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Introduction

The starting point of this work has been the theorem of Ihara [16], according to which every torsion-free subgroup G of $\mathbf{SL}_2(\mathbf{Q}_p)$ is a *free* group. This striking result was at the time (1966) the only one known concerning the structure of discrete subgroups of p-adic groups.

Ihara's proof is combinatorial; it uses, in a somewhat mysterious way, a decomposition of $\mathbf{SL}_2(\mathbf{Q}_p)$ as an amalgam of two copies of $\mathbf{SL}_2(\mathbf{Z}_p)$. But topology suggests a natural way to prove that a group G is free: it suffices to make G act freely ("without fixed points") on a tree X; the group G may then be identified with the fundamental group $\pi_1(G\backslash X)$ of the quotient graph $G\backslash X$, a group which is obviously free. Interpreted from this point of view, Ihara's proof amounts to taking for X the tree associated with the amalgam mentioned above; this tree, *the tree of* $\mathbf{SL}_2$ *over the field* $\mathbf{Q}_p$, then appears as a very special case of a *Bruhat-Tits building* ([37], [38]), the p-adic analogue of the symmetric homogeneous spaces of real Lie groups.

One is therefore led to clarify the connections between "trees", "amalgams" and "$\mathbf{SL}_2$". This is the subject of the present work, which is based on part of a course given at the Collège de France in 1968/69; the rest of this course has been published elsewhere ([34]).

There are two chapters.

Chapter I begins with the definition of amalgams and the normal form of their elements. We then pass to trees (§2), and more precisely to the following question: what can be said of a group G acting on a tree X when we know the quotient graph $G\backslash X$ as well as the stabilizers G_x ($x \in \operatorname{vert} X$) and G_y ($y \in \operatorname{edge} X$) of the vertices and edges? We first treat two special cases:

that where G acts freely, i.e. where the G_x and G_y reduce to $\{1\}$; the group G is then free; this case, which is that of Ihara's theorem, also gives a simple proof of Schreier's theorem, according to which a subgroup of a free group is free (§3);

that where the graph $G\backslash X$ is a segment $x \overset{y}{\text{———}} x'$, in which case G may be identified with the amalgam $G_x *_{G_y} G_{x'}$; moreover, every amalgam of two groups is obtained in this way, and uniquely so; one thus gets a convenient equivalence between "amalgams" and "groups acting on trees with a segment as fundamental domain".

The general case is the object of §5. In the oral course I confined myself to suggesting its possibility, without verification. The definitive form of the definitions and theorems, as well as the proofs, are due to Hyman Bass. The main result says, roughly, that one can reconstruct G from $G\backslash X$ and the stabilizers G_x and G_y: it is the "fundamental group" of a "graph of groups" carried by $G\backslash X$; conversely, every graph of groups is obtained in this way, in an essentially unique way. Here again, one can give a "normal form" for the elements of G. The case where $G\backslash X$ is a loop leads to "*HNN* groups".

The §6 studies the relations between "amalgams" and "fixed points". It shows that certain groups, such as $\mathbf{SL}_3(\mathbf{Z})$, $\mathbf{Sp}_4(\mathbf{Z})$, etc., always have fixed points when they act on trees; this proves that they are *not* amalgams. An earlier version of this result has already been published (Lect. Notes in Math. no. 372, Springer-Verlag, 1974, pp. 633–640).

Chapter II begins with the definition and main properties of the tree X attached to a vector space V of dimension 2 over a local field K. The *vertices* of this tree are classes of *lattices* of V, two lattices being in the same class if they are homothetic, i.e. if they have the same stabilizer in $\mathbf{GL}(V)$; two vertices are adjacent if they are represented by nested lattices whose quotient is of length 1; here one recognizes the notion of "lattice neighbours" which occurs in the classical definition of the Hecke operator T_p. (For example, when K is the field $\mathbf{Q}_p$ and V the Tate module of an elliptic curve E, the vertices of X correspond to the elliptic curves which are p-isogenic to E, and one recovers the usual description of p-isogenies by means of a tree.) Once X is defined, one can apply the results of Chap. I, and obtain without difficulty the theorem of Ihara cited above, as well as a theorem of Nagao [19] giving the structure of $\mathbf{GL}_2(k[t])$ as an amalgam of $\mathbf{GL}_2(k)$ and the Borel group $B(k[t])$. The latter result is generalized in §2 to the following situation: we replace $k[t]$ by the affine algebra A of a curve $C^{\text{aff}} = C - \{P\}$ with a single point P at infinity. The group $\Gamma = \mathbf{GL}_2(A)$ then acts on the tree X corresponding to the valuation defined by P, and the quotient $\Gamma\backslash X$ has a simple interpretation in terms of *vector bundles of rank* 2 *over* C. Using known results on such bundles, one gets structure theorems for $\Gamma\backslash X$, and hence also for Γ; moreover, one obtains information on the *homology* of Γ and its *Euler-Poincaré characteristic*.

We now mention a few questions, connected with those treated in the text, on which the reader may usefully consult the Bibliography:

the theory of *ends* of discrete groups, and *Stallings' theorem* ([6], [7], [8], [30], [35]);

analysis on trees, where the Hecke operator "sum over the neighbouring vertices" replaces the Laplacian ([13], [14]);

the *zeta function* of a discrete subgroup of $\mathbf{SL}_2(\mathbf{Q}_p)$ with compact quotient, cf. [15]; this can be given a simple interpretation in terms of trees and finite graphs;

modular forms relative to the group Γ of Chap. II, §2 and its congruence subgroups (articles to appear by D. Goss, G. Harder, W. Li, J. Weisinger);

Mumford's theory of *p-adic Schottky groups* and the algebraic curves with which they are associated ([17], [18]);

cohomological properties of *S-arithmetic groups* ([28], [29], [32], [33], [34]), in particular for function fields (G. Harder, Invent. Math., 42, 1977, pp. 135–175).

This work could not have been done without the friendly and efficient help of Hyman Bass – in writing as much as in completion of the results. I have great pleasure in thanking him.

I also thank S. C. Althoen and I. M. Chiswell for the corrections they have sent me.

Chapter I. Trees and Amalgams

§1 Amalgams

1.1 Direct limits

Let $(G_i)_{i \in I}$ be a family of groups, and, for each pair (i,j), let F_{ij} be a set of homomorphisms of G_i into G_j.

We seek a group $G = \varinjlim G_i$ and a family of homomorphisms $f_i \colon G_i \to G$ such that $f_j \circ f = f_i$ for all $f \in F_{ij}$, the group and the family being *universal* in the following sense:

(*) If H is a group and if $h_i \colon G_i \to H$ is a family of homomorphisms such that $h_j \circ f = h_i$ for all $f \in F_{ij}$, then there is exactly one homomorphism $h \colon G \to H$ such that $h_i = h \circ f_i$. (This amounts to saying $\mathrm{Hom}(G, H) \simeq \varprojlim \mathrm{Hom}(G_i, H)$, the inverse limit being taken relative to the F_{ij}.)

We then say that G is the *direct limit* of the G_i, relative to the F_{ij}.

Proposition 1. *The pair consisting of G and the family $(f_i)_{i \in I}$ exists and is unique up to unique isomorphism.*

Uniqueness follows in the usual manner from the universal property (or, what comes to the same thing, the fact that G represents the functor $H \mapsto \varprojlim \mathrm{Hom}(G_i, H)$). Existence is easy. One can, for example, define G by generators and relations; one takes the generating family to be the disjoint union of those for the G_i; as relations, on the one hand the xyz^{-1} where x, y, z belong to the same G_i and $z = xy$ in G_i, on the other hand the xy^{-1} where $x \in G_i$, $y \in G_j$ and $y = f(x)$ for at least one $f \in F_{ij}$.

Example. Take three groups A, G_1 and G_2 and two homomorphisms $f_1 \colon A \to G_1$, $f_2 \colon A \to G_2$. One says that the corresponding group G is obtained by *amalgamating* A in G_1 and G_2 by means of f_1 and f_2; we denote it by $G_1 *_A G_2$. One can have $G = \{1\}$ even though f_1 and f_2 are non-trivial (cf. exerc. 2).

Application (Van Kampen Theorem). Let X be a topological space covered by two open sets U_1 and U_2. Suppose that U_1, U_2 and $U_{12} = U_1 \cap U_2$ are arcwise connected. Let $x \in U_{12}$ be a basepoint. Then the fundamental group $\pi_1(X; x)$ is obtained by taking the three groups $\pi_1(U_1; x)$, $\pi_1(U_2; x)$ and $\pi_1(U_{12}; x)$ and amalgamating them according to the homomorphisms

$$\pi_1(U_{12}; x) \to \pi_1(U_1; x) \quad \text{and} \quad \pi_1(U_{12}; x) \to \pi_1(U_2; x).$$

(For a generalization to the case of a covering by a family of open sets, cf. R. Crowell, *Pac. J. Math.* 9, 1959, pp. 43–50.)

Exercises

1) Let $f_1: A \to G_1$ and $f_2: A \to G_2$ be two homomorphisms and let $G = G_1 *_A G_2$ be the corresponding amalgam. We define subgroups A^n, G_1^n and G_2^n of A, G_1 and G_2 recursively by the following conditions:

$$A^1 = \{1\}, \qquad G_1^1 = \{1\}, \qquad G_2^1 = \{1\}$$

$$A^n = \text{subgroup of } A \text{ generated by } f_1^{-1}(G_1^{n-1}) \text{ and } f_2^{-1}(G_2^{n-1})$$

$$G_i^n = \text{subgroup of } G_i \text{ generated by } f_i(A^n).$$

Let A^∞, G_i^∞ be the unions of the A^n, G_i^n respectively. Show that f_i defines an *injection* $A/A^\infty \to G_i/G_i^\infty$ and that G may be identified with the amalgam of G_1/G_1^∞ and G_2/G_2^∞ along A/A^∞.
It follows (using the results of no. 1.2) that the kernel of $A \to G$ is A^∞ and that the kernel of $G_i \to G$ is G_i^∞.

2) Let $A = \mathbf{Z}$, $G_1 = \mathbf{PSL}(2, \mathbf{Q})$ and $G_2 = \mathbf{Z}/2\mathbf{Z}$. We take $f_1: A \to G_1$ to be an injection and $f_2: A \to G_2$ to be a surjection. Show that $G_1 *_A G_2 = \{1\}$.

1.2 Structure of amalgams

Suppose we are given a group A, a family of groups $(G_i)_{i\in I}$ and, for each $i \in I$, an *injective* homomorphism $A \to G_i$. We identify A with its image in each of the G_i. We denote by $*_A\, G_i$ the direct limit (cf. no. 1.1) of the family (A, G_i) with respect to these homomorphisms, and call it the *sum*[1] of the G_i with A *amalgamated.*

Example. $A = \{1\}$; the corresponding group is denoted $*\, G_i$; it is the *free product* of the G_i.

We now define the notion of a *reduced word.* For all $i \in I$ we choose a set S_i of right coset representatives of G_i modulo A, and assume $1 \in S_i$; the map $(a, s) \mapsto as$ is then a bijection of $A \times S_i$ onto G_i mapping $A \times (S_i - \{1\})$ onto $G_i - A$.

Let $\boldsymbol{i} = (i_1, \ldots, i_n)$ be a sequence of elements of I (with $n \geqslant 0$) satisfying the following condition:

$$(T) \qquad i_m \neq i_{m+1} \quad \text{for} \quad 1 \leqslant m \leqslant n-1.$$

---

[1] Translator's note: This object is a sum in the sense of category theory, even though it is usually called a "product". We shall avoid calling it either as far as possible, and refer only to "amalgams", except in the case $A = \{1\}$, where the term "free product" is too standard to be changed.

A *reduced word of type* $\boldsymbol{i}$ is any family

$$m = (a; s_1, \ldots, s_n)$$

where $a \in A$, $s_1 \in S_{i_1}, \ldots, s_n \in S_{i_n}$ and $s_j \neq 1$ for all j.

Finally, we denote by f (resp. f_i) the canonical homomorphism of A (resp. G_i) into the group $G = *_A G_i$.

Theorem 1. *For all $g \in G$ there is a sequence $\boldsymbol{i}$ satisfying (T) and a reduced word $m = (a; s_1, \ldots, s_n)$ of type $\boldsymbol{i}$ such that*

$$(*) \qquad g = f(a) f_{i_1}(s_1) \cdots f_{i_n}(s_n).$$

Furthermore, $\boldsymbol{i}$ *and* m *are unique.*

Remark. Th. 1 implies that f and the f_i are *injective* (which is not evident *a priori*). We can then identify A and the G_i with their images in G and the reduced decomposition $(*)$ of an element $g \in G$ is then written

$$g = as_1 \cdots s_n \quad \text{with} \quad a \in A,\ s_1 \in S_{i_1} - \{1\}, \ldots, s_n \in S_{i_n} - \{1\}.$$

We likewise see that $G_i \cap G_j = A$ if $i \neq j$. In particular, the sets $S_i - \{1\}$ are pairwise disjoint in G.

Proof (following a method of van der Waerden, *Amer. J. Math.*, 70, 1948; see also Bourbaki, AI, §7).

Let $X_{\boldsymbol{i}}$ be the set of reduced words of type $\boldsymbol{i}$ and let X be the disjoint union of the $X_{\boldsymbol{i}}$. We are going to make G act on X; in view of the universal property of G it will suffice to make each G_i act, and to check that the action induced on A does not depend on i.

Suppose then that $i \in I$, and let Y_i be the set of reduced words of the form $(1; s_1, \ldots, s_n)$, with $i_1 \neq i$. The sets $A \times Y_i$ and $A \times (S_i - \{1\}) \times Y_i$ are sent into X by the maps

$$(a, (1; s_1, \ldots, s_n)) \mapsto (a; s_1, \ldots, s_n)$$

$$((a, s), (1; s_1, \ldots, s_n)) \mapsto (a; s, s_1, \ldots, s_n).$$

It is clear that this yields a *bijection* of $A \times Y_i \cup A \times (S_i - \{1\}) \times Y_i$ onto X. But $A \cup A \times (S_i - \{1\})$ is identified with G_i, whence we have a bijection

$$\theta_i \colon G_i \times Y_i \to X.$$

We then make G_i act on $G_i \times Y_i$ in the obvious way:

$$g' \cdot (g, y) = (g'g, y)$$

and transfer this to an action of G_i on X by means of θ_i; its restriction to A is given by

$$a' \cdot (a; s_1, \ldots, s_n) = (a'a; s_1, \ldots, s_n),$$

which does not depend on i.

We have now defined an action of G on X. Furthermore, if $m = (a; s_1, \ldots, s_n)$ is a reduced word and if g is its image in G according to the formula $(*)$, the transform by g of the identity word $e = (1;\ )$ (relative to the empty sequence $\boldsymbol{i} = \emptyset$) is m itself; this is checked by induction on n. If we denote the mapping $g \mapsto g \cdot e$ by $\alpha\colon G \to X$ and the mapping defined by $(*)$ by $\beta\colon X \to G$ then $\alpha \circ \beta = id$, whence the *injectivity* of β, that is, the *uniqueness* of the reduced decomposition (which is the non-trivial part of the theorem). We can then identify X with $\beta(X) \subset G$, and it remains to see that $X = G$. It suffices to prove that $G_iX \subset X$ for each i (since this implies $GX \subset X$, whence $X = G$ because $1 \in X$), but this is immediate.

Remark. It is possible to state th. 1 without involving the sets S_i of representatives. For each $i \in I$ we put $G_i' = G_i - A$. If $\boldsymbol{i} = (i_1, \ldots, i_n)$ we denote by $G_{\boldsymbol{i}}'$ the quotient of $G_{i_1}' \times \cdots \times G_{i_n}'$ under the action of A^{n-1} defined by the formula

$$(a_1, \ldots, a_{n-1})(g_1, \ldots, g_n) = (g_1 a_1^{-1}, a_1 g_2 a_2^{-1}, a_2 g_3 a_3^{-1}, \ldots, a_{n-1} g_n)$$

(We thus have $G_{\boldsymbol{i}}' = G_{i_1}' \times^A G_{i_2}' \times^A G_{i_3}' \times \cdots \times^A G_{i_n}'$.) The map $(g_1, \ldots, g_n) \mapsto f_{i_1}(g_1) \cdots f_{i_n}(g_n)$ induces a map $f_{\boldsymbol{i}}\colon G_{\boldsymbol{i}}' \to G$ on passing to the quotient. With this notation, we have:

Theorem 2. *The maps f and $f_{\boldsymbol{i}}$ define a bijection of the disjoint union of A and the $G_{\boldsymbol{i}}'$ onto G.*

This is merely a reformulation of th. 1. In fact, if $\boldsymbol{i}$ is non-empty, every element of $G_{\boldsymbol{i}}'$ admits exactly one representation in the form $(a, s_1, \ldots, s_n)$ with $a \in A$ and $s_m \in S_{i_m} - \{1\}$.

Special case. In the case of the *free product* $(A = \{1\})$ we simply have $S_i = G_i$ and $G_i' = G_i - \{1\}$.

In particular, if we take G_i to be the infinite cyclic group generated by an element x_i, so that G is the *free group* $F((x_i)_{i \in I})$ on the family of elements $(x_i)_{i \in I}$, theorems 1 and 2 give the existence and unicity of the decomposition of an element g of $F((x_i)_{i \in I})$ in the form

$$g = x_{i_1}^{m_1} \cdots x_{i_n}^{m_n}, \qquad i_1 \neq i_2, \ldots, i_{n-1} \neq i_n, \qquad m_1 \neq 0, \ldots, m_n \neq 0.$$

Generalizations. In the above theorems the crucial case is that of a family $(G_i)_{i \in I}$ with *two elements*; the case of any finite set then follows by induction (for example, $G_1 *_A G_2 *_A G_3$ can be obtained by two successive amalgams, as $(G_1 *_A G_2) *_A G_3$); the case of an infinite set reduces to the finite case by the direct limit process.

We can likewise (again proceeding inductively) make more complicated amalgams. For example, we may be given a family of groups $(G_i)_{i \in I}$, a set D of pairs of distinct elements of I and, for all $\{i,j\} \in D$ a group A_{ij} and *injections* $A_{ij} \to G_i$, $A_{ij} \to G_j$. The group

$$G = \varinjlim(G_i, A_{ij})$$

then has a structure theorem analogous to th. 1 *provided that the graph* having I as its set of vertices and D as its set of edges *has no circuit* (cf. §2). (The case where I is finite is treated by induction on Card(I), using the existence of a "terminal vertex"; the general case is deduced by the direct limit process, as above.)

Exercise. Let A be a ring, $(B_i)_{i \in I}$ a family of rings, and $A \to B_i$ homomorphisms. We define the amalgamated product $B = *_A B_i$ of the B_i with respect to A in the obvious way. We make the following hypothesis:

The homomorphisms $A \to B_i$ are injective; furthermore, for each $i \in I$ there is a sub-A-bimodule B'_i of B_i such that $B_i = A \oplus B'_i$.

Show that B is the direct sum of A and the sub-A-bimodules $B'_{\boldsymbol{i}}$ isomorphic to $B'_{i_1} \otimes_A B'_{i_2} \otimes_A \cdots \otimes_A B'_{i_n}$ (where $\boldsymbol{i} = (i_1, \ldots, i_n)$ runs over the set of finite sequences of elements of I satisfying the condition (T)). The proof is the same as for th. 1: make B act on the direct sum of A and the $B'_{\boldsymbol{i}}$.)

[A slightly more general case may be found in P. M. Cohn, *J. of Algebra*, 1, 1964, p. 47–69.]

1.3 Consequences of the structure theorem

We retain the notations of th. 1 and 2, and let G denote the group $*_A G_i$. The *type* of an element $g \in G$ is the sequence $\boldsymbol{i} = (i_1, i_2, \ldots, i_n)$ satisfying (T), so that the reduced decomposition of g is of type $\boldsymbol{i}$. We have $\boldsymbol{i} = \emptyset$ if and only if g belongs to A; with the exception of this case, the type of g can be characterized by the relation $g \in G'_{\boldsymbol{i}}$ (with the notation of th. 2); it does not depend on the choice of S_i.

The integer n is called the *length* of g; we denote it by $l(g)$. We have $l(g) \leqslant 1$ if and only if g belongs to one of the G_i.

Finally, an element g of length $\geqslant 2$ is called *cyclically reduced* if its type $\boldsymbol{i} = (i_1, \ldots, i_n)$ is such that $i_1 \neq i_n$.

Proposition 2. *a) Every element g of G is conjugate to a cyclically reduced element, or an element of one of the G_i.*

b) Every cyclically reduced element is of infinite order.

We prove a) by induction on $l(g)$. If $l(g) \geqslant 2$ and if g is not cyclically reduced, let $\boldsymbol{i} = (i_1, \ldots, i_n)$ be its type. We have $i_1 = i_n$. We can write g in the form

$$g = g_1 \cdots g_n \quad \text{with} \quad g_1 \in G'_{i_1}, \ldots, g_n \in G'_{i_n}$$

and then

$$g_1^{-1} g g_1 = g_2 \cdots g_{n-1}(g_n g_1) \quad \text{with} \quad g_n g_1 \in G_{i_1}.$$

We conclude that $g_1^{-1}gg_1$ is of length $n-1$ (if $g_ng_1 \notin A$) or of length $n-2$ (if $g_ng_1 \in A$). In view of the induction hypothesis, this element is conjugate to a cyclically reduced element or an element of one of the G_i, so the same is true of g.

As for b), if g is cyclically reduced of type $\boldsymbol{i} = (i_1, \ldots, i_n)$ it is clear that g^2 is of type $2\boldsymbol{i} = (i_1, \ldots, i_n, i_1, \ldots, i_n)$ and hence of length $2n$; more generally g^k ($k \geqslant 1$) is of length kn, and therefore not equal to 1.

Corollary 1. *Every element of G of finite order is conjugate to an element of one of the G_i.*

This follows from a) and b).

Remark. One can prove a slightly more precise result (by the same type of argument): every finite subgroup of G is conjugate to a subgroup of one of the G_i, cf. no. 4.3, th. 8.

Corollary 2. *If the G_i are torsion-free, so is G.*

(Recall that a group is said to be *torsion-free* if its elements $\neq 1$ are of infinite order.)

Proposition 3. *For all $i \in I$, let H_i be a subgroup of G_i. Suppose that $B = H_i \cap A$ is independent of i. Then the homomorphism $*_B H_i \to *_A G_i$ induced by the injections $H_i \to G_i$ is injective.*

(We can then identify the subgroup of G generated by the H_i with the amalgam $*_B H_i$.)

Let $i \in I$ and let T_i be a system of right coset representatives of H_i modulo B, including 1. Because $A \cap H_i = B$ we can extend T_i to a system S_i of right coset representatives of G_i modulo A. It is then clear that every reduced decomposition in $*_B H_i$ relative to T_i gives a reduced decomposition in $*_A G_i$ relative to S_i; whence the proposition. (Variant: the maps $H'_i \to G'_i$ are injective.)

Corollary. *If $H_i \cap A = \{1\}$ for all $i \in I$, the subgroup of G generated by the H_i can be identified with the free product $* H_i$.*

We now give an application to *free products*:

Proposition 4. *Let A and B be two groups, and let R be the kernel of the canonical homomorphism $A * B \to A \times B$. The group R is a free group, with free basis the set X of commutators $[a, b] = a^{-1}b^{-1}ab$ with $a \in A - \{1\}$, $b \in B - \{1\}$.*

Proof (following Magnus-Karrass-Solitar, [3], p. 196, exerc. 24). Let S be the subgroup of $A * B$ generated by X. If $a' \in A$ and $[a, b] \in X$ we have

$$a'^{-1}[a, b]a' = a'^{-1}a^{-1}b^{-1}aba' = [aa', b] \cdot [a', b]^{-1} \in S$$

whence $a'^{-1}Sa' = S$. In the same way one shows that $b'^{-1}Sb' = S$ if $b' \in B$, whence the fact that S is normal in $A * B$. It is clear that $(A * B)/S$ has the universal property which characterizes $A \times B$; so $S = R$ and we have that R is generated by X.

It remains to see that X is a free subset of R. Now we have the following lemma:

Lemma 1. *Let X be a subset of a group R. The following two conditions are equivalent:*

(i) *X is a free subset of R.*

(ii) *For every finite sequence $x_1, \ldots, x_n$ $(n \geqslant 1)$ of elements of $\{\pm 1\}$ such that we do not have $x_i = x_{i+1}$ and $\varepsilon_i = -\varepsilon_{i+1}$ for any i, the product $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$ is not equal to* 1.

Indeed, one knows (this follows easily from th. 1) that every element $\neq 1$ in the free group $F(X)$ with basis X can be written uniquely in the above form $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$. Then condition (ii) simply says that the homomorphism $F(X) \to R$ is injective.

In view of this, it will suffice to prove that the set X of the $[a, b]$ satisfies (ii). Suppose then that $[a_1, b_1], \ldots, [a_n, b_n] \in X$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$ are such that we do not simultaneously have $a_i = a_{i+1}$, $b = b_{i+1}$ and $\varepsilon_i = -\varepsilon_{i+1}$. We have to prove that the element

$$g = [a_1, b_1]^{\varepsilon_1} \cdots [a_n, b_n]^{\varepsilon_n}$$

is $\neq 1$. To be more precise, we are going to show that

(1) $l(g) \geqslant n + 3$.

(2) If $\varepsilon_n = +1$ (resp. -1) the reduced decomposition of g terminates in $a_n b_n$ (resp. in $b_n a_n$).

We proceed by induction on n, the case $n = 1$ being trivial. We assume, to fix ideas, that $\varepsilon_{n-1} = +1$; the reduced decomposition of

$$g' = [a_1, b_1]^{\varepsilon_1} \cdots [a_{n-1}, b_{n-1}]^{\varepsilon_{n-1}}$$

is then of the form

$$g' = s_1 \cdots s_p a_{n-1} b_{n-1} \quad \text{with} \quad p \geqslant n.$$

If $\varepsilon_n = +1$ we have

$$g = s_1 \cdots s_p a_{n-1} b_{n-1} a_n^{-1} b_n^{-1} a_n b_n$$

and this is a reduced decomposition of length $\geqslant n + 6$ which terminates with $a_n b_n$, whence (1) and (2) in this case. If $\varepsilon_n = -1$ we have

$$g = s_1 \cdots s_p a_{n-1} (b_{n-1} b_n^{-1}) a_n^{-1} b_n a_n.$$

If $b_{n-1} b_n^{-1} \neq 1$ this is a reduced decomposition of g of length $\geqslant n + 5$, terminating with $b_n a_n$, whence (1) and (2) in this case. If $b_{n-1} b_n^{-1} = 1$ we have $a_{n-1} \neq a_n$ (since $\varepsilon_{n-1} = -\varepsilon_n$); the decomposition

$$g = s_1 \cdots s_p (a_{n-1} a_n^{-1}) b_n a_n$$

is then reduced, of length $\geqslant n + 3$, and it indeed terminates with $b_n a_n$. Q.E.D.

Remark. One can also derive prop. 4 from results on graphs which will be proved later (cf. no. 4.3, exerc. 2).

Corollary. *The free product of two finite groups contains a free subgroup of finite index.*

Exercises

1) Let g, h be elements of $*_A G_i$ of length n, m respectively and of types $(i_1, \ldots, i_n)$, $(j_1, \ldots, j_m)$ respectively. Show that $l(gh) \leqslant n + m$ and that equality holds if and only if $i_n \neq j_1$, in which case gh is of type $(i_1, \ldots, i_n, j_1, \ldots, j_m)$.

2) Let H be a subgroup of $G = *_A G_i$, and assume that $A \cdot H = G$. Let $B = A \cap H$ and $H_i = G_i \cap H$. Show that H is generated by the H_i, and can be identified with $*_B H_i$.

1.4 Constructions using amalgams

Amalgams are often used to prove the non-triviality of groups defined by generators and relations. For example, they play an essential role in the work of G. Higman (*Proc. Royal Soc.* 262, 1961, pp. 455–475) characterizing the groups embeddable in finitely presented groups (they are those with "recursive" presentations).

We give only two elementary examples:

Proposition 5 (G. Higman, B. H. Neumann, H. Neumann). *Let A be a subgroup of a group G and $\theta: A \to G$ an injective homomorphism. Then there is a group G' containing G and an element s of G' such that $\theta(a) = sas^{-1}$ for all $a \in A$. Furthermore, if G is denumerable (or finitely generated, or torsion-free) we can choose G' to be a group with the same property.*

For each $n \in \mathbf{Z}$ we put $A_n = A$, $G_n = G$. Let H be the group obtained by amalgamating the G_n by means of the injections $A_n \to G_n$, $A_n \to G_{n+1}$ which are respectively θ and the canonical inclusion. (We can summarize this amalgam by the following diagram:

$$\begin{tikzcd}[column sep=small]
 & G & & G & & G & & G & \\
\cdots \arrow[ur] & & A \arrow[ul, "\theta"'] \arrow[ur, "\mathrm{id}"'] & & A \arrow[ul, "\theta"'] \arrow[ur, "\mathrm{id}"'] & & A \arrow[ul, "\theta"'] \arrow[ur, "\mathrm{id}"'] & & \cdots \arrow[ul]
\end{tikzcd}$$
.)

Let u_n be the canonical isomorphism $G_n \to G_{n+1}$. The u_n define an automorphism u of H ("shift"). Furthermore, if we identify G with G_0, and if $a \in A \subset G$, the element $u(a)$ of G_1 is equal to the element $\theta(a)$ of $G_0 = G$. Thus u extends θ. We then form the semi-direct product $G' = H \cdot S$ of an infinite cyclic group S generated by an element s, and the group H (with s acting on H through u). It is immediate that the pair (G', s) has the desired property (it is in fact *universal* for the property in question, cf. exerc. 2). Moreover, one checks that if G is denumerable, finitely generated, or without torsion, then the same is true of G'.

Remark. One sometimes says that G' is derived from (A, G, θ) by the *HNN construction.*

Corollary. *Every group G can be embedded in a group K enjoying the following property*:
$(*)$ *All the elements of K of the same order are conjugate.*
Furthermore, if G is denumerable (or torsion-free) we can choose K to be denumerable (or torsion-free).

First of all, if $x, y \in G$ have the same order, prop. 5 shows the existence of a group G_{xy} containing G in which x and y are conjugate (and we can take G_{xy} to be denumerable, or torsion-free, if G is). By repeating this operation and passing to the direct limit we construct a group $E(G)$, containing G, and such that all the elements of G of the same order are conjugate in $E(G)$. By applying this construction to $E(G)$ itself we obtain a group $E(E(G))$, on which the construction is repeated... The direct limit of the $E(E(\cdots E(G)\cdots))$ answers the question.

Remark. If K is torsion-free, condition $(*)$ just says that the elements of K other than 1 are conjugate to each other. In particular, K is a *simple* group (if it does not reduce to $\{1\}$).

Proposition 6 (G. Higman). *Let G be the group defined by the four generators x_1, x_2, x_3, x_4 and the four relations*:

$$x_2 x_1 x_2^{-1} = x_1^2, \qquad x_3 x_2 x_3^{-1} = x_2^2, \qquad x_4 x_3 x_4^{-1} = x_3^2, \qquad x_1 x_4 x_1^{-1} = x_4^2.$$

a) *Each subgroup of finite index in G is equal to G.*
b) *G is infinite.*

We first prove a). Every subgroup of finite index in G contains a normal subgroup of finite index (the intersection of the conjugates of the given subgroup, for example). It therefore suffices to prove that G does not have any finite non-trivial quotient. Now suppose $\bar{G}$ is such a quotient, and let n_i $(1 \leqslant i \leqslant 4)$ be the order of x_i in $\bar{G}$. Because $\bar{G} \neq \{1\}$, one of the n_i is > 1. Let p be the smallest prime divisor of any of the n_i and suppose for example that p divides n_1. The fact that

$$x_1^{2^{n_2}} = x_2^{n_2} x_1 x_2^{-n_2} = x_1 \text{ in } \bar{G}$$

implies that $2^{n_2} \equiv 1 (\operatorname{mod} n_1)$ and *a fortiori* $2^{n_2} \equiv 1 \pmod{p}$. This already implies $p \neq 2$. On the other hand, we have $2 \not\equiv 1 \pmod{p}$, so the order N of the image of 2 in the multiplicative group $(\mathbf{Z}/p\mathbf{Z})^*$ is such that $1 < N \leqslant p-1$. The congruence written above is equivalent to $n_2 \equiv 0 \pmod{N}$. If p' is a prime factor of N we then have $n_2 \equiv 0 \pmod{p'}$ and $p' \leqslant N \leqslant p-1$, which contradicts the minimal character of p, hence a).

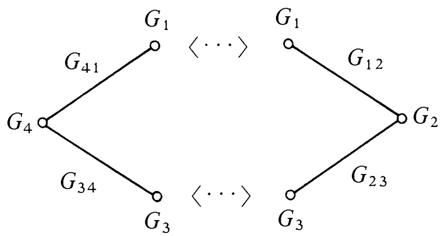
Proof of b): let G_{12} be the group defined by the generators x_1, x_2 and the relation $x_2 x_1 x_2^{-1} = x_1^2$. This is the semidirect product of $\mathbf{Z}[\frac{1}{2}]$ by $\mathbf{Z}$, where the latter acts on $\mathbf{Z}[\frac{1}{2}]$ by $n \mapsto 2n$. The subgroup G_1 (resp. G_2) of G_{12} generated by x_1 (resp. x_2)

is isomorphic to $\mathbf{Z}$, and we have $G_1 \cap G_2 = \{1\}$. We define similarly G_{23} (resp. G_{34}, G_{41}) and its subgroups G_2 and G_3 (resp. G_3, G_4 and G_4, G_1). By amalgamating the subgroup G_2 of G_{12} with the subgroup of the same name in G_{23} we obtain a group

$$G_{123} = G_{12} *_{G_2} G_{23}.$$

We define similarly

$$G_{341} = G_{34} *_{G_4} G_{41}.$$



In G_{123} the subgroup generated by G_1 and G_3 is isomorphic to the free group $F = G_1 * G_3$ (cf. cor. to prop. 3), and likewise in G_{341}. We can then form the amalgam $G_{123} *_F G_{341}$ and it is clear that this is the group G; its construction by successive amalgamations shows that it is infinite (it contains the free group on two generators).

Corollary. *There exists an infinite simple group generated by 4 elements.*

It suffices to take a simple quotient of G, which exists because G is finitely generated.

Problem. Is there an algebraic variety X over $\mathbf{C}$ whose fundamental group is isomorphic to the group G of prop. 6? If so, a) shows that X is *simply connected* in the algebraic sense, without being simply connected in the topological sense, because of b). (Note that, since G is finitely presented we can realize it as the fundamental group of a compact polyhedron and, if desired, of a compact differentiable manifold of dimension 4.)

Exercises

1) Show that the group defined by the presentation

$$x_2 x_1 x_2^{-1} = x_1^2, \qquad x_3 x_2 x_3^{-1} = x_2^2, \qquad x_1 x_3 x_1^{-1} = x_3^2$$

is trivial.

2) Show that the group G' constructed in the proof of prop. 5 is isomorphic to the quotient of $G * S$ by the normal subgroup generated by the elements of the form $s^{-1}a^{-1}s\theta(a)$ where a runs over A and s denotes a generator of the infinite cyclic group S.

1.5 Examples

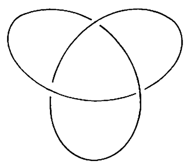
This concerns groups one encounters frequently and which are amalgams in a natural way.

1.5.1. The *infinite dihedral* group $\mathbf{D}_\infty$, isomorphic to $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$.

1.5.2. The "*trefoil knot*" group, defined by $\langle a, b; aba = bab\rangle$, is the sum of two copies of $\mathbf{Z}$ amalgamated by $\mathbf{Z} \xrightarrow{2} \mathbf{Z}$ and $\mathbf{Z} \xrightarrow{3} \mathbf{Z}$; in other words, it can be defined by $\langle x, y; x^2 = y^3\rangle$ (take $x = bab$ and $y = ab$, for example). This group appears in various guises:

(i) It is the inverse image of $\mathbf{SL}_2(\mathbf{Z})$ in the universal covering of $\mathbf{SL}_2(\mathbf{R})$.

(ii) It is the fundamental group of $\mathbf{R}^3 - N$ where N is the trefoil knot:



(iii) It is the group $B(3)$ of braids on three strings, i.e. the fundamental group of the space X_3 of three element subsets of $\mathbf{C}$.

(iv) It is the local fundamental group of the "ordinary cusp" singularity; or, what comes to the same thing, the fundamental group of $\mathbf{C}^2 - Y$, where Y is the affine curve with the equation $x^2 = y^3$.

1.5.3. The group $\mathbf{PSL}_2(\mathbf{Z}) = \mathbf{SL}_2(\mathbf{Z})/\{\pm 1\}$ is isomorphic to $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$. The group $\mathbf{SL}_2(\mathbf{Z})$ is isomorphic to $\mathbf{Z}/4\mathbf{Z} *_{\mathbf{Z}/2\mathbf{Z}} \mathbf{Z}/6\mathbf{Z}$ (the cyclic groups of orders 4 and 6 being generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$).

1.5.4. Let k be a commutative field and let $k[X]$ be the algebra of polynomials in one variable over k. We have

$$\mathbf{GL}_2(k[X]) \simeq \mathbf{GL}_2(k) *_{\mathrm{Tri}(k)} \mathrm{Tri}(k[X]),$$

where $\mathrm{Tri}(k)$ (resp. $\mathrm{Tri}(k[X])$) denotes the group of triangular matrices with coefficients in k (resp. $k[X]$). The group $\mathbf{SL}_2(k[X])$ admits an analogous decomposition.

In chapter II, no. 1.6, we shall return to these results which are due to H. Nagao [19].

1.5.5. The group $\mathbf{SL}_2(\mathbf{Z}[1/p])$ is the sum of two copies of $\mathbf{SL}_2(\mathbf{Z})$ amalgamated along the subgroup $\Gamma_0(p)$ consisting of the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $c \equiv 0 \pmod{p}$.

1.5.6. Let k be a field with a discrete valuation, with ring O_k and uniformiser π. Let $\Gamma = \mathbf{SL}_2(O_k)$ and let A be the subgroup of Γ consisting of the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $c \equiv 0 \pmod{\pi}$. The group $\mathbf{SL}_2(k)$ is isomorphic to $\Gamma *_A \Gamma$.

This result (like the preceding one) is due to Y. Ihara [16]; we return to these in chapter II, no. 1.4.

Exercises

1) Deduce 1.5.5 from 1.5.6 and exerc. 2 of no. 1.3.

2) Show that the curve in $\mathbf{R}^3$ with parametric equations

$$x = (2 + \cos 3t)\cos 2t$$

$$y = (2 + \cos 3t)\sin 2t$$

$$z = \sin 3t$$

is a trefoil knot.

3) Show the equivalence of the interpretations ii), iii), iv) of the trefoil knot group. (For ii) $\Rightarrow$ iv) construct a homeomorphism of $\mathbf{C}^2 - Y$ onto $\mathbf{R} \times (\mathbf{S}_3 - N)$. For iii) $\Rightarrow$ iv), define an isomorphism of X_3 onto $\mathbf{C} \times (\mathbf{C}^2 - Y)$.)

Define an "elliptic curve over X_3" by means of the equation $y^2 = (x - a)(x - b)(x - c)$. Deduce an interpretation of the homomorphism $B(3) \to \mathbf{SL}_2(\mathbf{Z})$ of i).

(For a generalization of this to the braid groups $B(n)$ and the calculation of the cohomology of the $B(n)$, see the notes published in 1968 by V. I. Arnold in *Uspekhi*, *Math. Zametki* and *Funkt. Anal.*)

§2 Trees

2.1 Graphs

Definition 1. *A graph Γ consists of a set $X = \operatorname{vert} \Gamma$, a set $Y = \operatorname{edge} \Gamma$ and two maps*

$$Y \to X \times X, \qquad y \mapsto (o(y), t(y))$$

and

$$Y \to Y, \qquad y \mapsto \bar{y}$$

which satisfy the following condition: for each $y \in Y$ we have $\bar{\bar{y}} = y$, $\bar{y} \neq y$ and $o(y) = t(\bar{y})$.

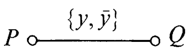
An element $P \in X$ is called a *vertex of Γ*; an element $y \in Y$ is called an (*oriented*) *edge*, and $\bar{y}$ is called the *inverse* edge. The vertex $o(y) = t(\bar{y})$ is called the *origin* of y, and the vertex $t(y) = o(\bar{y})$ is called the *terminus* of y. These two vertices are called the *extremities* of y. We say that two vertices are *adjacent* if they are the extremities of some edge.

There is an evident notion of *morphism* for graphs. We say that a morphism is *injective* if the corresponding maps on the vertices and edges are injective.

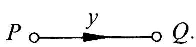
An *orientation* of a graph Γ is a subset Y_+ of $Y = \operatorname{edge} \Gamma$ such that Y is the disjoint union of Y_+ and $\bar{Y}_+$. It always exists. An *oriented graph* is defined, up to isomorphism, by giving the two sets X and Y_+ and a map $Y_+ \to X \times X$. The corresponding set of edges is $Y = Y_+ \coprod \bar{Y}_+$ where $\bar{Y}_+$ denotes a copy of Y_+.

Diagrams

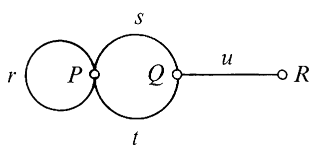
In practice a graph is often represented by a diagram, using the following convention: a point marked on the diagram corresponds to a vertex of the graph, and a line joining two marked points corresponds to a set of edges of the form $\{y, \bar{y}\}$. For example, the graph having 2 vertices P, Q and 2 edges y, $\bar{y}$ with $P = o(y)$, $Q = t(y)$ is represented by the diagram



or by



Likewise, the diagram



represents a graph with 3 vertices, P, Q, R and 8 edges $r, s, t, u, \bar{r}, \bar{s}, \bar{t}, \bar{u}$; furthermore,

r, s, t, u have the extremities $\{P, P\}$, $\{P, Q\}$, $\{P, Q\}$, $\{Q, R\}$ respectively. We have $o(r) = P = t(r)$ but the diagram does not tell whether P is the origin or the terminus of s.

Realization of a graph

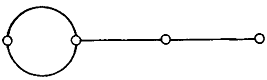
Let Γ be a graph and let $X = \operatorname{vert}\Gamma$, $Y = \operatorname{edge}\Gamma$. We form the topological space T which is the disjoint union of X and $Y \times [0, 1]$, where X and Y are provided with the discrete topology. Let R be the finest equivalence relation on T for which $(y, t) \equiv (\bar{y}, 1 - t)$, $(y, 0) \equiv o(y)$ and $(y, 1) \equiv t(y)$ for $y \in Y$ and $t \in [0, 1]$. The quotient space $\operatorname{real}(\Gamma) = T/R$ is called the *realization* of the graph Γ. (Realization is a functor which commutes with direct limits. Because a graph is a direct limit of its finite subgraphs, this shows that $\operatorname{real}(\Gamma)$ is a *CW-complex of dimension* $\leqslant 1$ in the sense of J. H. C. Whitehead.)

It is easy to see that the barycentric subdivision of $\operatorname{real}(\Gamma)$ is homeomorphic to the realization of the following graph Γ': $\operatorname{vert}\Gamma'$ is the union of X and the set of subsets of Y of the form $\{y, \bar{y}\}$ (the latter represents midpoints of the edges of Γ), we put $\operatorname{edge}\Gamma' = Y \times \{0, 1\}$ with $\overline{(y, \varepsilon)} = (\bar{y}, 1 - \varepsilon)$ if $\varepsilon = 0, 1$ and $o(y, 0) = o(y)$, $o(y, 1) = \{y, \bar{y}\}$. The graph Γ' is called the *barycentric subdivision of* Γ.

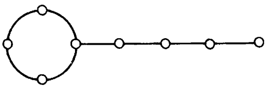
Here is an example:

Γ:

Γ':



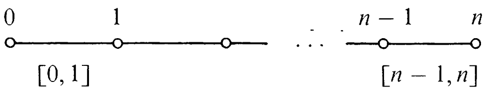
$(\Gamma')'$:



Paths

Let n be an integer $\geqslant 0$. Consider the oriented graph

Path_n:



It has $n + 1$ vertices $0, 1, \ldots, n$ and the orientation given by the n edges $[i, i + 1]$, $0 \leqslant i < n$ with $o([i, i + 1]) = i$ and $t([i, i + 1]) = i + 1$.

Definition 2. *A path (of length n) in a graph* Γ *is a morphism* c *of* Path_n *into* Γ.

For $n \geqslant 1$ the sequence $(y_1, \ldots, y_n)$ of edges $y_i = c([i - 1, i])$ such that $t(y_i) = o(y_{i+1})$, $1 \leqslant i < n$, determines c; we also denote it by c. If $P_i = c(i)$ we say that c is a path from P_0 to P_n and that P_0 and P_n are the extremities of the path.

A pair of the form $(y_i, y_{i+1}) = (y_i, \bar{y}_i)$ in the path is called a *backtracking*. It allows us to construct a path (of length $n - 2$) from P_0 to P_n, given (for $n > 2$) by the sequence $(y_1, \ldots, y_{i-1}, y_{i+2}, \ldots, y_n)$. By induction, we conclude that if there is a path from P to Q in Γ then *there is one without backtracking.*

The direct limit $\mathrm{Path}_\infty = \varinjlim \mathrm{Path}_n$ provides the notion of an *infinite path*. It is an infinite sequence $(y_1, y_2, \ldots)$ of edges such that $t(y_i) = o(y_{i+1})$ for all $i \geqslant 1$.

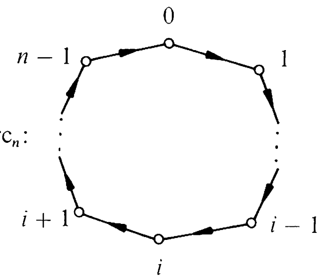
Definition 3. *A graph is said to be connected if any two vertices are the extremities of at least one path. The maximal connected subgraphs (under the relation of inclusion) are called the connected components of the graph.*

Remark. A graph is connected if and only if its realization is connected (or arcwise-connected, which comes to the same thing). More generally, the connected components of a graph correspond to those of its realization.

Circuits

Let n be an integer $\geqslant 1$. We consider the oriented graph

Circ_n:



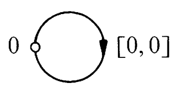
The set of vertices is $\mathbf{Z}/n\mathbf{Z}$, and the orientation is given by the n edges $[i, i+1]$ $(i \in \mathbf{Z}/n\mathbf{Z})$ with $o([i, i+1]) = i$ and $t([i, i+1]) = i + 1$.

Definition 4. *A circuit (of length n) in a graph is any subgraph isomorphic to* Circ_n.

Such a subgraph is defined by a path $(y_1, \ldots, y_n)$ without backtracking, such that the $P_i = t(y_i)$ $(1 \leqslant i \leqslant n)$ are distinct, and such that $P_n = o(y_1)$.

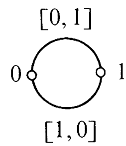
The case $n = 1$.

Circ_1:



We note that Circ_1 has an automorphism of order two which changes its orientation. A circuit of length 1 is called a *loop*.

The case $n = 2$.

Circ_2: 

Combinatorial graphs

Let (X, S) be a *simplicial complex of dimension* $\leqslant 1$. Recall that X is a set and S is a set of subsets of X with 1 or 2 elements, containing all the 1-element subsets. We associate with it a graph having X as its set of vertices, and edges equal to the pairs $(P, Q) \in X \times X$ such that $P \neq Q$ and $\{P, Q\} \in S$, with $\overline{(P, Q)} = (Q, P)$ and $o(P, Q) = P$, $t(P, Q) = Q$. In this graph, two edges with the same origin and same terminus are equal. This property is evidently equivalent to that of the following definition:

Definition 5. *A graph is called combinatorial if it has no circuit of length* $\leqslant 2$. (Cf. Bourbaki, [36], Annexe.)

Conversely, it is easy to see that every combinatorial graph Γ is derived (up to isomorphism) by the construction above from a simplicial complex (X, S) where $X = \operatorname{vert} \Gamma$ and S is the set of subsets $\{P, Q\}$ of X such that P and Q are either adjacent or equal.

Let Γ be a combinatorial graph. A set $\{P, Q\}$ of extremities of an edge y is called a *geometric edge* of Γ. The geometric edge $\{P, Q\}$ determines the set $\{y, \bar{y}\}$ of oriented edges. The structure of Γ is evidently determined by the set of its vertices and geometric edges.

