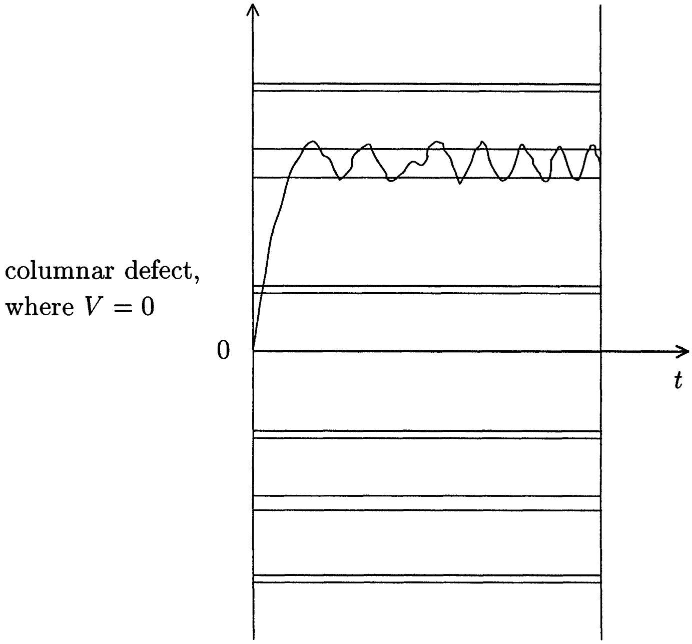
see (3.1.2) for the notation, and $R_{t,\omega}$ is a suitable random scale tending slowly to infinity, such that in particular for typical $\omega$:

$$R_{t,\omega} = o(\exp\{(\log t)^{1-\chi}\})\,, \quad \text{as } t \to \infty, \text{ with } \chi > 0, \text{ small enough}\,, \tag{1.13}$$

(one can also choose a deterministic scale $R_t = \exp\{(\log t)^{1-\chi}\}$, with $\chi > 0$, small enough, c.f. Remark 5.5). Loosely speaking, the *pinning effect* will be a statement of the sort:

(1.14) for $\mathbb{P}$-a.e. $\omega$, with $Q_{t,\omega}$ probability tending to 1, as $t$ tends to $\infty$, $Z_s, 0 \leq s \leq t$, comes within distance 1 of some near minimum $x$ of (1.11) and from then on never leaves $B(x, R_{t,\omega})$ up to time $t$.

The location of pinning in (1.14) need not be unique due to the presence of possible resonant pinning locations. We shall also obtain some less satisfactory estimates on how fast 'pinning' occurs. There are several difficulties attached to the proof of the pinning effect. For instance:



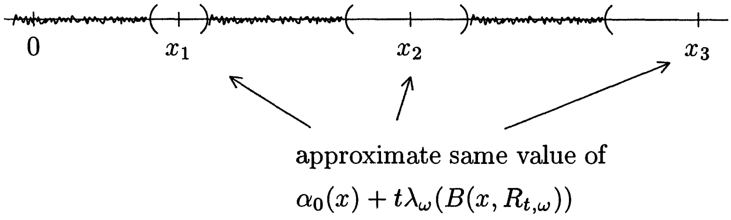

The location of pinning in (1.14) need not be unique due to the presence of possible resonant pinning locations. We shall also obtain some less satisfactory estimates on how fast 'pinning' occurs. There are several difficulties attached to the proof of the pinning effect. For instance:

(1.15) a) For typical $\omega$ and large $t$, the 'near minima' $x$ of (1.11) are

$$\text{such that } \alpha_0(x) << t\,\lambda_\omega(B(x, R_{t,\omega}))\,.$$

Thus informations on the location $x$ are hard to retrieve from the values of the function (1.11).

b) There can a priori be several competing near minima in (1.11), see also (1.10). These are 'resonances', and they can endanger the fact that the particle settles down. This is also related to the difficulties encountered in trying to derive time estimates, see the end of §5.




c) We should expect that for typical $\omega$ and large $t$, the probability $Q_{t,\omega}$ (non-pinning) is not so small (i.e. $>> \exp\{-t^\epsilon\}$, as $t \to \infty$, when $\epsilon > 0$). This can already be sensed at the level of the formal expression (1.9). Indeed one only needs to replace the function 1 by the characteristic function of a ball say of size 1, centered at a point $y$ at distance slightly larger than $R_{t,\omega}$ from some dominating $x_i$ in (1.10).

This makes it impractical to prove a pinning effect by devising 'a good lower bound on $\log S_{t,\omega}$' and a 'good upper bound' on $\log\,(E_0[$ nonpinning, $\exp\{-\int_0^t V(Z_s,\omega)ds\}])$'. Indeed both terms, in view of (4.5.10), should be equivalent to $-c(d,\nu)\, t(\log t)^{-2/d}$, and the respective upper and lower bounds would have to match to a precision higher than any small power $t^\epsilon$ of $t$. The dynamical point of view will turn out to be helpful here.

We close this section with lower estimates on expressions like $S_{t,\omega}$, which already capture the spirit of the random variational problem associated to (1.11). Throughout this chapter, we use the notation:

(1.16) $$R^U_{t,\omega} \stackrel{\text{def}}{=} R_t^{U,V(\cdot,\omega)}\,,$$

whenever $U$ is an open subset of $\mathbb{R}^d$, $t \geq 0$, $\omega \in \Omega$.

We now consider a nonempty bounded domain $U$ of $\mathbb{R}^d$, and the associated nonnegative normalized principal eigenfunction $\varphi$ of $R^U_{t,\omega}$, see the end of Chapter 3 §1. We denote by $\widetilde{U}$ the $\sqrt{d}$ open neighborhood of $U$, and further consider some $y \in U$, close to which $\varphi^2$ puts 'enough mass', in the sense that:

(1.17) $$\int_{y+[-1,1]^d} \varphi^2(x)\, dx \geq \frac{1}{2\,|\widetilde{U}|}\,.$$

Such $y$ always exists as can be seen by covering $U$ with boxes of unit size and using the equality $\int_U \varphi^2\, dx = 1$. The factor $\frac{1}{2}$ is of course unnecessary for this argument but is included in (1.17) in view of further applications.

**Theorem 1.1** *There exists a positive constant $c(d) > 0$, such that on the set of full $\mathbb{P}$-probability in (4.5.12), for large $t$, for any nonempty bounded domain $U \subseteq (-t,t)^d$, $x \in (-t,t)^d$ and $y \in U$, for which (1.17) holds:*

$$(1.18) \qquad \begin{aligned} &E_x\Big[\exp\Big\{-\int_0^u V(Z_s,\omega)ds\Big\}\Big] \ge \\ &\frac{c(d)}{|\widetilde{U}|}\, e_0(x,y,\omega)\, \exp\{-\lambda_\omega(U)\,u - \log t\}\,, \end{aligned}$$

*for $u \ge 2$, with the notation of (5.1.12).*

**Proof:** We apply Proposition 1.12 of Chapter 3, with $U' = \mathbb{R}^d$ and $A = y + [-1,1]^d$. Thus as soon as $u \ge 2$, the l.h.s. of (1.18) is bigger than:

$$(1.19) \qquad \begin{aligned} &\inf_{A\times A} r(2,\cdot,\cdot,\omega)\, \frac{1}{2\,|\widetilde{U}|}\, \exp\{-\lambda_\omega(U)\,u\} \\ &E_x\Big[\exp\Big\{-\int_0^{H_A} V(Z_s,\omega)ds\Big\},\, H_A < \infty\Big], \end{aligned}$$

where we used the notation of (5.1.10). Since $A \supset B(y) \overset{\text{def}}{=} \overline{B}(y,1)$, the rightmost term in (1.19) is bigger than $e_0(x,y,\omega)$. Furthermore:

$$\inf_{A\times A} r(2,\cdot,\cdot,\omega) \ge \inf_{[-1,1]^d\times[-1,1]^d} r_{(-2,2)^d,V=0}\,(2,\cdot,\cdot)\, \exp\{-2 \sup_{y+(-2,2)^d} V(\cdot,\omega)\}\,,$$

and (4.5.12) ensures that $\sup_{(-\ell,\ell)^d} V(\cdot,\omega) = o(\log \ell)$, as $\ell \to \infty$. Coming back to (1.19), our claim (1.18) readily follows. □

Applying (1.18) in the special case $x = 0$, $u = t$, yields a lower bound for $S_{t,\omega}$. Combined with the shape theorem 2.5 of Chapter 5, we see that the lower bound is close to the expression

$$c(d)\, t^{-d}\, \exp\{-\alpha_0(y) - t\lambda_\omega(U)\}\,,$$

with $y$ as in (1.17). Here choosing $U$ too large, for instance $U = (-t,t)^d$ is bad since one looses control on the size of $y$. One captures again the spirit of the variational problem corresponding to (1.11).

**Exercise:** (periodic potentials)
Assume that $\varphi$ is a $C^2$ 1-periodic positive function on $\mathbb{R}$ and $\frac{1}{2}\,\varphi'' = V\varphi$. For $t > 0$, define $Q_t$ as in (1.1) and

$$(1.20) \qquad \widetilde{Q}_t = \frac{1}{\varphi(0)}\, \exp\Big\{-\int_0^t V(Z_s)ds\Big\}\, \varphi(Z_t)\, P_0\,,$$

a) Show that $\widetilde{Q}_t$ is a probability and

$$(1.21) \qquad \frac{1}{c}\, \widetilde{Q}_t \le Q_t \le c\,\widetilde{Q}_t, \ \text{ for all } t > 0\,,$$

with $c = \frac{\sup \varphi}{\inf \varphi}$.

b) Show that $Z_s$, $0 \le s \le t$, under $\widetilde{Q}_t$ has same law as the solution of the stochastic differential equation

$$\begin{aligned} dZ_s &= d\beta_s + (\log\varphi)'(Z_s)ds\,, \\ Z_0 &= 0\,, \end{aligned}$$

with $\beta_.$ a Brownian motion.

c) Find a 1-periodic solution of

$$\tfrac{1}{2}\,\chi'' + (\log\varphi)'\,\chi' = -(\log\varphi)'\,, \tag{1.23}$$

and show that for $0 \le s \le t$,

$$Z_s + \chi(Z_s) = \chi(0) + \int_0^s (1+\chi')(Z_u)\,d\beta_u\,. \tag{1.24}$$

d) Show that

$$\lim_{M\to\infty}\ \varlimsup_{t\to\infty}\ Q_t\Big(\frac{1}{\sqrt{t}}\sup_{0\le s\le t}|Z_s| \le M\Big) = 1\,. \tag{1.25}$$

(Hint: use (1.21) and (1.24)).

This can be extended to higher dimension. Furthermore, one can show that $\frac{1}{\sqrt{t}}\,Z_{ut}$, $0 \le u \le 1$, under $\widetilde{Q}_t$ converges in law to a Brownian motion with covariance matrix $D$. This is an application of classical homogenization theory, see Bensoussan et al. [BLP78], Olla [Oll94]. □

## 6.2 From Weak to Strong Pinning

The proof of the pinning effect involves in essence two main steps. One step shows that for typical $\omega$, with $Q_{t,\omega}$ probability tending to 1 as $t$ tends to infinity a certain kind of 'weak pinning' takes place. The precise statement can be found below in (2.2) - (2.10). The proof of the fact that weak pinning takes place for 'typical $\omega$' involves a coarse graining technique on the path space $C(\mathbb{R}_+, \mathbb{R}^d)$ which is developed in §4 and applied in §5. The other step, which is the main object of this section shows that a 'weak pinning' property implies a 'strong pinning' property. It is convenient to work in this section with a set $\Omega_0$ of typical cloud configurations fulfilling the following assumptions:

(2.1) $\Omega_0$ is a set of full $\mathbb{P}$-measure where Theorem 4.4.6, Corollary 4.4.9, (4.5.12), Theorem 5.2.5 and Proposition 5.2.12 hold.

Essentially this means that for $\omega \in \Omega_0$, $\lambda_\ell(\omega) = \lambda_\omega((-\ell,\ell)^d)$ has the asymptotic behavior (4.4.52); the union of two disjoint open subsets of $(-\ell,\ell)^d$, $U, U'$ with $\lambda_\omega(U)$ and $\lambda_\omega(U')$ close to $c(d,\nu)(\log \ell)^{-2/d}$ has large diameter; $\sup_{(-\ell,\ell)^d} V(\cdot,\omega) = o(\log \ell)$, as $\ell \to \infty$; the shape theorem holds; and we have linear upper and lower bounds on the cost of large enough crossings of the path inside $(-\ell,\ell)^d$.

Our assumption throughout this section is that we have an $\omega \in \Omega_0$, for which

(2.2) $$\lim_{t\to\infty} Q_{t,\omega}(A) = 1, \quad \text{where}$$

(2.3) $$A = \bigcup_{x \in \mathbb{L}_{t,\omega}} A_x \cap \{T_{U(x)} \circ \theta_{H(x)} > t - H(x)\}, \quad \text{with for large } t$$

(2.4) $$\mathbb{L}_{t,\omega} \text{ a finite subset of } \Big(-\frac{t}{2}, \frac{t}{2}\Big)^d \text{ such that } |\mathbb{L}_{t,\omega}| \le t^d\,,$$

(2.5) $$\begin{array}{l}\text{for } x \in \mathbb{L}_{t,\omega},\ A_x \text{ is an } \mathcal{F}_{H(x)}\text{-measurable event included} \\ \text{in } \{H(x) \le t\}\,,\end{array}$$

(2.6) $$\text{for } x \in \mathbb{L}_{t,\omega},\ U(x) \text{ is an open subset of } (-t,t)^d \text{ containing } B(x)\,.$$

In particular on the event $A$, the path $Z_.$ comes before time $t$ within distance one of some point $x$ of $\mathbb{L}_{t,\omega}$, and from then on never leaves $U(x)$ up to time $t$. Furthermore, we assume that there exist $\Delta_{t,\omega} > 0$, $C_4(\omega) > 1$, such that:

(2.7) $$\text{as } t \to \infty,\ \log t = o(\Delta_{t,\omega}),\ \Delta_{t,\omega} = o(\sqrt{t})\,,$$

(2.8) $$\frac{1}{\Delta_{t,\omega}} \inf_{\mathbb{L}_{t,\omega}} d(x, U(x)^c) \to \infty\,,$$

(2.9) $$\text{for large } t, \text{ for all } x \in \mathbb{L}_{t,\omega},\ \lambda_\omega(U(x)) \ge \lambda_\omega(B(x,\Delta_{t,\omega})) - \frac{1}{t}\,,$$

(2.10) $$\begin{array}{l}\text{for any } L > 1, \text{ for large } t \text{ and any } x \in \mathbb{L}_{t,\omega}\,, \\ \lambda_\omega\Big(B(x, L\Delta_{t,\omega}) \backslash \overline{B}\Big(x, \frac{3}{2}\,\Delta_{t,\omega}\Big)\Big) \ge C_4\,\lambda_\omega(B(x,\Delta_{t,\omega}))\,.\end{array}$$

Some comments might be useful. The scale $\Delta_{t,\omega}$ is a 'small scale', which in our applications in §4 will grow slower than any power of $t$. The open sets $U(x)$ may have a 'large size' growing linearly with $t$. This is why the notion of pinning described by the event $B$ in (2.14) below, is a priori quite weak. In §4, we shall specify $C_4 \equiv C_4(d) = \kappa'(d, \frac{1}{2})$ (see Corollary 4.4.9) and (2.10) will be a consequence of the fact that we shall choose $\mathbb{L}_{t,\omega}$ and $\Delta_{t,\omega}$ so that for large $t$

$$\sup_{\mathbb{L}_{t,\omega}} \lambda_\omega(B(x, \Delta_{t,\omega})) \le \kappa\Big(d, \frac{1}{2}\Big)\, c(d,\nu)(\log t)^{-2/d} ,$$

where $\kappa$ is defined in Corollary 4.4.9. Of crucial importance for the results of this section will be assumption (2.9). We now define

$$L_0 = 2 + (3C_3 + 1)\, C_1^{-1}(1 - C_4^{-1})^{-1} \tag{2.11}$$

($C_1, C_3$ are defined in Theorem 5.2.11), as well as

$$\mathcal{N}(x) = B(x, L_0 \Delta_{t,\omega}), \quad \text{for } x \in \mathbb{L}_{t,\omega} . \tag{2.12}$$

The principal result of this section (see also (1.15) c)) is

**Theorem 2.1:** *(from weak to strong pinning)*
*Assume that $\omega \in \Omega_0$ is such that (2.2) - (2.10) holds, then*

$$\lim_{t\to\infty} Q_{t,\omega}(B) = 1, \quad \textit{with} \tag{2.13}$$

$$B = \bigcup_{x \in \mathbb{L}_{t,\omega}} A_x \cap \{T_{\mathcal{N}(x)} \circ \theta_{H(x)} > t - H(x)\} . \tag{2.14}$$

**Proof:** For $u > 0$, $x \in \mathbb{R}^d$, we define the probabilities on $C(\mathbb{R}_+, \mathbb{R}^d)$:

$$Q_{u,\omega,x} = \frac{1}{S_{u,\omega,x}} \exp\Big\{ - \int_0^u V(Z_s, \omega) ds \Big\}\, P_x \ \text{ with} \tag{2.15}$$

$$S_{u,\omega,x} = E_x\Big[\exp\Big\{ - \int_0^u V(Z_s, \omega) ds\Big\}\Big] , \ \text{the normalizing constant} . \tag{2.16}$$

The measures $Q_{u,\omega,x}$ will provide a dynamical point of view on the path measure $Q_{t,\omega}$. We begin with a reduction step:

**Lemma 2.2:** *The claim (2.13) follows from:*

$$\lim_{t\to\infty} t^d \sup_{x\in \mathbb{L}_{t,\omega}} \sup_{y \in B(x)} \sup_{0\le u \le t} Q_{u,\omega,y}(T_{\mathcal{N}(x)} \le u,\ T_{U(x)} > u) = 0 . \tag{2.17}$$

**Proof:** Since (2.2) holds and

$$Q_{t,\omega}(B) \ge Q_{t,\omega}(A) - Q_{t,\omega}(A \backslash B) ,$$

the claim (2.13) follows from:

$$\lim_{t\to\infty} Q_{t,\omega}(A\backslash B) = 0 . \tag{2.18}$$

For large $t$, $B$ is included in $A$ and

$$A\backslash B \subseteq \bigcup_{x \in \mathbb{L}_{t,\omega}} A_x \cap \{T_{U(x)} \circ \theta_{H(x)} > t - H(x) \geq T_{\mathcal{N}(x)}\}\,,$$

from which it follows that

$$(2.19)\quad Q_{t,\omega}(A\backslash B) \leq \sum_{\mathbb{L}_{t,\omega}} Q_{t,\omega}(A_x \cap \{T_{U(x)} \circ \theta_{H(x)} > t - H(x) \geq T_{\mathcal{N}(x)}\})$$

using (2.5) and the strong Markov property the last expression equals

$$\sum_{\mathbb{L}_{t,\omega}} \frac{1}{S_{t,\omega}}\, E_0\Big[A_x, \exp\Big\{-\int_0^{H(x)} V(Z_u,\omega)du\Big\}$$
$$\widetilde{E}_{Z_{H(x)}}\Big[\widetilde{T}_U > (t-H(x))_+ \geq T_{\mathcal{N}(x)},\ \exp\Big\{-\int_0^{(t-H(x))_+} V(\widetilde{Z}_u,\omega)du\Big\}\Big]\Big]$$

(the $\sim$ notations are here to avoid confusion between the various expectations)

$$\begin{aligned}
&= \sum_{\mathbb{L}_{t,\omega}} \frac{1}{S_{t,\omega}}\, E_0\Big[A_x,\ \exp\Big\{-\int_0^{H(x)} V(Z_u,\omega)du\Big\}\\
&\qquad S_{(t-H(x))_+,\omega,Z_{H(x)}}\ Q_{(t-H(x))_+,\omega,Z_{H(x)}}\\
&\qquad (\widetilde{T}_{U(x)} > (t-H(x))_+ \geq \widetilde{T}_{\mathcal{N}(x)})\Big]\\
&\overset{\binom{\text{strong Markov}}{\text{property}}}{=} \sum_{\mathbb{L}_{t,\omega}} E^{Q_{t,\omega}}[A_x,\ Q_{(t-H(x))_+,\omega,Z_{H(x)}}(\widetilde{T}_{U(x)} > (t-H(x))_+ \geq \widetilde{T}_{\mathcal{N}(x)})]\\
&\overset{(2.4)}{\leq} t^d \sup_{x\in\mathbb{L}_{t,\omega}}\ \sup_{y\in B(x)}\ \sup_{0\leq u\leq t} Q_{u,\omega,y}(T_{U(x)} > u \geq T_{\mathcal{N}(x)})\,.
\end{aligned}$$

This and (2.18) show that (2.13) follows from (2.17). □

Our next step is to prove a localization result for the nonnegative normalized principal Dirichlet eigenfunction $\varphi_x$ of $-\frac{1}{2}\,\Delta + V(\cdot,\omega)$ in $\mathcal{N}(x)$, for $x \in \mathbb{L}_{t,\omega}$. It is convenient to introduce the notation

$$(2.20)\qquad \mathcal{F}(x) = \overline{B}\Big(x, \frac{3}{2}\,\Delta_{t,\omega}\Big),\ \text{ for } x \in \mathbb{L}_{t,\omega}\,.$$

**Lemma 2.3:** *When $t$ is large, for all $x \in L_{t,\omega}$*

$$(2.21)\qquad \int_{B(x,2\Delta_{t,\omega})} \varphi_x^2(z)dz \geq \frac{1}{2}\,.$$

**Proof:** Since $\sup_{(-t,t)^d} V(\cdot,\omega) = o(\log t)$, as $t \to \infty$, when $t$ is large for all $x \in \mathbb{L}_{t,\omega}$:

$$\lambda_x \overset{\text{def}}{=} \lambda_\omega(\mathcal{N}(x)) \le \log t \,. \tag{2.22}$$

An application of (3.1.51) shows that consequently

$$0 \le \varphi_x \le e^{\log t} = t \,. \tag{2.23}$$

In view of (2.10), when $t$ is large, the formula (3.1.54) shows that for all $x \in \mathbb{L}_{t,\omega}$ and $y$ in the ring domain

$$\begin{aligned} \mathcal{V}(x) &= \mathcal{N}(x)\backslash\mathcal{F}(x) \\ &= \Big\{z \in \mathbb{R}^d,\ \tfrac{3}{2}\,\Delta_{t,\omega} < |z-x| < L_0\,\Delta_{t,\omega}\Big\}\,, \\ \varphi_x(y) &= E_y\Big[\varphi_x(Z_{T_x})\ \exp\Big\{\int_0^{T_x} \lambda_x - V(Z_s,\omega)ds\Big\},\ Z_{T_x} \in \mathcal{F}(x)\Big]\,, \end{aligned} \tag{2.24}$$

with $T_x = T_{\mathcal{V}(x)}$. In what follows we drop the subscript $x$ from the notation for simplicity. We then define

$$p = \frac{1+C_4}{2}\ (>1) \text{ and } q = \frac{p}{p-1} \text{ (the conjugate exponent of } p) \,. \tag{2.25}$$

The application of Hölder's inequality to (2.24) shows that for $y \in \mathcal{V}$:

$$\begin{aligned} 0 \le \varphi(y) \le \|\varphi\|_\infty\ & E_y\Big[\exp\Big\{\lambda p T - \int_0^T V(Z_s,\omega)ds\Big\}\Big]^{1/p} \\ & E_y\Big[\exp\Big\{-\int_0^T V(Z_s,\omega)ds\Big\},\ Z_T \in \mathcal{F}\Big]^{1/q} \,. \end{aligned} \tag{2.26}$$

Since $\lambda_\omega(\mathcal{V}) \overset{(2.10)}{\ge} \frac{C_4}{p}\cdot p\lambda$, with $\frac{C_4}{p} > 1$, it follows from the identity (3.1.43) and the estimate (3.1.20) that the second term of the rightmost member of (2.26) is smaller than $c(d,C_4)$. The first term of the rightmost member is smaller than $t$ by (2.23). Furthermore, observe that when $|y-x| \ge \Delta_{t,\omega}$ ($> C_2(\log t)$ for large $t$), the upper bound (5.2.73) shows that the last term in the rightmost member is smaller than $\exp\{-\frac{C_1}{q}\,\Delta_{t,\omega}\}$. As a result:

$$\int_{\mathcal{N}\backslash B(x,2\Delta_{t,\omega})} \varphi^2(y)dy \le (tc(d,C_4))^2\ e^{-\frac{2C_1}{q}\,\Delta_{t,\omega}}\,L_0^d\,\Delta_{t,\omega}^d\,|B(0,1)| \tag{2.27}$$

which tends to 0 as $t \to \infty$. This concludes the proof of (2.21). □

Let us mention that the principal eigenfunction $\psi_x$ of $U(x)$ in contrast to $\varphi_x$ might be 'localized' far away from $\mathcal{N}(x)$, due to the presence of other remote locations in $U(x)$ with low local principal eigenvalue. This fact is a priori quite dangerous since in the case of a given connected $U(x)$, an eigenfunction expansion argument shows that

$$\lim_{u\to\infty} Q_{u,\omega,x}(Z_u \in A \,|\, T_{U(x)} > u) = \frac{\int_A \psi_x}{\int_{U(x)} \psi_x} \,.$$

The particle might thus be tempted to end up far away from $\mathcal{N}(x)$ under $Q_{u,\omega,x}$, when $u$ is large. However the assumption (2.9) will ensure that $u$ smaller than $t$ is 'too short a time' to make it an 'attractive option' for the particle under the measure $Q_{u,\omega,y}$ to exit $\mathcal{N}(x)$.

In view of Lemma 2.2, the proof of our main claim (2.13) is reduced to checking that $Q_{u,\omega,y}(T_{\mathcal{N}(x)} \leq u,\ T_{U(x)} > u)$ tends uniformly to 0, when $t$ tends to infinity, in the manner specified in (2.17). To this end we shall derive lower bounds on $S_{u,\omega,y}$, and upper bounds on

$$H = E_y\Big[T_{\mathcal{N}(x)} \leq u,\ T_{U(x)} > u,\ \exp\Big\{-\int_0^u V(Z_s,\omega)ds\Big\}\Big] \,, \tag{2.28}$$

when $0 \leq u \leq t$, $x \in \mathrm{L}_{t,\omega}$, $y \in B(x)$.

