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Nevanlinna's Theory of Value Distribution

The Second Main Theorem and its Error Terms

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Preface

The Fundamental Theorem of Algebra says that a polynomial of degree d in one complex variable will take on every complex value precisely d times, provided the values are counted with their proper multiplicities. Toward the end of the nineteenth century, Picard [Pic 1879] generalized the Fundamental Theorem of Algebra by proving that a transcendental entire function – a sort of polynomial of infinite degree – must take on all but at most one complex value infinitely many times. However, there are a great many infinities, and after Picard's work, mathematicians tried to distinguish between "different" infinite degrees.

In hindsight, one recognizes that the key to progress was viewing the degree of a complex polynomial $P(z)$ as the rate at which the maximum modulus of $P(z)$ approaches infinity as $|z| \to \infty$. A decade after Picard's theorem, work of J. Hadamard [Had 1892a], [Had 1892b], [Had 1897] proved there was a strong connection between the growth order of an entire function and the distribution of the function's zeros. E. Borel [Bor 1897] then proved a connection between the growth rate of the maximum modulus of an entire function and the asymptotic frequency with which it must attain all but at most one complex value. Finally, R. Nevanlinna [Nev(R) 1925] found the right way to measure the "growth" of a meromorphic function and developed the theory of value distribution which now bears his name, and which is the subject of this book.

In the late 1920's, Nevanlinna organized his theory into the monograph [Nev(R) 1929]. That theory, including more recent developments, also makes up a substantial part of his his later monograph [Nev(R) 1970]. Another classic monograph, unfortunately now out of print, that discusses the theory in detail is that of W. Hayman [Hay 1964]. Many of those who grew up in the Russian speaking part of the world learned the theory from the excellent book by A. A. Gol'dberg and I. V. Ostrovskiĭ [GoOs 1970], a book which for some reason has yet to be translated into English. One of the reasons we have decided to write the present book is that the above mentioned books, though far from obsolete, are now somewhat out of date, and with the exception of [Nev(R) 1970], are becoming harder to find. We hasten to point out though that each of the above books contains many interesting topics and ideas we do not touch on here, and that we definitely do not see our present work as a replacement for any of these "classics."

R. Nevanlinna's original proof [Nev(R) 1925] of his "Second Main Theorem," the focus of our book, makes heavy use of special properties of logarithmic derivatives. Almost immediately after R. Nevanlinna proved the Second Main Theorem, F. Nevanlinna, R. Nevanlinna's brother, gave a "geometric" proof of the Second Main Theorem – see [Nev(F) 1925], [Nev(F) 1927], and the "note" at the end of [Nev(R) 1929]. This geometric viewpoint was later expanded upon by T. Shimizu [Shim 1929] and L. Ahlfors [Ahlf 1929], [Ahlf 1935a], and still later by S.-S. Chern [Chern 1960]. Both the "geometric" and "logarithmic derivative" approaches to the Second Main Theorem has certain advantages and disadvantages, so we will treat both approaches in detail.

Despite the geometric investigation and interpretation of the main terms in Nevanlinna's theorems, the so-called "error term" in Nevanlinna's Second Main Theorem was, until recently, largely ignored. Motivated by an analogy between Nevanlinna theory and Diophantine approximation theory, discovered independently by C. F. Osgood [Osg 1985]and P. Vojta [Vojt 1987], S. Lang recognized that the careful study of the error term in Nevanlinna's Second Main Theorem would be of interest in itself. To promote its study, Lang wrote the lecture notes volume [Lang 1990], where in the introduction he wrote:

> "... it seemed to me useful to give a leisurely exposition which might lead people with no background in Nevanlinna theory to some of the basic problems which now remain about the error term. The existence of these problems and the possibly rapid evolution of the subject ... made me wary of writing a book, ..."

As Lang predicted, a flurry of activity in the investigation of the error term, much of it involving the second author of this book, developed after [Lang 1990] appeared. Now things have calmed down on this front, and the error term in Nevanlinna's Second Main Theorem is very well understood and full of interesting geometric meaning. What remains to be done probably requires fundamentally new ideas, and so we felt this was a good time to write a book collecting together the existing work on error terms.

We have included a small sampling of applications for Nevanlinna's theory because we feel some exposure for the reader to the myriad of possible applications is essential to the reader's aesthetic appreciation of the field. However, applications are not the emphasis of this work, and indeed, many of the applications we discuss are due to Nevanlinna himself and already present in his 1929 monograph [Nev(R) 1929]. We do describe in some detail how knowledge of explicit error terms in Nevanlinna's theory provides a new look at some old theorems, but we have left it to others to write up to date accounts of applications to such subjects as differential equations, complex dynamics, and unicity theorems. For a more in depth overview of applications, the reader may want to read the book of Jank and Volkmann [JaVo 1985], and those particularly interested in complex differential equations may want to look at I. Laine's book [Laine 1993].

Because the connection between Nevanlinna theory and Diophantine approximation has been the motivation for so much of the current work in Nevanlinna theory and because of the considerable cross-fertilization between the two fields, we have included several sections discussing this connection in some detail, although lacking complete proofs. These sections assume no background at all in number theory, and we hope these sections will whet the appetites of a few die hard analysts enough that they will seek out more advanced treatments such as [Vojt 1987].

While including the state of the art in error terms, we have retained Lang's philosophy of providing a leisurely introduction to the field to those with no background in Nevanlinna theory, and this book will be easily accessible to those with only a basic course in one complex variable. Some readers may feel a bit uncomfortable at times because we did not hesitate in our use of the language of differential forms. Readers having difficulty with this language may want to consult a book such as [Spiv 1965]. We have two reasons for using this language of differential forms. First, we believe it is the language that most clearly and efficiently conveys the ideas behind some of the geometric arguments, and second, learning this language is absolutely essential to learning the several variable theory, a topic we plan to address in a future volume.

Finally, the reader should be aware that our bibliography is by no means a complete guide to the literature in the vast field of value distribution theory. Rather, we have tried to cite references that give the reader a good sense of the historical origins of the main ideas around the Second Main Theorem, and we have tried to provide a guide to the most recent work on the Second Main Theorem's error terms. We made no attempt to provide a complete history of each idea from its birth to today's state of the art error terms. Thus, our bibliography omits many important works that have advanced the field over the years.

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A Word on Notation

We use the standard notation of $\mathbf{Z}$, $\mathbf{R}$, and $\mathbf{C}$ to denote the integers, real numbers, and complex numbers, respectively. We use the notation $\mathbf{Z}_{>0}$ to denote those integers that are > 0. We use $\mathbf{C}^n$, $\mathbf{R}^n$, *etc.* to denote the n-th Cartesian products of these spaces.

The closed interval $\{x \in \mathbf{R} : a \leq x \leq b\}$ is denoted $[a, b]$, and the open interval is denoted (a, b). Half-open intervals are denoted $(a, b]$ or $[a, b)$.

A function is called C^∞ if derivatives of all orders exist. If f is a function of several variables, then C^∞ means all partial derivatives exist. A function is called C^k if all (partial) derivatives of order $\leq k$ exist and are continuous.

If X is a set in $\mathbf{R}^n$, in $\mathbf{C}^n$, or in some other space, we denote the closure of X by $\overline{X}$.

We write complex numbers either as $z = x + yi$ or $z = x + y\sqrt{-1}$. As we often want to keep the letter i free for summation indices, we tend use the $\sqrt{-1}$ notation, though the i notation looks better in exponents. If $z = x + y\sqrt{-1}$ is a complex number, then we use $\bar{z} = x - y\sqrt{-1}$ to denote the complex conjugate. The real part x is denoted $\operatorname{Re} z$, and the imaginary part y is denoted $\operatorname{Im} z$. Note that if X is a multi-point subset of $\mathbf{C}$, then $\overline{X}$ will be used to denote the closure of X in $\mathbf{C}$ as defined in the last paragraph, and it does *not* denote the image of X under complex conjugation.

We often write $f \equiv 0$ when we want to say that a function f is the zero function, and similarly the notation $f \not\equiv 0$ means that f is not the constant function 0. Purists will argue that the proper notation for this is $f = 0$ and $f \neq 0$ respectively, but it sometimes happens that some authors write $f(z) \neq 0$ to mean f never takes on the value 0, whereas others use this to mean f does not vanish at the point z. Thus it is convenient to have the notation $f \not\equiv 0$ to avoid this confusion.

As do the physicists, we use the symbols $\ll$ and $\gg$ to mean "much less than" and "much greater than." Thus, $r \gg 0$, means for all r sufficiently large.

We use big and little "*oh*" notation throughout for asymptotic statements, though in a slightly non-standard way. For example, if $f(t)$ and $g(t)$ are real-valued functions of a real variable t and if $h(t)$ is a real-valued function which is positive as $t \to t_0$, meaning that $h(t)$ is positive for t sufficiently near t_0 (or sufficiently large if $t_0 = \infty$), then if we write $f(t) \leq g(t) + O(h(t))$, and we are interested in what happens asymptotically as $t \to t_0$, then we mean that

$$\limsup_{t\to t_0} \frac{f(t)-g(t)}{h(t)} < \infty.$$

When we write $f(t) \le g(t) + o(h(t))$, then we mean

$$\limsup_{t\to t_0} \frac{f(t)-g(t)}{h(t)} \le 0.$$

Usually when we write such asymptotics, it will be clear from the context what asymptotic value t_0 we are interested in, and we will not state this explicitly. Most often, we will mean as $t \to \infty$. If we write $f(t) = g(t) + O(h(t))$, then we mean that both $f(t) \le g(t) + O(h(t))$ and $g(t) \le f(t) + O(h(t))$, and similarly for the little "*oh*" notation. Often $h(t)$ will be taken to be the constant function 1. So for example, the statement $f(t) \le g(t) + O(1)$, means that $f(t) - g(t)$ is bounded above for t sufficiently near t_0 (sufficiently large in the typical case when $t_0 = \infty$). Occasionally, we may write something like $f(t) \le g(t) + O_\varepsilon(h(t))$ to emphasize that

$$\limsup_{t\to t_0} \frac{f(t)-g(t)}{h(t)}$$

is bounded by a constant that may depend on some external parameter ε.

Constants which do not especially interest us will often be called C or some other unfancy name. Thus, the symbol C stands for many different constants throughout the book, and may even change its meaning within a single proof. Constants that we find interesting or want to refer back to at a later point are given names with subscripts, such as $c_{\ell m}$, or β_1. All such "long-term" constants appear in the Glossary of Notation so that their definitions can be easily located.

Theorems, lemmas, propositions, and so forth are numbered by chapter and section. Thus, Theorem 2.3.1 would refer to the first theorem in the third section of Chapter 2.

Especially important equations are also numbered by chapter and section. Equations that we want to refer back to several times within the context of a single proof, but never again later, are labeled by $(*)$, $(**)$, *etc.* Thus, there are many equation $(*)$'s, and a reference to equation $(*)$ always refers to the most recently occurring equation $(*)$.

Contents

Introduction

One of the first theorems we learn as mathematics students is the the Fundamental Theorem of Algebra, which says that a degree d polynomial of one complex variable will have d complex zeros, provided that the zeros are counted with multiplicity. If $P(z)$ is a degree d polynomial, then

$$\max\{|P(re^{i\theta})| : 0 \leq \theta \leq 2\pi\}$$

grows essentially like r^d as $r \to \infty$. Therefore, we can rephrase the Fundamental Theorem of Algebra as follows: a non-constant polynomial in one complex variable takes on every finite value an equal number of times counting multiplicity, and that number is determined by the order of growth of the maximum modulus of the polynomial on the circle of radius r centered at the origin as $r \to \infty$. A good way to sum up value distribution theory, otherwise known as Nevanlinna theory, is by saying that the main theorems in value distribution theory are generalizations of the Fundamental Theorem of Algebra to holomorphic and meromorphic functions. The purpose of this book is to describe this theory for meromorphic functions on the complex plane $\mathbf{C}$, or more generally for functions meromorphic in a disc in $\mathbf{C}$.

Before we begin to be precise about how the Fundamental Theorem of Algebra can be extended to meromorphic functions, we make a few quick observations about the differences between polynomials and transcendental functions. First, we note that the entire function e^z takes on many values infinitely often but never takes on the value 0. Thus, the most naive generalization of the Fundamental Theorem of Algebra that one might imagine is not true for entire functions. Second, transcendental functions take on values an infinite number of times, so we cannot really speak of the total number of times that a function takes on a value. Since a function meromorphic on the entire complex plane can have only finitely many zeros inside any finite disc, what we can and will speak of instead is the rate at which the number of zeros inside a disc of radius r grows as the radius tends to infinity.

Given a meromorphic function f and a value $a \in \mathbf{C} \cup \{\infty\}$, Nevanlinna theory studies the relationship between three associated functions: $N(f,a,r)$, $m(f,a,r)$, and $T(f,r)$. We will wait until §1.2 to give precise definitions of these functions and content ourselves here with the following informal descriptions. The function $N(f,a,r)$ is called the "counting function" because it counts, as a logarithmic average, the number of times f takes on the value a in the disc of radius r. The function $m(f,a,r)$ is called the "mean-proximity function," and it measures the percentage of the circle of radius r where the value of the function f is close to the value a. The function $T(f,r)$ is called the "characteristic" or "height" function; it essentially measures the area on the Riemann sphere covered by the image of the disc of radius r under the mapping f and does not depend on the value a. If f is an entire function, then the characteristic function measures the growth of the maximum modulus of the function f. The characteristic function plays the same role in Nevanlinna Theory that the degree of a polynomial plays in the Fundamental Theorem of Algebra.

In Chapter 1 we define these "Nevanlinna functions" precisely, and we prove:

First Main Theorem *If f is a non-constant meromorphic function on $\mathbf{C}$ and a is a point in $\mathbf{C} \cup \{\infty\}$, then*

$$T(f,r) - m(f,a,r) - N(f,a,r) = O(1)$$

as $r \to \infty$.

Since $m(f,a,r) \geq 0$ and $T(f,r)$ does not depend on a, the First Main Theorem says that the function f cannot take on the value a too often in the sense that the frequency with which f takes on the value a cannot be so high that the function $N(f,a,r)$ grows faster than $T(f,r)$. This is analogous to the statement that a polynomial of degree d takes on every value a *at most* d times. Actually, the First Main Theorem says even more in that it says the sum $m(f,a,r) + N(f,a,r)$ must be essentially independent of a. Thus, if f takes on the value a with a small enough frequency that $N(f,a,r)$ does not grow as fast as $T(f,r)$, then the function $m(f,a,r)$ must compensate, meaning that the image of f stays near the value a for sufficiently large arcs on large circles centered at the origin.

The subject of the later chapters is the deeper:

Second Main Theorem *If f is a non-constant meromorphic function on $\mathbf{C}$, and $a_1, \dots, a_q$ are distinct points in $\mathbf{C} \cup \{\infty\}$, then*

$$(q-2)T(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{ram}}(f,r) \leq o(T(f,r))$$

for a sequence of $r \to \infty$.

In fact, we will prove stronger and more precise forms of this theorem. The term $N_{\mathrm{ram}}(f,r)$ is positive (at least when $r \geq 1$) and measures how often the function f is ramified. Thus, the Second Main Theorem provides a lower bound on the sum of any finite collection of counting functions $N(f,a_j,r)$ for certain arbitrarily large radii r. Thus, taken together with the First Main Theorem, which provides an upper bound for the counting functions, we have our generalization of the Fundamental Theorem of Algebra.

The key results presented in this book, namely the First and Second Main Theorems, are essentially due to R. Nevanlinna [Nev(R) 1925] and F. Nevanlinna [Nev(F) 1927]. The Nevanlinnas gave precise, though not "best possible," estimates for the function implicit in the $o(T(f,r))$ on the right of the inequality in the Second Main Theorem. This function is known as the "error term." The fine structure of the error term was not considered especially interesting until relatively recently. In this book, we pay careful attention to the error terms, and the error terms we give are better than those obtained by the Nevanlinna brothers.