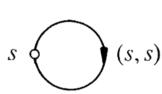
The graph $\Gamma(G, S)$

Let G be a group and let S be a subset of G. We let $\Gamma = \Gamma(G, S)$ denote the *oriented graph* having G as its set of vertices, $G \times S = (\operatorname{edge} \Gamma)_+$ as its orientation, with

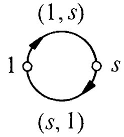
$$o(g, s) = g \quad \text{and} \quad t(g, s) = gs \quad \text{for each edge} \quad (g, s) \in G \times S.$$

The left multiplication by the elements of G defines an action of G on Γ which preserves orientation. Furthermore, G acts *freely* on the vertices and on the edges.

For example, let G be a *cyclic* group of order n generated by $S = \{s\}$. Here is the diagram for $\Gamma(G, S)$:

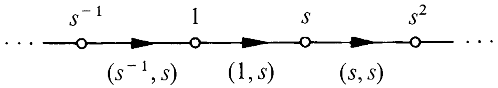
$n = 1$: 

$n = 2$:



$n < \infty$: $\Gamma(G, S) \cong \mathrm{Circ}_n$

$n = \infty$:



Proposition 7. *Let $\Gamma = \Gamma(G, S)$ be the graph defined by a group G and a subset S of G.*

(a) *For Γ to be connected, it is necessary and sufficient that S generate G.*

(b) *For Γ to contain a loop, it is necessary and sufficient that 1 belong to S.*

(c) *For Γ to be a combinatorial graph, it is necessary and sufficient that $S \cap S^{-1} = \emptyset$.*

For $g, g' \in G$ to be extremities of a path of length n it is necessary and sufficient that there exist $s_1, \ldots, s_n \in S \cup S^{-1}$ such that $g' = gs_1 \cdots s_n$; whence (a). (More precisely, the connected components of Γ correspond to the cosets gH of the subgroup H generated by S.) Assertions (b) and (c) are clear.

Exercises

1) A graph Γ is called *finite* if vert Γ and edge Γ are finite; it is called *locally finite* if every vertex is the extremity of a finite number of edges. Show that Γ is *finite* (resp. *locally finite*) if and only if its realization is a *compact* (resp. *locally compact*) space.

2) Show that a connected locally finite graph containing no injective infinite path is finite.

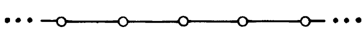
3) Let $\Gamma = \Gamma^{(0)}$ be a graph and, for $n > 0$, let $\Gamma^{(n)}$ be the barycentric subdivision of $\Gamma^{(n-1)}$. Show that $\Gamma^{(n)}$ contains no circuit of length $\leqslant n$. In particular, $\Gamma^{(2)}$ is combinatorial. (This allows all topological questions concerning graphs to be reduced to the case of combinatorial graphs.)

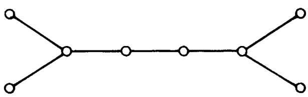
2.2 Trees

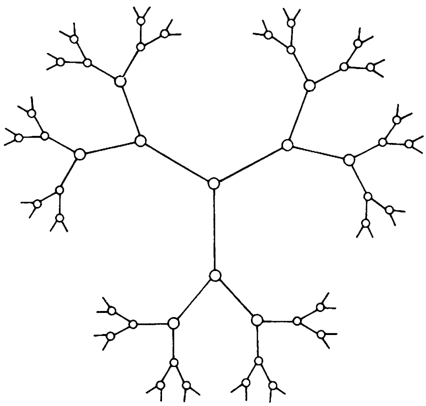
Definition 6. *A tree is a connected non-empty graph without circuits.*

In particular, a tree is a combinatorial graph.

Examples of trees







Geodesics in a tree

A *geodesic* in a tree is a path without backtracking.

Proposition 8. *Let P and Q be two vertices in a tree Γ. There is exactly one geodesic from P to Q and it is an injective path.*

Existence follows from the fact that Γ is connected.

Injectivity. Let $c\colon \mathrm{Path}_n \to \Gamma$ be a geodesic from $P = c(0)$ to $Q = c(n)$, and put $P_i = c(i)$. To show that c is injective it suffices to show that $P_0, \ldots, P_n$ are distinct (since Path_n is combinatorial). We can then assume that $n > 0$ and that c is defined by the sequence of edges $(y_1, \ldots, y_n)$. If the vertices are not distinct, let $j - i > 0$ be the minimal value such that $P_i = P_j$. Then $(y_{i+1}, \ldots, y_j)$ is a circuit, which contradicts the hypothesis that Γ is a tree.

Uniqueness. We can assume that $P \neq Q$ since a geodesic of length > 0 from P to P, being injective, would define a circuit.

Let $(y_1, \ldots, y_n)$ and $(w_1, \ldots, w_m)$ be two geodesics from P to Q. We have $y_n = w_m$ since otherwise the path $(y_1, \ldots, y_n, \bar{w}_m, \ldots, \bar{w}_1)$ would be a geodesic from P to P. Then the geodesics $(y_1, \ldots, y_{n-1})$ and $(w_1, \ldots, w_{m-1})$, which have the same terminus, must coincide by induction, whence the proposition.

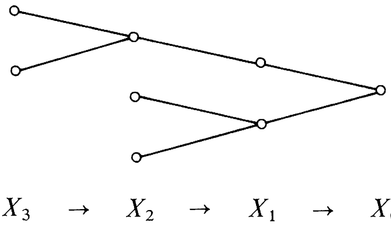
The length of the geodesic from P to Q is called the *distance* from P to Q, and is denoted by $l(P, Q)$. We have $l(P, Q) = 0$ if and only if $P = Q$, and $l(P, Q) = 1$ if and only if P and Q are adjacent.

Trees and inverse systems

Let P be a vertex of a tree Γ. For each integer $n \geqslant 0$ let X_n be the set of vertices Q of Γ such that $l(P, Q) = n$. If $Q \in X_n$, with $n \geqslant 1$, there is a single vertex Q' at distance $< n$ from P to which Q is adjacent; it is the vertex $o(y_n)$ where $(y_1, \ldots, y_n)$ is the geodesic from P to Q. This defines a map $f_n\colon Q \mapsto Q'$ of X_n into X_{n-1}, and hence an inverse system

$$(S_P) \qquad \cdots \to X_n \xrightarrow{f_n} X_{n-1} \to \cdots \to X_1 \to X_0 = \{P\}.$$

Knowledge of this system permits the reconstruction of the tree Γ; indeed, the set of vertices of Γ is the union X of the X_n, and the geometric edges are the $\{Q, f_n(Q)\}$ for $n \geqslant 1$ and $Q \in X_n$. Furthermore, every inverse system indexed by integers $\geqslant 1$ can be obtained in this way. We therefore have an equivalence between *pointed trees* and *inverse systems of sets* indexed by integers $\geqslant 1$:



The subtree generated by a set of vertices

Let Γ be a tree and let X' be a subset of $X = \text{vert}\,\Gamma$. Every subtree of Γ containing X' contains the geodesics with extremities in X'. Conversely, the vertices and edges of these geodesics form a subtree Γ' of Γ containing X'; it is the subtree *generated* by X'. If X' is finite then so is Γ', and it follows that Γ is the *direct union of its finite subtrees*: everything "of finite character" stated for trees can then be reduced to a statement for finite trees. For the latter we have a procedure of "dévissage" (cf. the remark after prop. 10) which often allows induction on the number of vertices.

Realization of a tree

Let P be a vertex of a tree Γ. If Q is a vertex at distance n from P, the subtree $\Gamma(P, Q)$ generated by $\{P, Q\}$ is isomorphic to Path_n, and canonically so if we make P correspond to the vertex 0 of Path_n. We can identify $\text{real}(\text{Path}_n)$ with the interval $[0, n]$. Consider the contraction of $\text{real}(\Gamma(P, Q))$ which corresponds to the contraction $x \to tx$ $(0 \leqslant t \leqslant 1)$ of the interval $[0, n]$ to 0.

Because Γ is the union of the subtrees $\Gamma(P, Q)$ ($Q \in \text{vert}\,\Gamma$) the realization of Γ is also the union of the subspaces $\text{real}(\Gamma(P, Q))$. Furthermore, it is easy to see that the contractions of these subspaces we have just defined are compatible. Hence we have a contraction of $\text{real}(\Gamma)$: *the realization of a tree is contractible.*

Terminal vertices

Let Γ be a *graph* and let $X = \text{vert}\,\Gamma$, $Y = \text{edge}\,\Gamma$. Let P be a vertex and let Y_P be the set of edges y such that $P = t(y)$. The cardinal n of Y_P is called the *index* of P. If $n = 0$ one says that P is *isolated*; if Γ is connected this is not possible unless $X = \{P\}$, $Y = \emptyset$. If $n \leqslant 1$ one says that P is a *terminal vertex* (or a *pending vertex*).

We let $\Gamma - P$ denote the subgraph of Γ with vertex set $X - \{P\}$ and edge set $Y - (Y_P \cup \bar{Y}_P)$.

Proposition 9. *Let P be a non-isolated terminal vertex of a graph Γ.*

(a) *Γ is connected if and only if $\Gamma - P$ is connected.*

(b) *Every circuit of Γ is contained in $\Gamma - P$.*
(c) *Γ is a tree if and only if $\Gamma - P$ is a tree.*

The hypothesis says that P is the terminus of a unique edge y. Assertion (a) is immediate. Also, every vertex belonging to a circuit is evidently of index $\geqslant 2$, whence (b). Assertion (c) follows from (a) and (b).

Bounded trees

The set of vertices of a tree Γ is a metric space under the distance l, whence the notions of the *diameter* of a tree and of a *bounded tree* (= tree of finite diameter). For example, every finite tree is bounded. If X' is a set of vertices of diameter n, the subtree generated by X' is evidently of diameter $\leqslant 3n$ (it is in fact of diameter n, cf. exerc. 2).

Proposition 10. *Let Γ be a tree of diameter $n < \infty$.*
(a) *The set $t(\Gamma)$ of terminal vertices of Γ is non-empty.*
(b) *If $n \geqslant 2$, $\operatorname{vert}\Gamma - t(\Gamma)$ is the vertex set of a subtree of diameter $n - 2$.*
(c) *If $n = 0$ we have $\Gamma \simeq \mathrm{Path}_0$ (diagram: $\circ$) and if $n = 1$ we have $\Gamma \simeq \mathrm{Path}_1$ (diagram: $\circ\!\!-\!\!\!-\!\!\!-\!\!\circ$).*

Assertion (c) is immediate, and (a) follows from (b) and (c). It therefore suffices to prove (b). Let $X' = \operatorname{vert}\Gamma - t(\Gamma)$. If $P, Q \in X'$, any point of the geodesic joining P to Q is non-terminal; we conclude that the subtree Γ' generated by X' has no vertices other than the elements of X'. Furthermore, if $l(P, Q) = m$, the geodesic joining P to Q can be extended ("in both directions") to a geodesic of length $m + 2$; whence $m + 2 \leqslant n$, which shows that $\operatorname{diam}(\Gamma') \leqslant n - 2$. On the other hand, because Γ is of diameter n, there is a geodesic of Γ of length n; by removing its first and last edges we obtain a geodesic of length $n - 2$ in Γ'; we then indeed have $\operatorname{diam}(\Gamma') = n - 2$.

Remark. The graph Γ' is preserved by all automorphisms of Γ. It follows immediately by induction on $\operatorname{diam}(\Gamma)$ that:

Corollary. *A tree of even finite diameter (resp. odd finite diameter), has a vertex (resp. geometric edge), which is invariant under all automorphisms.*

Remark. If Q is a vertex of a tree Γ_1, prop. 9 shows that the graph Γ obtained from Γ_1 by adjunction of a geometric edge $\{P, Q\}$ to a terminal vertex P is again a tree. Prop. 10(a) shows that every finite tree is obtained by repeated application of this procedure, beginning with a tree with a single vertex. This "dévissage" procedure is often convenient.

Exercises

1) Let Γ be a tree, P a vertex of Γ, and let (S_P) be the inverse system associated with P. Identify the inverse limit T_P of this system with the set C_P of infinite paths $y = (y_1, y_2, \ldots)$ without backtracking originating from P. Show that the topological space T_P is independent (up to a canonical homeomorphism) of the vertex P chosen. (If Q is another vertex of Γ and if

$y \in C_P$ show that there is an element $w \in C_Q$ and an integer d such that $y_i = w_{i+d}$ for i sufficiently large, and that these conditions determine w and d uniquely.)

[When Γ is locally finite, T_P is compact and totally discontinuous; it is the space of *ends* of the realization of Γ.]

2) Let Γ be a tree, and let $X = \operatorname{vert} \Gamma$.

a) Let $P, Q, R \in X$ and let P' be a vertex on the geodesic joining Q and R. Prove the inequality

$$l(P, P') \leqslant \sup(l(P, Q), l(P, R)).$$

b) Let X' be a subset of X of diameter n, and let Γ' be the tree generated by X'. Show using a) that the diameter of Γ' is equal to n.

3) Let Γ be a tree of finite diameter n. Show that all the geodesics of Γ of length n have the same middle vertex if n is odd and the same middle geometric edge if n is even. Deduce from this another proof of the cor. to prop. 10.

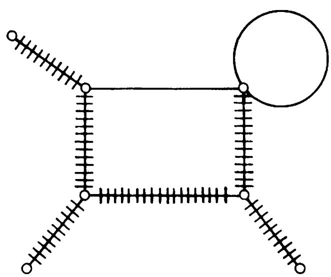
2.3 Subtrees of a graph

Let Γ be a non-empty graph. The set of subgraphs of Γ which are trees, ordered by inclusion, is evidently directed. By Zorn's lemma it has a maximal element; such an element is called a *maximal tree* of Γ.

Proposition 11. *Let Λ be a maximal tree of a connected non-empty graph Γ. Then Λ contains all the vertices of Γ.*

If not, then because Γ is connected, there would be an edge y with origin in Λ and terminus P outside Λ. Then by prop. 9(c) the subgraph derived from Λ by adjunction of the vertex P and the edges $y, \bar{y}$ would be a tree. This contradicts the maximal character of Λ and therefore $\operatorname{vert} \Lambda = \operatorname{vert} \Gamma$, whence the proposition.

Example



Proposition 12. *Let Γ be a connected graph with a finite number of vertices. Put*

$$s = \operatorname{Card}(\operatorname{vert} \Gamma), \qquad a = \tfrac{1}{2}\operatorname{Card}(\operatorname{edge} \Gamma).$$

Then $a \geqslant s - 1$ and equality holds if and only if Γ is a tree.

(Note that a is the number of "geometric edges" of Γ.)

We assume first that Γ is a tree, and show that $a = s - 1$. This property is true of a tree with a single vertex ($s = 1, a = 0$) and remains true when we adjoin a terminal vertex and a pair $\{y, \bar{y}\}$ of edges; it is therefore true for all finite trees (cf. remark at the end of no. 2.2).

We pass to the general case. The proposition is clear if Γ is empty. If not, let Γ' be a maximal tree of Γ. Because of prop. 11 we have $s(\Gamma) = s(\Gamma')$ and $a(\Gamma) \geqslant a(\Gamma')$, with equality if and only if $\Gamma = \Gamma'$; on the other hand, we have seen that $a(\Gamma') = s(\Gamma') - 1$, whence

$$a(\Gamma) = s(\Gamma) - 1 + (a(\Gamma) - a(\Gamma'))$$

and the proposition follows.

Remark. The Betti numbers B_i of the graph Γ are $B_0 = 1$, $B_1 = a(\Gamma) - a(\Gamma')$ and $B_i = 0$ for $i \geqslant 2$, if Γ is non-empty; if not $B_i = 0$ for all $i \geqslant 0$. The formula $a(\Gamma) = s(\Gamma) - 1 + (a(\Gamma) - a(\Gamma'))$ can then be written:

$$s(\Gamma) - a(\Gamma) = \sum_i (-1)^i B_i$$

which is a special case of the *Euler-Poincaré formula.*

Contraction of subtrees

Let Γ be a connected non-empty graph, and let Λ be a subgraph of Γ which is a disjoint union of a family Λ_i ($i \in I$) of trees. We are going to define a graph Γ/Λ such that $\mathrm{real}(\Gamma/\Lambda)$ is the quotient space of $\mathrm{real}(\Gamma)$ obtained by identification of each subspace $\mathrm{real}(\Lambda_i)$ to a point.

More precisely, the set of vertices of Γ/Λ is the quotient of $\mathrm{vert}\,\Gamma$ by the equivalence relation whose classes are the sets $\mathrm{vert}\,\Lambda_i$ and the elements of $\mathrm{vert}\,\Gamma - \mathrm{vert}\,\Lambda$. Its edge set is $\mathrm{edge}\,\Gamma - \mathrm{edge}\,\Lambda$, with the involution $y \mapsto \bar{y}$ induced by that on $\mathrm{edge}\,\Gamma$. Finally,

$$\mathrm{edge}(\Gamma/\Lambda) \to \mathrm{vert}(\Gamma/\Lambda) \times \mathrm{vert}(\Gamma/\Lambda)$$

is induced by

$$\mathrm{edge}\,\Gamma \to \mathrm{vert}\,\Gamma \times \mathrm{vert}\,\Gamma$$

by passing to quotients. It is easy to see that $\mathrm{real}(\Gamma/\Lambda)$ has the desired property.

Proposition 13. *The canonical projection* $\mathrm{real}(\Gamma) \to \mathrm{real}(\Gamma/\Lambda)$ *is a homotopy equivalence.*

We put $X = \mathrm{real}(\Gamma)$, $A = \mathrm{real}(\Lambda)$ and $A_i = \mathrm{real}(\Lambda_i)$; thus A is the disjoint union of the A_i ($i \in I$). We have seen in no. 2.2 that the realization of a tree is contractible. There is therefore a homotopy $h_t \colon A \to A$ $(0 \leqslant t \leqslant 1)$ such that $h_0 = 1_A$ and h_1 retracts each A_i to a point belonging to A_i.

Because A is a subcomplex of the CW-complex X, the pair (X, A) has the homotopy extension property (it is a "cofibration", cf. for example E. Spanier, *Algebraic Topology*, chap. 7, §6). Hence there is a homotopy $H_t\colon X \to X$ $(0 \leqslant t \leqslant 1)$ such that $H_0 = 1_X$ and such that H_t agrees with h_t on A. For $t = 1$ we obtain a map $H_1 : X \to X$ which factors through the quotient map $p\colon X \to Y = \mathrm{real}(\Gamma/\Lambda)$ of X by identification of each A_i with a point. This yields a map $f\colon Y \to X$ such that $H_1 = f \circ p$; in particular $f \circ p$ is homotopic to $H_0 = 1_X$.

It remains to show that $p \circ f$ is homotopic to 1_Y. We remark first that, because H_t leaves the A_i stable, it induces a homotopy $H'_t\colon Y \to Y$ on passing to the quotient. In the diagram

$$\begin{tikzcd} X \arrow[r, "H_1"] \arrow[d, "p"'] & X \arrow[d, "p"] \\ Y \arrow[r, "H'_1"'] \arrow[ur, "f"] & Y \end{tikzcd}$$

the square and the upper triangle commute. Since p is surjective the lower triangle commutes; then $p \circ f = H'_1$ is indeed homotopic to $H'_0 = 1_Y$, whence the proposition.

Corollary 1. *Let Γ be a connected non-empty graph. Then* $\mathrm{real}(\Gamma)$ *has the homotopy type of a bouquet of circles. Furthermore, Γ is a tree if and only if* $\mathrm{real}(\Gamma)$ *is contractible.*

We take Λ to be a maximal tree of Γ. According to prop. 11, Γ/Λ has a single vertex. Then $\mathrm{real}(\Gamma/\Lambda)$ is a CW-complex of dimension $\leqslant 1$ with a single zero-dimensional cell, hence it is a bouquet of circles. It is contractible if and only if the number B_1 of circles is zero. Because $B_1 = \frac{1}{2}\mathrm{Card}(\mathrm{edge}\,\Gamma - \mathrm{edge}\,\Lambda)$ we conclude that $B_1 = 0$ if and only if $\Gamma = \Lambda$, that is to say, if and only if Γ is a tree. In view of the proposition, this proves the corollary.

Remark. The number B_1 is the first Betti number of $\mathrm{real}(\Gamma)$. When $\mathrm{vert}\,\Gamma$ is finite, $B_1 = a(\Gamma) - s(\Gamma) + 1$, whence $a(\Gamma) = \frac{1}{2}\mathrm{Card}(\mathrm{edge}\,\Gamma)$ and $s(\Gamma) = \mathrm{Card}(\mathrm{vert}\,\Gamma)$, cf. prop. 12.

Corollary 2. *Under the hypotheses of prop. 13, Γ is a tree if and only if Γ/Λ is one.*

This follows from the proposition, together with cor. 1.

Remark. Cor. 2 is the main property of contractions that we shall use. It is easy to give a direct proof (cf. exerc. 2 and 3).

Exercises

1) Let Γ be a connected non-empty graph. If $P, Q \in \operatorname{vert} \Gamma$, let $l_\Gamma(P, Q)$ be the minimum length of paths joining P and Q. Let $P_0 \in \operatorname{vert} \Gamma$. Show that there is a maximal tree Λ of Γ such that $l_\Gamma(P_0, P) = l_\Lambda(P_0, P)$ for all $P \in \operatorname{vert} \Gamma$.

2) Assuming the hypotheses of prop. 13, suppose in addition that $\operatorname{vert} \Gamma$ is finite. Show directly that $a(\Gamma) - s(\Gamma) = a(\Gamma/\Lambda) - s(\Gamma/\Lambda)$. Deduce, by means of prop. 12, another proof of cor. 2 (first in the case where $\operatorname{vert} \Gamma$ is finite, then in the general case by taking limits).

3) With the hypotheses of prop. 13:
a) Show that, if Γ contains a circuit of length n, Γ/Λ contains a circuit of length $\leqslant n$.
b) Show that, if Γ/Λ contains a circuit of length n, Γ contains a circuit of length $\geqslant n$.
c) Deduce another proof of cor. 2 to prop. 13 from a) and b).

§3. Trees and free groups

3.1 Trees of representatives

Let X be a graph on which a group G acts. An *inversion* is a pair consisting of an element $g \in G$ and an edge y of X such that $gy = \bar{y}$; if there is no such pair we say G acts *without inversion*; this is the same as saying that there is an orientation of X preserved by G. (Example: G acts without inversion on the barycentric subdivision of X since the latter has a natural orientation, cf. no. 2.1.)

If G acts without inversion we can define the quotient graph $G\backslash X$ in an obvious way; the vertex set (edge set, respectively) of $G\backslash X$ is the quotient of vert X (edge X, respectively) under the action of G.

Proposition 14. *Let X be a connected graph, acted upon without inversion by a group G. Every subtree T' of $G\backslash X$ lifts to a subtree of X.*

Let Ω be the set of subtrees of X which project injectively into T'; it is a directed set under the relation of inclusion. Let T_0 be a maximal element of Ω, let T'_0 be its image in T', and suppose that $T'_0 \neq T'$. Then there is an edge y' of T' not belonging to T'_0. Since T' is connected we can assume that $o(y')$ is a vertex of T'_0; then the terminus P' of y' does not belong to vert T'_0 (otherwise the geodesic from $o(y')$ to P' in T'_0, followed by $\bar{y}'$, would give a circuit in T'). Let y be a lift of y'; since we are free to replace y by gy with $g \in G$, we can assume that $o(y)$ belongs to T_0. Let T_1 be the graph derived from T_0 by adjoining the vertex $P = t(y)$ and the edges $y, \bar{y}$. According to prop. 9(c), T_1 is a tree. But $T_1 \to T'$ is injective, which contradicts the maximality of T_0: whence the proposition.

A *tree of representatives* of X mod G is any subtree T of X which is the lift of a maximal tree in $G\backslash X$; because of prop. 11 of no. 2.3, every orbit of G in vert X contains exactly one element of vert X.

Exercise

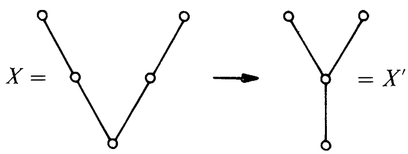
Consider the following property of a graph morphism $f\colon X \to X'$:

(*) Given $P \in \operatorname{vert} X$ and $y' \in \operatorname{edge} X'$ such that $f(P) = o(y')$, there is $y \in \operatorname{edge} X$ such that $P = o(y)$ and $f(y) = y'$.

(a) Show that (*) implies every subtree of X' which meets $f(X)$ lifts to a subtree of X.

(b) Check (*) for the morphism $X \to G\backslash X$ of prop. 14.

(c) Let f be the surjective morphism



Show that the tree X' does not lift to a tree in X.

3.2 Graph of a free group

Proposition 15. *Let $X = \Gamma(G, S)$ be the graph defined by a group G and a subset S of G (cf. no. 2.1). The following properties are equivalent*

(i) *X is a tree*

(ii) *G is a free group with basis S.*

Suppose that G is a free group with basis S. Then every element $g \in G$ can be written uniquely in reduced form

$$g = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}, \qquad s_i \in S, \qquad \varepsilon_i \in \{\pm 1\}$$

and $\varepsilon_i = \varepsilon_{i+1}$ if $s_i = s_{i+1}$ (cf. no. 1.3, lemma 1). The integer n is called the length of g and is denoted by $l(g)$. Let G_n be the set of elements of G of length n. If g is, as above, an element of G_n, $n \geqslant 1$, it is clear that g is adjacent in X to a unique element of G_{n-1}, namely

$$g = s_1^{\varepsilon_1} \cdots s_{n-1}^{\varepsilon_{n-1}}.$$

This yields a map $G_n \to G_{n-1}$ for each $n \geqslant 1$. Furthermore, we see that X is the graph defined by the inverse system

$$\cdots \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 = \{1\}$$

and it is therefore a tree (cf. no. 2.2).

Conversely, suppose that X is a tree. Prop. 7 of no. 2.1 shows that S generates G (since X is connected) and that $S \cap S^{-1} = \emptyset$ (since X is combinatorial). We show that S is a free family. If not, there is a non-trivial element $\hat{g}$ of the free group $F(S)$ with basis S whose image in G equals 1. Choose such an element $\hat{g}$ of minimum length n and let

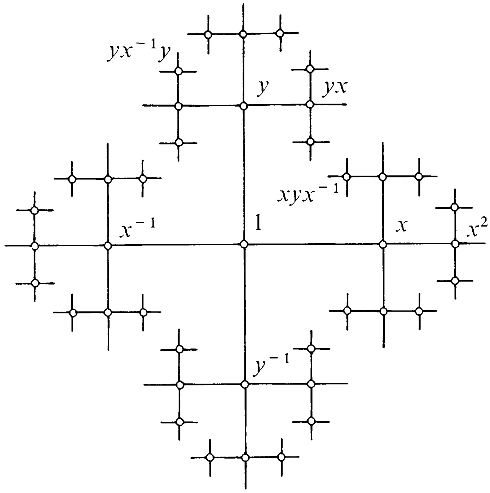
$$\hat{g} = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}$$

be its reduced decomposition in $F(S)$. The fact that $S \cap S^{-1} = \emptyset$ implies that $n \geqslant 3$. Let P_i $(0 \leqslant i \leqslant n)$ be the image of $s_1^{\varepsilon_1} \cdots s_i^{\varepsilon_i}$ in G. The minimality of $\hat{g}$ implies that $P_0, \ldots, P_{n-1}$ are all distinct. Furthermore, P_i is adjacent to P_{i+1} and $P_n = 1 = P_0$. Because $n \geqslant 3$ the geometric edges $\{P_i, P_{i+1}\}$ $(0 \leqslant i \leqslant n-1)$ and $\{P_n, P_0\}$ are all distinct. Thus $P_0, \ldots, P_{n-1}$ are the vertices of a circuit of length n in X. The existence of such a circuit contradicts the hypothesis that X is a tree, whence the proposition.

Example

$$S = \{x, y\}$$

$$G = F(S)$$



3.3 Free actions on a tree

We say that a group G acts *freely* on a graph X if it acts without inversion and no element $g \neq 1$ of G leaves a vertex of X fixed. For example, if S is a subset of a group G, the group G acts freely (by left multiplication) on the graph $\Gamma(G, S)$. Prop. 15 shows that, *if G is free, there is a tree upon which G acts freely.* In fact, this property characterizes free groups:

Theorem 4. *A group which acts freely on a tree is a free group.*

More precisely:

Theorem 4′. *Let G be a group which acts freely on a tree X. Choose a tree T of representatives of X* mod G (cf. no. 3.1) *and an orientation $Y_+ \subset$* edge X *preserved by G.*

a) *Let S be the set of elements $g \neq 1$ in G for which there is an edge $y \in Y_+$ with origin in T and terminus in gT. Then S is a basis for G.*

b) *If $X^* = G\backslash X$ has a finite number s of vertices, and if* $\mathrm{Card}(\mathrm{edge}\, X^*) = 2a$ *we have* $\mathrm{Card}(S) - 1 = a - s$.

Because G acts freely and because $T \to X^*$ is injective, the map $g \mapsto gT$ is a bijection of G onto the set of translates of T, and these translates are pairwise disjoint. In particular, we can form the graph $X' = X/G \cdot T$ (cf. no. 2.3) by contraction of each tree gT to a single vertex; we denote this vertex by (gT). By prop. 13 of no. 2.3, X' is a tree. Furthermore, the inverse of the bijection $g \mapsto (gT)$

can be considered as a bijection α: vert $X' \to$ vert $\Gamma(G, S) = G$, where $\Gamma(G, S)$ is the graph associated with G and S. We shall extend α to an isomorphism $\alpha: X' \to \Gamma(G, S)$. Thanks to prop. 15 (cf. no. 3.2), this will prove a).

Recall that edge X' = edge X − edge$(G \cdot T)$. We give X' the orientation $Y'_+ = Y_+ \cap$ edge X' induced by that of X. The morphism α will be a morphism of oriented graphs, and it then suffices to define $\alpha: Y'_+ \to G \times S = (\text{edge}\,\Gamma(G, S))_+$. Let $y \in Y'_+$ and let $(gT) = o(y)$, $(g'T) = t(y)$. From the fact that, in X, the edge y connects gT to $g'T$ we deduce that $s = g^{-1}g'$ belongs to S; we then set $\alpha(y) = (g, s)$. The surjectivity of $\alpha: Y'_+ \to G \times S$ is seen directly from the definition of S. The injectivity follows from the fact (previously remarked) that X' is a tree, and the fact that α: vert $X' \to$ vert $\Gamma(G, S)$ is injective; whence a).

b) Let W be the set of edges $y \in Y_+$ such that $o(y) \in T$ and $t(y) \notin T$. The proof of a) provides us with a bijection $W \to S$, whence Card W = Card S. On the other hand, let T^* be the image of T in $X^* = G\backslash X$; it is a maximal tree. We provide X^* with the orientation Y^*_+, the image of Y_+. It is immediate that Y^*_+ is the disjoint union of $Y^*_+ \cap$ edge T^* and W^*, the image of W, and that $W \to W^*$ is bijective. Then if the set vert X^* = vert T^* is finite we have

$$\begin{aligned}\text{Card}\, Y^*_+ - \text{Card vert}\, X^* &= \text{Card}\, W^* + (\text{Card}(\text{edge}\, T^*)_+ - \text{Card}(\text{vert}\, T^*))\\ &= \text{Card}\, W^* - 1 \quad (\text{by prop. 12 of no. 2.2})\\ &= \text{Card}\, S - 1\end{aligned}$$

whence b).

Topological interpretation

Because G acts freely on $\bar{X} = \text{real}(X)$ the latter is the universal covering of the quotient $\bar{X}^* = G\backslash\bar{X} = \text{real}(X^*)$, and G can be identified with the fundamental group $\pi_1(\bar{X}^*)$. Because T^* is a maximal tree of X^*, the quotient $\bar{X}^*/\text{real}(T^*) = \text{real}(X^*/T^*)$ is a bouquet of circles; by prop. 13 of no. 2.3 it has the same homotopy type as $\bar{X}^*$. The fundamental group $G = \pi_1(X^*)$ is then the free group with generators corresponding to the set of circles of the bouquet real(X^*/T^*). These circles, suitably oriented, correspond to the elements of W^*, and hence to the elements of S, as we have seen above; whence the theorem.

Exercise

Let S be a subset of a group G, generating G. Let $F(S)$ be the free group with basis S and let R be the kernel of $F(S) \to G$.

a) Show that R acts freely on the tree $\Gamma(F(S), S)$ and that the quotient graph is isomorphic to $\Gamma(G, S)$.

b) Deduce that R is isomorphic to the fundamental group of the realization of $\Gamma(G, S)$ (note that this gives another proof of prop. 15).

3.4 Application: Schreier's theorem

This is the following:

Theorem 5. *Every subgroup of a free group is free.*

Let G be a free group. We can make G act freely on a tree X (cf. prop. 15). If H is a subgroup of G, it is clear that H acts freely on X; it is then a free group by th. 4.

Notation. If G is free, the cardinality of a basis for G is independent of the set chosen (this can be seen, for example, by abelianizing G); we call this the *rank* of G, and denote it by r_G.

Corollary (Schreier index formula). *Let G be a free group and H a subgroup of finite index n in G. Then*

$$r_H - 1 = n(r_G - 1).$$

We put $G_1 = G$ and $G_2 = H$ and let Γ be a tree on which G acts freely. Let $\Gamma_i = G_i \backslash \Gamma$, $s_i = \text{Card vert}\, \Gamma_i$ and $a_i = \text{Card edge}\, \Gamma_i$ $(i = 1, 2)$. We have $s_2 = ns_1$ and $a_2 = na_1$. We can choose Γ so that s_1 is finite; for example the tree associated with a basis of G has $s_1 = 1$. The corollary follows then from the formula $r_{G_i} - 1 = \frac{1}{2}a_i - s_i$ $(i = 1, 2)$ (cf. th. 4′(b)).

Explicit form of Schreier's theorem

Proposition 16. *Let G be a free group with basis S, and let H be a subgroup of G.*

a) *One can choose a set T of representatives of $H\backslash G$ satisfying the following condition:*

(∗) *If $t \in T$ has the reduced decomposition*

$$t = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n} \cdot (s_i \in S, \quad \varepsilon_i = \pm 1 \quad \text{and} \quad \varepsilon_i = \varepsilon_{i+1} \quad \text{if} \quad s_i = s_{i+1})$$

then all the partial products $1, s_1^{\varepsilon_1}, s_1^{\varepsilon_1}s_2^{\varepsilon_2}, \ldots, s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}$ belong to T.

b) *Let T be as above and let $W = \{(t, s) \in T \times S, ts \notin T\}$. If $(t, s) \in W$, set $h_{t,s} = tsu^{-1}$ where $u \in T$ is such that $Hts = Hu$. Then*

$$R = \{h_{t,s}, (t, s) \in W\}$$

is a basis for H.

Let Γ be the oriented tree $\Gamma(G, S)$ on which G, and hence also H, acts freely, and leaves invariant the orientation $G \times S$ of Γ. If $t \in G$, the partial products in (∗) are none other than the vertices on the geodesic from 1 to t in Γ. Then a subset T of G satisfies the condition (∗) if and only if $T = \text{vert}\, \Lambda$ for a tree Λ of representatives of $\Gamma \bmod H$ which contains the vertex 1. In view of prop. 14, this proves a).

Let $T = \operatorname{vert} \Lambda$ be as above. By th. 4′, H has a generating set consisting of the $h \neq 1$ in H for which there is an edge $(t, s) \in G \times S$ with origin $t \in T$ and terminus $ts \in hT$. Because $h \neq 1$, the element $u = h^{-1}ts$ of T is different from 1. We then find precisely the edges (t, s) of W and the elements $h = h_{t,s}$ of H; whence the proposition.

Remark

We can choose T in a) so that each element t of T has minimum length in the class Ht (cf. no. 2.3, exerc. 1).

Example

Let $G = F(x, y)$ be the free group with basis $S = \{x, y\}$ with $x \neq y$, and let H be the kernel of the projection $G \to \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. By the corollary to th. 5 we have $r_H = 1 + 4(r_G - 1) = 5$. If we take the set of representatives $T = \{1, x, y, xy\}$, prop. 16 shows that H has the basis $\{h_1, h_2, h_3, h_4, h_5\}$ defined by the formulae:

$$xx = h_1 \cdot 1,\ yx = h_2 \cdot xy,\ yy = h_3 \cdot 1,\ xy \cdot x = h_4 \cdot y \text{ and } xy \cdot y = h_5 \cdot x.$$

We therefore obtain the basis

$$\{x^2, yxy^{-1}x^{-1}, y^2, xyxy^{-1}, xy^2x^{-1}\}.$$

Exercise

Let (S, W) be a presentation of a group G. (Recall what this means: W is a subset of the free group $F(S)$ with basis S, and we have an isomorphism $F(S)/\langle W \rangle \xrightarrow{\sim} G$, where $\langle W \rangle$ denotes the smallest normal subgroup of $F(S)$ containing W.)

Let H be a subgroup of G and let F' be its inverse image in $F(S)$; we have $G/H \simeq F/F'$. Let $T \subset F$ be a system of representatives of F/F' and let S' be a generating set for F'; let $W' = \bigcup_{t \in T} t^{-1} W t$. Show that $\langle W \rangle$ is the smallest normal subgroup of F' containing W'. Deduce that $\langle S', W' \rangle$ is a presentation of H. [When S, W and G/H are *finite*, the same is true of S' and W'; this shows that every subgroup of finite index in a finitely presented group is itself finitely presented.]

Appendix: Presentation of a group of homeomorphisms

The results of sections 3 and 4 can also be proved by using general theorems on groups of homeomorphisms. We recall these theorems:

Consider a connected, non-empty space X, acted upon by a group G (by homeomorphisms, of course). Let U be an open set of X such that $X = G \cdot U$ (i.e. the map $U \to G \backslash X$ is surjective) and let Σ denote the set of $g \in G$ such that $U \cap gU \neq \emptyset$. Then we have:

(1) *The set Σ generates G.* Indeed, let H be the subgroup of G generated by Σ. The space X is the union of the open sets $H \cdot U$ and $(G - H) \cdot U$; furthermore, these open sets are *disjoint* (because, if $h \cdot u = h' \cdot u'$ with $h \in H$ and $h' \in G - H$, we have $h^{-1}h' \cdot U \cap U \neq \emptyset$, whence $h^{-1}h' \in \Sigma$ and $h' \in H \cdot \Sigma = H$, which is absurd). Since X is connected, and $H \cdot U$ non-empty this implies $(G - H) \cdot U = \emptyset$, whence $G = H$.

Now let $F(\Sigma)$ be the free group with basis Σ; if $s \in \Sigma$ we let x_s denote the corresponding element of $F(\Sigma)$. By (1), the map $x_s \mapsto s$ extends to a surjective homeomorphism $\varepsilon: F(\Sigma) \to G$, and we want to describe the *kernel* R of ε (i.e. the set of "relations" between the elements of Σ.) The result is the following:

(2) *Suppose that* $\pi_0(X) = \pi_1(X) = 0$ *and* $\pi_0(U) = 0$ (in other words, that X is arcwise connected and simply connected, and U is arcwise connected). *Let W be the set of pairs (s, t) of elements of Σ such that $U \cap sU \cap stU \neq \emptyset$. The group R is the normal subgroup of $F(\Sigma)$ generated by the $x_s x_t (x_{st})^{-1}$ for $(s, t) \in W$.*

(Note that if $(s, t) \in W$ we have $s, t, st \in \Sigma$ so that x_s, x_t, x_{st} are defined.)

This result is due to A. M. Macbeath, *Ann. of Math.*, 79, 1964, pp. 473–488. [Analogous results may be found in M. Gerstenhaber, *Proc. Amer. Math. Soc.*, 4, 1953, pp. 745–750, H. Behr, *J. Crelle*, 211, 1962, pp. 116–122 and A. Weil, *Ann. of Math.*, 72, 1960, pp. 369–384.]

We briefly indicate a proof (for more details the reader is referred to Weil, *loc. cit.*): let $\tilde{G}$ be the quotient group of $F(\Sigma)$ by the subgroup $\tilde{R}$ generated by the $x_s x_t (x_{st})^{-1}$ for $(s, t) \in W$; since $\tilde{R}$ is evidently contained in R, we have a canonical projection $\tilde{G} \to G$.

If $s \in \Sigma$ let $\tilde{s}$ denote the image of x_s in $\tilde{G}$; the map $s \mapsto \tilde{s}$ is injective. Let $\tilde{X}$ be the topological space obtained from the disjoint union gU (where g runs over $\tilde{G}$) by identifying two points gu, $g'u'$ if and only if there is an $s \in \Sigma$ such that $g' = gs$, $u \in U \cap sU$, $u' \in U \cap s^{-1}U$ and $su' = u$. One checks (thanks to the fact that $\tilde{R}$ contains the $x_s x_t (x_{st})^{-1}$ for $(s, t) \in W$) that this is indeed an open equivalence relation. We have a natural projection $\pi: \tilde{X} \to X$ compatible with the actions of $\tilde{G}$ and G. Furthermore, if N denotes the kernel of $\tilde{G} \to G$, the inverse image of U under π can be identified with the product $N \times U$ (N being given the discrete topology): in particular, $\tilde{X}$ is a *covering* of X, with group N. The facts that U is arcwise connected and that Σ generates G imply in addition that $\tilde{X}$ is *arcwise connected.* Since X is simply connected, we conclude that $N = \{1\}$, whence the result.

Example

In the situation of th. 4′ we can apply (1) and (2) to the action of G on real(X); we take U to be a suitable open neighbourhood of real(T), for example the set of points whose distance from real(T) is $< \frac{2}{3}$. One then finds that the set Σ is $\{1\} \cup S \cup S^{-1}$ (in the notation of th. 4′) and that W is the set of pairs (s, t) of elements of Σ such that $s = 1$ or $t = 1$. It follows that G is indeed free with basis S.

§4. Trees and amalgams

In this section we make the convention (except where the contrary is expressly stated) that each group acting on a graph acts without inversion (cf. 3.1).

4.1 The case of two factors

Definition 7. *Let G be a group acting on a graph X. A fundamental domain of X* mod *G is a subgraph T of X such that $T \to G\backslash X$ is an isomorphism.*

If $G\backslash X$ is a tree it follows from prop. 14 of no. 3.1 that a fundamental domain exists. If X is a tree the converse is true:

Proposition 17. *Let G be a group acting on a tree X. A fundamental domain of X* mod *G exists if and only if $G\backslash X$ is a tree.*

It suffices to show that, if there is a fundamental domain T, the quotient $G\backslash X$ is a tree. But since X is connected and non-empty, so is $G\backslash X$; then T is a non-empty connected subgraph of a tree, i.e. a tree; whence the proposition.

A graph isomorphic to $\mathrm{Path}_1 = \overset{0}{\circ}\!\longrightarrow\!\overset{1}{\circ}$ is called a *segment*.

Theorem 6. *Let G be a group acting on a graph X, and let $T = \overset{P}{\circ}\!\xrightarrow{\;y\;}\!\overset{Q}{\circ}$ be a segment of X. Suppose that T is a fundamental domain of X* mod *G. Let G_P, G_Q and $G_y = G_{\bar{y}}$ be the stabilizers of the vertices and edges of T. The following properties are then equivalent:*

(1) *X is a tree.*

(2) *The homomorphism $G_P *_{G_y} G_Q \to G$ induced by the inclusions $G_P \to G$ and $G_Q \to G$ is an isomorphism.*

(Note that $G_y = G_P \cap G_Q$ is a subgroup of G_P and of G_Q; hence the amalgam $G_P *_{G_y} G_Q$ makes sense, cf. §1.)

Conversely, every amalgam of two groups acts on a tree with a segment as fundamental domain. More precisely:

Theorem 7. *Let $G = G_1 *_A G_2$ be an amalgam of two groups. Then there is a tree X (and only one, up to isomorphism) on which G acts, with fundamental domain a segment $T = \overset{P}{\circ}\!\xrightarrow{\;y\;}\!\overset{Q}{\circ}$, the vertices and edges of which have $G_P = G_1$, $G_Q = G_2$ and $G_y = A$ as their respective stabilizers.*

Proof of theorem 6

The theorem follows from the next two lemmas.

Lemma 2. *X is connected if and only if G is generated by $G_P \cup G_Q$.*

Let X' be the connected component of X containing T, let G' be the set of elements $g \in G$ such that $gX' = X'$, and let G'' be the subgroup of G generated by $G_P \cup G_Q$. If $h \in G_P \cup G_Q$ then the segments T and hT have a common vertex. We then have $hT \subset X'$, whence $hX' = X'$, i.e. $h \in G'$; this proves that G' contains G''. On the other hand, $G''T$ and $(G - G'')T$ are disjoint subgraphs of X, whose union is X. This implies that $G''T$ contains X', so that $G' \subset G''$, whence $G' = G''$. The graph X is connected if and only if $X = X'$, i.e. if $G = G' = G''$, whence the lemma.

Lemma 3. *X contains no circuit if and only if $G_P *_{G_y} G_Q \to G$ is injective.*

To say that X contains a circuit is the same as saying that there is a path $c = (w_0, \ldots, w_n)$, $n \geqslant 1$, in X without backtracking and such that $o(c) = t(c)$. We write w_i in the form $h_i y_i$ with $h_i \in G$ and $y_i = y$ or $\bar{y}$. Passing to $G\backslash X \cong T$ we see also that $\bar{y}_i = y_{i-1}$ $(1 \leqslant i \leqslant n)$. Let $P_i = o(y_i) = t(y_{i-1})$; we have $h_i = h_{i-1} g_i$ with $g_i \in G_{P_i}$ since

$$h_i P_i = h_i o(y_i) = o(h_i y_i) = t(h_{i-1} y_{i-1}) = h_{i-1} t(y_{i-1}) = h_{i-1} P_i$$

and $g_i \notin G_y$ since

$$\overline{h_i y_i} \neq h_{i-1} y_{i-1}.$$

The fact that $o(c) = t(c)$ is equivalent to $t(y_n) = P_0$, or again to

$$h_0 P_0 = h_n P_0 = h_0 g_1 \cdots g_n P_0, \quad \text{i.e.} \quad g_1 \cdots g_n \in G_{P_0}.$$

We conclude that X contains a circuit if and only if we can find a sequence $P_0, \ldots, P_n$ of vertices of T with $\{P_{i-1}, P_i\} = \{P, Q\}$ for all i and a sequence of elements $g_i \in G_{P_i} - G_y$ $(0 \leqslant i \leqslant n)$ such that $g_0 g_1 \cdots g_n = 1$. In view of th. 2 of no. 1.2, this amounts to saying that $G_P *_{G_y} G_Q \to G$ is not injective.

Proof of theorem 7

The uniqueness of X is clear: up to isomorphism, X is necessarily the following graph:

$$\operatorname{vert} X = (G/G_1) \coprod (G/G_2), \qquad \operatorname{edge} X = (G/A) \coprod \overline{(G/A)}$$

with the maps $o\colon G/A \to G/G_1$ and $t\colon G/A \to G/G_2$ being induced by the inclusions $A \to G_i$ $(i = 1, 2)$. If we put $P = 1 \cdot G_1$, $Q = 1 \cdot G_2$ and $y = 1 \cdot G_A$, the segment $T = \overset{P}{\circ}\xrightarrow{\;y\;}\overset{Q}{\circ}$ is a fundamental domain for the obvious action of G on X. Th. 6 then shows that X is a tree. Q.E.D.

Remarks

1) Ths. 6 and 7 establish an *equivalence* between "amalgams of two groups" and "actions on a tree with a segment as fundamental domain". The case of amalgams of more than two factors is analogous; we shall deal with it (thanks to the notion of a "tree of groups") in no. 4.5.

There is an analogous equivalence between

"*HNN* groups" (in the sense of no. 1.4)

and

"actions on a tree with a loop as quotient"

to which we shall return in §5.

2) The implication (1) $\Rightarrow$ (2) of th. 6 can also be proved by applying the results of the appendix to §3 to a sufficiently small open neighbourhood of the fundamental domain T.

3) The implication "X connected $\Rightarrow G$ is generated by $G_P \cup G_Q$" can be generalized as follows:

Lemma 4. *Let G be a group acting on a connected graph X, and let T be a tree of representatives of X* mod G (cf. no. 3.1). *Let Y be a subgraph of X containing T, each edge of which has an extremity in T, and such that $G \cdot Y = X$. For each edge y of Y with origin in T, let g_y be an element of G such that $g_y t(y) \in \operatorname{vert} T$. The group H generated by the elements g_y and the stabilizers G_P ($P \in \operatorname{vert} T$) is equal to G.*

It evidently suffices to show that $H \cdot (\operatorname{vert} T) = \operatorname{vert} X$. Since H contains the elements g_y, we have $\operatorname{vert} Y \subset H \cdot (\operatorname{vert} T)$, and it remains to see that $H \cdot Y = X$. Because X is connected, it suffices to show that an edge w with origin in $H \cdot Y$ belongs to $H \cdot Y$. By translating w by an element of H, if necessary, we can assume that $P = o(w)$ belongs to vert T. Since $G \cdot Y = X$, there is a $g \in G$ such that $gw \in Y$ and we must show that g belongs to H.