- *Lower bound on* $S_{u,\omega,y}$

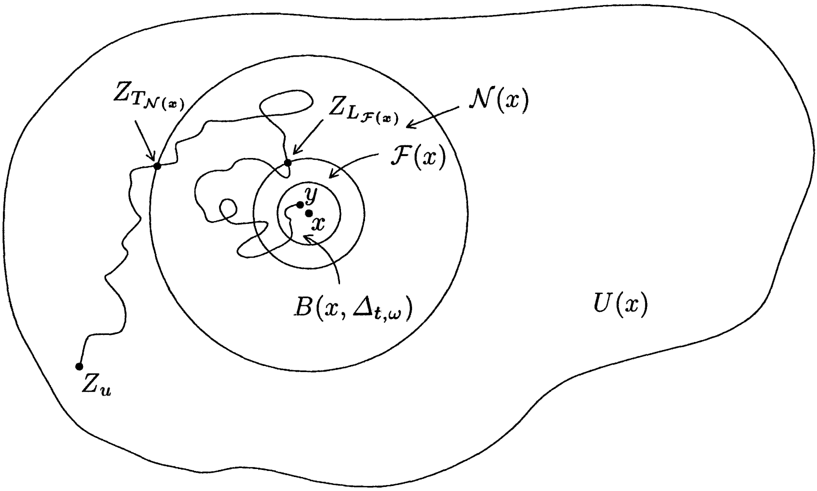
It follows from Lemma 2.3, that when $t$ is large for all $x \in \mathrm{L}_{t,\omega}$ we can find some $y_0 \in B(x, 2\Delta_{t,\omega})$ such that

$$\int_{y_0+[-1,1]^d} \varphi_x^2(z)dz \geq \frac{1}{2|B(x, 2\Delta_{t,\omega} + \sqrt{d})|} \,.$$

Applying Theorem 1.1, we thus see that for large $t$, for all $x \in \mathrm{L}_{t,\omega}$, $y \in B(x)$ and $u \in [0,t]$,

$$\begin{aligned} S_{u,\omega,y} &\geq \frac{1}{t^{d+1}}\, e_0(y, y_0, \omega)\ \exp\{-\lambda_\omega(\mathcal{N}(x))\,u\} \\ &\overset{(2.7)-(5.2.74)}{\geq} \frac{1}{t^{d+1}}\ \exp\{-3C_3\,\Delta_{t,\omega} - \lambda_\omega(\mathcal{N}(x))\,u\} \,. \end{aligned} \tag{2.29}$$

- *Upper bound on* $H$:




On the event $\{T_{U(x)} > u \geq T_{\mathcal{N}(x)}\}$, we consider $L_{\mathcal{F}(x)}$ the last time where $Z_.$ belongs to $\mathcal{F}(x)$ before exiting $\mathcal{N}(x)$. We can then introduce the random intervals $[\ell_1(w), \ell_1(w)+1)$ and $(\ell_2(w), \ell_2(w)+1]$ with $\ell_1(w) \leq \ell_2(w)$ integer valued, which contain respectively $L_{\mathcal{F}(x)}$ and $T_{\mathcal{N}(x)}$. This provides the following 'coarse graining scheme' of the event $\{T_{U(x)} > u \geq T_{\mathcal{N}(x)}\}$:

(2.30) $$\{T_{U(x)} > u \geq T_{\mathcal{N}(x)}\} \subseteq \bigcup_{0 \leq \ell_1 \leq \ell_2 \leq u} \mathcal{E}_{\ell_1,\ell_2}, \text{ where}$$

(2.31) $$\begin{aligned} \mathcal{E}_{\ell_1,\ell_2} &= \{T_{U(x)} > u,\ H_{\mathcal{F}(x)} \circ \theta_{\ell_1} < 1,\ T_{\mathcal{N}(x)\backslash\mathcal{F}(x)} \circ \theta_{\ell_1+1} \\ &\quad > (\ell_2 - \ell_1 - 1),\ T_{\mathcal{N}(x)} \circ \theta_{\ell_2} \leq 1 \wedge (u - \ell_2)\}, \\ &\quad \text{when } \ell_1 < \ell_2\,, \\ &= \{T_{U(x)} > u,\ H_{\mathcal{F}(x)} \circ \theta_{\ell_1} < T_{\mathcal{N}(x)} \circ \theta_{\ell_1} \leq 1 \wedge (u - \ell_1)\}, \\ &\quad \text{when } \ell_1 = \ell_2\,. \end{aligned}$$

As a result we find that for $x \in \mathrm{L}_{t,\omega}$, $y \in B(x)$, $u \in [0,t]$:

(2.32) $$H \leq (t+1)^2 \sup_{0 \leq \ell_1 \leq \ell_2 \leq u} E_y\Big[\mathcal{E}_{\ell_1,\ell_2}, \exp\Big\{-\int_0^u V(Z_s,\omega)ds\Big\}\Big]\,.$$

We shall now divide the collection of events $\mathcal{E}_{\ell_1,\ell_2}$, $0 \leq \ell_1 \leq \ell_2 \leq u$, into two groups:

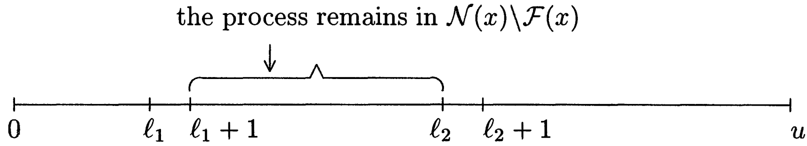
- the *slow crossings*, i.e. the $0 \leq \ell_1 \leq \ell_2 \leq u$ for which

(2.33) $$\lambda_\omega(\mathcal{N}(x)\backslash\mathcal{F}(x))(\ell_2 - \ell_1 - 1) \geq C_1(L_0 - 2)\,\Delta_{t,\omega}, \text{ and}$$

- the *fast crossings*, i.e. the $0 \leq \ell_1 \leq \ell_2 \leq u$ for which

$$\lambda_\omega(\mathcal{N}(x)\backslash\mathcal{F}(x))(\ell_2 - \ell_1 - 1) < C_1\,(L_0 - 2)\,\Delta_{t,\omega}\,. \tag{2.34}$$

We shall now derive upper bounds on the terms inside the supremum appearing in (2.32). For a slow crossing term, where in particular $\ell_2 > \ell_1$, we apply the simple Markov property at time $u \wedge (\ell_2 + 1)$, and then at time $\ell_1 + 1$.




Thus when $t$ is large, $x \in \mathrm{L}_{t,\omega}$, $y \in B(x)$, $0 \le u \le t$, for a slow crossing

$$\begin{aligned} &E_y\Big[\mathcal{E}_{\ell_1,\ell_2},\, \exp\Big\{-\int_0^u V(Z_s,\omega)ds\Big\}\Big] \le \\ &\|R^{U(x)}_{\ell_1,\omega}\|_{\infty,\infty}\|R^{\mathcal{N}(x)\backslash\mathcal{F}(x)}_{\ell_2-\ell_1-1,\omega}\|_{\infty,\infty}\|R^{U(x)}_{(u-\ell_2-1)_+,\omega}\|_{\infty,\infty}\,, \end{aligned} \tag{2.35}$$

(we have tacitly used the fact that $s \to (R^O_{s,\omega}\,1)(z)$ is a decreasing function). We can now apply the estimate (3.1.9) and observe that when $t$ is large for all $x \in \mathrm{L}_{t,\omega}$

$$\begin{aligned} &\lambda_\omega(U(x)) \le \lambda_\omega(\mathcal{N}(x)\backslash\mathcal{F}(x)) \le \log t\,, \quad \text{and} \\ &c(d)(1 + (t\log t)^{d/2}) \le t^d \end{aligned} \tag{2.36}$$

in view of (4.5.12). We thus find that the r.h.s. of (2.35) is smaller than:

$$\begin{aligned} &\exp\{3\log(t^d) - \ell_1\lambda_\omega(U(x)) - (\ell_2 - \ell_1 - 1)\,\lambda_\omega(\mathcal{N}(x)\backslash\mathcal{F}(x)) \\ &-(u - \ell_2 - 1)\lambda_\omega(U(x))\} \overset{(2.36)}{\le} \exp\{5\log(t^d) - u\lambda_\omega(U(x)) \\ &-(\ell_2 - \ell_1 - 1)(\lambda_\omega(\mathcal{N}(x)\backslash\mathcal{F}(x)) - \lambda_\omega(U(x)))\}\,. \end{aligned}$$

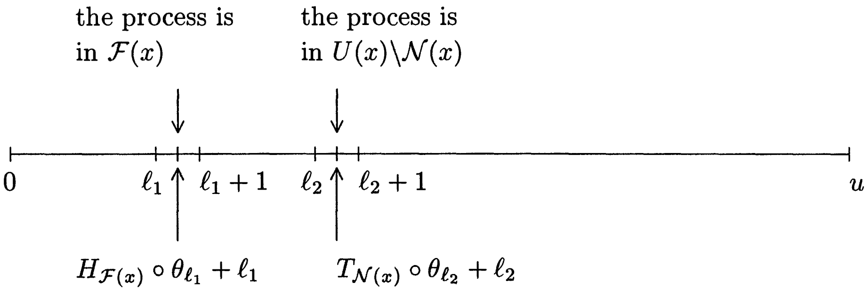
Observe now that

$$\begin{aligned} &\lambda_\omega(\mathcal{N}(x)\backslash\mathcal{F}(x)) - \lambda_\omega(U(x)) \ge \lambda_\omega(\mathcal{N}(x)\backslash\mathcal{F}(x)) - \lambda_\omega(B(x,\Delta_{t,\omega})) \\ &\overset{(2.10)}{\ge} (1 - C_4^{-1})\,\lambda_\omega(\mathcal{N}(x)\backslash\mathcal{F}(x))\,. \end{aligned} \tag{2.37}$$

In view of the definition of slow crossings in (2.33), we see that for large $t$, for all $x \in \mathrm{L}_{t,\omega}$, $y \in B(x)$, $0 \le u \le t$ and any slow crossing

$$\begin{aligned} &E_y\Big[\mathcal{E}_{\ell_1,\ell_2},\exp\Big\{-\int_0^u V(Z_s,\omega)ds\Big\}\Big] \le \exp\{5\log(1 + t^d) \\ &-u\lambda_\omega(U(x)) - C_1(1 - C_4^{-1})(L_0 - 2)\,\Delta_{t,\omega}\}\,. \end{aligned} \tag{2.38}$$

On the other hand for a fast crossing, we use the simple and strong Markov property respectively at times $u \wedge (\ell_2 + 1)$ and $\ell_1 + H_{\mathcal{F}(x)} \circ \theta_{\ell_1}$




As a result we find that when $t$ is large, $x \in \mathbb{L}_{t,\omega}$, $y \in B(x)$, $0 \le u \le t$, for a fast crossing:

$$\begin{aligned} &E_y\Big[\mathcal{E}_{\ell_1,\ell_2}, \exp\Big\{-\int_0^u V(Z_s,\omega)ds\Big\}\Big] \le \|R^{U(x)}_{\ell_1,\omega}\|_{\infty,\infty} \\ &\sup_{z \in \mathcal{F}(x)} E_z\Big[\exp\Big\{-\int_0^{T_{\mathcal{N}(x)}} V(Z_s,\omega)ds\Big\}\Big]\ \|R^{U(x)}_{(u-\ell_2-1)_+,\omega}\|_{\infty,\infty}\,. \end{aligned} \tag{2.39}$$

Applying now (5.2.73) and (2.7) to bound the cost of a crossing from $\mathcal{F}(x)$ to $\mathcal{N}(x)^c$, as well as (3.1.9) and (2.36) to bound the norms of the semigroups, we see that the above expression is smaller than

$$\begin{aligned} &\exp\{2\log(t^d) - \ell_1\lambda_\omega(U(x)) - C_1(L_0-2)\Delta_{t,\omega} \\ &-(u-\ell_2-1)\,\lambda_\omega(U(x))\} \\ &\le \exp\{4\log(t^d) - u\,\lambda_\omega(U(x)) \\ &-C_1(L_0-2)\Delta_{t,\omega} + \lambda_\omega(U(x))(\ell_2-\ell_1-1)\}\,. \end{aligned} \tag{2.40}$$

Observe that:

$$C_4\lambda_\omega(U(x)) \le C_4\lambda_\omega(B(x,\Delta_{t,\omega})) \overset{(2.10)}{\le} \lambda_\omega(\mathcal{N}(x)\backslash\mathcal{F}(x))\,.$$

We now take into account the definition of fast crossing in (2.34), when $\ell_2 \ge \ell_1 + 1$, and the fact that when $\ell_2 = \ell_1$, $\lambda_\omega(U(x))(\ell_2 - \ell_1 - 1) \le 0$. As a result, we find that when $t$ is large for $x \in \mathbb{L}_{t,\omega}$, $y \in B(x)$, $0 \le u \le t$, and any fast crossing

$$\begin{aligned} &E_y\Big[\mathcal{E}_{\ell_1,\ell_2}, \exp\Big\{-\int_0^u V(Z_s,\omega)ds\Big\}\Big] \le \exp\{4\log(1+t^d) \\ &-u\lambda_\omega(U(x)) - C_1(1-C_4^{-1})(L_0-2)\Delta_{t,\omega}\Big\}\,. \end{aligned} \tag{2.41}$$

Combining (2.38) and (2.32), we have shown:

(2.42) $H \leq (t+1)^2 \exp\{5\log(t^d) - u\lambda_\omega(U(x)) - C_1(1-C_4^{-1})(L_0-2)\Delta_{t,\omega}\}$ .

It now follows from the lower bound (2.29) and from (2.42) that for large $t$, $x \in \mathrm{L}_{t,\omega}$, $y \in B(x)$, $u \in [0,t]$,

$$\begin{aligned} & t^d\, Q_{u,\omega,y}(T_{U(x)} > u \geq T_{\mathcal{N}(x)}) \leq \\ (2.43) \qquad & t^{2d+1}(1+t)^2 \exp\{5\log(t^d) + [3C_3 - C_1(1-C_4^{-1})(L_0-2)]\,\Delta_{t,\omega} \\ & + u(\lambda_\omega(B(x,\Delta_{t,\omega})) - \lambda_\omega(U(x)))\}\ . \end{aligned}$$

Moreover the crucial assumption (2.9) implies that

$$(2.44) \qquad u(\lambda_\omega(B(x,\Delta_{t,\omega})) - \lambda_\omega(U(x))) \leq \frac{u}{t} \leq 1\ ,$$

and therefore due to our choice of $L_0$ in (2.11), the left of (2.43) is smaller than:

$$t^{2d+1}(1+t)^2 \exp\{5\log(t^d) + 1 - \Delta_{t,\omega}\} \underset{t\to\infty}{\overset{(2.7)}{\longrightarrow}} 0\ .$$

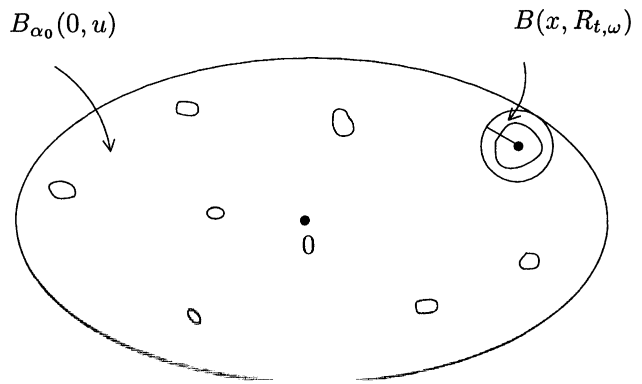
This finishes the proof of Theorem 2.1. □

## 6.3 Random Scales

We now come back to the variational problem (1.11). We shall replace the variational problem (1.11) by a simpler variational problem defined on $(0,\infty)$. In particular this will exempt us for the time being of specifying the scale $R_{t,\omega}$ in (1.11), or the meaning of the expression 'near minima'. The simpler random variational problem on the half-line consists in looking at minima of:

$$(3.1) \qquad u > 0 \longrightarrow u + t\lambda_\omega(B_{\alpha_0}(0,u))\ ,$$

where $B_{\alpha_0}(0,u)$ denotes the open ball centered at 0, with radius $u$ in the $\alpha_0$-norm.



The motivation for looking at (3.1) in place of (1.11) is the following. The principal eigenvalue $\lambda_\omega(B_{\alpha_0}(0,u))$ for large $u$ should be close to $\lambda_\omega(B(x,R_{t,\omega}))$ for some $x \in B_{\alpha_0}(0,u)$ with $\alpha_0(x)$ 'of order u', (see Proposition 4.4.3 as well as (5.15), (5.16) below). Thus for large $t$ the scale of minima of (1.11) and the scale of minima of (3.1) should qualitatively be of same order. On the other hand (3.1) is well defined, and turns out to be better behaved. It will signal the scale where 'the action ought to take place'. With the study of (3.1) we are handling a higher dimensional analogue of what we did in Chapter 3 §3. We begin with

**Lemma 3.1** *The function $\lambda_\omega(B_{\alpha_0}(0,u))$ is measurable in $\omega \in \Omega$, continuous decreasing in $u \in (0,\infty)$, tends to $\infty$ as $u$ tends to 0. There exists $\chi \in (0,1)$ such that for $\omega \in \Omega_0$ (see (2.1)), when $u$ is large:*

$$\frac{c(d,\nu)}{(\log u)^{\frac{2}{d}}} - \frac{1}{(\log u)^{\frac{2+\chi}{d}}} \le \lambda_\omega(B_{\alpha_0}(0,u)) \le \frac{c(d,\nu)}{(\log u)^{\frac{2}{d}}} + \frac{1}{(\log u)^{\frac{2+\chi}{d}}} \tag{3.2}$$

**Proof:** The first claim is analogous to (3.3.34). The only point which needs to be proved, is the right continuity in the $u$ variable (the left continuity being immediate by the second line of (3.1.2)). We shall use Theorem 3.1.11. We consider a sequence $u_n$ decreasing to $u \in (0,\infty)$, pick some $\tau > 0$, and in the notations of Theorem 3.1.11 choose $U_1 = B_{\alpha_0}(0,u) \subseteq U_2 = B_{\alpha_0}(0,u_n)$. We then define

$$S_1 = \tau + T_{U_1}\ , \tag{3.3}$$

and pick $\lambda \in (0, \lambda_\omega(U_1))$. Theorem 3.1.11 ensures in our situation that

$$\lambda \le \lambda_\omega(U_2) = \lambda_\omega(B_{\alpha_0}(0,u_n))\ , \tag{3.4}$$

as soon as:

$$\begin{aligned} &A \cdot C < 1, \quad \text{where} \\ &A = \sup_x\ 1 + \int_0^\infty \lambda e^{\lambda u}\,(R^{U_1}_{u,\omega}1)(x) \overset{(3.1.20)}{<} \infty\,, \quad \text{and} \\ &C = \sup_{x \notin U_1}\ E_x\Big[T_{U_2} > \tau,\ \exp\Big\{\lambda\tau - \int_0^\tau V(Z_s,\omega)ds\Big\}\Big]\,. \end{aligned} \tag{3.5}$$

Observe that the expectation vanishes when $x \notin U_2$. Thus

$$C \le \exp\{\lambda_\omega(U_1)\,\tau\}\ \sup_{U_2\backslash U_1}\ P_x[T_{U_2} > \tau]\,. \tag{3.6}$$

For each $x \in U_2\backslash U_1$, we can find a closed half space $H_x$ such that

$$x \in H_x \ \text{ and } \ H_x \cap B_{\alpha_0}(0,\alpha_0(x)) = \emptyset\,. \tag{3.7}$$

Therefore it follows that $(\frac{u_n}{u}\, H_x) \cap U_2 = \emptyset$ and

$$(3.8) \qquad P_x[T_{U_2} > \tau] \le P_0^1\Big[\sup_{0\le s\le \tau} Z_s \le \Big(\frac{u_n}{u} - 1\Big)\, \text{const}\Big] \underset{n\to\infty}{\longrightarrow} 0\,,$$

where $P_0^1$ stands for the one-dimensional Wiener measure and const $=$ $\sup\{|x|,\ x \in B_{\alpha_0}(0,u_1)\}$. As a result (3.5) is fulfilled when $n$ is large, and in view of (3.4) this completes the proof of the right continuity of $u \to \lambda_\omega(B_{\alpha_0}(0,u))$. Finally (3.2) is an immediate consequence of (4.4.52) and the fact that $\alpha_0$ is equivalent to the sup-norm on $\mathbb{R}^d$. □

We now define for $t > 0$, $\omega \in \Omega$, the value of the variational problem associated to (3.1):

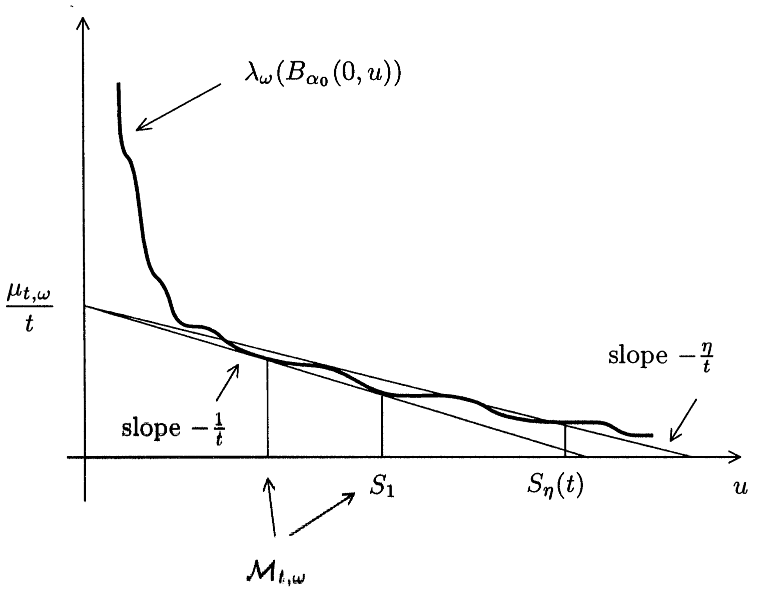
$$(3.9) \qquad \begin{aligned} \mu_{t,\omega} &= \inf_{u>0} \{u + t\lambda_\omega(B_{\alpha_0}(0,u))\} \\ &= t \inf_{u>0} \Big\{\frac{u}{t} + \lambda_\omega(B_{\alpha_0}(0,u))\Big\}\,. \end{aligned}$$

The second line of (3.9) shows that $\frac{\mu_{t,\omega}}{t}$ is a finite increasing concave function of $\frac{1}{t}$, which is thus continuous. The locus of minima of (3.1) is denoted by

$$(3.10) \qquad \mathcal{M}_{t,\omega} = \{u > 0,\ u + t\lambda_\omega(B_{\alpha_0}(0,u)) = \mu_{t,\omega}\}\,.$$

In view of Lemma 3.1, $\mathcal{M}_{t,\omega}$ is a compact subset of $(0,\infty)$. We can now introduce a family of random scales, which approximate from above the loci of minima of (3.1). For $\eta \in (0,1]$, $t > 0$, $\omega \in \Omega$, we define:

$$(3.11) \qquad S_\eta(t,\omega) = \inf\{v > 0,\ \eta u + t\lambda_\omega(B_{\alpha_0}(0,u)) > \mu_{t,\omega},\ \text{for}\ u \ge v\}\,.$$



We shall occasionally drop the subscript $\omega$ from the notation, or write $S_\eta$ in place of $S_\eta(t,\omega)$, when no confusion arises. We can now collect some useful properties of the random scales we have introduced.

**Proposition 3.2:** *For $\omega \in \Omega$, $S_\eta(t,\omega) \in (0,\infty)$ is a decreasing function of $\eta \in (0,1]$, moreover:*

$$\eta\, S_\eta(t,\omega) + t\lambda_\omega(B_{\alpha_0}(0, S_\eta(t,\omega))) = \mu_{t,\omega}\ , \tag{3.12}$$

$$S_1(t,\omega) = \max\ \mathcal{M}_{t,\omega} \ \textit{is a nondecreasing function of } t\ . \tag{3.13}$$

*For $0 < \eta < \eta' \le 1$, $t > 0$, $\omega \in \Omega$:*

$$\eta S_\eta(t,\omega) \ge \eta'\, S_{\eta'}(t,\omega)\ . \tag{3.14}$$

*There exits $\chi \in (0, \frac{1}{d})$ such that on the set of full $\mathbb{P}$ probability $\Omega_0$, (see (2.1)):*

$$\Big|\mu_{t,\omega} - c(d,\nu)\,\frac{t}{(\log t)^{\frac{2}{d}}}\Big| \le \frac{t}{(\log t)^{\frac{2}{d}+\chi}}\ , \quad \textit{for large } t\ , \tag{3.15}$$

$$\text{for } \eta \in (0,1],\ S_\eta(t,\omega) \le \frac{t}{(\log t)^{\frac{2}{d}+\chi}}\ , \quad \textit{for large } t\ , \tag{3.16}$$

$$S_1(t,\omega) \ge \frac{t}{\exp\{(\log t)^{1-\chi}\}}, \quad \textit{for large } t\ . \tag{3.17}$$

**Proof:** It is immediate from the definition (3.11) that $S_\eta(t,\omega) \in (0,\infty)$ and that it is decreasing in $\eta$. The continuity of the function $u > 0 \to \eta u + t\lambda_\omega(B_{\alpha_0}(0,u))$ implies (3.12), which in turn immediately implies (3.14). Furthermore from (3.12) follows that $S_1(t,\omega) \in \mathcal{M}_{t,\omega}$, and since $u + t\lambda_\omega(B_{\alpha_0}(0,u)) > \mu_{t,\omega}$, for $u > S_1(t,\omega)$, we see that $S_1(t,\omega) = \max \mathcal{M}_{t,\omega}$. Moreover, for $u \in (0, S_1(t,\omega))$,

$$0 > u - S_1(t,\omega) \ge t[\lambda_\omega(B_{\alpha_0}(0, S_1(t,\omega))) - \lambda_\omega(B_{\alpha_0}(0,u))]\ ,$$

and thus when $t' > t$, for $u \in (0, S_1(t,\omega))$:

$$u - S_1(t,\omega) > t'[\lambda_\omega(B_{\alpha_0}(0, S_1(t,\omega))) - \lambda_\omega(B_{\alpha_0}(0,u))]\ ,$$

which shows that $S_1(\cdot,\omega)$ is a nondecreasing function.

Let us then prove (3.15), (3.16) and (3.17). We pick $\omega$ in the set of full $\mathbb{P}$-probability where in particular (4.4.52) holds. In what follows, we assume that the constant $\chi$ which appears in (4.4.52), belongs to $(0,1)$. Observe that:

$$\begin{aligned} u + t\lambda_\omega(B_{\alpha_0}(0,u)) \;\; &\geq \;\; t, \text{ if } u \geq t\,, \\ & \qquad t\lambda_\omega(B_{\alpha_0}(0,t)), \text{ if } u \leq t\,. \end{aligned}$$

In view of (3.2), for large $t$,

$$\mu_{t,\omega} \geq t\lambda_\omega(B_{\alpha_0}(0,t)) \geq c(d,\nu)\,\frac{t}{(\log t)^{\frac{2}{d}}} - \frac{t}{(\log t)^{\frac{2+\chi}{d}}}\,. \tag{3.18}$$

On the other hand choosing $u = \frac{t}{(\log t)^{\frac{3}{d}}}$, and keeping in mind that $\chi \in (0,1)$, we see that for large $t$:

$$\begin{aligned} \mu_{t,\omega} \;\; &\leq \;\; \frac{t}{(\log t)^{\frac{3}{d}}} + c(d,\nu)\,\frac{t}{(\log t - \frac{3}{d}\,\log\log t)^{\frac{2}{d}}} \\ &\qquad + \frac{t}{(\log t - \frac{3}{d}\,\log\log t)^{\frac{2+\chi}{d}}} \\ &\leq \;\; c(d,\nu)\,\frac{t}{(\log t)^{\frac{2}{d}}} + 2\,\frac{t}{(\log t)^{\frac{2+\chi}{d}}}\,. \end{aligned} \tag{3.19}$$

The combination of (3.18) and (3.19) readily implies (3.15), with a different $\chi$. We now turn to the proof of (3.16). It follows from (3.19) that for large $t$, $S_\eta(t,\omega) < t$. Therefore, for large $t$, (3.12) implies that

$$\begin{aligned} \mu_{t,\omega} \;\; &= \;\; \eta\,S_\eta + t\lambda_\omega(B_{\alpha_0}(0,S_\eta)) \geq \eta S_\eta + t\lambda_\omega(B_{\alpha_0}(0,t)) \\ &\overset{(3.2)}{\geq} \;\; \eta\,S_\eta + c(d,\nu)\,\frac{t}{(\log t)^{\frac{2}{d}}} - \frac{t}{(\log t)^{\frac{2+\chi}{d}}}\,. \end{aligned} \tag{3.20}$$

This and (3.19) implies (3.16), (with a different $\chi$).

Let us finally prove (3.17). We choose $\chi' \in (0, \frac{\chi}{d})$. Observe that for $u \leq t\,\exp\{-(\log t)^{1-\chi'}\}$, when $t$ is large,

$$\begin{aligned} u + t\lambda_\omega(B_{\alpha_0}(0,u)) \;\; &\geq \;\; t\lambda_\omega(B_{\alpha_0}(0,t\,\exp\{-(\log t)^{1-\chi'}\})) \\ &\overset{(3.2)}{\geq} c(d,\nu)\,\frac{t}{[\log t(1-(\log t)^{-\chi'})]^{\frac{2}{d}}} - \frac{1}{[\log t(1-(\log t)^{-\chi'})]^{\frac{2+\chi}{d}}} \\ &\geq c(d,\nu)\,\frac{t}{(\log t)^{\frac{2}{d}}} + c'\,\frac{t}{(\log t)^{\frac{2}{d}+\chi'}} \;\overset{(3.19)}{>}\; \mu_{t,\omega}\,. \end{aligned}$$

This shows that for large $t$,

$$S_1(t,\omega) \geq t \ \exp\{-(\log t)^{1-\chi'}\} ,$$

which concludes the proof of (3.17). □

The random scales $S_\eta(t,\omega)$ are thus negligible compared to the value $\mu_{t,\omega}$ of the random variational problem (3.9). In the one-dimensional case, the same method as in the proof of Theorem 3.3.6 shows that the 'typical size of $S_\eta(t,\omega)$ is $\frac{t}{(\log t)^3}$', in the sense that:

$$\text{for } \eta \in (0,1] ,$$

(3.21)
$$\lim_{\rho\to 0} \ \varliminf_{t\to\infty} \ \mathbb{P}\Big[S_\eta(t,\omega) \in \Big[\rho \ \frac{t}{(\log t)^3} \ , \ \frac{1}{\rho} \ \frac{t}{(\log t)^3}\Big]\Big] = 1 ,$$

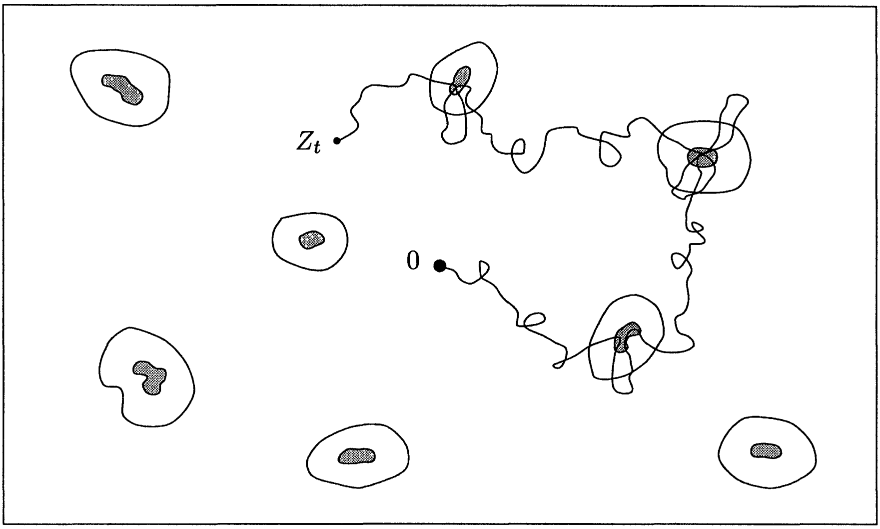
see also Section 5 of [Szn96a], I. It can also be shown parallel to (3.3.46) that

(3.22)
$$\lim_{\rho\to 0} \ \varliminf_{t\to\infty} \ \mathbb{P}\Big[ \ \Big|\mu_{t,\omega} - \frac{(\pi\nu)^2}{2} \ \frac{t}{(\log t)^2} \ \Big(1 + \frac{2\log\log t}{\log t}\Big)\Big| \leq \frac{1}{\rho} \ \frac{t}{(\log t)^3}\Big] = 1 .$$

In the higher dimensional case there are strong indications that the size of fluctuations of $\lambda_\omega(B_\alpha(0,u))$ for large $u$, is $(\log u)^{-\frac{2}{d}-1}$, see [Szn97b]. Thus the heuristics principle (3.3.54) leads to expect that for large $t$ the typical size of $S_\eta(t,\omega)$, in the sense of (3.21), is $\frac{t}{(\log t)^{\frac{2}{d}+1}}$. The comparison of the proven bounds (3.16), (3.17) with this expected behavior indicates that the random scales $S_\eta(t,\omega)$ are not too well understood in the higher dimensional case.