In many ways our presentation of this material resembles those of Nevanlinna [Nev(R) 1970], Hayman [Hay 1964], and Lang [Lang 1990]. Those familiar with the Russian work by A. A. Gol'dberg and I. V. Ostrovskiĭ [GoOs 1970] or the German work by Jank and Volkmann [JaVo 1985] will also recognize their influence on our presentation. The primary manner in which our presentation of the Second Main Theorem differs from those of Nevanlinna and Hayman is that we are careful in our treatment of the error terms, and of what is known as the "exceptional set."

There are several fundamentally different ways of proving the Second Main Theorem. Each approach has its own advantages and disadvantages, and the Second Main Theorems that result from each approach are not quite the same.

Chapter 2 contains the first of two fundamentally different proofs we will give for the Second Main Theorem. This proof is based on "negative curvature," a technique introduced by F. Nevanlinna into the theory almost from the very beginning [Nev(F) 1925], [Nev(F) 1927]. The error term estimate we give in Chapter 2 is essentially the work of P.-M. Wong [Wong 1989], and our presentation is quite similar to Chapter I of [Lang 1990]. In §2.6 we incorporate the work of Z. Ye [Ye 1991] in order to simplify the appearance of Wong's error term. Finally, in §2.8, we discuss how the Second Main Theorem is uniform over families of functions, and we are not aware of any other similar treatment.

In Chapter 3, we undertake a detailed study of Logarithmic derivatives. The main result of that chapter is:

Gol'dberg-Grinshtein Inequality *If f is a meromorphic function on $\mathbf{C}$, then for any r and ρ with $1 \leq r < \rho < \infty$,*

$$m(f'/f,\infty,r) \;\leq\; \max\left\{0, \log\frac{\rho(2T(f,\rho)+\beta_1)}{r(\rho-r)} + c_{gg}\right\}.$$

Here β_1 and c_{gg} are constants that will be defined in Chapter 3.

In Chapter 4, using ideas from Ye's refinement of Cartan's approach to the Second Main Theorem [Cart 1933], as in [Ye 1995], we use the precise estimate on $m(f'/f,r,\infty)$ provided by the Gol'dberg-Grinshtein estimates in Chapter 3 to prove the Second Main Theorem and give a careful analysis of the error term and exceptional set obtained by this method.

There is a third altogether different approach to proving the Second Main Theorem, initiated by A. Eremenko and M. Sodin [ErSo 1991]. This relatively new approach might be said to be a potential theoretic approach. Although this newer approach to the subject is very interesting and important, we have chosen not to discuss this method in this volume because, as yet, no one has succeeded in proving a version of the Second Main Theorem that includes the ramification term by this potential theoretic approach, nor has a detailed analysis of the error term coming from this approach been undertaken.

Although the applications of Nevanlinna's theory are not our focus, we feel that some detailed discussion of applications is essential in order that those readers new to the subject can gain an appropriate aesthetic and utilitarian appreciation for what often appears to be a very technical and obscure subject. Therefore, we have included a lengthy chapter, Chapter 5, on applications. We chose our applications only to illustrate the utility of the theory, and thus we made no effort to choose the most current or refined application of any given type. Our chosen applications are for the most part as in [Hay 1964] and [Nev(R) 1970], and many of them are due to R. Nevanlinna himself. In fact, those readers looking for more refined applications than those we give in Chapter 5 might well start by reading Nevanlinna's first monograph [Nev(R) 1929]. Our applications chapter does differ somewhat though from past treatments in that we have put some emphasis on what can be gained from precise knowledge of the error terms, something ignored by most previous authors.

The material covered in Chapter 7, where we discuss the sharpness of the error terms, the precise error terms of some classical special functions, and improvements to the error term for functions with restricted growth, has, up till now, only been available in the research literature.

The connection between Nevanlinna theory and Diophantine approximation theory has inspired a lot of recent work in both fields, and we have therefore including several sections, referred to as "number theoretic digressions," explaining and motivating in some detail this connection. We assume no background in number theory in these sections and hope that these sections will encourage even the most pure analysts to further explore this fascinating connection.

Given our emphasis on the Second Main Theorem and its error terms, we have not discussed anywhere in this volume the various other techniques for studying the distribution of values of meromorphic and entire functions. We instead refer the reader to the recent book [Rub 1996] for an introduction to some of these other techniques.

1 The First Main Theorem

As we mentioned in the introduction, the basis for Nevanlinna's theory are his two "main" theorems. This chapter discusses the first and easier of the two.

To understand where we are headed, looking at the concrete example of the transcendental entire function e^z is again helpful. We pointed out that e^z takes on all values other than zero infinitely often. Notice also that by periodicity each of these non-zero values is also taken on with the same asymptotic frequency. Of course e^z never attains the value 0; nor does it attain the value ∞. On the other hand, on every large circle centered at the origin, the function e^z spends most of its time close to one of these two omitted values. That is, fixing a small ε, the percentage of the circle of radius r where e^z is either smaller than ε or larger than $1/\varepsilon$ tends to 100% as the radius of the circle tends to infinity.

Nevanlinna's First Main Theorem will tell us that this behavior is typical. That is, if a meromorphic function takes on a particular value less often than "expected," then it must compensate for this by spending a lot of time "near" that value. As a consequence, Nevanlinna's First Main Theorem gives an upper bound (in terms of the growth of the function) on how often a meromorphic function can attain any value. This is analogous to the statement that a polynomial of degree d can take on any value at most d times. Note this last statement about polynomials is much easier to prove than the full Fundamental Theorem of Algebra. It is thus no surprise that Nevanlinna's First Main Theorem is much easier than his Second Main Theorem, which says that most values are taken on by a meromorphic function with the maximum asymptotic frequency allowed by the First Main Theorem.

1.1 The Poisson-Jensen Formula

We begin this chapter with this brief section which recalls the Poisson Formula for harmonic functions and applies this to the logarithm of the modulus of an analytic function to derive what is known as the "Poisson-Jensen Formula." The material in this first section is usually covered in a first course in complex analysis, and we could have chosen to regard the material in this section as a prerequisite. On the other hand, the Poisson Formula is at the very heart of Nevanlinna's theory of value distribution, and the First Main Theorem of Nevanlinna theory is really nothing other than the Poisson-Jensen formula dressed up in new notation. Thus, we felt a detailed treatment of this material here would be of value to the reader.

We begin with some general notation. We will use $\mathbf{C}$ to denote the complex plane. We will use r and R to denote positive real numbers, which will usually be radii of discs, we will always have $r < R$, and we will also allow $R = \infty$, whenever this makes sense. We will use $\mathbf{D}(r)$ and $\mathbf{D}(R)$ to denote open discs of radius r and R, respectively, each centered at the origin. When R is allowed to be infinite, $\mathbf{D}(R)$ will refer to the entire complex plane $\mathbf{C}$. We will consider the value distribution of meromorphic functions on $\mathbf{D}(R)$, and it will be convenient to consider a meromorphic function as a holomorphic map to what is known as the Riemann sphere or the projective line $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\} = \mathbf{P}^1$. To be more consistent with the higher dimensional theory, we prefer to refer to the Riemann sphere as the projective line $\mathbf{P}^1$.

If $R < \infty$, we will use $\overline{\mathbf{D}(R)}$ to denote the closure of the disc $\mathbf{D}(R)$ in the complex plane. If we say a function is harmonic, respectively holomorphic, or respectively meromorphic on the closed disc $\overline{\mathbf{D}(R)}$, then we mean that function should be harmonic, respectively holomorphic, or respectively meromorphic on some open neighborhood of $\overline{\mathbf{D}(R)}$.

We will now recall the Poisson and Poisson-Jensen Formulas. Let

$$\boxed{P(z,\zeta) = \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2} = \operatorname{Re}\left\{\frac{\zeta + z}{\zeta - z}\right\}.}$$

The function $P(z,\zeta)$ is called the **Poisson kernel**. Note that for fixed ζ and for $|z| < |\zeta|$,

$$\frac{\zeta + z}{\zeta - z}$$

is holomorphic, and hence $P(z,\zeta)$ being the real part of a holomorphic function is harmonic in z.

We being by showing how the Poisson kernel transforms under a Möbius automorphism of a disc.

Proposition 1.1.1 *Let*

$$T(w) = \frac{R^2(z - w)}{R^2 - \bar{z}w}.$$

Then, $T(w)$ is an automorphism of $\overline{\mathbf{D}(R)}$ that interchanges 0 and z. Moreover, when $|T(w)| = R$, then

$$\frac{dw}{2\pi i w} = P(z, T(w))\,\frac{d\zeta}{2\pi i \zeta},$$

where $P(z,\zeta)$ is the Poisson kernel and $\zeta = T(w)$.

Proof. One easily verifies that $T(w)$ is holomorphic on $\overline{\mathbf{D}(R)}$, and that when $|w| = R$, then $|T(w)| = R$. Hence, T is an automorphism of $\overline{\mathbf{D}(R)}$ that interchanges 0 and z.

Noting that T is its own inverse in the group of linear fractional transformations, we have that

$$w = T(\zeta) = \frac{R^2(z - \zeta)}{R^2 - \bar{z}\zeta}.$$

We then compute

$$\frac{dw}{2\pi i w} = \frac{1}{2\pi i}\left(\frac{-1}{z - \zeta} + \frac{\bar{z}}{R^2 - \bar{z}\zeta}\right)d\zeta = \left(\frac{-\zeta}{z - \zeta} + \frac{\bar{z}\zeta}{R^2 - \bar{z}\zeta}\right)\frac{d\zeta}{2\pi i \zeta}.$$

Note that when $|\zeta|^2 = R^2$,

$$\frac{-\zeta}{z - \zeta} + \frac{\bar{z}\zeta}{R^2 - \bar{z}\zeta} = \frac{-\zeta}{z - \zeta} + \frac{\bar{z}\zeta}{\zeta(\bar{\zeta} - \bar{z})} = \frac{R^2 - |z|^2}{|\zeta - z|^2} = \operatorname{Re}\left\{\frac{\zeta + z}{\zeta - z}\right\}. \quad \square$$

Our first theorem, called the Poisson Formula, allows us to express a harmonic function in a disc as an integral around the boundary circle.

Theorem 1.1.2 (Poisson Formula) *Let u be harmonic in the open disc $\mathbf{D}(R)$, $R < \infty$ and continuous on the closed disc $\overline{\mathbf{D}(R)}$. Let z be a point inside the disc. Then*

$$u(z) = \int_0^{2\pi} u\left(Re^{i\theta}\right)P(z, Re^{i\theta})\,\frac{d\theta}{2\pi},$$

where $P(z,\zeta)$ denotes the Poisson kernel.

Proof. First we will prove the theorem in the case that u is harmonic on the closed disc $\overline{\mathbf{D}(R)}$. Let $\zeta = T(w)$ as in Proposition 1.1.1. Because T is holomorphic on $\overline{\mathbf{D}(R)}$, we know $u(T(w))$ is harmonic on $\overline{\mathbf{D}(R)}$. By the mean value property of harmonic functions,

$$u(z) = u(T(0)) = \int_0^{2\pi} u\left(T\left(Re^{i\theta}\right)\right)\frac{d\theta}{2\pi} = \int_{|w|=R} u(T(w))\frac{dw}{2\pi i w}.$$

Letting $\zeta = Re^{i\theta}$, using Proposition 1.1.1, and recalling that $\frac{d\zeta}{i\zeta} = d\theta$,

$$u(z) = \int_{|w|=R} u(T(w))\,\frac{dw}{2\pi i w} = \int_0^{2\pi} u\left(Re^{i\theta}\right)P(z, Re^{i\theta})\,\frac{d\theta}{2\pi}.$$

In the case that u is not harmonic in a neighborhood of $\overline{\mathbf{D}(R)}$, all we need do is note that for $\rho < 1$, $u(\rho w)$ is harmonic on $\overline{\mathbf{D}(R)}$, and we can use what we have already proven to conclude that

$$u(\rho z) = \int_0^{2\pi} u\left(\rho Re^{i\theta}\right)\frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2}\,\frac{d\theta}{2\pi}.$$

The result follows by noting that $u(\rho w) \to u(w)$ uniformly on $|w| = R$ as $\rho \to 1$. $\square$

Corollary 1.1.3 *Let $P(z,\zeta)$ denote the Poisson kernel and let $R > 0$. Then*

$$\int_0^{2\pi} P(z, Re^{i\theta})\,\frac{d\theta}{2\pi} = 1.$$

Proof. Apply Theorem 1.1.2 to the constant function $u(z) \equiv 1$. □

We can also use the Poisson kernel to create harmonic functions in a disc if we are just given a continuous function on the boundary circle. This is known as "solving the Dirichlet Problem for the disc." To prove this, we need the following estimate.

Proposition 1.1.4 *Let $P(z,\zeta)$ denote the Poisson kernel. Given ζ_0 such that $|\zeta_0| > 0$, given $\varepsilon > 0$, and given $\delta > 0$, there exists a δ' such that for all ζ with $|\zeta - \zeta_0| \geq \delta$ and $|\zeta| = |\zeta_0|$, and all z with $0 < |z - \zeta_0| < \delta'$, we have $|P(z,\zeta)| \leq \varepsilon$.*

Proof. We have for z near ζ_0 and $|\zeta - \zeta_0| \geq \delta$ that

$$P(z,\zeta) = \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2} = \frac{|\zeta_0|^2 - |z|^2}{|(\zeta - \zeta_0) - (z - \zeta_0)|^2} \leq \frac{|\zeta_0|^2 - |z|^2}{(\delta - |z - \zeta_0|)^2}.$$

The term on the right clearly tends to zero as $z \to \zeta_0$. □

Theorem 1.1.5 (Solution to the Dirichlet Problem) *Let $\phi(Re^{i\theta})$ be a continuous function of θ for a fixed $R < \infty$. Then,*

$$u(z) \stackrel{\text{def}}{=} \int_0^{2\pi} \phi(Re^{i\theta}) P(z, Re^{i\theta})\,\frac{d\theta}{2\pi}$$

is a harmonic function of z in $\mathbf{D}(R)$, and for each θ_0

$$\lim_{z \to Re^{i\theta_0}} u(z) = \phi(Re^{i\theta_0}).$$

Proof. To check that u is harmonic, we differentiate under the integral sign with respect to z, which we can do because ϕ is continuous in θ and so are P and its z-derivatives since $|z| < R$. Because P is harmonic in z, so is u.