Since $gw \in Y$, we have $o(gw) \in \operatorname{vert} T$ or $t(gw) \in \operatorname{vert} T$. In the first case, P and gP are two vertices of T congruent mod G, hence they are equal and $g \in G_P$, whence $g \in H$. In the second case, the edge $y = gw$ has its origin in T; we then have $g_y t(y) \in \operatorname{vert} T$ and we deduce as in the first case that P and $g_y t(y) = g_y g P$ coincide, whence $g \in g_y^{-1} G_P$ and we again have $g \in H$.

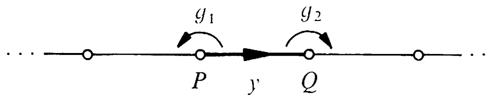
Remark. If G, X, T are given we can always find a subgraph satisfying the conditions of the lemma. When T is a fundamental domain we can even take $Y = T$ and all the g_y equal to 1.

Exercise

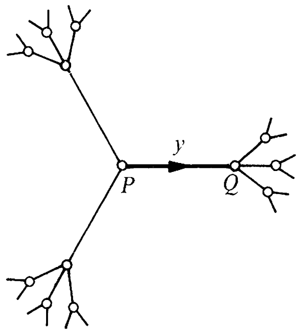
Let G be a group acting on a tree X *with inversion*. We assume that G acts *transitively* on vert X and edge X. Let $P \in \operatorname{vert} X$ and let y be an edge with origin P. Let G_P, G_y, $\tilde{G}_y$ be the stabilizers of P, y and $\{y, \bar{y}\}$; we have $G_y \subset \tilde{G}_y$ and $(\tilde{G}_y : G_y) = 2$. Show that the injections $G_P \to G$, $\tilde{G}_y \to G$ extend to an *isomorphism* $G_P *_{G_y} \tilde{G}_y \to G$. Conversely? (Apply th. 6 to the barycentric subdivision of X.)

4.2 Examples of trees associated with amalgams

(a) Dihedral group $\mathbf{D}_\infty$: $G_1 = \mathbf{Z}/2\mathbf{Z} = G_2$, $A = \{1\}$



(b) $G = \mathbf{Z}/3\mathbf{Z} * \mathbf{Z}/4\mathbf{Z}$



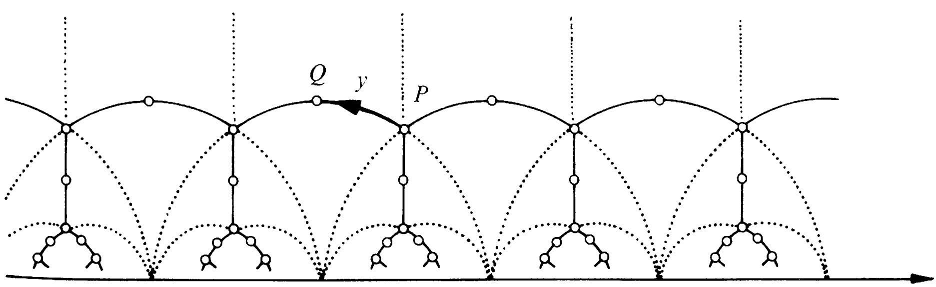
(c) $G = \mathbf{SL}_2(\mathbf{Z})$

This group acts in a well-known way on the half-plane $H = \{z|\mathrm{Im}(z) > 0\}$. Let y be the circular arc consisting of the points $z = e^{i\theta}$ with $\pi/3 \leqslant \theta \leqslant \pi/2$; its origin is the point $P = e^{\pi i/3}$ and its terminus is the point $Q = i$. Let X be the union of the transforms of y by G. One can show that X is a *tree* (or rather, the geometric realization of a tree) on which G acts with the segment PQ as fundamental domain. We have

$$G_P = \mathbf{Z}/6\mathbf{Z}, \qquad G_Q = \mathbf{Z}/4\mathbf{Z}, \qquad G_y = \mathbf{Z}/2\mathbf{Z},$$

so we recover the classical isomorphism between $\mathbf{SL}_2(\mathbf{Z})$ and

$$(\mathbf{Z}/4\mathbf{Z}) *_{\mathbf{Z}/2\mathbf{Z}} (\mathbf{Z}/6\mathbf{Z}).$$



Exercise

Let $j(z)$ be the modular function. Show that the tree X defined above is the set of points $z \in H$ such that $j(z) \in [0,1728]$.

4.3 Applications

In this section, G denotes an amalgam $G_1 *_A G_2$ of two groups. We let X denote the corresponding tree, cf. no. 4.1, th. 7.

Proposition 18. *Let Γ be a subgroup of $G = G_1 *_A G_2$ such that $\Gamma - \{1\}$ does not meet any conjugate of G_1 or G_2. Then Γ is a free group.*

The hypothesis on Γ is equivalent to saying that Γ *acts freely* on the tree X. The proposition then follows from th. 4 of no. 3.3.

(We leave to the reader the task of finding a basis for Γ by means of th. 4′ of no. 3.3.)

Theorem 8. *Every bounded subgroup of $G = G_1 *_A G_2$ is contained in a conjugate of G_1 or G_2.*

(A subset Σ of G is called *bounded* if there is a bound on the lengths of the reduced decompositions of elements of Σ.)

Corollary. *Every finite subgroup of G is contained in a conjugate of G_1 or G_2.*

Let $T = \overset{P}{\circ}\!\xrightarrow{\;y\;}\!\overset{Q}{\circ}$ be a fundamental domain of $X \bmod G$ such that $G_P = G_1$, $G_Q = G_2$, $G_y = A$. If $g \in G_1 \cup G_2$ then T, gT have a common vertex; this shows that, if Σ is a bounded subset of G, then the set $\Sigma \cdot (\operatorname{vert} T)$ is bounded (in terms of the metric on $\operatorname{vert} X$, cf. no. 2.2). Th. 8 then follows from the implication (b) $\Rightarrow$ (c) of prop. 19 below:

Proposition 19. *Let Γ be a group acting on a tree X. The following conditions are equivalent*:

(a) *For every bounded subset A of* $\operatorname{vert} X$, *$\Gamma \cdot A$ is bounded.*
(b) *There is $P \in$* $\operatorname{vert} X$ *such that $\Gamma \cdot P$ is bounded.*
(c) *There is a vertex of X invariant under Γ.*

The implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) are immediate. We show that (b) $\Rightarrow$ (c). Let Y be the subtree generated by the orbit $\Gamma \cdot P$. This is a bounded subtree, stable under Γ. By the cor. to prop. 10 of no. 2.2 there is a vertex or a geometric edge of Y invariant under Γ. But we have assumed that Γ contains no inversion. It follows that if Γ leaves a geometric edge $\{y, \bar{y}\}$ invariant then it leaves y and $\bar{y}$ invariant, and hence their extremities also. Whence (c).

Exercises

1) The hypotheses are those of prop. 18. Assume in addition that $\Gamma \backslash G/A$ is *finite*. Put:

$$a = \operatorname{Card}(\Gamma \backslash G/A), \qquad s_1 = \operatorname{Card}(\Gamma \backslash G/G_1), \qquad s_2 = \operatorname{Card}(\Gamma \backslash G/G_2),$$

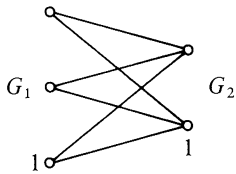
i) Show that A is of finite index in G_1 and G_2; if we put $n_1 = (G_1 : A)$, $n_2 = (G_2 : A)$, we have $s_1 = a/n_1$, $s_2 = a/n_2$.

ii) Show that the free group Γ is of finite rank, equal to

$$1 + a - s_1 - s_2 = 1 + a(1 - 1/n_1 - 1/n_2).$$

2) Assume that $A = \{1\}$ and let Γ denote the kernel of $G_1 * G_2 \to G_1 \times G_2$. Prop. 18 applies to Γ.

i) Show that $\Gamma\backslash X$ is isomorphic to a combinatorial graph with $G_1 \coprod G_2$ as its vertex set and $G_1 \times G_2$ as its set of geometric edges.



ii) Construct a tree of representatives of $X \bmod \Gamma$. Using th. 4′, show that Γ has a basis consisting of the commutators $g_1^{-1}g_2^{-1}g_1g_2$ with $g_1 \in G_1 - \{1\}$, $g_2 \in G_2 - \{1\}$ (cf. no. 1.3, prop. 4).

4.4 Limit of a tree of groups

Definition 8. *A graph of groups (G, T) consists of a graph T, a group G_P for each $P \in \operatorname{vert} T$, and a group G_y for each $y \in \operatorname{edge} T$, together with a monomorphism $G_y \to G_{t(y)}$ (denoted by $a \mapsto a^y$); one requires in addition that $G_{\bar{y}} = G_y$.*

In this section we consider the case where T is a *tree*, and we then say that (G, T) is a *tree of groups*. We let

$$G_T = \varinjlim (G, T)$$

denote the *direct limit* (in the sense of no. 1.1) of the tree of groups (G, T); we call it the "amalgam of the G_P along the G_y".

Examples

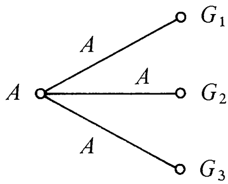
(a) Take T to be a segment $\overset{P}{\circ}\xrightarrow{\;y\;}\overset{Q}{\circ}$. We then have three groups G_P, G_Q and $G_y = G_{\bar{y}}$ and two monomorphisms $G_y \xrightarrow{\bar{y}} G_P$ and $G_y \xrightarrow{y} G_Q$; the group G_T is equal to $G_P *_{G_y} G_Q$.

(b) More generally, suppose that T is obtained by adjoining a vertex P and a geometric edge $\{y, \bar{y}\}$ to a tree T' (in other words, P is a terminal vertex of T, and $T' = T - P$, cf. no. 2.2). We then have

$$G_T = G_{T'} *_{G_y} G_P, \quad \text{whence} \quad G_{T'} = \varinjlim (G, T').$$

(c) Let $(G_i)_{i \in I}$ be a family of groups, let A be a group, and for each $i \in I$, let $f_i \colon A \to G_i$ be a monomorphism (cf. no. 1.2). Let 0 be an element not belonging to I; we form the tree T whose vertices are 0 and the elements of I, and whose edges are the pairs $(0, i)$ and $(i, 0)$ with $i \in I$.

We define a tree of groups (G, T) by putting $G_0 = A$, $G_y = A$ for each $y \in \operatorname{edge} T$ and defining the monomorphisms $A \to A$ and $A \to G_i$ in the obvious way.



It is clear that $G_T = \varinjlim (G, T)$ is the *amalgam* $*_A\, G_i$ defined in no. 1.2.

Convention. If (G, T) is a tree of groups with limit G_T we agree to *identify* the groups G_P and G_y with their images in G_T. This is legitimate because the homomorphisms $G_P \to G_T$ and $G_y \to G_T$ are *injective*. [Indeed it suffices to check this for a *finite* tree T; in this case, one argues by induction on the number of vertices of T, using (b) above.]

4.5 Amalgams and fundamental domains (general case)

Theorem 9. *Let (G, T) be a tree of groups. There is a graph X containing T and an action of $G_T = \varinjlim (G, T)$ on X which is characterized (up to isomorphism) by the following property:*

T is a fundamental domain for X mod *G_T and, for all $P \in \operatorname{vert} T$ (resp. all $y \in \operatorname{edge} T$) the stabilizer of P (resp. y) in G_T is G_P (resp. G_y).*

Moreover X is a tree.

(The graph X will be called the graph *associated* with (G, T).)

It is clear that $\operatorname{vert} X$ (resp. $\operatorname{edge} X$) is the disjoint union of the $G_T \cdot P \cong G_T/G_P$ for $P \in \operatorname{vert} T$ (resp. the $G_T \cdot y \cong G_T/G_y$ for $y \in \operatorname{edge} T$). The extremities are defined by means of the inclusions $G_y \to G_{o(y)}$ and $G_y \to G_{t(y)}$. This certainly defines a graph on which the group G_T acts in an obvious way, and all the assertions of the theorem are immediate, except the fact that X is a tree.

To prove this, we represent T as the direct limit of its finite subtrees (cf. no. 2.2), and G_T and X as the direct limits of the corresponding groups and graphs. We are therefore reduced to the case where T is finite. We now argue by induction on $n = \operatorname{Card} \operatorname{vert} T$. We can assume that $n > 1$ otherwise X and T coincide. Let $y \in \operatorname{edge} T$ be such that $P = t(y)$ is a terminal vertex of T (cf. no. 2.2). By prop. 9 of no. 2.2 T is obtained from the subtree $T' = T - P$ by adjunction of the vertex P and the edges $y, \bar{y}$. If we put $G_{T'} = \varinjlim (G, T')$ we have $G_T = G_{T'} *_{G_y} G_P$, cf. no. 4.4, example (b).

Let $X' = G_{T'}T'$; this is a subgraph of X. One sees easily that X' is the graph associated with (G, T'); it is a tree by the induction hypothesis. Moreover, the transforms gX', $g \in G_T/G_{T'}$, are pairwise disjoint. Let $\tilde{X}$ be the graph derived from X by contracting each of the gX' to a point (cf. no. 2.3). The group G_T acts on $\tilde{X}$; it has for fundamental domain the segment T/T', where $T/T' = \overset{(T')}{\circ}\xrightarrow{\;y\;}\overset{P}{\circ}$; the stabilizers of (T'), P and y are respectively $G_{T'}$, G_P and G_y. Since $G_{T'} *_G G_P \to G_T$ is an isomorphism, it follows from th. 6 of no. 4.1 that $\tilde{X}$ is a tree, and hence so is X (cf. no. 2.3, cor. 2 to prop. 13). Q.E.D.

Conversely, let G be a group acting on a graph X, with fundamental domain a tree T. Let (G, T) be the tree of groups whose G_P and G_y are the stabilizers in G of the vertices of T and its edges y (with the monomorphisms $G_y \to G_{t(y)}$ being the inclusions). Let $G_T = \varinjlim (G, T)$. The inclusions $G_P \to G$ extend to a homomorphism $G_T \to G$; by lemma 4 of no. 4.1, this homomorphism is *surjective* if X is connected.

On the other hand, if $\tilde{X}$ denotes the tree associated with (G, T), the identity map $T \to T$ extends uniquely to a morphism $\tilde{X} \to X$ which is equivariant with respect to $G_T \to G$.

Theorem 10. *With the above hypotheses and notations, the following properties are equivalent*:

(1) *X is a tree.*
(2) *$\tilde{X} \to X$ is an isomorphism.*
(3) *$G_T \to G$ is an isomorphism.*

The implications (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) follow from th. 9.

(2) $\Rightarrow$ (3): let $P \in \operatorname{vert} T$ and let $(G_T)_P$ (resp. G_P) be the corresponding stabilizer in G_T (resp. in G). By construction, the homomorphism $G_T \to G$ induces an isomorphism of $(G_T)_P$ onto G_P. On the other hand, if $\tilde{X} \to X$ is bijective, the kernel H of $G_T \to G$ is contained in $(G_T)_P$; we then have $H = \{1\}$. Since we already know that $G_T \to G$ is surjective, it follows that it is an isomorphism.

(1) $\Rightarrow$ (2): since $G_T \cdot T = \tilde{X}$ and $G \cdot T = X$, the morphism $\tilde{X} \to X$ is surjective. On the other hand, the homomorphism $G_T \to G$ induces isomorphisms between the stabilizers of the corresponding vertices (resp. corresponding edges) of $\tilde{X}$ and X. Hence $f\colon \tilde{X} \to X$ is *locally injective* (i.e. injective on the set of edges with a given origin). This suffices to make f *injective*, thanks to the following:

Lemma 5. *Let $f\colon \tilde{X} \to X$ be a locally injective morphism of a connected graph $\tilde{X}$ into a tree X. Then f is injective.*

Since $\tilde{X}$ is connected it suffices to show that, if c is an injective path in $\tilde{X}$, the path $f \circ c$ is also injective. Since X is a tree, it suffices to check that $f \circ c$ has no backtracking (cf. prop. 8 of no. 2.2). But the latter property follows immediately from the fact that c is injective and f is locally injective.

Remark. Let G be a group acting on a tree X. When $G\backslash X$ is a tree there is a fundamental domain T of $X \bmod G$ which is a tree (isomorphic to $G\backslash X$), and the structure of G is given by th. 10. The case where $G\backslash X$ is not a tree is more involved; it will be dealt with in §5.

Exercises

1) Prove the implication (1) $\Rightarrow$ (3) of th. 10 by using the results of the appendix of §3.

2) Extend prop. 18 and th. 8 to subgroups of $G_T = \varinjlim (G, T)$ where (G, T) is a tree of groups.

§5 Structure of a group acting on a tree

The main result of this section is th. 13 of no. 5.4, which gives the structure of a group G acting without inversion on a tree X. Two special cases have been treated in the preceding sections: that of a *free action* (G is then a free group, cf. no. 3.3) and that where the *quotient graph* $G\backslash X$ is a tree (in which case G is an amalgam of the stabilizers of the vertices of a tree of representatives of $G\backslash X$, no. 4.5).

In order to study the general case we need the notion[1] of the "fundamental group of a graph of groups", a notion which generalizes both the notions of free group and amalgam; this will be the object of nos. 5.1 and 5.2.

5.1 Fundamental group of a graph of groups

Let Y be a *connected, non-empty* graph and let (G, Y) be a *graph of groups*. (Recall, cf. no. 4.4, that this amounts to giving a group G_P for each $P \in \operatorname{vert} Y$ and, for each $y \in \operatorname{edge} Y$, a group G_Y together with a monomorphism $G_Y \to G_{t(y)}$ denoted by $a \mapsto a^y$; we also require $G_{\bar{y}} = G_y$.)

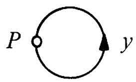
Examples

1) A *segment* of groups:

$$Y = \overset{P}{\circ} \xrightarrow{\quad y \quad} \overset{Q}{\circ}$$

$$\begin{tikzcd} & G_P \\ G_y = G_{\bar{y}} \arrow[ur, "\bar{y}"] \arrow[dr, "y"'] & \\ & G_Q \end{tikzcd}$$

2) A *loop* of groups:

$Y =$ 

$$\begin{tikzcd} G_y = G_{\bar{y}} \arrow[r, shift left, "\bar{y}"] \arrow[r, shift right, "y"'] & G_P \end{tikzcd}$$

The group $F(G, Y)$

This is the group generated by the groups G_P and the elements y of edge Y, subject to the relations

---

[1] due to H. Bass, as are the proofs of ths. 11, 12, 13. (For alternate proofs, see [42], [43], [44], [46].)

$$\bar{y} = y^{-1} \quad \text{and} \quad ya^y y^{-1} = a^{\bar{y}} \quad \text{if} \quad y \in \text{edge } Y, \quad a \in G_y.$$

More precisely, let Γ be the free product of the G_P and the free group with basis edge Y; the group $F(G, Y)$ is defined as the quotient of Γ by the normal subgroup generated by the elements $y\bar{y}$ and $ya^y y^{-1}(a^{\bar{y}})^{-1}$, $y \in \text{edge } Y$, $a \in G_y$.

Note that the above relations are equivalent to the relations

$$ya^y \bar{y} = a^{\bar{y}} \quad \text{if} \quad y \in \text{edge } Y, \quad a \in G_y.$$

Words of $F(G, Y)$

Let c be a path in Y whose origin is a vertex P_0. We let $y_1, \ldots, y_n$ denote the edges of c, where $n = l(c)$ and put

$$P_i = o(y_{i+1}) = t(y_i).$$

Definition 9. *A word of type c in $F(G, Y)$ is a pair (c, μ) where $\mu = (r_0, \ldots, r_n)$ is a sequence of elements $r_i \in G_{P_i}$. The element*

$$|c, \mu| = r_0 y_1 r_1 y_2 \cdots y_n r_n \quad \textit{of} \quad F(G, Y)$$

is said to be associated with the word (c, μ).

When $n = 0$ we have $|c, \mu| = r_0$.

[By abuse of notation, we do not distinguish between an element of one of the groups G_P and its image under the canonical homomorphism $G_P \to F(G, Y)$; indeed we shall see in no. 5.2 that this homomorphism is *injective*.]

The two definitions of the fundamental group of (G, Y)

a) Let P_0 be a vertex of Y. We let $\pi_1(G, Y, P_0)$ be the set of elements of $F(G, Y)$ of the form $|c, \mu|$, where c is a path whose extremities both equal P_0 (i.e. $o(c) = P_0 = t(c)$). One sees immediately that $\pi_1(G, Y, P_0)$ is a subgroup of $F(G, Y)$, called the *fundamental group* of (G, Y) at P_0. When G is the trivial graph of groups I (corresponding to $I_P = \{1\}$ for each vertex P of Y), the group $\pi_1(I, Y, P_0)$ coincides with the *fundamental group* (in the usual sense) $\pi_1(Y, P_0)$ of the graph Y at the point P_0 (more precisely, we obtain the *combinatorial definition* of this group). In the general case, the canonical morphism $G \to I$ extends to a homomorphism

$$\pi_1(G, Y, P_0) \to \pi_1(Y, P_0);$$

this homomorphism is surjective; its kernel is the normal subgroup of $\pi_1(G, Y, P_0)$ generated by the G_P.

b) Let T be a maximal tree of Y. The *fundamental group $\pi_1(G, Y, T)$ of (G, Y) at T* is, by definition, the quotient of $F(G, Y)$ by the normal subgroup generated by the elements y of edge T. Thus, if g_y denotes the image of y in $\pi_1(G, Y, T)$, the group $\pi_1(G, Y, T)$ is *generated by the groups G_P ($P \in$ vert Y) and the elements g_y ($y \in$ edge Y) subject to the relations*

$$g_y a^y g_y^{-1} = a^{\bar{y}}, \quad g_{\bar{y}} = g_y^{-1} \quad \text{if} \quad y \in \operatorname{edge} Y, \quad a \in G_y$$

$$g_y = 1 \quad \text{if} \quad y \in \operatorname{edge} T.$$

In particular, we have $a^y = a^{\bar{y}}$ if $y \in \operatorname{edge} T$, $a \in G_y$.

Remark. Let R be the smallest normal subgroup of $\pi_1(G, Y, T)$ containing the images of the G_P, $P \in \operatorname{vert} Y$. It follows from the definition above that the *quotient* $\pi_1(G, Y, T)/R$ is defined by the generators g_y, $y \in \operatorname{edge} Y$, and the relations

$$g_{\bar{y}} = g_y^{-1} \quad \text{as well as} \quad g_y = 1 \quad \text{if} \quad y \in \operatorname{edge} T.$$

This is the *fundamental group* (in the ordinary sense) $\pi_1(Y, T)$ of the graph Y relative to the maximal tree T; it is a free group, with a basis consisting of the g_y, $y \in A - (T \cap A)$, where A is an orientation of Y.

Examples

1) Suppose that $G_y = \{1\}$ for each $y \in \operatorname{edge} Y$, and let A be an orientation of Y. The group $\pi_1(G, Y, T)$ is then generated by the G_P ($P \in \operatorname{vert} Y$) and the elements g_y ($y \in A - T \cap A$), subject to no relations. We then have

$$\pi_1(G, Y, T) = (\underset{P}{*}\, G_P) * F$$

where F is the free group with basis $A - (T \cap A)$, in other words the fundamental group $\pi_1(Y, T)$, cf. the remark above.

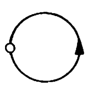
2) If $Y = \overset{P}{\circ}\!\xrightarrow{\;\;y\;\;}\!\overset{Q}{\circ}$ is a *segment* we have

$$\pi_1(G, Y, Y) = G_P *_{G_y} G_Q.$$

More generally, if Y is a *tree* we have

$$\pi_1(G, Y, Y) = \varinjlim\, (G, Y)$$

cf. no. 4.4; it is the amalgam of the G_P along the G_y.

3) $Y = P$  y is a *loop*. Put $A = G_y$. We have two monomorphisms

$$A \underset{y}{\overset{\bar{y}}{\rightrightarrows}} G_P\,;$$

the group $F(G, Y) = \pi_1(G, Y, P)$ is generated by G_P and an element $g = g_y$, with th defining relations:

$$g a^y g^{-1} = a^{\bar{y}} \quad \text{for each} \quad a \in A.$$

If we identify A with a subgroup of $G = G_P$ by $a \mapsto a^y$, and if we let θ denote the homomorphism $a \mapsto a^{\bar{y}}$ of A into G, we see that the group $\pi_1(G, Y, P)$ is none other than the *group derived from* (A, G, θ) *by the HNN construction*, cf. no. 1.4, proof of prop. 5. It follows from that proof that $\pi_1(G, Y, P)$ is the *semi-direct product* of the infinite cyclic group $g^{\mathbf{Z}}$ generated by g and the normal subgroup R generated by the conjugates $G_n = g^n G g^{-n}$ of G, $n \in \mathbf{Z}$; moreover, R is the *amalgamated sum* of the G_n as indicated in 1.4 *loc. cit.* [Note that $g^{\mathbf{Z}}$ can be interpreted as the fundamental group $\pi_1(Y, P)$ of the loop Y, and R as the group $\pi_1(G, \tilde{Y}, \tilde{Y})$ relative to the *universal covering* $\tilde{Y}$ of Y; for a generalization of all this see the exercise below.]

Proposition 20. *Let* (G, Y) *be a graph of groups, let* $P_0 \in \operatorname{vert} Y$ *and let* T *be a maximal tree of* Y. *The canonical projection* $p\colon F(G, Y) \to \pi_1(G, Y, T)$ *induces an isomorphism of* $\pi_1(G, Y, P_0)$ *onto* $\pi_1(G, Y, T)$.

(This shows in particular that $\pi_1(G, Y, P_0)$ is *independent of the choice of* P_0, up to an isomorphism which depends on the choice of T; similarly $\pi_1(G, Y, T)$ is *independent of the choice of* T. The situation is the same as for the usual fundamental group.)

Proof

If $P \in \operatorname{vert} Y$, let c_P be the geodesic in T joining P_0 to P. Let $y_1, \dots, y_n$ be the edges of c_P and put

$$\gamma_P = y_1 \cdots y_n \quad \text{in} \quad F(G, Y).$$

Likewise put

$$x' = \gamma_P x \gamma_P^{-1} \quad \text{if} \quad x \in G_P$$

$$y' = \gamma_{o(y)} y \gamma_{t(y)}^{-1} \quad \text{if} \quad y \in \operatorname{edge} Y.$$

If $y \in \operatorname{edge} T$ we have either $c_{t(y)} = (c_{o(y)}, y)$ or $c_{o(y)} = (c_{t(y)}, \bar{y})$; in both cases $y' = 1$. On the other hand we have $(\bar{y})'y' = 1$, and, if $a \in G_y$,

$$\begin{aligned} y'(a^y)'y'^{-1} &= \gamma_{o(y)} y \gamma_{t(y)}^{-1} \gamma_{t(y)} a^y \gamma_{t(y)}^{-1} \gamma_{t(y)} y^{-1} \gamma_{o(y)}^{-1} \\ &= \gamma_{o(y)} y a^y y^{-1} \gamma_{o(y)}^{-1} \\ &= \gamma_{o(y)} a^{\bar{y}} \gamma_{o(y)}^{-1} \\ &= (a^{\bar{y}})'. \end{aligned}$$

Since the elements x' and y' belong to $\pi_1(G, Y, P_0)$, the relations above show that there is a unique homomorphism

$$f\colon \pi_1(G, Y, T) \to \pi_1(G, Y, P_0)$$

such that $f(x) = x'$, $f(g_y) = y'$ if $x \in G_P$, $y \in \text{edge } Y$. We have $p(\gamma_P) = 1$, whence $p \circ f = Id$. On the other hand, let c be a closed path with origin P_0, edges $y_1, \ldots, y_n$ and vertices $P_i = o(y_{i+1}) = t(y_i)$; let (c, μ), with $\mu = (r_0, \ldots, r_n)$, be a word of type c, and let $r_0 y_1 r_1 y_2 \cdots y_n r_n$ be the element of $\pi_1(G, Y, P_0)$ associated with (c, μ), cf. def. 9. We have

$$r_i' = \gamma_{P_i} r_i \gamma_{P_i}^{-1}, \qquad y_i' = \gamma_{P_i} y_i \gamma_{P_{i+1}}^{-1}$$

whence

$$r_0' y_1' r_1' \cdots y_n' r_n' = \gamma_{P_0}(r_0 y_1 r_1 \cdots y_n r_n)\gamma_{P_0}^{-1} = r_0 y_1 \cdots y_n r_n$$

because $\gamma_{P_0} = 1$. We then have $f \circ p = Id$ and the proof is complete.

Exercise

Let (G, Y) be a non-empty connected graph of groups, and let T be a maximal tree of Y. Let $(\tilde{Y}, T)$ be the universal cover of Y relative to T; the graph $\tilde{Y}$ is a tree, on which the group $\Gamma = \pi_1(Y, T)$ acts freely. If $Q \in \text{vert } \tilde{Y}$ projects to $P \in \text{vert } Y$ we put $G_Q = G_P$; we define similarly G_y for $y \in \text{edge } \tilde{Y}$ as well as $G_y \to G_{t(y)}$; the result is a *tree of groups* $(G, \tilde{Y})$ on which Γ acts in a natural way.

Show that $\pi_1(G, Y, T)$ is canonically isomorphic to the semi-direct product of Γ and the group $\pi_1(G, \tilde{Y}, \tilde{Y}) = \varinjlim (G, \tilde{Y})$.

5.2 Reduced words

We retain the notation and hypotheses of no. 5.1. Let (c, μ) be a word of type c, where c is a path with origin P_0 and edges $y_1, \ldots, y_n$, and where $\mu = (r_0, \ldots, r_n)$, cf. def. 9.

Definition 10. *One says that (c, μ) is reduced if it satisfies the following condition:*

If $n = 0$ one has $r_0 \neq 1$; if $n \geqslant 1$ one has $r_i \notin G_{y_i}^{y_i}$ for each index i such that $y_{i+1} = \bar{y}_i$.

(We denote by G_y^y the image of G_y in $G_{t(y)}$ under the monomorphism $a \mapsto a^y$.)

In particular, every word whose type is a path of length $\geqslant 1$ without backtracking *is reduced.*

Theorem 11. *If (c, μ) is a reduced word, the associated element $|c, \mu|$ of $F(G, Y)$ is $\neq 1$.*

Corollary 1. *The homomorphisms $G_P \to F(G, Y)$ are injective.*

This is the special case where c is of zero length.

Corollary 2. *If (c, μ) is reduced, and if $l(c) \geqslant 1$ one has $|c, \mu| \notin G_{P_0}$, where $P_0 = o(c)$.*

If we had $|c, \mu| = x \in G_{P_0}$ the reduced word (c, μ') where

$$\mu' = (x^{-1}r_0, r_1, \ldots, r_n)$$

would be such that $|c, \mu'| = 1$, which would contradict the theorem.

Corollary 3. *Let T be a maximal tree of Y, and let (c, μ) be a reduced word whose type c is a closed path* (i.e. $o(c) = t(c)$). *Then the image of $|c, \mu|$ in $\pi_1(G, Y, T)$ is $\neq 1$.*

Let $P_0 = o(c)$ be the origin of c; we have $|c, \mu| \in \pi_1(G, Y, P_0)$ and th. 11 says that $|c, \mu| \neq 1$. The corollary then follows from the fact that the canonical projection

$$f\colon F(G, Y) \to \pi_1(G, Y, T)$$

induces an isomorphism of $\pi_1(G, Y, P_0)$ onto $\pi_1(G, Y, T)$, cf. prop. 20.

Preliminaries to the proof of theorem 11

We are going to use a technique of "dévissage" which will reduce the problem to the case where Y is either a segment or a loop.

Let Y' be a connected non-empty subgraph of Y and let $(G|Y', Y')$ be the restriction of the graph of groups (G, Y) to Y'. *We assume that th. 11 is true for* $(G|Y', Y')$; by cor. 1 above this implies in particular that the homomorphism $G_P \to F(G|Y', Y')$ is *injective* for each $P \in \operatorname{vert} Y'$.

Let $W = Y/Y'$ be the graph derived from Y by *contraction* of Y' to a vertex, denoted by (Y'); we have

$$\operatorname{vert} W = (\operatorname{vert} Y - \operatorname{vert} Y') \cup \{(Y')\}$$

and

$$\operatorname{edge} W = \operatorname{edge} Y - \operatorname{edge} Y';$$

if $y \in \operatorname{edge} W$ we define $o_W(y)$ and $t_W(y)$ in the obvious way:

$$o_W(y) = \begin{cases} o(y) & \text{if} \quad o(y) \notin \operatorname{vert} Y' \\ (Y') & \text{otherwise} \end{cases}$$

$$t_W(y) = \begin{cases} t(y) & \text{if} \quad t(y) \notin \operatorname{vert} Y' \\ (Y') & \text{otherwise.} \end{cases}$$

Let (H, W) be the graph of groups defined as follows:

if $P \in \operatorname{vert} Y - \operatorname{vert} Y'$, put $H_P = G_P$;

if $P = (Y')$, put $H_P = F(G|Y', Y')$;

if $y \in \operatorname{edge} W$, put $H_y = G_y$ and define $H_y \to H_{t(y)}$ in the obvious way (it is injective, in view of the hypothesis on Y').

The projection $(G, Y) \to (H, W)$ induces a homomorphism of $F(G, Y)$ into $F(H, W)$.

Lemma 6. *The morphism $F(G, Y) \to F(H, W)$ is an isomorphism.*

This is essentially trivial; it amounts to saying that one can construct $F(G, Y)$ in two steps; first take the generators and relations relative to Y', which gives $H_{(Y')} = F(G|Y', Y')$, then adjoin the other generators and the other relations, which correspond to the construction of $F(H, W)$. [We leave to the reader the task of writing down a formal proof, for example by defining a morphism $F(H, W) \to F(G, Y)$ inverse to the morphism $F(G, Y) \to F(H, W)$.]

We are now going to associate with each word (c, μ) of (G, Y) a word (c', μ') of (H, W) so that $|c', \mu'|$ is the image of $|c, \mu|$ under the above isomorphism. The idea is simply to replace each "piece" of (c, μ) in Y' by the corresponding element in $H_{(Y')}$. More precisely, let $(P_0, \dots, P_n)$ be the sequence of vertices of c and let $(y_1, \dots, y_n)$ be the sequence of its edges; let $\mu = (r_0, \dots, r_n)$. If $0 \leqslant i \leqslant j \leqslant n$ we let (c_{ij}, μ_{ij}) be the subword of (c, μ) where c_{ij} is the subpath of c consisting of $(P_i, y_i, P_{i+1}, \dots, P_j)$ and where $\mu_{ij} = (r_i, \dots, r_j)$. If c_{ij} is contained in Y' we let r_{ij} denote the element of $H_{(Y')} = F(G/Y', Y')$ associated with (c_{ij}, μ_{ij}). We define an increasing sequence of integers

$$0 \leqslant i_0 \leqslant j_0 < i_1 \leqslant j_1 < \cdots < i_m \leqslant j_m \leqslant n$$

by the following conditions:

(i) each path $c_{i_a j_a}$ $(0 \leqslant a \leqslant m)$ is contained in Y',

(ii) each vertex (resp. edge) of c which is contained in Y' belongs to one of the $c_{i_a j_a}$.

Thus the intermediate paths $c_{j_a i_{a+1}}$ are of length $\geqslant 1$, and none of their vertices (apart from the extremities) belong to Y'; these paths therefore define paths in W, which we also denote by $c_{j_a i_{a+1}}$.

Let (c', μ') be the word of (H, W) defined by:

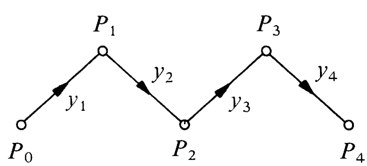
$$c' = (\dots, c_{j_{a-1} i_a}, c_{j_a i_{a+1}}, \dots)$$

$$\mu' = (\dots, \mu_{j_{a-1}+1, i_a - 1}, r_{i_a j_a}, \mu_{j_a+1, i_{a+1}-1}, \dots).$$

(Here we make the convention that $\mu_{h,k}$ is the empty sequence if $h > k$.)

Note that the path c' commences with $c_{i_o j_o}$ if $i_o = 0$ and with $c_{o i_o}$ if not; similarly μ' commences with $r_{i_o j_o}$ if $i_o = 0$, and with $\mu_{o, i_o - 1}$ if not. The situation is analogous for the "ends" of c' and μ'.

Example. Suppose that $n = 4$ and that the only vertices and the only edges of c in Y' are P_0, P_1, P_2 and y_2. We then have $c' = (y_1, y_3, y_4)$, $\mu' = (r_0, r_1 y_2 r_2, r_3, r_4)$.



It is clear that $|c, \mu|$ and $|c', \mu'|$ correspond to each other under the isomorphism of lemma 6. Moreover:

Lemma 7. *If (c, μ) is a reduced word of (G, Y) then (c', μ') is a reduced word of (H, W).*

Suppose that $l(c') = 0$, i.e. that c' reduces to a vertex P. If $P \neq (Y')$ we have $l(c) = 0$ and $c = P$; we have $\mu = (r_0)$ with $r_0 \neq 1$ in G_P, and since $H_P = G_P$ we see that (c', μ') is reduced. If $P = (Y')$ the path c is contained in Y' and we have $|c, \mu| \neq 1$ because th. 11 is assumed to apply to $(G|Y', Y')$; hence $|c', \mu'|$ is indeed $\neq 1$.

Now suppose that $l(c') \geqslant 1$ and let $w_1, \ldots, w_p$ be the edges of c'. We have to show that $w_{h+1} = \bar{w}_h \Rightarrow r'_h \notin H^{w_h}_{w_h}$, where $\mu' = (r'_0, \ldots, r'_p)$. Let $P = t(w_h)$. If $P \neq (Y')$ the above implication follows from the fact that (c, μ) is reduced. When $P = (Y')$ we distinguish two cases:

(α) (w_h, r'_h, w_{h+1}) is of the form (y_i, r_i, y_{i+1}) where i is an index such that $y_{i+1} = \bar{y}_i$. We have $r_i \notin G^{y_i}_{y_i}$ because (c, μ) is reduced. But r'_h is the image of r_i in $H_{(Y')}$ and the hypothesis on $(G|Y', Y')$ implies that $G_{t(y_i)} \to H_{(Y')}$ is injective; moreover, this homomorphism transforms $G^{y_i}_{y_i}$ into $H^{w_h}_{w_h}$. Hence r'_h indeed does not belong to $H^{w_h}_{w_h}$.

(β) (w_h, r'_h, w_{h+1}) is of the form $(y_{i_a}, r_{i_a j_a}, y_{j_a+1})$, with $i_a < j_a$ and $r_{i_a j_a} = |c_{i_a j_a}, \mu_{i_a j_a}| \in F(G|Y', Y')$. Since $l(c_{i_a j_a}) \geqslant 1$, cor. 2 to th. 11, applied to $(G|Y', Y')$, shows that $r_{i_a j_a} \notin G_Q$, where

$$Q = o(c_{i_a j_a}) = t(y_{i_a}).$$

In particular, $r_{i_a j_a}$ does not belong to the subgroup $H^{w_h}_{w_h}$ of G_Q.

Lemma 8. *If th. 11 is true for (H, W) then it is also true for (G, Y).*

This follows from lemmas 6 and 7.

Proof of theorem 11

1) *The case where Y is a segment* $\overset{P_{-1}}{\circ}\xrightarrow{\;y\;}\overset{P_1}{\circ}$.

The element $|c, \mu|$ is of the form

$$r_0 y^{e_1} r_1 y^{e_2} \cdots y^{e_n} r_n$$

with $e_i = \pm 1$, $e_{i+1} = -e_i$, $r_0 \in G_{P_{-e_1}}$, $r_i \in G_{P_{e_i}} - G^{y^{e_i}}_y$; if $n = 0$ we have $r_0 \neq 1$.

Let ϕ be the canonical homomorphism

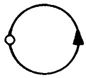
$$F(G, Y) \to \pi_1(G, Y, Y) = G_{P_{-1}} *_{G_y} G_{P_1}.$$

We have $\phi(|c, \mu|) = r_0 r_1 \cdots r_n$ and it follows from th. 1 of no. 1.2 that $r_0 r_1 \cdots r_n \neq 1$, whence $|c, \mu| \neq 1$.

2) *The case where Y is a tree.*

A standard direct limit argument reduces us to the case where Y is finite. We then argue by induction on $n = \frac{1}{2}\mathrm{Card}(\mathrm{edge}\, Y)$, the case $n = 0$ being trivial. If

$n \geqslant 1$ we take Y' to be a segment contained in Y; by 1), th. 11 applies to Y'. On the other hand, the quotient graph $W = Y/Y'$ is a tree (cf. no. 2.3, cor. 2 to prop. 13) and $\frac{1}{2}\mathrm{Card}(\text{edge } W) = n - 1$. The induction hypothesis then applies to (H, W) and lemma 8 gives our conclusion.

3) *The case where Y is a loop* 0  y.

We have determined (cf. no. 5.1, example 3) the structure of $F(G, T)$. This group is the semi-direct product of the infinite cyclic group generated by y and the normal subgroup R generated by G_0. Moreover, if we put $G_n = y^n G_0 y^{-n}$ and $A = G_y$, the group R is the sum of the G_n amalgamated according to the homomorphisms

$$G_{n-1} \leftarrow A \rightarrow G_n$$

$$y^{n-1} a^{\bar{y}} y^{1-n} \leftarrow\!\shortmid\, a \mapsto y^n a^y y^{-n} \quad \text{(cf. no. 1.4).}$$

The element $|c, \mu|$ is of the form

$$r_0 y^{e_1} r_1 y^{e_2} \cdots y^{e_n} r_n$$

with $r_i \in G_0$, $e_i = \pm 1$ and $r_i \notin A^{y^{e_i}}$ if $e_{i+1} = -e_i$.

If $\Sigma e_i \neq 0$ we have $|c, \mu| \notin R$, whence $|c, \mu| \neq 1$. Assume then that $\Sigma e_i = 0$ and put

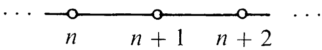
$$d_i = e_1 + \cdots + e_i \quad \text{and} \quad s_i = y^{d_i} r_i y^{-d_i};$$

we have

$$|c, \mu| = s_0 s_1 \cdots s_n \quad \text{with} \quad s_i \in G_{d_i}, \qquad d_0 = d_n = 0$$

$$d_{i+1} - d_i = e_{i+1} = \pm 1, \quad \text{and} \quad s_i \notin y^{d_i} A^{y^{e_i}} y^{-d_i} \quad \text{if} \quad d_{i+1} = d_{i-1}.$$

Then let T be the tree whose vertex set is $\mathbf{Z}$ and whose geometric edges are $\{n, n+1\}$, $n \in \mathbf{Z}$.



Let (K, T) be the tree of groups defined by the G_n and the $G_{n-1} \leftarrow A \rightarrow G_n$ as above. We have $R = \pi_1(K, T, T)$ and $s_0 \cdots s_n$ is associated with a reduced word of (K, T) whose type is a *closed path* (since $d_0 = d_n = 0$). By 2), we can apply cor. 3 of th. 11 to (K, T) and conclude that $s_0 \cdots s_n$, and hence also $|c, \mu|$, is $\neq 1$.

4) *General case.*

As in case 2) above we may assume that Y is finite, and then argue by induction on $n = \frac{1}{2}\mathrm{Card}(\text{edge } Y)$, the case $n = 0$ being trivial. If $n \geqslant 1$ we choose a subgraph Y' of Y with two edges; this is either a segment or a loop and th. 11 is applicable to $(G|Y', Y')$. On the other hand, the induction hypothesis says that th. 11 is also applicable to (H, W), where $W = Y/Y'$; the conclusion then follows by lemma 8.

Remark. One could likewise reduce the general case to the case of a tree by passing to the *universal covering* of Y, and using the exercise of no. 5.1.

Exercise

Let c be a path with vertices $P_0, \ldots, P_n$ and edges $y_1, \ldots, y_n$. Let (c, μ) and (c, μ') with

$$\mu = (r_0, \ldots, r_n), \qquad \mu' = (r_0, \ldots, r_n)$$

be two words of type c. We say that (c, μ) and (c, μ') are *equivalent* if there is a family $(a_1, \ldots, a_n)$, $a_i \in G_{y_i}$ such that

$$r'_0 = r_0 a_1^{\bar{y}_1}, \qquad a_i^{y_i} r'_i = r_i a_{i+1}^{\bar{y}_{i+1}} \quad (1 \leqslant i \leqslant n-1) \quad \text{and} \quad a_n^{y_n} r'_n = r_n.$$

(a) Show that if (c, μ) and (c, μ') are equivalent we have

$$|c, \mu| = |c, \mu'| \quad \text{in} \quad F(G, Y).$$

Moreover, if (c, μ) is reduced, so is (c, μ').

(b) Conversely, assume that $|c, \mu| = |c, \mu'|$ and that (c, μ) and (c, μ') are reduced. Show that (c, μ) and (c, μ') are equivalent.

(Argue by induction on n. Put

$$d = (c, \bar{c}) = (y_1, \ldots, y_n, \bar{y}_n, \ldots, \bar{y}_1)$$

$$v = (r'_0, \ldots, r'_{n-1}, r'_n r_n^{-1}, r_{n-1}^{-1}, \ldots, r_0^{-1})$$

so that (d, v) is a word of type d and $|d, v| = 1$. By th. 11, (d, v) is not reduced. Show that this implies $r'_n r_n^{-1} \in G_{y_n}^{y_n}$ and use the induction hypothesis.)

(c) Let $P_0 \in \operatorname{vert} Y$ and let $g \in \pi_1(G, Y, P_0)$, $g \neq 1$. Show that there is a closed path c with origin P_0 and a reduced word (c, μ) of type c such that $|c, \mu| = g$. Show that these conditions determine c uniquely, and (c, μ) up to equivalence (same argument as for (b)).

5.3 Universal covering relative to a graph of groups

We take

(G, Y):	a graph of groups, with Y connected and non-empty
T:	a maximal tree of Y
A:	an orientation of Y.

If $y \in \operatorname{edge} Y$, put

$$e(y) = \begin{cases} 0 & \text{if} \quad y \in A \\ 1 & \text{if} \quad y \notin A \end{cases}$$

and let $|y|$ denote the one of the two edges y, $\bar{y}$ which belongs to A.

Definition of the universal covering

We propose to construct the following objects:

- a graph $\tilde{X} = \tilde{X}(G, Y, T)$;
- an action of $\pi = \pi_1(G, Y, T)$ on $\tilde{X}$;
- a morphism $p: \tilde{X} \to Y$ which induces an isomorphism $\pi\backslash\tilde{X} \xrightarrow{\sim} Y$;
- sections $\operatorname{vert} Y \to \operatorname{vert} \tilde{X}$ and $\operatorname{edge} Y \to \operatorname{edge} \tilde{X}$ of p (these sections will be denoted $P \mapsto \tilde{P}$ and $y \mapsto \tilde{y}$).

If $P \in \operatorname{vert} Y$ we also require the stabilizer $\pi_{\tilde{P}}$ of $\tilde{P}$ in π to equal G_P. Likewise, if $y \in \operatorname{edge} Y$, the stabilizer $\pi_{\tilde{y}}$ must equal the subgroup G_w^w of $G_{t(w)}$, where $w = |\tilde{y}|$.

These conditions force $\operatorname{vert} \tilde{X}$ to be the disjoint union of the orbits $\pi \cdot \tilde{P} \simeq \pi/\pi_{\tilde{P}}$, and $\operatorname{edge} \tilde{X}$ to be the disjoint union of the orbits $\pi \cdot \tilde{y} \simeq \pi/\pi_{\tilde{y}}$. We therefore *put*

$$\operatorname{vert} \tilde{X} = \coprod_{P \in \operatorname{vert} Y} \pi/\pi_{\tilde{P}}, \qquad \operatorname{edge} \tilde{X} = \coprod_{y \in \operatorname{edge} Y} \pi/\pi_{\tilde{y}}$$

where

$$\pi_{\tilde{P}} = G_P, \qquad \pi_{\tilde{y}} = G_w^w \quad (w = |\tilde{y}|)$$

are the groups defined above, and we let $\tilde{P}$ (resp. $\tilde{y}$) denote the image of 1 in $\pi/\pi_{\tilde{P}}$ (resp. $\pi/\pi_{\tilde{y}}$). It remains to define $\overline{g\tilde{y}}$, $o(g\tilde{y})$ and $t(g\tilde{y})$ for $g \in \pi$, $y \in \operatorname{edge} Y$. We take

$$\overline{g\tilde{y}} = g\tilde{\bar{y}}$$

$$o(g\tilde{y}) = g g_y^{-e(y)} \widetilde{o(y)}$$

$$t(g\tilde{y}) = g g_y^{1-e(y)} \widetilde{t(y)}$$

(g_y denotes the image of y in π, cf. no. 5.1). Of course, it is necessary to check that these expressions are legitimate. For the first, this follows from the fact that $\pi_{\tilde{y}} = \pi_{\tilde{\bar{y}}}$. For the second we have to show that if $h \in \pi_{\tilde{y}}$ then

$$h g_y^{-e(y)} \widetilde{o(y)} = g_y^{-e(y)} \widetilde{o(y)}$$

i.e.

$$g_y^{e(y)} \pi_{\tilde{y}} g_y^{-e(y)} \subset \pi_{\widetilde{o(y)}} = G_{o(y)}.$$

However, if $e(y) = 0$, we have $|y| = y$ and $\pi_{\tilde{y}} = G_{\bar{y}}^{\bar{y}} \subset G_{o(y)}$ whence the relation above. If $e(y) = 1$ we have $\pi_{\tilde{y}} = G_y^y$ and the relation $g_y a^y g_y^{-1} = a^{\bar{y}}$ ($a \in G_y$) shows that $g_y \pi_{\tilde{y}} g_y^{-1} = G_{\bar{y}}^{\bar{y}} \subset G_{o(y)}$, which gives the relation sought. Note also that the equation

$$\pi_{\tilde{y}} = g_y^{e(y)-1} G_y^y g_y^{1-e(y)}$$

is valid without any hypothesis on $e(y)$.