## 6.4 Coarse Graining of Path Space

The object of this section is to introduce a coarse graining scheme (see end of Chapter 4 §1) on the space $C(\mathbb{R}_+,\mathbb{R}^d)$. This coarse graining scheme will be the main ingredient in Section 5 for the proof of a weak pinning property which combined with the results of Section 2 will provide the proof of the pinning property. It will also be applied in Section 5 to relate the finer asymptotic behavior of $S_{t,\omega}$ with the value $\mu_{t,\omega}$ of the random variational problem (3.9).




The general idea to produce this coarse graining scheme is to first construct a *clearings and forest picture* of the random environment, and then keep track of a *system of excursions* of the path $Z_{[0,t]}$, in and out of the clearings. The clearings and forest picture will use the method of enlargement of obstacles developed in Chapter 4. The system of excursions will enable to define a collection of elementary 'strategies' of the path and we shall derive uniform estimates on the cost of each such strategy in terms of the various crossings involved and of the time spent in the respective clearings.

### A) The Forest and Clearings Picture

We shall use here the notions developed in Chapter 4. To this end we select some fixed admissible collection of parameters (see (4.3.66)), which will be used in the construction of the density set and the bad set of the method of enlargement of obstacles. Throughout we view $n$ as a function of $t \geq 2$, via the requirement:

(4.1) $$2^{n(t)} \leq t < 2^{n(t)+1} \, .$$

We consider $(\log 2^n)^{1/d}$ as the natural scale of clearings in which we are interested, and introduce

(4.2) $$\epsilon \equiv \epsilon_n = (\log 2^n)^{-1/d} \, .$$

The method of enlargement of obstacles in Chapter 4 enables to associate to each $t \geq 2$ and $\omega \in \Omega$ a partitioning of $\mathbb{R}^d$ into the sets $\mathcal{D}_{n,\omega}$, $\mathcal{B}_{n,\omega}$, $(\mathcal{D}_{n,\omega} \cup \mathcal{B}_{n,\omega})^c$, where in the notations of (4.2.13), (4.3.47), (4.3.48),

$$\mathcal{D}_{n,\omega} = \epsilon_n^{-1}\, \mathcal{D}_{\epsilon_n}(\omega_{\epsilon_n}),\ \mathcal{B}_{n,\omega} = \epsilon_n^{-1}\, \mathcal{B}_{\epsilon_n}(\omega_{\epsilon_n}),\ \text{and} \tag{4.3}$$

$$\omega_{\epsilon_n} = \sum_i \delta_{\epsilon_n x_i},\ \text{for}\ \omega = \sum_i \delta_{x_i}\ . \tag{4.4}$$

The sets $\mathcal{D}_{n,\omega}$ and $\mathcal{B}_{n,\omega}$ are simply scaled versions of the density and bad sets of Chapter 4, and the unit scale of Chapter 4 now corresponds to $(\log 2^n)^{1/d}$. In this light it is also convenient to introduce for $q \in \mathbb{Z}^d$ and $t \geq 2$ the notation

$$\begin{aligned} C_{n,q} &= \epsilon_n^{-1}\, C_q \quad \text{(see (4.1.10) for the notation)} \\ &= \{z \in \mathbb{R}^d,\ q_i(\log 2^n)^{1/d} \leq z_i < (q_i+1)(\log 2^n)^{1/d}, \\ &\qquad i = 1,\dots,d\}\ . \end{aligned} \tag{4.5}$$

For each $\eta \in [\frac{1}{2}, 1)$, $t \geq 2$, $\omega \in \Omega_0$ (the set of full $\mathbb{P}$-measure introduced in (2.1)), we associate a clearing and forest picture in the following fashion: we choose a fixed $r(d,\nu,\eta) \in (0, \frac{1}{4})$ such that in the notations of Proposition 4.2.4,

$$\frac{c_2(d)}{r^2} > 4\, \frac{c(d,\nu)}{1-\eta}\ , \tag{4.6}$$

and in the notations of Theorem 4.2.6

$$r < r_0(d, M = 2c(d,\nu)) \ \text{and}\ \frac{c_3(d)}{4r} > 2\ , \tag{4.7}$$

(in fact (4.6) ensures that $r < r_0(d,M)$, see (4.2.70) where $r_0(d,M)$ is defined). Thus a box $C_{n,q}$, $q \in \mathbb{Z}^d$, is an *n-clearing box* if

$$|C_{n,q} \backslash \mathcal{D}_{n,\omega}| \geq r^d\, |C_{n,q}|\ , \tag{4.8}$$

and an *n-forest box* if

$$|C_{n,q} \backslash \mathcal{D}_{n,\omega}| < r^d\, |C_{n,q}|\ . \tag{4.9}$$

For simplicity we shall speak of clearing boxes and forest boxes, when this causes no confusion. The closure of the union of clearing boxes is denoted by

$$\mathcal{A}_{\eta,t,\omega} = \bigcup_{q: C_{n,q}\ \text{clearing box}} \overline{C_{n,q}}\ . \tag{4.10}$$

For $\eta \in [\frac{1}{2}, 1)$, $\omega \in \Omega_0$ and large $t$, Proposition 4.2.4 and (4.6) imply that:

$$\lambda_\omega(\mathcal{A}^c_{\eta,t,\omega}) \geq \lambda_f \stackrel{\text{def}}{=} \frac{4c(d,\nu)}{(1-\eta)(\log 2^n)^{2/d}}\,, \tag{4.11}$$

(this estimate will be instrumental in the proof of Proposition 4.4 below), on the other hand (4.7) and Theorem 4.2.6 show the important eigenvalue estimate:

$$\begin{aligned}
&\text{for all open set } U \subseteq \mathbb{R}^d,\\
&\lambda_\omega(U') \geq \lambda_\omega(U) \geq \lambda_\omega(U') \wedge \frac{2c(d,\nu)}{(\log 2^n)^{2/d}} - \frac{1}{t}\,, \quad \text{with}\\
&U' = U \cap \bigcup_{\substack{q: C_{n,q} \cap U \neq \emptyset \\ C_{n,q} \text{ clearing box}}} \epsilon_n^{-(d+1)} \text{ open } \|\cdot\|\text{-neighborhood of } C_{n,q}\,.
\end{aligned} \tag{4.12}$$

(One applies Theorem 4.2.6 with $U_2 = \epsilon_n U$, $U_1 = \epsilon_n U'$ and $\mathcal{A} = \epsilon_n(\bigcup_{\substack{q: C_{n,q} \cap U \neq \emptyset \\ C_{n,q} \text{ clearing box}}} \overline{C_{n,q}}))$.

We can then introduce the size of the neighborhoods of the clearing boxes we shall consider when defining in subsection B) the system of excursions in and out of the clearings. We define $R = R(\eta, t, \omega)$ as the smallest integer such that:

$$\frac{R}{\epsilon_n} > \max\left\{\frac{t}{(\log t)^{2/d}}\,\frac{(\log t)^{2-\eta}}{S_1(t,\omega)}\,,\ \epsilon_n^{-(d+1)}\right\}. \tag{4.13}$$

As a result of (3.17) we have for some $\chi \in (0, \frac{1}{d})$ and all $\omega \in \Omega_0$,

$$\frac{R}{\epsilon_n} \leq \exp\{(\log t)^{1-\chi}\}, \text{ for large } t\,, \tag{4.14}$$

and in fact, if the conjectured asymptotic behavior of $S_1$ holds true (see after (3.21)), one should expect that in an 'appropriate sense' $\frac{R}{\epsilon_n}$, for large $t$, is comparable to $(\log t)^{2+1-\eta}$. We also define

(4.15) $\quad \mathcal{O}$ the open $R\epsilon_n^{-1}$ neighborhood in the $\|\cdot\|$-distance of $\mathcal{A}_{\eta,t,\omega}$.

Just as in (4.4.27), we consider:

$$M_0(d,\nu,\eta) = 2\Big(\Big[\frac{4d}{\nu r^d}\Big]\Big\} + 1\Big), \quad \mu = \Big(\frac{\nu r^d}{4d}\Big) \wedge \frac{1}{2}\,. \tag{4.16}$$

The interest of these quantities lies in:

**Proposition 4.1:** *For each $\eta \in [\frac{1}{2}, 1)$, there exists a set $\Omega_\eta \subseteq \Omega_0$, of full $\mathbb{P}$-measure, such that when $\omega \in \Omega_\eta$, for large $t$, any closed $\|\cdot\|$-ball of radius $t^\mu$ intersecting*

$$\mathcal{T} \stackrel{\text{def}}{=} (-2^n, 2^n)^d \tag{4.17}$$

*intersects at most $M_0$ clearing boxes, and any connected component of $\mathcal{O}$ intersecting $\mathcal{T}$ has $\|\cdot\|$-diameter no larger than*

$$\Delta_{\eta,t,\omega} = 3\sqrt{d}\, M_0\, R\epsilon_n^{-1} . \tag{4.18}$$

**Proof:** It follows from (4.4.30) that for large $t$

$$\begin{aligned} &\mathbb{P}[\text{all blocks in } \epsilon_n^{-1}\,\mathcal{K} \text{ contain at most } M_0, n\text{-clearing boxes}] \\ &\geq 1 - 2^{-nd} , \end{aligned} \tag{4.19}$$

provided $\mathcal{K}$ as in (4.4.28) stands for the collection of blocks of the form $q + [0, [2^{n\mu}]]^d$ intersecting $(-2^n, 2^n)^d$. The same argument showing (4.4.31) together with Borel-Cantelli's lemma imply that on a set of full $\mathbb{P}$-measure, for large $t$, any connected component of $\mathcal{O}$ intersecting $(-2^n(\log 2^n)^{1/d}, 2^n(\log 2^n)^{1/d})^d$ has $\|\cdot\|$-diameter smaller than $(2R+1)\,\epsilon_n^{-1}\, M_0$. Our claims now easily follow. □

Finally as an immediate consequence of (4.12) and the fact that $R\epsilon_n^{-1} \geq \epsilon_n^{-(d+1)}$, we see that for $\eta \in [\frac{1}{2}, 1)$, $\omega \in \Omega_0$, and large $t$, for any open set $U$ in $\mathbb{R}^d$,

$$\lambda_\omega(U) \geq \inf_{\mathcal{C}} \lambda_\omega(\mathcal{C}) \wedge \frac{2c(d,\nu)}{(\log 2^n)^{2/d}} - \frac{1}{t} , \tag{4.20}$$

where the infimum runs over the connected components $\mathcal{C}$ of $\mathcal{O}$ containing a clearing box intersecting $U$. This estimate will be crucial when trying to check the important assumption (2.9), which is part of the 'weak pinning' property.

## B) Excursion Systems and Coarse Graining

We now choose some fixed $\eta \in [\frac{1}{2}, 1)$ and $\omega \in \Omega_\eta \subseteq \Omega_0$ (see Proposition 4.1). To define excursions of the process $Z_.$ in and out of clearings, it is convenient to introduce

$$\widetilde{A}_{\eta,t,\omega} = \bigcup_{q \in \frac{1}{\sqrt{d}}\mathbb{Z}^d \cap A_{\eta,t,\omega}} B(q) . \tag{4.21}$$

It now follows from the fact that $\mathcal{A}_{\eta,t,\omega}$ is a union of boxes $\overline{C}_{n,q}$, that for $t \geq 2$:

$$\mathcal{A}_{\eta,t,\omega} \subseteq \widetilde{A}_{\eta,t,\omega} \subseteq \text{ closed 1-neighborhood of } \mathcal{A}_{\eta,t,\omega} \,. \tag{4.22}$$

We shall now drop the subscript $\eta, t, \omega$ for notational simplicity. We consider the successive excursions of the path $Z_.$ in $\widetilde{A}$ and out of $\mathcal{O}$:

$$\begin{aligned} R_1 &= \inf\{u \geq 0,\ Z_u \in \widetilde{A}\} \leq \infty\,, \\ D_1 &= \inf\{u \geq R_1,\ Z_u \notin \mathcal{O}\} \leq \infty, \quad \text{and by induction for } k \geq 1\,, \\ R_{k+1} &= R_1 \circ \theta_{D_k} + D_k\,, \\ D_{k+1} &= D_1 \circ \theta_{D_k} + D_k\,, \quad \text{so that} \end{aligned} \tag{4.23}$$

$$0 \leq R_1 \leq D_1 \leq R_2 \leq \ldots \leq R_k \leq D_k \leq \ldots \leq \infty\,,$$

and all these inequalities with may be the exception of the first one are strict when the l.h.s. is finite. We further introduce

$$N_t = \sum_{i \geq 1} 1_{\{R_i \leq t\}}\,, \tag{4.24}$$

which measures the number of successive returns to $\widetilde{A}$ before time $t$. To keep track of successive displacements of $Z_.$ at $\|\cdot\|$-distance $t^\mu$ (see Proposition 4.1), we consider the sequence of iterates of

$$\begin{aligned} H^1 &= \inf\{s \geq 0,\ \|Z_s - Z_0\| \geq t^\mu\}\,, \\ H^{i+1} &= H^1 \circ \theta_{H^i} + H^i, \ \text{ for } i \geq 1\,. \end{aligned} \tag{4.25}$$

We can thus define

$$I_t = \sum_{i \geq 1} 1_{\{H_i \leq t\}}\,, \tag{4.26}$$

which counts the number of completed successive displacements at $\|\cdot\|$-distance $t^\mu$ performed by $Z_.$ up to time $t$.

We are going to introduce a coarse graining scheme on the path space $C(\mathbb{R}_+, \mathbb{R}^d)$, (see the end of Chapter 4 §1). The essential part will be the event

$$E = \Big\{T_T > t,\ 1 \leq N_t \leq C_5 \,\frac{S_1(t,\omega)}{(\log t)^{2-\eta}}\,,\ I_t < [t^{1-\mu}]\Big\}\,, \tag{4.27}$$

where the constant $C_5(d, \nu, W)$ is defined via:

$$C_1 \cdot C_5 = 2c(d, \nu), \quad (\text{see (5.2.73) for the definition of } C_1)\,. \tag{4.28}$$

**Lemma 4.2:** *(E is essential)*

$$\overline{\lim_{t\to\infty}} \frac{(\log t)^{2/d}}{t} \log Q_{t,\omega}(E^c) < 0 . \tag{4.29}$$

**Proof:** Observe that

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log P_0[T_{\mathcal{T}} \le t] < 0 , \tag{4.30}$$

furthermore if $K = [C_5\, S_1(t,\omega)(\log t)^{\eta-2}]$,

$$E_0\Big[T_{\mathcal{T}} > t,\ N_t \ge C_5\, \frac{S_1}{(\log t)^{2-\eta}},\ \exp\Big\{ - \int_0^t V(Z_s,\omega)ds\Big\}\Big] \le$$
$$E_0\Big[R_K < T_{\mathcal{T}},\ \exp\Big\{ - \int_0^{R_K} V(Z_s,\omega)ds\Big\}\Big] ,$$

and since $Z_{D_i}$ and $Z_{R_{i+1}}$ lie at distance $\ge R\epsilon_n^{-1} - 1 \ge C_2 \log 2^n$, it follows from successive applications of the strong Markov property and (5.2.73) that

$$\begin{aligned} &E_0\Big[T_{\mathcal{T}} > t,\ N_t \ge C_5\, \frac{S_1}{(\log t)^{2-\eta}},\ \exp\Big\{ - \int_0^t V(Z_s,\omega)ds\Big\}\Big] \\ &\le \quad \exp\{-(K-1)(R\epsilon_n^{-1} - 1)\} \\ &\overset{(4.13)-(4.28)}{\le} \quad \exp\Big\{ - 2c(d,\nu)\, \frac{t}{(\log t)^{2/d}}\, (1+o(1))\Big\} ,\quad \text{as } t\to\infty . \end{aligned} \tag{4.31}$$

By a similar argument,

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log E_0[T_{\mathcal{T}} > t,\ I_t \ge [t^{1-\mu}]] < 0 . \tag{4.32}$$

Observe also that since

$$\begin{aligned} E_0\Big[N_t = 0,\ \exp\Big\{ - \int_0^t V(Z_s,\omega)ds\Big\}\Big] &= R_{t,\omega}^{\tilde{A}^c}\, 1(0) \\ &\overset{(4.22)-(3.1.9)}{\le} \quad c(d)(1 + (\lambda_\omega(\mathcal{A}^c)\, t)^{d/2})\, e^{-\lambda_\omega(\mathcal{A}^c)\, t} \\ &\overset{(4.11)}{\le} \quad \tilde{c}(d)(1 + (\lambda_f t)^{d/2})\, e^{-\lambda_f t} \\ &\le \quad \exp\Big\{ - 2c(d,\nu)\, \frac{t}{(\log t)^{2/d}}\, (1+o(1))\Big\} , \end{aligned} \tag{4.33}$$

as $t \to \infty$, where we used the notation

$$\tilde{c}(d) = c(d) \sup_{u\ge 0, h\ge 0} \frac{1+(u+h)^{d/2}}{1+u^{d/2}}\, e^{-h} \in (1,\infty) . \tag{4.34}$$

On the other hand, since $\omega \in \Omega_\eta \subseteq \Omega_0$, we know that

$$S_{t,\omega} \geq \exp\Big\{ - c(d,\nu)\, \frac{t}{(\log t)^{2/d}}\, (1+o(1))\Big\}, \quad \text{as } t \to \infty\,,$$

(see for instance (3.15) and (5.3) below). The combination of this lower bound and of the upper bounds (4.30) - (4.33) proves (4.29). □

We now come to the definition of the covering $\mathcal{G}_{\eta,t,\omega}$ of the essential part $E$ of the path space $C(\mathbb{R}_+, \mathbb{R}^d)$. We shall write $\mathcal{G}$ for $\mathcal{G}_{\eta,t,\omega}$ when this causes no confusion, and we shall use the notation

$$R_i^- = [R_i],\ D_i^- = [D_i], \text{ for } i \geq 1\,. \tag{4.35}$$

The collection $\mathcal{G}$ consists of two type of events depending on whether $t$ belongs to an interval $[D_k, R_{k+1})$ (first type) or to an interval $[R_k, D_k)$ (second type).

Events of the first type have the form

$$\begin{aligned} G = \Big\{&D_N \leq t < R_{N+1} \wedge T_{\mathcal{T}},\ Z_{R_1} \in B(x_1),\\ &Z_{D_1} \in B(y_1), \dots, Z_{R_N} \in B(x_N),\\ &Z_{D_N} \in B(y_N),\ Z_t \in B(x_{N+1}),\\ &R_1^- = r_1,\ D_1^- = d_1, \dots, R_N^- = r_N,\ D_N^- = d_N\Big\}, \text{ with} \end{aligned} \tag{4.36}$$

$$1 \leq N \leq C_5\, \frac{S_1}{(\log t)^{2-\eta}}\,, \tag{4.37}$$

$$x_i, y_j \in \Big(\frac{1}{\sqrt{d}}\, \mathbb{Z}^d\Big) \cap \mathcal{T}, \text{ for } 1 \leq i \leq N+1,\ 1 \leq j \leq N\,, \tag{4.38}$$

(4.39) for $i \in [1,N]$, $x_i \in \mathcal{A}$ and $B(y_i)$ intersects the connected component of $\mathcal{O}$ containing $x_i$ ,

(4.40) the number of distinct connected components of $\mathcal{O}$ containing some $x_i$, $i \in [1,N]$, is smaller than $M_0\, t^{1-\mu}$ ,

$$0 \leq r_1 \leq d_1 \leq \ldots \leq r_N \leq d_N \leq t \text{ are integers}\,. \tag{4.41}$$

Events of the second type have the form:

$$G = \Big\{R_N \le t < D_N \wedge T_T,\ Z_{R_1} \in B(x_1),\ Z_{D_1} \in B(y_1), \dots, Z_{R_N} \in B(x_N),\ Z_{D_N} \in B(y_N),\ R_1^- = r_1,\ D_1^- = d_1, \dots, R_N^- = r_N\Big\}, \tag{4.42}$$

and the parameters $N, x_1, y_1, \dots, x_N, r_1, d_1, \dots, r_N$, satisfy analogous constraints to (4.37) - (4.41), with obvious modifications.

We view each event $G$ of $\mathcal{G}$ as describing a strategy of the path. The 'cost' of each such strategy will be the quantity:

$$\text{cost}(G) = \Big(-\log e_0(0, x_1, \omega)\big) \vee (\lambda_f\, \tau_0) + \sum_k a_0(y_k, x_{k+1}, \omega) \vee (\lambda_f\, \tau_k\Big) + \sum_k t_k\, \lambda_\omega(\mathcal{C}(x_k))\,, \tag{4.43}$$

where the first sum on the r.h.s. of (4.43) runs over $k \in [1, N]$, if $G$ is of first type and $k \in [1, N-1]$, if $G$ is of second type, and the second sum runs in both cases over $k \in [1, N]$. Moreover we use here the following notations:

(4.44) for $x \in \mathcal{A}$, $\mathcal{C}(x)$ is the connected component of $\mathcal{O}$ containing $x$ ,

$$\begin{aligned} \tau_k &= (r_{k+1} - d_k - 1)_+,\ 1 \le k < N\,, \\ \tau_N &= (t - d_N - 1)_+,\ \text{for } G \text{ of first type}\,, \\ &= 0,\ \text{for } G \text{ of second type}\,, \\ \tau_0 &= r_1\,, \end{aligned} \tag{4.45}$$

$$\begin{aligned} t_k &= (d_k - r_k - 1)_+,\ 1 \le k < N\,, \\ t_N &= (d_N - r_N - 1)_+,\ \text{for } G \text{ of first type}\,, \\ t_N &= (t - r_N - 1)_+,\ \text{for } G \text{ of second type}\,, \end{aligned} \tag{4.46}$$

$\lambda_f$ is defined in (4.11) and $e_0(\cdot,\cdot,\omega)$ as well as $a_0(\cdot,\cdot,\omega)$ are defined in Chapter 5 §1. The next theorem together with Lemma 4.2 constitute the coarse graining scheme on the path space. However the fact that the quantity cost$(G)$ is truly useful only comes in Proposition 4.4, and uses in a crucial way the constraint (4.40). Recall that $\eta \in [\frac{1}{2}, 1)$ and $\omega \in \Omega_\eta$ are kept fixed throughout this subsection.

**Theorem 4.3:**

$$E \subseteq \bigcup_{\mathcal{G}} G, \ \textit{for large } t\,, \tag{4.47}$$

(4.48) $$|\mathcal{G}| = \exp\{o(S_1(t,\omega))\}, \quad as\ t \to \infty\,,$$

(4.49) $$\overline{\lim_{t\to\infty}} \frac{1}{S_1(t,\omega)} \sup_{G\in\mathcal{G}} \Big\{ \log E_0\Big[G, \exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big] + \mathrm{cost}(G)\Big\} \le 0\,.$$

**Proof:**

*Proof of (4.47):* For large $t$, the requirements $T_{\mathcal{T}} > t$, $I_t < [t^{1-\mu}]$, together with Proposition 4.1 imply that the sequence $Z_{R_j}$, $1 \le j \le N_t$, does not meet more than $M_0[t^{1-\mu}]$ clearing boxes. Moreover each $Z_{R_i}$, $1 \le i \le N_t$, belongs to some $B(x_i) \subseteq \widetilde{\mathcal{A}}$, for some $x_i \in (\frac{1}{\sqrt{d}}\,\mathbb{Z}^d) \cap \mathcal{A} \cap \mathcal{T}$. Once these remarks are made, it is straightforward to check (4.47).

*Proof of (4.48):* Observe that

$$\Big|\Big(\frac{1}{\sqrt{d}}\,\mathbb{Z}^d\Big) \cap \mathcal{T}\Big| \le c(d)\, t^d\,.$$

Thus for a given $N \in [1,\ C_5\ \frac{S_1}{(\log t)^{2-\eta}}]$, the number of possibilities for $x_1, y_2, \dots, x_N$, and possibly $y_N$, $x_{N+1}$, is no larger than:

$$(c(d)\,t^d)^{2N+1} \le \exp\Big\{3C_5\ \frac{S_1}{(\log t)^{2-\eta}}\ \log(c(d)\,t^d)\Big\} = \exp\{o(S_1)\}\,.$$

Similarly the number of possibilities for $r_1, d_1, \dots, r_N$, and possibly $d_N$ is no larger than:

$$(t+1)^{2N} \le \exp\Big\{2\,C_5\ \frac{S_1}{(\log t)^{2-\eta}}\ \log(1+t)\Big\} = \exp\{o(S_1)\}\,.$$

Since the number of possibilities for $N$ is sublinear in $t$, the total number of events of first type or of second type is $\exp\{o(S_1)\}$, as $t \to \infty$.

*Proof of (4.49):* We consider the case of a large $t$ and an arbitrary event $G$ of first type. Applying the strong Markov property at time $D_N$, we find

(4.50) $$\begin{aligned} &E_0\Big[G, \exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big] = \\ &E_0\Big[D_N \le t \wedge T_{\mathcal{T}}, Z_{R_1} \in B(x_1), \dots, D_N^- = d_N, \\ &\exp\Big\{-\int_0^{D_N} V(Z_s,\omega)ds\Big\} \\ &\widetilde{E}_{Z_{D_N}}\Big[\widetilde{R}_1 \wedge \widetilde{T}_{\mathcal{T}} > t - D_N,\ \widetilde{Z}_{t-D_N} \in B(x_{N+1}), \\ &\exp\Big\{-\int_0^{t-D_N} V(\widetilde{Z}_s,\omega)ds\Big\}\Big]\Big]\,, \end{aligned}$$

with obvious notations. We bound the inner expectation in the r.h.s. of (4.50) in two ways. On the one hand it is smaller than:

$$e_0(Z_{D_N}, x_{N+1}, \omega) \leq \exp\{-a_0(y_N, x_{N+1}, \omega) + \log t\}\,, \tag{4.51}$$

on the event $\{Z_{D_N} \in B(y_N)\}$, as can be seen by either using (5.2.22) and (4.5.12), when $|y_N - x_{N+1}| > 3$, or the fact that the r.h.s. of (4.51) is larger than 1 when $|y_N - x_{N+1}| \leq 3$, by (4.5.12) and (5.2.32).