It remains to check that

$$\lim_{z \to Re^{i\theta_0}} u(z) = \phi(Re^{i\theta_0}).$$

We have by Corollary 1.1.3

$$\begin{aligned} |u(z) - \phi(Re^{i\theta})| &= \left| \int_0^{2\pi} (\phi(Re^{i\theta}) - \phi(Re^{i\theta_0}))\, P(z, Re^{i\theta})\,\frac{d\theta}{2\pi} \right| \\ &\leq \int_0^{2\pi} |\phi(Re^{i\theta}) - \phi(Re^{i\theta_0})|\, P(z, Re^{i\theta})\,\frac{d\theta}{2\pi}, \end{aligned}$$

since $P(z,\zeta)$ is always positive. Let $\varepsilon > 0$. By the continuity of ϕ, there exists a $\delta > 0$ such that $|\phi(Re^{i\theta}) - \phi(Re^{i\theta_0})| < \varepsilon$ for those θ with $|Re^{i\theta_0} - Re^{i\theta}| < \delta$. Hence

$$\int_{\{\theta : |Re^{i\theta} - Re^{i\theta_0}| < \delta\}} |\phi(Re^{i\theta}) - \phi(Re^{i\theta_0})|\, P(z, Re^{i\theta})\,\frac{d\theta}{2\pi} \leq \int_0^{2\pi} \varepsilon P(z, Re^{i\theta})\,\frac{d\theta}{2\pi} = \varepsilon.$$

On the other hand, from Proposition 1.1.4, there exists a δ' such that if

$$0 < |z - Re^{i\theta_0}| < \delta' \quad \text{and} \quad |Re^{i\theta_0} - Re^{i\theta}| \geq \delta,$$

then $|P(z, Re^{i\theta})| < \varepsilon$. Thus, for these z,

$$\begin{aligned} &\int_{\{\theta : |Re^{i\theta} - Re^{i\theta_0}| \geq \delta\}} |\phi(Re^{i\theta}) - \phi(Re^{i\theta_0})|\, P(z, Re^{i\theta})\,\frac{d\theta}{2\pi} \\ &\qquad \leq \varepsilon \int_0^{2\pi} |\phi(Re^{i\theta}) - \phi(Re^{i\theta_0})|\,\frac{d\theta}{2\pi}. \end{aligned}$$

The proof is completed by noting that

$$\int_0^{2\pi} |\phi(Re^{i\theta}) - \phi(Re^{i\theta_0})|\,\frac{d\theta}{2\pi}$$

is bounded by the continuity of ϕ. □

If f is analytic and free from zeros, then $\log|f|$ is harmonic, and we can therefore apply the Poisson Formula to it. However, since we are interested in the distribution of the zeros of f, we will want to see what happens in the case that f has zeros. This is what is called the "Poisson-Jensen" Formula.

Before giving the precise statement of the Poisson-Jensen Formula, we introduce some additional notation. Given a non-constant meromorphic function f and a point a in the complex plane, we can always write $f(z) = (z-a)^m g(z)$, where m is an integer, and g is analytic and non-vanishing in a neighborhood of a. The integer m is called the **order** of f at the point a and is denoted $\mathrm{ord}_a f$. Note that $\mathrm{ord}_a f > 0$ if and only if f has a zero at a, and $\mathrm{ord}_a f < 0$ if and only if f has a pole at a. We will refer to the non-zero complex number $g(a)$ as the **initial Laurent coefficient** of f at a, denoted $\mathrm{ilc}(f,a)$, because it is the first non-vanishing coefficient of the Laurent expansion of f expanded about a.

Theorem 1.1.6 (Poisson-Jensen Formula) *Let $f \not\equiv 0, \infty$ be meromorphic on $\overline{\mathbf{D}(R)}$. Let $a_1, \ldots, a_p$ denote the zeros of f in the open disc $\mathbf{D}(R)$, each zero repeated according to its multiplicity, and let $b_1, \ldots, b_q$ denote the poles of f in*

$\mathbf{D}(R)$, *also repeated according to multiplicity. For any* z *in* $\mathbf{D}(R)$ *which is not a zero or pole of* f, *we have*

$$\begin{aligned}\log|f(z)| &= \int_0^{2\pi} \frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\log|f(Re^{i\theta})|\frac{d\theta}{2\pi} \\ &\quad - \sum_{i=1}^{p}\log\left|\frac{R^2-\bar{a}_i z}{R(z-a_i)}\right| + \sum_{i=1}^{q}\log\left|\frac{R^2-\bar{b}_i z}{R(z-b_i)}\right| \\ &= \int_0^{2\pi}\frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\log|f(Re^{i\theta})|\frac{d\theta}{2\pi} \\ &\quad - \sum_{\zeta\in\mathbf{D}(R)}(\mathrm{ord}_\zeta f)\log\left|\frac{R^2-\bar{\zeta}z}{R(z-\zeta)}\right|.\end{aligned}$$

If $z=0$, *this simplifies to*

$$\begin{aligned}\log|f(0)| &= \int_0^{2\pi}\log|f(Re^{i\theta})|\frac{d\theta}{2\pi} - \sum_{i=1}^{p}\log\left|\frac{R}{a_i}\right| + \sum_{i=1}^{q}\log\left|\frac{R}{b_i}\right| \\ &= \int_0^{2\pi}\log|f(Re^{i\theta})|\frac{d\theta}{2\pi} - \sum_{\zeta\in\mathbf{D}(R)}(\mathrm{ord}_\zeta f)\log\left|\frac{R}{\zeta}\right|.\end{aligned}$$

Corollary 1.1.7 *Let* $R, f, a_1,\ldots,a_p,$ *and* $b_1,\ldots,b_q$ *be as in Theorem 1.1.6. Then, for any* z *which is not a zero or pole of* f,

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \int_0^{2\pi}\frac{2Re^{i\theta}}{(Re^{i\theta}-z)^2}\log|f(Re^{i\theta})|\frac{d\theta}{2\pi} \\ &\quad + \sum_{i=1}^{p}\left(\frac{\bar{a}_i}{R^2-\bar{a}_i z}+\frac{1}{z-a_i}\right) - \sum_{i=1}^{q}\left(\frac{\bar{b}_i}{R^2-\bar{b}_i z}+\frac{1}{z-b_i}\right) \\ &= \int_0^{2\pi}\frac{2Re^{i\theta}}{(Re^{i\theta}-z)^2}\log|f(Re^{i\theta})|\frac{d\theta}{2\pi} \\ &\quad + \sum_{\zeta\in\mathbf{D}(R)}(\mathrm{ord}_\zeta f)\left(\frac{\bar{\zeta}}{R^2-\bar{\zeta}z}+\frac{1}{z-\zeta}\right).\end{aligned}$$

Proof of Corollary 1.1.7. Note that

$$\frac{f'(z)}{f(z)} = \frac{\partial}{\partial z}\log f(z) = 2\frac{\partial}{\partial z}\log|f(z)|,$$

where the last equality follows from the Cauchy-Riemann equations since

$$\log|f(z)| = \frac{1}{2}[\log f(z) + \log\overline{f(z)}].$$

The corollary then follows by differentiating the equation in the theorem and moving the derivative inside the integral. □

Corollary 1.1.8 *Let* $f \not\equiv 0,\infty$ *be meromorphic on* $\overline{\mathbf{D}(R)}$. *Let* $a_1,\ldots,a_p$ *denote the non-zero zeros of* f *in* $\mathbf{D}(R)$, *repeated according to multiplicity, and let* $b_1,\ldots,b_q$ *denote the non-zero poles of* f *in* $\mathbf{D}(R)$, *repeated according to multiplicity. Then,*

$$\begin{aligned}\log|\mathrm{ilc}(f,0)| &= \int_0^{2\pi}\log|f(Re^{i\theta})|\frac{d\theta}{2\pi} \\ &\quad - \sum_{i=1}^{p}\log\left|\frac{R}{a_i}\right| + \sum_{i=1}^{q}\log\left|\frac{R}{b_i}\right| - (\mathrm{ord}_0 f)\log R \\ &= \int_0^{2\pi}\log|f(Re^{i\theta})|\frac{d\theta}{2\pi} - \sum_{\substack{\zeta\in\mathbf{D}(R)\\ \zeta\neq 0}}(\mathrm{ord}_\zeta f)\log\left|\frac{R}{\zeta}\right| \\ &\quad - (\mathrm{ord}_0 f)\log R. \end{aligned} \tag{1.1.9}$$

Remark. Equation (1.1.9) in Corollary 1.1.8 as well as the second equation in Theorem 1.1.6 are referred to simply as the **Jensen Formula**.

Proof of Corollary 1.1.8. Simply apply Theorem 1.1.6 (the Poisson-Jensen Formula) to the function $f(w)w^{-\mathrm{ord}_0 f}$. □

Proof of Theorem 1.1.6. First of all, note that it suffices to prove the theorem when f has no zeros or poles on the circle of radius R. Indeed, because zeros and poles on the circle of radius R do not cause the integrals on the right hand sides of the formulas to diverge, we can consider the function $f(\rho w)$, which will not have any zeros on the circle of radius R for all ρ sufficiently close to, but less than 1. The theorem follows from this case by letting $\rho \to 1$ and Lebesgue dominated convergence.

Now we assume f has no zeros or poles on the circle of radius R. Consider the function

$$F(w) = f(w)\frac{\prod_{i=1}^{p}\left(\frac{R^2-\bar{a}_i w}{R(w-a_i)}\right)}{\prod_{i=1}^{q}\left(\frac{R^2-\bar{b}_i w}{R(w-b_i)}\right)}.$$

We have chosen F so that it has no zeros or poles in the closure of $\mathbf{D}(R)$ and so that $|F(w)| = |f(w)|$ when $|w| = R$. The function $\log|F(w)|$ is therefore harmonic in an open neighborhood of $\overline{\mathbf{D}(R)}$. The proof of the theorem is completed by applying the Poisson Formula to $\log|F|$. □

If we look closely at equation (1.1.9), we can start to see our generalization of the Fundamental Theorem of Algebra. The left-hand side of equation (1.1.9) is just a constant. The right-hand side of the equation has two different types of terms. The first term on the right is an integral over the circle of radius R involving the absolute value of f. The other term is a sum over the zeros and poles of f. If f is holomorphic or meromorphic on the whole plane $\mathbf{C}$, then as $R \to \infty$, the left-hand side of this equation does not change, so that means although all the terms on the right-hand side might tend to infinity, if they do, then they always essentially cancel each other out.

1.2 The Nevanlinna Functions

At the end of the last section, we noted that equation (1.1.9) can be viewed as the first step toward generalizing the Fundamental Theorem of Algebra to meromorphic functions. In this section, we explore this in more detail. We begin by defining the Nevanlinna functions and then state the first fundamental relationship between them, known as the "First Main Theorem" of Nevanlinna theory. Although the First Main Theorem is really just a restatement of the Jensen Formula (Corollary 1.1.8), it is this formulation which makes clear that Jensen's Formula is a weak generalization of the Fundamental Theorem of Algebra.

As we proceed to define the Nevanlinna functions, f will always be a meromorphic function on $\mathbf{D}(R)$, where R might be ∞, and $r < R$. *Note that most of our definitions will not make any sense for certain constant functions f, and thus we exclude any such constant functions from consideration.*

Counting Functions

First, we define the **counting functions.** The **unintegrated counting function** $n(f,\infty,r)$ simply counts the number of poles the function f has on the closed disc $\overline{\mathbf{D}(r)}$, each pole counted according to its multiplicity. Note that because f is meromorphic on some larger disc, this number is always finite. If a is a complex number, we define $n(f,a,r)$ to be

$$\boxed{n(f,a,r) = n\left(\frac{1}{f-a},\infty,r\right),}$$

which is simply the number of times f takes on the value a on $\overline{\mathbf{D}(r)}$. We choose to count the a-points of f in the closed disc $\overline{\mathbf{D}(r)}$ instead of in the open disc $\mathbf{D}(r)$ because we can then use $n(f,0,r)$ to denote the multiplicity with which f takes on the value a at $z=0$.

We then define the **integrated counting function** $N(f,a,r)$ by

$$\boxed{N(f,a,r) = n(f,a,0)\log r + \int_0^r [n(f,a,t) - n(f,a,0)]\frac{dt}{t}.}$$

So, in particular,

$$N(f,0,r) = (\mathrm{ord}_0^+ f)\log r + \sum_{\substack{z\in\mathbf{D}(r)\\ z\neq 0}} (\mathrm{ord}_z^+ f)\log\left|\frac{r}{z}\right|,$$

where $\mathrm{ord}_z^+ f = \max\{0, \mathrm{ord}_z f\}$ is just the multiplicity of the zero. Note that the points z on the circle $|z| = r$ where $f(z) = 0$ do not contribute to $N(f,0,r)$ since in that case $\log|r/z| = 0$. Thus we see that the integrated counting function is very similar to the terms in the right hand side of equation (1.1.9). In fact, with this new notation, we can rewrite Corollary 1.1.8 as

Corollary 1.2.1 *Let $f \not\equiv 0,\infty$ be meromorphic on $\overline{\mathbf{D}(r)}$. Then,*

$$\log|\mathrm{ilc}(f,0)| = \int_0^{2\pi} \log|f(re^{i\theta})|\frac{d\theta}{2\pi} + N(f,\infty,r) - N(f,0,r).$$

The unintegrated counting function $n(f,a,r)$ of course appears to be the more "natural" of the two counting functions to consider, and those new to the field often wonder why $N(f,a,r)$ is used so often instead of $n(f,a,r)$. One reason is that it is N and not n that naturally appears in Corollary 1.2.1. Another advantage of N over n is that it is a continuous function of r. We will say a bit more on this in §1.11.

We will also have occasion to discuss **truncated counting functions.** We let $n^{(k)}(f,\infty,r)$ denote the number of poles f has in the closed disc of radius r, but this time we only count each pole with a maximum multiplicity of k. In other words, if z_0 is a pole of f in the disc with multiplicity $\leq k$, then it is counted according to its multiplicity. If, however, z_0 is a pole of f in the disc with multiplicity greater than k, then we count z_0 as if it only had multiplicity k. We similarly define $n^{(k)}(f,a,r)$. Note in particular that $n^{(1)}(f,a,r)$ counts the number of times f takes on a without counting multiplicity at all. As the reader might now expect, the integrated truncated counting function $N^{(k)}(f,a,r)$ is then defined by

$$N^{(k)}(f,a,r) = n^{(k)}(f,a,0)\log r + \int_0^r [n^{(k)}(f,a,t) - n^{(k)}(f,a,0)]\frac{dt}{t}.$$

Mean Proximity Functions

We will now explain the significance of the integral term in Corollary 1.2.1. To begin with, it is useful to introduce the notion of a Weil function. Given a point a in $\mathbf{P}^1$, a **Weil function** with singularity at a is a continuous map

$$\lambda_a : \mathbf{P}^1 \setminus \{a\} \to \mathbf{R}$$

which has the property that in some open neighborhood U of a in $\mathbf{P}^1$, there is a continuous function α on U such that

$$\lambda_a(z) = -\log|z-a| + \alpha(z),$$

where z is a holomorphic local coordinate on U. Thus, Weil functions are almost continuous functions, except that they have a certain specified logarithmic singularity at the point a. Notice that the difference between any two Weil functions with the same singular point will be a continuous function on $\mathbf{P}^1$ and will therefore be bounded since $\mathbf{P}^1$ is compact. Also notice that for a meromorphic function $f(z)$ on $\mathbf{D}(R)$, the composition of f with a Weil function $\lambda_a(f(z))$ is large precisely when $f(z)$ is close to a. Given a Weil function λ_a, one defines a **mean proximity function**

$$m(f,\lambda_a,r) = \int_0^{2\pi} \lambda_a(f(re^{i\theta}))\,\frac{d\theta}{2\pi}.$$

The mean proximity function measures how close f is, on average, to a on the circle of radius r. Note that if we take two different Weil functions with the same singularity a, then the mean proximity functions we get from each Weil function will differ by a bounded amount as $r \to R$.

One has many choices for Weil functions, but many things do not depend on the specific choice of Weil function. Two specific types of Weil functions come up often in Nevanlinna theory, each type having its own advantages. R. Nevanlinna used the following Weil functions in his first proof of the his main theorems.

$$\begin{aligned} \lambda_a(z) &= \log^+ \frac{1}{|z-a|} && \text{if } a, z \neq \infty \\ \lambda_a(\infty) &= 0 && \text{if } a \neq \infty \\ \lambda_a(z) &= \log^+ |z| && \text{if } a = \infty, \end{aligned}$$

where for a positive real number x, $\log^+ x = \max\{0, \log x\}$. Hence, we define

$$\boxed{\begin{aligned} m(f,a,r) &= \int_0^{2\pi} \log^+ \left|\frac{1}{f(re^{i\theta})-a}\right| \frac{d\theta}{2\pi} \quad a \neq \infty \quad \text{and} \\ m(f,\infty,r) &= \int_0^{2\pi} \log^+ |f(re^{i\theta})|\,\frac{d\theta}{2\pi}. \end{aligned}}$$

When looking at things from an "analytic" viewpoint, we will find it convenient to use Nevanlinna's choice of Weil function as above, but when we wish to look at things "geometrically," we will prefer a different choice of Weil function, which we now describe.