It remains to show that the third formula is legitimate, i.e. that we have

$$g_y^{e(y)-1}\pi_{\tilde{y}}g_y^{1-e(y)} \subset \pi_{\widetilde{t(y)}} = G_{t(y)}.$$

This follows by the same type of computation (we can also replace y by $\bar{y}$, which reduces us to the previous case).

We have now defined the graph $\tilde{X}$ as well as the action of π on $\tilde{X}$; we have $\pi\backslash\tilde{X} = Y$. Moreover, if $y \in \operatorname{edge} T$, we have $g_y = 1$, whence $o(\tilde{y}) = \widetilde{o(y)}$, $t(\tilde{y}) = \widetilde{t(y)}$, and we see that $P \mapsto \tilde{P}$, $y \mapsto \tilde{y}$ defines a *lifting* $T \xrightarrow{\sim} \tilde{T} \subset \tilde{X}$ of T into $\tilde{X}$.

Theorem 12. *Let (G, Y) be a connected, non-empty graph of groups, let T be a maximal tree of Y and let A be an orientation of Y. Then the graph $\tilde{X} = \tilde{X}(G, Y, T)$ constructed above is a tree.*

We first show that $\tilde{X}$ is *connected.* If $y \in \operatorname{edge} Y$ we have $o(\tilde{y}) = \widetilde{o(y)}$ if $e(y) = 0$ and $t(\tilde{y}) = \widetilde{t(y)}$ if $e(y) = 1$; so one of the extremities of $\tilde{y}$ belongs to the tree $\tilde{T}$. This shows that the smallest subgraph W of $\tilde{X}$ containing all the $\tilde{y}, y \in \operatorname{edge} Y$, is connected; moreover, $\pi \cdot W = \tilde{X}$. It then suffices to show that there is a subset S of π, generating π, and such that $W \cup sW$ is connected for all $s \in S$ (this will imply, by induction on n, that

$$W \cup s_1 W \cup s_1 s_2 W \cup \cdots \cup s_1 \cdots s_n W$$

is connected for any $s_1, \ldots, s_n \in S \cup S^{-1}$). We take S to be the union of the $G_{\tilde{P}}$ ($P \in \operatorname{vert} Y$) and the $\{g_y\}, y \in \operatorname{edge} Y$. If $s \in G_{\tilde{P}}$ the graphs W and sW have a common vertex $\tilde{P}$ and $W \cup sW$ is certainly connected; likewise W and $g_y W$ have a common vertex, namely

$$o(\tilde{y}) = o(g_y\tilde{y}) \quad \text{if} \quad e(y) = 1$$

and

$$t(\tilde{y}) = t(g_y\tilde{y}) \quad \text{if} \quad e(y) = 0.$$

To show that $\tilde{X}$ is a tree, it now suffices to prove that $\tilde{X}$ *does not contain any closed path of length $n > 0$ without backtracking.*

Let $\tilde{c}$ be such a path, let $(s_1\tilde{y}_1, \ldots, s_n\tilde{y}_n)$ be the sequence of its edges, and let $(P_0, \ldots, P_n)$ be the sequence of vertices of the projection c of $\tilde{c}$ in Y; we have $P_0 = P_n$. If we put $e_i = e(y_i)$ and $g_i = g_{y_i}$ we have

$$t(s_n\tilde{y}_n) = s_n g_n^{1-e_n}\tilde{P}_0 = s_1 g_1^{-e_1}\tilde{P}_0 = o(s_1\tilde{y}_1)$$

$$t(s_1\tilde{y}_1) = s_1 g_1^{1-e_1}\tilde{P}_1 = s_2 g_2^{-e_2}\tilde{P}_1 = o(s_2\tilde{y}_2)$$

$$\cdots$$

$$t(s_{n-1}\tilde{y}_{n-1}) = s_{n-1} g_{n-1}^{1-e_{n-1}}\tilde{P}_{n-1} = s_n g_n^{-e_n}\tilde{P}_{n-1} = o(s_n\tilde{y}_n).$$

Putting $q_i = s_i g_i^{-e_i}$, this gives

$$q_n g_n r_n = q_1$$

$$q_1 g_1 r_1 = q_2$$

$$\cdots$$

$$q_{n-1} g_{n-1} r_{n-1} = q_n$$

with $r_i \in G_{P_i} = \pi_{\tilde{P}_i}$. Whence

$$g_1 r_1 = q_1^{-1} q_2,\ g_2 r_2 = q_2^{-1} q_3,\ \ldots,\ g_n r_n = q_n^{-1} q_1$$

and by multiplying we obtain the relation

$$(*) \qquad g_1 r_1 g_2 r_2 \cdots g_n r_n = 1.$$

Now let (c, μ) be the word of type c defined by the sequence $\mu = (1, r_1, \ldots, r_n)$. We will show that (c, μ) is *reduced.*

Indeed, suppose that $y_{i+1} = \bar{y}_i$, so we have $g_{i+1} = g_i^{-1}$ and $e_{i+1} = 1 - e_i$. The formula

$$(s_i g_i^{-e_i}) \cdot g_i r_i = s_{i+1} g_{i+1}^{-e_{i+1}}$$

then shows that $r_i = g_i^{e_i - 1}(s_i^{-1} s_{i+1}) g_i^{1-e_i}$. We have to prove that $r_i \notin G_{y_i}^{y_i}$, i.e., that

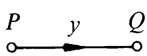
$$s_i^{-1} s_{i+1} \notin g_i^{1-e_i} G_{y_i}^{y_i} g_i^{e_i - 1}.$$

But the latter group is equal to $\pi_{\tilde{y}_i}$, and we certainly have $s_i \pi_{\tilde{y}_i} \neq s_{i+1} \pi_{\tilde{y}_i}$ because $s_{i+1} \tilde{y}_{i+1} = s_{i+1} \tilde{y}_i$ is different from $s_i \tilde{y}_i$ (it is here that we use the hypothesis that $\tilde{c}$ has no backtracking).

Because c is closed and (c, μ) is reduced, cor. 3 to th. 11 of no. 5.2 shows that the image of $|c, \mu|$ in $\pi_1(G, Y, T)$ is $\neq 1$. Since this image is $g_1 r_1 \cdots g_n r_n$, formula $(*)$ is contradicted and the theorem is proved.

Examples

1) When all the G_P equal $\{1\}$ we have $\pi = \pi_1(Y, T)$ and $\tilde{X}$ is the *universal covering* of Y (in the usual sense) relative to T.

2) When Y is a segment 

, $\tilde{X}$ is the tree associated with the amalgam $\pi = G_P *_{G_y} G_Q$, cf. no. 4.1.

5.4 Structure theorem

Let G be a group which acts without inversion on a connected nonempty graph X. We shall see (cf. th. 13 below) that, if X is a tree, then G can be identified with the fundamental group of a certain graph of groups (G, Y), where $Y = G\backslash X$.

We begin with the construction of (G, Y). Let T be a maximal tree of Y and let $j: T \to X$ be a lifting of T (cf. no. 3.1, prop. 14). Let A be an orientation of Y and put

$$e(y) = \begin{cases} 0 & \text{if } \ y \in A \\ 1 & \text{if } \ y \notin A \quad (\text{i.e. if } \bar{y} \in A). \end{cases}$$

We shall extend j to a section j: edge $Y \to$ edge X such that $j\bar{y} = \overline{jy}$; it suffices to define jy for $y \in A -$ edge T; in this case we choose jy so that $o(jy) \in \operatorname{vert} jT$; we then have $o(jy) = jo(y)$. We also choose $\gamma_y \in G$ so that $t(jy) = \gamma_y jt(y)$; this is possible because $t(jy)$ and $jt(y)$ have the same projection $t(y)$ in vert Y. We then extend $y \mapsto \gamma_y$ to all of edge Y by the formulae

$$\gamma_{\bar{y}} = \gamma_y^{-1} \quad \text{and} \quad \gamma_y = 1 \quad \text{if} \quad y \in \text{edge } T.$$

For each $y \in$ edge Y we have

$$o(jy) = \gamma_y^{-e(y)} jo(y)$$

$$t(jy) = \gamma_y^{1-e(y)} jt(y).$$

Let G_Q (resp. G_z) denote the stabilizer of a vertex Q (resp. edge z) of X. The graph of groups (G, Y) is then defined by the formulae

$$G_P = G_{jP} \quad \text{for} \quad P \in \text{vert } Y$$

$$G_y = G_{jy} \quad \text{for} \quad y \in \text{edge } Y,$$

the homomorphism $G_y \to G_{t(y)}$ being $a \mapsto a^y = \gamma_y^{e(y)-1} a \gamma_y^{1-e(y)}$. (The latter definition is legitimate since $\gamma_y^{e(y)-1} G_{jy} \gamma_y^{1-e(y)} \subset G_{jt(y)}$ for all $y \in$ edge Y.) Let $\phi: \pi_1(G, Y, T) \to G$ be the homomorphism defined by the inclusions $G_P \to G$ and by $\phi(g_y) = \gamma_y$. Let

$$\psi: \quad \tilde{X}(G, Y, T) \to X$$

be the map defined by

$$\psi(g\tilde{P}) = \phi(g) jP \quad \text{and} \quad \psi(g\tilde{y}) = \phi(g) jy.$$

One checks that ψ is a graph morphism and that ψ is ϕ-equivariant.

Let W be the smallest subgraph of X which contains jy for all $y \in$ edge Y. Each edge of W has an extremity in jT and we have $G \cdot W = X$. Moreover, W is contained in $\psi(\tilde{X})$ and ϕ induces isomorphisms between the stabilizers of corresponding vertices and edges in $\tilde{X}$ and X. Applying lemma 4 of no. 4.1 we see that ϕ (and hence also ψ) is *surjective*; in addition, the fact that ϕ induces isomorphisms between the stabilizers of the vertices and edges shows that ψ is *locally injective* (cf. no. 4.5).

Theorem 13. *With the above notation and hypotheses, the following properties are equivalent*:

(1) *X is a tree.*

(2) *$\tilde{X} \xrightarrow{\psi} X$ is an isomorphism.*

(3) *$\pi_1(G, Y, T) \xrightarrow{\phi} G$ is an isomorphism.*

(The most interesting assertion is the implication (1) ⇒ (3): if X is a tree, G is generated by the G_P ($P \in \operatorname{vert} Y$) and the γ_y ($y \in \operatorname{edge} Y$), the latter being subject to the relations of no. 5.1

$$\gamma_y a^y \gamma_y^{-1} = a^{\bar{y}}, \qquad \gamma_{\bar{y}} = \gamma_y^{-1} \quad \text{and} \quad \gamma_y = 1 \quad \text{if} \quad y \in \operatorname{edge} T.)$$

Proof

The implication (1) ⇒ (2) follows from lemma 5 of no. 4.5 because we already know that ψ is surjective and locally injective; the implication (2) ⇒ (1) follows from th. 12 of no. 5.3.

On the other hand, let N be the kernel of ϕ and let $P \in \operatorname{vert} Y$. We have $N \cap G_{\tilde{P}} = \{1\}$ because ϕ defines an isomorphism of $G_{\tilde{P}}$ onto G_{jP}. If n is a non-trivial element of N, the two vertices $\tilde{P}$ and $n\tilde{P}$ of $\tilde{X}$ are distinct and have the same image jP in X; hence (2) ⇒ (3). The implication (3) ⇒ (2) is clear.

Corollary 1. *Suppose that X is a tree. Let R be the subgroup of G generated by the G_P, $P \in \operatorname{vert} X$. Then R is a normal subgroup of G, and G/R can be identified with the fundamental group of the graph $Y = G\backslash X$.*

It is clear that R is the smallest normal subgroup of G containing the $G_P = G_{jP}$ for $P \in \operatorname{vert} Y$. The assertion about G/R then follows from (1) ⇒ (3) combined with the *Remark* of no. 5.1.

Corollary 2. *Suppose that X is a tree and that $Y = G\backslash X$ is a loop. Then G is an HNN group.*

This follows from (1) ⇒ (3) together with example 3 of no. 5.1.

Exercises

1) Give a direct proof of cor. 1 to th. 13.

2) Suppose that X is a tree. Show the equivalence of:
(a) Y is a tree;
(b) G is generated by the G_P, $P \in \operatorname{vert} X$;
(c) G is generated by the G_P, $P \in \operatorname{vert} Y$.

3) With the hypotheses of th. 13, show that the kernel N of ϕ is isomorphic to the fundamental group of X.

5.5 Application: Kurosh's theorem

Let $H = *_A H_i$ be the sum of a family of groups $(H_i)_{i\in I}$ amalgamated along a common subgroup A (cf. no. 1.2). Let G be a subgroup of H. If $x \in H/H_i$ the subgroup xH_ix^{-1} is well-defined; we put $G_{i,x} = G \cap xH_ix^{-1}$. The group $G_{i,x}$ is the stabilizer of x under the natural action of G on H/H_i.

Theorem 14. *Suppose that $G - \{1\}$ does not meet any conjugate of A. One can then find:*

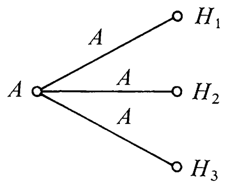
(a) *a free subgroup F of G,*

(b) *for each $i \in I$, a subset X_i of H/H_i which is a system of coset representatives for $G\backslash H/H_i$,*

such that $$G = \left(\mathop{*}_{i\in I,\ x\in H_i} G_{i,x} \right) * F.$$

(In other words, G is the free product of its intersections with the conjugates of the H_i (suitably indexed) and a free group.)

Let (H, T) be the tree of groups defined by the H_i and A (cf. no. 4.4, example 3). We have $H = \varinjlim (H, T)$.



Let X be the corresponding tree (cf. th. 9). The group H acts on X with T as fundamental domain; the stabilizers of the edges are the conjugates of A; those of the vertices are the conjugates of the H_i and of A.

Since G is a subgroup of H, it acts on X and we can apply th. 13. We then see that $G = \pi_1(G, Y, T)$, where $Y = G\backslash X$ and T is a maximal tree of Y, a lifting of which is chosen in X. The hypothesis that $G - \{1\}$ meets no conjugate of A is equivalent to saying that $G_y = \{1\}$ for each $y \in \text{edge } Y$. By example 1 of no. 5.1 we then have

$$G = \pi_1(G, Y, T) \simeq (\mathop{*}_P G_P) * F$$

where F is a free group (isomorphic to $\pi_1(Y, T)$), and where P runs over the set of vertices of T. But we have

$$\text{vert } X \cong H/A \coprod_{i\in I} H/H_i$$

and

$$\text{vert } T \cong G\backslash H/A \coprod G\backslash H/H_i.$$

The lifting of T into X then defines the system of representatives

$$X_A \subset H/A \quad \text{and} \quad X_i \subset H/H_i$$

of $G\backslash H/A$ and $G\backslash H/H_i$ respectively; if x belongs to X_A (resp. X_i) the corresponding group G_P is $G \cap xAx^{-1}$ (resp. $G \cap xH_ix^{-1}$). Whence the theorem, since the $G \cap xAx^{-1}$ reduce to $\{1\}$.

Remarks

1) The hypothesis $G \cap xAx^{-1} = \{1\}$ for each $x \in H$ is satisfied when $A = \{1\}$, i.e. *when H is the free product of the H_i*; th. 14 then reduces to a theorem of Kurosh.

2) One can sharpen the statement of the theorem (as we did in 3.4 for Schreier's theorem) by giving:

a) conditions the X_i must satisfy (they express the fact that T must be connected);

b) a procedure for constructing a basis for F (the elements of this basis are the g_y of no. 5.1, where y runs through A – edge T, where A is an orientation of $Y = G\backslash X$).

3) When $G - \{1\}$ meets one of the conjugates of A it is still true that $G \simeq \pi_1(G, Y, T)$, but the structure of this π_1 is less simple (cf. no. 5.1).

Exercises

The hypotheses and notations are those of th. 14.

1) Let R be the subgroup of G generated by the $G \cap xH_ix^{-1}$, $i \in I$, $x \in H$. Show that R is the smallest normal subgroup of G which contains the $G_{i,x}$, $i \in I$, $x \in X_i$. Deduce that G/R is isomorphic to F.

2) Suppose that I and $G\backslash H/A$ are finite and put

$$a = \mathrm{Card}(G\backslash H/A), \qquad x_i = \mathrm{Card}(G\backslash H/H_i).$$

Show that the rank r_F of the free group F is given by the formula

$$r_F = 1 - a + \sum_{i \in I} (a - x_i).$$

3) Under the following hypotheses

a) $A = \{1\}$, i.e. H is the free product of the H_i,

b) G is *indecomposable*, i.e. any decomposition $G = B * C$ implies $B = \{1\}$ or $C = \{1\}$,

c) G is not isomorphic to $\mathbf{Z}$,

show that G is contained in a conjugate of one of the H_i (apply th. 14).

§6 Amalgams and fixed points

We say that a group G *is an amalgam* if it can be written $G \simeq G_1 *_A G_2$ with $G_1 \neq A \neq G_2$. In what follows we show that some groups, for example $\mathbf{SL}_3(\mathbf{Z})$, are *not* amalgams; as we shall see, this amounts to proving that there are necessarily *fixed points* when these groups act on trees.

Convention. Each group acting on a tree is assumed to act *without inversion*, cf. 3.1.

6.1 The fixed point property for groups acting on trees

Let G be a group acting (without inversion) on a tree X. The set X^G of fixed points of G in X is a subgraph of X; if P and Q are two vertices of X^G, the geodesic joining P to Q is fixed by G and therefore contained in X^G; it follows that, if X^G is non-empty, it is a tree. We are interested in the groups G with the property:

(FA) *$X^G \neq \emptyset$ for any tree X on which G acts.*

This property is "almost" equivalent to that of not being an amalgam. More precisely:

Theorem 15. *Suppose that G is denumerable. Then G has property* (FA) *if and only if the following three conditions are satisfied*:

(i) *G is not an amalgam*;

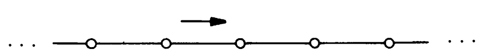
(ii) *G has no quotient isomorphic to* $\mathbf{Z}$;

(iii) *G is finitely generated.*

Proof

(FA) $\Rightarrow$ (i): If G is an amalgam $G_1 *_A G_2$ with $G_1 \neq A$ and $G_2 \neq A$ there is a tree X on which G acts, with a segment PQ as fundamental domain, the stabilizers of P, Q being G_1, G_2 respectively, cf. no. 4.1, th. 7. Since G is distinct from G_1 and G_2, we then have $X^G = \emptyset$, which contradicts (FA).

(FA) $\Rightarrow$ (ii): If G has a quotient isomorphic to $\mathbf{Z}$ we can make it act by translations on the doubly infinite chain



and this contradicts (FA).

(FA) $\Rightarrow$ (iii): Since G is denumerable, it is the union of an increasing sequence $G_1 \subset G_2 \subset \cdots \subset G_n \subset \cdots$ of finitely generated subgroups. We form a graph X whose vertex set is the disjoint union of the sets G/G_n, two vertices being joined by an edge if and only if they belong to two consecutive sets G/G_n and G/G_{n+1} and correspond under the canonical map $G/G_n \to G/G_{n+1}$. One checks that X is a tree; moreover, G acts in an obvious way on X. If G has property (FA) there is a vertex P

of X invariant under G; if $P \in G/G_n$ this implies that $G_n = G$, so G is finitely generated.

Conversely, suppose that G has properties (i), (ii) and (iii) and that it acts on a tree X. If $T = G \backslash X$ denotes the quotient of X by G, the fundamental group $\pi_1(T)$ of the graph T is isomorphic to a quotient of G (no. 5.4, cor. 1 to th. 13). Since $\pi_1(T)$ is a free group this is impossible, by (ii), unless $\pi_1(T) = \{1\}$. Therefore, T is a tree, and we can lift it to a subtree of X, cf. no. 3.1. The group G can then be identified with the group $G_T = \varinjlim (G, T)$, limit of the tree of groups defined by the groups G_P and G_y which fix the vertices P and edges y of T, cf. no. 4.5, th. 10. It follows that G is the union of the groups $G_{T'} = \varinjlim (G, T')$ where T' runs through the set of *finite* subtrees of T. Since G is finitely generated, there is a T' such that $G = G_{T'}$; we choose a minimal T' with this property. If T' reduces to a single vertex P we have $G = G_P$ and G has a fixed point. If not, T' has a terminal vertex P, and $T'' = T' - \{P\}$ is a tree (no. 2.2, prop. 9); if y denotes the unique edge which joins P to T'' we have

$$G = G_{T'} = G_{T''} *_A G_P, \quad \text{where} \quad A = G_y.$$

In view of the minimality hypothesis on T', we have $G_{T''} \neq G$ and $G_P \neq G$, which shows that G is an amalgam and contradicts hypothesis (i).

Remarks

1) When G is not denumerable, th. 15 remains valid provided condition (iii) is replaced by:

(iii′) *G is not the union of a strictly increasing sequence of subgroups.*

Examples of (non-denumerable) groups satisfying conditions (i), (ii) and (iii′) have been constructed by J. Tits and S. Koppelberg (*C. R. Acad. Sci. Paris*, 279, 1974, pp. 583–585).

2) More complete results are obtained by taking account of the fixed points of G not only on the *vertices*, but also on the *ends* of the trees on which it acts. Cf. no. 6.5, exerc. 2, as well as H. Bass [9] and J. Tits [12].

6.2 Consequences of property (FA)

Proposition 21. *Let G be a group with property* (FA). *If G is contained in an amalgam $G_1 *_A G_2$, then G is contained in a conjugate of G_1 or of G_2.*

This simply restates the fact that G has a fixed point in the tree associated with the amalgam $G_1 *_A G_2$, cf. no. 4.1.

Proposition 22. *Let G be a denumerable group with property* (FA), *and let $\rho: G \to \mathbf{GL}_2(k)$ be a linear representation of G of degree 2 over a commutative field k. Then for each $s \in G$ the eigenvalues of $\rho(s)$ are integral over $\mathbf{Z}$.*

(When k is of characteristic 0 these eigenvalues are then *algebraic integers*; when k is of characteristic $\neq 0$ they are *roots of unity*.)

Let k_ρ be the subfield of k generated by the coefficients of the matrices $\rho(s)$ for $s \in G$. By th. 15, G is finitely generated, so k_ρ is finitely generated over the prime field $\mathbf{Q}$ or $\mathbf{F}_p$. Let v be a discrete valuation of k_ρ, and let O_v be the corresponding valuation ring. We let X_v denote the tree associated with v (chap. II, §1), a tree on which $\mathbf{GL}_2(k_\rho)$ acts. Let $\mathbf{GL}_2(k_\rho)^0$ be the kernel of the homomorphism

$$v \circ \det : \mathbf{GL}_2(k) \to \mathbf{Z} \quad \text{cf. chap. II, no. 1.2, p. 75.}$$

Condition (ii) of th. 15 shows that $\rho(G)$ is contained in $\mathbf{GL}_2(k_\rho)^0$, which acts without inversion on X_v. Since G has property (FA), there is a vertex of X_v invariant under G. This implies (chap. II, no. 1.3) that $\rho(G)$ is contained in a conjugate of $\mathbf{GL}_2(O_v)$. Thus for each $s \in G$ the coefficients of the characteristic polynomial of s belong to the intersection of the O_v. But it is known that this intersection is equal to the set of elements of k_ρ which are integral over $\mathbf{Z}$ (cf. for example Grothendieck, *EGA* II, p. 140, cor. 7.1.8); the eigenvalues of the $\rho(s)$ are then integers over $\mathbf{Z}$.

Exercise

Let A be a subgroup of a group H, and let $\theta : A \to H$ be an injective homomorphism of A into H; let $\tilde{H}$ be the group derived from (A, H, θ) by the *HNN* construction, cf. nos. 1.4 and 5.1. Show that every subgroup of $\tilde{H}$ which has property (FA) is contained in a conjugate of H.

6.3 Examples

6.3.1. A finitely generated *torsion* group has property (FA).

In view of th. 15, it suffices to check that such a group cannot be an amalgam $G_1 *_A G_2$. But this is clear, because if we take $s_1 \in G_1 - A$ and $s_2 \in G_2 - A$ the element $s_1 s_2$ is cyclically reduced (in the sense of no. 1.3) and hence has infinite order, cf. prop. 2.

6.3.2. If G has property (FA), so has every *quotient* of G.

This is clear.

6.3.3. Let H be a *normal subgroup* of G. If H and G/H have property (FA), then so has G.

Indeed, if the group G acts on a tree X, the group G/H acts on the tree X^H, so there is a fixed point.

6.3.4. Let G' be a *subgroup of finite index* in G. If G acts on a tree X and if $X^{G'} \neq \emptyset$, then $X^G \neq \emptyset$.

Indeed, let H be a normal subgroup of finite index in G contained in G' (for example, the intersection of the conjugates of G'). We have $X^H \neq \emptyset$ and G/H acts on the tree X^H; since G/H is finite, it has a fixed point; whence $X^G \neq \emptyset$.

In particular, *if G' has property* (FA), *then the same is true of* G.

6.3.5. On the other hand, it is not true that if G has property (FA) then so have its subgroups of finite index. Here is a counter-example (we shall see others in no. 6.5, exerc. 3 and 4): take G to be the Schwarz group defined by the two generators a and b and the relations

$$a^A = b^B = (ab)^C = 1$$

where A, B, C are integers $\geqslant 2$. The group G has property (FA). Indeed, if G acts on a tree, the elements a, b and ab each have a fixed point (because they are of finite order, cf. 4.3, prop. 19), and cor. 1 to prop. 26 of no. 6.5 below shows that there is a fixed point common to a and b, and hence fixed by G. However, if A, B, C are such that $1/A + 1/B + 1/C \leqslant 1$ it is well known that G contains a subgroup H of finite index isomorphic to the fundamental group of a compact orientable surface of genus $\geqslant 1$; such a subgroup H has a quotient isomorphic to $\mathbf{Z}$, and therefore it does not satisfy (FA).

6.3.6. We shall see other examples of groups with property (FA) in no. 6.6.

6.4 Fixed points of an automorphism of a tree

The geodesic joining two subtrees

If P and Q are two vertices of a tree we let P–Q denote the geodesic joining P to Q (no. 2.2, prop. 8).

Lemma 9. *Let T_1 and T_2 be two disjoint subtrees of a tree X, and let n be the distance between T_1 and T_2* (i.e. the infimum of the $l(P, Q)$ for $P \in \operatorname{vert} T_1$ and $Q \in \operatorname{vert} T_2$).

(a) *There is exactly one pair* (P_1, P_2) *in* $\operatorname{vert} T_1 \times \operatorname{vert} T_2$ *such that* $l(P_1, P_2) = n$.

(b) $l(Q_1, Q_2) = l(Q_1, P_1) + n + l(P_2, Q_2)$ *if* $Q_1 \in \operatorname{vert} T_1$ *and* $Q_2 \in \operatorname{vert} T_2$.

(c) *Every subtree of X meeting both T_1 and T_2 contains the geodesic P_1–P_2.* (We say that P_1–P_2 is the geodesic *joining* T_1 to T_2.)

Let $P_1 \in \operatorname{vert} T_1$ and $P_2 \in \operatorname{vert} T_2$ be such that $l(P_1, P_2) = n$. The vertices of P_1–P_2 other than the extremities belong neither to T_1 nor to T_2. It follows that if $Q_1 \in \operatorname{vert} T_1$, $Q_2 \in \operatorname{vert} T_2$ the path obtained by juxtaposing the geodesics Q_1–P_1, P_1–P_2 and P_2–Q_2 is without backtracking. We then have

$$l(Q_1, Q_2) = l(Q_1, P_1) + n + l(P_2, Q_2)$$

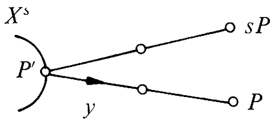
which proves (b); the assertions (a) and (c) follow immediately.

Automorphisms with fixed points

Let s be an automorphism (without inversion) of a tree X. We say that *s has a fixed point* if the subgraph X^s of X consisting of the points fixed by s is non-empty, in which case it is a tree (6.1).

Proposition 23. *Suppose that s has a fixed point. Let $P \in \operatorname{vert} X$, let n be the distance between P and X^s, and let P–P' be the geodesic joining P to X^s. Then the geodesic P–sP is obtained by juxtaposing the geodesics P–P' and P'–$sP = s(P'$–$P)$.*

This is clear if $n = 0$ since then $P = P' = sP$. If $n \geqslant 1$ we let y denote the edge of P–P' with origin P'. We have $sy \neq y$ otherwise the extremity of y would belong to X^s. It follows that the path obtained by juxtaposing P–P' and $s(P'–P) = P'–sP$ is without backtracking; it is therefore the geodesic joining P to sP.



Corollary 1. $l(P, sP) = 2n$.

Corollary 2. *The midpoint of the geodesic P–sP is fixed by s.*

Indeed this midpoint is none other than P', which belongs to X^s.

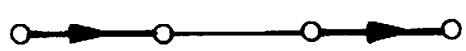
Corollary 3. *Suppose $n \geqslant 1$. Let P_1, P_2 be the vertices of P–sP at distance 1 from P, sP respectively. Then $sP_1 = P_2$.*

This follows from the fact that s transforms P–P' into sP–P'.

Let y and y' be two edges. Put

$$a = l(o(y), o(y')), \qquad b = l(t(y), t(y')).$$

Then $b - a = 0, 2$ or -2. We say that y and y' are

coherent if $b = a$ 

convergently incoherent if $b = a - 2$ 

divergently incoherent if $b = a + 2$ 

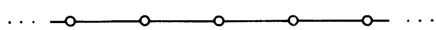
(In other words, y and y' are coherent if and only if they are oriented in the same way in the unique geodesic which contains them.)

With this terminology, cor. 3 is equivalent to:

Corollary 4. *If $y \in \operatorname{edge} X$ is not fixed by s then y and sy are incoherent.*

Automorphisms without fixed points

A doubly infinite chain



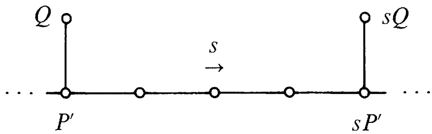
will be called a *straight path*.

Proposition 24 (Tits). *Suppose the automorphism s has no fixed points. Put*

$$m = \operatorname*{Inf}_{P \in \operatorname{vert} X} l(P, sP) \quad \text{and} \quad T = \{P \in \operatorname{vert} X \mid l(P, sP) = m\}.$$

Then:

i) *T is the vertex set of a straight path of X.*
ii) *s induces a translation of T of amplitude m.*
iii) *Every subtree of X stable under s and s^{-1} contains T.*
iv) *If a vertex Q of X is at a distance d from T then $l(Q, sQ) = m + 2s$*

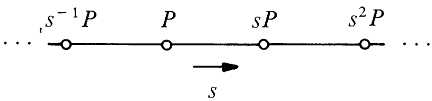


$(m = 4,\ d = 1)$

Proof

Let $P \in T$ and let $P_0 = P, P_1, \ldots, P_m = sP$ be the vertices of the geodesic $c = P\text{–}sP$. The geodesics c and sc define a path without backtracking joining P and s^2P. If not, we would have $P_{m-1} = sP_1$; but this is impossible when $m = 1$ because s acts without inversion, and it is impossible when $m \geqslant 2$ because the distance $l(P_1, sP_1)$ would then be $m - 2 < m$. It follows immediately by induction that the geodesics $s^n c$ $(n \in \mathbf{Z})$ form a straight path

(T_P)



stable under s, and that s is a translation of amplitude m on T_P. If Q is a vertex of X at distance d from T_P, and if P' is the vertex of T_P nearest to Q (cf. lemma 9), the path consisting of the geodesics $Q\text{–}P'$, $P'\text{–}sP'$ and $sP'\text{–}sQ$ is without backtracking. We then have $l(Q, sQ) = d + m + d = m + 2d$. In particular, we cannot have $l(Q, sQ) = m$ unless $d = 0$; whence the fact that $T = T_P$, which proves i), ii) and iv). Finally, if a subtree X' of X is stable under s and s^{-1}, and if we choose Q in X', the geodesic $Q\text{–}sQ$ is contained in X'; with the notation above, the same is true for $P'\text{–}sP'$, and hence also for T_P, which is the union of the $s^n(P'\text{–}sP')$.

Corollary. *Let $y \in \operatorname{edge} X$. Then y and sy are coherent if and only if y is an edge of the straight path T associated with s.*

This follows, for example, from iv).

Combining props. 23 and 24 we obtain:

Proposition 25. *Let s be an automorphism of a tree X. The following properties are equivalent*:

(a) *s has no fixed points;*
(b) *there is an edge y of X such that y and sy are coherent and distinct;*
(c) *there is a straight path X stable under s, on which s induces a translation of non-zero amplitude.*

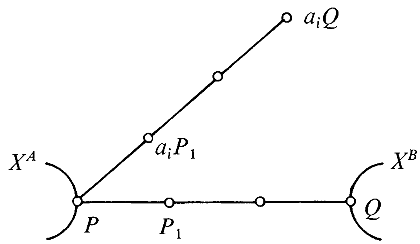
An automorphism with these properties is sometimes called *hyperbolic* (this terminology arises from the case of $\mathbf{SL}_2$, cf. chap. II, no. 1.3).

Exercise

With the notation of prop. 24, let Γ be the group generated by s. Show that Γ acts freely on X, and that the quotient graph $\Gamma\backslash X$ contains exactly one circuit, namely $\Gamma\backslash T$; this circuit is of length m. The injection $\Gamma\backslash T \to \Gamma\backslash X$ is a homotopy equivalence.

6.5 Groups with fixed points (auxiliary results)

Proposition 26. *Let G be a group generated by elements a_i, b_j and let A, B be the subgroups of G generated by the a_i and b_j respectively. Assume that G acts on a tree X so that $X^A \neq \emptyset$, $X^B \neq \emptyset$ and so that, for each pair (i, j) the automorphism $a_i b_j$ has a fixed point. Then G has a fixed point* (i.e. $X^G \neq \emptyset$).



We have $X^G = X^A \cap X^B$. Suppose that the trees X^A and X^B are disjoint, and let P–Q be the geodesic joining them, with $P \in \operatorname{vert} X^A$, $Q \in \operatorname{vert} X^B$ (cf. lemma 9). Let P_1 be the vertex of this geodesic at distance 1 from P. We have $P_1 \notin X^A$ and there is an index i such that $a_i P_1 \neq P_1$. The path obtained by juxtaposing the geodesics Q–P and P–$a_iQ = a_i(P$–$Q)$ is therefore without backtracking: it is the geodesic Q–a_iQ, with midpoint P. But we have $a_iQ = a_ib_jQ$ for all j; since a_ib_j has a fixed point, cor. 2 to prop. 23 shows that the midpoint P of Q–a_ib_jQ is fixed by a_ib_j. We then have $a_ib_jP = P$, i.e. $b_jP = a_i^{-1}P = P$, which shows that P is fixed by all the b_j, contrary to the fact that $P \notin X^B$.

Corollary 1. *Let a, b and c be three automorphisms of a tree such that $abc = 1$. If a, b, c have fixed points, then they have a common fixed point.*

This follows from prop. 26 applied to $a_i = a$, $b_j = b$.

Corollary 2. *Suppose that G is generated by a finite number of elements $s_1, \ldots, s_m$ such that the s_i and the s_is_j have fixed points. Then G has a fixed point.*

This is proved by induction on m, applying prop. 26 with

$$a_1 = s_1, \ldots, a_{m-1} = s_{m-1} \quad \text{and} \quad b_1 = s_m.$$

Corollary 3. *If G is finitely generated, and if each of its elements has fixed points, then so has G.*

This follows from cor. 2.

Remark. We can also deduce cor. 2 from cor. 1: let X_i be the subtree of X fixed by s_i; cor. 1 shows that $X_i \cap X_j \neq \emptyset$ for each pair (i,j), and we can apply the following lemma:

Lemma 10. *Let $X_1, \ldots, X_m$ be subtrees of a tree X. If the X_i meet pairwise, then their intersection is non-empty.*

Proof. Arguing by induction on m, we can assume that

$$Y = X_1 \cap X_2 \cap \cdots \cap X_{m-1}$$

is non-empty. If Y does not meet X_m, let P–Q be the geodesic joining Y to X_m. If $i \leqslant m-1$, the tree X_i meets both Y and X_m, and therefore contains P–Q by lemma 9c). But then P–$Q \subset Y$, which is absurd.

The case of nilpotent groups

Recall that a group is called *nilpotent* if one of the terms of its lower central series equals $\{1\}$ (cf. *Bourbaki*, A–I, §6).

Proposition 27. *Let G be a finitely generated nilpotent group acting on a tree X. Only two cases are possible* (and are mutually exclusive):

(a) *G has a fixed point.*

(b) *There is a straight path T stable under G on which G acts by translations by means of a non-trivial homomorphism $G \to \mathbf{Z}$.*

Suppose first that we have case (b). Choose an element s of G whose action on T is a non-trivial translation. By prop. 25, s has no fixed point; this shows that (a) and (b) are exclusive. Moreover, prop. 24 shows that T is unique: it is the intersection of the subtrees of X stable under s and s^{-1}.

We now choose a composition sequence

$$\{1\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$$

such that the successive quotients G_i/G_{i-1} are cyclic, and argue by induction on n. The case $n = 0$ is trivial. Suppose then that $n \geqslant 1$ and apply the induction hypothesis to the group $H = G_{n-1}$. If H has a fixed point, the result follows from prop. 26 applied to the action of the cyclic group G/H on the tree X^H. If H has no fixed point, the straight path T stable under H is stable under G (because H is a normal subgroup of G) and we obtain a homomorphism $G \to \mathrm{Aut}(T)$ whose image contains a non-trivial group of translations.. This image is then either the infinite dihedral group or the group $\mathbf{Z}$ (acting by translations). But the former case is impossible, since the infinite dihedral group is not nilpotent. Therefore only the second case remains, i.e. case (b).

Corollary 1. *If G is generated by elements which have fixed points, then G has a fixed point.*

Suppose that we are in case (b) and that G is generated by a family (s_i). Since $G \to \mathbf{Z}$ is non-trivial, at least one of the s_i has an image $\neq 0$ in $\mathbf{Z}$; by prop. 25 such an element s_i cannot have a fixed point.

Corollary 2. *Let G' be the commutator subgroup of G, and let s be an element of G such that $s^n \in G'$ for some integer $n \geqslant 1$. Then s has a fixed point.*

This is clear if G has a fixed point. If not, we are in case (b) and the hypothesis on s implies that its image under the homomorphism $G \to \mathbf{Z}$ is zero. The element s therefore leaves the straight path T fixed.

Exercises

1) Let $(X_i)_{i \in I}$ be a finite family of subtrees of a tree X. Let N be the *nerve* of this family, i.e. the simplicial complex whose vertex set is I, a subset J of I being a simplex if and only if $\bigcap_{j \in J} X_j \neq \emptyset$. Show that N has the same homotopy type as the union of the X_i. Deduce that each connected component of N is *contractible*. (This gives another proof of lemma 10.)

2) (Tits) Let G be a group acting on a tree X. Suppose that each element of G has fixed points, but that G has none. If $P \in \mathrm{vert}\, X$ let $s \in G$ be such that $sP \neq P$ and let P_1 be the vertex of the geodesic P–sP at distance 1 from P. Show that P_1 does not depend on the choice of s. Let f: $\mathrm{vert}\, X \to \mathrm{vert}\, X$ be the map $P \mapsto P_1$; one has $f \circ s = s \circ f$ for each $s \in G$. Show that for each $P \in \mathrm{vert}\, X$ the $f^n P$ $(n = 1, 2, \ldots)$ tend to an *end* of X which is independent of P and *fixed* by G.

3) (Generalization of 6.3.5). Let W be a Coxeter group, defined by a finite matrix (m_{ij}) such that $m_{ij} \neq \infty$ for each (i, j), cf. Bourbaki, [36], §1; let W^+ be the subgroup of W consisting of the elements of even length (*loc. cit.*, exerc. 9). Show that W and W^+ have property (FA) (apply cor. 2 of prop. 26).

4) Let G be a group and N a nilpotent normal subgroup. Suppose that there is no subgroup N' of N which is normal in G and such that N/N' is isomorphic to $\mathbf{Z}$. Show that, if G acts on a tree X, then N has a fixed point in X (use prop. 27). Deduce that if G/N has property (FA) then so has G.

(Example: take $N = \mathbf{Z}^2$ and let G be the semidirect product of a cyclic group C of order 3, 4 or 6 with N, where the action of C on N is non-trivial. Since $G/N = C$ is finite, we deduce that G has property (FA), while N, which is of finite index in G, has not.)

5) Let $\omega = (1 + \sqrt{-3})/2$ be a primitive 6th root of unity, and let $G = \mathbf{SL}_2(\mathbf{Z}[\omega])$. Put

$$x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} \omega & 0 \\ 0 & \omega' \end{pmatrix}, \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{with} \quad \omega' = \omega^{-1}.$$

a) Show that x and y generate the Borel subgroup $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ of G and that x, y, w generate G.

b) Show that B has property (FA) (use the preceding exercise) and that xw and yw are of finite order.

c) Deduce, by means of prop. 26, that G has property (FA).

d) Same questions for the group $\mathbf{GL}_2(\mathbf{Z}[i])$.

6.6 The case of $\mathbf{SL}_3(\mathbf{Z})$

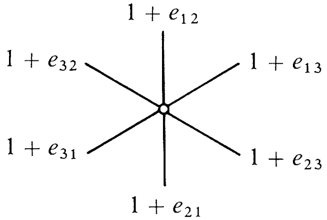
Theorem 16. *The group* $\mathbf{SL}_3(\mathbf{Z})$ *has property* (FA).

In view of th. 15 this implies:

Corollary. *The group* $\mathbf{SL}_3(\mathbf{Z})$ *is not an amalgam.*

Proof of th. 16

If $i, j \in \{1, 2, 3\}$ we let e_{ij} denote the elementary matrix whose entries are zero with the exception of a 1 in the ith row and jth column. It is well known that $\mathbf{SL}_3(\mathbf{Z})$ is generated by the $1 + e_{ij}$ with $i \neq j$. It is convenient to order these six matrices as follows:



(This corresponds to the fact that the "roots" of $\mathbf{SL}_3$ form a regular hexagon.) We are then led to put

$$z_0 = z_6 = 1 + e_{12}, \qquad z_1 = 1 + e_{13}, \qquad z_2 = 1 + e_{23}, \qquad z_3 = 1 + e_{21},$$

$$z_4 = 1 + e_{31}, \qquad z_5 = 1 + e_{32} \quad \text{and} \quad z_{i+6} = z_i \quad \text{for each } i.$$

We have the following properties:

(i) z_i commutes with z_{i+1} and z_{i-1}.

(ii) the commutator $[z_{i-1}, z_{i+1}]$ equals z_i^{-1} or z_i according as i is even or odd.

In particular, $\mathbf{SL}_3(\mathbf{Z})$ is generated by $\{z_1, z_3, z_5\}$. Moreover, for each $i \in \mathbf{Z}/6\mathbf{Z}$, the elements z_{i-1} and z_{i+1} generate a nilpotent group B_i, and z_i belongs to the commutator subgroup of B_i.

Now suppose that $\mathbf{SL}_3(\mathbf{Z})$ acts on a tree. Cor. 2 to prop. 27, applied to the group B_i, shows that z_i has a fixed point. Since this is true for each $i \in \mathbf{Z}/6\mathbf{Z}$, we see that B_i is generated by elements which have fixed points, and it therefore has a fixed point by cor. 1 to prop. 27. In particular, $z_{i-1}z_{i+1}$ has a fixed point. Cor. 2 of prop. 26, applied to $\{z_1, z_3, z_5\}$, then shows that $\mathbf{SL}_3(\mathbf{Z})$ has a fixed point.

Remark. Instead of using cor. 2 to prop. 26 we could also apply lemma 10 to the three trees formed by the fixed points of z_1, z_3, z_5: these trees meet in pairs, and therefore have a non-empty intersection.

Generalizations

1) Let N be an integer $\geqslant 1$ and let G_N be the subgroup of $\mathbf{SL}_3(\mathbf{Z})$ generated by z_1^N, z_2^N, z_3^N. The same argument as above shows that G_N has property (FA). But:
a) every subgroup of finite index in $\mathbf{SL}_3(\mathbf{Z})$ contains a G_N (this is immediate);
b) G_N is of finite index in $\mathbf{SL}_3(\mathbf{Z})$ (this has been proved by Tits, *C. R. Acad. Sci. Paris*, 283, 1976, p. 693–698).
So in view of 6.3.4 this implies:

Theorem 16′ (Margulis-Tits). *Every subgroup of finite index in* $\mathbf{SL}_3(\mathbf{Z})$ *has property* (FA).

2) Th. 16 and 16′ extend to the groups $G(A)$, where G is a simple "Chevalley" group of rank $\geqslant 2$, and A is the ring of integers (or more generally, "S-integers") of an algebraic number field (Tits, *loc. cit.*). In particular, the groups

$$\mathbf{SL}_n\left(\mathbf{Z}\left[\frac{1}{N}\right]\right),\ n \geqslant 3,\ \mathbf{Sp}_{2n}\left(\mathbf{Z}\left[\frac{1}{N}\right]\right),\ n \geqslant 2, \ldots,\ \mathbf{E}_8\left(\mathbf{Z}\left[\frac{1}{N}\right]\right)$$

have property (FA).

3) For $\mathbf{SL}_2$ the situation is different. It is clear that $\mathbf{SL}_2(\mathbf{Z})$ does not have property (FA). It is the same with $\mathbf{SL}_2(A)$ when A is the ring of integers of an imaginary quadratic field not isomorphic to $\mathbf{Q}(\sqrt{-1})$ or $\mathbf{Q}(\sqrt{-3})$, since such a group has a quotient isomorphic to $\mathbf{Z}$ (cf. [20], th. 9, p. 519). On the other hand, if K is an algebraic number field not isomorphic to $\mathbf{Q}$ nor to an imaginary quadratic field, one can show that every arithmetic subgroup of $\mathbf{SL}_2(K)$ has property (FA); this applies notably to the groups $\mathbf{SL}_2(\mathbf{Z}[\sqrt{D}])$ and their subgroups of finite index (D a square-free integer > 1).

4) Margulis has recently obtained general results on discrete subgroups of semi-simple groups which contain th. 16′ and its generalizations mentioned above as special cases.

Chapter II. $\mathbf{SL}_2$

§1 The tree of $\mathbf{SL}_2$ over a local field

Notation

The letter K denotes a field with a *discrete valuation* v; recall that v is a homomorphism of K^* onto $\mathbf{Z}$, such that

$$v(x+y) \geqslant \operatorname{Inf}(v(x), v(y)) \quad \text{for} \quad x, y \in K,$$

with the convention that $v(0) = +\infty$.

Let $\mathcal{O}$ denote the valuation ring of K, i.e. the set of $x \in K$ such that $v(x) \geqslant 0$. We choose a uniformizer π, i.e., an element $\pi \in K^*$ such that $v(\pi) = 1$ and let k denote the residue field $\mathcal{O}/\pi\mathcal{O}$. For each $x \in K^*$ we have

$$x\mathcal{O} = \mathcal{O}x = \pi^{v(x)}\mathcal{O} = \{y \in K \mid v(y) \geqslant v(x)\}.$$

In particular, all ideals of $\mathcal{O}$ are two-sided.

We let V denote a (right) vector space of dimension 2 over K.

1.1 The tree

The class of a lattice

A *lattice* of V is any finitely generated $\mathcal{O}$-submodule of V which generates the K-vector space V; such a module is free of rank 2. If $x \in K^*$, and if L is a lattice of V, Lx is also a lattice of V (use the fact that $\mathcal{O}x = x\mathcal{O}$). Thus the group K^* acts on the set of lattices; we call the orbit of a lattice under this action its *class*; two lattices belonging to the same class are called *equivalent*. The set of lattice classes is denoted by X.