On the other hand, since $\lambda_\omega(\widetilde{\mathcal{A}}^c) \geq \lambda_\omega(\mathcal{A}^c) \overset{(4.11)}{\geq} \lambda_f$, as in (4.33), (4.34), the inner expectation in the r.h.s. of (4.50) is smaller than:

$$\begin{aligned} &\widetilde{c}(d)(1+t^d)\ \exp\{-\lambda_f\,\tau_N\} \text{ on the set } \{D_N^- = d_N\} \text{ with} \\ &\tau_N \overset{(4.45)}{=} (t - d_N - 1)_+\,. \end{aligned} \tag{4.52}$$

Thus for large $t$ and an arbitrary $G$ of first type:

$$\begin{aligned} &E_0\Big[G, \exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big] \leq \\ &E_0\Big[D_N \leq T_{\mathcal{T}}, Z_{R_1} \in B(x_1), \dots, D_N^- = d_N, \\ &\exp\Big\{-\int_0^{D_N} V(Z_s,\omega)ds\Big\}\Big]\ \exp\{-(\lambda_f\,\tau_N) \vee a_0(y_N, x_{N+1}, \omega) \\ &+ \log\widetilde{c}(d) + \log(1+t^d)\}\,. \end{aligned} \tag{4.53}$$

Applying the strong Markov property at time $R_N$, the expectation in the r.h.s. of (4.53) is smaller than:

$$\begin{aligned} &E_0\Big[R_N < T_{\mathcal{T}},\ Z_{R_1} \in B(x_1), \dots, Z_{R_N} \in B(x_N), \\ &R_1^- = r_1, \dots, R_N^- = r_N\,,\ \exp\Big\{-\int_0^{R_N} V(Z_u,\omega)du\Big\} \\ &\widetilde{E}_{Z_{R_N}}\Big[D_1 \wedge \widetilde{T}_{\mathcal{T}} \geq (d_N - R_N)_+\,, \\ &\exp\Big\{-\int_0^{(d_N - R_N)_+} V(\widetilde{Z}_u,\omega)du\Big\}\Big]\Big] \end{aligned} \tag{4.54}$$

and the inner expectation of the last term, on the event $\{R_N^- = r_N,\ Z_{R_N} \in B(x_N)\}$ by (3.1.9) and (4.5.12) is smaller than

$$c(d)(1+t^d)\ \exp\{-\lambda_\omega(C(x_N))\,t_N\}\,, \tag{4.55}$$

with $t_N \overset{(4.46)}{=} (d_N - r_N - 1)_+$, and $C(x_N)$ the connected component of $\mathcal{O}$ containing $x_N$.

If we now iterate these bounds, we see by induction that for large $t$ and any $G$ of first type

$$\begin{aligned}
&\log\Big(E_0\Big[G, \exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big]\Big) \le \\
(4.56)\qquad &-\Big\{(-\log e_0(0,x_1,\omega)) \vee (\lambda_f\, \tau_0) + \sum_1^N a_0(y_k, x_{k+1},\omega) \vee (\lambda_f\, \tau_k) \\
&+ \sum_1^N t_k\, \lambda_\omega(\mathcal{C}(x_k))\Big\} + (2N+1)[\log \widetilde{c}(d) + \log(1+t^d)]\,,
\end{aligned}$$

with $\widetilde{c}(d)$ as in (4.34).

Since by (4.37), $N \le C_5\, S_1(\log t)^{\eta-2}$, we also see that

$$(4.57)\qquad (2N+1)(\log \widetilde{c}(d) + \log(1+t^d)) = o(S_1)\,.$$

We have entirely analogous estimates for events $G$ of second type, and this proves our claim (4.49). $\square$

The above theorem is only useful if interesting lower bounds can be derived on the quantity $\mathrm{cost}(G)$ introduced in (4.43). This is precisely the role of:

**Proposition 4.4:**

$$\begin{aligned}
(4.58)\qquad &\varliminf_{t\to\infty} \frac{1}{S_1(t,\omega)} \inf_{G\in\mathcal{G}} \Big\{\mathrm{cost}(G) - \Big[\frac{(1+3\eta)}{4} \sup_i a_0(x_i) \\
&+ t \min_{1\le k\le N} \lambda_\omega(\mathcal{C}(x_k)) \wedge \frac{2c(d,\nu)}{(\log t)^{2/d}}\Big]\Big\} \ge 0\,.
\end{aligned}$$

**Proof:** For $a,b \ge 0$, $a \vee b \ge \frac{1+\eta}{2}\, a + \frac{1-\eta}{2}\, b$. Thus for $G \in \mathcal{G}$,

$$\begin{aligned}
(4.59)\qquad \mathrm{cost}(G) \ \ge\ &\Big(\frac{1+\eta}{2}\Big)\Big[-\log e_0(0,x_1,\omega) + \sum_k a_0(y_k,x_{k+1},\omega)\Big] \\
&+ \frac{2c}{(\log t)^{2/d}}\Big(\sum_k \tau_k\Big) + \min_k(\lambda_\omega(\mathcal{C}(x_k))\Big(\sum_k t_k\Big)\,.
\end{aligned}$$

Since $N \le C_5\, S_1(\log t)^{\eta-2}$, we have from (4.45) - (4.46)

$$(4.60)\qquad t - 2C_5\, \frac{S_1}{(\log t)^{2-\eta}} \le \sum_0^N \tau_i + \sum_1^N t_k \le t\,.$$

It follows from (4.59)-(4.60) that

$$\begin{aligned}\mathrm{cost}(G) \geq \Big(\frac{1+\eta}{2}\Big)\Big[-\log e_0(0,x_1,\omega) + \sum_k a_0(y_k,x_{k+1},\omega)\Big]\\ + t \min_{1\le k\le N} \lambda_\omega(\mathcal{C}(x_k)) \wedge \frac{2c}{(\log t)^{2/d}} - 4C_5\, c(d,\nu)\frac{S_1}{(\log t)^{2-\eta+\frac{2}{d}}}\,.\end{aligned} \tag{4.61}$$

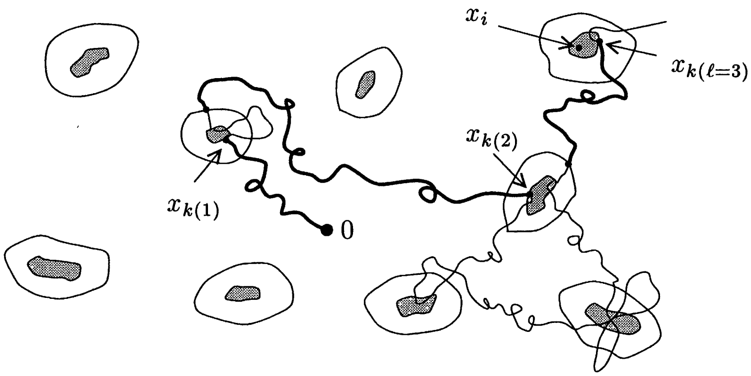
We shall now derive a lower bound on

$$-\log e_0(0,x_1,\omega) + \sum_k a_0(y_k,x_{k+1},\omega)\,. \tag{4.62}$$

The key observation to derive such a lower bound is the following consequence of (4.40):

(4.63) for any $i \in [2, N+1]$, when $G$ is of the first type, or $i \in [2, N]$ when $G$ is of the second type, one can extract a sequence $x_{k(j)}$, $1 \leq j \leq \ell$, with $\ell \leq M_0\, t^{1-\mu} + 1$, such that:

- $k(1) = 1$, and for $j \in [2,\ell]$, $x_{k(j-1)}$ and $x_{k(j)-1}$ belong to the same connected component of $\mathcal{O}$.
- $x_{k(\ell)}$ equals $x_i$ or belongs to the same connected component of $\mathcal{O}$, when $x_i \in \mathcal{O}$.




The construction of the extracted sequence goes as follows. We first pick $k(1) = x_1$. If $x_1 = x_i$ or both lie in the same connected component of $\mathcal{O}$, we are finished. Otherwise, we look at the largest integer $k \leq i$ (in fact $k < i$) such that $x_k \in \mathcal{C}(x_1)$, and we define $k(2) = k+1$. If $x_{k(2)} = x_i$ or both lie in the same connected component of $\mathcal{O}$, we are finished. Otherwise we proceed as before. The construction stops after $\ell$ steps, with $\ell$ no larger than 1+

number of connected components of $\mathcal{O}$ visited by the sequence $x_j$, $1 \le j \le N$ or $N+1$. In view of (4.40), this yields $\ell \le M_0\, t^{1-\mu} + 1$.

Coming back to the expression (4.62), we see it is bigger than

$$\begin{aligned}
&-\log e_0(0,x_1,\omega) + \sum_{j=2}^{\ell} a_0(y_{k(j)-1}, x_{k(j)}, \omega) \\
&= -\log e_0(0,x_1,\omega) + \sum_{j=2}^{\ell} a_0(x_{k(j-1)}, x_{k(j)}, \omega) + a_0(x_{k(\ell)}, x_i, \omega) \\
&\quad + \sum_{j=2}^{\ell} [a_0(y_{k(j)-1}, x_{k(j)}, \omega) - a_0(x_{k(j-1)}, x_{k(j)}, \omega)] \\
&\quad - a_0(x_{k(\ell)}, x_i, \omega)\,.
\end{aligned} \tag{4.64}$$

Using the subadditivity at hand (see (5.2.2), (5.2.6)), the r.h.s. of (4.64) is bigger than:

$$-\log e_0(0,x_i,\omega) - \sum_{j=2}^{\ell} a_0(x_{k(j-1)}, y_{k(j)-1}, \omega) - a_0(x_{k(\ell)}, x_i, \omega)\,. \tag{4.65}$$

When $t$ is large, all connected components of $\mathcal{O}$ intersecting $\mathcal{T}$ have diameter smaller than $\Delta_{\eta,t,\omega} (> C_2 \log 2^{n+1})$, by Proposition 4.1. We can therefore apply the upper bound on cost of crossings of (5.2.74) and see that for large $t$ for all $G \in \mathcal{G}$, and $i \in [1, N+1]$ or $[1,N]$, (depending on whether $G$ is of first type or second type), the expression in (4.65) is bigger than:

$$\begin{aligned}
&-\log e_0(0,x_i,\omega) - (M_0\, t^{1-\mu} + 1)(\Delta_{\eta,t,\omega} + 1) \\
&= -\log e_0(0,x_i,\omega) + o(S_1)\,,
\end{aligned} \tag{4.66}$$

in view of (3.17), (4.13), (4.18). This and (4.61) shows that for large $t$ uniformly in $G \in \mathcal{G}$,

$$\begin{aligned}
\mathrm{cost}(G) \ge &\left(\frac{1+\eta}{2}\right) \sup_i \; -\log e_0(0,x_i,\omega) \\
&+ t\Big(\min_{1\le k\le N} \lambda_\omega(\mathcal{C}(x_k)) \wedge \frac{2c(d,\nu)}{(\log t)^{2/d}}\Big)
\end{aligned} \tag{4.67}$$

and using the shape theorem 5.2.5, uniformly in $G \in \mathcal{G}$:

$$\ge \frac{(1+3\eta)}{4} \sup_i \alpha_0(x_i) + t \min_{1\le k\le N} \lambda_\omega(\mathcal{C}(x_k)) \wedge \frac{2c(d,\nu)}{(\log t)^{2/d}} + o(S_1(t,\omega))\,.$$

This concludes the proof of (4.51). □

## 6.5 Applications

In this section, we shall apply the coarse graining scheme developed in Section 4 to obtain a finer asymptotics of $S_{t,\omega}$ and to conclude the proof of the pinning effect.

### A) Finer Asymptotics for $S_{t,\omega}$

We know from (4.5.10) that

$$\text{(5.1)} \qquad \mathbb{P}\text{-a.s.}, \; S_{t,\omega} = \exp\{-c(d,\nu)\, \frac{t}{(\log t)^{2/d}}(1+o(1))\}, \quad \text{as } t \to \infty .$$

We shall now obtain a more refined asymptotics of $S_{t,\omega}$ involving the value $\mu_{t,\omega}$ of the random variational problem on the half-line introduced in (3.1):

$$\mu_{t,\omega} = \inf_{u>0} \{u + t\,\lambda_\omega(B_{\alpha_0}(0,u))\} .$$

We know from Proposition 3.4 that for $\omega \in \Omega_0$ (see (2.1)),

$$\begin{aligned} &\mu_{t,\omega} = S_1(t,\omega) + t\,\lambda_\omega(B_{\alpha_0}(0, S_1(t,\omega))), \quad \text{and} \\ &\left|\mu_{t,\omega} - c(d,\nu)\, \frac{t}{(\log t)^{2/d}}\right| \le \frac{t}{(\log t)^{\frac{2}{d}+\chi}}, \quad \text{for large } t . \end{aligned}$$

**Theorem 5.1:** *On a set of full* $\mathbb{P}$*-measure, for any* $\eta \in [\frac{1}{2}, 1)$,

$$\text{(5.2)} \qquad S_{t,\omega} = \exp\Big\{ -\mu_{t,\omega} + o\Big(S_1\Big(\frac{t}{\eta}, \omega\Big)\Big)\Big\}, \quad \textit{as } t \to \infty .$$

**Proof:** The proof involves and 'adequate' lower bound and upper bound on $S_{t,\omega}$. We begin with lower bound:

**Lemma 5.2:**

$$\text{(5.3)} \qquad \textit{For } \omega \in \Omega_0,\; S_{t,\omega} \ge \exp\{-\mu_{t,\omega} + o(S_1(t,\omega))\}, \; \textit{as } t \to \infty .$$

**Proof:** We apply Theorem 1.1, with $U = B_{\alpha_0}(0, S_1)$. Thus for large $t$ we can find $y \in B_{\alpha_0}(0, S_1)$ such that

$$\text{(5.4)} \qquad S_{t,\omega} \ge \frac{c(d)}{t^{d+1}} \exp\{-\lambda_\omega(B_{\alpha_0}(0,S_1))\,t\}\; e_0(0,y,\omega) .$$

From the shape theorem 5.2.5, we also know that

$$\text{(5.5)} \qquad \sup_{y \in B_{\alpha_0}(0,S_1)} |-\log e_0(0,y,\omega) - \alpha_0(y)| = o(S_1), \quad \text{as } t \to \infty .$$

Coming back to (5.4), we see that

$$S_{t,\omega} \geq \frac{c(d)}{t^{d+1}} \exp\{-\lambda_\omega(B_{\alpha_0}(0,S_1))\,t - S_1 + o(S_1)\}\,,$$

which yields (5.3). □

We now turn to the upper bound. It is convenient to define in the notations of Proposition 4.1 the set of full $\mathbb{P}$-measure

$$\Omega_1 = \bigcap_{\eta\in[\frac{1}{2},1)\cap\mathbb{Q}} \Omega_\eta \subseteq \Omega_0\,. \tag{5.6}$$

From now on we keep a fixed choice of $\eta \in [\frac{1}{2},1)\cap\mathbb{Q}$ and $\omega \in \Omega_1$. Then for large $t$

$$\begin{aligned} S_{t,\omega} &\overset{(4.29)}{\leq} 2E_0\Big[E, \exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big] \\ &\overset{(4.47)}{\leq} |\mathcal{G}_{\eta,t,\omega}| \sup_{G\in\mathcal{G}_{\eta,t,\omega}} E_0\Big[G, \exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big] \\ &\overset{(4.48)-(4.49)}{\leq} \exp\{o(S_1) - \inf_{G\in\mathcal{G}_{\eta,t,\omega}} \operatorname{cost}(G)\} \\ &\overset{(4.57)}{\leq} \exp\Big\{o(S_1) - \inf_{G\in\mathcal{G}_{\eta,t,\omega}} \Big[\eta \max_i \alpha_0(x_i) \\ &\qquad + t\Big(\min_{1\leq k\leq N} \lambda_\omega(\mathcal{C}(x_k)) \wedge \frac{2c(d,\nu)}{(\log t)^{2/d}}\Big)\Big]\Big\}\,. \end{aligned} \tag{5.7}$$

Since $t$ is large, the diameter of each $\mathcal{C}(x_k)$ is smaller than $\Delta_{\eta,t,\omega}$ (see Proposition 4.1), and thus for any $G \in \mathcal{G}_{\eta,t,\omega}$ and $x_k$, $1 \leq k \leq N$:

$$\mathcal{C}(x_k) \subset B_{\alpha_0}(0, \max_i \alpha_0(x_i) + \text{const}\ \Delta_{\eta,t,\omega})\,, \tag{5.8}$$

where the constant solely depends on $\alpha_0(\cdot)$. Thus for large $t$

$$\begin{aligned} S_{t,\omega} \leq\ & \max\Big\{\exp\Big\{o(S_1) - 2c(d,\nu)\,\frac{t}{(\log t)^{2/d}}\Big\}\,, \\ & \exp\{o(S_1) - \inf_{u>0}\ \{\eta u + t\,\lambda_\omega(B_{\alpha_0}(0,u))\}\}\Big\}\,. \end{aligned} \tag{5.9}$$

Note that the minimum of the function

$$u > 0 \to \eta u + t\,\lambda_\omega(B_{\alpha_0}(0,u))$$

is achieved at $S_1(\frac{t}{\eta},\omega)(\leq S_\eta(t,\omega))$, and equals $\eta\mu_{\frac{t}{\eta},\omega}$. Furthermore, we have

$$\mu_{t,\omega} - \inf_{u>0}\ \{\eta u + t\,\lambda_\omega(B_{\alpha_0}(0,u))\} \leq (1-\eta)\,S_1\Big(\frac{t}{\eta},\omega\Big)\,: \tag{5.10}$$

Keeping in mind that $\mu_{t,\omega} \sim c(d,\nu)\ t(\log t)^{-2/d}$, as $t \to \infty$, we see that for large $t$:

$$S_{t,\omega} \le \exp\Big\{o(S_1) + (1-\eta)\, S_1\Big(\frac{t}{\eta}, \omega\Big) - \mu_{t,\omega}\Big\}\,. \tag{5.11}$$

Using the fact that $S_1(\cdot,\omega)$ is nondecreasing, see (3.13), this and (5.3) implies our claim (5.2). □

Let us mention that more can be said in the one-dimensional case. Indeed we know from (3.22) that the quantity $\mu_{t,\omega}$ for large $t$ 'with arbitrarily high $\mathbb{P}$ probability' lies within order $\frac{t}{(\log t)^3}$ of $\frac{(\pi\nu)^2}{2}\ \frac{t}{(\log t)^2}\ (1 + \frac{2\log\log t}{\log t})$.

## B) Pinning Effect

We shall now give a second application of the coarse graining scheme on path space to prove a weak pinning property. This together with Section 2 will yield a proof of the pinning effect. We keep the notation $\Omega_1$ for the set of full $\mathbb{P}$-probability introduced in (5.6). For $\eta_0 \in [\frac{1}{2}, 1)$, $\omega \in \Omega_1$, $t \ge 2$, we define

$$R_{\eta_0,t,\omega} = \max\Big\{\frac{t}{(\log t)^{2/d}}\ \frac{(\log t)^{2-\eta_0}}{S_1(t)}\,,\ (\log t)^{1+\frac{2-\eta_0}{d}}\Big\}\,. \tag{5.12}$$

This is a 'small scale' since by (3.17), there exists a $\chi \in (0, \frac{1}{d})$ such that for all $\omega \in \Omega_1$:

$$R_{\eta_0,t,\omega} \le \exp\{(\log t)^{1-\chi}\},\ \text{ for large } t\,, \tag{5.13}$$

moreover, when $\eta \in (\eta_0, 1)$, and $\omega \in \Omega_1$,

$$\Delta_{\eta,t,\omega} = o(R_{\eta_0,t,\omega}),\ \text{(in the notations of (4.18))}\,. \tag{5.14}$$

We now want to define a notion of near minima of (see also (1.11)):

$$x \in \mathbb{R}^d \longrightarrow \alpha_0(x) + t\,\lambda_\omega(B(x, R_{\eta_0,t,\omega}))\,.$$

Note that the quantities $\mu_{t,\omega}$ and $\inf_x\{\alpha_0(x) + t\,\lambda_\omega(B(x, R_{\eta_0,t,\omega}))\}$ are very close when $\omega \in \Omega_1$ and $t$ is large. Indeed on the one hand the eigenvalue estimate (4.20) applied to $U = B_{\alpha_0}(0, S_1(t,\omega))$, together with (4.18), (5.14) enable to deduce that for large $t$

$$\inf_x\{\alpha_0(x) + t\lambda_\omega(B(x, R_{\eta_0,t,\omega}))\} \le \mu_{t,\omega} + 1\,. \tag{5.15}$$

On the other hand it follows from an argument analogous to (5.8) that for large $t$, (5.16) $$\mu_{t,\omega} \le \inf_x \{\alpha_0(x) + t\,\lambda_\omega(B(x, R_{\eta_0,t,\omega}))\} + c\,R_{\eta_0,t,\omega}\ ,$$

where $c$ solely depends on $\alpha_0(\cdot)$. We now introduce a skeleton of $\eta_0$-near minima via:

(5.17) $$\begin{aligned}\mathcal{L}_{\eta_0,t,\omega} = \ & \Big\{x \in \frac{1}{\sqrt{d}}\,\mathbb{Z}^d,\ \alpha_0(x) + t\,\lambda_\omega(B(x, R_{\eta_0,t,\omega})) \\ & \le \mu_{t,\omega} + (1-\eta_0)\,S_{\eta_0}(t,\omega)\Big\}\ .\end{aligned}$$

**Theorem 5.3:** *(Pinning effect)*
*For $\omega$ in the set of full $\mathbb{P}$-measure $\Omega_1$ and $\eta_0 \in [\frac{1}{2}, 1)$,*

(5.18) $$\lim_{t\to\infty} Q_{t,\omega}(C) = 1\ ,$$

*where $C$ is the event:*

(5.19) $$C = \bigcup_{x \in \mathcal{L}_{\eta_0,t,\omega}} \{H(x) \le t\} \cap \{T_{B(x,R_{\eta_0,t,\omega})} \circ \theta_{H(x)} > t - H(x)\}\ .$$

**Proof:** We choose some fixed $\omega \in \Omega_1$, $\frac{1}{2} \le \eta_0 < \eta < 1$, $\eta$ rational and apply the coarse graining scheme of §4 with the value $\eta$. We also define

(5.20) $$\mathrm{L}_{\eta,t,\omega} = \Big\{x \in \mathcal{A} \cap \frac{1}{d}\,\mathbb{Z}^d : \alpha_0(x) + t\,\lambda_\omega\,(\mathcal{C}(x)) \le \mu_{t,\omega} + (1-\eta)\,S_\eta\Big\}\ ,$$

with $\mathcal{C}(x)$ defined as (4.44). For $x \in \mathrm{L}_{\eta,t,\omega}$, we introduce

(5.21) $$U(x) = \mathcal{T} \backslash \bigcup_{\mathcal{C}:\lambda_\omega(\mathcal{C}) < \lambda_\omega(\mathcal{C}(x))} \mathcal{C} \cap \mathcal{A}\ ,$$

where $\mathcal{C}$ runs over the set of connected components of $\mathcal{O}$. Furthermore we define the event:

(5.22) $$A = \bigcup_{x \in \mathrm{L}_{\eta,t,\omega}} \{H(x) \le t\} \cap \{T_{U(x)} \circ \theta_{H(x)} > t - H(x)\}\ .$$

We shall establish a weak pinning property in the precise sense explained in (2.2) - (2.10). We begin with

**Lemma 5.4:**

(5.23) $$\lim_{t\to\infty} Q_{t,\omega}(A) = 1\ .$$

**Proof:** We introduce the subclass $\mathcal{G}'$ of $\mathcal{G} = \mathcal{G}_{\eta,t,\omega}$:

(5.24) $$\begin{aligned}\mathcal{G}' = \Big\{ & G \in \mathcal{G}; \\ & \frac{(1+3\eta)}{4}\,\sup_i \alpha_0(x_i) + t \min_{1\le k\le N} \lambda_\omega(\mathcal{C}(x_k)) \le \mu_{t,\omega} + \frac{(1-\eta)}{5}\,S_1\Big\}\ .\end{aligned}$$

The first step is to show

$$\lim_{t\to\infty} Q_{t,\omega}\Big(\bigcup_{G\in\mathcal{G}\backslash\mathcal{G}'} G\Big) = 0\,. \tag{5.25}$$

To this end we observe that for large $t$,

$$\begin{aligned}
& Q_{t,\omega}\Big(\bigcup_{G\in\mathcal{G}\backslash\mathcal{G}'} G\Big) \\
\leq \quad & \frac{|\mathcal{G}|}{S_{t,\omega}} \sup_{G\in\mathcal{G}\backslash\mathcal{G}'} E_0\Big[G, \exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big] \\
\overset{(4.48)-(5.3)}{\leq} \quad & \exp\{o(S_1) + \mu_{t,\omega}\} \sup_{\mathcal{G}\backslash\mathcal{G}'} \\
& E_0\Big[G, \exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big] \\
\overset{(4.49)-(4.57)}{\leq} \quad & \exp\Big\{o(S_1) + \mu_{t,\omega} - \inf_{\mathcal{G}\backslash\mathcal{G}'} \Big[\frac{(1+3\eta)}{4} \sup_i \alpha_0(x_i) \\
& + t \min_{1\leq k\leq N} \lambda_\omega(\mathcal{C}(x_k)) \wedge \frac{2c(d,\nu)}{(\log t)^{2/d}}\Big]\Big\}\,,
\end{aligned} \tag{5.26}$$

In view of (5.24) and since $\mu_{t,\omega} \overset{(3.15)}{\sim} c(d,\nu)\, t(\log t)^{-2/d}$, as $t\to\infty$, the above expression for large $t$ is smaller than

$$\max\Big\{\exp\Big\{-\frac{c(d,\nu)\,t}{2(\log t)^{2/d}}\Big\}, \exp\Big\{o(S_1) - \frac{(1-\eta)}{5}\, S_1\Big\}\Big\} \underset{t\to\infty}{\longrightarrow} 0\,.$$

This shows (5.25). Together with Lemma 4.2 and (4.47) this implies that

$$\lim_{t\to\infty} Q_{t,\omega}\Big(\bigcup_{\mathcal{G}'} G\Big) = 1\,. \tag{5.27}$$

Now for $G\in\mathcal{G}$, we consider $x_*(G)$ and $i_*(G)$ where

$$\begin{aligned}
x_*(G) = \ & x_{i_*(G)}, \text{ with } i_*(G) \text{ the smallest integer in } [1,N] \\
& \text{such that } \lambda_\omega(\mathcal{C}(x_{i_*(G)})) = \min_{1\leq k\leq N} \lambda_\omega(\mathcal{C}(x_k))\,.
\end{aligned} \tag{5.28}$$

We shall drop $G$ from the notation when this causes no confusion. Note that on the event $G$, see (4.36), (4.42), the path $Z_.$ has never entered the connected component of $\widetilde{\mathcal{A}}$ (see (4.21)) containing $B(x_*)$, during the time interval $[0, R_{i_*})$. Thus

$$G \subseteq \{H(x_{i_*}) = R_{i_*}\}\,.$$

Moreover on the event $G$, during $[R_{i_*}, t]$, the path $Z_.$ does not enter any connected component of $\mathcal{A}$ included in a connected component $\mathcal{C}$ of $\mathcal{O}$ with $\lambda_\omega(\mathcal{C}) < \lambda_\omega(\mathcal{C}(x_{i_*}))$. It now follows from the inclusion $G \subseteq \{T_{\mathcal{T}} > t\}$, and the definition (5.21) that

$$G \subseteq \{H(x_*) \le t\} \cap \{T_{U(x_*)} \circ \theta_{H(x_*)} > t - H(x_*)\} \,. \tag{5.29}$$

Our claim (5.23) will thus follow from (5.27) when we prove that:

$$\text{for large } t, \text{ for any } G \in \mathcal{G}', \; x_*(G) \in \mathrm{L}_{\eta,t,\omega} \,. \tag{5.30}$$