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ be two points in $\mathbf{C}$. If we use stereographic projection to identify the complex plane with the sphere of radius $1/2$ centered at the origin in $\mathbf{R}^3$, minus the north pole, and if $(\rho_j, \theta_j, \zeta_j)$, $j = 1, 2$ are the cylindrical coordinate representations of the images of z_j, then we have the following relations:

$$\rho_j = \frac{r_j}{1+r_j^2} \qquad \zeta_j = \frac{r_j^2 - 1}{4(1+r_j^2)}.$$

Now, the square of the standard Euclidean distance in $\mathbf{R}^3$ between the two points $(\rho_j, \theta_j, \zeta_j)$ is given by

$$\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos(\theta_1 - \theta_2) + \zeta_1^2 + \zeta_2^2 - 2\zeta_1\zeta_2.$$

Writing ρ_j and ζ_j in terms of r_j and simplifying, one sees that the square of the distance between the images of z_1 and z_2 in $\mathbf{R}^3$ is given by

$$\frac{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}{(1+r_1^2)(1+r_2^2)} = \frac{|z_1 - z_2|^2}{(1+|z_1|^2)(1+|z_2|^2)}.$$

We therefore define the (square of the) **chordal distance** between two points in the complex plane to be

$$\|z_1, z_2\|^2 = \frac{|z_1 - z_2|^2}{(1+|z_1|^2)(1+|z_2|^2)}.$$

The term "chordal distance" comes from the fact that we derived this formula by measuring the length of the chord of the sphere connecting z_1 and z_2 when thought of as points on the sphere. We note that it is not obvious by just looking at the formula for $\|z_1, z_2\|$ that this distance satisfies the triangle inequality. However, since the chordal distance comes directly from the Euclidean distance in $\mathbf{R}^3$, it clearly must satisfy the triangle inequality. We can continuously extend our notion of chordal distance to $\mathbf{P}^1$ by saying that

$$\|z, \infty\|^2 = \frac{1}{1+|z|^2}.$$

We now define another Weil function associated to a by

$$\mathring{\lambda}_a(z) = -\log \|z, a\|,$$

and this is the other Weil function we will choose to work with.

Note that unlike the Weil function used by Nevanlinna, this more "geometric" Weil function $\mathring{\lambda}_a$ is smooth, and even real analytic, away from the point a. In Ahlfors's collected works [Ahlf 1982, pp. 56], Ahlfors says "rightly or wrongly" he was initially "somewhat disturbed" by the non-smoothness of the Weil function λ_∞ used by Nevanlinna. He explains that his experimentation with $\mathring{\lambda}_\infty$ as an alternative was what led him [Ahlf 1929] to discover the geometric interpretation of Nevanlinna's characteristic function (to be defined below). This interpretation had previously been discovered by T. Shimizu [Shim 1929] and will be explained in detail in §1.11.

Since we have identified the Riemann sphere with the sphere of radius $1/2$ in $\mathbf{R}^3$, we always have $\mathring{\lambda}_a(z) \geq 0$ since $\|z, a\| \leq 1$. We will take

$$\boxed{\mathring{m}(f,a,r) = \int_0^{2\pi} -\log \|f(re^{i\theta}), a\|\,\frac{d\theta}{2\pi}}$$

to be our definition of the "geometric" **mean proximity function** to a of f, which we distinguish from our previous definition by writing a small circle above the m.

Since working with explicit constants is one of our goals in this book, we record here the explicit constant relating Nevanlinna's definition of mean proximity to this other mean proximity function.

Proposition 1.2.2 *For all $x > 0$,*

$$\log^+ x \le \frac{1}{2}\log(1+x^2) \le \log^+ x + \frac{\log 2}{2}.$$

Also, given $a \in \mathbf{P}^1$ and a meromorphic function $f \not\equiv a$ on $\mathbf{D}(R)$, we have for all $r < R$,

$$m(f,a,r) \le \mathring{m}(f,a,r) \le m(f,a,r) + \frac{\log 2}{2}.$$

Proof. The inequalities

$$\log^+ x \le \frac{1}{2}\log(1+x^2) \le \log^+ x + \frac{\log 2}{2}.$$

are elementary, and the other inequality follows from this and the definitions. □

We also record here the following useful observation, which we will use to relate the various proximity functions. For those already familiar with Nevanlinna theory, this observation is what is used in what is often known as the "product into sum" estimate. We will say more about this when we discuss the Second Main Theorem.

Proposition 1.2.3 *Let $a_1, \dots, a_q$ be q distinct points in $\mathbf{P}^1$. Then, for all w in $\mathbf{P}^1$,*

$$\prod_{j=1}^{q} \|w, a_j\| \ge \left(\frac{1}{2}\right)^{q-1} \left(\min_j \prod_{i \ne j} \|a_i, a_j\|\right) \min_j \|w, a_j\|.$$

Moreover, if λ_{a_j} are Weil functions with singularities at the points a_j such that there exist non-negative constants C_1 and C_2 so that for all $w \ne a_j$ in $\mathbf{P}^1$,

$$-C_1 \le -\log\|a_j, w\| - \lambda_{a_j}(w) \le C_2,$$

then for all $w \ne a_j$ in $\mathbf{P}^1$, we have

$$\sum_{j=1}^{q} \lambda_{a_j}(w) \le \max_j \lambda_{a_j}(w) + \log \max_j \prod_{i \ne j} \|a_i, a_j\|^{-1} + (q-1)\log 2 + qC_1 + C_2.$$

Proof. The point is that w cannot be close to all the points at the same time. More precisely, since $\|z, w\|$ is a metric on $\mathbf{P}^1$, the triangle inequality implies that there is at most one index j such that

$$\|w, a_j\| < \frac{1}{2} \min_{i \ne j} \|a_i, a_j\|.$$

For all other indices i, again by the triangle inequality,

$$\|w, a_i\| \ge \frac{1}{2}\|a_j, a_i\|.$$

Thus, the first statement. The second follows easily from the first. □

Because Proposition 1.2.3 will be used on several occasions, it is convenient to introduce some notation. Given distinct points $a_1, \dots, a_q$, we define

$$\boxed{\dot{\mathcal{D}}(a_1, \dots, a_q) = \log \max_j \prod_{i \ne j} \|a_i, a_j\|^{-1} + (q-1)\log 2.}$$

Height or Characteristic Functions

Finally, we define the last of our Nevanlinna functions. The **Nevanlinna height** or **Nevanlinna characteristic function** with respect to a is defined by

$$\boxed{T(f,a,r) = m(f,a,r) + N(f,a,r),}$$

and the "geometric" **height** or **characteristic function** with respect to a is defined by

$$\boxed{\mathring{T}(f,a,r) = \mathring{m}(f,a,r) + N(f,a,r) + c_{\text{fmt}}(f,a),}$$

where $c_{\text{fmt}}(f,a)$ is a constant that does not depend on r and is defined by

$$c_{\text{fmt}}(f,a) = \begin{cases} \log\|f(0), a\| & \text{if } f(0) \ne a \\ \log|\text{ilc}(f-a,0)| + 2\log\|a,\infty\| & \text{if } f(0) = a, a \ne \infty \\ -\log|\text{ilc}(f,0)| & \text{if } f(0) = a = \infty. \end{cases} \tag{1.2.4}$$

The geometric characteristic function is often called the **Ahlfors-Shimizu characteristic function** for reasons that we will explain in §1.11. The constant c_{fmt} is known as the **First Main Theorem constant**, for reasons which will become apparent below.

The height or characteristic function T (or $\mathring{T}$), the mean proximity function m (or $\mathring{m}$), and the counting function N are the three main **Nevanlinna functions.** Nevanlinna theory can be described as the study of how the growth of these three functions is interrelated. We would like to point out that there is not universal agreement on what the order of the three arguments to each of these functions should be, and many authors use a subscript for one or more of the arguments. We have decided we would like to avoid subscripts for typographical reasons, and we have ordered the arguments the way we have because one often thinks of having one function f at a time, a finite number of values a, and an infinite set of radii r. Our arguments thus appear in order of increasing cardinality.

1.3 The First Main Theorem

This brings us to the main result of this section. Nevanlinna's First Main Theorem says that the height function T does not really depend on a. More precisely,

Theorem 1.3.1 (First Main Theorem) *Let $a \in \mathbf{C}$, and let $f \not\equiv a, \infty$ be a meromorphic function in $\mathbf{D}(R)$, $R \le \infty$. Then,*

$$\Big|T(f,a,r) - T(f,\infty,r) + \log|\text{ilc}(f-a,0)|\Big| \le \log^+|a| + \log 2,$$

and $\mathring{T}(f,a,r) = \mathring{T}(f,\infty,r)$.

Remarks. The equality between the $\mathring{T}$ explains the the definition of c_{fmt}. This equality is one of the advantages of the Weil function λ coming from the chordal metric. For the Nevanlinna characteristic T the best one can say is the difference is bounded, as in the statement of the theorem, not that the difference is actually constant.

Proof. We first prove the equality between the $\mathring{T}$. Directly from the definition of the chordal distance,

$$\begin{aligned} m(f,a,r)&-m(f,\infty,r)\\ &= \int_0^{2\pi} -\log|f(re^{i\theta})-a|\,\frac{d\theta}{2\pi} + \frac{1}{2}\int_0^{2\pi} \log(1+|f(re^{i\theta})|^2)\,\frac{d\theta}{2\pi}\\ &\quad +\frac{1}{2}\log(1+|a|^2) - \frac{1}{2}\int_0^{2\pi}\log(1+|f(re^{i\theta})|^2)\frac{d\theta}{2\pi}\\ &= \int_0^{2\pi} -\log|f(re^{i\theta})-a|\,\frac{d\theta}{2\pi} - \log\|a,\infty\|. \end{aligned}$$

Corollary 1.2.1 applied to $f-a$ gives us that

$$\int_0^{2\pi} -\log|f(re^{i\theta})-a|\,\frac{d\theta}{2\pi} = N(f-a,\infty,r)-N(f-a,0,r) - \log|\mathrm{ilc}(f-a,0)|.$$

Clearly $N(f-a,0,r) = N(f,a,r)$, and since f and $f-a$ have the same poles, $N(f-a,\infty,r) = N(f,\infty,r)$. Therefore, if $f(0)\neq a,\infty$, then by the definition of $\mathring{T}$ and c_{fmt},

$$\begin{aligned} \mathring{T}(f,a,r)&-\mathring{T}(f,\infty,r)\\ &= \mathring{m}(f,a,r) - \mathring{m}(f,\infty,r) + N(f,a,r) - N(f,\infty,r)\\ &\quad + \log\|f(0),a\| - \log\|f(0),\infty\|\\ &= -\log\|a,\infty\| - \log|f(0)-a| + \log\|f(0),a\| - \log\|f(0),\infty\|\\ &= 0. \end{aligned}$$

If $f(0)=a$ or ∞, then the definition of c_{fmt} was chosen so that we would get a similar cancellation. This we leave as an exercise to the reader.

Using Corollary 1.2.1 with the m functions as we did with the $\mathring{m}$ functions, we get

$$\begin{aligned} m(f,a,r)&-m(f,\infty,r)\\ &= \int_0^{2\pi}\log^+\left|\frac{1}{f(re^{i\theta})-a}\right|\frac{d\theta}{2\pi} - \int_0^{2\pi}\log^+|f(re^{i\theta})|\,\frac{d\theta}{2\pi}\\ &= \int_0^{2\pi}\log^+\left|\frac{1}{f(re^{i\theta})-a}\right|\frac{d\theta}{2\pi} - \int_0^{2\pi}\log^+|f(re^{i\theta})-a|\,\frac{d\theta}{2\pi}\\ &\quad + \int_0^{2\pi}\log^+|f(re^{i\theta})-a|\,\frac{d\theta}{2\pi} - \int_0^{2\pi}\log^+|f(re^{i\theta})|\,\frac{d\theta}{2\pi}\\ &= N(f,\infty,r) - N(f,a,r) - \log|\mathrm{ilc}(f-a,0)|\\ &\quad + \int_0^{2\pi}\left[\log^+|f(re^{i\theta})-a| - \log^+|f(re^{i\theta})|\right]\frac{d\theta}{2\pi}. \end{aligned}$$

For the inequality involving the T functions, note that if x and y are positive real numbers, then

$$\log^+(x+y)\le\log^+(2\max\{x,y\})\le\log^+x+\log^+y+\log 2.$$

Thus, if w is any complex number,

$$\Big|\log^+|w-a|-\log^+|w|\Big|\le\log^+|a|+\log 2.$$

This then gives us

$$\begin{aligned} &\Big|T(f,a,r)-T(f,\infty,r)+\log|\mathrm{ilc}(f-a,0)|\Big|\\ &\qquad\le\int_0^{2\pi}\Big|\log^+|f(re^{i\theta})-a|-\log^+|f(re^{i\theta})|\Big|\,\frac{d\theta}{2\pi}\le\log^+|a|+\log 2.\ \square \end{aligned}$$

Note that the First Main Theorem is really just Jensen's Formula written with different notation, and is thus not particularly deep. Given the First Main Theorem, one often writes just $T(f,r)$ instead of $T(f,a,r)$ since any two of these differ by a term bounded independent of r. To be definite, we will take

$$\boxed{T(f,r)=T(f,\infty,r),}\quad\text{and similarly}\quad\boxed{\mathring{T}(f,r)=\mathring{T}(f,\infty,r).}$$

We began this chapter by promising that it would contain a generalization of the Fundamental Theorem of Algebra. Let's see where we are in that regard. If it were true that $N(f,a,r)$ were essentially independent of a, then we would indeed have a generalization of the Fundamental Theorem of Algebra, because this would say that the rate of growth of the set of points where f is equal to a is independent of a, or loosely speaking f takes on all values equally often. We have already mentioned that this cannot be true because, for example, e^z is never 0. In fact, if one allows $a=\infty$, this is not even true for polynomials, since polynomials have no poles. What the First Main Theorem tells us though is that

$$m(f,a,r)+N(f,a,r)$$

does not depend on a, except for a bounded term independent of r. Thus, if we measure the growth of not only the set of points where f is equal to a, but also where f is "close to" a, then this combination grows in such a way that it is essentially independent of a, and this is our first generalization of the Fundamental Theorem of Algebra. Of course, this is not a strict generalization in the sense that it in no way implies the Fundamental Theorem of Algebra. This also explains why the function $m(f,a,r)$ is sometimes called the **compensation function** because it "compensates" for the fact that the most naive generalization of the Fundamental Theorem of Algebra is not true for meromorphic functions. We point out that since $m(f,a,r)$ is always positive, the First Main Theorem gives us an *upper bound* on the number of times f takes on the value a. Thus, it can be regarded as an analog of the statement that a polynomial of degree d has *at most* d zeros.

Looking back at our example of e^z, we notice that although e^z is never 0, it is close to zero (meaning that it is less than ε in absolute value, ε a positive number much less than one) on nearly half of each large circle centered at the origin. On the other hand, if a is a non-zero finite value, then e^z hits a regularly, e^z being a periodic function, but e^z is close to a, meaning that $e^z - a$ is less than ε in absolute value, only on a very tiny arc of each large circle centered at the origin. Notice that for e^z, the counting function N dominates the sum $m(f,a,r) + N(f,a,r)$ for most values of a, and only for the special values of 0 and ∞ is the mean proximity function m significant. Those readers already familiar with Picard's Theorem, which states that a non-constant meromorphic function on the complex plane can omit at most two values in $\mathbf{P}^1$, might already suspect that for most a, the counting function is the dominant term. This is indeed the case, and later when we discuss the Second Main Theorem, we will turn toward proving deeper results in that direction.