The distance between two classes

Let L and L' be two lattices of V. By the invariant factor theorem there is an $\mathcal{O}$-basis $\{e_1, e_2\}$ of L and integers a, b such that $\{e_1\pi^a, e_2\pi^b\}$ is an $\mathcal{O}$-basis for L'. Moreover, the set $\{a, b\}$ does not depend on the choice of bases for L, L'. We have $L' \subset L$ if and only if a and b are $\geqslant 0$, in which case L/L' is isomorphic to $(\mathcal{O}/\pi^a\mathcal{O}) \oplus (\mathcal{O}/\pi^b\mathcal{O})$.

Replacing L, L' by $Lx, L'y$ (with $x, y \in K^*$) has the effect of replacing $\{a, b\}$ by $\{a+c, b+c\}$ where $c = v(y/x)$. The integer $|a-b|$ therefore depends only on the classes Λ and Λ' of L and L'; we denote it by $d(\Lambda, \Lambda')$ and call it the *distance* between Λ and Λ'.

If L is given, then each class $\Lambda' \in X$ has exactly one representative L' satisfying the following equivalent conditions:

(i) $L' \subset L$ and L' is maximal (in Λ') with this property;
(ii) $L' \subset L$ and $L' \not\subset L\pi$;
(iii) $L' \subset L$ and L/L' is monogenic.

For such an L we have

$$L/L' \simeq \mathcal{O}/\pi^n\mathcal{O}, \quad \text{where} \quad n = d(\Lambda, \Lambda').$$

In particular:

$d(\Lambda, \Lambda') = 0 \Leftrightarrow \Lambda = \Lambda'$

$d(\Lambda, \Lambda') = 1 \Leftrightarrow$ there are representatives $L' \subset L$ of Λ' and Λ such that $L/L' \simeq k$

(i.e. $l(L/L') = 1$, where l denotes length).

The tree of V

Two elements Λ, Λ' of X are called *adjacent* if $d(\Lambda, \Lambda') = 1$. In this way one defines a *combinatorial graph* structure on X.

Theorem 1. *The graph X is a tree.*

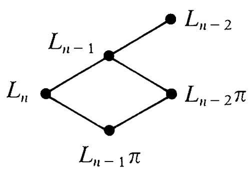
We first show that X is *connected.* If Λ and Λ' are two vertices of X, choose lattices L and L' representing Λ and Λ', with $L' \subset L$. A Jordan-Hölder sequence for L/L' gives a sequence of lattices

$$L' = L_n \subset L_{n-1} \subset \cdots \subset L_0 = L$$

such that $l(L_{i-1}/L_i) = 1$ for $1 \leqslant i \leqslant n$. The classes $\Lambda_0, \ldots, \Lambda_n$ of these lattices define a path in X between the extremities Λ and Λ'. Whence the fact that X is connected.

To prove that X is a tree, it now suffices to show that, if $\Lambda_0, \Lambda_1, \ldots, \Lambda_n$ $(n \geqslant 1)$ is the sequence of vertices in a path without backtracking in X, then $\Lambda_0 \neq \Lambda_n$. In fact we shall show (by induction on n) that $d(\Lambda_0, \Lambda_n) = n$.

From what we have said above, we can find representatives L_i of the Λ_i such that $L_{i+1} \subset L_i$ and $l(L_i/L_{i+1}) = 1$. We have $l(L_0/L_n) = n$ and we want to show $L_n \not\subset L_0\pi$. By the induction hypothesis we have $L_{n-1} \not\subset L_0\pi$. The lattices L_n and $L_{n-2}\pi$ are the inverse images of two lines in the k-plane $L_{n-1}/L_{n-1}\pi$. These lines are distinct: if not, $\Lambda_{n-2}, \Lambda_{n-1}, \Lambda_n$ would correspond to a backtracking in the given path.



We then have

$$L_{n-1} = L_n + L_{n-2}\pi$$

whence

$$L_{n-1} \equiv L_n \pmod{L_0\pi}$$

and we conclude that indeed $L_n \not\subset L_0\pi$. Q.E.D.

Remarks

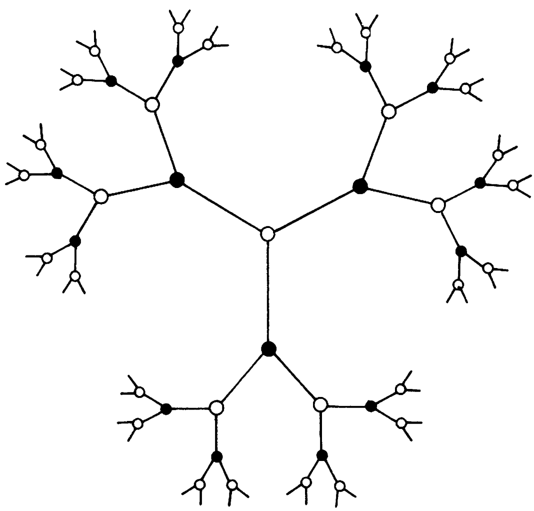
1) The proof shows in addition that $d(\Lambda, \Lambda')$ coincides with the *distance* between the vertices Λ and Λ' in the tree X (in the sense of chap. I, no. 2.2).

2) A construction related to the one above has been used (in arbitrary dimension) by H. Behr; cf. *J. Crelle*, 211, 1962, p. 123–135.

3) The tree X is a special case of a *Bruhat-Tits building* (cf. 1.7 below, and [38], no. 10.2).

Completion. Let $\hat{K}$ be the completion of K with respect to v; its valuation ring is $\hat{\mathcal{O}} = \varprojlim \mathcal{O}/\pi^n\mathcal{O}$. We put $\hat{V} = V \otimes_K \hat{K}$. If we associate with each lattice L of V the lattice $\hat{L} = L \otimes_{\mathcal{O}} \hat{\mathcal{O}}$ of $\hat{V}$ we obtain a *bijection* of the set of lattices of V onto the set of lattices of $\hat{V}$; this bijection induces an isomorphism of the tree of V onto that of $\hat{V}$ (which permits us, if we so desire, to confine ourselves *to the case where K is complete*).

Diagram (residue field of two elements):



(The notations ● and ○ correspond to a partition of the vertices into two classes such that vertices in the same class are an *even* distance apart.)

Projective lines

Let L_0 be a lattice of V, and let $\Lambda_0 \in X$ be its class. Each vertex Λ of X is represented by a unique lattice $L \subset L_0$ such that $L_0/L \simeq \mathcal{O}/\pi^n\mathcal{O}$, where $n = d(\Lambda_0, \Lambda)$. The $\mathcal{O}/\pi^n\mathcal{O}$-module $L_0/L_0\pi^n$ is free of rank 2, and $L/L_0\pi^n$ is a direct factor of rank 1. Thus we see that *the vertices of X at distance n from Λ_0 correspond bijectively to direct factors of $L_0/L_0\pi^n$ of rank* 1, *i.e. to points of the projective line* $\mathbf{P}(L_0/L_0\pi^n) \simeq \mathbf{P}_1(\mathcal{O}/\pi^n\mathcal{O})$.

For $n = 1$, this means that the *edges with origin Λ_0 correspond bijectively to the points of* $\mathbf{P}(L_0/L_0\pi) \simeq \mathbf{P}_1(k)$; if $q = \mathrm{Card}(k)$, the number of these edges is $q + 1$.

An infinite path with origin Λ_0 and without backtracking is called an *end* of X. The preceding shows that *the ends of X correspond to the points of*

$$\varprojlim \mathbf{P}(L_0/L_0\pi^n) = \mathbf{P}(\hat{L}_0) \simeq \mathbf{P}_1(\hat{\mathcal{O}});$$

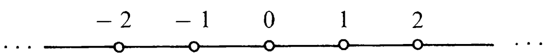
note also that $\mathbf{P}(\hat{L}_0)$ can be canonically identified with $\mathbf{P}(\hat{V})$, the set of lines of $\hat{V}$. (In particular, the set of ends of X "does not depend" on the origin Λ_0 chosen, cf. chap. I, no. 2.2.) More precisely, if D is a line of $\hat{V}$, one associates it with the end defined by the sequence of lattices

$$\hat{L}_n = \hat{L}_0\pi^n + (\hat{L}_0 \cap D);$$

conversely, D is the unique line contained in the intersection of the $\hat{L}_n$.

Straight paths

A *straight path* is any path isomorphic to



A straight path of X is associated with the pair of ends consisting of the paths $(0, 1, 2, \ldots)$ and $(0, -1, -2, \ldots)$. One checks without difficulty that this yields the bijections:

[straight paths of the tree X]

$\leftrightarrow$ [pairs of distinct ends]

$\leftrightarrow$ [decompositions of $\hat{V}$ into the direct sum of two lines].

Exercises

1) Show that X depends only on $\mathrm{Card}(k)$, up to isomorphism.

2) Suppose $\mathrm{Card}(k)$ is finite and put on the group $G = \mathrm{Aut}(X)$ the pointwise convergence topology.

a) Show that G is a locally compact topological group.

b) Let G_Λ be the stabilizer of the vertex Λ of X. Show that G_Λ is an open profinite subgroup of G.

c) Let G^+ be the subgroup of index 2 in G consisting of the automorphisms s such that $d(\Lambda, s\Lambda) \equiv 0 \pmod 2$ for each $\Lambda \in X$ (or for *one* $\Lambda \in X$, which comes to the same thing). Show that each compact subgroup of G^+ is contained in a G_Λ, and that the G_Λ are the maximal compact subgroups of G^+.

(Tits [11] has proved that G^+ is a simple group.)

3) (Hecke operators) Suppose that $q = \operatorname{Card}(k)$ is finite.

a) Show that the number of vertices of X at a distance $n \geqslant 1$ from a given vertex is $q^{n-1}(q+1)$.

b) Let Θ_n denote the correspondence[1] which associates with each vertex $\Lambda \in X$ the formal sum of the vertices Λ' of X such that $d(\Lambda, \Lambda') = n$. Put

$$T_0 = \Theta_0 = Id$$

$$T_1 = \Theta_1$$

$$T_n = \sum_{0 \leqslant i \leqslant n/2} \Theta_{n-2i} = \Theta_n + T_{n-2}.$$

Show that

$$\Theta_1\Theta_1 = \Theta_2 + (q+1)\Theta_0$$

$$\Theta_1\Theta_n = \Theta_{n+1} + q\Theta_{n-1} \quad (n \geqslant 2)$$

and

$$T_1T_n = T_{n+1} + qT_{n-1} \quad (n \geqslant 1).$$

Deduce that the T_n and the Θ_n are polynomials in T_1. Prove the identity

$$\sum_{n=0}^{\infty} T_n x^n = \frac{1}{1 - T_1 x + qx^2}$$

where x is an indeterminate.

4) (Generalization to dimension $n \geqslant 2$). Let E be a K-vector space of finite dimension n, and let X be the space of lattices of E (up to homothety). A finite subset σ of X is called a simplex if one can find lattices $L_0, \ldots, L_i$ representing the elements of σ so that $L_0 \supset L_1 \supset \cdots \supset L_0\pi$.

a) Show that this defines a structure of *simplicial complex* of dimension $n-1$ on the set X, and that this complex is isomorphic to the Bruhat-Tits building for $\mathbf{SL}_n(K)$, cf. [38], no. 10.2.

---

[1] A *correspondence* T on a set X is an endomorphism of the free $\mathbf{Z}$-module with basis X; to define T it suffices to give its values on the elements P of X:

$$T(P) = \sum_{Q \in X} n_{PQ} Q,$$

where the n_{PQ} are integers which are almost all zero (for P fixed).

b) Let w be a *valuation* on E, i.e. a map of E into $\mathbf{R} \cup \{+\infty\}$ such that

$$w(x) = +\infty \Leftrightarrow x = 0$$

$$w(x\lambda) = w(x) + v(\lambda) \qquad \text{if} \quad x \in E, \quad \lambda \in K$$

$$w(x + y) \geqslant \operatorname{Inf}(w(x), w(y)) \quad \text{if} \quad x, y \in E.$$

If $a \in \mathbf{R}$, let $E^w_a = \{x | x \in E,\ w(x) \geqslant a\}$; this is a lattice of E.

For each lattice L of E, the set of the $a \in \mathbf{R}$ such that $E^w_a = L$ is a half-open interval I^w_L (possibly empty); its length i^w_Λ depends only on the image Λ of L in X. Show that the set of Λ such that $i^w_\Lambda \neq 0$ is a simplex of X; let P_w be the point in the geometric realization real(X) of X whose barycentric coordinates are the i^w_Λ. Show that each point of real(X) can be obtained in this way, and that we have $P_w = P_{w'}$ if and only if $w' - w$ is constant. (Thus we obtain an *interpretation* of the points of real(X), which is that of Goldman-Iwahori, cf. [38], pp. 238–239.)

1.2 The groups GL(V) and SL(V)

We let $\mathbf{GL}(V)$ denote the group $\operatorname{Aut}(V)$ of K-automorphisms of V; it is isomorphic to $\mathbf{GL}_2(K)$; its centre is C^*, where C denotes the centre of the field K.

We let $\mathbf{SL}(V)$ be the subgroup of $\mathbf{GL}(V)$ defined by one of the following equivalent properties (cf. for example E. Artin, *Geometric Algebra*, chap. IV):

i) It is the subgroup of $\mathbf{GL}(V)$ generated by the unipotent elements.

ii) It is the commutator subgroup $(\mathbf{GL}(V), \mathbf{GL}(V))$ of $\mathbf{GL}(V)$.

iii) It is the kernel of the *Dieudonné determinant*

$$\det\colon \quad \mathbf{GL}(V) \to K^*/(K^*, K^*)$$

(Recall that if $s \in \mathbf{GL}(V)$ is represented by an invertible matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$$\det(s) \equiv \begin{cases} ad - aca^{-1}b & (\text{if } a \neq 0) \\ -cb & (\text{if } a = 0) \end{cases}$$

cf. Artin, *loc. cit.*)

Remark. Suppose that K is *finite over its centre* C. For each $n \geqslant 1$, the algebra of matrices $\mathbf{M}_n(K)$ is an algebra with centre C; the *reduced norm* $\operatorname{Nrd}\colon \mathbf{M}_n(K) \to C$ is defined (Bourbaki, A-VIII); it induces a homomorphism $N_n\colon \mathbf{GL}_n(K) \to C^*$. On passing to the quotient, N_1 defines a homomorphism

$$N\colon \quad K^*/(K^*, K^*) \to C^*$$

and we have $N_n = N \circ \det$ for each n. The kernel $\mathbf{SL}'_n(K)$ of N_n contains $\mathbf{SL}_n(K)$, and one may ask whether $\mathbf{SL}'_n(K)$ *is equal to* $\mathbf{SL}_n(K)$, or, what amounts to the same, whether the homomorphism N is *injective* (the "Tannaka-Artin problem"). This is true when C is locally compact (Nakayama-Matsushima, *Proc. Imp. Acad. Tokyo*, 1943), but false in general (Platonov, *Izv. Akad. Nauk*, 1976).

The groups $\mathbf{GL}(V)^+$ *and* $\mathbf{GL}(V)^0$

The valuation v: $K^* \to \mathbf{Z}$ is trivial on (K^*, K^*); if $s \in \mathbf{GL}(V)$ we can then speak of $v(\det(s))$, and we obtain a surjective homomorphism

$$v(\det): \quad \mathbf{GL}(V) \to \mathbf{Z}.$$

We shall let $\mathbf{GL}(V)^0$ denote the kernel of this homomorphism and let $\mathbf{GL}(V)^+$ denote the kernel of the composite $\mathbf{GL}(V) \to \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$. We have

$$\mathbf{SL}(V) \subset \mathbf{GL}(V)^0 \subset \mathbf{GL}(V)^+ \subset \mathbf{GL}(V).$$

Characterization of $v(\det(s))$ *in terms of lattices*

If L_1 and L_2 are two lattices of V we put

$$\chi(L_1, L_2) = l(L_1/L_3) - l(L_2/L_3) \quad \text{with} \quad L_3 \subset L_1 \cap L_2;$$

this integer does not depend on the choice of the auxiliary lattice L_3.

Proposition 1. *Let L be a lattice of class Λ, and let $s \in \mathbf{GL}(V)$. One has*

$$\chi(L, sL) = v(\det(s)).$$

Choose a basis $\{e_1, e_2\}$ of L and integers a, b such that $\{e_1\pi^a, e_2\pi^b\}$ is a basis of sL. The matrix of s is then the product of the diagonal matrix s_1 with diagonal terms π^a, π^b and a matrix $s_2 \in \mathbf{GL}_2(\mathcal{O})$. The formulae given above show that $v(\det(s_2)) = 0$ and $v(\det(s_1)) = a + b$. It then remains to check that $\chi(L, sL) = a + b$, which is immediate.

Corollary. *We have* $d(\Lambda, s\Lambda) \equiv v(\det(s)) \pmod 2$.

Indeed, in the above notation we have

$$d(\Lambda, s\Lambda) = |a - b| \equiv a + b \equiv v(\det(s)) \pmod 2.$$

Exercises

1) Prove directly that $s \mapsto \chi(L, sL)$ is a homomorphism d of $\mathbf{GL}(V)$ into $\mathbf{Z}$. Determine $d(s)$ when s is defined by one of the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

with respect to an $\mathcal{O}$-basis of L. Deduce another proof of prop. 1 (use the fact that $\mathbf{GL}_2(K)$ is generated by the above matrices).

2) Suppose that K is locally compact. Interpret $\chi(L_1, L_2)$ and $v(\det(s))$ in terms of a Haar measure on V.

1.3 Action of $\mathbf{GL}(V)$ on the tree of V; stabilizers

The group $\mathbf{GL}(V)$ acts on the tree X. Moreover, the cor. to prop. 1 shows that an element $s \in \mathbf{GL}(V)$ belongs to $\mathbf{GL}(V)^+$ if and only if s preserves each of the two classes of vertices of X (two vertices being in the same class if and only if their distance is even). In particular, $\mathbf{GL}(V)^+$ *acts without inversion* on X.

On the other hand, $\mathbf{GL}(V)$ acts *with inversion*: if e_1, e_2 is a basis for V, let L and L' be the lattices of the bases (e_1, e_2), $(e_1, e_2\pi)$, and let Λ, Λ' be the corresponding vertices of X. The element s with matrix $\begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$ transforms L into L' and L' into $L\pi$; it therefore transforms the edge $\Lambda\Lambda'$ into its inverse $\Lambda'\Lambda$.

Stabilizers of the vertices

Let G be a subgroup of $\mathbf{GL}(V)$, and let L be a lattice of class Λ. We let G_L (resp. G_Λ) denote the subgroup of G consisting of the elements s such that $sL = L$ (resp. $s\Lambda = \Lambda$); similarly, if $\Lambda\Lambda'$ is an edge of X we denote its stabilizer in G by $G_{\Lambda\Lambda'}$.

By definition, G_Λ is the set of $s \in G$ for which there is an $x \in K^*$ with $sL = Lx$.

Lemma 1. *If G is contained in $\mathbf{GL}(V)^0$ then $G_L = G_\Lambda$.*

Indeed, suppose that we have $sL = Lx$, as above. One checks immediately that $\chi(L, Lx) = 2v(x)$. By prop. 1 we then have $v(\det(s)) = 2v(x)$, whence $v(x) = 0$ if s belongs to $\mathbf{GL}(V)^0$; this means that x is an invertible element of $\mathcal{O}$, so that $Lx = L$ and hence $s \in G_L$.

(This applies in particular to the group $\mathbf{SL}(V)$.)

Bounded subgroups

Recall that a subgroup G of $\mathbf{GL}(V)$ is called *bounded* if it is a bounded subset of the vector space $\mathrm{End}(V)$. If we identify $\mathbf{GL}(V)$ with $\mathbf{GL}_2(K)$, this means that there is an integer d such that $v(s_{ij}) \leqslant d$ for each $s = (s_{ij}) \in G$.

Proposition 2. *If G is a subgroup of $\mathbf{GL}(V)^0$, the following conditions are equivalent*:

(i) *G is bounded* (in the above sense).

(ii) *There is a $\Lambda \in X$ such that the orbit $G \cdot \Lambda$ is bounded* (in terms of the distance on the tree X).

(iii) *There is a lattice of V stable under G.*

(iv) *G leaves a vertex of X fixed.*

The equivalence (iii)$\Leftrightarrow$(iv) follows from lemma 1 above. The equivalence (ii)$\Leftrightarrow$(iv) has been proved in chap. I (no. 4.3, prop. 19). The implications (iii)$\Rightarrow$(i)$\Rightarrow$(ii) are immediate.

(Of course, it is easy to prove the implication (i)$\Rightarrow$(iii) directly: one chooses a lattice L_0 of V, and puts $L = \sum_{s \in G} sL_0$; using the fact that G is bounded, one sees that L is a lattice of V, and it is obviously stable under G.)

Corollary. *The maximal bounded subgroups of G are the stabilizers of the vertices of X.*

Stabilizers of the edges: Iwahori subgroups

Let $\Lambda\Lambda'$ be an edge of X, representing the lattices L, L' with $L' \subset L$ and $l(L/L') = 1$. Let G be a subgroup of $\mathbf{GL}(V)^0$. By lemma 1, *the stabilizer* $G_{\Lambda\Lambda'}$ is $G_L \cap G_{L'}$. This is the subgroup of G_L consisting of the elements s whose images in $\mathbf{GL}(L/L\pi) \simeq \mathbf{GL}_2(k)$ leave the line L'/L stable; we call it the Iwahori subgroup of G relative to $\Lambda\Lambda'$.

(Matrix interpretation: take $L = e_1\mathcal{O} \oplus e_2\mathcal{O}$, $L' = e_1\mathcal{O} \oplus e_2\pi\mathcal{O}$; then $G_{\Lambda\Lambda'}$ consists of the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $c \equiv 0 \pmod{\pi}$. It is the inverse image of the triangular subgroup $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ of $\mathbf{GL}_2(k)$.)

Stabilizers of ends: Borel subgroups

Let b be an *end* of X, represented by an infinite path $\Lambda_0, \Lambda_1, \ldots$ without backtracking. We say that an element $s \in \mathbf{GL}(V)$ leaves b invariant if there is an integer d such that $s\Lambda_i = \Lambda_{i+d}$ for all sufficiently large i (cf. chap. I, no. 2.2, exerc. 1). If D_b denotes the line of $\hat{V}$ which corresponds to the end b (no. 1.1), this is equivalent to $sD_b = D_b$. The stabilizer of b in $\mathbf{GL}(V)$ is then the intersection of $\mathbf{GL}(V)$ with the Borel subgroup of $\mathbf{GL}(\hat{V})$ corresponding to D_b. If K is complete we then have identity between "stabilizers of the ends of X" and "Borel subgroups".

(Matrix interpretation: if we choose a basis e_1, e_2 such that D_b is generated by e_1, we have $sD_b = D_b$ if and only if s is of the form $\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$ and the integer d corresponding to s is $v(\delta) - v(\alpha)$.)

Stabilizers of straight paths: Cartan subgroups

Let

$$T = \cdots \;—\underset{\Lambda_{-1}}{\circ}—\!—\!—\underset{\Lambda_0}{\circ}—\!—\!—\underset{\Lambda_1}{\circ}—\!—\!—\underset{\Lambda_2}{\circ}—\; \cdots$$

be a straight path of X. We have seen in no. 1.1 that T corresponds to a decomposition of $\hat{V}$ into a direct sum of two lines D_1, D_2. It follows that the stabilizer $N_T = \mathbf{GL}(\hat{V})_T$ of T in $\mathbf{GL}(\hat{V})$ is the set of s which transforms $\{D_1, D_2\}$ into $\{D_1, D_2\}$. This group contains as a subgroup of index 2 the "Cartan" subgroup $H_T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ consisting of s such that $sD_1 = D_1$, $sD_2 = D_2$.

One easily checks that the map $N_T \to \mathrm{Aut}(T)$ is surjective (the group $\mathrm{Aut}(T)$ is the infinite dihedral group $\mathbf{D}_\infty$). An element $s \in N_T$ belongs to H_T if and only if its image in $\mathrm{Aut}(T)$ is a translation (i.e. if s respects the orientation of T). When such an s is written in matrix form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ the amplitude of the translation is $v(a) - v(d)$, once T is suitably oriented (when this amplitude is non-zero, s then defines a *hyperbolic* automorphism of X, in the sense of chap. I, no. 6.4, and T is the corresponding straight path).

Exercises

1) Let ϕ be the canonical homomorphism $\mathbf{GL}(V) \to \mathrm{Aut}(X)$.
   a) Show that ϕ is not surjective.
   b) Show that the kernel of ϕ is the centre C^* of $\mathbf{GL}(V)$.

2) Suppose that K is commutative, and identify $\mathbf{PGL}(V) = \mathbf{GL}(V)/K^*$ with the group of K-automorphisms of the simple algebra $A = \mathrm{End}_K(V)$. Let S be a subset of $\mathbf{PGL}(V)$. Prove the equivalence of the following properties:
   a) S is bounded in the vector space $\mathrm{End}_K(A)$.
   b) There is a vertex Λ of X such that $S \cdot \Lambda$ is a bounded subset of X (in the sense of chap. I, no. 2.2).
   If S is a subgroup of $\mathbf{PGL}(V)$, these properties are equivalent to:
   c) There is either a vertex or a geometric edge of X which is stable under S.

1.4 Amalgams

Notation

L and L' are two lattices of V, with $L' \subset L$ and $l(L/L') = 1$; we denote their classes by Λ and Λ'; these are *adjacent* vertices in the tree X.

G denotes a *subgroup* of $\mathbf{GL}(V)^+$; we let G_L, $G_{L'}$, $G_{LL'}$, G_Λ, $G_{\Lambda'}$, $G_{\Lambda\Lambda'}$ be the stabilizers of L, L', the pair (L, L'), Λ, Λ' and the pair (Λ, Λ') in G. If G is contained in $\mathbf{GL}(V)^0$ we have

$$G_\Lambda = G_L, \qquad G_{\Lambda'} = G_{L'} \quad \text{and} \quad G_{\Lambda\Lambda'} = G_{LL'}$$

cf. no. 1.3, lemma 1.

Theorem 2. *Under the following hypothesis:*
(D) – The closure of G in $\mathbf{GL}(V)$ *contains* $\mathbf{SL}(V)$,
the segment Λ ○———○ Λ' is a fundamental domain for the action of G on the tree X (in the sense of chap. I, no. 4.1).

Proof

a) *Action on the vertices*

Let X^+ (resp. X^-) be the set of vertices of X at an even distance from Λ (resp. Λ'). The partition (X^+, X^-) is invariant under G and we have to show that, for each $\Lambda_1 \in X^+$ (resp. $\Lambda_1 \in X^-$), there is a $g \in G$ such that $g\Lambda = \Lambda_1$ (resp. $g\Lambda' = \Lambda_1$). Suppose that we have $\Lambda_1 \in X^+$ (the case of X^- is analogous), and let $2n$ be the distance from Λ_1 to Λ. Using the invariant factor theorem, we see that there is a representative L_1 of Λ_1 and a basis $\{e_1, e_2\}$ of L such that L_1 has basis $\{e_1\pi^n, e_2\pi^{-n}\}$. Let s be the automorphism of V with matrix $\begin{pmatrix} \pi^n & 0 \\ 0 & \pi^{-n} \end{pmatrix}$ with respect to $\{e_1, e_2\}$. We have $s \in \mathbf{SL}(V)$ and $sL = L_1$; since the closure of G contains $\mathbf{SL}(V)$, and since $\mathbf{GL}_2(\mathcal{O})$ is *open* in $\mathbf{GL}_2(K)$ there is an element $g \in G$ such that

$$g = s \cdot u \quad \text{with} \quad u \in \mathrm{Aut}(L) \simeq \mathbf{GL}_2(\mathcal{O}).$$

We then have $gL = sL = L_1$, whence $g\Lambda = \Lambda_1$.

b) *Action on the edges*

In view of the above, it suffices to prove that G_L acts *transitively* on the edges with origin Λ, or, what amounts to the same, on the set P_L of sub-lattices L_1 of L such that $l(L/L_1) = 1$. Take a basis $\{e_1, e_2\}$ of L; the lattices L_1 in question are each derived from a particular one L_0 by the automorphisms with matrices

$$x_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \qquad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{with} \quad a \in \mathcal{O}.$$

Let U be the subgroup of $\mathbf{GL}_2(\mathcal{O})$ consisting of the matrices congruent to 1 (mod π); this is an open subgroup. In view of the hypothesis (D), there are then $g_a, h \in G$ such that

$$g_a = x_a u_a \quad \text{with} \quad u_a \in U, \quad \text{and} \quad h = wu \quad \text{with} \quad u \in U.$$

We have $g_a L = hL = L$ and $g_a L_0 = x_a L_0$, $hL_0 = wL_0$; this shows that G_L indeed acts transitively on P_L.

(*Variant*: use the density of G to prove that the image of the homomorphism $G_L \to \mathrm{Aut}(L/L\pi) \to \mathbf{PGL}_2(k)$ contains $\mathbf{PSL}_2(k)$; then remark that $\mathbf{PSL}_2(k)$ acts *transitively* on the projective line $\mathbf{P}_1(k)$.)

Theorem 3. *If the hypothesis* (D) *is satisfied, the group G is the sum of the subgroups G_Λ and $G_{\Lambda'}$, amalgamated along their intersection $G_{\Lambda\Lambda'}$.*

(More briefly: $G = G_\Lambda *_{G_{\Lambda\Lambda'}} G_{\Lambda'}$.)

This follows from the theorem above and th. 6 of chap. I, no. 4.1.

Corollary 1 (Ihara). *One has $\mathbf{SL}_2(K) = \mathbf{SL}_2(\mathcal{O}) *_\Gamma \mathbf{SL}_2(\mathcal{O})$ where Γ is the subgroup of $\mathbf{SL}_2(\mathcal{O})$ consisting of the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $c \equiv 0$ (mod π).*

One applies th. 3 to $G = \mathbf{SL}(V)$, bearing in mind that $G_\Lambda = G_L$, $G_{\Lambda'} = G_{L'}$ and $G_{\Lambda\Lambda'} = G_{LL'}$ because $G \subset \mathbf{GL}(V)^0$; the groups G_L and $G_{L'}$ may be identified with $\mathbf{SL}_2(\mathcal{O})$.

[The two injections of Γ into $\mathbf{SL}_2(\mathcal{O})$ which are used to define the amalgam are

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b\pi \\ \pi^{-1}c & \pi^{-1}d\pi \end{pmatrix}.]$$

Examples

1) The group $\mathbf{SL}_2(\mathbf{Q}_p)$ is an amalgam of two copies of $\mathbf{SL}_2(\mathbf{Z}_p)$.

2) More generally, let A be a *dense* subring of K. The group $G = \mathbf{SL}_2(A)$ is dense in $\mathbf{SL}_2(K)$; indeed, its closure contains the additive subgroups $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ and it is well known that these generate $\mathbf{SL}_2(K)$. We can then apply ths. 2 and 3 to the group G.

Corollary 2. *If p is a prime number one has*

$$\mathbf{SL}_2(\mathbf{Z}[1/p]) = \mathbf{SL}_2(\mathbf{Z}) *_{\Gamma_0(p)} \mathbf{SL}_2(\mathbf{Z})$$

where $\Gamma_0(p)$ is the subgroup of $\mathbf{SL}_2(\mathbf{Z})$ consisting of the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $c \equiv 0 \pmod p$.

This is the particular case $K = \mathbf{Q}$, $v = p$-adic valuation, $A = \mathbf{Z}[1/p]$ and $A \cap \mathcal{O} = \mathbf{Z}$.

Other examples

a) Cor. 2 can be generalized: if n is an integer prime to p, the group $\mathbf{SL}_2(\mathbf{Z}[1/pn])$ is the sum of two copies of $\mathbf{SL}_2(\mathbf{Z}[1/n])$ amalgamated along the subgroup of the $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \equiv 0 \pmod p$. Thus we have a "dévissage" procedure which expresses the groups $\mathbf{SL}_2(\mathbf{Z}[1/p_1 \cdots p_k])$ as amalgams of certain congruence subgroups of $\mathbf{SL}_2(\mathbf{Z})$.

b) The same method applies to groups of "S-integral points" of groups $\mathbf{SL}_2$ over number fields or function fields: they can be expressed as amalgams of groups of "integral points".

c) Ihara has studied the discrete subgroups G of $\mathbf{SL}_2(\mathbf{R}) \times \mathbf{SL}_2(\mathbf{Q}_p)$ whose projection on each factor is dense (and injective). Th. 3 applies to such a group G: we have $G = G_L *_{G_{LL'}} G_{L'}$. Moreover, G_L, $G_{L'}$ and $G_{LL'}$ are *discrete* subgroups of $\mathbf{SL}_2(\mathbf{R})$; they have a compact quotient (resp. a quotient of finite volume) if and only if the same is true of G in $\mathbf{SL}_2(\mathbf{R}) \times \mathbf{SL}_2(\mathbf{Q}_p)$.

Presentation of $\mathbf{SL}_2(\mathbf{Z}[\frac{1}{2}])$ *and* $\mathbf{SL}_2(\mathbf{Z}[\frac{1}{3}])$

Let $G = G_1 *_A G_2$ be an amalgam. Let $\langle g_{1\alpha}; r_{1\beta} = 1\rangle$ be a *presentation* of G_1 by a family of generators $\{g_{1\alpha}\}$ and a family of relators $\{r_{1\beta}\}$; similarly, let $\langle g_{2\gamma}; r_{2\delta} = 1\rangle$ be a presentation of G_2. Let $\{a_\varepsilon\}$ be a generating family for A; for each ε let $a_\varepsilon = m_{1\varepsilon}((g_{1\alpha}))$ be an expression for $a_\varepsilon \in G_1$ in terms of the $g_{1\alpha}$ ($m_{1\varepsilon}$ being an element of the free group associated with the family $g_{1\alpha}$); similarly, let $a_\varepsilon = m_{2\varepsilon}((g_{2\gamma}))$ be an expression for $a_\varepsilon \in G_2$ in terms of the $g_{2\gamma}$. Then it is clear that

$$\langle g_{1\alpha}, g_{2\gamma};\ r_{1\beta} = r_{2\delta} = 1,\ m_{1\varepsilon} = m_{2\varepsilon}\rangle$$

is a *presentation* of $G = G_1 *_A G_2$.

We apply this method to the group $G = \mathbf{SL}_2(\mathbf{Z}[1/p])$ by taking G_1 to be the group $\mathbf{SL}_2(\mathbf{Z})$, A to be the group $\Gamma_0(p)$ and G_2 to be the group of the $\begin{pmatrix} a & p^{-1}b \\ pc & d \end{pmatrix}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z})$, which is a group isomorphic to $\mathbf{SL}_2(\mathbf{Z})$.

It is well known that $\mathbf{SL}_2(\mathbf{Z})$ has the presentation

$$\langle S, T;\ S^4 = 1,\ (ST)^3 = S^2 \rangle \quad \text{with} \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The group G_2 then has the presentation

$$\langle S_p, T_p;\ S_p^4 = 1,\ (S_pT_p)^3 = S_p^2 \rangle$$

with $S_p = \begin{pmatrix} 0 & p^{-1} \\ -p & 0 \end{pmatrix}$ and $T_p = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}$.

It remains to exhibit a generating family for $\Gamma_0(p)$, so that the $m_{1\varepsilon}$, $m_{2\varepsilon}$ can be equated. We confine ourselves to the cases $p = 2$ and $p = 3$, which are particularly simple: the group $\Gamma_0(p)$ is then generated by the three elements $Y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $T_p = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}$ [this can be checked in various ways, for example by looking at the corresponding fundamental domains in the Poincaré half-plane; however, when $p \geqslant 5$, $\Gamma_0(p)$ is not generated by Y, S^2 and T_p: the minimum number of elements in a generating family for $\Gamma_0(p)/\{\pm 1\}$ is $2k + 3$, where k is the integer part of $(p-1)/12$].

We have

$$Y^{-1} = STS^{-1} = S_pT_p^pS_p^{-1}, \qquad T_p = T^p, \qquad S^2 = S_p^2.$$

Thus we see that, for $p = 2, 3$, $\mathbf{SL}_2(\mathbf{Z}[1/p])$ admits the following presentation:

$$\langle S, T, S_p;\ S^4 = 1,\ S^2 = S_p^2 = (ST)^3 = (S_pT^p)^3,\ STS^{-1} = S_pT^{p^2}S_p^{-1} \rangle.$$

If we put $U_p = S^{-1}S_p = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}$ this gives the presentation:

$$\langle S, T, U_p;\ S^4 = 1,\ S^2 = (SU_p)^2 = (ST)^3 = (SU_pT^p)^3,\ U_p^{-1}TU_p = T^{p^2} \rangle,$$

due to Behr-Mennicke (*Can. J. Math.*, 20, 1968, pp. 1432–1438).

Exercises

1) Show that $\mathbf{GL}(V)$ acts transitively on the set of vertices and edges of X, and contains an inversion. Deduce a decomposition of $\mathbf{GL}(V)$ as an amalgam (cf. chap. I, no. 4.1, exerc.).

2) Define correspondences T_n^+ and T_n^- ($n \geqslant 0$) on X by putting (cf. no. 1.1, exerc. 3):

$$T_n^+ P = T_n P, \qquad T_n^- P = 0 \quad \text{if} \quad P \in X^+$$

$$T_n^+ P = 0, \qquad T_n^- P = T_n P \quad \text{if} \quad P \in X^-.$$

Show that the T_n^+ and the T_n^- commute with each other, and with the elements of $\mathbf{GL}(V)^+$. Conversely, show that each correspondence on X which commutes with a group G satisfying the hypothesis (D) is a linear combination of the T_n^+ and the T_n^-.

3) Exhibit a presentation of $\mathbf{SL}_2(\mathbf{Z}[\frac{1}{6}])$ in terms of the generating family $\{S, T, U_2, U_3\}$.

1.5 Ihara's theorem

Theorem 4. *Let Γ be a subgroup of $\mathbf{GL}(V)^0$ not containing any bounded subgroup $\neq \{1\}$. Then Γ is a free group.*

By prop. 2 of no. 1.3 the hypothesis on Γ is equivalent to saying that Γ acts *freely* on the tree X; it is then a free group, cf. chap. I, no. 3.3, th. 4. (*Variant*: use the decomposition of $\mathbf{GL}(V)^0$ as an amalgam, combined with prop. 18 of chap. I, no. 4.3.)

Remark. The statement in Ihara [16] is slightly more general; one may also prove it by means of trees, cf. exerc. 2.

The locally compact case

Suppose that K is *locally compact*, which is equivalent to saying that K is complete and its residue field k is finite. Put $q = \mathrm{Card}(k)$. For simplicity we restrict ourselves to subgroups of the group $G = \mathbf{SL}_2(K)$. We put on G the *unique Haar measure μ such that*

$$\mu(G_L) = q - 1$$

where $G_L = \mathbf{SL}_2(\mathcal{O})$ is the stabilizer of the lattice $L = \mathcal{O} \oplus \mathcal{O}$ (note that G_L is an open compact subgroup of G).

Theorem 5. *Let Γ be a discrete torsion-free subgroup of G.*

a) *Γ is a free group.*

b) *The following conditions are equivalent:*

b$_1$) *$\mu(G/\Gamma)$ is finite.*

b$_2$) *G/Γ is compact.*

b$_3$) *The graph $\Gamma \backslash X$ is finite.*

c) *If the conditions* b) *are satisfied, Γ is finitely generated and its rank r_Γ is given by the formula*

$$r_\Gamma - 1 = \mu(G/\Gamma).$$

Since K is locally compact, "bounded" is equivalent to "relatively compact". If H is a bounded subgroup of Γ, H is then relatively compact; since Γ is discrete, this implies that H is compact and discrete, i.e. finite, and since Γ is torsion-free, we have $H = \{1\}$. On the other hand, if $V = K^2$, we have

$$\mathbf{SL}_2(K) = \mathbf{SL}(V) \subset \mathbf{GL}(V)^0.$$

We can thus apply th. 4, and we see that Γ is *free*, which proves assertion a).

Let L' be the lattice $\mathcal{O} \oplus \mathcal{O}\pi$, let $G_{L'}$ be its stabilizer in G and let $G_{LL'} = G_L \cap G_{L'}$, so that $G = G_L *_{G_{LL'}} G_{L'}$. Since $G_{LL'}$ is of index $q+1$ in G_L and $G_{L'}$, we have

$$\mu(G_{LL'}) = \frac{q-1}{q+1} \quad \text{and} \quad \mu(G_{L'}) = q - 1 = \mu(G_L).$$

Let S (resp. S', S'') be a set of double coset representatives for $\Gamma\backslash G/G_L$ (resp. $\Gamma\backslash G/G_{L'}$, $\Gamma\backslash G/G_{LL'}$). It is clear that G/Γ is compact if and only if S (resp. S', S'') is finite; since the set of vertices (resp. geometric edges) of $\Gamma\backslash X$ is in bijective correspondence with $S \coprod S'$ (resp. with S''), this shows the equivalence of b$_2$) and b$_3$). On the other hand, $\Gamma\backslash G$ is the disjoint union of the orbits sG_L ($s \in S$) of G_L; since Γ is torsion-free, we have $sG_Ls^{-1} \cap \Gamma = \{1\}$, which shows that $G_L \to sG_L \subset \Gamma\backslash G$ is injective; the orbit sG_L then has measure equal to that of G_L, i.e. $q-1$. We conclude that $\mu(G/\Gamma) = \mu(\Gamma\backslash G)$ is finite if and only if S is finite, which shows the equivalence of b$_1$) and b$_2$). In addition, we see that if G/Γ is compact then

$$\mu(G/\Gamma) = (q-1)\,\mathrm{Card}(S)$$

and, by an analogous argument:

$$\mu(G/\Gamma) = (q-1)\,\mathrm{Card}(S') = \frac{q-1}{q+1}\,\mathrm{Card}(S'').$$

By th. 4′ of chap. I, no. 3.3 we have

$$\begin{aligned} 1 - r &= \mathrm{Card}(\mathrm{vert}\,\Gamma\backslash X) - \tfrac{1}{2}\mathrm{Card}(\mathrm{edge}\,\Gamma\backslash X) \\ &= \mathrm{Card}(S) + \mathrm{Card}(S') - (\mathrm{Card}(S'') \\ &= \mu(G/\Gamma)(q-1)^{-1}(1 + 1 - (q+1)) = -\mu(G/\Gamma) \end{aligned}$$

which completes the proof.

Remark. If the conditions b) are satisfied, c) shows that $\mu(G/\Gamma)$ is equal to the opposite of the Euler-Poincaré characteristic of the group Γ; this means that $-\mu$ is the *Euler-Poincaré measure* of G, cf. [34], §3.

Examples of discrete subgroups of $\mathbf{SL}_2(\mathbf{Q}_p)$ *with compact quotient*

Let L be a totally real algebraic number field with a discrete valuation v such that the corresponding completion L_v is isomorphic to $\mathbf{Q}_p$ (e.g. $L = \mathbf{Q}$). Let S_∞ be the set of real places of L. Choose a quaternion field D on L which decomposes at v (i.e. $D \otimes L_V \simeq \mathbf{M}_2(L_V)$) and which does not decompose at any real place.

If L' is a commutative L-algebra let $G(L')$ denote the group of elements of $D \otimes_L L'$ of reduced norm 1; the functor $L' \mapsto G(L')$ is representable by an algebraic group G over L which is an anisotropic form of $\mathbf{SL}_2$. Let $S = \{v\} \cup S_\infty$, and let Γ be the group of S-*units* of D (relative to a basis for D over L); the group Γ is a discrete subgroup of

$$G_S = \prod_{s \in S} G(L_s) = G(L_v) \times G_{S_,}.$$

In view of the hypotheses we have made, $G_{S_,}$ is compact. On the other hand, Godement's criterion shows that G_S/Γ is compact. It follows that Γ may be identified with a *discrete subgroup with compact quotient* of $G(L_v) \simeq \mathbf{SL}_2(\mathbf{Q}_p)$; this is the example sought. (Note that Γ can have torsion; however, there are subgroups of finite index in Γ which are torsion-free: suitable congruence subgroups, for example.)

Relations between "finite volume" and "compactness"

The argument used for the equivalence $\mathrm{b}_1) \Leftrightarrow \mathrm{b}_2)$ can be extended to all p-adic Lie groups, as Tamagawa has remarked. More precisely, let G be a locally compact unimodular group, let U be an open compact subgroup of G, let Γ be a discrete subgroup of G, and let μ be a Haar measure on G. Let S be a system of double coset representatives for $\Gamma \backslash G/U$ and, for each $s \in S$, let $g(s)$ be the order of the *finite* group $\Gamma_s = \Gamma \cap sUs^{-1}$. The homogeneous space $\Gamma \backslash G$ is the disjoint union of the $sU \simeq (s^{-1}\Gamma_s s)\backslash U$. Whence

$$\mu(sU) = \mu(U)/g(s)$$

and

$$\mu(G/\Gamma) = \mu(U) \sum_{s \in S} \frac{1}{g(s)}.$$

The following are thus equivalent:

i) G/Γ has *finite volume*;

ii) The series $\sum_{s \in S} 1/g(s)$ is *convergent*.

These properties are evidently implied by:

iii) G/Γ is *compact* (i.e. S is *finite*).

Moreover, one sees that, if i) is satisfied, property iii) is equivalent to:

iv) The integers $g(s)$ are *bounded*.

Examples

a) Γ is *torsion-free*. The $g(s)$ are all equal to 1, and we have i) $\Leftrightarrow$ iii): this is the case considered in th. 5.

b) The group G is a *Lie group* over $\mathbf{Q}_p$; we can then choose U in such a way that it is a torsion-free pro-p-group (using the exponential map, for example). The $g(s)$ are again equal to 1, and we have the equivalence i) $\Leftrightarrow$ iii).

On the other hand, in the equal-characteristic case, there are discrete subgroups Γ such that G/Γ is of finite volume but non-compact; we give such an example in the next section.

Exercises

1) Suppose that K is locally compact. Show that each discrete torsion-free subgroup of $\mathbf{PGL}_2(K)$ is free (same method as for th. 4).

2) (Ihara) Let G be a group, U a subgroup of G and X the homogeneous space G/U. Let P_0 denote the image in X of the neutral element of G. Let $(P, Q) \mapsto d(P, Q)$ be a map of $X \times X$ into $\mathbf{N}$ with the following properties:

i) $d(gP, gQ) = d(P, Q)$ for all $P, Q \in X$ and $g \in G$;

ii) d is symmetric, and $d(P, Q) = 0 \Leftrightarrow P = Q$;

iii) For each $n \geqslant 0$, the number d_n of elements $Q \in X$ such that $d(P_0, Q) = n$ is finite.

Put $q = d_1 - 1$. Two points P, Q of X are called *adjacent* if $d(P, Q) = 1$; this defines a structure of combinatorial graph on X, invariant under G.

a) For each integer $n \geqslant 0$, let Θ_n denote the correspondence which transforms $P \in X$ into the sum of the points $Q \in X$ such that $d(P, Q) = n$. Show the equivalence of the following properties:

iv) X is a *tree*, and d is the corresponding distance function;

v) The Θ_n satisfy the formulae (cf. no. 1.1, exerc. 3):

v_1) $\Theta_1\Theta_1 = \Theta_2 + (q + 1)\Theta_0$

v_2) $\Theta_1\Theta_n = \Theta_{n+1} + q\Theta_{n-1}$ $(n \geqslant 2)$

b) Suppose that i), ..., iv) are satisfied. Let Γ be a subgroup of G, not containing any element of order 2, and such that $\Gamma \cap gUg^{-1} = \{1\}$ for each $g \in G$. Show that Γ acts freely and without inversion on the tree X; and deduce that it is a free group.

1.6 Nagao's theorem

This concerns the structure of $\mathbf{GL}_2(k[t])$, where $k[t]$ is the ring of polynomials in one indeterminate t over a commutative field k. If R is a commutative ring, we put

$$B(R) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \cap \mathbf{GL}_2(R) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| \, a, d \in R^*, \; b \in R \right\}.$$

For example:

$$B(k[t]) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| \, a, d \in k^*, \; b \in k[t] \right\}.$$

Theorem 6. *The group* $\mathbf{GL}_2(k[t])$ *is the sum of the subgroups* $\mathbf{GL}_2(k)$ *and* $B(k[t])$ *amalgamated along their intersection* $B(k)$:

$$\mathbf{GL}_2(k[t]) = \mathbf{GL}_2(k) *_{B(k)} B(k[t]).$$

(There is an analogous statement for $\mathbf{SL}_2$, cf. exerc. 2.)

This result has been proved by H. Nagao [19] by a direct method. We give below a different proof, based on the action of the group $\mathbf{GL}(k[t])$ on a suitable tree; this proof, less simple than that of Nagao, has the advantage of being adaptable to rings other than $k[t]$, cf. §2.