As a first step observe that when $t$ is large and $G \in \mathcal{G}'$

$$\begin{aligned} &\frac{(1+3\eta)}{4}\,(\alpha_0(x_*) + c\,\Delta_{\eta,t,\omega}) + t\,\lambda_\omega(B_{\alpha_0}(0, \alpha_0(x_*) + c\,\Delta_{\eta,t,\omega})) \\ &\le \mu_{t,\omega} + \frac{1-\eta}{5}\, S_1 + c\,\Delta_{\eta,t,\omega} \,, \end{aligned}$$

where $c$ is a positive constant depending on $\alpha_0(\cdot)$ as in (5.16). This implies that for large $t$, $\alpha_0(x_*) \le S_\eta(t,\omega)$, since otherwise the above inequality together with (3.12) would force

$$\frac{(1-\eta)}{4}\, S_\eta(t,\omega) \le \frac{1-\eta}{5}\, S_1 + c\,\Delta_{\eta,t,\omega}$$

which is false when $t$ is large. Coming back to (5.24) we see that for large $t$ and $G \in \mathcal{G}'$,

$$\begin{aligned} \alpha_0(x_*) + t\,\lambda_\omega(\mathcal{C}(x_*)) &\le \mu_{t,\omega} + \frac{3}{4}\,(1-\eta)\,S_\eta + \frac{(1-\eta)}{5}\,S_1 \\ &\le \mu_{t,\omega} + (1-\eta)\,S_\eta \,, \end{aligned} \tag{5.31}$$

which yields (5.30) and finishes the proof of the Lemma 5.4. □

We shall now see that a weak pinning property holds, once we choose in the notations of §2: $\mathrm{L}_{t,\omega} = \mathrm{L}_{\eta,t,\omega}$, $A_x = \{H(x) \le t\}$, $U(x)$ as in (5.21) for $x \in \mathrm{L}_{\eta,t,\omega}$, and $\Delta_{t,\omega} = \Delta_{\eta,t,\omega}$. Lemma 5.4 takes care of (2.2) - (2.3), furthermore (2.4) - (2.7) are immediate to check. We choose $C_4 = \kappa'(d, \frac{1}{2})$ (in the notation of Corollary 4.4.9). Since for large $t$ any $x \in \mathrm{L}_{\eta,t,\omega}$ is such that

$$\lambda_\omega(B(x, \Delta_{t,\omega})) \le \lambda_\omega(\mathcal{C}(x)) \le \kappa\Big(d, \frac{1}{2}\Big)\,\frac{c(d,\nu)}{(\log t)^{2/d}} \,, \tag{5.32}$$

it follows from Corollary 4.4.9 that (2.10) holds. It further follows from Corollary 4.4.9 and (5.32) that for large $t$, for any $x \in \mathrm{L}_{\eta,t,\omega}$, any connected component $\mathcal{C}$ of $\mathcal{O}$ with $\lambda_\omega(\mathcal{C}) < \lambda_\omega(\mathcal{C}(x))$ is such that

$$\mathrm{diam}(\mathcal{C}(x) \cup \mathcal{C}) \ge t^{1/2} \,, \tag{5.33}$$

which together with Proposition 4.1 implies that assumption (2.8) holds. Finally when $t$ is large, for any $x \in \mathbb{L}_{\eta,t,\omega}$,

$$\begin{aligned} \lambda_\omega(U(x)) &\overset{(4.20)}{\geq} \inf_{\mathcal{C}} \lambda_\omega(\mathcal{C}) \wedge \frac{2c(d,\nu)}{(\log 2^n)^{2/d}} - \frac{1}{t} \\ &\geq \lambda_\omega(\mathcal{C}(x)) - \frac{1}{t}, \end{aligned}$$

since $\mathcal{C}$ runs over all connected components of $\mathcal{O}$ containing a clearing box intersecting $U(x)$, and by construction (see (5.21)) for all such terms $\lambda_\omega(\mathcal{C}) \geq \lambda_\omega(\mathcal{C}(x))$, and $\lambda_\omega(\mathcal{C}(x)) \overset{(5.32)}{\leq} 2c(d,\nu)(\log 2^n)^{-2/d}$. This proves that (2.9) holds. We can thus apply Theorem 2.1 which shows that

$$\lim_{t\to\infty} Q_{t,\omega}(B) = 1, \quad \text{with} \tag{5.34}$$

$$B = \bigcup_{x \in \mathbb{L}_{\eta,t,\omega}} \{H(x) \leq t\} \cap \{T_{B(x,L_0\Delta_{\eta,t,\omega})} \circ \theta_{H(x)} > t - H(x)\}\,.$$

Since for large $t$, $\mathbb{L}_{\eta,t,\omega} \subseteq \mathcal{L}_{\eta_0,t,\omega}$ and $L_0\,\Delta_{\eta,t,\omega} \leq R_{\eta_0,t,\omega}$, we see that $B \subseteq C$, and thus our claim (5.18) follows. □

**Remark 5.5:**

**1)** One obtains immediate corollaries of (5.18) by replacing $R_{\eta_0,t,\omega}$ or $S_{\eta_0}(t,\omega)$ with larger quantities in the definitions (5.17) of the skeleton of near minima and (5.19) of the pinning event. In particular in view of (5.13) and (5.16), one can state a pinning effect with a deterministic

$$\begin{aligned} R(t) &= \exp\{(\log t)^{1-\chi}\}, \ \ \chi > 0, \ \text{small, or with} \\ S(t) &= \frac{t}{(\log t)^{\frac{2}{d}+\chi}}, \ \ \chi > 0, \ \text{small}\,. \end{aligned}$$

**2)** In the one-dimensional case, more can be said. We know that the random scales $S_\eta$, $\eta \in (0,1]$ are comparable to the scale $t(\log t)^{-3}$, in the sense of (3.21). This implies in turn a control on the size of the 'small scale' $R_{\eta_0,t,\omega}$. Indeed choosing $\epsilon \in (0,\frac{1}{2})$ and $\eta_0 \in (1-\epsilon, 1)$ we have for any $\rho \in (0,1)$

$$\varliminf_{t\to\infty} \mathbb{P}[R_{\eta_0,t,\omega} \leq (\log t)^{2+\epsilon}] \geq \varliminf_{t\to\infty} \mathbb{P}\Big[R_{\eta_0,t,\omega} \leq \frac{1}{\rho}\,(\log t)^{2+1-\eta_0}\Big]$$

but the r.h.s. of the above inequality tends to 1 as $\rho$ converges to 0, in view of (3.21) and (5.12), thus

$$\lim_{t\to\infty} \mathbb{P}[R_{\eta_0,t,\omega} \leq (\log t)^{2+\epsilon}] = 1, \quad \text{for } \eta_0 \in (1-\epsilon,1)\,. \tag{5.35}$$

As for the sets $\mathcal{L}_{\eta_0,t,\omega}$, they 'live' in the asymptotic scale $t(\log t)^{-3}$. Indeed $\mathcal{L}_{\eta_0,t,\omega} \subseteq \mathcal{L}_{\frac{1}{2},t,\omega}$, for $\eta_0 \in [\frac{1}{2},1)$, and for $\omega \in \Omega_1$, $\mathcal{L}_{\frac{1}{2},t,\omega} \subseteq B_{\alpha_0}(0,S_\eta(t,\omega))$

for large $t$, provided $\eta < \frac{1}{2}$. This together with (3.21) and a similar argument as in the proof of (3.3.47) of Theorem 3.6 shows that for $\eta_0 \in [\frac{1}{2}, 1)$

$$\text{(5.36)} \qquad \lim_{\rho \to 0} \varlimsup_{t \to \infty} \mathbb{P}\Big[\mathcal{L}_{\eta_0,t,\omega} \subset t(\log t)^{-3} \Big(\Big[-\frac{1}{\rho}, -\rho\Big] \cup \Big[\rho, \frac{1}{\rho}\Big]\Big)\Big] = 1 .$$

We can then introduce the event

$$C^{\epsilon,\rho} = \bigcup_{x \in \mathbb{Z} \cap [\frac{t}{(\log t)^3}([-\frac{1}{\rho},-\rho] \cup [\rho, \frac{1}{\rho}])]} \{H(x) \leq t\} \cap \{T_{(x-(\log t)^{2+\epsilon}, x+(\log t)^{2+\epsilon})} \circ \theta_{H(x)} > t - H(x)\} ,$$

where the path is pinned in scale $t(\log t)^{-3}$ within an interval of length $2(\log t)^{2+\epsilon}$. As a result of Theorem 5.3 (5.35) and (5.36), one deduces that for any $\epsilon > 0$,

$$\text{(5.37)} \qquad \lim_{\rho \to 0} \varliminf_{t \to \infty} \mathbb{E}[Q_{t,\omega}(C^{\epsilon,\rho})] = 1 .$$

□

**Complement: time estimates**

We shall now discuss estimates on how fast pinning occurs. We introduce a new random scale measuring the mutual distance between points of low local eigenvalue in the ball $B_{\alpha_0}(0, S_{\frac{1}{2}})$:

$$\text{(5.38)} \qquad \begin{aligned} D(t,\omega) = \ & \inf\{|x - x'|;\ x, x' \in B_{\alpha_0}(0, S_{\frac{1}{2}}(t,\omega)),\ |x - x'| > 3t^{\frac{1}{3}} \\ & \lambda_\omega(B(x, t^{\frac{1}{3}})),\ \lambda_\omega(B(x', t^{\frac{1}{3}})) \leq (\mu_{t,\omega} + S_{\frac{1}{2}}(t,\omega))/t\} , \end{aligned}$$

for $t > 0$, $\omega \in \Omega$. When no confusion occurs we shall simply write $D$ for $D(t,\omega)$. We know from Proposition 3.2 that $\mu_{t,\omega} + S_{\frac{1}{2}} \sim c(d,\nu)\, t(\log t)^{-2/d}$, when $\omega \in \Omega_0$, and $t \to \infty$. It thus follows from Theorem 4.4.7 that

$$\text{(5.39)} \qquad \mathbb{P}\text{–a.s.},\ \forall \zeta \in (0,1),\ t^\zeta = o(D(t,\omega)),\ \text{ as } t \to \infty .$$

Thus $D$ defines a 'large scale'. What is less clear and can for the time being only be substantiated in the one-dimensional situation is whether $D$ and $S_{\frac{1}{2}}$ are typically comparable. This difficulty is also related to (1.15) b) discussed in the introduction. For $\eta_0 \in [\frac{1}{2}, 1)$ and $\omega \in \Omega_1$, we define (see (5.6) and (5.2.73)):

$$\text{(5.40)} \qquad T_0 = 8\,t\, \frac{(1-\eta_0)}{1 \wedge (C_1\, D(t,\omega)/S_{\frac{1}{2}}(t,\omega))}, \quad \text{as well as}$$

$$\text{(5.41)} \quad \widetilde{C} = \bigcup_{x \in \mathcal{L}_{\eta_0,t,\omega}} \{H(x) \leq t \wedge T_0\} \cap \{T_{B(x, R_{\eta_0,t,\omega})} \circ \theta_{H(x)} > t - H(x)\} .$$

In contrast to the event $C$ in (5.19), $\widetilde{C}$ requires pinning to occur before time $T_0$. This is of course only of interest when $T_0 < t$, see (5.68) below.

**Theorem 5.6:** *For $\eta_0 \in [\frac{1}{2}, 1)$ and $\omega$ in the set of full $\mathbb{P}$-measure $\Omega_1$,*

$$\lim_{t\to\infty} Q_{t,\omega}(\widetilde{C}) = 1 . \tag{5.42}$$

**Proof:** The proof is analogous to that of Theorem 5.3. We choose some fixed $\omega \in \Omega_1$, $\frac{1}{2} \le \eta_0 < \eta < 1$, $\eta$ rational, and apply the coarse graining scheme of §4, with the value $\eta$. Our claim will follow once we show

$$\lim_{t\to\infty} Q_{t,\omega}(\widetilde{A}) = 1, \quad \text{where} \tag{5.43}$$

$$\widetilde{A} = \bigcup_{x\in \mathbb{L}_{\eta,t,\omega}} \{H(x) \le t \wedge T_0\} \cap \{T_{U(x)} \circ \theta_{H(x)} > t - H(x)\} . \tag{5.44}$$

We introduce some further notations complementing (5.28). For $G \in \mathcal{G}$, we write, see (5.28)

$$t_*(G) = r_{i_*(G)} , \tag{5.45}$$

so that $G \subseteq \{t_*(G) \le H(x_*(G)) \le t_*(G) + 1\}$,

$$\begin{aligned} &\lambda_{**}(G) = \\ &\min_{1\le i<i_*(G)} \lambda_\omega(C(x_i)) \wedge \frac{2c(d,\nu)}{(\log t)^{2/d}} \Big(\text{i.e. } \frac{2c(d,\nu)}{(\log t)^{2/d}} \text{ if } i_*(G) = 1\Big) , \end{aligned} \tag{5.46}$$

$$x_{**}(G) = \begin{cases} x_j , & \text{if } j = \min\{1 \le i < i_*(G),\, \lambda_\omega(C(x_i)) = \lambda_{**}(G)\} \\ & \text{and this set is not empty} , \\ 0 , & \text{otherwise} . \end{cases} \tag{5.47}$$

When this causes no confusion, we shall drop $G$ from the notation. With a variation on the argument used to prove Proposition 4.4, now splitting terms appearing in cost$(G)$ strictly before $i_*(G)$ and after $i_*(G)$, and keeping in mind that for large $t$, $\lambda_*(G) \overset{(5.31)}{\le} 2c(d,\nu)(\log t)^{-2/d}$, we have

$$\begin{aligned} &\varliminf_{t\to\infty} \frac{1}{S_1(t,\omega)} \inf_{\mathcal{G}'} \Big\{\text{cost}(G) - \Big[\frac{(1+\eta)}{2} (-\log e_0(0, x_{**}, \omega) \\ &+ a_0(x_{**}, x_*, \omega)) + t_*\, \lambda_{**} + (t - t_*)\lambda_*\Big]\Big\} \ge 0 . \end{aligned} \tag{5.48}$$

From (5.2.6), we also know that

$$-\log e_0(0, x_{**}, \omega) + a_0(x_{**}, x_*, \omega) \ge -\log e_0(0, x_*, \omega) .$$

With the help of the shape theorem 5.2.5 we obtain two distinct lower bounds on $\mathrm{cost}(G)$, $G \in \mathcal{G}'$. Namely, when $t$ is large uniformly in $G \in \mathcal{G}'$:

$$\mathrm{cost}(G) \geq L_1 \vee L_2 + o(S_1), \quad \text{where} \tag{5.49}$$

$$\begin{aligned} L_1 &= \frac{(1+3\eta)}{4}\, \alpha_0(x_*) + t_*\,\lambda_{**} + (t-t_*)\,\lambda_*\,, \\ L_2 &= \frac{(1+3\eta)}{4}\, (\alpha_0(x_{**}) + a_0(x_{**}, x_*, \omega)) + t_*\,\lambda_{**} + (t-t_*)\,\lambda_*\,. \end{aligned}$$

We consider the subclass $\mathcal{G}''$ of $\mathcal{G}'$:

$$\mathcal{G}'' = \left\{G \in \mathcal{G}';\ t_*(G) > \tfrac{T_0}{2}\right\}, \tag{5.50}$$

since $T_0 \to \infty$, and $G \subseteq \{H(x_*) \leq t_* + 1\}$, the same argument used in the proof of Lemma 5.4, see (5.29) - (5.30), shows that (5.43) follows from

$$\lim_{t\to\infty} Q_{t,\omega}\Big(\bigcup_{\mathcal{G}''} G\Big) = 0\,. \tag{5.51}$$

As a result of (5.49) and (5.3), using the coarse graining scheme as in (5.26), we see that for large $t$

$$\begin{aligned} Q_{t,\omega}\Big(\bigcup_{\mathcal{G}''} G\Big) \leq\ & \max\Big\{\exp\Big\{-\tfrac{1}{2}\, c(d,\nu)\, \frac{t}{(\log t)^{2/d}}\Big\}, \\ & \exp\Big\{\inf_{\mathcal{G}''} L_1 \vee L_2 + \mu_{t,\omega} + o(S_1)\Big\}\Big\}. \end{aligned} \tag{5.52}$$

For large $t$, any $\mathcal{C}(x_*(G))$ and $\mathcal{C}(x_{**}(G))$ are respectively contained in $B(x_*(G),\, R_{\eta_0,t,\omega})$ and $B(x_{**}(G),\, R_{\eta_0,t,\omega})$, and as in (5.16), uniformly in $G \in \mathcal{G}''$:

$$\mu_{t,\omega} \leq \alpha_0(x_*) + t\,\lambda_* + o(S_1)\,, \tag{5.53}$$

$$\mu_{t,\omega} \leq \alpha_0(x_{**}) + t\,\lambda_{**} + o(S_1)\,, \tag{5.54}$$

the second inequality being obvious when $\lambda_{**} = 2c(d,\nu)(\log t)^{-2/d}$ since $\mu_{t,\omega} \sim c(d,\nu)(\log t)^{-2/d}$.

We shall now seek an upper bound on

$$-\inf_{\mathcal{G}''} L_1 \vee L_2 + \mu_{t,\omega}\,. \tag{5.55}$$

The rough idea is that either the gap $\lambda_{**} - \lambda_*(\geq 0)$ is 'large', and in this case it does not pay to spend time around $x_{**}$, or the gap is 'small', but then by

(5.38) $x_*$ and $x_{**}$ are 'well separated' and it does not pay to make the extra move from $x_{**}$ to $x_*$. More precisely, for large $t$, uniformly in $G \in \mathcal{G}''$, either

$$t_*(\lambda_{**} - \lambda_*) \geq 2(1-\eta)\, S_\eta, \quad \text{and then} \tag{5.56}$$

$$\begin{aligned} -(L_1 \vee L_2) + \mu_{t,\omega} &\overset{(5.53)}{\leq} -L_1 + \alpha_0(x_*) + t\,\lambda_* + o(S_1) \\ &\leq \frac{3}{4}\,(1-\eta)\,\alpha_0(x_*) - t^*(\lambda_{**} - \lambda_*) + o(S_1) \\ &\leq -\frac{5}{4}\,(1-\eta)\,S_\eta + o(S_1)\,, \end{aligned} \tag{5.57}$$

since $\alpha_0(x_*) \leq S_\eta$ (see after (5.30)). Or

$$t_*(\lambda_{**} - \lambda_*) \leq 2(1-\eta)\, S_\eta\,, \tag{5.58}$$

and then since $G \in \mathcal{G}''$,

$$\frac{T_0}{2}\,(\lambda_{**} - \lambda_*) \leq 2(1-\eta)\, S_\eta \;\; \text{and thus}$$

$$\begin{aligned} \lambda_{**} &\leq \lambda_* + 4(1-\eta)\,\frac{S_\eta}{t} \times \frac{t}{T_0} \overset{(5.40)}{\leq} \lambda_* + \frac{S_\eta}{2t} \\ &\overset{(5.31)}{\leq} \frac{1}{t}\,\Big(\mu_{t,\omega} + \frac{1}{2}\,S_\eta\Big) + \frac{S_\eta}{2t} \leq \frac{1}{t}\,\Big(\mu_{t,\omega} + S_{\frac{1}{2}}\Big)\,. \end{aligned} \tag{5.59}$$

From Corollary 4.4.9, we deduce that for large $t$ and any $G \in \mathcal{G}''$ fulfilling (5.58):

$$\mathrm{diam}(\mathcal{C}(x_*),\, \mathcal{C}(x_{**})) \geq t^{\frac{1}{2}}\,,$$

since diam $\mathcal{C}(x_*)$ and diam $\mathcal{C}(x_{**})$ are smaller than $R_{\eta_0,t,\omega} < t^{\frac{1}{3}}$, the definition (5.38) implies

$$|x_* - x_{**}| \geq D\,. \tag{5.60}$$

Furthermore our application of (5.2.73) also shows that:

$$a_0(x_*, x_{**}, \omega) \geq C_1\, D\,. \tag{5.61}$$

We thus find that

$$\begin{aligned} -(L_1 \vee L_2) + \mu_{t,\omega} \leq\; & -\frac{(1+3\eta)}{4}\,(\alpha_0(x_{**}) + C_1\, D) \\ & -t_*\,\lambda_{**} - (t - t_*)\lambda_* + \mu_{t,\omega}\,, \end{aligned} \tag{5.62}$$

and either $\alpha_0(x_{**}) \geq 2S_\eta (\geq 2\alpha_0(x_*))$ and

$$-(L_1 \vee L_2) + \mu_{t,\omega} \overset{(5.53)}{\leq} -\frac{\eta}{4}\,S_\eta + o(S_1)\,, \tag{5.63}$$

or $\alpha_0(x_{**}) \le 2S_\eta$ and

$$-(L_1 \vee L_2) + \mu_{t,\omega} \overset{(5.54)}{\le} \frac{3}{2}\,(1-\eta)\,S_\eta + (t-t_*)(\lambda_{**}-\lambda_*) - C_1\,D + o(S_1)$$
$$\overset{(5.58)}{\le} (1-\eta)\,S_\eta\Big[\frac{2t}{t^*} - \frac{1}{2}\Big] - C_1\,D\,,$$

and since $G \in \mathcal{G}''$,

$$\le (1-\eta)\,S_\eta\Big[\frac{4t}{T_0} - \frac{1}{2}\Big] - C_1\,D\,.$$

Observe that

$$C_1\,D \ge \frac{C_1\,D \wedge S_{\frac{1}{2}}}{S_{\frac{1}{2}}}\cdot S_{\frac{1}{2}} \ge 8(1-\eta)\,S_\eta\,\frac{t}{T_0}\,, \tag{5.64}$$

thus we find

$$-(L_1 \vee L_2) + \mu_{t,\omega} \le -\frac{1}{2}\,(1-\eta)\,S_\eta + o(S_1)\,. \tag{5.65}$$

Collecting (5.57), (5.63), (5.65), we have shown that

$$\overline{\lim_{t\to\infty}}\ \frac{1}{S_\eta}\,(\inf_{\mathcal{G}''} -L_1 \vee L_2 + \mu_{t,\omega}) < 0\,, \tag{5.66}$$

which together with (5.52) implies (5.51). This concludes our proof of Theorem 5.6. □

In the one-dimensional case, it can be proven, see [Szn96a], I §5, that

$$\lim_{\rho\to 0}\ \underline{\lim_{t\to\infty}}\ \mathbb{P}\Big[\rho\,\frac{t}{(\log t)^3} < D(t,\omega)\Big] = 1\,. \tag{5.67}$$

This together with (3.21) and the fact that $D \le \text{const}\ S_{\frac{1}{2}}$ shows that $D$ and $S_{\frac{1}{2}}$ are comparable. In particular

$$\lim_{\rho\to 0}\ \underline{\lim_{t\to\infty}}\ \mathbb{P}\Big[\frac{S_{\frac{1}{2}}}{(C_1\,D)\wedge S_{\frac{1}{2}}} \le \frac{1}{\rho}\Big] = 1\,. \tag{5.68}$$

Analogous to what was explained in Remark 5.5, we can introduce for $\epsilon, \rho \in (0,1)$, the event:

$$\begin{aligned}\widetilde{C}^{\epsilon,\rho} = &\bigcup_{x\in\mathbb{Z}\cap[\frac{t}{(\log t)^3}[-\frac{1}{\rho},-\rho]\cup[\rho,\frac{1}{\rho}])]} \{H(x) \le \epsilon t\}\\ &\cap \{T_{(x-(\log t)^{2+\epsilon},\ x+(\log t)^{2+\epsilon})} \circ \theta_{H(x)} > t - H(x)\}\,,\end{aligned} \tag{5.69}$$

where the path is pinned in time $\epsilon t$ in scale $t(\log t)^{-3}$ within an interval of length $2(\log t)^{2+\epsilon}$. As an application of Theorem 5.6, (5.35), (5.36), (5.68) we have

**Corollary 5.7:** $(d = 1)$

*For any* $\epsilon > 0$,

$$\lim_{\rho \to 0} \varliminf_{t \to \infty} E[Q_{t,\omega}(\widetilde{C}^{\epsilon,\rho})] = 1 . \tag{5.70}$$

Further applications of the coarse graining method of Section 4 can be given. One can for instance show in any dimension that for typical $\omega$, and any $\eta \in [\frac{1}{2}, 1)$, the path $Z_.$ does not leave the ball $B_{\alpha_0}(0, S_\eta(t,\omega))$ up to time $t$ with $Q_{t,\omega}$ probability tending to 1 as $t$ tends to infinity, see [Szn96a], I. The coarse graining method can also be used for the proof of the quenched large deviation principle of $Z_t$ in critical scale $t(\log t)^{-2/d}$, see Chapter 7, §2.

## 6.6 Notes and References

The material in this chapter is adapted with some modifications from [Szn96a]. Qualitatively similar models appear in the physical literature in the study of 'flux lines in dirty-high-temperature superconductors', see Krug-Halpin Healy [KH93], Section 4.6.3 of the review article of Krug [Kru97], and Nelson-Vinokur [NV92]. This offers an alternative 'physical interpretation' of quenched Brownian motion in a Poissonian potential, which does not refer to trapping problems. On the mathematical side a discrete version of the model is analyzed in Khanin et al. [KMSS94].

# 7. Overview, further Results and Problems

In this chapter we give an overview of known results for Brownian motion and Poissonian obstacles, and discuss some of the connections with other topics of the literature on random media. We also mention a number of currently open problems.

## 7.1 Normalizing Constants and Diagonal Behavior

We begin by recalling the basic asymptotics for the normalizing constants associated to Brownian motion in a soft Poissonian potential. The respective normalizing constants for the quenched and annealed path measures in $d$-dimensional space, cf. Introduction, are:

$$S_{t,\omega} = E_0\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big], \quad \text{(quenched)}\,, \tag{1.1}$$

$$S_t = \mathbb{E} \otimes E_0\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big], \quad \text{(annealed)}\,. \tag{1.2}$$

In terms of the nonnegative bounded solution $u_\omega(t,x)$ of

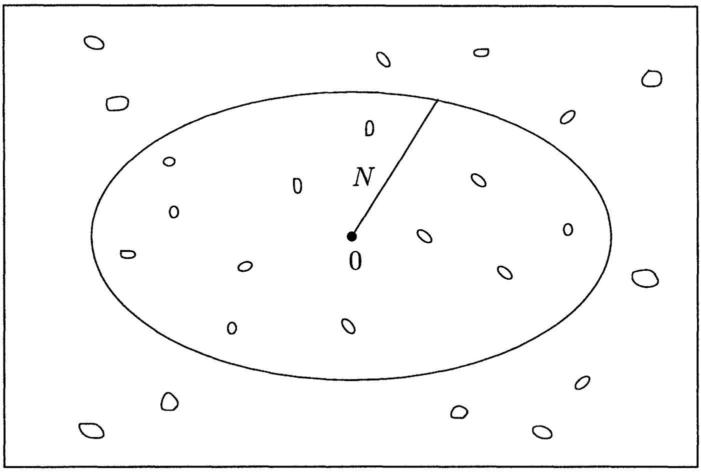
$$\begin{cases} \partial_t\, u_\omega & = \ \frac{1}{2}\, \Delta u_\omega - V(\cdot,\omega)\, u_\omega, \quad \text{on} \ (0,\infty)\times \mathbb{R}^d \\ u_\omega(0,x) & = \ 1\,, \end{cases} \tag{1.3}$$

these quantities can respectively be represented as:

$$S_{t,\omega} = u_\omega(t,0), \quad \text{and} \quad S_t = \mathbb{E}[u_\omega(t,0)], \quad \text{(see (1.4.55))} \tag{1.4}$$

The intuitive content of the difference between quenched and annealed points of views is captured by the formula :

$$\mathbb{P}\text{-a.s.}, \quad S_t = \lim_{N\to\infty} \frac{1}{|B(0,N)|} \int_{B(0,N)} u_\omega(t,x)\,dx, \quad \text{(see (4.5.5))}\,, \tag{1.5}$$




which for typical cloud configurations expresses $S_t$ as the limit of averages for $u_\omega(t,x)$ over a large ball centered at the origin, in contrast to $S_{t,\omega}$ which is the value at the origin $u_\omega(t,0)$. The respective principal asymptotic behaviors of the normalizing constants can be found in Theorem 4.5.1 and 4.5.3:

$$
(1.6)\qquad
\begin{aligned}
\mathbb{P}\text{-a.s.},\ S_{t,\omega} &= \exp\Big\{-c(d,\nu)\,\frac{t}{(\log t)^{2/d}}\,(1+o(1))\Big\},\\
&\quad \text{as } t\to\infty,\ \text{ where}\\
c(d,\nu) &= \lambda(B(0,R_0)),\ \text{with } R_0=\Big(\frac{d}{\nu\omega_d}\Big)^{1/d},\\
&= \lambda_d\Big(\frac{d}{\nu\omega_d}\Big)^{-2/d},
\end{aligned}
$$

$$
(1.7)\qquad
\begin{aligned}
S_t &= \exp\{-\widetilde{c}(d,\nu)\,t^{\frac{d}{d+2}}(1+o(1))\},\ \text{ as } t\to\infty,\ \text{ where}\\
\widetilde{c}(d,\nu) &= \inf_{U\text{ open}}\{\nu|U|+\lambda(U)\}\\
&= (\nu\omega_d)^{\frac{2}{d+2}}\Big(\frac{d+2}{2}\Big)\Big(\frac{2\lambda_d}{d}\Big)^{\frac{d}{d+2}},\\
&\quad \text{choosing } U=B(0,\widetilde{R}_0),\ \text{with } \widetilde{R}_0=\Big(\frac{2\lambda_d}{d\nu\omega_d}\Big)^{\frac{d}{d+2}}.
\end{aligned}
$$

Here $\nu>0$, denotes the intensity of the Poisson cloud, $\lambda(U)$ the principal Dirichlet eigenvalue of $-\frac{1}{2}\Delta$ in $U$, $\omega_d$ the volume of the unit ball in $\mathbb{R}^d$, and $\lambda_d$ its principal Dirichlet eigenvalue $\lambda(B(0,1))$.