In summary, the First Main Theorem gives us an upper bound on $N(f,a,r)$ and hence on the number of times f takes on the value a, whereas the more elusive lower bounds on $N(f,a,r)$ will have to wait until Chapter 2 and Chapter 4 where we prove the Second Main Theorem.

1.4 Ramification and Wronskians

We now discuss "ramification" and introduce one last piece of notation. Although we will not prove anything about ramification until Chapters 2 and 4, we would like to introduce the concept and the notation here.

A point z_0 in $\mathbf{D}(R)$ is called a **ramification point** for a non-constant meromorphic function $f\colon \mathbf{D}(R) \to \mathbf{P}^1$ if f is not locally a topological covering map at the point z_0. If $f(z_0) \neq \infty$, then f has a series expansion at z_0 of the form

$$f(z) = A + B(z-z_0)^p + (z-z_0)^{p+1}g(z),$$

where $B \neq 0$ and $g(z)$ is analytic in a neighborhood of z_0. One easily sees that f is locally a topological covering map at z_0 precisely when $p = 1$, in which case f is locally a one sheeted covering by the inverse function theorem. If $p > 1$, then locally in a neighborhood of z_0, the map f is a p to 1 cover, *except at* z_0. If $p > 1$,

then the integer $(p-1)$ is called the **ramification index** of f at z_0. Note that if f is analytic, then f ramifies precisely at the zeros of f', and if z_0 is a ramification point then its ramification index is given by $\mathrm{ord}_{z_0} f'$. If $f(z_0) = \infty$, then one must look at the series expansion of $1/f$ at z_0, and in that case, the ramification index of f at z_0 is given by $\mathrm{ord}_{z_0}(1/f)'$. We emphasize that the ramification index is always non-negative, even at a pole of f.

Let f be a meromorphic function and write $f = f_1/f_0$, where f_0 and f_1 are holomorphic without common factors. We regard the fact that any meromorphic function on $\mathbf{D}(R)$, $R \leq \infty$ can be written as the quotient of two holomorphic functions without common zeros to be a fundamental fact from complex variables, and we will make use of this fact throughout without further comment. Those unfamiliar with this result should see, for example, Chapter XIII of [Lang 1993]. Then, away from the poles of f (*i.e.* the zeros of f_0), we have that

$$f' = \frac{f_0 f_1' - f_0' f_1}{(f_0)^2},$$

and this is zero precisely when $f_0 f_1' - f_0' f_1 = 0$. At the poles of f, which are not zeros of f_1 since we have assumed f_0 and f_1 to be without common factors,

$$\left(\frac{1}{f}\right)' = \frac{f_0' f_1 - f_0 f_1'}{(f_1)^2},$$

and again the zeros of $(1/f)'$ are given precisely by $f_0' f_1 - f_0 f_1' = 0$. Thus, the **Wronskian**

$$W = W(f_0, f_1) = \begin{vmatrix} f_0 & f_1 \\ f_0' & f_1' \end{vmatrix} = f_0 f_1' - f_0' f_1$$

of f_0 and f_1 is a useful function since its zeros are precisely the ramification points of f, and $\mathrm{ord}_{z_0} W$ is exactly the ramification index of f at z_0. The zeros of W together with their multiplicities are referred to as the **ramification divisor** of f. Note that W has no poles.

If P is a polynomial, then the ramification divisor for P is given by the zeros of P', counted with multiplicity, and so the degree ($=$ the number of points counted with multiplicity) of the ramification divisor of P is one less than the degree of P itself. In some sense, one of the consequences of the Second Main Theorem in Nevanlinna theory will be a generalization of this observation to entire functions. For meromorphic functions, looking only at polynomials is actually misleading. Instead, we should look at rational functions. Let $R = P/Q$ be a non-constant rational function, where P and Q are polynomials without common factors. Let p be the degree of P and let q be the degree of Q. The ramification divisor of R is determined by the zeros of the Wronskian of P and Q, which is a polynomial of degree at most $(p+q)-1$. Thus, the number of points, counted with multiplicity, in the ramification divisor cannot be more than a little less than the *sum* of the number of zeros and poles of R, again counted with multiplicity. It is a generalization of this statement to meromorphic functions that will be part of the content of the Second Main Theorem.

Of course the ramification divisor of a holomorphic or meromorphic function can have infinitely many points, and so we do not measure its size by counting the total number of points, but rather its growth, just as we did with the zeros and a-points (a in $\mathbf{P}^1$) of the function itself. We thus introduce the unintegrated and integrated counting functions associated to the ramification divisor. Let f be a meromorphic function on $\mathbf{D}(R)$, and let W be the associated Wronskian. Then for $r < R$, define

$$n_{\rm ram}(f,r) = n(W,0,r) \quad \text{and} \quad N_{\rm ram}(f,r) = N(W,0,r).$$

The following alternative to Wronskians for computing the ramification term will also be useful.

Proposition 1.4.1 *Let f be meromorphic on $\mathbf{D}(R)$. Then, for $r < R$,*

$$n_{\rm ram}(f,r) = n(f',0,r) + 2n(f,\infty,r) - n(f',\infty,r),$$

and

$$N_{\rm ram}(f,r) = N(f',0,r) + 2N(f,\infty,r) - N(f',\infty,r).$$

Proof. If z_0 is a ramification point of f, and if f is analytic at z_0, then z_0 is a zero of f', and moreover the ramification index of f at z_0 is the order of vanishing of f' at z_0. Clearly, such z_0 are not poles of f nor f'. On the other hand, if z_0 is both a pole of f and a ramification point with ramification index $p-1$, then locally at z_0, f looks like

$$f(z) = A\left(\frac{1}{z-z_0}\right)^p + \left(\frac{1}{z-z_0}\right)^{p-1} g(z),$$

where g is analytic in a neighborhood of z_0, and A is a non-zero constant. Thus,

$$f'(z) = -Ap\left(\frac{1}{z-z_0}\right)^{p+1} + \left(\frac{1}{z-z_0}\right)^p h(z),$$

where h is analytic in a neighborhood of z_0. Therefore f has a pole of order p at z_0 and f' has a pole of order $p+1$ at z_0. Hence,

$$2\,\mathrm{ord}_{z_0} f - \mathrm{ord}_{z_0} f' = 2p - (p+1) = p-1,$$

and this is the ramification index at z_0. □

Remark. We conclude this section by pointing out that R. Nevanlinna used the notation N_1 for ramification, and that Nevanlinna's notation is more widely used than the notation $N_{\rm ram}$ that we use here. We prefer the $N_{\rm ram}$ notation because we feel it is more descriptive notation, and therefore easier to remember what it stands for. Moreover, we feel that the use of the N_1 notation makes it harder for the reader to distinguish when one means the ramification counting function from when one means the counting function truncated to order 1.

1.5 Nevanlinna Functions for Sums and Products

As complicated functions are often built by piecing together simpler functions, in this brief section, we discuss how the Nevanlinna functions of sums and products are related to the Nevanlinna functions of the summands and factors.

Proposition 1.5.1 *Let $f_1, \ldots, f_p$ be meromorphic functions $\not\equiv \infty$. We have the following inequalities*

$$\begin{aligned}
m(\textstyle\sum f_i,\infty,r) &\le \textstyle\sum m(f_i,\infty,r) + \log p,\\
m(\textstyle\prod f_i,\infty,r) &\le \textstyle\sum m(f_i,\infty,r),\\
N(\textstyle\sum f_i,\infty,r) &\le \textstyle\sum N(f_i,\infty,r),\\
N(\textstyle\prod f_i,\infty,r) &\le \textstyle\sum N(f_i,\infty,r),\\
T(\textstyle\sum f_i,r) &\le \textstyle\sum T(f_i,r) + \log p,\\
T(\textstyle\prod f_i,r) &\le \textstyle\sum T(f_i,r).
\end{aligned}$$

Remark. Similar statements are true for $\mathring{m}$ and $\mathring{T}$ since these differ from m and T by bounded terms.

Proof. The inequalities involving the mean-proximity functions are obvious once one notes that for positive real numbers $x_1, \ldots, x_p$,

$$\log^+ \sum_{i=1}^p x_i \le \log^+(p \max x_i) \le \log^+ \max x_i + \log p \le \sum_{i=1}^p \log^+ x_i + \log p.$$

The inequalities involving the counting functions follow from the observation that the only way a point z_0 can be a pole of a sum or product is if it is a pole of at least one of the summands or factors. The height inequality then follows simply by adding the proximity and counting inequalities together. □

We now verify that the height or characteristic function is unchanged, up to a bounded term, after post-composition by a linear fractional transformation.

Proposition 1.5.2 *If f is a non-constant meromorphic function, and if*

$$g = \frac{af+b}{cf+d},$$

for constants a, b, c, and d with $ad - bc \neq 0$, then

$$T(g,r) = T(f,r) + O(1).$$

Proof. If $c = 0$, then this is trivial by Proposition 1.5.1. If $c \neq 0$, note that if we set

$$f_1 = f + \frac{d}{c}, \quad f_2 = cf_1, \quad f_3 = \frac{1}{f_2}, \quad \text{and } f_4 = \frac{(bc - ad)f_3}{c},$$

then $g = f_4 + a/c$, and so by repeated use of Proposition 1.5.1, and the First Main Theorem (Theorem 1.3.1), we see that

$$\begin{aligned} T(g,r) &= T(f_4,r) + O(1) = T(f_3,r) + O(1) \\ &= T(f_2,r) + O(1) \\ &= T(f_1,r) + O(1) = T(f,r) + O(1). \quad \square \end{aligned}$$

1.6 Nevanlinna Functions for Some Elementary Functions

We now compute the Nevanlinna functions for some familiar meromorphic functions.

Rational Functions

Consider

$$f(z) = c\frac{z^p + a_{p-1}z^{p-1} \ldots + a_0}{z^q + b_{q-1}z^{q-1} \ldots + b_0},$$

where the numerator and denominator have no common factors, and $c \neq 0$. If $p > q$, then $f(z) \to \infty$ as $z \to \infty$, and so $m(f,a,r) = O(1)$ as $r \to \infty$ for all finite a. Also, the equation $f(z) = a$ has p roots, counting multiplicity, for all finite a, making $N(f,r,a) = p\log r + O(1)$ as $r \to \infty$. Hence,

$$T(f,r) = m(f,a,r) + N(f,a,r) + O(1) = p\log r + O(1).$$

The equation $f(z) = \infty$ has q solutions, so

$$N(f,\infty,r) = q\log r + O(1),$$

and hence by the First Main Theorem (Theorem 1.3.1),

$$m(f,\infty,r) = (p-q)\log r + O(1).$$

Turning things upside-down, we have if $q > p$, that

$$T(f,r) = q\log r + O(1).$$

For $a \neq 0$, we have

$$N(f,a,r) = q\log r + O(1) \text{ and } m(f,a,r) = O(1).$$

Also,

$$m(f,0,r) = (q-p)\log r + O(1), \text{ and } N(f,0,r) = p\log r + O(1).$$

On the other hand, if $p = q$, then

$$T(f,r) = q\log r + O(1).$$

Also, for $a \neq c$,

$$m(f,a,r) = O(1), \text{ and } N(f,a,r) = q\log r + O(1).$$

Furthermore, if k denotes the order of vanishing of $f - c$ at ∞, then

$$m(f,c,r) = k\log r + O(1) \text{ and } N(f,c,r) = (q-k)\log r + O(1).$$

In all cases, $\boxed{T(f,r) = d\log r + O(1),}$ where d is the degree of the rational map.

In §5.1 we will take up a discussion of representing certain meromorphic functions, specifically those of finite order, as certain infinite products. When we do so, we will derive the converse to what we have discussed here. Namely, if we know $T(f,r) = O(\log r)$, then f must be a rational function.

The Exponential Function e^z

Now consider the function $f(z) = e^z$. In this case,

$$m(f,\infty,r) = \int_0^{2\pi} \log^+|e^{re^{i\theta}}| \frac{d\theta}{2\pi} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r\cos\theta \frac{d\theta}{2\pi} = \frac{r}{\pi}.$$

Since f is entire, $N(f,\infty,r) = 0$ and so $T(f,r) = r/\pi$. For a not equal to zero or infinity, $f(z) = a$ has solutions which are $2\pi i$ periodic. Thus, there are about $2t/2\pi$ roots in the disc of radius t, and hence

$$N(f,a,r) = \int_0^r \frac{t}{\pi}\frac{dt}{t} + O(\log r) = \frac{r}{\pi} + O(\log r).$$

Hence, $m(f,a,r) = O(\log r)$.

In fact, using a more refined analysis, we can see:

Proposition 1.6.1 *Let $f(z) = e^z$. If $a \neq 0$ is a non-zero complex number, then*

$$T(f,a,r) = \frac{r}{\pi} + O(1), \quad N(f,a,r) = \frac{r}{\pi} + O(1), \text{ and } m(f,a,r) = O(1).$$

If $a = 0$ or ∞, then

$$T(f,a,r) = \frac{r}{\pi}, \quad N(f,a,r) = 0, \text{ and } m(f,a,r) = \frac{r}{\pi}.$$

Here, as usual, the $O(1)$ terms are bounded functions as $r \to \infty$.

Proof. We have already proven this in the case $a = 0, \infty$. and simply restated it here for the convenience of the reader. That

$$T(f,a,r) = \frac{r}{\pi} + O(1)$$

for all other a then follows immediately from the First Main Theorem (Theorem 1.3.1). Also, that $m(f,a,r) = O(1)$ for $a \neq 0, \infty$ will follow from the First Main Theorem and the statement, $N(f,a,r) = 2r/2\pi + O(1)$.

We will now do the estimate for $N(f,1,r)$. Exactly the same method gives the same estimate for $N(f,a,r)$ for any $a \neq 0, \infty$, but the notation is a little more straightforward in the case $a = 1$, so we leave the other cases as an exercise for the reader. The function e^z is equal to 1 at the points $\pm 2\pi k\sqrt{-1}$, where k is a non-negative integer. Thus,

$$N(f,1,r) = \log r + 2 \sum_{0<k\leq r/2\pi} \log \frac{r}{2\pi k},$$

where the sum runs over positive integers $k \leq r/2\pi$. If we let $\lfloor r/2\pi \rfloor$ denote the largest integer $\leq r/2k$, then we easily see that this can be written

$$N(f,1,r) = \log r + 2\left(\left\lfloor \frac{r}{2\pi} \right\rfloor \log \frac{r}{2\pi} - \log \left\lfloor \frac{r}{2\pi} \right\rfloor !\right).$$

We now recall that Stirling's formula tells us that for integers j,

$$\lim_{j\to\infty} \frac{j!}{j^j e^{-j}\sqrt{2\pi j}} = 1.$$

We can use this to replace the $\log\lfloor r/2\pi \rfloor !$ term in our expression for $N(f,1,r)$ to conclude that $N(f,1,r)$ is equal to

$$\log r + 2\left(\left\lfloor \frac{r}{2\pi} \right\rfloor \log \frac{r}{2\pi} - \left\lfloor \frac{r}{2\pi} \right\rfloor \log \left\lfloor \frac{r}{2\pi} \right\rfloor + \left\lfloor \frac{r}{2\pi} \right\rfloor - \frac{1}{2} \log \left\lfloor \frac{r}{2\pi} \right\rfloor\right) + O(1).$$

Now,

$$\log \frac{r}{2\pi} - \log \left\lfloor \frac{r}{2\pi} \right\rfloor \leq \log \left(\frac{\lfloor r/2\pi \rfloor + 1}{\lfloor r/2\pi \rfloor}\right) \leq \log \left(1 + \frac{1}{\lfloor r/2\pi \rfloor}\right) \leq \frac{1}{\lfloor r/2\pi \rfloor} + O(1).$$

Thus the term

$$\left\lfloor \frac{r}{2\pi} \right\rfloor \left(\log \frac{r}{2\pi} - \log \left\lfloor \frac{r}{2\pi} \right\rfloor\right)$$

appearing in our expression for $N(f,1,r)$ is bounded. Moreover, the $\log\lfloor r/2\pi \rfloor$ cancels the $\log r$ term out front, up to a bounded term, and the only thing that remains is the $2\lfloor r/2\pi \rfloor$ term, which is of course equal to r/π, up to a bounded term. □

The Trigonometric Functions $\sin z$ **and** $\cos z$.