Notation

Let $K = k(t), \Gamma = \mathbf{GL}_2(k[t])$. We put on K the discrete valuation v corresponding to the "point at infinity":

$$v\left(\frac{a}{b}\right) = \deg(b) - \deg(a) \quad \text{if} \quad a, b \in k[t], \quad b \neq 0.$$

The corresponding valuation ring $\mathcal{O}$ consists of the a/b such that

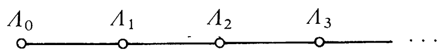
$$\deg(b) \geqslant \deg(a),$$

and we choose the element $1/t$ as uniformizer. The completion $\hat{K}$ of K is the field of formal series $k((1/t))$.

We put $V = K^2$; the canonical basis of V is denoted $\{e_1, e_2\}$. If n is an integer $\geqslant 0$ we let L_n denote the lattice with basis $\{e_1 t^n, e_2\}$, and Λ_n the corresponding vertex in the tree X associated with V. We have $L_n \subset L_{n+1}$ and

$$L_{n+1}/L_n \simeq \mathcal{O}/t^{-1}\mathcal{O} \simeq k.$$

The Λ_n are the vertices of a path T without backtracking:



Likewise, we put

$$\Gamma_0 = \mathbf{GL}_2(k)$$

and, for $n \geqslant 1$,

$$\Gamma_n = \begin{pmatrix} * & d^0 \leqslant n \\ 0 & * \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| \, a, d \in k^*, \, b \in k[t], \, \deg(b) \leqslant n \right\}.$$

Proposition 3. (a) *The* Λ_n *are pairwise inequivalent* mod Γ.

(b) Γ_n *is the stabilizer of* Λ_n *in* Γ.

(c) Γ_0 *acts transitively on the set of edges with origin* Λ_0.

(d) *For* $n \geqslant 1$, Γ_n *leaves the edge* $\Lambda_n \Lambda_{n+1}$ *fixed and acts transitively on the set of edges with origin* Λ_n *distinct from* $\Lambda_n \Lambda_{n+1}$.

Let $s = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of Γ transforming Λ_n into Λ_{n+m}. By hypothesis, there is an integer h such that $sL_n = L_{n+m}t^{-h}$; since $v(\det(s)) = 0$, prop. 1 shows that $m = 2h$. By writing down what it means for s to map L_n into $L_{n+2h}t^{-h}$ we obtain the conditions:

$$\deg(a) \leqslant h, \qquad \deg(b) \leqslant n + h, \qquad \deg(c) \leqslant n - h, \qquad \deg(d) \leqslant -h.$$

These conditions on a and d cannot be realized simultaneously if $h \neq 0$, which proves (a). For $h = 0$, $n = 0$ the four conditions above mean that a, b, c, d are scalars, i.e. that $s \in \Gamma_0$; for $h = 0$, $n \geqslant 1$ they mean that a, d are scalars, that $c = 0$, and that $\deg(b) \leqslant n$, i.e. that $s \in \Gamma_n$. Whence (b).

The assertion (c) follows from the fact that the set of edges with origin Λ_0 may be identified with the projective line $\mathbf{P}_1(k)$, and that $\mathbf{GL}_2(k)$ acts transitively on $\mathbf{P}_1(k)$.

The inclusion $\Gamma_n \subset \Gamma_{n+1}$ shows that Γ_n leaves the edge $\Lambda_n \Lambda_{n+1}$ fixed. On the other hand, the action of Γ_n on the k-plane $L_n/L_n t^{-1}$ is given by the homomorphism

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ 0 & d^n \end{pmatrix} \quad \text{where} \quad b = b_0 + b_1 t + \cdots + b_n t^n, \quad b_i \in k.$$

Its image in $\mathbf{GL}_2(k)$ is the triangular subgroup $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$; this subgroup acts on the projective line $\mathbf{P}_1(k)$ by fixing one point (corresponding to the edge $\Lambda_n \Lambda_{n+1}$) and permuting the others transitively. Whence (d).

Corollary. *The path* $T = \overset{\Lambda_0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{\Lambda_1}{\circ}\!\!-\!\!\!-\!\!\!- \cdots$ *is a fundamental domain of* X mod Γ.

Note first that Γ is contained in $\mathbf{GL}(V)^0$, hence acts without inversion on X, and the quotient graph $X' = \Gamma \backslash X$ is defined. In view of (a), the projection $X \to X'$ defines an isomorphism of T onto a subgraph T' of X', and we have to show that T' is *equal* to X'. Since X' is connected, it suffices to prove that T' is *open* in X', i.e. that each edge y of X' whose origin P belongs to T' is contained in T'. By hypothesis, P is the image of a vertex Λ_n of T, and y is the image of an edge $\tilde{y}$ of X with origin Λ_n. If $n = 0$, (c) shows that $\tilde{y}$ is congruent to $\Lambda_0 \Lambda_1$ mod Γ, and hence that y belongs to edge T'. If $n \geqslant 1$, (d) shows that $\tilde{y}$ is congruent mod Γ to either $\Lambda_n \Lambda_{n+1}$ or $\Lambda_n \Lambda_{n-1}$, and we again have $y \in \text{edge } T'$. Q.E.D.

Remark. The bijectivity of vert $T \to$ vert $\Gamma \backslash X$ can also be deduced from the fact (due to Grothendieck) that every vector bundle over the projective line $\mathbf{P}_1$ is a direct sum of line bundles; we return to this in §2.

Proof of theorem 6

Since T is a fundamental domain of X mod Γ, th. 10 of chap. I, no. 4.5 shows that Γ may be identified with the direct limit of the *tree of groups* defined by T and the Γ_n. In other words, Γ is the sum of the groups Γ_n amalgamated along the $\Gamma_n \cap \Gamma_{n+1}$. This amalgam can be made in two steps:

i) Construct the amalgam of the Γ_n, $n \geqslant 1$. Since the Γ_n form an increasing sequence, we simply get their *union*, i.e. the group $B(k[t])$.

ii) Amalgamate $\Gamma_0 = \mathbf{GL}_2(k)$ with $B(k(t))$ along $B(k)$. This gives

$$\Gamma = \mathbf{GL}_2(k) *_{B(k)} B(k[t]). \qquad \text{Q.E.D.}$$

Remark. Suppose k is *finite*. The group $\mathbf{SL}_2(k[t])$ is then a *discrete* subgroup of the locally compact group $\mathbf{SL}_2(\hat{K}) = \mathbf{SL}_2(k((1/t)))$, and it is easy to see (cf. exerc. 6) that the quotient is *of finite volume* but *non-compact.*

Exercises

1) Extend th. 6 and prop. 3 to the case of a non-commutative field k.

2) Put $SB(R) = \mathbf{SL}_2(R) \cap B(R)$. Show that

$$\mathbf{SL}_2(k[t]) = \mathbf{SL}_2(k) *_{SB(k)} SB(k[t]).$$

(Use th. 6, as well as exerc. 2 of chap. I, no. 1.3.)

3) (Nagao) Show that $\mathbf{GL}_2(k[t])$ and $\mathbf{SL}_2(k[t])$ are not finitely generated groups. (Use the fact that the additive group $k[t]$ is not a finitely generated module over the algebra of the multiplicative group k^*.)

4) Take $k = \mathbf{F}_2$. Show that $\mathbf{GL}_2(k[t])$ can be defined by the generating family

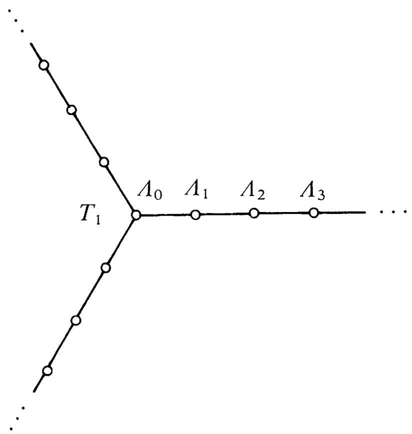
$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \ldots, \quad T_n = \begin{pmatrix} 1 & t^n \\ 0 & 1 \end{pmatrix}, \ldots$$

and the relations

$$S^2 = 1, \qquad (ST_0)^3 = 1, \qquad T_n^2 = 1 \quad (n \geqslant 0), \qquad T_nT_m = T_mT_n \quad (n, m \geqslant 0).$$

5) Let Γ_1 be the kernel of $\Gamma \to \mathbf{GL}_2(k)$ given by $t \mapsto 0$. Let

$$C = \Gamma_1 \cap B(k[t]) = \begin{pmatrix} 1 & tk[t] \\ 0 & 1 \end{pmatrix}.$$



Let S be a set of representatives of $\mathbf{GL}_2(k)/B(k) \simeq \mathbf{P}_1(k)$ and let T_1 be the subgraph of X which is the union of the sT, $s \in S$. Show that T_1 is a tree, and that it is a fundamental domain of Γ_1 in X. Deduce that Γ_1 is the *free product* of the sCs^{-1}, as s runs through S.

6) Suppose that k is finite. Put $q = \mathrm{Card}(k)$, $\Gamma^0 = \mathbf{SL}_2(k[t])$, $G = \mathbf{SL}_2(\hat{K})$ and $g(n) = \mathrm{Card}(\Gamma_n \cap \Gamma^0)$ for $n \geqslant 0$. One has $g(0) = q(q^2 - 1)$ and

$$g(n) = (q-1)q^{n+1} \quad \text{for} \quad n \geqslant 1.$$

Put on G the measure μ defined in no. 1.5. Show that

$$\begin{aligned} \mu = (G/\Gamma^0) &= (q-1) \sum_{n \text{ even}} \frac{1}{g(n)} \\ &= (q-1) \sum_{n \text{ odd}} \frac{1}{g(n)} \\ &= 1/(q^2-1). \end{aligned}$$

1.7 Connection with Tits systems

Tits systems

Recall (cf. Bourbaki, *Gr. et Alg. de Lie*, [36], §2) that a *Tits system* (or a "BN-pair") is a quadruplet (G, B, N, S) where G is a group, B and N subgroups of G, and S a subset of $W = N/(B \cap N)$ satisfying the following axioms:

(T1) *The set $B \cup N$ generates G and $B \cap N$ is a normal subgroup of N.*

(T2) *The set S generates $W = N/(B \cap N)$ and consists of elements of order* 2.

(If $w \in W$ we let $C(w)$ denote the double coset BwB; similarly we put ${}^wB = wBw^{-1}$.)

(T3) *One has $C(s)C(w) \subset C(w) \cup C(sw)$ for $s \in S$ and $w \in W$.*

(T4) *For each $s \in S$ one has ${}^sB \not\subset B$.*

The group W is called the *Weyl group* of (G, B, N, S); the pair (W, S) is a *Coxeter system* (*loc. cit.*, th. 2). The group G is the disjoint union of the double cosets $C(w)$, $w \in W$ ("Bruhat decomposition").

If S' is a subset of S, we let $W_{S'}$ denote the subgroup of W generated by S', and we put

$$G_{S'} = BW_{S'}B = \bigcup_{w \in W_{S'}} C(w).$$

One knows (*loc. cit.*, th. 3) that $S' \mapsto G_{S'}$ is a bijection of the set of subsets of S onto the set of subgroups of G containing B. One says that $G_{S'}$ is the *standard parabolic subgroup* of type S', and that $\mathrm{Card}(S')$ is its *rank*.

The Tits system with Weyl groups of type $\circ\overset{\infty}{\text{———}}\circ$

Let (G, B, N, S) be a Tits system, and suppose that (W, S) is of type $\tilde{A}_1$, i.e., that S consists of two elements $\{s_1, s_2\}$ and that W is infinite (in which case W is an infinite dihedral group generated by s_1 and s_2). Let G_1 and G_2 be the standard parabolic subgroups corresponding to the subsets $\{s_1\}$ and $\{s_2\}$ of S. We have

$$G_1 = B \cup C(s_1), \qquad G_2 = B \cup C(s_2).$$

Theorem 7. *The group G is the sum of the groups G_1 and G_2 amalgamated along their intersection B*:

$$G = G_1 *_B G_2.$$

The subgroup G' of G generated by G_1 and G_2 contains B, and hence is of the form $G_{S'}$, with $S' \subset S$. Since S' contains s_1 and s_2, we have $S' = S$, whence $G' = G$.

It remains to show that if $\tilde{g}$ is an element $\neq 1$ in $G_1 *_B G_2$ its image g in G is $\neq 1$. Put $I = \{1, 2\}$ and let

$$(i_1, \ldots, i_n) \in I^n, \qquad i_j \neq i_{j+1} \quad (1 \leqslant j < n), \qquad n = l(g),$$

be the type of the reduced word representing $\tilde{g}$ (cf. chap. I, no. 1.2). We distinguish two cases:

a) $n = 0$.

Then $\tilde{g} \in B$, and since $B \to G$ is injective, we certainly have $g \neq 1$.

b) $n \geqslant 1$.

Let $w = s_{i_1} \cdots s_{i_n} \in W$. Since $W = \{1, s_1\} * \{1, s_2\}$, this is an element of W of length n; in particular $w \neq 1$. On the other hand, we can write $\tilde{g}$ in the form

$$\tilde{g} = bp_1 \cdots p_n \quad \text{with} \quad b \in B \quad \text{and} \quad p_j \in G_{i_j} - B, \qquad 1 \leqslant j \leqslant n.$$

We have $p_j \in C(s_{i_j})$. By cor. 1 to th. 2 of Bourbaki (*loc. cit.*) we deduce that g belongs to $C(w)$. Since $w \neq 1$, this implies $g \neq 1$. Q.E.D.

Remark. Let X be the *building* defined by (G, B, N, S), cf. Bourbaki, *loc. cit.*, exerc. 10. Th. 7 is equivalent to saying that X is a *tree* on which G acts without inversion, with a segment x_1x_2 as fundamental domain, the stabilizers of x_1 and x_2 being G_1 and G_2 respectively.

Examples

1) Take

$$G = \mathbf{SL}_2(K)$$

$$B = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathcal{O}),\ c \equiv 0 \pmod{\pi}\right\}$$

$$N = G \cap \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \cup G \cap \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} = \text{normalizer of standard torus}$$

$$S = \{s_1, s_2\} \quad \text{where} \quad s_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix}.$$

One checks that (G, B, N, S) is a Tits system (this is a very special case of the results of Iwahori-Matsumoto, *Publ. Math. I.H.E.S.*, [39] – see also Bourbaki, *loc. cit.* exerc. 21). Since s_1s_2 is of infinite order, this is a Tits system of type $\circ\overset{\infty}{\text{———}}\circ$. The parabolic subgroups G_1 and G_2 are:

$$G_1 = \mathbf{SL}_2(\mathcal{O})$$

$$G_2 = \left\{\begin{pmatrix} a & \pi^{-1}b \\ \pi c & d \end{pmatrix}, \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathcal{O})\right\}.$$

By th. 7 we have $\mathbf{SL}_2(K) = G_1 *_B G_2$: this recovers cor. 1 to th. 3. The *building* attached to the Tits system (G, B, N, S) may be identified with the *tree* X of no. 1.1.

2) Following Bruhat-Tits, one obtains analogous results when one starts with a simple and simply connected algebraic group $\mathcal{G}$ of relative rank 1 over a local field K (assumed commutative, with perfect residue field). The group $G = \mathcal{G}(K)$ has a Tits system of type $\circ\overset{\infty}{\text{———}}\circ$ and the corresponding building X is a *tree*. A good part of what has been done in the preceding sections can be generalized; for example, if K is locally compact, every discrete torsion-free subgroup of G is a *free* group (the proof is the same as that for Ihara's theorem).

Tits systems and amalgams: general case

Let G be a group, and let $(G_i)_{i \in I}$ be a family of subgroups of G. We say (by abuse of language) that G is the *sum of the* G_i *amalgamated along their intersections* if G is the *direct limit* (cf. chap. I, no. 1.1) of the system formed by the G_i, the $G_i \cap G_j$, and the inclusions

$$\begin{array}{ccc} & & G_i \\ & \nearrow & \\ G_i \cap G_j & & \\ & \searrow & \\ & & G_j \end{array} \quad .$$

Theorem 8 (Tits). *Let (G, B, N, S) be a Tits system; for each $s \in S$, let $G_s = G_{\{s\}}$ be the corresponding standard parabolic subgroup. Then G is the sum of N and the G_s $(s \in S)$ amalgamated along their intersections.*

Proof (from a letter of Tits, 6/13/1968 – see also [41], §13).

First assume only that (G, B, N, S) has the properties (T_1), (T_2), (T_4) and that (W, S) is a Coxeter system. We have:

Lemma 2. *Property (T_3) is equivalent to the conjunction of the following two:*
(T_5) *For each $s \in S$ one has $C(s)C(s) = B \cup C(s)$.*
(T_6) *If $s \in S$ and $w \in W$ are such that $l(sw) > l(w)$ one has*

$$C(s)C(w) = C(sw).$$

(We let $l(w)$ denote the *length* of w in the Coxeter group W, cf. Bourbaki, *loc. cit.*)

The implication $(\mathrm{T}_3) \Rightarrow \{(\mathrm{T}_5)$ and $(\mathrm{T}_6)\}$ is proved in Bourbaki. Conversely, suppose (T_5) and (T_6) hold, and let $s \in S$, $w \in W$. We have to prove that

$$C(s)C(w) \subset C(w) \cup C(sw).$$

This is clear if $l(sw) > l(w)$, by (T_6). If not, we have $l(sw) < l(w)$. Applying (T_6) to s and sw we find $C(s)C(sw) = C(w)$. Whence

$$\begin{aligned} C(s)C(w) &= C(s)C(s)C(sw) \\ &\subset (B \cup C(s))C(sw) \quad [\text{by } (\mathrm{T}_5)] \\ &\subset C(sw) \cup C(s)C(sw) \\ &\subset C(sw) \cup C(w). \end{aligned}$$

Lemma 3. *Suppose that property (T_5) of lemma 2 holds. Let $s \in S$, $w \in W$ be such that $l(sw) > l(w)$. Put*

$$B_s = {}^sB \cap B, \qquad B_w = {}^wB \cap B, \qquad B_{sw} = {}^{sw}B \cap B \quad \textit{and} \quad {}^sB_w = sB_ws^{-1} = sB_ws.$$

Then the properties

(T$_6$) $C(s)C(w) = C(sw)$

and

(T$_7$) $B_s B_w = B$.

are equivalent. Moreover, these properties imply:

(T$_8$) $B_{sw} = {}^sB_w \cap B$.

If (T$_6$) holds, we have $C(s)C(w) \cap C(w) = \emptyset$. Multiplying on the right by w^{-1}, this gives

$$C(s) \cdot {}^wB \ \cap \ B \cdot {}^wB = \emptyset, \quad \text{hence} \quad C(s) \cap {}^wB = \emptyset.$$

By (T$_5$), we have ${}^sB \subset B \cap C(s)$. Whence:

$${}^sB \cap {}^wB \subset (B \cup C(s)) \cap {}^wB = B \cap {}^wB = B_w,$$

and, conjugating by s,

$$B \cap {}^{sw}B \subset {}^sB_w, \quad \text{i.e.} \quad B_{sw} \subset {}^sB_w.$$

Whence $B_{sw} \subset {}^sB_w \cap B$; since the opposite inclusion is trivial, this shows that $(T_6) \Rightarrow (T_8)$. On the other hand, the formula $C(s)C(w) = C(sw)$ shows that $B \subset sBs \cdot wBw^{-1}$, and hence that each $b \in B$ can be written $b = xy$ with $x \in sBs$, $y \in wBw^{-1}$. We have

$$y = x^{-1}b \in sBsB \cap wBw^{-1} \subset (B \cup C(s)) \cap {}^wB = B_w$$

by what we have seen above. We conclude that y belongs to B, and hence x also, so we have $b \in B_s \cdot B_w$, whence (T$_7$).

Conversely, suppose that (T$_7$) holds. We have

$$B = B_s B_w \subset {}^sB\, {}^wB,$$

i.e. $B \subset sBs \cdot wBw^{-1}$, or $sBw \subset BswB$, which is equivalent to (T$_6$).

We now return to the case where (G, B, N, S) is a *Tits system* and let $\tilde{G}$ denote the sum of N and the G_s ($s \in S$), amalgamated along their intersections. To avoid confusion, we let $\tilde{N}$, $\tilde{B}$, $\tilde{G}_s$ denote the images of N, B, G_s in $\tilde{G}$; the canonical projection $\phi\colon \tilde{G} \to G$ induces an *isomorphism* of $\tilde{N}$ onto N, of $\tilde{B}$ onto B, of $\tilde{G}_s$ onto G_s, of $\tilde{N} \cap \tilde{G}_s$ onto $N \cap G_s$. (Note that the intersection of G_s and $G_{s'}$ reduces to B for $s \neq s'$.) We identify $\tilde{W} = \tilde{N}/(\tilde{B} \cap \tilde{N})$ with W.

Lemma 4. *$(\tilde{G}, \tilde{B}, \tilde{N}, S)$ is a Tits system.*

We have to check the analogues $(\tilde{T}_1), \dots, (\tilde{T}_4)$ of the axioms $(T_1), \dots, (T_4)$ for (G, B, N, S). The axioms $(\tilde{T}_1)$, $(\tilde{T}_3)$ and $(\tilde{T}_4)$ are immediate. On the other hand, lemmas 2 and 3 allow us to replace $(\tilde{T}_3)$ by $(\tilde{T}_5)$ and $(\tilde{T}_7)$. But the axiom $(\tilde{T}_5)$:

$$\tilde{C}(s)\tilde{C}(s) = \tilde{B} \cup \tilde{C}(s)$$

takes place inside the group $\tilde{G}_s$; since $\phi\colon \tilde{G}_s \to G_s$ is an isomorphism, this axiom is certainly satisfied.

It remains to prove

($\tilde{\mathrm{T}}_7$) $\tilde{B}_s\tilde{B}_w = \tilde{B}$ for $s \in S$, $w \in W$ and $l(sw) > l(w)$.

We first show, by induction on $l(w)$, that $\phi(\tilde{B}_w) = B_w$. This is true if $w = 1$. It then remains to prove that, if it is true for w and if $s \in S$ is such that $l(sw) > l(w)$, then it is true for sw. By (T_8) we have

$$B_{sw} = {}^sB_w \cap B = {}^s\phi(\tilde{B}_w) \cap \phi(\tilde{B}).$$

But $\tilde{B}$, $\tilde{B}_w$ and s are found in $\tilde{G}_s$, which is mapped isomorphically onto G_s; hence ${}^s\phi(\tilde{B}_w) \cap \phi(\tilde{B}) = \phi({}^s\tilde{B}_w \cap \tilde{B})$ i.e.

$$B_{sw} = {}^sB_w \cap B = \phi({}^s\tilde{B}_w \cap \tilde{B}).$$

Since we evidently have ${}^s\tilde{B}_w \cap \tilde{B} \subset \tilde{B}_{sw}$, this shows that $\phi(\tilde{B}_{sw})$ contains B_{sw}; the opposite inclusion is trivial.

This being so, we have $\phi(\tilde{B}_s\tilde{B}_w) = \phi(\tilde{B}_s)\phi(\tilde{B}_w) = B_sB_w = B$. Since $\phi\colon \tilde{B} \to B$ is an isomorphism, this implies that $\tilde{B}_s\tilde{B}_w = \tilde{B}$, and concludes the proof of the lemma.

Th. 8 is now immediate. Indeed, since $(\tilde{G}, \tilde{B}, \tilde{N}, S)$ is a Tits system, $\tilde{G}$ is the disjoint union of the double cosets $\tilde{C}(w)$, $w \in W$, and ϕ maps $\tilde{C}(w)$ into $C(w)$. If $x \in \mathrm{Ker}(\phi)$ we then have

$$x \in \tilde{C}(1) = \tilde{B}$$

and since the restriction of ϕ to $\tilde{B}$ is injective, this implies $x = 1$. Thus ϕ is *injective*; the fact that it is *surjective* follows simply from the fact that G is generated by B and the G_s.

Corollary 1. *Suppose that S is non-empty and that, if s, t are distinct elements of S, the order of st in W is infinite. Then $G = *_B\, G_s$.*

In view of the structure of Coxeter groups (Bourbaki, *loc. cit.*) the hypothesis implies that W is the free product of the subgroups $W_s = \{1, s\}$. If $H = B \cap N$, we have $W = N/H$, hence $N = *_H\, N_s$ with $N_s = W_sH = N \cap G_s$, and we can suppress N in the amalgam which gives G.

Note that, when $\mathrm{Card}(S) = 2$, we recover th. 7.

Corollary 2. *If $\mathrm{Card}(S) \geqslant 2$ the group G is the sum of its standard parabolic subgroups of rank 2, amalgamated along their intersections.*

The argument is the same as above; we use the fact that W is the sum of the $W_{S'}$ (with $\mathrm{Card}(S') = 2$), amalgamated along their intersections.

Corollary 3. *If* Card(S) *is finite* $\geqslant 3$, *the group* G *is the sum of its proper maximal standard parabolic subgroups, amalgamated along their intersections* (or along the standard parabolic subgroups of rank 1, which amounts to the same).

This follows from cor. 2.

Exercises

1) Let (G, B, N, S) be a Tits system of type $\circ\overset{\infty}{\rule{3em}{0.4pt}}\circ$, let $G = G_1 *_B G_2$ be the decomposition of G given by th. 7, and let X be the corresponding tree. Let x_i $(i = 1, 2)$ be the vertex of X with stabilizer G_i; the stabilizer of the edge x_1x_2 is B. Prove the following properties:

a) The number of edges of X with a given origin is $\geqslant 3$.

b) For each $n \geqslant 0$ and each $i = 1, 2$ the group G_i acts *transitively* on the set of the $x \in \operatorname{vert} X$ such that $d(x, x_i) = n$.

2) Let X be a tree on which a group G acts, and let x_1x_2 be a segment which is a fundamental domain of $X \bmod G$. Let G_1, G_2 and B be the stabilizers of x_1, x_2 and x_1x_2. Suppose that the properties (a) and (b) of the preceding exercise hold. Let $s_i \in G_i - B$ $(i = 1, 2)$. Show that each double coset of G modulo B contains exactly one element of the form

$$s_{i_1} \cdots s_{i_n} \quad \text{with} \quad i_j \in \{1, 2\} \quad \text{and} \quad i_j \neq i_{j+1} \quad \text{for} \quad 1 \leqslant j < n.$$

Deduce that B is a *Tits subgroup* of G (in the sense of Bourbaki, *loc. cit.*, exerc. 3) and that the corresponding Weyl group is infinite dihedral.

§2 Arithmetic subgroups of the groups $\mathbf{GL}_2$ and $\mathbf{SL}_2$ over a function field of one variable

Notation

Let k_0 be a commutative field, and C a smooth projective curve over k_0, absolutely connected, of genus g. We let K denote the function field of C; it is an extension of k_0 of transcendence degree 1, and k_0 is algebraically closed in K.

The closed points of C correspond to the discrete valuations of K which are trivial on k_0. If Q is such a point, let v_Q denote the corresponding valuation, $\mathcal{O}_Q$ its local ring, $\kappa(Q)$ its residue field, and $\deg(Q)$ the degree of $\kappa(Q)$ over k_0. Let e be the *gcd* of the $\deg(Q)$; when k_0 is finite or algebraically closed we have $e = 1$.

We *choose* a closed point P of C and put

$$v = v_P, \qquad \mathcal{O} = \mathcal{O}_P, \qquad k = \kappa(P), \qquad d = \deg(P),$$

so that $(K, v, \mathcal{O}, k)$ satisfies the hypotheses of §1.

Let $C^{\text{aff}} = C - \{P\}$; this is an affine curve having P as its unique "point at infinity". Its affine algebra A is a Dedekind domain whose principal ideals correspond to the points of C distinct from P. We have $A^* = k_0^*$.

We put $V = K^2$, so that $\mathbf{GL}(V) = \mathbf{GL}_2(K)$. We are going to study the subgroup $\Gamma = \mathbf{GL}_2(A)$ of $\mathbf{GL}(V)$ and its subgroups of finite index, in particular $\mathbf{SL}_2(A)$ when k_0 is finite. To do this, we use the action of Γ on the *tree* X of classes of $\mathcal{O}$-lattices of V, cf. §1.

Example

When C is the projective line, and P its point at infinity, we have $k = k_0$, $A = k[t]$, $K = k(t)$. This case has been studied in no. 1.6. We return to it in no. 2.4.

2.1 Interpretation of the vertices of $\Gamma\backslash X$ as classes of vector bundles of rank 2 over C

Let L be an $\mathcal{O}$-lattice of $V = K^2$. We associate with L:

a) the vertex x_L defined by L in the tree X, cf. no. 1.1. (Recall that each vertex of X is of the form x_L, for a suitable lattice L, and $x_L = x_{L'}$ if and only if there is an $\alpha \in K^*$ such that $L' = \alpha L$ – note that, since K is commutative, we may write scalars on the *left*.)

b) the *coherent subsheaf* E_L of the constant sheaf V over C characterized by the following two properties:

b₁) at each point Q of C^{aff} the localization $(E_L)_Q$ of E_L is $(\mathcal{O}_Q)^2$;

b₂) the localization $(E_L)_P$ of E_L at P is L.

Existence and uniqueness of such a sheaf are immediate.

Since E_L is locally free of rank 2, we can interpret it as a *vector bundle of rank* 2 *over* C; this is the usual identification of a vector bundle with its sheaf of germs of sections. Its fibre at P is $L/\pi L$, where π is a uniformizer of $\mathcal{O}$. In view of b₁) one has:

i) The restriction of E_L to C^{aff} is trivial.

Moreover:

ii) If $\alpha \in K^*$, and if $n = v(\alpha)$, one has $E_{\alpha L} = I_P^{\otimes n} \otimes E_L$, where I_P is the sheaf of ideals at the point P on C;

iii) If L' is an $\mathcal{O}$-lattice of V, the bundles E_L and $E_{L'}$ are isomorphic if and only if there is an $s \in \Gamma = \mathbf{GL}_2(A)$ such that $L' = sL$.

(More generally, the homomorphisms of E_L into $E_{L'}$ are given by the matrices $s \in \mathbf{M}_2(A)$ such that $sL \subset L'$.)

We say that two vector bundles E and E' on C are *I_P-equivalent* if there is an $n \in \mathbf{Z}$ such that E' is isomorphic to $I_P^{\otimes n} \otimes E$. Properties i), ii), iii) above imply:

Proposition 4. *The correspondence $x_L \leftarrow\!\dashv L \mapsto E_L$ induces a bijection of* vert$(\Gamma\backslash X)$ *onto the set of I_P-equivalence classes of vector bundles of rank 2 over C whose restrictions to C^{aff} are trivial.*

Remarks

1) This correspondence identifies the *group* Aut(E) *of automorphisms* of a bundle E with the *stabilizer in Γ* of a vertex corresponding to E.

2) The *edges* correspond to pairs (E, E') of sheaves such that E' is a subsheaf of E and the quotient sheaf E/E' is zero outside P, and of length 1 (as an $\mathcal{O}$-module) at P; note that such a subsheaf is characterized by giving a *line* of the fibre $E(P)$ of E at P.

3) The *standard lattice* $\mathcal{O}^2$ corresponds to the *trivial bundle* $\mathbf{1}^2$ of rank 2 (where $\mathbf{1}$ denotes the trivial bundle of rank 1). In view of 1), its stabilizer in Γ is $\mathbf{GL}_2(k_0)$.

4) The group Γ is contained in the group denoted $\mathbf{GL}(V)^0$ in section 1.2. It follows that it preserves the partition of vert(X) into *even* vertices (i.e. those at an even distance from the standard vertex defined by $\mathbf{1}^2$), and *odd* vertices; we can then speak of even or odd vertices of $\Gamma\backslash X$; two adjacent vertices are of different parity.

5) There are analogous results for the group $\Gamma_1 = \mathbf{SL}_2(A)$, cf. exerc. 2.

The determinant of a bundle

If E is a vector bundle of rank 2 over C we put

$$\det(E) = \textstyle\bigwedge^2 E.$$

It is a vector bundle of rank 1 (i.e. a line bundle, i.e. an invertible sheaf). Its *degree* (= degree of a rational section) will be denoted by deg(E). We have:

Lemma 5. *The following conditions are equivalent:*

i) *The restriction of E to C^{aff} is trivial.*

i′) *The restriction of* det(E) *to C^{aff} is trivial.*

i″) *There is an $n \in \mathbf{Z}$ such that* det(E) *is isomorphic to $I_P^{\otimes n}$.*

The equivalence i) $\Leftrightarrow$ i′) follows from the fact that C^{aff} is of dimension 1 (one can also remark that the restriction of E to C^{aff} corresponds to a projective A-module of rank 2, and use the classification of such modules). The equivalence i′) $\Leftrightarrow$ i″) is immediate.

Example. If L is an $\mathcal{O}$-lattice of V, we have $\det(E_L) = I_P^{\otimes n}$ with $n = \chi(\mathcal{O}^2, L)$, cf. no. 1.2. It follows that

$$\deg(E_L) = -n\deg(P) = -nd.$$

Remark. Let E be a bundle satisfying the conditions of lemma 5. The formula

$$\det(I_P^{\otimes m} \otimes E) = I_P^{\otimes 2m} \otimes \det(E)$$

shows that we can choose m so that the determinant of the bundle $I_P^{\otimes m} \otimes E$ is isomorphic to $\mathbf{1}$ or to I_P. Hence:

Proposition 5. *The correspondence of prop. 4 induces a bijection of* $\mathrm{vert}(\Gamma\backslash X)$ *onto the set of isomorphism classes of vector bundles E of rank 2 over C such that* $\det(E)$ *is isomorphic to* $\mathbf{1}$ *or to I_P.*

The vertices of $\Gamma\backslash X$ corresponding to bundles E such that $\det(E) \simeq \mathbf{1}$ are the *even* vertices, in the sense of remark 4) above; the others are the *odd* vertices.

Exercises

1) Put $\tilde{\Gamma} = \mathbf{PGL}_2(A) = \Gamma/k_0^*$, $\Gamma_1 = \mathbf{SL}_2(A)$, $\tilde{\Gamma}_1 = \Gamma_1/\{\pm 1\}$, so that $\tilde{\Gamma}_1$ may be identified with a subgroup of $\tilde{\Gamma}$, which in turn is embedded in $\mathbf{PGL}_2(K)$. Show that $\det\colon \Gamma \to k_0^*$ induces an isomorphism $\tilde{\Gamma}/\tilde{\Gamma}_1 \to k_0^*/k_0^{*2}$ on passing to the quotient. Deduce that k_0^*/k_0^{*2} acts on $\Gamma_1\backslash X$, and that the quotient is $\Gamma\backslash X$. Deduce:

a) A vertex of $\Gamma\backslash X$, corresponding to a bundle E, is the image of *a single vertex* in $\Gamma_1\backslash X$ if and only if, for each $\alpha \in k_0^*$, there is an $s \in \mathrm{Aut}(E)$ such that $\det(s) = \alpha$.

b) An edge of $\Gamma\backslash X$, corresponding to a pair of sheaves $E \supset E'$, is the image of *a single edge* in $\Gamma_1\backslash X$ if and only if, for each $\alpha \in k_0^*$, there is an $s \in \mathrm{Aut}(E)$ such that $\det(s) = \alpha$ and $s(E') = E'$.

c) If each element of k_0 is a square, the map $\Gamma_1\backslash X \to \Gamma\backslash X$ is an isomorphism.

2) The notations are those of exerc. 1.

a) Show that the elements of $\mathrm{vert}(\Gamma_1\backslash X)$ can be interpreted as classes of pairs (E,f) where E is a bundle of rank 2 and f an isomorphism $\det(E) \xrightarrow{\sim} I_P^{\otimes n}$ with $n = -\deg(E)/d$; two pairs (E,f) and (E',f') are in the same class if and only if one can find an integer m such that (E',f') is isomorphic to the pair (E_m, f_m) derived from (E,f) by tensor product with $I_P^{\otimes m}$ (this is meaningful, because $\det(E_m) = I_P^{\otimes 2m} \otimes \det(E)$).

b) Give a similar interpretation of the elements of $\mathrm{edge}(\Gamma_1\backslash X)$ as classes of triplets (E, f, E'), where (E,f) is as in a), and E' is a subsheaf of E such that the sheaf E/E' is zero outside of P, and of length 1 at P (one can, if one prefers, substitute for E' a line of the fibre $E(P)$ of E at P).

c) Use these interpretations to recover the results of exerc. 1.

3) Let H be a finitely generated sub-A-module of V which is a projective A-module of rank 2, and let $\Gamma_H = \mathbf{GL}(H)$. Identify H with a vector bundle of rank 2 over the curve C^{aff}.

a) Define a bijection of $\mathrm{vert}(\Gamma_H\backslash X)$ onto the set of I_P-equivalence classes of vector bundles of rank 2 over C whose restrictions to C^{aff} are isomorphic to H (same method as for prop. 4).

b) Extend to Γ_H the results proved for Γ in this section.

4) Prove a result analogous to prop. 4 for an affine curve with any finite number of "points at infinity" $P_1, \ldots, P_h$, the tree X being replaced by the product of the trees relative to the P_i (cf. Stuhler [21]).

5) Let $x \in \mathrm{vert}(\Gamma\backslash X)$, and let E be a bundle corresponding to x. Then x is a *terminal vertex* of $\Gamma\backslash X$ (cf. chap. I, no. 2.2) if and only if the group $\mathrm{Aut}(E)$ acts transitively on the set of k-lines of the fibre $E(P)$ of E at P.

6) Make $\mathbf{GL}_2(k_0)$ act on the projective line $\mathbf{P}_1(k)$ by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot z = (\alpha z + \beta)/(\gamma z + \delta).$$

a) Let N be the cardinal of the set of orbits. Show that:

$$N = 1 \Leftrightarrow d = 1 \quad (\text{recall that } d = [k:k_0])$$

$$N = 2 \Leftrightarrow d = 2 \text{ or } 3$$

$$N = 2 + \mathrm{Card}(k_0) \quad \text{if} \quad d = 4.$$

b) Let x_1 be the standard vertex of $\Gamma\backslash X$. The cardinality of the set of edges of $\Gamma\backslash X$ with origin x_1 equals N. Deduce that x_1 is a terminal vertex of $\Gamma\backslash X$ if and only if $d = 1$.

2.2 Bundles of rank 1 and decomposable bundles

This concerns well-known results, in which the choice of the "point at infinity" P does not matter. We review them quickly. For more details, the reader can refer to [21], [22], ..., [27].

Bundles of rank 1

The classes of vector bundles of rank 1 over C form a group $\mathrm{Pic}(C)$ with the tensor product as composition law. One has an exact sequence

$$0 \to J(k_0) \to \mathrm{Pic}(C) \xrightarrow{\deg} e\mathbf{Z} \to 0$$

where e denotes the *gcd* of the degrees of closed points of C and J is the *Jacobian variety* of C.

When k_0 is a finite field of q_0 elements, the group $J(k_0)$ is finite. More precisely, a theorem of Weil gives

$$\mathrm{Card}(J(k_0)) \leqslant (1 + q_0^{1/2})^{2g}$$

where $g = \dim J$ is the *genus* of C. Moreover, $e = 1$.

Subbundles of rank 1 of the bundles of rank 2

Let E be a vector bundle of rank 2 over C. Let F be a subsheaf of E of rank 1; one checks easily that F is contained in a unique maximal subsheaf $\bar{F}$ of rank 1, which is locally a direct factor of E; one may define $\bar{F}$, for example, as the intersection of E with the line of the generic fibre of E containing F. When $F = \bar{F}$, one says that F is a *subbundle* of rank 1 of the bundle E; the quotient sheaf E/F is then a bundle of rank 1. We put

$$N(E;F) = \deg(F) - \deg(E/F) = 2\deg(F) - \deg(E) \quad \text{and} \quad N(E) = \operatorname*{Sup}_F N(E;F).$$

Proposition 6. (i) *One has* $-2(g+e-1) \leqslant N(E) < \infty$.

(ii) *If E contains a subbundle F of rank* 1 *such that* $N(E;F) > 0$, *then this bundle is unique; in particular one has* $N(E) = N(E;F)$.

We recall the proof:

For (i) we remark that $N(E)$ does not change when we replace E by $X \otimes E$, where X is a bundle of rank 1. Since this operation changes the degree of E by a multiple of $2e$ we can assume that

$$2g - 2 < \deg(E) \leqslant 2g + 2e - 2.$$

By the *Riemann-Roch theorem* we have

$$\dim H^0(C,E) \geqslant \deg(E) + 2 - 2g > 0$$

where $H^0(C,E)$ is the space of *sections* of E over C (when E is of the form E_L of no. 2.1 we have $H^0(C,E) = A^2 \cap L$). This shows that E contains a subsheaf F of rank 1 isomorphic to the trivial bundle $\mathbf{1}$. If $\bar{F}$ denotes the subbundle of E defined by F, we have $\deg(\bar{F}) \geqslant \deg(F) = 0$, whence

$$N(E;\bar{F}) \geqslant -\deg(E) \geqslant -2(g-e-1)$$

and consequently

$$N(E) \geqslant -2(g+e-1).$$

On the other hand, if $\bar{F}$ is a subbundle of E of rank 1, the Riemann-Roch theorem shows that

$$\deg(F') \leqslant g - 1 + \dim H^0(C,F') \leqslant g - 1 + \dim H^0(C,E),$$

and hence $\deg(F')$ is *bounded* (this also follows from (ii)). Whence $N(E) < \infty$.

For (ii), suppose that E contains another subbundle F_1 of rank 1 such that $N(E;F_1) > 0$. After replacing E by $F_1^{-1} \otimes E$, we can assume that $F_1 \simeq \mathbf{1}$, i.e. that F_1 is generated by a section s of E which is everywhere non-zero. Since $N(E;F_1) > 0$, we have $\deg(E) < 0$, whence $\deg(E/F) < 0$ because $N(E;F) > 0$. It follows that

$H^0(C, E/F) = 0$, and the exact sequence

$$0 \to H^0(C, F) \to H^0(C, E) \to H^0(C, E/F)$$

shows that s is contained in F. Whence $F_1 \subset F$, and consequently $F_1 = F$, because F_1 is a subbundle of E of rank 1.

Remarks

1) The fact that $N(E)$ is finite shows that E has a subbundle F *of maximum degree*; when $N(E) > 0$ this subbundle is *unique*, by (ii).

2) The integer $N(E; F)$ is invariant by extension of the ground field. This is not so for $N(E)$, cf. exerc. 2, 3.

3) Suppose that k_0 is algebraically closed. One says, with Mumford, that E is *semistable* if $N(E) \leqslant 0$ and that E is *stable* if $N(E) < 0$, cf. [26], [27] where one finds information on the "moduli schemes" of such bundles.

(Note that Mumford uses "unstable" for "non-semistable". This may lead to annoying confusions...)

Decomposable bundles

Let E be a vector bundle of rank 2 over C. One says that E is *decomposable* if it is the direct sum of two subbundles of rank 1. Such a decomposition is unique, up to an automorphism of E, cf. for example Atiyah, *Bull. S.M.F.*, 84 (1956), p. 307–317.

Remark

Let $E = F \oplus F'$ be a decomposable bundle, with $\deg(F) \geqslant \deg(F')$. There are two cases:

a) F and F' are *isomorphic*. We have $E \simeq F \oplus F = F \otimes (\mathbf{1} \oplus \mathbf{1})$. The endomorphism ring of E is the algebra of matrices $\mathbf{M}_2(k_0)$, and each decomposition of E into a direct sum is derived from the decomposition given by an element of $\mathbf{GL}_2(k_0) = \mathrm{Aut}(E)$. We have $N(E) = 0$.

b) F and F' are *not isomorphic*. We then have $H^0(C, F' \otimes F^{-1}) = 0$, because $F' \otimes F^{-1}$ is a bundle of rank 1 which is non-trivial and of degree $\leqslant 0$. This implies that the endomorphisms of E can be represented by triangular matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $a, d \in k_0$ and $b \in H^0(C, F \otimes F'^{-1})$. In particular, F is stable by every endomorphism of E. If $E = F_1 \oplus F_2$ is another decomposition of E, one of the F_i is equal to F and the other derived from F' by a matrix $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with $b \in H^0(C, F \otimes F'^{-1})$ as above. One checks easily that F is a subbundle of E of maximum degree, i.e. that $N(E) = N(E; F)$. When $\deg(F) > \deg(F')$ F itself is the unique subbundle of E of degree $> \frac{1}{2}\deg(E)$, cf. prop. 6 (ii).

We now show that every bundle E whose invariant $N(E)$ is sufficiently large is decomposable:

Proposition 7. *Let F be a subbundle of rank 1 of a bundle E of rank 2. If $N(E; F) > 2g - 2$ then F is a direct factor of E.*

We put $F' = E/F$, so that E is an extension of F' by F. Such extensions are classified by the elements of the cohomology group $H^1(C, F \otimes F'^{-1})$. Since

$$\deg(F \otimes F'^{-1}) = N(E; F) > 2g - 2$$

this group is zero by the "duality theorem". The extension E is therefore trivial.

Corollary. *Every bundle E of rank 2 such that $N(E) > 2g - 2$ is decomposable.*

Example. If C is the projective line we have $g = 0$, $e = 1$, and $N(E) \geqslant 0$ by prop. 6; the corollary above then shows that E is decomposable; this recovers the theorem of Grothendieck [23] cited in section 1.6.

The situation is different for a curve of genus 0 without rational points, cf. exerc. 1.

Exercises

1) Suppose that $g = 0$, in which case C is the "Severi-Brauer variety" associated with a quaternion algebra D over k_0. One has $e = 1$ if D is a matrix algebra, i.e. if C is isomorphic to the projective line $\mathbf{P}_1$; if D is a field one has $e = 2$. We let T denote the tangent bundle of C, and E a non-trivial extension of T by $\mathbf{1}$; we have an exact sequence

$$0 \to \mathbf{1} \to E \to T \to 0.$$

a) Show that $\mathrm{End}(E) = D$.

b) If $e = 1$, E is isomorphic to $F \oplus F$, where F is a bundle of rank 1 such that $\deg(F) = 1$. One has $N(E) = 0$.

c) If $e = 2$, E is indecomposable and $\mathbf{1}$ is a subbundle of rank 1 of maximal degree in E. One has $N(E) = N(E; F) = -2$.

Each indecomposable bundle of rank 2 is isomorphic to a tensor product of E with a bundle of rank 1.

2) Let k'_0 be an extension of k_0, and C' the curve derived from C by extension of the scalars from k_0 to k'_0; if $E, F, \ldots$ are bundles over C we denote the corresponding bundles over C' by $E', F', \ldots$.

a) If E is of rank 2 one has $N(E') \geqslant N(E)$. Use exerc. 1 to construct an example where $N(E) = -2$ and $N(E') = 0$.

b) Let H be a subbundle of E of rank 1 such that $N(E', H) > 0$. Show that there is a subbundle F of E such that $H = F'$. (When k'_0 is a Galois extension of k_0, this follows from prop. 6 (ii), because the latter shows that H is stable under $\mathrm{Gal}(k'_0/k_0)$. As Raynaud has remarked, the same method applies in the general case, provided that one extends prop. 6 (ii) to the case where C is defined over a local Artin ring, and uses Grothendieck's descent theory.)

Deduce that $N(E') > 0$ implies $N(E) = N(E')$.

3) The notation being that of exerc. 2, suppose that k'_0 is a separable quadratic extension of k_0; let $\lambda \mapsto \bar{\lambda}$ denote the non-trivial automorphism of k'_0, and its extension to C', $\mathrm{Pic}(C')$,

etc. Let H be a bundle of rank 1 over C whose class h in $\mathrm{Pic}(C')$ does not belong to the image of $\mathrm{Pic}(C)$. Let E be the bundle of rank 2 over C obtained from H by restriction of scalars from k'_0 to k_0 (as a sheaf, E is the direct image of H under the canonical morphism $C' \to C$).

a) Show that E is indecomposable, and that E' decomposes into $H \oplus \bar{H}$. Deduce that $N(E) < 0$ and $N(E') = 0$.

b) The field k'_0 embeds naturally in $\mathrm{End}(E)$. Show that

$$\mathrm{End}(E) = k'_0$$

if and only if $h \neq \bar{h}$ in $\mathrm{Pic}(C')$. When $h = \bar{h}$, the ring $\mathrm{End}(E)$ is a field of quaternions with centre k_0; recover the results of exerc. 1 from this point of view.

4) Suppose k_0 is finite.

a) Let F and F' be two bundles of rank 1 over C. Show that the extensions of F' by F are finite in number, up to isomorphism.

b) Let $c \in \mathrm{Pic}(C)$. Show that the classes of indecomposable bundles E of rank 2 such that $\det(E)$ is of class c are finite in number (use prop. 6 and 7 to show that $N(E)$ does not take more than a finite number of values, and so reduce to a)).

c) Deduce that the subset of $\mathrm{vert}(\Gamma\backslash X)$ which corresponds to indecomposable bundles is finite.