The annealed asymptotics (1.7) goes back to Donsker-Varadhan [DV75c], who used large deviation theory for occupation local times of Brownian motion on a torus to derive (1.7). In this book both asymptotics are derived through the analysis of principal Dirichlet eigenvalues of $-\frac{1}{2}\Delta+V(\cdot,\omega)$ in large boxes, with the help of the method of enlargement of obstacles. The intuitive picture underlying the quenched asymptotics is that the Brownian path in (1.1) goes to some obstacle free ball of radius of order $R_0(\log t)^{1/d}$ occurring within distance slightly smaller than $t$ from the origin and spends the remaining time up to $t$ in this ball. Alternatively for the annealed asymptotics, an atypical obstacle-free ball of radius of order $\widetilde{R}_0\, t^{\frac{1}{d+2}}$ containing the origin is assumed to occur and the Brownian particle remains in this ball up to time $t$. Understanding up to what point these intuitive pictures do hold is of course quite another matter. We shall now describe some generalisations and sharpenings of (1.6), (1.7).

**Annealed asymptotics:**

A sharper version of (1.7) can be found in Theorem 4.5.6. However the question of finding for large $t$ an equivalent of $S_t$ in 'simple terms', rather than of $\log S_t$, is until now open. The asymptotic behavior (1.7) holds for hard obstacles modelled on a nonpolar compact set $C$ as well. In this case $S_t$ can be reexpressed in terms of the Wiener sausage $W_t^C = \bigcup_{0\le s\le t} Z_s - C$:

(1.8) $$S_t = E_0[\exp\{-\nu\,|W_t^C|\}], \text{ see (4.5.26)}\,.$$

The asymptotics (1.7) can be generalized to higher moments of $u_\omega(0,t)$, see (4.5.92):

(1.9) $$\begin{aligned} S_{t,p} &= \exp\{-p^{\frac{d}{d+2}}\,\widetilde{c}(d,\nu)(1+o(1))\}, \text{ as } t\to\infty, \text{ with} \\ S_{t,p} &\overset{\text{def}}{=} \mathbb{E}[u_\omega(t,0)^p], \;\; p\ge 1 \text{ integer}\,. \end{aligned}$$

The situation of slightly rarefied traps, i.e. when the intensity of the Poisson cloud has the form $\frac{\nu}{t^\mu}$, with $\mu\in(0,\frac{2}{d})$ is discussed in Theorem 4.5.7. In this case

(1.10) $$S_{t,\mu} = \exp\{-t^{\frac{d-2\mu}{d+2}}\,\widetilde{c}(d,\nu)(1+o(1))\}, \text{ as } t\to\infty\,,$$

with $S_{t,\mu}$ as in (4.5.93). By a scaling argument the situation of slightly rarefied traps can be viewed to be equivalent to that of 'not too rapidly shrinking traps', see (4.5.103), Bolthausen [Bol90], Sznitman [Szn90b], and also Bolthausen-den Hollander [BdH94]. The value $\mu=\frac{2}{d}$ is 'critical'; some results and heuristics for the principal asymptotics of $S_{t,\frac{2}{d}}$ are discussed at the end of Chapter 4. For more rarefied traps, i.e. when $\mu>\frac{2}{d}$, $S_{t,\mu}$ is expected to have similar principal asymptotic behavior as $\exp\{-\frac{\nu}{t^\mu}\,E_0[|W_t^C|]\}$, see also (4.5.102), (4.5.106).

If one replaces in (1.7) Brownian motion by an elliptic diffusion in divergence form with periodic coefficients, the principal asymptotic behavior of $S_t$ is the same as for the 'homogenized process', which is Brownian motion with a suitable constant covariance matrix $\overline{a}$. The asymptotics (1.7) holds with

$\widetilde{c}(d,\nu)$ replaced by $\widetilde{c}(d,\nu\sqrt{\det \overline{a}})$, see [Szn90a], II. Analogues of (1.7) for certain left invariant diffusions on stratified nilpotent Lie groups which enjoy a scaling property can also be found in [Szn90a], II. However the method employed uses a projection argument on a large compact quotient space, which restricts the generality of the result, cf. the discussion preceding (4.5.64).

For the simple random walk in the presence of killing traps independently thrown at each site of $\mathbb{Z}^d$, with probability $1-e^{-\nu}$, $\nu>0$, the annealed asymptotics corresponding to (1.7) has been derived by Donsker-Varadhan [DV75c]. Antal provided another proof in [Ant95], after developing a version of the method of enlargement of obstacles adapted to this discrete setting. Parallel to (1.8) one has

$$\begin{aligned} &S_t^{\text{disc}} = \\ (1.11)\qquad &E_0\left[\exp\{-\nu \text{ number of sites visited by the walk up to time } t\}\right], \end{aligned}$$

and the constant $\widetilde{c}_{\text{disc}}(d,\nu)$ playing the role of $\widetilde{c}(d,\nu)$ in (1.7) is simply obtained by replacing $\lambda_d$ with $\frac{1}{2d}\,\lambda_d$ in the definition of $\widetilde{c}(d,\nu)$, (Brownian motion with covariance matrix $\frac{I}{2d}$ can be viewed as the homogenized process). The case where 'bonds' replace 'sites' in (1.11) is also treated by Antal [Ant95]. The associated constant now corresponds to replacing $\nu$ with $\nu d$ and $\lambda_d$ with $\frac{1}{2d}\,\lambda_d$ in the definition of $\widetilde{c}(d,\nu)$.

The extension of (1.7) to Brownian motion on the Sierpinski gasket was studied by Pietruska-Paluba [PP91], who employed a version of the method of enlargement of obstacles. The obstacles are 'killing' points of a Poisson cloud with intensity measure $\nu\mu$, with $\mu$ the $x^{d_f}$-Hausdorff measure on the gasket and $d_f = \frac{\log 3}{\log 2}$ the 'Hausdorff dimension' of the gasket. It is shown in [PP91] that for suitable $c_1, c_2 > 0$, (independent of $\nu$):

$$\begin{aligned} &-c_2\,\nu^{\frac{2}{(d_s+2)}} \le \underline{\lim}_{t\to\infty}\; t^{-\frac{d_s}{(d_s+2)}} \log S(t) \le \\ (1.12)\qquad &\overline{\lim}_{t\to\infty}\; t^{-\frac{d_s}{d_s+2}} \log S(t) \le -c_1\,\nu^{\frac{2}{(d_s+2)}}, \end{aligned}$$

where $d_s = 2\,\frac{\log 3}{\log 5}$, is the 'spectral dimension' of the gasket, (which governs the singularity of the heat kernel on the diagonal). A better understanding of the variational problem

$$\inf_{U \text{ open}} \{\nu\mu(U) + t\lambda(U)\},$$

with $\lambda(U)$ the principal Dirichlet eigenvalue of $U$ for the Sierpinski Laplacian, would be desirable to improve (1.12). This question is related to a Faber-Krahn like problem:

What are the open sets with minimal $\lambda(U)$ and $\mu(U) = u$, as $u > 0$ varies?

Asymptotics like (1.7) also extend beyond the setting of Poissonian obstacles. The method of enlargement of obstacles is well adapted to handle such generalizations. It is for instance possible to replace the Poisson point process with a fairly general Gibbsian cloud, see [Szn93b]; this includes the possibility of certain hard core situations where balls centered at the points of the cloud are constrained not to overlap. It can be shown that for such Gibbsian clouds:

$$(1.13) \qquad S_t = \exp\{-t^{\frac{d}{d+2}}\, \widetilde{c}(d,p)(1+o(1))\}, \quad \text{as } t \to \infty\,,$$

where $p$ is the 'pressure' and $\widetilde{c}(\cdot,\cdot)$ is as in (1.7). Interestingly no assumption regarding the presence of phase transitions for the Gibbsian cloud is required for (1.13) to hold (the occurrence of such phase transitions for Gibbsian clouds is currently quite poorly understood).

**Quenched asymptotics:**

Unlike (1.7), the asymptotic behavior (1.6) does not readily extend to the hard obstacle situation. Qualitatively the possibility of such an extension strongly depends on whether the complement of the obstacle set 'percolates', i.e. has an unbounded component or not. If the complement of the obstacle set does not percolate, then (1.6) clearly breaks down: when the origin does not belong to the obstacle set, $S_{t,\omega}$ has an exponential decay in $t$ at a rate which is the principal Dirichlet eigenvalue of $-\frac{1}{2}\,\Delta$ in the connected component of the complement of the obstacle set containing the origin. On the other hand for hard ball obstacles in $\mathbb{R}^d$, $d \geq 2$, in a highly percolating regime of the complement of the obstacle set, it was shown in [Szn93a] that the asymptotics (1.6) persists on a set of positive $\mathbb{P}$-measure.

In the discrete setting, a sharper result was proved by Antal [Ant94]. For i.i.d. traps distributed along the edges of $\mathbb{Z}^d$, $d \geq 2$, with occurrence probability $1-e^{-\nu}$, Antal showed that when $e^{-\nu}$ is bigger than the critical probability for bond percolation on $\mathbb{Z}^d$, and the origin 'belongs to the infinite trap free cluster',

$$(1.14) \quad P_0[T > t] = \exp\Big\{ - c_{\text{disc}}(d,\nu)\, \frac{t}{(\log t)^{2/d}}\,(1+o(1))\Big\}, \quad \text{as } t \to \infty\,,$$

where $T$ denotes the first time at which the simple random walk crosses a 'killing bond', and $c_{\text{disc}}(d,\nu)$ is obtained by replacing $\lambda_d$ with $\frac{1}{2d}\,\lambda_d$ in the definition of $c(d,\nu)$ in (1.6). The upper bound part uses the method of enlargement of obstacles in a discrete setting, see [Ant95], and the lower bound part uses estimates on the distance inside the infinite cluster, see Antal-Pisztora [AP96].

Returning to the basic situation of soft Poissonian obstacles, it is possible to sharpen (1.6). This sharpening involves the random variational problem on the half-line, cf. Chapter 6 §3,

$$\mu_{t,\omega} = \inf_{u>0} \{u + t\, \lambda_\omega(B_{\alpha_0}(0,u))\}\,, \tag{1.15}$$

where $\lambda_\omega(B_{\alpha_0}(0,u))$ is the principal Dirichlet eigenvalue of $-\frac{1}{2}\,\Delta + V(\cdot,\omega)$ in the ball of radius $u$ in the $\alpha_0$-norm centered at the origin. The $\alpha_0$-norm is one of the quenched Lyapunov exponents of Chapter 5, see also §2 below. It is shown in Theorem 6.5.1 that

$$\mathbb{P}\text{-a.s., for } \eta \in (0,1),\ S_{t,\omega} = \exp\Big\{ - \mu_{t,\omega} + o\Big(S_1\Big(\frac{t}{\eta}\,,\,\omega\Big)\Big)\Big\}\,, \tag{1.16}$$

where $S_1(t,\omega)$ is a 'random scale', see (6.3.11), (6.3.16), which in particular is such that $S_1(t,\omega) \le \frac{t}{(\log t)^{\frac{2}{d}+\chi}}$ for large $t$, for a suitable deterministic $\chi > 0$. More is known in dimension 1, see (6.3.21), (6.3.22). The variational problem (1.16) captures the competition between distance to the origin and decrease of the principal eigenvalue $\lambda_\omega(B_{\alpha_0}(0,u))$.

**Diagonal behavior:**

The respective quenched and annealed large $t$ behavior on the diagonal of the kernel $r_\omega(t,\cdot,\cdot) = r_{\mathbb{R}^d,V(\cdot,\omega)}(t,\cdot,\cdot)$ (see (1.3.4) for the notation) of $e^{t(\frac{1}{2}\Delta - V)}$ are governed by similar asymptotics as (1.6), (1.7):

$$\mathbb{P}\text{-a.s. } r_\omega(t,0,0) = \exp\Big\{ - c(d,\nu)\, \frac{t}{(\log t)^{2/d}}\, (1 + o(1))\Big\},\ \text{ as } t \to \infty\,, \tag{1.17}$$

$$L(t) = \mathbb{E}[r_\omega(t,0,0)] = \exp\{-\tilde{c}(d,\nu)\, t^{\frac{d}{d+2}} (1 + o(1))\},\ \text{ as } t \to \infty\,, \tag{1.18}$$

see (4.5.24), (4.5.83). Further references in the literature are Carmona-Lacroix [CL91], Pastur-Figotin [PF92] and [Szn90a] in the case of (1.18), as well as [Szn95c] for (1.17). An important application of (1.18) concerns the so-called Lifshitz tail behavior of the density of states, which we now describe. One introduces the normalized counting measure

$$\ell^N_\omega(d\lambda) = \frac{1}{|B(0,N)|} \sum_{i \ge 1} \delta_{\lambda^{i,N}_\omega}\,, \tag{1.19}$$

where $\lambda^{i,N}_\omega$, $i \ge 1$ are the Dirichlet eigenvalues counted with multiplicity of $-\frac{1}{2}\,\Delta + V(\cdot,\omega)$ in $B(0,N)$. One can show, see (4.5.88), [CL91], [PF92], [Szn90a], that

$$\begin{aligned} &\mathbb{P}\text{-a.s., } \ell^N_\omega \text{ converges vaguely to a deterministic measure} \\ &\text{(the density of states) } \ell(d\lambda) \text{ on } [0,\infty), \\ &\text{and } L(t) = \int_0^\infty e^{-\lambda t} d\ell(\lambda), \text{ for } t > 0\,. \end{aligned} \tag{1.20}$$

An application of the exponential Tauberian theorem of de Bruinj-Minlos-Povzner, see Bingham-Goldie-Teugels [BGT87], p. 254, shows that

$$\ell([0,\lambda]) = \exp\Big\{ -\nu \Big|B\Big(0, \sqrt{\tfrac{\lambda_d}{\lambda}}\Big)\Big| \, (1+o(1))\Big\}, \text{ as } \lambda \to 0\,. \tag{1.21}$$

This strongly departs from the case with no obstacles (i.e. $V=0$), where the usual Weyl asymptotics governs the small $\lambda$ behavior of $\ell_{V=0}([0,\lambda])$:

$$\ell_{V=0}([0,\lambda]) \sim \frac{1}{(2\pi)^{d/2}\,\Gamma(\frac{d}{2}+1)}\,\lambda^{d/2}, \text{ as } \lambda \to 0\,. \tag{1.22}$$

The asymptotics (1.21) is the Lifshitz tail of the density of states. It measures the rarefaction of small eigenvalues of $-\frac{1}{2}\,\Delta + V(\cdot,\omega)$ in large boxes, due to the presence of obstacles, (see I.M. Lifshitz [Lif65], Section 3). The intuitive content of (1.21) is best captured after noticing that the expression $\exp\{-\nu\,|B(0,\sqrt{\frac{\lambda_d}{\lambda}})|\}$ represents the probability that the Poisson cloud puts no point in a ball of radius $\sqrt{\frac{\lambda_d}{\lambda}}$, i.e. a ball with principal Dirichlet eigenvalue $\lambda$ for $-\frac{1}{2}\,\Delta$. It would be desirable to devise a proof of (1.21) avoiding the Laplace transform and Tauberian step, and analyzing directly $\ell([0,\lambda])$ for small $\lambda$. One possible strategy might be to construct a clearing and forest picture in the spirit of Chapter 4, choosing $\lambda^{-1/2}$ as unit size for clearings, and then to compare $\ell([0,\lambda])$ with the similar quantity for the density of states obtained when imposing Dirichlet boundary conditions on the forest part.

**Magnetic Schrödinger operator:**

A version of the method of enlargement of obstacles was recently used by L. Erdös [Erd98] to study the Lifshitz tail effect for the magnetic Schrödinger operator:

$$H = \frac{1}{2}\,(-i\nabla - A)^2 + V(\cdot,\omega) \text{ in } \mathbb{R}^2\,, \tag{1.23}$$

with $A = (-\frac{Bx_2}{2}, \frac{Bx_1}{2})$ a vector potential generating the magnetic field $B = \operatorname{curl} A$, and $V(\cdot,\omega)$ Poissonian obstacles (soft or hard). The presence of the magnetic field induces some 'low lying eigenvalues' in the sense that the principal Dirichlet eigenvalue for $\frac{1}{2}\,(-i\nabla - A)^2$ in a disk of radius $R$ behaves as $\frac{B}{2} + \exp\{-\frac{B}{2}\,R^2(1+o(1))\}$, as $R\to\infty$, with $\frac{B}{2}$ the 'first Landau level', that is the bottom of the $L^2$-spectrum of $H$ in the absence of $V$. If $\ell(d\lambda)$ denotes the density of states associated to $H$ in (1.23), Erdös showed for instance in the case of hard obstacles made of disks of radius $a$, that

$$\ell\Big(\frac{B}{2} + [0,\lambda]\Big) \sim \frac{1}{\lambda^{\frac{2\pi\nu}{B}+o(1)}}\,, \text{ as } \lambda \to 0\,. \tag{1.24}$$

As in the case of (1.21), the proof goes through the analysis of the large $t$ behavior of the Laplace transform $L(t)$ of $\ell$. The relevant scale of 'clearings' when using the method of enlargement of obstacles turns out to be $\sqrt{\log t}$, which is much smaller than the natural scale $t^{\frac{1}{d+2}}$ showing up in the analysis of (1.18) or (1.7).

**Parabolic problem for the Anderson model:**

The investigation of the quenched and annealed large time asymptotic behavior of solutions of parabolic problems like:

$$(1.25) \qquad \begin{cases} \partial_t u &= \kappa\, \Delta u + \xi(x,\omega)\, u \ \text{ on } \ \mathbb{R}_+ \times \mathbb{Z}^d \\ u(0,x) &= 1\,, \end{cases}$$

with $\Delta$ the discrete Laplacian, $\kappa$ a positive constant, and $\xi(x)$, $x \in \mathbb{Z}^d$, i.i.d. unbounded variables, has been discussed from a physical point of view as a model of intermittency in Zeldovitch et al. [ZMRS88]. From a mathematical perspective the problem has been investigated in the works of Gärtner-Molchanov [GM90], [GM98], Carmona-Molchanov [CM95]. A 'true' unboundedness of the $\xi(\cdot,\omega)$ variables is usually required. As a result the quenched and annealed asymptotics of $u(t,0)$ are dominated by 'pockets', where $\xi(\cdot,\omega)$ takes large values, with typical length scale independent of $t$. This should be contrasted with (1.6) and (1.7), where the natural length scales of 'pockets' are respectively of order $(\log t)^{1/d}$ and $t^{\frac{1}{d+2}}$. Dependence on $t$ of the pockets occurs when the variables $\xi$ are 'almost bounded from above'. The critical behavior for tails of distributions of the $\xi$-variables corresponds to the double exponential distribution:

$$(1.26) \qquad P[\xi(0) > r] = \exp\{-e^{r/\rho}\}, \ \ r \in \mathbb{R}, \ \ \rho > 0\,.$$

Loosely speaking for fatter tails, when '$\rho = \infty$', 'pockets' consist of one single site. This is for instance the case of the Gaussian distribution. When $\xi(x,\omega)$ are i.i.d. $N(0,\sigma^2)$-distributed, then:

$$(1.27) \qquad \mathbb{P}\text{-a.s.}, \ \ u(t,0) = \exp\{\sqrt{2d\sigma^2 \log t}\, t - 2d\kappa\, t + o(1)\}, \ \ \text{as } \ t \to \infty\,,$$

for the quenched asymptotics, whereas

$$(1.28) \qquad \mathbb{E}[u(t,0)] = \exp\Big\{\frac{\sigma^2}{2}\, t^2 - 2d\, \kappa t + o(1)\Big\}, \ \ \text{as } \ t \to \infty\,,$$

for the annealed asymptotics, see [GM98].

Intuitively in (1.27) $\sqrt{2d\sigma^2 \log t}$ 'corresponds' to the $\mathbb{P}$-a.s. behavior of maxima of $\xi$ in a box of size $t$, whereas $\exp\{-2d\kappa t\}$ represents the probability for the simple random walk (in the Feynman-Kac representation of $u$) to remain at a high peak for a duration $t$. A similar interpretation can be given for (1.28).

On the other hand, in the case of the double exponential distribution, pockets consist of more than one site. It is shown in Gärtner-Molchanov [GM98], that

$$\text{(1.29)} \qquad \mathbb{P}\text{-a.s.,} \quad u(t,0) = \exp\Big\{\rho \log(\log t^d)\, t - 2d\kappa\chi\Big(\frac{\rho}{\kappa}\Big)\, t + o(t)\Big\}, \quad \text{as } t \to \infty\,,$$

in the quenched case and

$$\text{(1.30)} \qquad \mathbb{E}[u(t,0)] = \exp\Big\{H(t) - 2d\kappa\chi\Big(\frac{\rho}{\kappa}\Big)\, t + o(t)\Big\}, \quad \text{as } t \to \infty\,,$$

in the annealed case. Here $\rho \log(\log t^d)$ 'governs' the $\mathbb{P}$-a.s. maximum of $\xi(\cdot)$ in a box of size $t$, ($\psi(u) = \rho \log u$ is the inverse function of $r \to \log \frac{1}{\mathbb{P}[\xi > r]}$), and $H(t) = \mathbb{E}[e^{t\xi(0)}]$. The function $\chi(\cdot)$ is increasing $[0,1]$-valued and $2d\kappa\chi(\frac{\rho}{\kappa})$ is the minimal value for the bottom of $L^2$-spectra of $-\kappa\Delta + \psi(\cdot)$, with $\psi(\cdot) \geq 0$ such that $\Sigma\, e^{-\frac{\psi(x)}{\rho}} = 1$. This variational problem has a qualitatively similar role as the Faber-Krahn inequality (i.e. balls achieve minimum principal Dirichlet eigenvalue of the Laplacian among open sets of a given volume), which intervenes in (1.6), (1.7).

**Random walk in random environment and neutral pockets**

Qualitatively similar asymptotics to (1.6) and (1.7) are known to occur in the context of one-dimensional random walks in random environment, with positive or zero drift, when one investigates the slow regime of the walk, c.f. Dembo-Peres-Zeitouni [DPZ96], Gantert-Zeitouni [GZ97], Pisztora-Povel-Zeitouni [PPZ97], Pisztora-Povel [PP97].

Here the environment consists of i.i.d. variables, $\omega(x)$, $x \in \mathbb{Z}$, which take values in $[\frac{1}{2}, 1)$ and equal $\frac{1}{2}$ with positive $\mathbb{P}$-probability smaller than one. The numbers $\omega(x)$ and $1 - \omega(x)$ respectively determine the probability of performing a jump to site $x+1$ or site $x-1$, when the walk is in $x$ and evolves in the environment $\omega$. The quenched point of view is described by the laws $\mathbb{P}_{x,\omega}$ of the associated Markov chain starting in $x$. One also defines the annealed measures $P_x = \int P_{x,\omega}\, \mathbb{P}(d\omega)$. The above mentioned assumptions on the variables $\omega(x)$ correspond to the 'positive or zero drift' situation. In particular this ensures the existence of a nonvanishing velocity of the walk $(X_n)_{n \geq 0}$, cf. Solomon [Sol75]:

$$\text{(1.31)} \qquad \mathbb{P}_0\text{-a.s.,} \quad \frac{X_n}{n} \underset{n\to\infty}{\overset{\mathbb{P}\text{-a.s.}}{\longrightarrow}} v_\infty = \frac{1 - \mathbb{E}[\rho]}{1 + \mathbb{E}[\rho]}, \quad \text{with } \rho = \frac{1-\omega(0)}{\omega(0)}\,.$$

The behavior in the slow regime of the walk exhibits some similarities with Brownian motion among Poissonian obstacles. Heuristically, the slowdown of the walk both in the quenched and annealed situation makes use of the presence of large neutral intervals occurring in the medium, which play the role of resting places. Precise large deviation estimates on the occurrence of slow velocities have been derived in Pisztora-Povel [PP97] for the quenched measure and Pisztora-Povel-Zeitouni [PPZ97] for the annealed measure. In particular it is shown in these references that:

$$\begin{aligned} &\mathbb{P}\text{-a.s., for } 0 \le a < b < v_\infty,\ P_{0,\omega}\Big[\frac{X_n}{n} \in [a,b]\Big] = \\ (1.32)\qquad &\exp\Big\{-\frac{n}{(\log n)^2}\,(I(b)+o(1))\Big\},\ \text{as } n \to \infty, \text{ with} \\ &I(v) = \frac{1}{8}\Big(\pi \log \mathbb{P}\Big[\omega(0)=\frac{1}{2}\Big]\Big)^2\Big(1-\frac{v}{v_\infty}\Big), \text{ for } 0 \le v \le v_\infty\,, \end{aligned}$$

$$\begin{aligned} &\text{for } 0 \le a < b < v_\infty,\ P_0\Big[\frac{X_n}{n} \in [a,b]\Big] = \\ &\exp\{-n^{\frac{1}{3}}(J(b)+o(1))\},\ \text{ as } n \to \infty, \text{ with} \\ (1.33)\qquad &J(v) = \inf_{s>0}\Big\{\Big|\log \mathbb{P}\Big[\omega(0)=\frac{1}{2}\Big]\Big|\, s + \frac{\pi^2}{8s^2}\Big(1-\frac{v}{v_\infty}\Big)\Big\} \\ &\phantom{J(v)} = \frac{3}{2}\Big|\frac{\pi^2}{2}\log \mathbb{P}\Big[\omega(0)=\frac{1}{2}\Big]\Big|^{\frac{2}{3}}\Big(1-\frac{v}{v_\infty}\Big)^{\frac{1}{3}}, \\ &\text{for } 0 \le v \le v_\infty\,. \end{aligned}$$

## 7.2 Off-diagonal Behavior

We shall now review several results which in a loose sense are qualitatively related to the behavior for large $t$ of $r_\omega(t,0,x)$ and $\mathbb{E}[r_\omega(t,0,x)]$, when $x$ is 'far away from the origin'. As it turns out, it is useful to first get rid of the $t$-dependence of the problem.

**Shape theorems:**

An important role is played by the quantities, cf. (5.1.3):

$$(2.1)\qquad \begin{aligned} &e_\lambda(x,y,\omega) = E_x\Big[\exp\Big\{-\int_0^{H(y)} (\lambda+V)(Z_s,\omega)ds\Big\},\ H(y)<\infty\Big]\,, \\ &\lambda \ge 0,\ x,y \in \mathbb{R}^d,\ \omega \in \Omega\,, \end{aligned}$$

where $H(y)$ denotes the entrance time of $Z_.$ in $B(y)$ the closed ball of radius 1 centered at $y$. These functions have a natural supermultiplicative property, cf. (5.2.6):

$$(2.2)\qquad e_\lambda(x,y,\omega) \ge e_\lambda(x,z,\omega)\ \inf_{B(z)} e_\lambda(\cdot,y,\omega)\,,$$

which enables one to use the subadditive ergodic theorem. Moreover these functions contain substantial information on large deviation properties of $Z_t$ under the path measures $Q_{t,\omega}$ and $Q_t$ (cf. Introduction), as described further below. Closely related to (2.2) is the fact, cf. (5.2.3), that for $\mathbb{P}$-a.e. $\omega$,

$$(2.3) \qquad d_\lambda(x,y,\omega) = \max\Big(-\inf_{B(x)} \log e_\lambda(\cdot,y,\omega),\ -\inf_{B(y)} \log e_\lambda(\cdot,x,\omega)\Big)\,,$$

defines a distance function on $\mathbb{R}^d$, which induces the usual topology. However these metrics are rather poorly understood at the moment.

Shape theorems describe the respective quenched and annealed exponential decay of the functions $e_\lambda$ over large distances. They are proved by patching together applications in several directions of $\mathbb{R}^d$ of the subadditive ergodic theorem in the quenched situation, and of direct subadditivity in the annealed situation. As shown in Theorems 5.2.5 and 5.3.4, one can construct two families of norms $\beta_\lambda(\cdot) \le \alpha_\lambda(\cdot)$ on $\mathbb{R}^d$, such that:

$$(2.4) \qquad \begin{array}{l} \mathbb{P}\text{-a.s. for } M>0,\ \lim\limits_{x\to\infty}\ \sup\limits_{0\le\lambda\le M} \dfrac{1}{|x|}\Big|-\log e_\lambda(0,x,\omega)-\alpha_\lambda(x)\Big| = 0\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{(quenched shape theorem)}\,, \end{array}$$

$$(2.5) \qquad \begin{array}{l} \text{for } M>0,\ \lim\limits_{x\to\infty}\ \sup\limits_{0\le\lambda\le M} \dfrac{1}{|x|}\Big|-\log(\mathbb{E}[e_\lambda(0,x,\omega)])-\beta_\lambda(x)\Big| = 0\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{(annealed shape theorem)}\,. \end{array}$$

A further advantage of the functions $e_\lambda$ stems from the fact that $e_\lambda(\cdot,y,\omega)$ can be interpreted as an equilibrium potential, cf. (2.3.26). This provides an efficient tool to show robustness of the shape theorems (2.4), (2.5), namely:

(2.6) One can replace $e_\lambda(0,x,\omega)$ by either one of $e_\lambda(x,0,\omega)$, $\exp\{-d_\lambda(0,x,\omega)\}$ and $g_\lambda(0,x,\omega)$ (the $\lambda$-Green function) in statements (2.4), (2.5) .