We can use the same analysis based on Stirling's formula that we used to prove Proposition 1.6.1 to conclude that for all finite a,

$$N(\sin z, a, r) + O(1) = N(\cos z, a, r) + O(1) = \frac{2r}{\pi} + O(1).$$

Since $\sin z$ and $\cos z$ can be expressed as linear combinations of e^{iz} and e^{-iz}, we have

$$T(\sin z, r) + O(1) = T(\cos z, r) + O(1) \leq \frac{2r}{\pi} + O(1).$$

It follows that

$$T(\sin z, r) + O(1) = T(\cos z, r) + O(1) = \frac{2r}{\pi} + O(1),$$

and

$$m(\sin z, a, r) + O(1) = m(\cos z, a, r) + O(1) = O(1).$$

1.7 Growth Order and Maximum Modulus

In this section we define the "growth order" of a meromorphic function f in terms of the growth of the Nevanlinna height (or characteristic) function $T(f,r)$ defined in §1.2. In the case that f is holomorphic, we also compare the growth of $T(f,r)$ to the growth of the maximum modulus of f. In this section, we only consider non-constant meromorphic functions f defined on all of $\mathbf{C}$. We define the **(growth) order** ρ and **lower (growth) order** λ of f by

$$\rho = \limsup_{r\to\infty} \frac{\log T(f,r)}{\log r} \quad \text{and} \quad \lambda = \liminf_{r\to\infty} \frac{\log T(f,r)}{\log r}.$$

If $\rho = \infty$, then f is said to be of **infinite order**, and if $\rho < \infty$, then f is said to be of **finite order**. When $0 < \rho < \infty$ one further distinguishes the growth of $T(f,r)$ as follows. Let

$$C = \limsup_{r\to\infty} \frac{T(f,r)}{r^\rho}.$$

One says that f has **maximal type** if $C = +\infty$, one says f has **mean type** if $0 < C < \infty$, and one says f has **minimal type** if $C = 0$. In addition, one says that f belongs to the **convergence class** if $\int_{r_0}^\infty T(f,t)/t^{\rho+1}\,dt$ converges. We remark that if f belongs to the convergence class, then f is also of minimal type.

To put things in perspective, we now compute the growth orders of the elementary functions we discussed in § 1.6. In § 1.6, we saw that if f was a rational function, then $T(f,r) = O(\log r)$. Therefore, rational functions have order 0. If $f = e^z$, then we saw that $T(f,r) = r/\pi + O(1)$. Hence, e^z is of order 1, mean type, and the same is true for $\sin z$ and $\cos z$. The function e^{e^z} is an example of a function with infinite order.

We now consider only the case of holomorphic functions f. Define the **maximum modulus function**

$$M(f,r) = \max_{|z|=r} |f(z)|.$$

We then have the following fundamental relationship between M and T.

Proposition 1.7.1 *Let f be a holomorphic function in an open neighborhood of $|z| \le R < \infty$. If $0 \le r < R$, then*

$$T(f,r) \le \log^+ M(f,r) \le \frac{R+r}{R-r} T(f,R)$$

Proof. The first inequality follows from the fact that since f is holomorphic,

$$T(f,r) = \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi}.$$

Note that the second inequality is trivial if $M(f,r) \le 1$. If $M(f,r) > 1$, then choose φ_0 so that $|f(re^{i\varphi_0})| = M(f,r)$. From the Poisson-Jensen Formula, Theorem 1.1.6, we then have

$$\begin{aligned}
\log^+ M(f,r) &= \log M(f,r) = \log |f(re^{i\varphi_0})| \\
&\le \int_0^{2\pi} \frac{R^2 - r^2}{|Re^{i\theta} - re^{i\phi_0}|^2} \log |f(Re^{i\theta})| \frac{d\theta}{2\pi} \\
&\le \frac{R+r}{R-r} \int_0^{2\pi} \log^+ |f(Re^{i\theta})| \frac{d\theta}{2\pi} \\
&= \frac{R+r}{R-r} T(f,R). \quad \square
\end{aligned}$$

The significance of this proposition is that for a holomorphic function f, we can define the growth order and growth type with $T(f,r)$ as we have done here, or in the more classical way, by using $\log M(f,r)$, and in either case we get the same result.

1.8 A Number Theoretic Digression: The Product Formula

A lot of current research in Nevanlinna theory is motivated by the deep analogy between Nevanlinna theory and Diophantine approximation. Although in this book we will never go very deeply into number theory or this analogy, we will from time to time make brief digressions into number theory to point out certain aspects of this important connection between the two fields. In this section, we make our first such digression to explain how the Jensen Formula can be considered the Nevanlinna theory analogue to what is known as the "product formula" in number theory.

Let F be a field. By an **absolute value** on F, we mean a real-valued function $|\ |$ on F satisfying the following three conditions:

AV 1. $|a| \ge 0$, and $|a| = 0$ if and only if $a = 0$.

AV 2. $|ab| = |a||b|$.

AV 3. $|a+b| \le |a| + |b|$.

The inequality in condition **AV 3** is known as the **triangle inequality.** Two absolute values $|\ |_1$ and $|\ |_2$ are called **equivalent** if there is a positive constant λ such that $|\ |_1 = |\ |_2^{\lambda}$. Over the rational number field $\mathbf{Q}$, we have the following absolute values. We have the standard Archimedean absolute value $|\ |$, also called "the absolute value at infinity" and sometimes denoted $|\ |_\infty$, which is defined by $|x|_\infty = x$ if $x \ge 0$, and $|x|_\infty = -x$ if $x < 0$. Given a prime number p, we can also define a **p-adic absolute value** as follows. Given a rational number x, write $x = p^e a/b$, where e is an integer and a and b are integers relatively prime to p. Then define $|x|_p = p^{-e}$. It is left to the reader to verify that $|\ |_p$ satisfies the three conditions necessary to be an absolute value. In fact, $|\ |_p$ satisfies a strong form of **AV 3**, namely

AV 3′. $|a+b|_p \le \max\{|a|_p, |b|_p\}$.

An absolute value that satisfies this strong form of the triangle inequality, **AV 3′**, is called **non-Archimedean,** and an absolute value that does not satisfy **AV 3′** is called **Archimedean.** A theorem of A. Ostrowski [Ostr 1918] states that any absolute value on $\mathbf{Q}$ is equivalent to one of the following: a p-adic absolute value for some prime p, the standard Archimedean absolute value $|\ |_\infty$, or the trivial absolute value $|\ |_0$ defined by $|x|_0 = 1$ for all $x \ne 0$. See, for example, [CaFr 1967] for the proof of Ostrowski's Theorem.

Now notice that if x is a non-zero rational number, and p is a prime that does not divide the numerator or denominator of x, then $|x|_p = 1$. Consider the product

$$\prod_p |x|_p,$$

where we include the case $p = \infty$ in the product. This product is well-defined because only finitely many terms in the product are different from 1. Now it is an easy exercise to check that for any non-zero x this product is in fact equal to one. Written additively, this so-called **product formula** reads

$$0 = \sum_p \log |x|_p = \sum_{p \text{, finite}} \log |x|_p + \log |x|_\infty.$$

Although the product formula is trivial to verify on $\mathbf{Q}$, the **Artin-Whaples Product Formula** says that over any number field F (a **number field** is just a finite extension of $\mathbf{Q}$), we can choose absolute values $|\ |_v$ in the various equivalence classes v of absolute values on F so that

$$0 = \sum_v \log|x|_v = \sum_{v \text{ non-Archimedean}} \log|x|_v + \sum_{v \text{ Archimedean}} \log|x|_v. \qquad (1.8.1)$$

We remark that for the non-Archimedean v, we can choose exactly one absolute value per equivalence class, and each equivalence class is counted precisely once. However, for the Archimedean equivalence classes, we count each equivalence class with multiplicity either one or two. This is a technical detail which can be ignored for now; we will return to it in §6.1.

C. F. Osgood [Osg 1981] [Osg 1985] noticed similarities between the structure of some proofs in Nevanlinna theory and the proofs of some Diophantine approximation results from number theory. Later, but independently from Osgood, P. Vojta [Vojt 1987] explored the connections between Nevanlinna theory and number theory in much greater depth, creating a complete dictionary translating between Nevanlinna theory and Diophantine approximation theory. We will discuss Vojta's dictionary in greater detail in Chapter 6.

One of Vojta's observations was that the Artin-Whaples Product Formula, equation (1.8.1), is the analogue of the Jensen Formula, equation (1.1.9). To better appreciate this, we can define the following absolute values on the field of meromorphic functions defined on $\overline{\mathbf{D}(r)}$. Let f be a function meromorphic on $\overline{\mathbf{D}(r)}$. If z is a non-zero point in the interior of $\mathbf{D}(r)$, we can define

$$|f|_{z,r} = \left|\frac{r}{z}\right|^{-\mathrm{ord}_z f}.$$

Notice that these absolute values are completely analogous to the p-adic absolute values on $\mathbf{Q}$. If $z = 0$, then we define

$$|f|_{0,r} = r^{-\mathrm{ord}_0 f}.$$

The analogues to the Archimedean absolute values are given by the "absolute values"

$$|f|_{\theta,r} = \left|f\left(re^{i\theta}\right)\right|,$$

as θ varies between 0 and 2π. Note that technically these are not really absolute values because they do not satisfy condition **AV 1** and because they can be infinite at poles of f. However, if $f \not\equiv 0, \infty$, then the zeros and poles of f are discrete, so the $|f|_{\theta,r}$ are well-defined and non-zero for all but finitely many θ. We can then re-write equation (1.1.9) as

$$\log|\mathrm{ilc}(f,0)| = \sum_{z\in\mathbf{D}(r)} \log|f|_{z,r} + \int_0^{2\pi} \log|f|_{\theta,r}\,\frac{d\theta}{2\pi}.$$

This looks exactly like the Artin-Whaples Product Formula, except that on the left-hand side we have a constant instead of zero, and on the right-hand side we have an infinite number of Archimedean absolute values, so we have to integrate them instead of adding them up.

1.9 Some Differential Operators on the Plane

As usual, let $z = x + y\sqrt{-1}$ be the complex variable on the plane, where x and y are the standard Cartesian coordinates on the plane. Let $\partial/\partial x$ and $\partial/\partial y$ be the usual partial differential operators on the plane. The following differential operators will be useful:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left[\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}\right] \quad \text{and} \quad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left[\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right].$$

The operator $\partial/\partial\bar z$ goes back at least to H. Poincaré [Poin 1899] but was perhaps first used systematically as we shall here by W. Wirtinger [Wirt 1927]. We will therefore refer to these operators as **Wirtinger derivatives**. Note that in terms of the Wirtinger derivatives, the Cauchy-Riemann equations can be written $\partial f/\partial \bar z = 0$. Associated to the Wirtinger derivatives, we have the differential operators ∂ and $\bar\partial$. If f is a C^1 function, then ∂f and $\bar\partial f$ are the 1-forms:

$$\partial f = \frac{\partial f}{\partial z}dz \quad \text{and} \quad \bar\partial f = \frac{\partial f}{\partial \bar z}d\bar z,$$

and the operators ∂ and $\bar\partial$ are extended to forms of all degrees in the natural way. Most often we will prefer to work with the operators:

$$\boxed{d = \partial + \bar\partial \quad \text{and} \quad d^c = \frac{\sqrt{-1}}{4\pi}(\bar\partial - \partial).}$$

Note that d is just the standard exterior derivative, and that both operators are real in the sense that if η is a real-valued form, then so are $d\eta$ and $d^c\eta$. The operator d^c was first introduced by J. Carlson and P. Griffiths in [CaGr 1972], and the constant out front was chosen to help minimize the number of constants appearing in important formulas. We also note at this point that if f is a C^2 function, then

$$dd^c f = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial f = \frac{\sqrt{-1}}{2\pi}\frac{\partial^2 f}{\partial z\partial \bar z}dz\wedge d\bar z = \frac{1}{4\pi}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)dx\wedge dy,$$

and so f is harmonic if and only if $dd^c f = 0$. Here we have used the standard notation $\wedge$ for the **wedge product**. We also remark that in polar coordinates (r,θ), the operator d^c has the form:

$$\boxed{d^c f = \frac{1}{2}\left(r\frac{\partial f}{\partial r}\frac{d\theta}{2\pi} - \frac{\partial f}{\partial \theta}\frac{dr}{2\pi r}\right).} \qquad (1.9.1)$$

In terms of the holomorphic coordinate z, any 1-form η can be written as

$$\eta = g(z,\bar{z})dz + h(z,\bar{z})d\bar{z},$$

where g and h are functions on the plane. The **pull back** of a 1-form η by a differentiable function f, denoted $f^*\eta$, is then given by

$$f^*\eta = (g \circ f)\left[\frac{\partial f}{\partial z} + \frac{\partial \bar{f}}{\partial z}\right] dz + (h \circ f)\left[\frac{\partial f}{\partial \bar{z}} + \frac{\partial \bar{f}}{\partial \bar{z}}\right] d\bar{z}.$$

Similarly, any 2-form Ω can be written

$$\Omega = g(z,\bar{z})dz \wedge d\bar{z},$$

and pull back by a function f takes the form

$$f^*\Omega = (g \circ f)\left[\frac{\partial f}{\partial z} \cdot \frac{\partial \bar{f}}{\partial \bar{z}} - \frac{\partial \bar{f}}{\partial z} \cdot \frac{\partial f}{\partial \bar{z}}\right] dz \wedge d\bar{z} = (g \circ f)J(f)dz \wedge d\bar{z},$$

where $J(f)$ denotes the determinant of the Jacobian matrix of f. Note in particular that if $\eta = g(z)dz$ is a holomorphic 1-form and f is a holomorphic function, then $f^*\eta = (g \circ f)f'dz$, and that if $\Omega = h(z,\bar{z})dz \wedge d\bar{z}$ is a 2-form and f is again holomorphic, then

$$f^*\Omega = (h \circ f)|f'|^2 dz \wedge d\bar{z}.$$

Combining the operator dd^c with the logarithm function will be important in various contexts, so we state here a proposition containing some useful formulas. Note that the operator $dd^c \log$ is an important operator, called the **first Chern operator**, because it is related to the first Chern class associated to a line bundle. However, we prefer not to discuss this relationship in the one variable setting so as to keep the one variable discussion as simple as possible.