5) (following Harder-Narasimhan [25]). The notations and hypotheses are those of the preceding exercise. Put $q_0 = \mathrm{Card}(k_0)$ and let $\zeta_C(s)$ denote the *zeta-function* of the curve C, which we write in the form

$$\zeta_C(s) = Z_C(T) \quad \text{where} \quad T = q_0^{-s}.$$

It is known that Z_C is a rational function of T; one has

$$Z_C(T) = P_C(T)/(1 - T)(1 - q_0 T)$$

where $P_C(T)$ is a unitary polynomial of degree $2g$.

Let $c \in \mathrm{Pic}(C)$. Consider the series

$$M_c = \sum_{\det(E) = c} 1/\mathrm{Card}(\mathrm{Aut}(E)),$$

where the summation is over the classes of bundles E of rank 2 such that $\det(E)$ is of class c. It follows from exerc. 4 that M_c is finite.

The Tamagawa number of $\mathbf{SL}_2$ (and of its "forms") is known to be equal to 1. Use this fact to prove:

$$M_c = \zeta_C(-1)/(q_0 - 1) = P_C(q_0)/(q_0 - 1)^2(q_0^2 - 1).$$

In particular, M_c does not depend on c.

Check the case $g = 0$ directly, cf. no. 1.6, exerc. 6.

2.3 Structure of $\Gamma\backslash X$

Bundles of rank 1 over C^{aff}

Every bundle of rank 1 over C^{aff} extends to a bundle of rank 1 over C, defined up to a tensor product by a power of I_P. It follows that the group

$$\mathrm{Pic}(A) = \mathrm{Pic}(C^{\mathrm{aff}})$$

is the quotient of $\mathrm{Pic}(C)$ by the cyclic subgroup generated by the class of I_P. Since $\deg(I_P) = -d$ we get an exact sequence

$$0 \to J(k_0) \to \mathrm{Pic}(A) \to e\mathbf{Z}/d\mathbf{Z} \to 0 \quad \text{(cf. no. 2.2)}.$$

Moreover, for each $c \in \mathrm{Pic}(A)$ there is a unique element $\bar{c}$ of $\mathrm{Pic}(C)$ which extends c, and which is such that

$$0 \leqslant \deg(\bar{c}) < d.$$

We let F_c denote a bundle of rank 1 over C of class $\bar{c}$; such a bundle is determined up to isomorphism; we put $f_c = \deg(F_c) = \deg(\bar{c})$.

The cusp associated with an element of $\mathrm{Pic}(A)$

Let $c \in \mathrm{Pic}(A)$. We are going to construct a subgraph Δ_c of $\Gamma\backslash X$, which we shall call the *cusp* associated with c.

Let F_c be the bundle of rank 1 defined above, and let $n \in \mathbf{Z}$. We put

$$F'_{c,n} = I_P^{\otimes n} \otimes F_c^{-1} \quad \text{and} \quad E_{c,n} = F_c \oplus F'_{c,n},$$

so that $E_{c,n}$ is a decomposable bundle of rank 2 such that $\det(E_{c,n}) = I_P^{\otimes n}$. By lemma 5, its restriction to C^{aff} is trivial; it therefore defines a *vertex* $x_{c,n}$ of $\Gamma\backslash X$, cf. prop. 4.

Proposition 8. *The vertices $x_{c,n}$ ($c \in \mathrm{Pic}(A)$, $n \geqslant 1$) are pairwise distinct.*

We first remark that if $n \geqslant 1$ then $\deg(F_c) > \deg(F'_{c,n})$ which shows that F_c is the unique subbundle of rank 1 of maximal degree in $E_{c,n}$, cf. prop. 6. Now suppose that we have

$$x_{c,n} = x_{d,m} \quad (c, d \in \mathrm{Pic}(A), \quad n \geqslant 1, \quad m \geqslant 1).$$

By prop. 4, this means that there is an integer q such that $E_{c,n}$ is isomorphic to $I_P^{\otimes q} \otimes E_{d,m}$. Consequently we have the isomorphisms

$$F_c \simeq I_P^{\otimes q} \otimes F_d \quad \text{and} \quad F'_{c,n} \simeq I_P^{\otimes q} \otimes F'_{d,m}.$$

In view of the definition of F_c, we deduce first that $c = d$, $q = 0$, then that $m = n$. Whence the proposition.

We now concern ourselves with the *edges* with origin $x_{c,n}$ in $\Gamma\backslash X$. The canonical inclusion $I_P^{\otimes n} \subset I_P^{\otimes (n+1)}$ defines an inclusion of $E_{c,n}$ in $E_{c,n+1}$, and hence an edge $y_{c,n}$ with origin $x_{c,n}$ and terminus $x_{c,n+1}$. Similarly, we have an edge $\bar{y}_{c,n-1}$ with origin $x_{c,n}$ and terminus $x_{c,n-1}$.

Proposition 9. *If $(n-1)d > 2g-2-2f_c$ the edges $y_{c,n}$ and $\bar{y}_{c,n-1}$ are the only edges of $\Gamma\backslash X$ with origin $x_{c,n}$.*

We first remark that, if E is a vector bundle of rank 2 corresponding to a vertex x of $\Gamma\backslash X$, then the edges of $\Gamma\backslash X$ with origin x correspond to the *orbits* of the group $\mathrm{Aut}(E)$ acting on *the space of k-lines* of the *fibre* $E(P)$ of E at P, which is a vector space of dimension 2 over k. When we take $x = x_{c,n}$, $E = E_{c,n}$ the fibre of E at P is the direct sum of two lines:

$$E_{c,n}(P) = D \oplus D',$$

where D is the fibre $F_c(P)$ of F_c and D' the fibre $F'_{c,n}(P)$ of $F'_{c,n}$. The line D corresponds to the edge $y_{c,n}$, the line D' to the edge $\bar{y}_{c,n-1}$. If we put $\Gamma_{c,n} = \mathrm{Aut}(E_{c,n})$, prop. 9 amounts to saying that each line of $D \oplus D'$ is $\Gamma_{c,n}$-conjugate to D or D'. In fact:

Lemma 6. *If $(n-1)d > 2g-2-2f_c$ each line of $D \oplus D'$ distinct from D is $\Gamma_{c,n}$-conjugate to D'.*

We first remark that the group $\Gamma_{c,n}$ contains the automorphisms of the form $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ where s is a section of the bundle of rank 1

$$H = Hom(F'_{c,n}, F_c) \simeq F_c^{\otimes 2} \otimes I_P^{\otimes(-n)}$$

which is of degree $nd + 2f_c$. Such an automorphism acts on the fibre $D \oplus D'$ by the unipotent matrix $\begin{pmatrix} 1 & s(P) \\ 0 & 1 \end{pmatrix}$, where $s(P) = H(P)$ denotes the value of the section s at P. Since the supplementaries of D are conjugate to each other via the matrices $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ with $\lambda \in H(P)$, we see that lemma 6 is a consequence of the following:

Lemma 7. *If H is a bundle of rank 1 such that*

$$\deg(H) > 2g-2+d,$$

the homomorphism

$$\text{“value at } P\text{”}: \quad H^0(C, H) \to H(P)$$

is surjective.

Indeed, we have an exact sequence

$$H^0(C, H) \to H(P) \to H^1(C, I_P \otimes H)$$

and the duality theorem shows that $H^1(C, I_P \otimes H) = 0$ because

$$\deg(I_P \otimes H) > 2g-2.$$

(Note that the hypothesis on $\deg(H)$ is indeed satisfied in the case of lemma 6, since $\deg(H) = nd + 2f_c > 2g - 2 + d$.)

Put now

$$m = \operatorname{Sup}(2g - 2 + d, 3d - 2).$$

Let n_c be the greatest integer such that $2f_c + dn_c \leqslant m$. We have

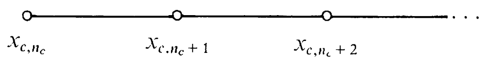
i) $n_c \geqslant 1$.

Indeed, we have $2f_c + dn_c > m - d \geqslant 2d - 2$. Since $f_c \leqslant d - 1$, this implies $dn_c > 0$, whence $n_c \geqslant 1$.

ii) $dn_c > 2g - 2 - 2f_c$.

Indeed, we have $2f_c + dn_c > m - d \geqslant 2g - 2$.

We now define the cusp Δ_c to be the subgraph of $\Gamma\backslash X$ with the $x_{c,n}$ $(n \geqslant n_0)$ as vertices and the $y_{c,n}; \bar{y}_{c,n}$ $(n \geqslant n_0)$ as edges. It is an *infinite path*



with the vertex x_{c,n_c}, which we denote simply by x_c, as origin. The inequality i), together with prop. 8, shows that the Δ_c are *pairwise disjoint*. The inequality ii), together with prop. 9, shows that Δ_c *does not meet the rest of* $\Gamma\backslash X$ *except at its origin* x_c.

Structure of $\Gamma\backslash X$

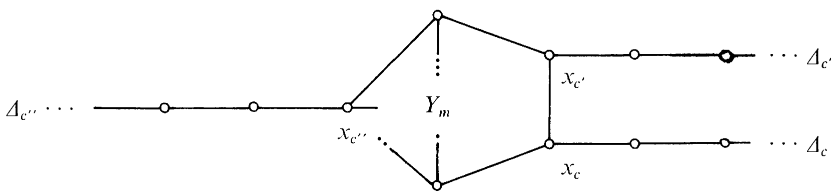
Let Y_m be the subgraph of $\Gamma\backslash X$ whose vertices correspond to bundles E such that $N(E) \leqslant m$, and whose edges are the edges of $\Gamma\backslash X$ whose extremities both belong to $\operatorname{vert}(Y_m)$. Since $N(E_{c,n_c}) = 2f_c + dn_c$ is $\leqslant m$, the x_c are vertices of Y_m.

Theorem 9. (a) *The graph* $\Gamma\backslash X$ *is the union of the subgraphs* Y_m *and* Δ_c, *with* $c \in \operatorname{Pic}(A)$.

(b) *One has* $\operatorname{vert}(Y_m) \cap \operatorname{vert}(\Delta_c) = \{x_c\}$ *and* $\operatorname{edge}(Y_m) \cap \operatorname{edge}(\Delta_c) = \emptyset$.

(c) *There is a constant* $l = l(g, d, e)$ *such that each vertex of* $\Gamma\backslash X$ *is at a distance* $\leqslant l$ *from one of the cusps* Δ_c.

One can express (a) and (b) by saying that the graph $\Gamma\backslash X$ is obtained by *glueing* the cusps Δ_c to the graph Y_m at their extrēmities x_c:



This construction is analogous to that of the fundamental domain of a fuchsian group.

Proof of (a)

Let x be a vertex of $\Gamma\backslash X$, and let E be a bundle of rank 2 corresponding to x. If $N(E) \leqslant m$ we have $x \in \operatorname{vert}(Y_m)$ by definition of Y_m. If $N(E) \geqslant m$ there is a subbundle F of E of rank 1 such that $N(E;F) \geqslant m$; since $m > 2g-2$, prop. 7 shows that F is a direct factor in E; we have $E = F \oplus F'$. Let $c \in \operatorname{Pic}(A)$ be the class of the restriction of F to C^{aff}. After tensoring E by a power of I_P, we can assume that $F = F_c$. Since $\det(E)$ is a power of I_P (cf. lemma 5), it follows that F' is isomorphic to $I_P^{\otimes n} \otimes F_c^{-1}$, with $n \in \mathbf{Z}$, i.e. that $E \simeq E_{c,n}$, $x = x_{c,n}$. The hypothesis $N(E;F) \geqslant m$ means that $2f_c + nd \geqslant m$; in view of the definition of n_c, this implies $n \geqslant n_c$ and shows that x is a vertex of Δ_c. We thus have

$$\operatorname{vert}(\Gamma\backslash X) = \operatorname{vert}(Y_m) \cup \bigcup_c \operatorname{vert}(\Delta_c).$$

Now let y be an edge of $\Gamma\backslash X$. If the two extremities of y belong to Y_m, we have $y \in \operatorname{edge}(Y_m)$ by definition of Y_m. If one of the extremities of y does not belong to Y_m, it is a vertex $x_{c,n}$ of one of the Δ_c, and we have $x_{c,n} \neq x_c$, whence $n \geqslant n_c + 1$ and

$$(n-1)d \geqslant n_c d > 2g - 2 - 2f_c, \quad \text{cf. ii).}$$

In view of prop. 9 this implies $y \in \operatorname{edge}(\Delta_c)$. We thus have

$$\operatorname{edge}(\Gamma\backslash X) = \operatorname{edge}(Y_m) \cup \bigcup_c \operatorname{edge}(\Delta_c),$$

which completes the proof of (a).

Proof of (b)

Let $x_{c,n} \in \operatorname{vert}(Y_m) \cap \operatorname{vert}(\Delta_c)$. The fact that $x_{c,n}$ belongs to Y_m means that $N(E_{c,n}) \leqslant m$, i.e. that $2f_c + dn \leqslant m$; in view of the definition of n_c, and the fact that $n \geqslant n_c$, this implies $n = n_c$ and we see that $\operatorname{vert}(Y_m) \cap \operatorname{vert}(\Delta_c)$ indeed reduces to $\{x_c\}$.

The formula $\operatorname{edge}(Y_m) \cap \operatorname{edge}(\Delta_c) = \emptyset$ is obvious because no edge of Δ_c can have both its extremities equal to x_c.

Proof of (c)

We will show that one can take

$$l = \left[\frac{m + 2g + 2e - 2}{d}\right]$$

(recall that $[\alpha]$ denotes the greatest integer $\leqslant \alpha$).

Let $x \in \operatorname{vert}(\Gamma\backslash X)$ and let E be a bundle of rank 2 corresponding to x. Let $n = N(E)$ and let

$$0 \to F \to E \to F' \to 0$$

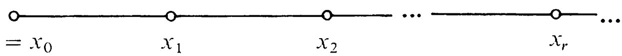
be an exact sequence, where F and F' are bundles of rank 1 such that

$$\deg(F) - \deg(F') = n.$$

Let E_1 be the subsheaf of E which is the inverse image under $E \to F'$ of the subsheaf $I_P \cdot F'$ of F'. The exact sequence

$$0 \to F \to E_1 \to I_P \cdot F' \to 0$$

shows that $N(E_1) \geqslant n + d$. On the other hand, it is clear that the vertex x_1 of $\Gamma\backslash X$ corresponding to E_1 is *adjacent* to the vertex x. Iterating this construction, we obtain a *path*



of $\Gamma\backslash X$ where x_r corresponds to a bundle E_r such that $N(E_r) \geqslant n + rd$. Let $r = l + 1$. We have $rd > m + 2g + 2e - 2$, whence

$$n + rd > m, \qquad \text{cf. prop. 6 (i).}$$

In view of (a), this implies that x_r is a vertex of one of the cusps Δ_c, and it is not the origin x_c of this cusp. Since x_{r-1} is adjacent to x_r, we also have $x_{r-1} \in \operatorname{vert}(\Delta_c)$, which shows that the distance from x to Δ_c is indeed $\leqslant r - 1 = l$.

Corollary 1. *Each vertex of Y_m is at a distance $\leqslant l$ from one of the x_c.*

This follows from (c), taking into account that each path of $\Gamma\backslash X$ joining a vertex of Y_m to a vertex of Δ_c goes through the origin x_c of Δ_c.

Corollary 2. *The inclusion $Y_m \to \Gamma\backslash X$ is a homotopy equivalence.*

This follows from (a) and (b) by "retraction of the cusps".

Corollary 3. *The graph Y_m is connected.*

This follows from cor. 2 and the fact that $\Gamma\backslash X$ is connected, because it is a quotient of the tree X.

Corollary 4. *If k_0 is finite, so is Y_m.*

Put $q = \operatorname{Card}(k)$. There are at most $q + 1$ edges of $\Gamma\backslash X$ with a given origin, and hence at most

$$1 + (q + 1) + q(q + 1) + \cdots + q^{l-1}(q + 1) = q^l + 2q^{l-1} + \cdots + 2q + 2$$

vertices of $\Gamma\backslash X$ at a distance $\leqslant l$ from a given x_c. Since the x_c are finite in number, cor. 1 shows that $\operatorname{vert}(Y_m)$ is finite, and it is obviously the same for $\operatorname{edge}(Y_m)$.

Remarks

1) The cor. 4 can also be deduced from exerc. 4 of no. 2.2.

2) Let us mention, without proof, the following result due to H. Bass:

d) *There is a constant $l' = l'(g, d, e)$ such that any two vertices of Y_m are at a distance $\leqslant l'$ from each other.*

In other words, the *diameter* of Y_m is finite, and bounded by l'.

Cusps and rational ends

Let D be a line of the K-vector space $V = K^2$. We associate with D:

a) the *end* b_D of the tree X defined in section 1.1;

b) the *class* $c = c(D)$ in Pic(A) of the A-module $D_A = A^2 \cap D$; this makes sense because D_A is projective of rank 1.

Proposition 10. *Up to a finite graph, the image of the end b_D in $\Gamma\backslash X$ is equal to the cusp Δ_c associated with c.*

Let L_0 be the standard lattice $\mathcal{O}^2$, and let L_n ($n \geqslant 0$) be the sublattice of L_0 defined by $L_n = L_0 \cap D + \pi^n L_0$, where π is a uniformizer of $\mathcal{O}$.

The vertices x_n of X corresponding to the L_n form an infinite chain without backtracking which represents the end b_D, cf. no. 1.1. Let F denote the coherent subsheaf of the constant sheaf D defined by:

$$F_Q = \mathcal{O}_Q^2 \cap D = (D_A)_Q \quad \text{if } Q \text{ is a point of } C^{\text{aff}}$$

$$F_P = L_0 \cap D.$$

It is a locally free sheaf of rank 1, whose restriction to C^{aff} is of class c. The quotient E_L/F is a locally free subsheaf of rank 1 of the constant sheaf V/D. For each n we have an exact sequence:

$$0 \to F \to E_{L_n} \to I_P^{\otimes n} \cdot F' \to 0.$$

When n is sufficiently large, prop. 7 shows that this exact sequence splits; we then have

$$E_{L_n} \simeq F \oplus I_P^{\otimes n} \otimes F'.$$

In view of the definition of F_c given at the beginning, there are integers r, s such that

$$F \simeq I_P^{\otimes r} \otimes F_c \quad \text{and} \quad F' \simeq I_P^{\otimes s} \otimes F_c^{-1}.$$

We deduce that E_{L_n} is I_P-equivalent to the bundle denoted above by $E_{c,n+s-r}$; i.e., we have $x_n = x_{c,n+s-r}$, whence the proposition.

Remarks

1) It follows from elementary properties of modules over Dedekind domains (Bourbaki, AC. VII, §4, no. 10) that the map $D \mapsto c(D)$ has the following properties:

i) $c(D) = c(D') \Leftrightarrow D$ and D' are Γ-conjugate;
ii) for each $c \in \mathrm{Pic}(A)$ there is a line D such that $c = c(D)$.
Thus we recover the bijection:

$$\text{cusps of } \Gamma\backslash X \leftrightarrow \text{elements of } \mathrm{Pic}(A).$$

2) The ends of X correspond to points of the projective line $\mathbf{P}_1(\hat{K})$ over the *completion* $\hat{K}$ of K, cf. no. 1.1; note that it is only the *K-rational points* which yield the cusps of $\Gamma\backslash X$. The situation is analogous to that of the group $\mathbf{SL}_2(\mathbf{Z})$, where the cusps are given by the *$\mathbf{Q}$-rational points* of the projective line $\mathbf{P}_1(\mathbf{R})$.

Exercises

1) If x is a vertex of X let $n(x)$ denote the invariant $N(E)$ of a bundle E corresponding to x.
a) Show that, if x and x' are adjacent, then $|n(x') - n(x)| \leqslant d$.
b) Suppose that $n(x) > 0$. Show, using prop. 6 (ii), that there is a unique $x' \in \mathrm{vert}(X)$ which is adjacent to x and such that $n(x') = n(x) + d$. By iterating this construction one obtains a sequence of vertices $x, x', x'', \ldots$ which form a path without backtracking; the end defined by this path is the unique end of X invariant under the stabilizer of x in Γ.

2) Suppose that k_0 is finite of characteristic p. Put

$$q_0 = \mathrm{Card}(k_0), \qquad q = q_0^d = \mathrm{Card}(k).$$

Let Γ' be a subgroup of finite index in Γ. If x (resp. y) is a vertex (resp. geometric edge) of $\Gamma'\backslash X$ let Γ'_x (resp. Γ'_y) denote the stabilizer in Γ' of a representative of x (resp. y).
a) Show that the series

$$\sum_{x \in \mathrm{vert}(\Gamma'\backslash X)} 1/\mathrm{Card}(\Gamma'_x) \quad \text{and} \quad \sum_{y \in \mathrm{edge}(\Gamma'\backslash X)} 1/\mathrm{Card}(\Gamma'_y)$$

are convergent. (Reduce to the case where $\Gamma' = \Gamma$, and use cor. 4 to th. 9.)
b) Show that

$$\sum_{x\,\mathrm{even}} 1/\mathrm{Card}(\Gamma'_x) = \sum_{x\,\mathrm{odd}} 1/\mathrm{Card}(\Gamma'_x) = \frac{q+1}{2} \sum_{y} 1/\mathrm{Card}(\Gamma'_y).$$

Let $M_{\Gamma'}$ denote the common value of these series. Show that $M_{\Gamma'} = (\Gamma : \Gamma') \cdot M_c$ where

$$M_c = \zeta_C(-1)/(q_0 - 1) = P_C(q_0)/(q_0 - 1)^2(q_0^2 - 1)$$

is the number defined in exerc. 5 of no. 2.2.
c) Put

$$\chi(\Gamma') = \sum_{x \in \mathrm{vert}(\Gamma'\backslash X)} 1/\mathrm{Card}(\Gamma'_x) - \sum_{y \in \mathrm{geom\,edge}(\Gamma'\backslash X)} 1/\mathrm{Card}(\Gamma'_y).$$

Then $\chi(\Gamma') = -(q-1)M_{\Gamma'}$ and $\chi(\Gamma'') = (\Gamma' : \Gamma'')\chi(\Gamma')$ if $\Gamma'' \subset \Gamma'$.

d) Show that

$$\chi(\Gamma) = \sum_{\substack{x \in \mathrm{vert}(Y_m) \\ x \neq x_c}} 1/\mathrm{Card}(\Gamma_x) - \sum_{y \in \mathrm{geom\,edge}(Y_m)} 1/\mathrm{Card}(\Gamma_y)$$

where the first summation is over the vertices of Y_m distinct from the x_c, $c \in \mathrm{Pic}(A)$. (Note that, if x is a vertex of one of the cusps Δ_c, and y is the geometric edge joining x to the succeeding vertex of Δ_c, we have $\Gamma_x = \Gamma_y$, so that the contributions of x and y in $\chi(\Gamma)$ cancel.)

e) If Γ' is contained in $\mathbf{SL}_2(K)$, $\chi(\Gamma')$ is equal to $-\mu(\mathbf{SL}_2(\hat{K})/\Gamma')$ where μ is the normalized Haar measure on $\mathbf{SL}_2(\hat{K})$, cf. no. 1.5. In particular, for $\Gamma' = \mathbf{SL}_2(A)$ we have

$$\chi(\mathbf{SL}_2(A)) = -\mu(\mathbf{SL}_2(\hat{K})/\mathbf{SL}_2(A)) = -(q-1)\zeta_C(-1).$$

(For a generalization of this formula to all simply connected split simple groups, see [34], p. 158 footnote.)

f) Suppose that Γ' is *without p'-torsion*, in other words that all elements of finite order in Γ' have order a power of p (i.e. are unipotent). Show that $(\Gamma:\Gamma')$ is divisible by $(q_0-1)(q_0^2-1)$. (Make the finite group $\mathbf{SL}_2(k_0)$ act on the homogeneous space Γ/Γ' and observe that all the orbits have order a multiple of $(q_0-1)(q_0^2-1)$.) Deduce that $\chi(\Gamma')$ is an *integer*.

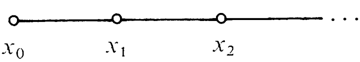
2.4 Examples

2.4.1. *The affine line*: $A = k[t]$, $g = 0$, $e = d = 1$

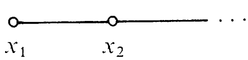
The curve C is the projective line $\mathbf{P}_1$ and the point P is of degree 1; we have $k = k_0$; this is the case already considered in no. 1.6. All the bundles E are decomposable; they can be represented by

$$E_n = \mathbf{1} \oplus I_P^{\otimes n} \qquad (n = 0, 1, \ldots).$$

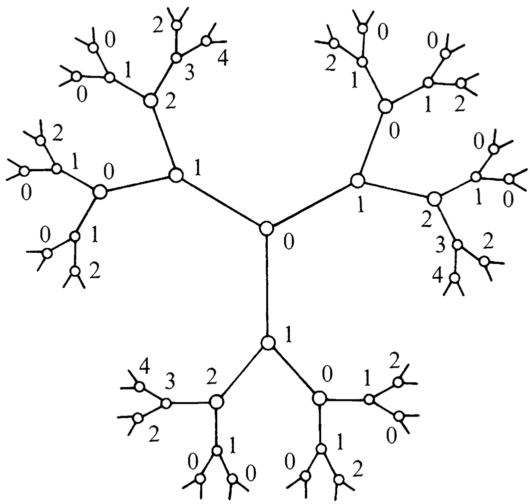
The graph $\Gamma\backslash X$ is the infinite path



whose vertices x_n correspond to E_n. The integer m of no. 2.3 is equal to 1, and the graph Y_m is the segment x_0 —— x_1. There is just a single cusp Δ_c, because $A = k[t]$ is a principal ideal ring; it is the infinite path:



The diagram below represents X for $k = \mathbf{F}_2$; the number n next to a vertex means that the vertex is of *type* n, i.e. its image in $\Gamma\backslash X$ is x_n.



Note that, if a vertex x is of type 0, all its neighbours are of type 1. On the other hand, if x is of type $n \geqslant 1$, one of its neighbours is of type $n+1$, and the others are of type $n-1$; there is only one infinite path with origin x on which the "type" increases; this path corresponds to a rational end, namely the unique end of X fixed by the stabilizer Γ_x of x in Γ. (For the behaviour of paths corresponding to an irrational end, see exerc. 1.)

Remark

If one replaces $\Gamma = \mathbf{GL}_2(k[t])$ by its subgroup $\Gamma_1 = \mathbf{SL}_2(k[t])$, one sees easily that the quotient graph $\Gamma\backslash X$ does not change (cf. no. 2.1, exerc. 1, or no. 1.6, exerc. 2). This result generalizes as follows:

Let S be a simple, simply connected algebraic k-group, X_S the Bruhat-Tits building of S over the local field $\hat{K} = k((1/t))$ (cf. [37], [38], [39]), and put $\Gamma_S = S(k[t])$. *Then the natural action of Γ_S on X_S admits a fundamental domain which is an apartment of X_S* in the sense of Bruhat-Tits [28], 7.4.12 (cf. [47]). This allows Γ_S to be written as an amalgam of certain subgroups, as for $S = \mathbf{SL}_2$, cf. no. 1.6.

2.4.2. *The case $g = 0$, $e = 1$, $d \geqslant 2$*

The curve C is the projective line $\mathbf{P}_1$ and the point P is of degree $d \geqslant 2$. The field k is an extension of k_0 of degree d; it can be generated by one element. A typical case is that where $k_0 = \mathbf{R}$, $d = 2$, $k = \mathbf{C}$, $A = \mathbf{R}[u, v]/(u^2 + v^2 - 1)$.

We have $\operatorname{Pic}(C) = \mathbf{Z}$; $\operatorname{Pic}(A) = \mathbf{Z}/d\mathbf{Z}$. If F is a bundle of rank 1 and degree 1 the representatives F_c of the elements c of $\operatorname{Pic}(A)$ (cf. no. 2.3) are the $F^{\otimes c}$, $0 \leqslant c < d$. Every bundle of rank 2 is isomorphic to one of the bundles

$$E_{a,b} = F^{\otimes a} \oplus F^{\otimes b} \quad \text{with} \quad a, b \in \mathbf{Z}.$$

The restriction of such a bundle to C^{aff} is trivial if and only if

$$a + b \equiv 0 \pmod d;$$

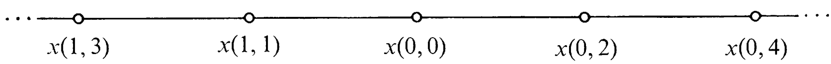
in that case we let $x(a, b)$ denote the corresponding vertex of $\Gamma\backslash X$. Every vertex of $\Gamma\backslash X$ is obtained in this way, and we have

$$x(a, b) = x(a', b') \Leftrightarrow \begin{cases} a - a' = b - b' \equiv 0 \pmod d \\ \text{or} \\ a - b' = b - a' \equiv 0 \pmod d. \end{cases}$$

The vertex $x(a, b)$ is connected to the vertex $x(a + d, b)$ by a canonical edge $y(a, b)$, and it is connected to $x(a - d, b)$ by the edge $\bar{y}(a - d, b)$; if $|b - a| \geqslant d - 1$ one checks by means of prop. 9 that $y(a, b)$ and $\bar{y}(a - d, b)$ are the only edges of $\Gamma\backslash X$ with origin $x(a, b)$. To determine the graph $\Gamma\backslash X$ completely, it remains to make the list of the other edges, cf. exerc. 2. We just give the result for $d = 2, 3, 4$:

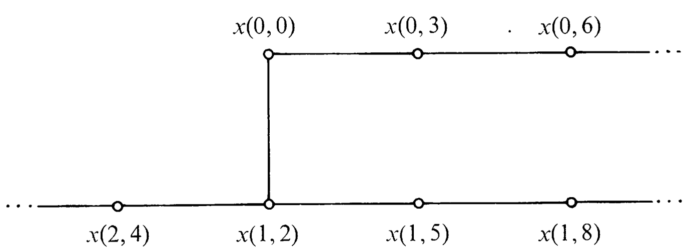
a) *the case* $d = 2$

The vertices are the $x(0, 2n)$ and $x(1, 2n - 1)$ with $n = 0, 1, \ldots$. Apart from the canonical edges $y(a, b)$ and their inverses, there are only two more: the pair connecting the vertices $x(0, 0)$ and $x(1, 1)$. This can be seen, for example, by determining the edges issuing from $x(0, 0)$, cf. no. 2.1, exerc. 6. The graph $\Gamma\backslash X$ is then:



b) *the case* $d = 3$

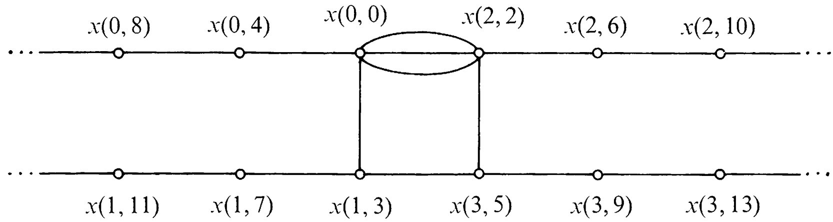
The vertices are the $x(0, 3n)$, $x(1, 3n + 2)$, $x(2, 3n + 4)$, with $n = 0, 1, \ldots$. Apart from the canonical edges, one again finds only one more pair, which connects the vertices $x(0, 0)$ and $x(1, 2)$. The graph $\Gamma\backslash X$ is:



c) *the case* $d = 4$

The vertices are the $x(0, 4n)$, $x(1, 4n + 3)$, $x(2, 4n + 2)$, $x(3, 4n + 5)$, with $n = 0, 1, \ldots$. From the vertex $x(1, 3)$ start, not only the two canonical edges (to $x(1, 7)$ and $x(3, 5)$, but also another edge to $x(0, 0)$; similarly $x(3, 5)$ has an edge connecting it to $x(2, 2)$. The vertex $x(0, 0)$ is connected to each of the vertices $x(0, 4)$ and $x(1, 3)$ by single edges, but it is connected to $x(2, 2)$ by q_0 edges, where

$q_0 = \mathrm{Card}(k_0)$; similarly $x(2,2)$ is connected to $x(2,6)$ and $x(3,5)$ by single edges, but to $x(0,0)$ by q_0 edges. All the other edges are canonical. The graph $\Gamma\backslash X$ is:



In contrast to the preceding, *this graph is not a tree.*

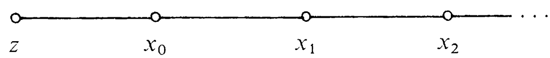
2.4.3. *The case* $g = 0$, $e = 2$, $d = 2$

The curve C is a curve of genus 0 without rational points. The point P is of degree 2. We can realize C as a plane conic, P being the intersection of this conic with a line defined over k_0. A typical case is that where $k_0 = \mathbf{R}$, $k = \mathbf{C}$, $A = \mathbf{R}[u, v]/(u^2 + v^2 + 1)$.

We have $\mathrm{Pic}(A) = 2\mathbf{Z}/2\mathbf{Z} = 0$: there is only one cusp. In $\Gamma\backslash X$, we have vertices x_n $(n = 0, 1, \ldots)$ which correspond to the decomposable bundles

$$E_n = \mathbf{1} \oplus I_P^{\otimes n}.$$

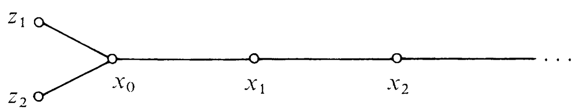
There is another vertex z, which corresponds to an indecomposable bundle E whose endomorphism ring is the field of quaternions D associated with C (cf. no. 2.2, exerc. 1). The graph $\Gamma\backslash X$ is an infinite path:



Remark

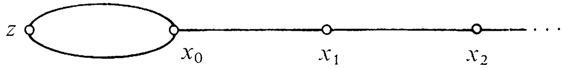
If we replace Γ by $\Gamma_1 = \mathbf{SL}_2(A)$, the graph $\Gamma\backslash X$ is replaced by a graph $\Gamma_1\backslash X$ whose left part (that relative to z —— x_0) is somewhat different: the vertex z is replaced by a family of vertices corresponding to the elements of $k_0^*/\mathrm{Nrd}\, D^*$, and the edge joining z to x_0 is replaced by a family of edges corresponding to the elements of $k_0^*/N_{k/k_0}k^*$, cf. no. 2.1, exerc. 1. For example:

a) If $k_0 = \mathbf{R}, k = \mathbf{C}, D = \mathbf{H}$ (Hamilton's quaternions), the vertex z and the edge z —— x_0 are doubled; the graph $\Gamma_1\backslash X$ is



Note the three edges issuing from x_0, which correspond to the three orbits of $\mathbf{SL}_2(\mathbf{R})$ on the complex projective line $\mathbf{P}_1(\mathbf{C})$: the upper half-plane, the real axis, and the lower half-plane.

b) If k_0 is a p-adic field $\mathbf{Q}_p$, only the edge z ——— x_0 is doubled; the graph $\Gamma_1\backslash X$ is:



2.4.4. *The ring* $A = \mathbf{F}_2[u, v]/(u^2 + u + v^3 + v + 1)$

The curve C defined in characteristic 2 by the equation

$$(*) \qquad u^2 + u = v^3 + v + 1$$

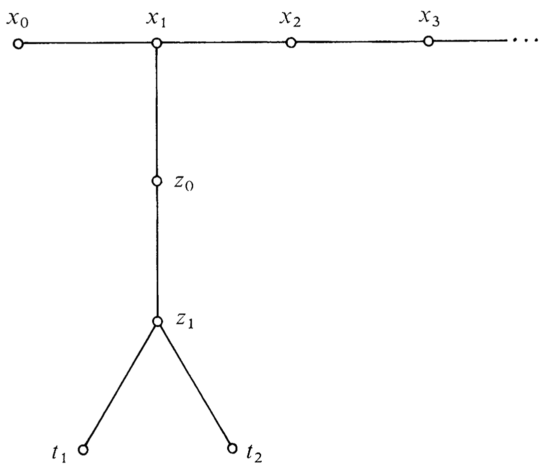
is an elliptic curve: we have $g = 1$. Its point at infinity P is of degree 1: we have $e = d = 1$, $k = k_0 = \mathbf{F}_2$. Since the Jacobian variety J of C is equal to C, we have $J(k_0) = C(k_0) = 0$ (indeed, the equation $(*)$ has no solution in $\mathbf{F}_2$). It follows that $\mathrm{Pic}(A)$ reduces to 0: the ring A is principal, there is only one cusp. Up to I_P-equivalence, one finds that the bundles of rank 2 are the following:

a) the decomposable bundles $E_n = \mathbf{1} \oplus I_P^{\otimes n}$ $(n = 0, 1, \ldots)$; let x_n be the corresponding vertices of $\Gamma\backslash X$;

b) a non-trivial extension of $\mathbf{1}$ by $\mathbf{1}$ (resp. of I_P by $\mathbf{1}$); let z_0 (resp. z_1) be the corresponding vertex;

c) two bundles obtained by restriction of scalars (cf. no. 2.2, exerc. 3) from bundles H and $H^{\otimes 2}$ of rank 1 over the curve C' derived from C by extension of the base field to $\mathbf{F}_4$ (one has $\mathrm{Pic}(C') = \mathbf{Z}/5\mathbf{Z} \times \mathbf{Z}$, and one takes H to be a bundle whose class is of order 5); let t_1 and t_2 denote the corresponding vertices.

The graph $\Gamma\backslash X$ has the following structure:



Exercises

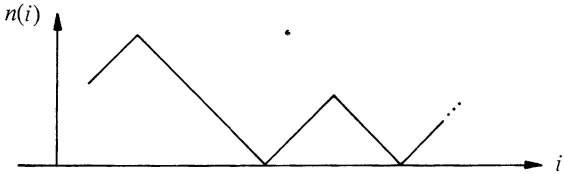
1) The hypotheses are those of 2.4.1. Let

$$\Delta = Q_1 \text{———} Q_2 \text{———} Q_3 \text{———} \cdots$$

be an infinite path without backtracking in X. We let $n(i)$ denote the type of Q_i; it is an integer $\geqslant 0$; knowledge of the $n(i)$ determines the image of Δ in $\Gamma\backslash X$.

a) One has $n(i+1) = n(i) \pm 1$. Show that if $n(i+1) = n(i) - 1$, then $n(i+j) = n(i) - j$ for each j such that $0 \leqslant j \leqslant n(i)$.

The graph of the function $n(i)$, extended to $\mathbf{R}_+$ by linearity, has the following shape



b) Deduce that only two cases are possible:

b$_1$) There is an $N \in \mathbf{Z}$ such that $n(i) = i + N$ for i sufficiently large.

b$_2$) There are an infinity of values of i such that $n(i) = 0$.

c) Let $\delta \in \mathbf{P}_1(\hat{K})$ be the *end* defined by Δ. Show that b$_1$) is equivalent to $\delta \in \mathbf{P}_1(K)$; i.e., that b$_2$) occurs if and only if δ is irrational.

2) The hypotheses are those of 2.4.2. Let $f_d(t)$ denote the unitary irreducible polynomial of degree d which defines the point P; one has

$$k = k_0[t]/(f_d).$$

a) Show that the ring A consists of the rational functions $g(t)/f_d(t)^n$, with n an integer $\geqslant 0$, $g \in k_0[t]$ and $\deg(g) \leqslant nd$.

b) Let ∞ denote the point at infinity of $\mathbf{P}_1$, and I_∞ the corresponding sheaf of ideals. Let F be the sheaf I_∞^{-1} (sheaf of germs of functions having at most a single pole at infinity), so that the sections of $F^{\otimes n}$ are polynomials in t of degree $\leqslant n$. If a, b, a', b' are integers, the vector space $H\begin{pmatrix} a & a' \\ b & b' \end{pmatrix}$ of homomorphisms of the sheaf $E_{a,b}$ into the sheaf $E_{a',b'}$ may be identified with the group of matrices $\phi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, with $\alpha, \beta, \gamma, \delta \in k_0[t]$, $\deg(\alpha) \leqslant a' - a$, $\deg(\beta) \leqslant a' - b$, $\deg(\gamma) \leqslant b' - a$, $\deg(\delta) \leqslant b' - b$.

In particular, the group $G\begin{pmatrix} a \\ b \end{pmatrix}$ of invertible elements of $H\begin{pmatrix} a & a \\ b & b \end{pmatrix}$ is none other than $\mathrm{Aut}(E_{a,b})$.

c) Suppose that a, b, a', b' are such that

$$a' + b' = a + b + d \equiv 0 \pmod{d}$$

and let $\Phi\begin{pmatrix} a & a' \\ b & b' \end{pmatrix}$ denote the subset of $H\begin{pmatrix} a & a' \\ b & b' \end{pmatrix}$ consisting of the matrices ϕ such that $\det(\phi) = \lambda f_d$ with $\lambda \in k_0^*$; this set is stable under right multiplication by an element of $G\begin{pmatrix} a \\ b \end{pmatrix}$, and under left multiplication by an element of $G\begin{pmatrix} a' \\ b' \end{pmatrix}$. Show that the elements of the quotient

$$G\begin{pmatrix} a' \\ b' \end{pmatrix} \Big\backslash \Phi\begin{pmatrix} a & a' \\ b & b' \end{pmatrix} \Big/ G\begin{pmatrix} a \\ b \end{pmatrix}$$

correspond bijectively to the *edges* of $\Gamma\backslash X$ with origin $x(a,b)$ and terminus $x(a',b')$. Deduce that $x(a,b)$ and $x(a',b')$ are adjacent in $\Gamma\backslash X$ if and only if either $\{a,b\}\cap\{a',b'\}\neq\emptyset$ (the case corresponding to a canonical edge) or $\mathrm{Inf}(a',b')\geqslant \mathrm{Sup}(a,b)$.

d) Use c) to show that $x(0,0)$ is adjacent to all the $x(a,b)$ such that $a+b=d$ and $a,b\geqslant 0$.

e) Suppose d is even. Show that $E\mapsto F^{\otimes d/2}\otimes E$ defines an automorphism of $\Gamma\backslash X$ of order 2.

3) Let k be a finite field, and C an elliptic curve over k whose group $C(k)$ of k-points reduces to a single element.

a) Let π be the Frobenius endomorphism of C, which we identify in the usual way with an integer of an imaginary quadratic field. One has $\pi+\bar{\pi}=\pi\cdot\bar{\pi}=q$, with $q=\mathrm{Card}(k)$. Deduce that $q\leqslant 4$.

b) Show that, up to isomorphism, there are only three possibilities:

(i) $k=\mathbf{F}_2$, C defined by $u^2+u=v^3+v+1$

(ii) $k=\mathbf{F}_3$, C defined by $u^2=v^3-v-1$

(iii) $k=\mathbf{F}_4$, C defined by $u^2+u=v^3+\rho$, with $\rho\notin\mathbf{F}_2$.

c) Case (i) is that of 2.4.4. Determine $\Gamma\backslash X$ in cases (ii) and (iii) [the only difference is that instead of 2 vertices t_1, t_2, one finds 3 vertices in case (ii) and 4 vertices in case (iii)].

4) Put $\Gamma_1=\mathbf{SL}_2(A)$. Show that, in the case 2.4.2, the canonical map

$$\mathrm{vert}(\Gamma_1\backslash X)\to\mathrm{vert}(\Gamma\backslash X)$$

is bijective; give examples where the map

$$\mathrm{edge}(\Gamma_1\backslash X)\to\mathrm{edge}(\Gamma\backslash X)$$

is not injective. (Use exerc. 1 of no. 2.1.)

5) Suppose k_0 is finite. Show that, with the notations of exerc. 2 of no. 2.3, one has

$$\chi(\Gamma)=-(q-1)/(q_0-1)^2(q_0^2-1)\quad\text{in cases 2.4.1, 2.4.2}$$

$$\chi(\Gamma)=-5/3\quad\text{in case 2.4.4.}$$

6) Determine $\Gamma\backslash X$ when $g=1$ and k_0 is algebraically closed (use [22]).

2.5 Structure of Γ

We can apply the results of chap. I, no. 5.4 to the pair (X,Γ): once a maximal tree T of $\Gamma\backslash X$ is chosen, together with a lift j of T into X and of $\mathrm{edge}(\Gamma\backslash X)$ into $\mathrm{edge}(X)$, the group Γ may be identified with the *fundamental group*

$$\pi_1(\underline{\Gamma},\Gamma\backslash X,T)$$

of a certain *graph of groups* $(\underline{\Gamma},\Gamma\backslash X)$ carried by $\Gamma\backslash X$, cf. chap. I, th. 13. Recall that, if $x\in\mathrm{vert}(\Gamma\backslash X)$, the group $\underline{\Gamma}_x$ attached to x is the stabilizer of the representative jx

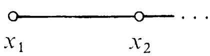
of x in vert(X); if jx is defined by a lattice L, we thus have $\underline{\Gamma}_x = \mathrm{Aut}(E_L)$, cf. no. 2.1. Similarly, if $y \in \mathrm{edge}(\Gamma\backslash X)$, the group $\underline{\Gamma}_y$ is the stabilizer of the edge jy of X. When k_0 is finite, all the groups $\underline{\Gamma}_x$ and $\underline{\Gamma}_y$ are finite.

Examples

The situation is particularly simple when $\Gamma\backslash X$ is a *tree*, because Γ is then the limit of a *tree of groups* carried by $\Gamma\backslash X$, cf. chap. I, no. 4.4. Thus:

a) in the case 2.4.1 ($A = k[t]$, $g = 0$, $e = d = 1$), where $\Gamma\backslash X$ is:

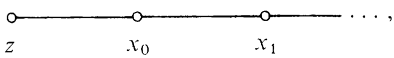
we have $\underline{\Gamma}_{x_0} = \mathbf{GL}_2(k)$, $\underline{\Gamma}_y = B(k)$, and the limit of the subtree of groups formed by the cusp



is the Borel group $B(A)$. We therefore recover *Nagao's theorem*:

$$\Gamma = \mathbf{GL}_2(k) *_{B(k)} B(A), \quad \text{cf. no. 1.6, th. 6.}$$

b) in the case 2.4.3 ($G = 0$, $e = d = 2$), where $\Gamma\backslash X$ is



we similarly find

$$\Gamma = D^* *_h \mathbf{GL}_2(k_0) *_{B(k_0)} B(A)$$

where D is the field of quaternions attached to the curve C, and where h is the group k^*, viewed as Cartan subgroup both of $\mathbf{GL}_2(k_0)$ and of D^*.

When $A = \mathbf{R}[u, v]/(u^2 + v^2 + 1)$ this gives

$$\Gamma = \mathbf{H}^* *_{\mathbf{C}^*} \mathbf{GL}_2(\mathbf{R}) *_{B(\mathbf{R})} B(A).$$

We now return to the general case, and revive the notation $Y_m, \Delta_c, x_c, \ldots$ of no. 2.3. If $x = x_{c,n}$ is a vertex of the cusp Δ_c, and y the edge joining x to the succeeding vertex $x' = x_{c,n+1}$, we have seen that $\underline{\Gamma}_x = \underline{\Gamma}_y$, so that we have a sequence of *embeddings*:

$$\underline{\Gamma}_{x_c} \subset \cdots \subset \underline{\Gamma}_x \subset \underline{\Gamma}_{x'} \subset \cdots$$

and the limit Γ_c of the tree of groups defined on Δ_c by $(\underline{\Gamma}, \Gamma\backslash X)$ is simply the *union* of the increasing sequence formed by the $\underline{\Gamma}_x$; one can also characterize Γ_c as the intersection of Γ with the *Borel subgroup* of $\mathbf{GL}_2(K)$ formed by the elements which stabilize the *end* of X defined by the lift $j\Delta_c$ of Δ_x (cf. nos. 1.3 and 2.3). Either characterization shows that Δ_c is a semi-direct product of the group $k_0^* \times k_0^*$ by a group U_c which is a projective A-module of rank 1 (and in particular a k_0-vector space of infinite dimension).

Now let Λ_m denote the fundamental group of the restriction of the graph of groups $(\underline{\Gamma}, \Gamma\backslash X)$ to Y_m. Put $\Phi_c = \underline{\Gamma}_{x_c}$. The fact that x_c is a vertex of both Y_m and Δ_c shows that we have *canonical injections*:

$$\Phi_c \to \Lambda_m \quad \text{and} \quad \Phi_c \to \Gamma_c.$$

Theorem 10. *The group Γ is the sum of the groups Γ_c $(c \in \operatorname{Pic}(A))$ and Λ_m amalgamated along their common subgroups Φ_c according to the above injections.*

This follows from th. 9 and the easily checked fact that, if a graph of groups $\underline{G}$ is obtained by "glueing" two graphs of groups $\underline{G}_1$ and $\underline{G}_2$ by a tree of groups $\underline{G}_{12}$, then one has

$$\pi_1(\underline{G}) = \pi_1(\underline{G}_1) *_{\pi_1(\underline{G}_{12})} \pi_1(\underline{G}_2).$$

Corollary. *The group Γ is not finitely generated.*

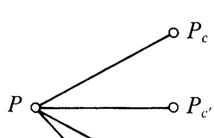
Indeed, let Δ_c be a cusp and let Γ'_c be the group obtained by amalgamating Λ_m and the $\Gamma_{c'}$, for $c' \neq c$. By th. 10, we have

$$\Gamma = \Gamma'_c *_{\Phi_c} \Gamma_c.$$

But Γ_c is the union of the strictly increasing sequence of the groups $\underline{\Gamma}_x$, with $x \in \operatorname{vert}(\Delta_c)$, and these subgroups all contain Φ_c. It follows that Γ is the union of the strictly increasing sequence of the $\Gamma'_c *_{\Phi_c} \underline{\Gamma}_x$, and it is therefore not finitely generated.