The word 'shape' refers to the closed unit balls of the norms $\alpha_\lambda(\cdot)$ and $\beta_\lambda(\cdot)$. These balls describe in first approximation the respective low level sets of $e_\lambda(0,\cdot,\omega)$ and $\mathbb{E}[e_\lambda(0,\cdot,\omega)]$ after adequate rescaling, see Corollary 5.2.9, and 5.3.6. The terminology is inherited from first passage percolation theory, (see Hammersley-Welsh [HW10], Kesten [Kes86]). The connection of (2.4), (2.5) with the shape theorems of first passage percolation is best explained in a discrete setting. If $Z_.$ denotes the simple random walk on $\mathbb{Z}^d$, and $V(x,\omega)$, $x \in \mathbb{Z}^d$, are i.i.d. non-negative variables, one can define, in analogy with (2.1):

$$(2.7) \qquad e_\lambda(x,y,\omega) = E_x\Big[\exp\Big\{-\sum_{n=0}^{H(y)-1} (\lambda + V(Z_n,\omega))\Big\},\ H(y)<\infty\Big]\,,$$

for $\lambda \geq 0$, $x, y \in \mathbb{Z}^d$, with $H(y)$ the entrance time of $Z_.$ in $y$. Likewise one can consider the site 'passage time':

$$(2.8) \qquad \begin{aligned} T_\lambda(x, y, \omega) = \inf \Big\{ \sum_{i=0}^{n-1} \lambda + V(z_i, \omega);\ z_0 \dots z_n \text{ nearest neighbor} \\ \text{path from } x \text{ to } y \Big\}, \text{ for } \lambda \geq 0,\ x, y \in \mathbb{Z}^d\ , \end{aligned}$$

and one has the immediate comparison:

$$(2.9) \qquad T_\lambda(x, y, \omega) \leq -\log e_\lambda(x, y, \omega)\ .$$

Under the assumption $\mathbb{E}[V(0,\omega)^d] < \infty$, Zerner proved in [Zer98] the analogue of (2.4). Further he obtained a reinforcement of (2.4), which is very useful for applications to wave front propagations in random media, see [LT97] and (2.31) below:

$$(2.10) \qquad \mathbb{P}\text{-a.s., for } c > 0, \quad \lim_{c(|x| \vee |y|) \leq |x-y| \to \infty} \frac{-\log e_\lambda(x, y, \omega)}{\alpha_\lambda(y - x)} = 1\ ,$$

with the additional assumption that $\lambda > 0$, when $d = 2$.

The connection with first passage percolation comes through a semiclassical limit. Namely one picks $\beta > 0$, and denotes by $\alpha^\beta_{\lambda,\text{disc}}(\cdot)$ the norm coming out of the analogue of (2.4), with $e_\lambda$ as in (2.6) except that $(\lambda + V)(\cdot)$ is replaced by $\beta(\lambda + V)(\cdot)$. Further one denotes by $\mu_\lambda(\cdot)$ the first passage percolation semi-norms defined by the shape theorem relative to $T_\lambda$, (when $\lambda = 0$, $\mu_0(\cdot)$ can be degenerate, see Kesten [Kes86]). Then it can be shown that:

$$(2.11) \qquad \frac{\alpha^\beta_{\lambda,\text{disc}}(\cdot)}{\beta} \downarrow \mu_\lambda(\cdot), \quad \text{as } \beta \to \infty\ , \quad \text{c.f. [Zer98]}\ .$$

In the original setting of Poissonian obstacles, Wüthrich proved a qualitatively similar result in [Wüt97a], when $\lambda > 0$, and $W(\cdot)$ is continuous:

$$(2.12) \qquad \frac{\alpha^\beta_\lambda(\cdot)}{\sqrt{\beta}} \to \gamma_\lambda(\cdot), \quad \text{as } \beta \to \infty\ .$$

Here $\gamma_\lambda(\cdot)$ denotes the norm on $\mathbb{R}^d$, resulting from the shape theorem associated to the random Riemannian distance function:

$$(2.13) \qquad \begin{aligned} \rho_\lambda(x, y, \omega) = \inf \Big\{ \int_0^1 \sqrt{2(\lambda + V(z_s, \omega))}\ |\dot{z}_s|\, ds; \\ \text{with } z \text{ a } C^1\text{-curve such that } z(0) = x, z(1) = y \Big\}\ . \end{aligned}$$

Note however that in the continuous time and space setting, the rescaling factor in (2.12) is $\beta^{-\frac{1}{2}}$ instead of $\beta^{-1}$ in (2.11). The proof uses estimates of

Agmon [Agm82] on the exponential decay of solutions of elliptic P.D.E.s to derive adequate lower bounds on $-\log e_\lambda^\beta$.

**Lyapunov exponents**

In dimension bigger than one, no explicit formulas are known for the norms $\alpha_\lambda(\cdot)$, $\beta_\lambda(\cdot)$. However when $W(\cdot)$ is rotationally invariant, the Lyapunov norms are proportional to the Euclidean norm, but the proportionality factor is unknown. Unlike the case of first passage percolation, or when obstacles are absent, (i.e. $V = 0$, so that $\alpha_\lambda(\cdot) = \beta_\lambda(\cdot) = \sqrt{2\lambda}\ |\cdot|)$, $\alpha_0(\cdot)$ and $\beta_0(\cdot)$ are always norms in the context of soft Poissonian obstacles. In particular, we see from (2.13) that $g_0(0,x,\omega)$ for typical $\omega$, and $\mathbb{E}[g_0(0,x,\omega)]$, decay exponentially as $x$ tends to infinity. This fact is close to preoccupations of localization theory (see Carmona-Lacroix [CL91], Chapter 9), since 0 is precisely the bottom of the $L^2$-spectrum of $-\frac{1}{2}\,\Delta + V(\cdot,\omega)$ in $\mathbb{R}^d$.

In the one-dimensional situation, cf. (5.2.56), and in a slightly different context motivated by wave propagations in random media, cf. Freidlin [Fre85], p. 502, one can find the following formula for the quenched Lyapunov exponents:

$$\alpha_\lambda(1) = \mathbb{E}\Big[-\log\Big(E_0\Big[\exp\Big\{-\int_0^{H_{\{1\}}}(\lambda+V)(Z_s,\omega)ds\Big\}\Big]\Big)\Big]\,, \tag{2.14}$$

with $H_{\{1\}}$ the entrance time of $Z_.$ in 1. From this formula it can be deduced that $\alpha_\lambda(1)$ has an analytic extension to the half plane $Re\,\lambda > 0$, see also [Szn94b], p. 1682. In the higher dimensional case analyticity properties of $\alpha_\lambda$ are unknown.

For the annealed situation, it is shown in [Szn94b], that for continuous $W(\cdot)$:

$$\begin{aligned}\beta_\lambda(1) = \inf_{\mathbb{Q}}\Big\{&\mathbb{E}^{\mathbb{Q}}\Big[-\log\Big(E_0\Big[\exp\Big\{-\int_0^{H_{\{1\}}}(\lambda+V)(Z_s,\omega)ds\Big\}\Big]\Big)\Big]\\ &+H(\mathbb{Q}|\mathbb{P})\Big\}\,,\end{aligned} \tag{2.15}$$

where $\mathbb{Q}$ runs over laws of stationary point processes on $\mathbb{R}$ and $H(\mathbb{Q}|\mathbb{P})$ denotes the entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$. Further in the one-dimensional situation, for a singular $W$ of the form $W(\cdot) = c\delta_{\{0\}}$, with $c > 0$, Povel [Pov98] proved a generalization of the annealed shape theorem and showed that for all $c > 0$:

$$\beta^c_{\lambda=0}(1) = \nu - |\lambda_1^c| \in (0,\nu)\,, \tag{2.16}$$

where $\nu$ stands for the intensity of the Poisson cloud and $\lambda_1^c$ is the (negative) principal eigenvalue of $-\frac{1}{2}\,\Delta - \nu e^{-c|x|^2}$ in $\mathbb{R}^2$. In particular this shows that

$$\beta^c_{\lambda=0}(1) < \beta^{\text{hard obstacles}}_{\lambda=0}(1) = \nu, \text{ for all } c > 0\,.$$

It is unknown whether for soft obstacles, the inequality $\beta_0(1) < \nu$, always holds. This question turns out to be of importance for certain large deviation properties of $Z_t$ under the annealed path measure $Q_t$, see (2.28) below.

In arbitrary dimension, it is possible to show that when $W(\cdot)$ is 'sizeable', then $\alpha_\lambda(\cdot) > \beta_\lambda(\cdot)$, see [Szn95a], I, Theorem 1.4. Heuristically the proof uses the fact that in the hard obstacle case, when the complement of the obstacle set does not percolate, the 'natural $\alpha_\lambda$' explodes whereas $\beta_\lambda$ remains finite. On the other hand it is expected that when the dimension is large enough $\alpha_\lambda = \beta_\lambda$, if $W(\cdot)$ is 'small'. This belief is in part based on what happens in the case of time dependent potentials.

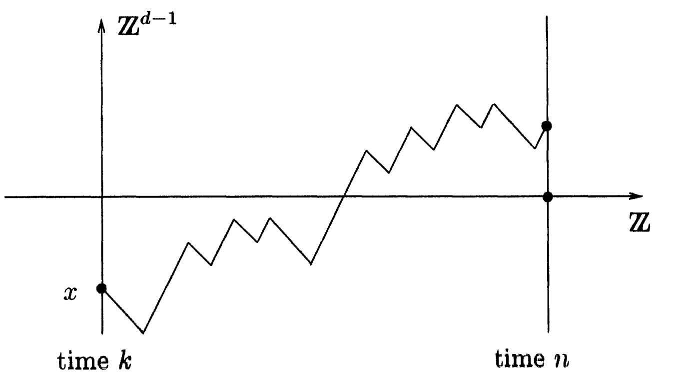
**Time dependent potentials:**

There are several variants in the literature, cf. Bolthausen [Bol89], Carmona-Molchanov [CM94], Imbrie [Imb88], Imbrie-Spencer [IS88], Kifer [Kif97], Sinai [Sin95], but one typical setting is as follows. One has time dependent potentials $V(i,z)$, $(i,z) \in \mathbb{Z} \times \mathbb{Z}^{d-1}$, which are i.i.d. variables, and one considers for $n \geq k$, $x, y \in \mathbb{Z}^d$,

$$S(k,x;n) = E_{k,x}\left[\exp\left\{\sum_{i=k}^{n} V(i,Z_i)\right\}\right], \quad \text{as well as} \tag{2.17}$$

$$S(k,x;n,y) = E_{k,x}\left[\exp\left\{\sum_{i=k}^{n} V(i,Z_i)\right\},\ Z_n = y\right] \tag{2.18}$$

where $Z_.$ denotes the simple random walk on $\mathbb{Z}^{d-1}$, starting in $x$ at time $k$. The associated path measures define the so-called directed polymers in random environment, see for instance [Bol89], [IS88], [Sin95], and (3.22) below. As a whole, the situation of high dimension and nearly constant potentials is better understood.



Specifically when $d-1 \geq 3$, and one assumes

$$\Lambda = \mathbb{E}[e^V] < \infty, \quad \text{together with} \tag{2.19}$$

$$\mathbb{E}\Big[\Big(\frac{e^V}{\Lambda} - 1\Big)^2\Big] \ \text{small enough}\,, \tag{2.20}$$

it can be shown by a martingale argument, see [Bol89] and also [Sin95], that

$$\frac{1}{\Lambda^{n+1}}\, S(0,0;n) \ \underset{L^2}{\overset{\mathbb{P}\text{-a.s.}}{\longrightarrow}} \ \varphi > 0\,. \tag{2.21}$$

Since $\Lambda^{n+1} = \mathbb{E}[S(0,0;n)]$, this implies that $S(0,0;n)$ has $\mathbb{P}$-a.s. the same asymptotic exponential growth rate as $\mathbb{E}[S(0,0;n)]$, namely $\log \Lambda$. These asymptotic growth rates which are qualitatively similar to the quenched and annealed point to hyperplane Lyapunov exponents of Corollary 5.2.11 and 5.3.7, thus coincide in the high-dimensional-low-disorder regime. Further Sinai showed in [Sin95] that:

$$S(k,x;n,y) = \Lambda^{n-k+1} p(n-k, y-x)[\varphi \circ \tau_{k,x} \cdot \psi \circ \tau_{n,y} + R^{n,y}_{k,x}]\,, \tag{2.22}$$

where $p(\cdot,\cdot)$ is the transition kernel of the simple random walk on $\mathbb{Z}^{d-1}$, $\tau$ denotes the space-time shift on potentials, $\varphi$ is defined in (2.21), $\psi$ is defined analogously to $\varphi$ after reversing the direction of time and

$$\lim_{n\to\infty} \ \sup_{\|y-x\|\leq A\sqrt{n-k}} \ \mathbb{E}[|R^{n,y}_{k,x}|] = 0, \quad \text{for any } A > 0\,. \tag{2.23}$$

The proof uses the $L^2(\mathbb{P})$-convergent perturbation expansion

$$\begin{aligned} &S(k,x;n,y) = \sum_{r\geq 0} \ \sum_{\substack{0\leq k_1<\dots<k_r\leq n \\ z_1,\dots,z_n}} \\ &p(k_1-k, z_1-x)\, h(k_1,z_1,\omega) \dots h(k_r,z_r,\omega)\, p(n-k_r, y-z_r), \\ &\text{with } h(k,z,\omega) = \frac{e^{V(k,z)}}{\Lambda} - 1\,. \end{aligned} \tag{2.24}$$

The main contribution in the asymptotic regime considered in (2.23) comes from terms where $r \leq \text{const}\log(n-k)$, and the sequence $x, z_1, \dots, z_r, y$ first stays near $x$, has one single large jump and then stays near $y$.

It is natural to wonder whether in the small-$W(\cdot)$-large-dimension regime for Poissonian soft obstacles, the asymptotic behavior for $\lambda > 0$, of $g_\lambda(0,y,\omega)$, as $y \to \infty$, exhibits similar features. The role of $p(\cdot)$ in (2.22) might then be replaced by $(-\frac{1}{2}\,\Delta + \lambda_*)^{-1}(\cdot,\cdot)$, with $\lambda_* > \lambda$ an adequate constant. This would hint at the possibility that $\alpha_\lambda(\cdot) = \beta_\lambda(\cdot) = \sqrt{2\lambda_*}\ |\cdot|$, in this situation. Coming back to time dependent potentials, let us mention that estimates on exponential growth rates in the strong (Gaussian) potential regime can be found in Carmona-Molchanov [CM94] and Carmona-Viens-Molchanov [CVM96].

**Large deviations:**

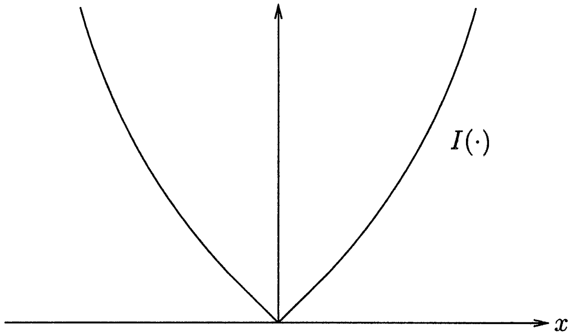
As mentioned at the beginning of this section, shape theorems and Lyapunov exponents are an important step in the derivation of several large deviation principles governing the position of $Z_t$ under the quenched and annealed path measures $Q_{t,\omega}$ and $Q_t$ (cf. Introduction). In the one-dimensional setting a similar strategy was used by Gärtner and Freidlin, who then applied their results to wave front propagation in one-dimensional random media, see Chapter 7 of Freidlin [Fre85] and also below. Specifically the above mentioned large deviation results for Brownian motion in a soft Poissonian potential are of the following nature.

In the quenched situation the scales of displacements $t(\log)^{-2/d}$ and $t$, play a special role. It will be convenient to write $\psi(t) << \varphi(t)$ in place of $\psi = o(\varphi)$, as $t \to \infty$, in what follows. For displacements from the origin, of order $\varphi(t)$, with either $\varphi(t) = t(\log t)^{-2/d}$, or $t(\log t)^{-2/d} << \varphi(t) << t$, or $\varphi(t) = t$, one has cf. [Szn94b], [Szn95c] and also Theorem 5.4.2:

$\mathbb{P}$-a.s. under $Q_{t,\omega}$, $Z_t/\varphi(t)$ satisfies a large deviation principle at rate $\varphi(t)$, with rate function

$$\text{(2.25)} \quad \begin{aligned} &\text{i)} \ \ \alpha_0(x), \text{ if } \varphi(t) = t(\log t)^{-2/d}\,, \\ &\text{ii)} \ \ \alpha_0(x), \text{ if } t(\log t)^{-2/d} << \varphi << t\,, \\ &\text{iii)} \ \ I(x) \stackrel{\text{def}}{=} \sup_{\lambda \geq 0} \,(\alpha_\lambda(x) - \lambda), \text{ if } \varphi(t) = t\,. \end{aligned}$$

As far as proofs are concerned, the most delicate parts correspond to the upper bounds in i) for the critical scale $t(\log t)^{-2/d}$, see [Szn95c], and the lower bounds in iii) for the other critical scale $t$, see [Szn94b] or Theorem 5.4.2. Incidentally the combination of Lemma 6.4.2 and Theorem 6.4.3, where a coarse graining of excursions systems of the path between clearings and forest is devised, provides up to minor changes a proof of the upper bound part of i). The function $I(x)$ turns out to be nonanalytic, for it coincides on a neighborhood of the origin with $\alpha_0(x)$, cf. [Szn95b], and is asymptotic to $\frac{|x|^2}{2}$ at infinity.




The large deviation principles (2.25) can be applied to investigate the off-diagonal behavior of $r_\omega(t,0,\cdot)$, see [Szn95c]:

$$\begin{aligned}
&\mathbb{P}\text{-a.s., for } v \neq 0,\ -\log r_\omega(t,0,\varphi(t)\,v) \ \sim \\
&-c(d,\nu)\,\frac{t}{(\log t)^{2/d}},\ \text{ if } \ \varphi << \frac{t}{(\log t)^{2/d}}\,, \\
&-(c(d,\nu)+\alpha_0(v))\,\frac{t}{(\log t)^{2/d}},\ \text{ if } \ \varphi(t) = \frac{t}{(\log t)^{2/d}}\,, \\
&-\alpha_0(v)\,\varphi(t),\ \text{ if } \ \frac{t}{(\log t)^{2/d}} << \varphi << t\,, \\
&\quad \text{and for } 0<\epsilon<1,\ \varphi(\cdot-\epsilon\varphi(\cdot)) \sim \varphi(\cdot)\,, \\
&-I(v)\,t,\ \text{ if } \ \varphi(t)=t\,, \\
&-\frac{1}{2}\,v^2\,\frac{\varphi(t)^2}{t},\ \text{ if } \ t << \varphi\,.
\end{aligned} \tag{2.26}$$

Here $c(d,\nu)$ is the same constant as in (1.6). The special roles of the scales $t(\log t)^{-2/d}$ and $t$, are displayed in (2.26). Intuitively for displacements at distance of order $t(\log t)^{-2/d}$, survival costs and travel costs come on the same footing. On the other hand for displacements of order $t$, the time $t$ may become too short and the cost of the displacement involves the norms $\alpha_\lambda(\cdot)$ with $\lambda > 0$, when $v$ is sizeable.

In the annealed situation the critical scales are instead $t^{\frac{d}{d+2}}$, and $t$, see [Szn95a], and Theorem 5.4.5. The analogue of (2.25) is now:

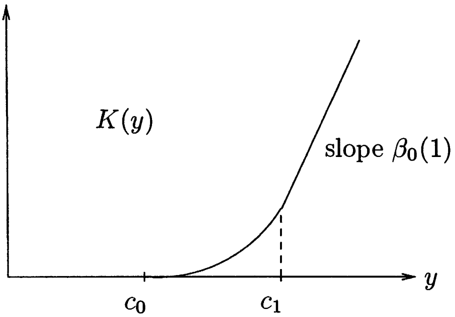
$$\begin{aligned}
&Z_t/\varphi(t) \text{ satisfies a large deviation principle at rate} \\
&\varphi(t), \text{ with rate function} \\
&\text{i) } \beta_0(x), \text{ if } \varphi(t) = t^{\frac{d}{d+2}} \text{ and } d \geq 2, \\
&\text{ii) } \beta_0(x), \text{ if } t^{\frac{d}{d+2}} << \varphi << t, \\
&\text{iii) } J(x) \stackrel{\text{def}}{=} \sup_{\lambda \geq 0}\,(\beta_\lambda(x) - \lambda), \text{ if } \varphi(t) = t.
\end{aligned} \tag{2.27}$$

The case i) of the critical scale $t^{1/3}$ in dimension one turns out to be singular. This comes from the fact that the clearings which are used as resting places by the process, have size of order $t^{\frac{1}{d+2}}$, which coincides with the scale of the displacement $t^{\frac{d}{d+2}}$, when $d = 1$. Thus in dimension one the clearings contribute in a sizeable fashion to the off-diagonal displacement in scale $t^{1/3}$ of the particle. As shown by Povel [Pov97b],

$Z_t/t^{1/3}$ satisfies a large deviation principle at rate $t^{1/3}$ with rate function:

$$K(y) = \begin{cases} 0, & |y| \in [0, c_1]\,, \\ \nu|y| + \dfrac{\pi^2}{2y^2} - \tilde{c}(1,\nu), & |y| \in [c_0, c_1]\,, \\ \nu c_1 + \dfrac{\pi^2}{2c_1^2} + \beta_0(1)(|y| - c_1) - \tilde{c}(1,\nu), & |y| \geq c_1\,, \end{cases} \tag{2.28}$$

here $\tilde{c}(1,\nu)$ is defined in (1.7), $c_0 = (\frac{\pi^2}{\nu})^{1/3}$, is the unique minimum of $\Psi(\ell) = \nu\ell + \frac{\pi^2}{2\ell^2}$, $\ell > 0$, and $c_1 = (\frac{\pi^2}{\nu - \beta_0(1)})^{1/3} \leq \infty$, is the location where $\Psi'(\ell) = \beta_0(1)$.




When $W(\cdot)$ is small it can be shown that $\beta_0(1) < \nu = \beta_0^{\text{hard obstacle}}(1)$, so that $c_1$ is finite. In fact (2.16) suggests that this is always the case. Just as in (2.26), the large deviation principles (2.27) - (2.28) can be used to derive the principal asymptotic behavior when $\varphi(t) \to \infty$, of $\log \mathbb{E}[r_\omega(t, 0, \varphi(t)v)]$, as $t \to \infty$, see [Szn95a], [Pov97b].

The above quenched and annealed large deviation theorems also have natural applications to the study of Brownian motion with a constant drift $h$ in a Poissonian potential. A special interest of this situation is the presence of a transition between a 'ballistic behavior' of the path for large $h$ and a 'localized behavior' for small $h$, see Corollary 5.4.8, 5.4.10, Eisele-Lang [EL87], Grassberger-Procaccia [GP88], Povel [Pov97b], [Pov95]. The transition between localized and ballistic behaviors occurs when $\alpha_0^*(h)$ (quenched case) or

$\beta_0^*(h)$ (annealed case) crosses the value 1. Here $\alpha_0^*(\cdot)$ and $\beta_0^*(\cdot)$ denote the dual norms to $\alpha_0(\cdot)$ and $\beta_0(\cdot)$.

In the discrete time and space setting, Zerner [Zer98] showed a reinforcement of (2.25) iii) with the help of (2.9). If $Q_{n,x,\omega}$ denotes the quenched path measure starting at $x$, and if one assumes (with no loss of generality) that 0 is the infimum of the support of the law of the nonnegative i.i.d. variables $V(y,\omega)$, $y \in \mathbb{Z}^d$, it is proved in [Zer97] under the assumption $\mathbb{E}[V(0,\omega)^d] < \infty$, that:

(2.29) On a set of full $\mathbb{P}$-probability, for any $x \in \mathbb{R}^d$, $\frac{Z_n}{n}$ under $Q_{n,[nx],\omega}$ satisfies a large deviation principle with rate function $I(\cdot - x)$ where $I(\cdot) = \sup_{\lambda \geq 0} (\alpha_\lambda(\cdot) - \lambda)$ ,

here $[nx]$ is 'the closest point' in $\mathbb{Z}^d$ to $nx$. Having a moving initial point for the path turns out to be quite handy for applications to problems of wave propagations.

**Wave front propagation in random media:**

Wave front propagation in a continuous one-dimensional random medium is for instance extensively discussed in Chapter 7 of Freidlin [Fre85]. In a discrete multidimensional context, Lee and Torcaso [LT97] investigated the large $t$ behavior of the solution $u$ with values in [0,1] of the lattice K.P.P. equation:

$$\begin{aligned} &\partial_t u = \Delta_{\text{disc}}\, u + (A - V(x,\omega))\, u(1-u), \; t > 0, \; x \in \mathbb{Z}^d, \\ &u(0,x) = \delta_0(x) \, , \end{aligned} \tag{2.30}$$

here $A$ is a positive constant and $V(x,\omega)$ are i.i.d. nonnegative variables such that $\mathbb{E}[V^d(0,\omega)] < \infty$, and 0 is the infimum of the support of the law of $V(0,\omega)$. With the help of Zerner's results [Zer98], Lee and Torcaso showed that for $x \neq 0$, on a set of full $\mathbb{P}$-measure:

$$\lim_{t\to\infty} \frac{1}{t} \log u(t,[tx]) = -(I(x) - A)_+ \, , \tag{2.31}$$

where the notation $[tx]$ is the same as in (2.29), $I(\cdot) = \sup_{\lambda \geq 0}(\alpha_\lambda(\cdot) - \lambda)$, and $\alpha_\lambda(\cdot)$ are the Lyapunov norms associated to the nonnegative potentials $V$. It is further conjectured that $u(t,[tx]) \overset{\mathbb{P}-\text{a.s.}}{\longrightarrow} 1$, as $t \to \infty$, when $I(x) < A$. Loosely speaking the wave front refers to the location where the solution drops from values close to 1 to values close to 0. Intuitively the above result shows that up to scaling by a factor $\frac{1}{t}$, the wave front lies near the level set $\{I(\cdot) = A\}$, when $t$ is large.

**Random walks in random environments**

The strategy of proving a large deviation principle via the derivation of shape theorems constructing Lyapunov exponents can be applied to multidimensional random walks in random environments. This is particularly interesting since general results for random walks in random environments are rare as soon as $d \geq 2$.

Specifically, the random environment is given by a family of $2d$-dimensional i.i.d. random vectors $(\omega(x,e))_{|e|=1}$, with positive components adding to 1. These components determine the probability of performing a jump from $x$ to $x+e$, for the random walk in the random environment $\omega$. The law of this discrete Markov chain starting from $x$ is denoted below by $P_{x,\omega}$. Zerner constructed in [Zer97] Lyapunov exponents $\alpha_\lambda(\cdot)$, $\lambda \geq 0$, via analogous shape theorems to (2.4) and (2.10). Here $e_\lambda$ has to be replaced with the quantity $E_{x,\omega}[\exp\{-\lambda H(y)\},\ H(y) < \infty]$, and the shape theorem assumes the ellipticity condition: $-\log \omega(0,e)$ has finite $d$-moments. In this new setting, $\alpha_\lambda(\cdot)$ is convex, homogeneous of degree one, but need not be symmetric, further $\alpha_0(x)$ may vanish for nonzero $x$.

Under the additional assumption that 0 belongs to the convex hull of the support of the law of $\sum_{|e|=1} \omega(0,e)\, e$, (the 'nestling property' in the terminology of [Zer97]), Zerner showed the analogue of the quenched large deviation principle (2.29):

$$\begin{aligned}(2.32)\qquad &\mathbb{P}\text{-a.s. for } x \in \mathbb{R}^d,\ \tfrac{Z_n}{n} \text{ under } P_{[nx],\omega} \text{ satisfies a large} \\ &\text{deviation principle with rate function } I(\cdot - x),\ \text{ where} \\ &I(\cdot) = \sup_{\lambda \geq 0} \left(\alpha_\lambda(\cdot) - \lambda\right).\end{aligned}$$

The nestling property as it turns out is equivalent to the fact that the probability of return to the origin has a subexponential decay:

$$(2.33) \qquad \mathbb{P}\text{-a.s.},\ \overline{\lim_n}\ \frac{1}{n} \log \mathbb{P}_{0,\omega}[Z_n = 0] = 0\,.$$

It ensures that one only needs to consider Lyapunov exponents $\alpha_\lambda(\cdot)$ with nonnegative $\lambda$, when proving the lower bound part of the large deviation principle (2.32).