Proposition 1.9.2 *Let u be a C^2 function. Then*

$$\mathbf{dd^c1} \qquad u^2 dd^c \log u = u dd^c u - du \wedge d^c u,$$

$$\mathbf{dd^c2} \qquad dd^c \log(1+u) = \frac{dd^c u}{1+u} - \frac{du \wedge d^c u}{(1+u)^2},$$

$$\mathbf{dd^c3} \qquad = \frac{dd^c u}{(1+u)^2} + \frac{u^2 dd^c \log u}{(1+u)^2},$$

$$\mathbf{dd^c4} \qquad = \frac{du \wedge d^c u}{u(1+u)^2} + \frac{u dd^c \log u}{1+u}.$$

Proof. Equations $\mathbf{dd^c1}$ and $\mathbf{dd^c2}$ follow by direct computation using the fact that d and d^c are derivations, and hence they satisfy the Leibniz Rule. Equation $\mathbf{dd^c3}$ follows by combining $\mathbf{dd^c1}$ and $\mathbf{dd^c2}$, and $\mathbf{dd^c4}$ follows by again using $\mathbf{dd^c1}$. □

1.10 Theorems of Stokes, Fubini, and Green-Jensen

We began this chapter with the Poisson and Poisson-Jensen formulas. We will look at these again, but this time in the context of Stokes's Theorem (or Green's Theorem). We first recall Stokes's Theorem in the form we will need.

Theorem 1.10.1 (Stokes's Theorem) *Let η be a 1-form which is C^1 in a neighborhood of the disc of radius t, $\mathbf{D}(t)$, except possibly at a discrete set of points Z. Assume that $d\eta$ is absolutely integrable on $\mathbf{D}(t)$, and that η is absolutely integrable on $\partial\mathbf{D}(t)$. Denote by $S(Z,\varepsilon)$ the formal sum of small circles of radius ε around the points in Z. Then*

$$\int_{\mathbf{D}(t)} d\eta = \int_{\partial\mathbf{D}(t)} \eta - \lim_{\varepsilon\to 0} \int_{S(Z,\varepsilon)} \eta.$$

Remark. The formula in Stokes's Theorem consists of three different integral terms, each of which we will give a name according to the following table:

interior term	boundary term	singular term
$\int_{\mathbf{D}(t)} d\eta$	$\int_{\partial\mathbf{D}(t)} \eta$	$\lim_{\varepsilon\to 0} \int_{S(Z,\varepsilon)} \eta$

We also recall here Fubini's Theorem in a global form.

Theorem 1.10.2 (Fubini) *Let $\pi\colon X \to Y$ be a fibering of real manifolds, meaning a C^∞ map of manifolds which is locally isomorphic to a product. Let $q = \dim Y$ and $\dim X = p + q$, so p is the dimension of a fiber. Let η be a p-form on X and let ξ be a q-form on Y. Assume that $\pi^*\xi \wedge \eta$ is absolutely integrable on X. Then,*

$$\int_X \pi^*\xi \wedge \eta = \int_{y\in Y} \left(\int_{\pi^{-1}(y)} \eta \right) \xi(y).$$

Proof. By using a partition of unity, the proof reduces to a local statement which is nothing more than the more familiar Fubini Theorem over a product space. □

Before stating the next theorem, we recall that $d\theta$ can be considered as a 1-form on $\mathbf{C} \setminus \{0\}$. Namely,

$$\boxed{d\theta = d\tan^{-1}\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2+y^2} = 2\pi d^c \log|z|^2.}$$

Theorem 1.10.3 (Green-Jensen Formula) *Let u be a function that is C^2 in a neighborhood of $\mathbf{D}(r)$ except at a discrete set of singularities Z, and assume that u is continuous at 0. Assume further that u satisfies the following three conditions:*

GJ 1. *u is absolutely integrable on $\partial\mathbf{D}(r)$.*

GJ 2. *$du \wedge \frac{d\theta}{2\pi}$ is absolutely integrable on $\mathbf{D}(r)$.*

GJ 3. *For every non-zero a in Z, $\lim\limits_{\epsilon\to 0} \int\limits_{|z-a|=\epsilon} u \frac{d\theta}{2\pi} = 0$.*

GJ 4. *$dd^c u$ is absolutely integrable on $\mathbf{D}(r)$.*

Then

$$\int_0^r \frac{dt}{t} \int\limits_{\mathbf{D}(t)} dd^c u + \int_0^r \frac{dt}{t} \lim_{\epsilon\to 0} \int\limits_{S(Z,\epsilon,t)} d^c u = \frac{1}{2}\int_0^{2\pi} u(re^{i\theta})\frac{d\theta}{2\pi} - \frac{1}{2}u(0),$$

where $S(Z,\epsilon,t)$ is the portion of $S(Z,\epsilon)$ inside $\mathbf{D}(t)$.

Proof. We will evaluate the integral

$$\int\limits_{\mathbf{D}(r)} du \wedge \frac{d\theta}{2\pi}$$

two different ways: by using Stokes's Theorem and by using Fubini's Theorem. We remark that the integral is convergent by condition **GJ 4**.

If 0 is in Z, let $Z' = Z \setminus \{0\}$, and otherwise let $Z' = Z$. To apply Stokes's Theorem, note that

$$\int\limits_{\mathbf{D}(r)} d\left(u\frac{d\theta}{2\pi}\right) = \lim_{\epsilon\to 0} \int\limits_{\mathbf{D}(r)-\left(\mathbf{D}(\epsilon)\cup D(Z',\epsilon)\right)} d\left(u\frac{d\theta}{2\pi}\right),$$

where $D(Z',\epsilon)$ denotes the formal sum of open discs of radius ϵ centered around the points of Z'. Because $d(d\theta) = 0$,

$$du \wedge \frac{d\theta}{2\pi} = d\left(u\frac{d\theta}{2\pi}\right)$$

and applying Stokes's Theorem (Theorem 1.10.1), we get

$$\int\limits_{\mathbf{D}(r)} d\left(u\frac{d\theta}{2\pi}\right) = \int\limits_{\partial\mathbf{D}(r)} u\frac{d\theta}{2\pi} - \lim_{\epsilon\to 0}\left[\int\limits_{\partial\mathbf{D}(\epsilon)} u\frac{d\theta}{2\pi} + \int\limits_{S(Z',\epsilon)} u\frac{d\theta}{2\pi}\right].$$

Now,

$$\lim_{\epsilon\to 0} \int\limits_{\partial\mathbf{D}(\epsilon)} u\frac{d\theta}{2\pi} = u(0)$$

since u is assumed to be continuous at 0. The term

$$\lim_{\epsilon\to 0} \int\limits_{S(Z',\epsilon)} u\frac{d\theta}{2\pi}$$

vanishes by assumption **GJ 3**.

Before applying Fubini, note that for any two C^2 functions α and β, we have, for degree reasons,

$$d\alpha \wedge d^c\beta = d\beta \wedge d^c\alpha.$$

So in particular, away from singularities,

$$du \wedge \frac{d\theta}{2\pi} = du \wedge d^c \log|z|^2 = d\log|z|^2 \wedge d^c u.$$

The disc $\mathbf{D}(r)$ is fibered over the interval $(0,r)$ by the map $z \mapsto |z|$, and the form $d\log|z|^2$ is the pull back of the form $2dt/t$ on the interval $(0,r)$ under this mapping. Therefore, when we apply Fubini (Theorem 1.10.2), we get

$$\int\limits_{\mathbf{D}(r)} du \wedge \frac{d\theta}{2\pi} = 2\int_0^r \frac{dt}{t} \int\limits_{\partial\mathbf{D}(t)} d^c u.$$

Thus,

$$\int_0^r \frac{dt}{t} \int\limits_{\partial\mathbf{D}(t)} d^c u = \frac{1}{2} \int\limits_{\partial\mathbf{D}(r)} u(re^{i\theta})\frac{d\theta}{2\pi} - \frac{1}{2}u(0).$$

Since for almost all t, the set of singularities Z does not intersect the boundary of $\mathbf{D}(t)$, we can therefore again apply Stokes's Theorem (Theorem 1.10.1) to the left hand side of the above equality to complete the proof. □

1.11 The Geometric Interpretation of the Ahlfors-Shimizu Characteristic

With these preliminaries out of the way, we begin to explore the geometric significance of the Ahlfors-Shimizu characteristic function $\mathring{T}(f,r)$. The **Fubini-Study** or **spherical area** form ω on $\mathbf{P}^1$ is defined in terms of a local coordinate z by

$$\boxed{\omega = \frac{\sqrt{-1}}{2\pi}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}.}$$

Note that this form is well-defined on all of $\mathbf{P}^1$ because it remains invariant if we replace z by $1/w$. The form ω is called the spherical area form because if D is a subset of $\mathbf{P}^1$ and if we think of P^1 as a sphere in $\mathbf{R}^3$, then

$$\int_D \omega$$

is proportional to the area of D thought of as a subset of the sphere in $\mathbf{R}^3$. When we discussed the chordal distance in §1.2, we identified $\mathbf{P}^1$ with the sphere in $\mathbf{R}^3$ of radius $1/2$, which has total area π. Here, the form ω is normalized so that

$$\int_{\mathbf{P}^1} \omega = 1,$$

so this is not entirely consistent with what we did in §1.2. Nonetheless, this discrepancy by a factor of π in our normalization of the area form turns out to be convenient in that introducing the factor of π here once and for all keeps various factors of π from entering into important formulas.

The **Euclidean** form Φ on the complex plane $\mathbf{C}$ is defined by

$$\boxed{\Phi = \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z} = \frac{1}{\pi} dx \wedge dy = 2r\, dr \wedge \frac{d\theta}{2\pi},}$$

where (x, y) and (r, θ) are the standard Cartesian and polar coordinates on the plane. If D is a subset of the complex plane, then

$$\int_D \Phi$$

is, again up to a factor of $1/\pi$, the area of D considered as a subset of the plane.

If $f = f_1/f_0$ is a meromorphic function with f_0, f_1 holomorphic without common zeros, then let

$$\boxed{(f^\sharp)^2 = \frac{|f'|^2}{(1+|f|^2)^2} = \frac{|W|^2}{(|f_0|^2 + |f_1|^2)^2},}$$

where $W = f_0 f_1' - f_0' f_1$ is the Wronskian of f_0 and f_1, which is holomorphic. Note that $|f_0|^2 + |f_1|^2$ is a positive real analytic function, and hence $(f^\sharp)^2$ is also real analytic. Furthermore, we have

$$\boxed{f^*\omega = (f^\sharp)^2 \Phi.}$$

Since $(f^\sharp)^2$ measures the ratio of the forms $f^*\omega$ and Φ, we see that $(f^\sharp)^2$ measures how much f locally distorts area as it maps from the Euclidean plane to $\mathbf{P}^1$, where area on $\mathbf{P}^1$ is spherical area. For this reason, the quantity $f^\sharp$ is called the **spherical derivative** of the mapping f.

The form $f^*\omega$ is what is known as a "pseudo area form." Away from the zeros of $f^\sharp$, $f^*\omega$ is an actual area form, but because $f^*\omega$ is zero when $f^\sharp$ is zero, it is not a true area form, and hence the term "pseudo."

The following proposition is an example of what is known as a "curvature computation."

Proposition 1.11.1 *We have*

(*i*) $$f^*\omega = (f^\sharp)^2 \Phi = dd^c \log(|f_0|^2 + |f_1|^2).$$

Away from the poles of f,

(*ii*) $$dd^c \log(1 + |f|^2) = dd^c \log(|f_0|^2 + |f_1|^2).$$

Away from points z *where* $f^\sharp(z) = 0$, *which are precisely the ramification points of* f.

(*iii*) $$-dd^c \log f^\sharp = dd^c \log(|f_0|^2 + |f_1|^2).$$

Remark. The equation $f^*\omega = -dd^c \log f^\sharp$ is a manifestation of the fact that $\mathbf{P}^1$ with the Fubini-Study metric has constant curvature $+1$. Notice that with positive curvature, one gets a negative sign in equations involving $dd^c \log$. Conversely, with negative curvature, one does not get a minus sign in $dd^c \log$ equations. Thus, in many ways, it would have been more convenient had the notions of positive and negative curvature been reversed.

Proof. Because

$$1 + |f|^2 = \frac{|f_0|^2 + |f_1|^2}{|f_0|^2}$$

and because $dd^c \log |f_0| = 0$, we get (ii). Since $(f^\sharp)^2 = |W|^2/(|f_0|^2 + |f_1|^2)^2$, and $dd^c \log |W| = 0$, (iii) follows similarly. To get (i), we apply equation $\mathbf{dd^c 3}$ of Proposition 1.9.2 with $u = |f|^2$. Away from the poles of f,

$$dd^c |f|^2 = |f'|^2 \Phi \quad \text{and} \quad dd^c \log |f|^2 = 0.$$

Combining this with $dd^c \log |f_0| = 0$ and (ii) gives us (i). □

Proposition 1.11.2 *Let* g *be holomorphic on an open neighborhood of the closed disc of radius* r, *and assume* $g(0) \neq 0$. *Let* Z *denote the zeros of* g *inside* $\mathbf{D}(r)$, *and let* $S(Z, \varepsilon)$ *denote the formal sum of small circles of radius* ε *centered at each of the points in* Z. *Then,* $\log |g|^2$ *satisfies the* **GJ** *conditions of Theorem 1.10.3, and moreover*

$$\lim_{\varepsilon \to 0} \int_{S(Z,\varepsilon)} d^c \log |g|^2 = n(g, 0, r).$$

Proof. If g has no zeros on the circle of radius r, then condition **GJ 1** is clearly satisfied. Since g has at most finitely many zeros on the circle of radius r, the question of integrability is local. We leave it as an exercise to the reader to verify that for $a \neq 0$, the function $\log|z-a|$ is integrable on the circle of radius $|a|$. From this, **GJ 1** easily follows.

Because $dd^c \log|g|^2 = 0$ away from singularities, condition **GJ 4** is trivially satisfied.

Let w be a local coordinate around a zero of g, so that $w = 0$ corresponds to the zero. Then, we can write

$$g(w) = w^m h(w),$$

where h is a holomorphic function with $h(0) \neq 0$.

For **GJ 2**, note that in terms of z,

$$d\log|g(z)|^2 \wedge \frac{d\theta}{2\pi} = \frac{\sqrt{-1}}{4\pi}\left(\frac{g'(z)}{g(z)}\frac{1}{\bar z} + \frac{\overline{g'(z)}}{\overline{g(z)}}\frac{1}{z}\right) dz \wedge d\bar z.$$

Thus,

$$\left| d\log|g|^2 \wedge \frac{d\theta}{2\pi}\right| \leq \frac{1}{2\pi}\left|\frac{g'}{g}\right| \frac{1}{|z|}|dz \wedge d\bar z|.$$

Near $z = 0$, $|g'/g|$ is bounded since we assume that $g(0) \neq 0$. Converting to polar coordinates,

$$\frac{1}{|z|} = \frac{1}{r} \qquad \text{and} \qquad |dz \wedge d\bar z| = 2r\,dr\,d\theta,$$

and so $d\log|g|^2 \wedge d\theta$ is absolutely integrable in a neighborhood of 0. The other possible problems could be the zeros of g. In terms of the local coordinate w,

$$\frac{g'(w)}{g(w)} = \frac{m}{w} + \frac{h'(w)}{h(w)}.$$

Because $h(0) \neq 0$, the term $|h'/h|$ is bounded in a neighborhood of $w = 0$. The term $1/|z|$ is bounded in a neighborhood of $w = 0$, and so converting to polar coordinates $w = \rho e^{i\varphi}$ around $w = 0$, we again see $d\log|g|^2 \wedge d\theta$ is absolutely integrable.