Another proof of th. 10

If $c \in \operatorname{Pic}(A)$ let $\Delta_{c,1}$ denote the subgraph of Δ_c obtained by removing x_c as well as the two edges (inverses of each other) with origin or terminus x_c. By th. 9, $\operatorname{vert}(\Gamma\backslash X)$ is the disjoint union of $\operatorname{vert}(Y_m)$ and the $\operatorname{vert}(\Delta_{c,1})$. Let X_m (resp. X_c) denote the inverse image of Y_m (resp. $\Delta_{c,1}$) in X. Let $\bar{X}$ be the tree obtained by *contracting to a point* each of the connected components of X_m and of the X_c (cf. chap. I, no. 2.3). The group Γ acts on $\bar{X}$, and the quotient $T = \Gamma\backslash\bar{X}$ is the graph derived from $\Gamma\backslash X$ by contracting Y_m to a point P and each of the $\Delta_{c,1}$ to a point P_c. In view of th. 9 the only edges of T are those which connect P to the P_c; in particular, T is a tree:



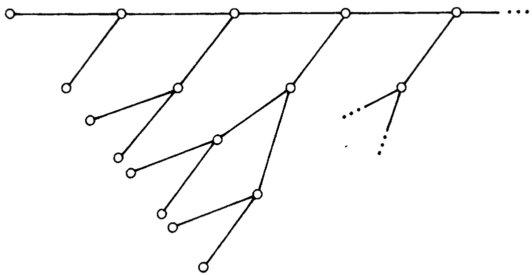
By th. 9 of chap. I, it follows that Γ is the limit of a *tree of groups* (Σ, T) carried by T. Moreover, one easily checks that the group Σ_P attached to the vertex P is Λ_m, that attached to P_c is Γ_c, and that attached to the edge P——P_c is Φ_c. Th. 10 follows.

Exercises

1) Let E be a connected component of one of the X_c.

a) Show that E contains exactly one end of X, and that this end is rational, i.e. it belongs to $\mathbf{P}_1(K)$. Conversely, each rational end can be represented by exactly one such component E.

b) Suppose that $\mathrm{Card}(k) = 2$. Show that E is isomorphic to the following tree:



2) (after Quillen). The hypotheses are those of 2.4.1, i.e. we have $A = k[t]$. Let Z_0 denote the set of k-vector subspaces v of A^2 with the following property:

($*$) the canonical homomorphism $A \otimes_k v \to A^2$ is bijective. (Such a subspace is a "k-form" of the A-module A^2.)

a) Let L be an $\mathcal{O}$-lattice of $V = K^2$. Show that the bundle E_L associated with L is trivial if and only if there is a $v \in Z_0$ such that $L = \mathcal{O} \cdot v$, in which case $v = A^2 \cap L$. Deduce a bijection between Z_0 and the set of vertices of X of type 0 in the sense of no. 2.4.1.

b) Let $Z_1 = \mathbf{P}_1(K)$ and let Z be the disjoint union of Z_0 and of Z_1. If $D \in Z_1$ and $v \in Z_0$ we say that D and v are adjacent if D (viewed as a K-line of K^2) is of the form $K \otimes_k d$, where d is a k-line of v. This defines a combinatorial graph structure on Z. Show that this graph is a *tree*, isomorphic to that denoted by $\bar{X}$ on the preceding page. The group Γ acts on Z with a segment as fundamental domain; thus one recovers Nagao's decomposition:

$$\Gamma = \mathbf{GL}_2(k) *_{B(k)} B(k[t]).$$

3) In case 2.4.4, show that Γ is the free product of $\mathfrak{S}_3 *_{\mathbf{Z}/2\mathbf{Z}} A$ and two copies of $\mathbf{Z}/3\mathbf{Z}$.

2.6 Auxiliary results

Proposition 11. *Let $\underline{G} = (G, Y)$ be a graph of groups with the following properties:*

(i) *the graph Y is finite;*

(ii) *for each $x \in \mathrm{vert}(Y)$ the group G_x associated with x is finite.*

Let T be a maximal tree of Y, and let $G = \pi_1(G, Y, T)$ be the fundamental group of $\underline{G}$ at T (chap. I, no. 5.1). *Then*:

a) *G is finitely generated.*

b) *Every torsion-free subgroup of G is free.*

c) *G has torsion-free subgroups of finite index.*

Let us say that a group has *virtually* a property (P) if it has a subgroup of finite index which enjoys (P). With this terminology we have:

Corollary. *The group G is virtually free of finite rank.*

In particular, it is a group of (WFL) *type*, and of *virtual cohomological dimension* $\leqslant 1$, in the sense of [34], no. 1.8.

Remark. A. Karrass, A. Pietrowski and D. Solitar (*J. Australian Math. Soc.*, 1973) have proved a *converse* to prop. 11: every virtually free group of finite rank is the fundamental group of a graph of groups with properties (i) and (ii). This result generalizes the theorem of Stallings ([7], 0.2), according to which every torsion-free group which is virtually free of finite rank is free. There are analogous results, due to D. Cohen, G. P. Scott and M. J. Dunwoody, for non-finitely generated groups. See [44], [45].

Proof of prop. 11

The assertion (a) follows from the definition of $\pi_1(G, Y, T)$ by generators and relations.

The assertion (b) follows from the following lemma, which generalizes prop. 18 of chap. I:

Lemma 8. *Let H be a subgroup of G such that $H - \{1\}$ does not meet any conjugate of the G_x, $x \in \mathrm{vert}(Y)$. Then H is a free group.*

Indeed, H acts freely on the *tree* $\tilde{X}$ which is the universal covering of (G, Y, T) because the stabilizers of the vertices of $\tilde{X}$ are the conjugates of the G_x, cf. chap. I, no. 5.3; we then apply th. 4 of chap. I, no. 3.3.

To prove (c), we choose an integer $n \geqslant 1$ which is a multiple of the orders of the G_x, $x \in \mathrm{vert}(Y)$. Let E be a finite set of n elements, and let $\mathfrak{S}_E$ be the group of permutations of E. If F is a finite group, we say that a homomorphism $f\colon F \to \mathfrak{S}_E$ is *regular* if the corresponding action of F on E is free, in other words if $f(x) \cdot e = e$ (with $x \in F$, $e \in E$) implies $x = 1$.

Lemma 9. *Suppose that the order of F divides n. Then*

(α) *There is a regular homomorphism $f\colon F \to \mathfrak{S}_E$, and such a homomorphism is unique up to conjugation by an element of $\mathfrak{S}_E$.*

(β) *If F' is a subgroup of F, each regular homomorphism of F' into $\mathfrak{S}_E$ extends to a regular homomorphism of F into $\mathfrak{S}_E$.*

(γ) *If F' and F'' are two subgroups of F, and ϕ an isomorphism of F' onto F'', then, for each regular homomorphism $f\colon F \to \mathfrak{S}_E$, there is an $s \in \mathfrak{S}_E$ such that $f(\phi(x)) = s \cdot f(x) \cdot s^{-1}$ for each $x \in F'$.*

Assertion (α) means that, up to isomorphism, there is exactly one F-set of n elements on which F acts freely, which is obvious. Assertions (β) and (γ) follow from (α).

Lemma 10. *There is a homomorphism $f\colon G \to \mathfrak{S}_E$ such that the restriction of f to each of the G_x, $x \in \operatorname{vert}(Y)$, is a regular homomorphism.*

In view of the definition of G (chap. I, no. 5.1), it suffices to construct:

$$\text{regular homomorphisms} \quad f_x\colon \ G_x \to \mathfrak{S}_E, \qquad x \in \operatorname{vert}(Y)$$

$$\text{elements } x_y \text{ of } \mathfrak{S}_E, \qquad y \in \operatorname{edge}(Y)$$

satisfying the compatibility conditions:

$$s_y \cdot f_x(a^y) \cdot s_y^{-1} = f_{x'}(a^{\bar{y}}) \quad \text{if} \quad y \in \operatorname{edge}(Y), \quad x = t(y), \quad x' = o(y), \quad a \in G_y$$

$$s_y = 1 \quad \text{if} \quad y \in \operatorname{edge}(T).$$

To do this we first construct the f_x and the s_y on the tree T; this is done by induction on the number of vertices of T, using lemma 9(β). The existence of the s_y for $y \notin \operatorname{edge}(Y)$ then follows from lemma 9(γ).

Let H be the kernel of a homomorphism $f\colon G \to \mathfrak{S}_E$ with the properties of lemma 10. It is clear that H is of finite index in G. On the other hand, for each $x \in \operatorname{vert}(Y)$, the restriction of f to G_x is injective. We then have $H \cap G_x = \{1\}$, and, since H is a normal subgroup of G, this implies that $H - \{1\}$ does not meet any conjugates of the G_x. By lemma 8, H is free, hence *a fortiori* torsion-free, which proves (c).

Residually finite groups

Let G be a group. Recall ([3], p. 116, 414) that G is said to be *residually finite* if it is separated for the topology of subgroups of finite index, i.e. if for each $x \in G - \{1\}$ there is a subgroup H of finite index in G which does not contain x; note that one can then choose H to be a normal subgroup of G: it suffices to replace it by the intersection of its conjugates.

Every free group is residually finite (cf. for example Bourbaki, A.I, p. 150, exerc. 34). Every subgroup of a product of residually finite groups is residually finite.

Proposition 12. *Let $\underline{G} = (G, Y)$ be a graph of groups with the following properties:*

(i) *the graph Y is finite;*

(ii′) *for each $x \in \operatorname{vert}(Y)$, the group G_x is residually finite;*

(ii″) *for each $y \in \operatorname{edge}(Y)$, the group G_y is finite.*

Let T be a maximal tree of Y, and let G be the fundamental group of (G, Y, T). Then:

(a) *G is residually finite;*

(b) *for each $x \in \operatorname{vert}(Y)$, the topology of subgroups of finite index in G induces on G_x the topology of subgroups of finite index in G_x.*

(Note that the properties (ii′) and (ii″) are weaker than the property (ii) of prop. 11.)

Let $\mathscr{R}$ be the set of families $R = (R_x)_{x \in \operatorname{vert}(Y)}$, where R_x is a normal subgroup of finite index in G_x such that $R_x \cap G_y = \{1\}$ for each edge y with extremity x (which allows us to identify G_y with a subgroup of G_x/R_x). The G_x/R_x and the G_y then form a graph of groups which satisfies the hypotheses of prop. 11. Let $G_R = \pi_1(G/R, Y, T)$ and let f_R denote the canonical homomorphism of G onto G_R. The f_R define a homomorphism

$$f\colon\ G \to \prod_{R \in \mathscr{R}} G_R.$$

Let us show that f is *injective*. For this, we choose a vertex P_0 of Y and identify G (resp. G_R) with the fundamental group of (G, Y) (resp. of $(G/R, Y)$) at P_0, cf. chap. I, prop. 20. Let g be an element of G distinct from 1, and represent g in the form $|c, \mu|$ where (c, μ) is a *reduced word* (chap. I, no. 5.2). It follows from the hypotheses (i), (ii') and (ii'') that one can choose R so that $(f_R(c), \mu)$ is a reduced word of G_R. In view of th. 11 of chap. I, this implies that $f_R(g) \neq 1$, which indeed shows that f is injective. But by the cor. to prop. 11, each of the G_R is virtually free, hence residually finite. It follows that G is residually finite, whence (a).

Now let H be a subgroup of finite index in one of the G_x. Choose $R \in \mathscr{R}$ such that $R_x \subset H$. In view of prop. 11, there is a subgroup H_1 of finite index in G_R which is torsion-free, and hence meets G_x/R_x only in the neutral element. The group $H_2 = f_R^{-1}(H_1)$ is then a subgroup of finite index in G such that $H_2 \cap G_x \subset H$. This proves (b).

Remark

Prop. 12 no longer remains valid when the hypothesis (ii'') is dropped, cf. Bourbaki, A.I, p. 150, exerc. 35.

Exercises

1) Any free product of residually finite groups is residually finite.

2) Let $G = \pi_1(G, Y, T)$. Suppose that the G_x, $x \in \operatorname{vert}(Y)$, are finite, but do not assume that Y is finite. Show that every torsion-free subgroup of G is free. Prove the equivalence of the following properties:

c) G has torsion-free subgroups of finite index;

c') the orders of the G_x are bounded.

(The implication c) $\Rightarrow$ c') is immediate. To prove c') $\Rightarrow$ c), use the same method as in the proof of prop. 11.)

3) We assume the hypotheses of prop. 11. Let $\chi(G)$ denote the *Euler-Poincaré characteristic* of G in the sense of Wall (cf. [34], p. 99); if H is a free subgroup of finite index in G, and if r_H denotes the rank of H, one has

$$\chi(G) = (1 - r_H)/(G:H).$$

Show that

$$\chi(G) = \sum_{x \in \operatorname{vert}(Y)} 1/\operatorname{Card}(G_x) - \sum_{y \in \operatorname{geom\,edge}(Y)} 1/\operatorname{Card}(G_y)$$

(For a generalization of this result, see [34], p. 102, prop. 14.)

4) The notations are those of no. 2.5. Suppose that k_0 is finite. Show that

$$\chi(\Gamma) = \chi(\Lambda_m) - \sum_{c \in \mathrm{Pic}(A)} \chi(\Phi_c),$$

where $\chi(\Gamma)$ is defined as in exerc. 2 of no. 2.3, and $\chi(\Lambda_m)$ and $\chi(\Phi_c)$ are the Euler-Poincaré characteristics of Λ_m and of Φ_c in the sense of Wall. (Use the preceding exercise, part d) of exerc. 2 of no. 2.3, and the fact that

$$\chi(\Phi_c) = 1/\mathrm{Card}(\Phi_c)$$

because Φ_c is finite.)

5) Give an example of a strictly increasing sequence of finite groups whose union G has no subgroup of finite index other than itself. Deduce that prop. 12 does not extend to the case of infinite graphs, even when all the G_x are finite.

2.7 Structure of Γ: case of a finite field

In this section we assume that k_0 is *finite*. The graph Y_m is then finite, by cor. 4 to th. 9. It is the same for the groups $\underline{\Gamma}_x$, and in particular the groups Φ_c appearing in th. 10. As for Λ_m, we have:

Theorem 11. *The group Λ_m is virtually free of finite rank* (in the sense of no. 2.6).

This follows from the cor. to prop. 11, and the properties of Y_m and the $\underline{\Gamma}_x$ just recalled.

Congruence subgroups and subgroups of finite index

Among the subgroups of finite index in Γ one has the *congruence subgroups*. Recall (cf. [20]) that H is a "congruence" subgroup of Γ if there is an ideal $\mathfrak{a} \neq 0$ of A such that H contains the kernel of

$$\mathbf{GL}_2(A) \to \mathbf{GL}_2(A/\mathfrak{a}).$$

(A similar definition applies to $\mathbf{SL}_2$ as well.) The topology defined on Γ by the congruence subgroups is separated; *a fortiori*, Γ is residually finite.

Let $S_a(\Gamma)$ (resp. $S(\Gamma)$) denote the set of congruence subgroups (resp. subgroups of finite index) in Γ. One has

$$S_a(\Gamma) \subset S(\Gamma)$$

and the "congruence subgroup problem" is to see whether equality holds. The answer is negative:

Theorem 12. *The set $S_a(\Gamma)$ is denumerable, and the set $S(\Gamma)$ of continuum power.*

(So there are "many" subgroups of finite index which are not congruence subgroups.)

The first assertion follows from the fact that the set of ideals of A is denumerable and, if $\mathfrak{a}$ is an ideal $\neq 0$, there are only a finite number of subgroups of Γ which contain the kernel of $\mathbf{GL}_2(A) \to \mathbf{GL}_2(A/\mathfrak{a})$.

On the other hand, since Γ is denumerable, the set of its subsets is of continuum power, $\mathfrak{c}$. We then have

$$\operatorname{Card} S(\Gamma) \leqslant \mathfrak{c}$$

and it remains to prove the opposite inequality. To do this we choose a cusp Δ_c, and let Γ_c denote the corresponding subgroup of Γ, cf. no. 2.5. Let $S(\Gamma_c)$ denote the set of subgroups of finite index in Γ_c.

Lemma 11. *One has* $\operatorname{Card} S(\Gamma_c) = \mathfrak{c}$.

By construction, Γ_c contains a subgroup U_c of finite index which is a k_0-vector space of denumerably infinite dimension. The dual of this space is of dimension $\mathfrak{c}$. Hence the set of hyperplanes of U_c is of continuum power, which proves the lemma.

Lemma 12. *The topology of subgroups of finite index in Γ induces on Γ_c the topology of its subgroups of finite index.*

This follows from prop. 12, applied to the tree of groups (Σ, T) defined at the end of no. 2.5.

Now consider the subset S of $S(\Gamma) \times S(\Gamma_c)$ formed by the pairs (H, H') such that

$$H \in S(\Gamma), \qquad H' \in S(\Gamma_c) \quad \text{and} \quad H \cap \Gamma_c \subset H'.$$

By lemma 12, the map $\mathrm{pr}_2 : S \to S(\Gamma_c)$ is surjective. In view of lemma 11, we then have $\operatorname{Card} S \geqslant \mathfrak{c}$. On the other hand, the map $\mathrm{pr}_1 : S \to S(\Gamma)$ is obviously surjective, and its fibres are finite; we thus have $\operatorname{Card} S(\Gamma) = \operatorname{Card} S$, cf. Bourbaki, E III, p. 50, prop. 4, whence $\operatorname{Card} S(\Gamma) \geqslant \mathfrak{c}$, and the proof is complete.

Remark. The proof of th. 12 given in [20], §3, is somewhat different: it avoids using graphs of groups, but uses instead *homological properties* of the subgroups of finite index in Γ (cf. next section).

2.8 Homology

We retain the hypotheses and notation of nos. 2.5 and 2.7; we let p denote the characteristic of k_0.

Review of homology

If a group G acts simplicially on a contractible simplicial complex X there is a *spectral sequence* which relates the homology groups of G to those of the stabilizers of the simplexes of X, cf. [34], p. 95. When X is a *tree*, on which G acts without inversion, this spectral sequence takes a particularly simple form, which we now recall.

First choose an orientation $\mathrm{edge}^+(X)$ of X invariant under G; this is possible since G acts without inversion. Define the group $C_i(X)$ of i-dimensional *chains* of X by:

$$C_i(X) = 0 \quad \text{if} \quad i \neq 0, 1$$

$$C_0(X) = \text{free abelian group with basis } \mathrm{vert}(X)$$

$$C_1(X) = \text{free abelian group with basis } \mathrm{edge}^+(X).$$

We have two homomorphisms

$$\partial\colon \quad C_1(X) \to C_0(X) \quad \text{given by} \quad \partial(y) = t(y) - o(y)$$

$$\varepsilon\colon \quad C_0(X) \to \mathbf{Z} \qquad \text{given by} \quad \varepsilon(x) = 1.$$

The group G acts on the $C_i(X)$ and commutes with ∂ and ε (provided that it is made to act trivially on $\mathbf{Z}$). Thus we obtain a sequence of G-modules

$$0 \to C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} \mathbf{Z} \to 0$$

which is *exact* because X is a tree.

Now, if M is any G-module, this exact sequence yields an exact sequence of G-modules

$$0 \to C_1(X) \otimes M \to C_0(X) \otimes M \to M \to 0,$$

whence, by passing to homology, an exact sequence:

$$\cdots \to H_i(G, C_1(X) \otimes M) \to H_i(G, C_0(X) \otimes M) \to H_i(G, M)$$

$$\to H_{i-1}(G, C_1(X) \otimes M) \to \cdots$$

Let Σ_0 (resp. Σ_1) denote a system of representatives of $\mathrm{vert}(X)$ (resp. $\mathrm{edge}^+(X)$) modulo the action of G; if x (resp. y) is a vertex (resp. edge) of X, let G_x (resp. G_y) denote its stabilizer in G. If we let $\mathbf{Z}[G/H]$ denote the free abelian group with the elements of the homogeneous space G/H as basis, we have G-module isomorphisms:

$$C_0(X) = \coprod_{x \in \Sigma_0} \mathbf{Z}[G/G_x], \qquad C_1(X) = \coprod_{y \in \Sigma_1} \mathbf{Z}[G/G_y].$$

But, by *Shapiro's lemma*, $H_i(G, \mathbf{Z}[G/H] \otimes M)$ may be identified with $H_i(H, M)$. We then have

$$H_i(G, C_0(X) \otimes M) = \coprod_{x \in \Sigma_0} H_i(G_x, M)$$

$$H_i(G, C_1(X) \otimes M) = \coprod_{y \in \Sigma_1} H_i(G_y, M),$$

and we obtain:

Proposition 13. *If G acts on a tree X, one has, for each G-module M, an exact homology sequence*

$$\cdots \to H_{i+1}(G, M) \to \coprod_{y \in \Sigma_1} H_i(G_y, M) \to \coprod_{x \in \Sigma_0} H_i(G_x, M) \to H_i(G, M) \to \cdots$$

and an exact cohomology sequence

$$\cdots \to H^i(G, M) \to \prod_{x \in \Sigma_0} H^i(G_x, M) \to \prod_{y \in \Sigma_1} H^i(G_y, M) \to H^{i+1}(G, M) \to \cdots .$$

Remarks

1) The homomorphisms $H_i(G_x, M) \to H_i(G, M)$ occurring in this exact sequence are those induced by the injections $G_x \to G$; as for the

$$H_i(G_y, M) \to H_i(G_x, M),$$

they are zero if x is not an extremity of y, and induced (resp. opposite to induced) by the injections $G_y \to G_x$ if $x = t(y)$ (resp. $x = o(y)$). The situation is dual for cohomology.

2) One can write the exact sequence of prop. 13 in the form

$$0 \to H_0(G \backslash X, \mathscr{H}_i) \to H_i(G, M) \to H_1(G \backslash X, \mathscr{H}_{i-1}) \to 0$$

where $H_r(G \backslash X, \mathscr{H}_s)$ denotes the rth homology group of the graph $G \backslash X$ evaluated with respect to the *cosheaf* $\mathscr{H}_s$ formed by the $H_s(G_x, M)$ and the $H_s(G_y, M)$, cf. [20], no. 3.3, lemma 10.

Example (Lyndon, *Ann. of Math.* 52, 1950)

Suppose that G is an amalgam $G_1 *_{G_{12}} G_2$, and apply prop. 13 to the action of G on the corresponding tree (chap. I, no. 4.1). We obtain the *exact homology sequence*

$$\cdots \to H_{i+1}(G, M) \to H_i(G_{12}, M) \to H_i(G_1, M) \oplus H_i(G_2, M) \to H_i(G, M) \to \cdots$$

and the *exact cohomology sequence*

$$\cdots \to H^i(G, M) \to H^i(G_1, M) \oplus H^i(G_2, M) \to H^i(G_{12}, M) \to H^{i+1}(G, M) \to \cdots .$$

A more "topological" way to present this is to say that the classifying space

$BG = K(G, 1)$ is obtained by glueing BG_1 and BG_2 along BG_{12} (cf. J. H. C. Whitehead, *Mathematical Works*, vol. II, p. 199–200), which allows the Mayer-Vietoris exact sequence to be applied to BG.

Homology of Γ

The method above applies to Γ by making it act, either on the tree X (which has the advantage that the $\underline{\Gamma}_x$ and $\underline{\Gamma}_y$ are finite) or on the tree $\bar{X}$ defined at the end of no. 2.5. The second point of view leads, for example, to the exact sequence:

$$\cdots \to H_{i+1}(\Gamma, M) \to \coprod_c H_i(\Phi_c, M) \to H_i(\Lambda_m, M) \oplus \coprod_c H_i(\Gamma_c, M)$$

$$\to H_i(\Gamma, M) \to \cdots,$$

where the summation is over the elements c of $\mathrm{Pic}(A)$.

Using known properties of the homology of finite groups, of their direct limits, and of free groups, one obtains various results about the $H_i(\Gamma, M)$. For example:

Proposition 14. *Suppose that the Γ-module M is a finitely-generated abelian group. Let $\mathscr{F}$ denote the class of finite abelian groups and $\mathscr{D}_p$ that of the groups which are the direct sum of a finite group and a denumerable p-primary torsion group. Then*:

a) *For $i \leqslant 2$, the canonical homomorphism*

$$H_i(\Lambda_m, M) \oplus \coprod_c H_i(\Gamma_c, M) \to H_i(\Gamma, M)$$

is $\mathscr{F}$-bijective (i.e. its kernel and cokernel belong to $\mathscr{F}$). *For $i = 1$, it is $\mathscr{F}$-injective, and its cokernel is finitely generated.*

b) *For $i \leqslant 1$* (resp. *for $i \geqslant 2$*), *$H_i(\Lambda_m, M)$ is finitely generated* (resp. *is finite*).

c) *One has $H_i(\Gamma_c, M) \in \mathscr{D}_p$ for $i \geqslant 1$ and $c \in \mathrm{Pic}(A)$.*

Assertion a) follows from the exact sequence written above, together with the fact that the $H_i(\Phi_c, M)$ are finite for $i \geqslant 1$, and finitely generated for $i = 0$.

Assertion b) follows from the fact that Λ_m is virtually free of finite rank, cf. th. 11.

If $c \in \mathrm{Pic}(A)$, let U_c denote the subgroup of unipotent elements of Γ_c; it is a direct limit of finite abelian groups of type $(p, \ldots, p)$; it follows that, for $i \geqslant 1$, the $H_i(U_c, M)$ are denumerable p-primary torsion groups. On the other hand, the quotient Γ_c/U_c is isomorphic to $k_0^* \times k_0^*$, hence finite of an order prime to p. Using the spectral sequence of group extensions, we deduce that, for $i \geqslant 1$, the group $H_i(\Gamma_c, M)$ is canonically isomorphic to

$$H_i(\Gamma_c/U_c, H_0(U_c, M)) \oplus H_0(\Gamma_c/U_c, H_i(U_c, M)).$$

The first factor of this sum is finite of order prime to p; the second is a denumerable p-primary torsion group; whence c).

Corollary 1. *The homomorphism* $\coprod_c H_i(\Gamma_c, M) \to H_i(\Gamma, M)$ *is* $\mathcal{F}$*-bijective for* $i \geqslant 2$; *for* $i = 1$ *it is* $\mathcal{F}$*-injective, and its cokernel is finitely generated.*

This follows from a) and b).

Corollary 2. *One has* $H_i(\Gamma, M) \in \mathcal{D}_p$ *for* $i \geqslant 2$.

This follows from c) and cor. 1.

Corollary 3. *If* M *is of finite order prime to* p, *all the* $H_i(\Gamma, M)$ *are finite.*

This follows from cor. 1 and 2 and the fact that, if an integer n kills M, it also kills all the $H_i(\Gamma, M)$.

Corollary 4. *One has* $H_i(\Gamma, \mathbf{Q}) = 0$ *for* $i \geqslant 2$, *and* $H_1(\Gamma, \mathbf{Q})$ *is a* $\mathbf{Q}$*-vector space of finite dimension, isomorphic to* $H_1(\Lambda_m, \mathbf{Q})$.

We have $H_i(\Gamma, \mathbf{Q}) = H_i(\Gamma, \mathbf{Z}) \otimes \mathbf{Q}$. By cor. 2, applied to $M = \mathbf{Z}$, $H_i(\Gamma, \mathbf{Z})$ is a torsion group for $i \geqslant 2$; hence $H_i(\Gamma, \mathbf{Q}) = 0$ in this case.

For $i = 1$, the exact sequence written before prop. 14 shows that the homomorphism $H_1(\Lambda_m, \mathbf{Q}) \to H_1(\Gamma, \mathbf{Q})$ is an isomorphism.

Remark. We can also deduce cor. 4 from prop. 13 applied to the tree X; indeed, since the stabilizers of the vertices and edges are finite, their homology with coefficients in $\mathbf{Q}$ is 0 in dimension $\geqslant 1$, and equal to $\mathbf{Q}$ in dimension 0. This gives isomorphisms

$$H_i(\Gamma, \mathbf{Q}) \simeq H_i(\Gamma \backslash X, \mathbf{Q}) \qquad (i = 0, 1, \dots)$$

between the homology groups of Γ and those of the graph $\Gamma \backslash X$; in particular, we have $H_i(\Gamma, \mathbf{Q}) = 0$ for $i \geqslant 2$. The same argument, applied to Λ_m, shows that $H_i(\Lambda_m, \mathbf{Q})$ may be identified with $H_i(Y_m, \mathbf{Q})$; since the inclusion $Y_m \to \Gamma \backslash X$ is a homotopy equivalence (cf. cor. 2 to th. 9) we finally obtain isomorphisms

$$H_i(\Gamma, \mathbf{Q}) \simeq H_i(\Lambda_m, \mathbf{Q}) \qquad (i = 0, 1, \dots)$$

whence cor. 4.

Subgroups of finite index in Γ

Let G be a subgroup of finite index in Γ. We make G act on the projective line $\mathbf{P}_1(K)$, i.e. on the *rational ends* of X. Since the orbits of Γ in $\mathbf{P}_1(K)$ are finite in number, the same is true for the *orbits of* G. Let Σ be a system of representatives of these orbits, and, for each $\sigma \in \Sigma$, let G_σ denote the stabilizer of σ in G. The G_σ are representatives (modulo G-conjugation) of the intersections of G with the Borel subgroups of $\mathbf{GL}_2(K)$ (or with the Γ_c, which amounts to the same). *The results proved above for* Γ *and the* Γ_c *are also valid for* G *and the* G_σ. This can be seen in two ways, which we just sketch.

a) Using the finite morphism $G\backslash X \to \Gamma\backslash X$, one shows that $G\backslash X$, just as $\Gamma\backslash X$, is obtained by *glueing the "cusps"* Δ_σ $(\sigma \in \Sigma)$ *to a finite graph* Y_G; the stabilizer of Δ_σ is conjugate to G_σ. One can then repeat the arguments of this section, replacing $(\Gamma, \Gamma_c, Y_m, \ldots)$ by $(G, G_\sigma, Y_G, \ldots)$ and one sees that *prop. 14 and its corollaries remain valid.*

b) One uses *Shapiro's lemma*

$$H_i(G, M) = H_i(\Gamma, \operatorname{Ind}_G^\Gamma M)$$

where $\operatorname{Ind}_G^\Gamma M = \mathbf{Z}[\Gamma] \otimes_{\mathbf{Z}[G]} M$ is the Γ-module *induced* by the G-module M. Then, applying prop. 14 and its corollaries to $\operatorname{Ind}_G^\Gamma M$, one gets the desired results.

Exercises

1) The notations are those of prop. 14. If $c \in \operatorname{Pic}(A)$, let U_c' denote the subgroup of U_c consisting of the elements which act trivially on M. Show that U_c' is of finite index in U_c. If this index is p^a, show that p^{a+1} kills all the $H_i(U_c, M)$ for $i \geqslant 1$ (remark that p kills $H_i(U_c', M)$ and use a transfer argument). Deduce the existence of an integer $n \geqslant 1$ which kills the $H_i(\Gamma_c, M)$ for $i \geqslant 1$, and the $H_i(\Gamma, M)$ for $i \geqslant 2$; show that $H_1(\Gamma, M)$ is the direct sum of a finitely generated abelian group and a denumerable group killed by a power of p.

2) The notations are those at the end of no. 2.8. In particular, G denotes a subgroup of finite index in Γ. If F is a group, we let F^{ab} denote the largest abelian quotient of F; one knows that $F^{\mathrm{ab}} = H_1(F, \mathbf{Z})$.

a) Let $\alpha: \coprod G_\sigma^{\mathrm{ab}} \to G^{\mathrm{ab}}$ be the homomorphism induced by the injections of the G_σ into G. Show, using prop. 14, that $\operatorname{Ker}(\alpha)$ is finite, and that $\operatorname{Coker}(\alpha)$ is finitely generated (cf. [20], no. 3.3).

b) Let $\sigma \in \Sigma$. Show that G_σ is the semi-direct product of a finite abelian group R_σ of order prime to p, and a unipotent group U_σ isomorphic to a denumerably infinite $\mathbf{F}_p$-vector space. Show that $G_\sigma^{\mathrm{ab}} = R_\sigma \times U_\sigma$ if R_σ is contained in the group k_0^* of homotheties, and that $G_\sigma^{\mathrm{ab}} = R_\sigma$ otherwise.

c) Prove the equivalence of the following properties:

c$_1$) the group G^{ab} is finitely generated;

c$_2$) the groups G_σ^{ab} are finite;

c$_3$) for each line D of V there is an element s of G, of order prime to p, which is not a homothety, and leaves D stable.

When these properties do not hold, G^{ab} is the product of a finitely generated group with an $\mathbf{F}_p$-vector space of infinite dimension (cf. [20], *loc. cit.*).

3) a) Let l be a prime number, and let G be a subgroup of Γ without l-torsion (i.e. without elements of order l). Show that, for each G-module M, and each $i \geqslant 2$, the multiplication by l in $H_i(G, M)$ is bijective. (Make G act on the tree X, and note that the stabilizers of the vertices and edges are of finite order prime to p.)

b) Let $\mathfrak{a}$ be an ideal of A which is $\neq 0$ and let $G_\mathfrak{a}$ be the kernel of the homomorphism

$$\mathbf{GL}_2(A) \to \mathbf{GL}_2(A/\mathfrak{a}).$$

Show that, if $\mathfrak{a} \neq A$, $G_\mathfrak{a}$ is without l-torsion for each $l \neq p$.

2.9 Euler-Poincaré characteristic

The hypotheses and notations are those of nos. 2.5, 2.7 and 2.8. Let G denote a subgroup of finite index in Γ.

Definition of the Euler-Poincaré characteristic $\chi(G)$ of G

The definition of Wall ([34], p. 99) does not apply, since G contains no subgroup of finite index "of type (FL)". It is nevertheless natural to *define* $\chi(G)$, by analogy with prop. 14 of [34], by means of the *formula*

$$(*) \qquad \chi(G) = \sum_x 1/\mathrm{Card}(G_x) - \sum_y 1/\mathrm{Card}(G_y)$$

where x (resp. y) runs through a system of representatives of the vertices (resp. geometric edges) of $G\backslash X$, and G_x (resp. G_y) denotes the stabilizer of x (resp. y) in G. The two series occurring in $(*)$ converge (no. 2.3, exerc. 2). One has

$$\chi(G) = \chi(\Gamma) \cdot (\Gamma : G),$$

and one can calculate $\chi(\Gamma)$ by means of the *value of the zeta function of the curve C at the point* -1 (*loc. cit.*).

Here is another point of view which leads to the same result: one knows that, under suitable hypotheses ([34], prop. 15) the Euler-Poincaré characteristic of an amalgam is given by the formula

$$\chi(G_1 *_{A_1} G_2 *_{A_2} \cdots * G_n) = \sum \chi(G_i) - \sum \chi(A_i).$$

Let us agree to apply this formula to the decomposition of Γ as an amalgam given by th. 10:

$$\chi(\Gamma) = \chi(\Lambda_m) - \sum_c \chi(\Phi_c) + \sum_c \chi(\Gamma_c)$$

and agree in addition that $\chi(\Gamma_c) = 0$ (which is natural because Γ_c is the union of finite subgroups of orders tending to $+\infty$, and $\chi(F) = 1/\mathrm{Card}(F)$ if F is finite). Thus we arrive at the following definition of $\chi(\Gamma)$:

$$\chi(\Gamma) = \chi(\Lambda_m) - \sum_c \chi(\Phi_c)$$

where, this time, the terms on the right-hand side are Euler-Poincaré characteristics in the sense of Wall. This definition leads to the *same* value of $\chi(\Gamma)$ as that given above, cf. no. 2.6, exerc. 4.

Homological interpretation of $\chi(G)$

Consider the case where G is *without p'-torsion*, i.e. where the elements of G of finite order have order a power of p. One sees easily that there are such subgroups (for example congruence subgroups, cf. no. 2.8, exerc. 3). The number $\chi(G)$ is then an *integer* < 0 (no. 2.3, exerc. 2). It is this integer that we now interpret.

To do this, let $\mathbf{Z}[\mathbf{P}_1(K)]$ be the free abelian group with basis $\mathbf{P}_1(K)$, and let $\varepsilon\colon \mathbf{Z}[\mathbf{P}_1(K)] \to \mathbf{Z}$ be the *augmentation* homomorphism, characterized by $\varepsilon(\delta) = 1$ for all $\delta \in \mathbf{P}_1(K)$. Let St denote the kernel of ε. We have the exact sequence

$$0 \to \mathrm{St} \to \mathbf{Z}[\mathbf{P}_1(K)] \xrightarrow{\varepsilon} \mathbf{Z} \to 0.$$

The group $\mathbf{GL}_2(K)$ acts in a natural way on St, its "Steinberg module"; *a fortiori*, G acts on St.

Theorem 13. *The Steinberg module* St *is a finitely generated projective* $\mathbf{Z}[G]$*-module, which is stably free of rank* $-\chi(G)$.

In other words, there are free $\mathbf{Z}[G]$ modules L_0 and L_1, of finite ranks l_0 and l_1, such that:

$$\mathrm{St} \oplus L_0 \simeq L_1 \quad \text{and} \quad \chi(G) = l_0 - l_1.$$

Proof

Let us say that a vertex (resp. edge) of X is *G-stable* if its stabilizer is $\{1\}$, and *G-unstable* if not. Let X_∞ be the subgraph of X consisting of the G-unstable vertices and edges, and put

$$S_0 = \mathrm{vert}(X) - \mathrm{vert}(X_\infty), \qquad S_1 = \mathrm{edge}^+(X) - \mathrm{edge}^+(X_\infty)$$

where $\mathrm{edge}^+(X)$ is an orientation of X invariant under G. The group G acts freely on S_0 and S_1; it follows that the $\mathbf{Z}[G]$-modules

$$L_0 = \mathbf{Z}[S_0] \quad \text{and} \quad L_1 = \mathbf{Z}[S_1]$$

are free of ranks $l_0 = \mathrm{Card}(G\backslash S_0)$ and $l_1 = \mathrm{Card}(G\backslash S_1)$. Th. 13 then follows from the more precise statement:

Theorem 13′. (a) *The l_i are finite* ($i = 0, 1$).

(b) *The* $\mathbf{Z}[G]$*-modules* $\mathrm{St} \oplus L_0$ *and* L_1 *are isomorphic.*

(c) *One has* $\chi(G) = l_0 - l_1$.

Proof of (a)

We first remark that, if a vertex x of X is G-stable, then its stabilizer in Γ has at most N elements, where $N = (\Gamma : G)$. But the structure of $\Gamma\backslash X$ given in no. 2.3 shows that such vertices are finite in number, modulo the action of Γ, hence modulo the action of G. This proves that $l_0 = \mathrm{Card}(G\backslash S_0)$ is finite; an analogous argument shows the finiteness of l_1.

Proof of (b)

We can interpret L_0 (resp. L_1) as the group of 0-dimensional (resp. 1-dimensional) chains of X modulo X_α :

$$L_0 = C_0(X \bmod X_\alpha), \qquad L_1 = C_1(X \bmod X_\alpha).$$

The exact sequence for relative homology gives:

$$0 \to H_1(X_\alpha) \to H_1(X) \to H_1(X \bmod X_\alpha) \to H_0(X_\alpha) \to H_0(X)$$
$$\to H_0(X \bmod X_\alpha) \to 0.$$

Since $H_1(X) = 0$ and $H_0(X) = \mathbf{Z}$ one gets that $H_0(X \bmod X_\alpha) = 0$ and one has an exact sequence

$$0 \to H_1(X \bmod X_\alpha) \to H_0(X_\alpha) \to \mathbf{Z} \to 0.$$

In other words, $H_1(X \bmod X_\alpha)$ may be identified with the kernel $\tilde{H}_0(X_\alpha)$ of the augmentation homomorphism $H_0(X_\alpha) \to \mathbf{Z}$; it is the "reduced 0th homology group" of the graph X_α.

Lemma 13. *One has* $H_0(X_\alpha) \simeq \mathbf{Z}[\mathbf{P}_1(K)]$ *and* $\tilde{H}_0(X_\alpha) \simeq \mathrm{St}$.

Assuming this lemma, and using the definition of relative homology, we have an exact sequence

$$0 \to H_1(X \bmod X_\alpha) \to C_1(X \bmod X_\alpha) \to C_0(X \bmod X_\alpha) \to H_0(X \bmod X_\alpha) \to 0,$$

which can be written

$$0 \to \mathrm{St} \to L_1 \to L_0 \to 0.$$

Since L_0 is free, this exact sequence *splits*, which proves (b).

Proof of lemma 13 .

If $x \in \mathrm{vert}(X_\alpha)$, the stabilizer G_x of x is a non-trivial finite p-subgroup of G, hence of $\mathbf{GL}_2(K)$. Such a subgroup fixes exactly one *line* D_x of the space $V = K^2$, i.e. it fixes exactly one *rational end* $b_x \in \mathbf{P}_1(K)$ of X. Let

$$\mathrm{Path}_x\colon \quad x \text{———} x_1 \text{———} x_2 \text{———} \cdots$$

denote the unique path without backtracking with origin x and end b_x. Since G_x fixes x and b_x, it also fixes Path_x; we thus have $\mathrm{Path}_x \subset X_\alpha$. The map $x \mapsto b_x$ induces a *bijection of the set of connected components of* X_α *onto* $\mathbf{P}_1(K)$:

(i) If x and x' are in the same connected component of X_α we have $b_x = b_{x'}$. Indeed, it suffices to prove this when x and x' are adjacent. Suppose then that y is the edge joining them. We have $G_y \neq \{1\}$, so G_y fixes exactly one point of $\mathbf{P}_1(K)$; but, since G_y is contained in G_x and $G_{x'}$, it fixes b_x and $b_{x'}$. We therefore have $b_x = b_{x'}$.

(ii) If x and x' are two vertices of X_∞ such that $b_x = b_{x'}$, then x and x' belong to the same connected component of X_∞. Indeed, the paths Path_x and $\mathrm{Path}_{x'}$ meet, because they define the same end; their union is connected, and contains x and x'.

(iii) Every rational end of X is of the form b_x. This can be checked, for example, by means of the description of $\Gamma\backslash X$ given in no. 2.3.

Lemma 13 is now clear, because $H_0(X_\infty)$ is the free abelian group whose basis is the set of connected components of X_∞.

Proof of (c)

The sum $(*)$ above, which defines $\chi(G)$, can be decomposed into its part relative to X_∞ and its part relative to $X \bmod X_\infty$. But if x (resp. y) is not in X_∞ we have $G_x = \{1\}$ (resp. $G_y = \{1\}$). The contribution of $X \bmod X_\infty$ to $\chi(G)$ is therefore

$$\mathrm{Card}(G\backslash S_0) - \mathrm{Card}(G\backslash S_1) = l_0 - l_1.$$

Thus we have to prove that *the contribution of X_∞ to $\chi(G)$ is zero*, i.e. that

$$\sum_{x \in \mathrm{vert}(G\backslash X_\infty)} 1/\mathrm{Card}(G_x) - \sum_{y \in \mathrm{geom\,edge}(G\backslash X_\infty)} 1/\mathrm{Card}(G_y) = 0.$$

To do this we shall prove that the terms on the left *cancel in pairs*: if x is a vertex of X_∞, we associate with it the geometric edge y_x which connects it to the next vertex x_1 of Path_x. Since G_x fixes Path_x, it fixes y_x; we thus have $G_x \subset G_{y_x}$, whence $G_x = G_{y_x}$, and the contributions of x and y_x to $\chi(G)$ cancel. It remains to check that *$x \mapsto y_x$ is a bijection of* $\mathrm{vert}(X_\infty)$ *onto* $\mathrm{geom\,edge}(X_\infty)$. To do this, let $y \in \mathrm{geom\,edge}(X_\infty)$, let b_y be the unique rational end fixed by G_y, and let Path_y be the smallest path without backtracking containing y and with end b_y. The origin x_y of this path is one of the two extremities of y (it is the extremity of y "furthest from b_y"). Thus we obtain a map $y \mapsto x_y$ of $\mathrm{geom\,edge}(X_\infty)$ into $\mathrm{vert}(X_\infty)$, and one checks that it is the inverse of the map $x \mapsto y_x$ defined above; this map is therefore a bijection, which completes the proof of (c).

Remark. This "contraction towards infinity" has been used for similar purposes by Quillen (unpublished) and Stuhler [21].

Interpretation of $\chi(G)$ in terms of relative homology

First recall what the *relative homology groups* of G modulo a non-empty family of subgroups G_σ are (for more details, see H. Trotter, *Ann. of Math.* 76, 1962): one defines a G-module R by the exact sequence

$$0 \to R \to \coprod \mathbf{Z}[G/G_\sigma] \to \mathbf{Z} \to 0$$

and, if M is a G-module, one puts:

$$H_i(G \bmod G_\sigma, M) = \mathrm{Tor}_{i-1}^{\mathbf{Z}[G]}(R, M) = H_{i-1}(G, R \otimes M).$$

The Tor exact sequence, together with Shapiro's lemma, gives an *exact sequence for relative homology*:

$$\cdots \to \coprod H_i(G_\sigma, M) \to H_i(G, M) \to H_i(G \bmod G_\sigma, M) \to \coprod H_{i-1}(G_\sigma, M) \to \cdots$$

We apply this by taking the G_σ to be the stabilizers of a system of representatives of $G\backslash \mathbf{P}_1(K)$, cf. the end of no. 2.8; because G has no p'-torsion, the G_σ can be characterized as the representatives of G-conjugation classes of the *maximal unipotent subgroups* of G. By construction, we have

$$\mathbf{Z}[\mathbf{P}_1(K)] = \coprod \mathbf{Z}[G/G_\sigma]$$

so that the module R above is none other than the module St of th. 13. We have

$$H_i(G \bmod G_\sigma, M) = \mathrm{Tor}_{i-1}^{\mathbf{Z}[G]}(\mathrm{St}, M)$$

for each G-module M. Hence:

Theorem 14. (a) *One has $H_i(G \bmod G_\sigma, M) = 0$ for $i \neq 1$.*

(b) *If M is finitely generated over $\mathbf{Z}$ one has*

$$H_1(G \bmod G_\sigma, M) \simeq M^{\mu(G)}, \quad \textit{where} \quad \mu(G) = -\chi(G).$$

The first assertion follows from the fact that

$$\mathrm{Tor}_{i-1}(\mathrm{St}, M) = 0 \quad \text{for} \quad i \neq 1,$$

since St is $\mathbf{Z}[G]$-projective.

To prove (b) we remark that, if we put $H(M) = H_1(G \bmod G_\sigma, M)$, we have

$$H(M) = \mathrm{St} \otimes_{\mathbf{Z}[G]} M,$$

whence by th. 13:

$$H(M) \oplus M^{l_0} \simeq M^{l_1}, \quad \text{with} \quad l_1 - l_0 = \mu(G).$$

Since M is finitely generated, a simple "invariant factor" argument shows that

$$H(M) \simeq M^{l_1 - l_0},$$

whence (b).

Corollary. *One has*

$$\chi(G) = \Sigma(-1)^i \operatorname{rank} H_i(G \bmod G_\sigma, \mathbf{Z}) = -\operatorname{rank} H_1(G \bmod G_\sigma, \mathbf{Z})$$
$$= \Sigma(-1)^i \dim H_i(G \bmod G_\sigma, \mathbf{Q}) = -\dim H_1(G \bmod G_\sigma, \mathbf{Q}).$$

Exercises

1) Show that, if two subgroups of finite index in Γ are isomorphic, they have the same Euler-Poincaré characteristic.

2) Make the same hypotheses on G as in ths. 13 and 14. Let $b(G)$ denote the first Betti number of $G\backslash X$, and $h(G)$ the number of elements of $G\backslash \mathbf{P}_1(K)$.

a) Show that $\chi(G) = 1 - b(G) - h(G)$.

b) Let $H_c^i(G\backslash X, \mathbf{Q})$ denote the ith group of cohomology with *compact supports* of the graph $G\backslash X$, with coefficients in $\mathbf{Q}$. Show that $H_c^i(G\backslash X, \mathbf{Q}) = 0$ for $i \neq 1$ and that $\dim H_c^1(G\backslash X, \mathbf{Q}) = -\chi(G)$.

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