**Fluctuations:**

In the physics literature related to growing surfaces and solids far from equilibrium, see Krug-Spohn [KS91], questions related to fluctuations around the limiting shape have been extensively analyzed. Mathematical advances came from the works of Alexander [Ale93], [Ale97] Kesten [Kes93], and Newman-Piza [NP95] in the context of first passage percolation. However fluctuations around the limiting shape are still rather poorly understood.

For Brownian motion in a Poissonian potential, typical examples of 'growing surfaces' are provided by balls of large radius in the $d_\lambda(\cdot,\cdot,\omega)$ metrics with center at the origin, or low level sets of the functions $e_\lambda(0,\cdot,\omega)$, or $g_\lambda(0,\cdot,\omega)$. Upper bounds on fluctuations are somewhat easier to derive than lower bounds. In the case of truncated Poissonian potentials, i.e. when $V$ is replaced by:

$$(2.34) \qquad V_M(x,\omega) = \Big(\int V(x-y)\,\omega(dy)\Big) \wedge M,\ x \in \mathbb{R}^d,\ \omega \in \Omega\,,$$

it was shown in [Szn96b], using a martingale technique as in Kesten [Kes93], that:

$$(2.35) \qquad \begin{array}{l} \text{for } |y| > 4,\ \mathbb{P}[|d_\lambda(0,\cdot,\omega) - \mathbb{E}[d_\lambda(0,y,\omega)]| \ge u\sqrt{|y|}] \\ \le c\,\exp\{-c'u\}\,,\ \ 0 \le u \le c''|y|, \text{ when } d \ge 3 \text{ or } \lambda > 0\,. \end{array}$$

When $d = 1$ or $2$, and $\lambda = 0$, similar estimates hold with $\sqrt{|y|}$ replaced by $\sqrt{|y|}\,\log|y|$. Fluctuations of $d_\lambda(0,\cdot,\omega)$ around $\alpha_\lambda(\cdot)$, were further investigated in the case of a rotationally invariant $W(\cdot)$ (hence $\alpha_\lambda(\cdot)$ is proportional to the Euclidean norm). It was shown in [Szn96b] that when $d \ge 3$ or $\lambda > 0$:

$$(2.36) \qquad \mathbb{P}\text{-a.s., } |d_\lambda(0,y,\omega) - \alpha_\lambda(y)| \le c\sqrt{|y|}\,\log|y|, \text{ when } |y| \text{ is large}\,.$$

In the case of dimension $d = 1$ or $2$, and $\lambda = 0$, $\log|y|$ is replaced by $(\log|y|)^2$. The bounds in (2.35) and (2.36) are far from the expected size of fluctuations, which for instance, when $d = 2$, and $\lambda > 0$, are supposed to be of rough order $|y|^{1/3}$ instead of $|y|^{1/2}$, and in high dimension might remain bounded, see Krug-Spohn [KS91].

Lower bounds on fluctuations are more difficult to obtain. At the level of variance analysis, following a qualitatively similar strategy as Newman-Piza [NP95], Wüthrich showed in [Wüt96] that for rotationally invariant $W(\cdot)$, when $d = 2$ and $\lambda \ge 0$,

$$(2.37) \qquad \chi \ge \frac{1}{8}\,,$$

provided the exponent $\chi$ is defined via

$$\chi = \sup\{\kappa \ge 0,\ \exists c > 0,\ \text{ with } Var(d_\lambda(0,y,\omega)) \ge c|y|^{2\kappa}, \text{ for } |y| \ge 1\}\,.$$

With a slightly different exponent $\chi^{(2)}$, measuring fluctuations around a median $M_\lambda(0,y)$ of $d_\lambda(0,y,\omega)$:

$$\chi^{(2)} = \inf\{\kappa \ge 0,\ \lim_{r\to\infty} \mathbb{P}[\sup_{x \in \overline{B}(0,r)} |d_\lambda(0,x,\omega) - M_\lambda(0,x)| \le r^\kappa] = 1\}\,,$$

Wüthrich was able to show in [Wüt97b] that when $d = 2$ and $\lambda \geq 0$:

$$\max(\chi, \chi^{(2)}) \geq \frac{1}{5} \,. \tag{2.38}$$

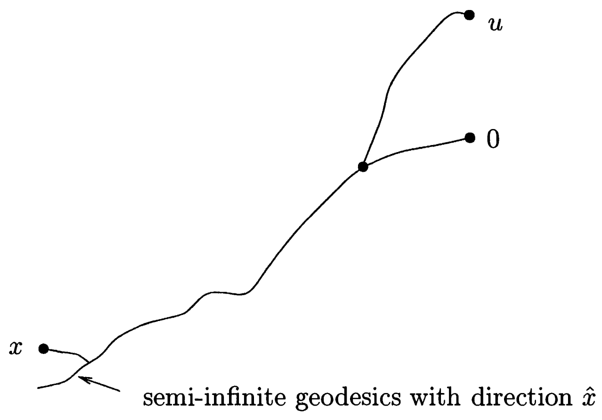
Fluctuation properties of $d_\lambda$ have a further interest. As discussed in §3 below, they are closely related to transversal fluctuations of crossing Brownian motion in a Poissonian potential.

**Geodesics:**

In the context of two-dimensional first passage percolation, when the passage times across bonds are i.i.d., strictly positive and exponentially integrable variables with continuous distribution, assuming a uniform curvature of the limiting shape (which unfortunately is presently unknown), Newman [New95], see also [New97], constructed a 'Busemann function', to borrow a terminology from hyperbolic geometry. Namely for $\mu$ a continuous probability on $S^1$, he showed that for $\mu$-almost every $\hat{x}$,

$$\mathbb{P}\text{-a.s.} \lim_{\substack{x\to\infty \\ \frac{x}{\|x\|}\to\hat{x}}} [T(u,x,\omega) - T(0,x,\omega)] = H_{\hat{x}}(u,\omega),\ u \in \mathbb{Z}^2 \,, \tag{2.39}$$

with $T(z, z', \omega)$ the passage time between $z$ and $z'$ in $\mathbb{Z}^2$. The proof uses semi-infinite minimizing geodesics, relative to the distance function $T(\cdot,\cdot,\omega)$. The argument displays a certain (weak) hyperbolicity of the model. Most of the minimizing path from 0 to $x$ or $u$ to $x$ goes through the semi-infinite geodesics with asymptotic directions $\hat{x}$, respectively beginning in 0 and $u$, and these semi-infinite geodesics are shown to coincide after a certain point.




In the context of Brownian motion among Poissonian obstacles and of the distance functions $d_\lambda(\cdot,\cdot,\omega)$, in the high potential or low dimension regime, it would be of great interest to develop similar results. An appropriate notion of quasi-geodesics is missing so far.

## 7.3 Asymptotic Behavior under Weighted Path Measures

We now discuss some results and open problems about the typical behavior of the canonical process under various path measures which naturally arise in, or relate to the subject of Brownian motion and Poissonian obstacles. We begin with the

**Pinning effect:**

The canonical path $Z_s, 0 \le s \le t$, under the quenched path measure (c.f. Introduction or Chapter 6 §1):

$$Q_{t,\omega} = \frac{1}{S_{t,\omega}} \exp\Big\{ - \int_0^t V(Z_s,\omega)ds\Big\} P_0 \,, \tag{3.1}$$

can be represented in a 'dynamical fashion', as the solution of the stochastic differential equation:

$$\begin{cases} dZ_s &= d\beta_s + \dfrac{\nabla u_\omega}{u_\omega}\,(t-s, Z_s)ds,\ 0 \le s \le t\,, \\ Z_0 &= 0\,, \end{cases} \tag{3.2}$$

with $\beta_.$ a $d$-dimensional Brownian motion, and $u_\omega$ the solution of the Cauchy problem (1.3), cf. Chapter 6 §1. It is explained in the Introduction, and described at length in Chapter 6, that a 'pinning effect' governs the behavior of the path $Z_.$ under $Q_{t,\omega}$, when $t$ is large and $\omega$ typical. In particular one can choose a small $\chi > 0$, so that for large $t$ the path gets attracted to the 'near minima' of the function:

$$x \longrightarrow F_t(x,\omega) = \alpha_0(x) + t\lambda_\omega(B(x,R_t))\,, \tag{3.3}$$

with $R_t = \exp\{(\log t)^{1-\chi}\}$ a 'small scale', and $\lambda_\omega(U)$ the principal Dirichlet eigenvalue of $-\frac{1}{2}\,\Delta + V(\cdot,\omega)$ in $U$, cf. Theorem 6.5.3 and Remark 6.5.5. Indeed one can define a skeleton of near minima of $F_t(\cdot,\omega)$:

$$\mathcal{L}_{t,\omega} = \Big\{x \in \frac{1}{\sqrt{d}}\,\mathbb{Z}^d,\ F_t(x,\omega) \le \mu_{t,\omega} + \frac{t}{(\log t)^{\chi+\frac{2}{d}}}\Big\}\,, \tag{3.4}$$

with $\mu_{t,\omega}$ defined in (1.15). It can be seen that when $t$ is large $\mu_{t,\omega}$ is very close to $\min_x\ F_t(x,\omega)$, see (6.5.15) - (6.5.16), and the near minima of $F_t(\cdot,\omega)$ lie almost at distance $t$ from the origin. Further one has

$$\mathbb{P}\text{-a.s.}\ \lim_{t\to\infty} Q_{t,\omega}(C) = 1\,, \tag{3.5}$$

provided $C$ denotes the event

$$(3.6) \qquad C = \left\{ \begin{array}{l} Z_. \text{ comes before time } t, \text{ within distance 1 of some} \\ x \in \mathcal{L}_{t,\omega} \text{ from which it then does not move further} \\ \text{away than distance } R_t \text{ up to time } t \end{array} \right\}.$$

As a matter of fact more precise estimates can be derived, replacing $R_t$ by the random scale $R_{1-\epsilon,t,\omega}$, see (6.5.12), and $t(\log t)^{-\chi-\frac{2}{d}}$ by $\epsilon\, S_1(t,\omega)$, see (6.3.11), with $\epsilon > 0$ arbitrarily small. One can also derive some informations on the time at which pinning occurs, cf. Theorem 6.5.6. Unfortunately when $d > 1$, the estimates on the above random scales are so far rather primitive. The Ansatz relating the size of fluctuations of principal eigenvalues of $-\frac{1}{2}\,\Delta + V(\cdot,\omega)$ in a large block and the scale $S_1(t,\omega)$, see (3.3.54) and [Szn97b], leads to expect that 'typically' $S_1$ has size $t(\log t)^{-1-\frac{2}{d}}$ and $R_{1-\epsilon,t,\omega}$ is smaller than $(\log t)^{2+\epsilon'}$, with $\epsilon' > \epsilon$.

This can only be proved in the one-dimensional situation, and yields in this case a statement about the pinning of the path $Z_.$ within time $\epsilon t$, in scale $\frac{t}{(\log t)^3}$, within an interval of length $(\log t)^{2+\epsilon}$, cf. Corollary 6.5.7.

**Confinement property:**

We now turn to the annealed situation, in the case of hard obstacles. That is we consider a nonpolar compact set $K$, see Chapter 2 §4, and the behavior for large $t$ of the path $Z_.$ under the annealed survival probability:

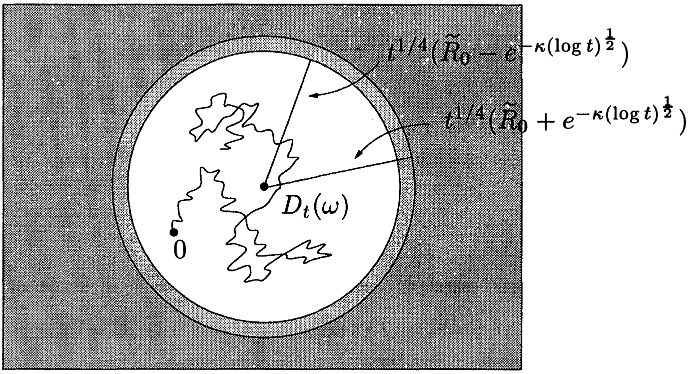
$$(3.7) \qquad Q_t(dw, d\omega) = \mathbb{P} \otimes P_0[\,\cdot\,|\, T > t]\,,$$

with $T$ the entrance time in the obstacle set $\bigcup_i x_i + K$, and $\omega = \sum_i \delta_{x_i}$. The annealed measure $Q_t$ favors the selection of a starting point $Z_0$ of the path, in a 'suitably good neighborhood', where survival to trapping is less difficult to achieve, and where the path tends to remain. As it turns out, the path is 'confined in scale $t^{\frac{1}{d+2}}$':

$$(3.8) \qquad \text{the laws of } \big(\sup_{0\le s\le t} |Z_s|\big)\, t^{-\frac{1}{d+2}} \text{ under } Q_t, t \ge 1, \text{ are tight}\,.$$

This is easily seen in dimension 1, but substantially more delicate in higher dimension. Specifically it was shown in [Szn91b] that

(3.9) when $d = 2$, there exists a $\kappa > 0$ and a measurable map $D_t(\omega)$ with values in $B(0, t^{\frac{1}{4}}\,(\widetilde{R}_0(d=2,\nu) + e^{-\kappa(\log t)^{\frac{1}{2}}}))$, such that with $Q_t$-probability tending to 1 as $t$ goes to infinity, $Z_{[0,t]}$ is included in $B(D_t(\omega), t^{\frac{1}{4}}(\widetilde{R}_0 + e^{-\kappa(\log t)^{\frac{1}{2}}}))$, and no obstacle falls in $B(D_t(\omega),\ t^{\frac{1}{4}}(\widetilde{R}_0 - e^{-\kappa(\log t)^{\frac{1}{2}}}))$.




It can also be shown that 'many obstacles' fall outside the larger disk. Thus under $Q_t$, the process $Z_.$ tends to live in a circular clearing of radius $\sim \widetilde{R}_0\, t^{\frac{1}{4}}$ containing the origin. The strategy of the proof is to show that in a large window of size $Lt^{\frac{1}{4}}$, a suitable coarse grained picture $t^{\frac{1}{4}}\, U_\epsilon$, $(\epsilon = t^{-\frac{1}{4}})$, of the complement of the obstacle set with overwhelming $Q_t$-probability satisfies:

$$\nu\, |U_\epsilon| + \lambda(U_\epsilon) \le \widetilde{c}(2,\nu) + \exp\{-\kappa'(\log t)^{\frac{1}{2}}\}\,. \tag{3.10}$$

The minimum of the functional $U \to \nu|U| + \lambda(U)$, see (1.7), is achieved for $U$ a disk of radius $\widetilde{R}_0$, and defines the constant $\widetilde{c}(2,\nu)$. Very much in the same fashion as in Theorem 4.4.1, it can be shown with the help of the Bonnesen isoperimetric inequality that $U_\epsilon$ is 'close' to a disk of radius $\widetilde{R}_0$. The proof then shows that with $Q_t$-probability close to 1, the process does not perform up to time $t$ a noticeable excursion outside this disk, and a slightly smaller concentric disk does not receive any obstacles. The argument for this last step resembles the proof of (4.4.2). The corrections $e^{-\kappa(\log t)^{\frac{1}{2}}}$ in (3.9), (3.10), could probably be sharpened to $t^{-\chi}$, with $\chi > 0$ small, with the help of the version of the method of enlargement of obstacles described in Chapter 4.

In the case of the simple random walk on $\mathbb{Z}^2$, Bolthausen proved in [Bol94] a version of the statement (3.7), with a rather different strategy using a refined version of Donsker-Varadhan's approach [DV75a], [DV75c]. The proof in [Szn91b] strongly relies on Bonnesen's isoperimetric inequality, which is special to the two-dimensional situation. However recently, Povel with the help of an isoperimetric inequality of Hall [Hal92] was able to prove in [Pov97a] the confinement property, in higher dimension. He showed that when $d \ge 3$, there exists $\chi > 0$ such that

$$\lim_{t\to\infty} Q_t[\sup_{0\le u\le t} |Z_u| \le 2t^{\frac{1}{d+2}}(\widetilde{R}_0 + t^{-\chi})] = 1\,. \tag{3.11}$$

Apart from (3.9) or (3.11), some further information on the fashion in which the clearing is distributed can be obtained, cf. Schmock [Sch90], when $d = 1$, [Szn91b], when $d = 2$, and the remark in Povel [Pov97a], when $d \geq 3$:

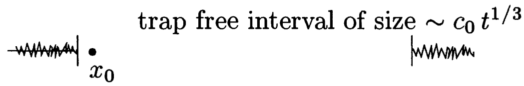
(3.12) as $t \to \infty$, $t^{-\frac{1}{d+2}} Z_{\cdot t^{\frac{2}{d+2}}}$ converges in law under $Q_t$ to the mixture with weight $\psi(x)/\int \psi$ of the laws of Brownian motion starting from 0 conditioned not to exit the time ball $B(x, \widetilde{R}_0)$, where $\psi$ is the principal Dirichlet eigenfunction of $-\frac{1}{2}\Delta$ in $B(0, \widetilde{R}_0)$.

**Path measure with constant drift:**

Povel investigated in the one-dimensional case the behavior of the canonical path $Z_.$ under the annealed path measure:

$$Q_t^h = \mathbb{P} \otimes P_0^h[\,\cdot\,|\,T > t\,]\,, \tag{3.13}$$

with $P_0^h$ the law of Brownian motion with a constant drift $h \in (0, \nu)$, and $T$ the entrance time in the support of the Poissonian cloud configuration $\omega$ with intensity $\nu$. The assumption $h \in (0, \nu)$ means that the drift is subcritical (cf. Theorem 5.4.9 and Corollary 5.4.10); this is the 'subballistic situation'. The presence of the drift makes it interesting to keep track of both a macroscopic and microscopic description of the path.




For the macroscopic point of view, which is a one-dimensional analogue of (3.12) in the presence of a drift, it is shown in [Pov95] that when $h \in (0, \nu)$ and $t$ tends to infinity:

(3.14) $t^{-\frac{1}{3}} X_{\cdot t^{\frac{2}{3}}}$ converges in law under $Q_t^h$ to the law of Brownian motion starting from 0, conditioned not to exit the interval $(0, c_0)$ with $c_0 = \left(\frac{\pi^2}{\nu - h}\right)^{\frac{1}{3}}$.

Loosely speaking, as an effect of the drift, the origin tends to be located near the left end of the trap free interval containing the origin. On the other hand the 'microscopic point of view' is illustrated by the following result of [Pov95], showing that when $h \in (0, \nu)$ and $t$ tends to infinity:

$Z_.$ under $Q_t^h$ converges in law to the mixture of Bessel-3 processes $X_.^a$

$$\begin{cases} dX_u^a &= d\beta_u + \dfrac{1}{\beta_u + a}\, du, \quad u \geq 0\,, \\ X_0^a &= 0\,, \end{cases} \tag{3.15}$$

which start in 0 and never hit $-a$, and the density of the mixture is $h^2 a\, e^{-ha}$.

In the absence of drift (i.e. when $h = 0$), it can be seen that $Z_.$ under $Q_t$ converges to Wiener measure, as $t$ tends to infinity. Intuitively the process does not feel the boundary of the trap free clearing in macroscopic time. The situation is different in the presence of a subcritical drift $h \in (0, \nu)$, and gives rise to both nontrivial macroscopic and microscopic behaviors. It would be quite interesting to study higher dimensional generalizations of such results. Let us also mention that the case of 'supercritical drifts' in arbitrary dimension, leading to ballistic behavior of the path is poorly understood. For results in this context see Theorem 5.4.9 and Corollary 5.4.10, Grassberger-Procaccia [GP88], Eisele-Lang [EL87], in the annealed case, and Theorem 5.4.7 and Corollary 5.4.8 in the quenched case.

**Crossing Brownian motion:**

Quenched an annealed path measures for Brownian crossings in a Poissonian potential were introduced in (6) and (7) of the Introduction. We merely discuss the quenched situation where the path measure of interest is:

$$\hat{P}^{\lambda}_{x,\omega} = \frac{1_{\{H(x)<\infty\}}}{e_\lambda(0,x,\omega)} \exp\Big\{ - \int_0^{H(x)} (\lambda + V)(Z_s, \omega) ds \Big\} P_0 \,, \tag{3.16}$$

with $e_\lambda$ and $H(x)$ defined in (2.1), and $\lambda \geq 0$. The measure $\hat{P}^{\lambda}_{x,\omega}$ describes the behavior of the Brownian path in the potential $\lambda + V(\cdot, \omega)$, conditioned to perform a (long) crossing from 0 to $x$. It is convenient to work with a rotationally invariant $W(\cdot)$ and a truncated Poissonian potential

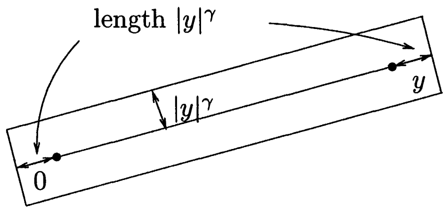
$$V(x,\omega) = \Big( \sum_i W(x - x_i) \Big) \wedge M \,, \tag{3.17}$$

in particular this leads to Lyapunov norms $\alpha^M_\lambda(\cdot)$ which are homothetic to the Euclidean norm. The behavior of the path $Z_.$ under $\hat{P}^{\lambda}_{x,\omega}$ and under another closely related path measure (corresponding to a point to hyperplane problem), was studied by Wüthrich in [Wüt97b] and [Wüt97c]. There is a natural interplay of 'transversal fluctuations of the path' and fluctuations of the distance function $d_\lambda(0, y, \omega)$, qualitatively similar to what happens in first passage percolation, cf. Newman-Piza [NP95].

The strength of transversal fluctuations of the path $Z_.$ under the measure $\hat{P}^{\lambda}_{x,\omega}$ for large $x$, can be measured with the help of the exponent $\xi_1$ introduced by Wüthrich in [Wüt97b]:

$$\xi_1 = \inf\{\gamma \geq 0, \lim_{x\to\infty} \mathbb{E}[\hat{P}^{\lambda}_{x,\omega}[A(x,\gamma)]] = 1\} \,, \tag{3.18}$$

where $A(y,\gamma)$ denotes the event that the path $Z_.$ up to $H(x)$ does not exit the truncated cylinder:




As for fluctuations of the distance function $d_\lambda(0,y,\omega)$, Wüthrich introduced in [Wüt97b] two exponents $\chi^{(1)}$ and $\chi^{(2)}$, with

$$\chi^{(1)} = \inf\{\kappa \geq 0,\ \lim_{y\to\infty} \mathbb{P}[\,|d_\lambda(0,y,\omega) - M_\lambda(0,y)| \leq |y|^\kappa] = 1\}\,, \tag{3.19}$$

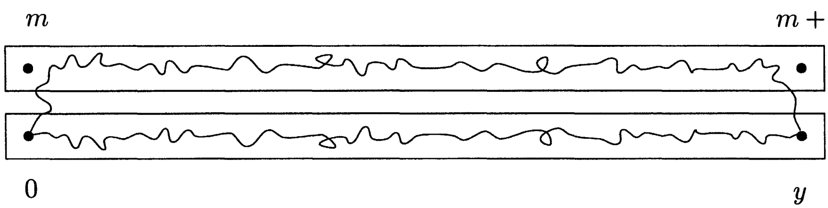
with $M_\lambda(0,y)$ a medium of $d_\lambda(0,y,\omega)$ and $\chi^{(2)}$ defined below (2.37), so that:

$$\chi^{(1)} \leq \chi^{(2)} \Big(\leq \frac{1}{2}\Big)\,.$$

He then showed that:

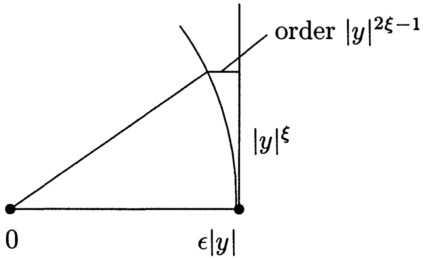
$$\frac{3}{4} \geq \frac{\chi^{(2)}+1}{2} \geq \xi_1 \geq \frac{\chi^{(1)} + \frac{\chi^{(1)}}{\chi^{(2)}}}{2}\,. \tag{3.20}$$

This goes in the direction of a prototypical scaling relation '$\chi = 2\xi - 1$', linking transversal fluctuations of the path to fluctuations of the distance function $d_\lambda$, cf. Krug-Spohn [KS91]. Heuristically it corresponds to the following picture:




the cylinders have width of order $|y|^\xi$

Loosely speaking the conjectured relation $\chi = 2\xi - 1$ corresponds to a balance between the gain of moving out to a 'better' cylinder of radius $|y|^\xi$ where the cost of crossing is smaller by an amount $|y|^\chi$, against the extra cost of the detour which due to the curvature of the limit shape (a ball here) is of order $|y|^{2\xi-1}$.




Of course this has some similar flavor with the Ansatz (3.3.54) which is supposed to govern the random scales discussed at the end of Chapter 3 or in §3 of Chapter 6. In the two dimensional case, it is believed that '$\chi = \frac{1}{2}$' and '$\xi = \frac{2}{3}$', in a broad variety of situations, cf. Krug-Spohn [KS91].

On a slightly different model where $H(y)$ in (3.16) is replaced by the entrance time in the hyperplane $\{x_1 = L\}$, and $\lambda$ is strictly positive, Wüthrich was also able to show in [Wüt97c] that the path $Z_.$ is 'superdiffusive', when $d = 2$, and that the analogous exponent $\widetilde{\xi}_1$ to (3.18), for the transversal fluctuations satisfies

$$(3.21) \qquad \widetilde{\xi}_1 \geq \frac{3}{5} \,.$$

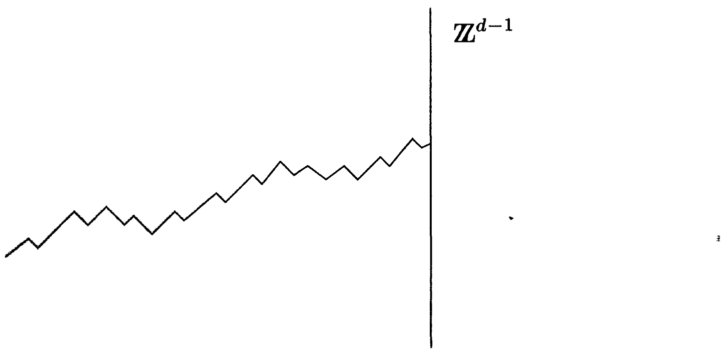
The strategy used to prove this result is similar to the one employed by Licea-Newman-Piza [LNP96] in the context of first passage percolation.

**Directed polymers in random potentials:**

In the notations of (2.17) the path measure of interest is

$$(3.22) \qquad Q_{k,x;n} = \frac{1}{S(k,x;n)} \exp\Big\{\sum_{i=k}^{n} V(i,Z_i)\Big\}\, P_{k,x}\,, \; k \leq n, \; x \in \mathbb{Z}^{d-1} \,,$$

where $P_{k,x}$ denotes the simple random walk on $\mathbb{Z}^{d-1}$, starting at $x$ at time $k$.



As already mentioned in §2 in the discussion of time dependent potentials, the case of $d \geq 4$ and 'small disorder' is better understood, cf. Bolthausen [Bol89], Imbrie-Spencer [IS88], Sinai [Sin95], and certain central limit theorems governing the law of $\frac{Z_n}{\sqrt{n}}$ can be derived. For instance when $d \geq 4$ and $h(i,x) = \frac{e^{V(i,x)}}{\Lambda} - 1$ take the value $\pm\epsilon$ with probability $\frac{1}{2}$, it is shown in [Bol89] that provided $\epsilon$ is small:

$$\text{(3.23)}\quad \begin{array}{l} \mathbb{P}\text{-a.s., as } n \to \infty,\ \frac{Z_n}{\sqrt{n}} \text{ converges in law under } Q_{0,0;n} \text{ to a} \\ \text{centered Gaussian distribution with covariance matrix } \frac{1}{d-1} \cdot I\,. \end{array}$$

On the other hand the case of low dimension or high potential is very poorly understood. In the case $d = 2$, it is believed based on physical predictions around the KPZ-equation, cf. Krug-Spohn [KS91], that the path $Z_.$ under $Q_{0,0;n}$ 'lives' in scale $n^{\frac{2}{3}}$, (as mentioned above $\frac{2}{3}$ is also the value conjectured for the exponent governing transversal fluctuations of crossing Brownian motion, when $d = 2$).

The simpler random sign model where $d = 2$, and $V(i,x) = \sigma X_i \operatorname{sign}(x)$, with $\sigma$ a positive constant, $X_i$ i.i.d. variables and $\mathbb{P}[X_i = \pm 1] = \frac{1}{2}$, has been studied by Sinai in [Sin93]. It is shown that in an appropriate sense, the path tends to remain 'close' to the 'natural interface $\{y = 0\}$', under the path measure $Q_{k,0;0}$, as $k \to -\infty$. One can wonder whether some environment dependent 'quasigeodesics' can be defined and play a similar role for instance in the case of the two dimensional directed polymer.

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