For condition **GJ 3**,

$$\begin{aligned}\int_{|w|=\epsilon} \log|g|^2 \frac{d\theta}{2\pi} &= \int_{|w|=\epsilon} \log|w|^{2m} \frac{d\theta}{2\pi} + \int_{|w|=\epsilon} \log|h|^2 \frac{d\theta}{2\pi} \\ &= \log\epsilon^{2m} \int_{|w|=\epsilon} \frac{d\theta}{2\pi} + \int_{|w|=\epsilon} \log|h|^2 \frac{d\theta}{2\pi}.\end{aligned}$$

Since $w = 0$ is not at $z = 0$, by Stokes's Theorem,

$$\int_{|w|=\epsilon} \frac{d\theta}{2\pi} = \int_{|w|\leq\epsilon} \frac{d(d\theta)}{2\pi} = 0.$$

Also, since $\log|h|^2$ is C^∞ near $w = 0$,

$$\lim_{\epsilon\to 0} \int_{|w|=\epsilon} \log|h|^2 \frac{d\theta}{2\pi} = 0.$$

Thus, $\log|g|^2$ satisfies **GJ 3**.

We now compute

$$\lim_{\epsilon\to 0} \int_{S(Z,\epsilon)} d^c \log|g|^2.$$

To finish the proof of the proposition, it suffices to show

$$\lim_{\epsilon\to 0} \int_{|w|=\epsilon} d^c \log|g|^2 = \mathrm{ord}_{w=0}\, g.$$

Because

$$\log|g|^2 = \log|w|^{2m} + \log|h|^2,$$

and because $\log|h|^2$ is C^∞ at $w = 0$,

$$\lim_{\epsilon\to 0} \int_{|w|=\epsilon} d^c \log|g|^2 = \lim_{\epsilon\to 0} \int_{|w|=\epsilon} d^c \log|w|^{2m}.$$

Using the polar coordinate formula for d^c, equation (1.9.1), we get

$$\int_{|w|=\epsilon} d^c \log|w|^{2m} = \int_0^{2\pi} m\frac{d\varphi}{2\pi} = m. \qquad \square$$

Combining Proposition 1.11.1 and Proposition 1.11.2 gives us a geometric interpretation of the height or characteristic function, which was first discovered by Ahlfors [Ahlf 1929] and Shimizu [Shim 1929], and this is why we have attached their names to $\mathring{T}$.

Theorem 1.11.3 (Ahlfors-Shimizu) *Let f be a non-constant function, meromorphic in an open neighborhood of $\mathbf{D}(r)$, and let ω be the Fubini-Study form on $\mathbf{P}^1$. Then,*

$$\mathring{T}(f, a, r) = \mathring{T}(f, \infty, r) = \int_0^r \frac{dt}{t} \int_{\mathbf{D}(t)} f^*\omega = \int_0^r \frac{dt}{t} \int_{\mathbf{D}(t)} (f^\sharp)^2 \Phi.$$

We point out here that the geometric significance of this theorem is that the Ahlfors-Shimizu height function measures, as a logarithmic average, the growth of the spherical area on $\mathbf{P}^1$ covered by the map f. Hence the First Main Theorem says that up to the constant $c_{fmt}(f,a)$, the sum $m(f,a,r)+N(f,a,r)$ is equal to the natural geometric quantity

$$\int_0^r \frac{dt}{t}\int_{\mathbf{D}(t)} f^*\omega.$$

Proof of Theorem 1.11.3. By Proposition 1.11.1,

$$\int_0^r \frac{dt}{t}\int_{\mathbf{D}(t)} f^*\omega = \int_0^r \frac{dt}{t}\int_{\mathbf{D}(t)} (f^\sharp)^2\Phi = \int_0^r \frac{dt}{t}\int_{\mathbf{D}(t)} dd^c \log(1+|f|^2).$$

Write $f = f_1/f_0$, where f_1 and f_2 are holomorphic without common zeros. For now, assume that $f(0)\neq\infty$. Recall that

$$\log(1+|f|^2) = \log(|f_0|^2+|f_1|^2) - \log|f_0|^2.$$

The function $\log(|f_0|^2+|f_1|^2)$ is C^∞, and since $f_0(0)\neq 0$, the function $\log|f_0|^2$, and hence the function $\log(1+|f|^2)$, satisfies the conditions of Theorem 1.10.3 by Proposition 1.11.2. Applying Theorem 1.10.3, we get

$$\begin{aligned}\int_0^r \frac{dt}{t}\int_{\mathbf{D}(t)} dd^c \log(1+|f|^2) &= \frac{1}{2}\int_0^{2\pi} \log(1+|f(re^{i\theta})|^2)\frac{d\theta}{2\pi}\\ &\quad -\frac{1}{2}\log(1+|f(0)|^2)\\ &\quad -\int_0^r \frac{dt}{t}\lim_{\epsilon\to 0}\int_{S(Z,\epsilon,t)} d^c\log(1+|f|^2).\end{aligned}$$

Directly from the definitions

$$\frac{1}{2}\int_0^{2\pi} \log(1+|f(re^{i\theta})|^2)\frac{d\theta}{2\pi} = \mathring{m}(f,\infty,r),$$

and

$$-\frac{1}{2}\log(1+|f(0)|^2) = \log\|f(0),\infty\|.$$

Because $\log(|f_0|^2+|f_1|^2)$ is C^∞, it does not contribute to the limit as $\epsilon\to 0$, and so the final term becomes

$$\int_0^r \frac{dt}{t}\lim_{\epsilon\to 0}\int_{S(Z,\epsilon,t)} d^c\log|f_0|^2 = N(f,\infty,r)$$

by Proposition 1.11.2. Putting these together gives us the theorem when 0 is not a pole of f. In the case that 0 is a pole, then let $g = 1/f$. Noting that ω is invariant under $z\mapsto 1/z$, and applying Theorem 1.3.1, we get

$$\begin{aligned}\mathring{T}(f,\infty,r) &= \mathring{T}(f,0,r) = \mathring{T}(g,\infty,r)\\ &= \int_0^r \frac{dt}{t}\int_{\mathbf{D}(t)} g^*\omega = \int_0^r \frac{dt}{t}\int_{\mathbf{D}(t)} f^*\omega. \qquad \square\end{aligned}$$

Since

$$\int_{\mathbf{D}(t)} f^*\omega$$

is a nondecreasing, nonnegative function of t, an easy consequence of the Ahlfors-Shimizu interpretation of the height is that $T(f,r)$ is a convex, strictly increasing function of $\log r$. Recall that a real valued function $\chi(t)$ of a real variable t is called **convex** (sometimes called "concave up") if for every pair of real numbers x and y and every t in the interval $[0,1]$, one has

$$\chi(tx+(1-t)y)\leq t\chi(x)+(1-t)\chi(y),$$

or in other words that every secant line for the graph of ϕ lies above the graph itself. That $N(f,a,r)$ is a nondecreasing convex function of $\log r$ is immediate from its definition (since $n(f,a,t)$ is a nonnegative nondecreasing function of t.) However, the proximity function $\mathring{m}(f,a,r)$ need not be nondecreasing nor a convex function of $\log r$. Thus, that $\mathring{T}$ is a convex nondecreasing function of $\log r$ is not immediate from the definition

$$\mathring{T}(f,a,r) = m(f,a,r)+N(f,a,r)+O(1).$$

We will see in §1.13 that the Nevanlinna characteristic $T(f,r)$ is also a convex function of $\log r$.

1.12 Why N and $\mathring{T}$ are Used in Nevanlinna Theory Instead of n and A

We saw in §1.11 that

$$\mathring{T}(f,r) = \int_0^r \frac{dt}{t}\int_{\mathbf{D}(t)} f^*\omega.$$

People first learning Nevanlinna theory often wonder about the logarithmic integration with respect to dt/t. If one lets

$$A(f,r) = \int_{\mathbf{D}(r)} f^*\omega,$$

then $A(f,r)$ measures the amount of spherical area covered by the image of $\mathbf{D}(r)$ under f. This seems like it might be a more natural quantity than $T(f,r)$, and similarly the unintegrated counting function $n(f,a,r)$ seems more natural than the integrated form $N(f,a,r)$. After all, if f is a degree d polynomial, then the Fundamental Theorem of Algebra says

$$d = \lim_{r\to\infty} n(f,a,r) = \lim_{r\to\infty} A(f,r)$$

for all a in $\mathbf{C}$. Thus, one might think that comparing $n(f,a,r)$ to $A(f,r)$ for an arbitrary meromorphic function is a more natural generalization of the Fundamental Theorem of Algebra than the comparison between $N(f,a,r)$ and $T(f,a,r)$ given by the First Main Theorem. However, the natural occurrence of $N(f,a,r)$ in the Jensen Formula (Corollary 1.2.1) indicates that working with $N(f,a,r)$ may in fact be more natural than working with $n(f,a,r)$. Once one accepts that integrating $n(f,a,t)$ against dt/t is a natural thing to do, doing the same with $A(f,r)$ to get $T(f,r)$ is clearly the natural quantity with which to compare $N(f,a,r)$.

As it turns out, comparing $n(f,a,r)$ and $A(f,r)$ is a far more subtle business than comparing $N(f,a,r)$ and $T(f,r)$. Because $\mathring{m}(f,a,r)$ is positive, Theorem 1.3.1 implies that $N(f,a,r) \le \mathring{T}(f,r) - c_{\mathrm{fmt}}(f,a)$, and we interpreted this informally as saying that $\mathring{T}$ provides an upper bound on how often f can attain the value a. An analog of the First Main Theorem would be an inequality of the form $n(f,a,r) \le A(f,r) + O(1)$. However, J. Miles [Miles 1971] gave an example of a meromorphic function such that

$$\limsup_{r\to\infty} \frac{n(f,0,r)}{A(f,r)} = \infty.$$

In fact, in [Gol 1978], Gol'dberg constructed an entire function f such that for every a in $\mathbf{C}$,

$$\limsup_{r\to\infty} \frac{n(f,a,r)}{A(f,r)} = \infty.$$

Thus, unlike with N and T, one cannot bound n in terms of A for all sufficiently large r.

There are weaker statements that one can prove. For example, if we replace lim sup by lim inf, then the First Main Theorem implies that if f is a meromorphic function on $\mathbf{C}$, then

$$n(f,a,r) \le A(f,r) + o(1) \quad \text{for a sequence of } r \to \infty. \tag{1.12.1}$$

Indeed, if inequality (1.12.1) were false, then $n(f,a,r)$ would be greater than $A(f,a,r)$ for all r sufficiently large. Integrating this inequality against dt/t would then contradict the First Main Theorem. Inequality (1.12.1) could be called the "unintegrated First Main Theorem." One disadvantage of inequality (1.12.1) is that the sequence of $r\to\infty$ for which the inequality holds may depend on the point a. This problem is addressed by a theorem of W. Hayman and F. Stewart [HayStew 1954] (see also [Hay 1964]).

Theorem 1.12.2 (Hayman-Stewart) *Let f be a non-constant meromorphic function on $\mathbf{C}$ and set*

$$n(f,r) = \sup_{a\in\mathbf{P}^1} n(f,a,r).$$

Then,

$$1 \le \liminf_{r\to\infty} \frac{n(f,r)}{A(f,r)} \le e.$$

Hayman and Stewart asked whether the constant e in Theorem 1.12.2 could be reduced to 1, and then this would have been an analog of the First Main Theorem for n and A. However, S. Toppila [Topp 1976] constructed a meromorphic function f such that for all r sufficiently large,

$$n(f,r) \ge \frac{80}{79} A(f,r).$$

The constant e in Theorem 1.12.2 is not best possible and has recently been improved by J. Miles [Miles 1998].

1.13 Relationships Among the Nevanlinna Functions on Average

Recall that the First Main Theorem (Theorem 1.3.1), said that for a function f meromorphic on $\mathbf{C}$,

$$T(f,a,r) = m(f,a,r) + N(f,a,r)$$

is essentially independent of the value a, in that for fixed f, and $r\to\infty$, changing a changes T by a bounded term. We also promised we would see later that for "most" values of a, N is the dominant term in the sum, and this was indeed the case in the specific examples we considered in §1.6. In the next chapter, we will begin in earnest our journey toward Nevanlinna's Second Main Theorem, which makes this statement precise in a very strong way. In the present section we content ourselves with a weaker statement, first proved by H. Cartan [Cart 1929]. Cartan's theorem allows us to conclude that N is the dominant term for "most" a, and it follows almost immediately from the Jensen formula.

Theorem 1.13.1 (Cartan) *Let $f(z)$ be a meromorphic function in $\mathbf{D}(R)$, and let $0 < r < R$. Then,*

$$T(f,r) = \int_0^{2\pi} N(f,e^{i\varphi},r)\,\frac{d\varphi}{2\pi} + C,$$

where $C = \log^+|f(0)|$ if $f(0) \ne \infty$, and $C = \log|\mathrm{lc}(f,0)|$ if $f(0) = \infty$. Furthermore,

$$\int_0^{2\pi} m(f,e^{i\varphi},r)\,\frac{d\varphi}{2\pi} \le \log 2.$$

Since the First Main Theorem tells us that $N(f,a,r) \le T(f,r) + O(1)$, one should think of Theorem 1.13.1 as saying that the proximity function $m(f,a,r)$ is insignificant for most values a on the unit circle, and that $T(f,r)$ essentially equals $N(f,a,r)$ for most values a on the circle. Note that there is nothing particularly special about the unit circle. Indeed, since T remains invariant up to a bounded term if f is replaced by $L \circ f$, where L is a Möbius transformation (Proposition 1.5.2), a similar inequality is true if the unit circle is replaced by any circle or line in the plane. In fact, Frostman [Fros 1934] has shown that a similar result holds if the circle is replaced by any set of positive capacity.

Since $\mathring{T}$ differs from T by a bounded amount, we get similar results for T and m, up to bounded terms.

Because N is convex function of $\log r$, and we have expressed T as an integral of N, we immediately get the following corollary.

Corollary 1.13.2 *The function $T(f,r)$ is a convex function of* $\log r$.

We begin the proof of Theorem 1.13.1 with a Proposition.

Proposition 1.13.3 *For any $a \in \mathbf{C}$,*

$$\int_0^{2\pi} \log|a - re^{i\varphi}| \frac{d\theta}{2\pi} = \max\{\log|a|, \log r\}$$

Proof. The case $a = 0$ is obvious, so we assume $a \neq 0$. We apply the Jensen Formula (Corollary 1.2.1), to the function $a - z$ when $|z| = r$. If $|a| > r$, we get

$$\log|a| = \int_0^{2\pi} \log|a - re^{i\varphi}| \frac{d\varphi}{2\pi},$$

and if $0 < |a| \le r$, we get

$$\log|a| = \int_0^{2\pi} \log|a - re^{i\varphi}| \frac{d\varphi}{2\pi} + \log\frac{|a|}{r}. \quad \square$$

Proof of Theorem 1.13.1. If $f(0) \neq \infty$, then for all φ for which $f(0) \neq e^{i\varphi}$, Corollary 1.2.1 applied to $f(z) - e^{i\varphi}$ gives us

$$\log|f(0) - e^{i\varphi}| = \int_0^{2\pi} \log|f(re^{i\theta}) - e^{i\varphi}| \frac{d\theta}{2\pi} + N(f,\infty,r) - N(f,e^{i\varphi},r).$$

If $f(0) = \infty$, then we get instead

$$\log|\mathrm{ilc}(f,0)| = \int_0^{2\pi} \log|f(re^{i\theta}) - e^{i\varphi}| \frac{d\theta}{2\pi} + N(f,\infty,r) - N(f,e^{i\varphi},r)$$

for all φ. In either case, we integrate with respect to φ. When $f(0) \neq \infty$, we get

$$\int_0^{2\pi} \log|f(0) - e^{i\varphi}| \frac{d\varphi}{2\pi} = \int_0^{2\pi}\int_0^{2\pi} \log|f(re^{i\theta}) - e^{i\varphi}| \frac{d\theta}{2\pi}\frac{d\varphi}{2\pi}$$
$$+ N(f,\infty,r) - \int_0^{2\pi} N(f,e^{i\varphi},r)\frac{d\varphi}{2\pi}.$$

Interchanging the order of integration in the double integral, and making use of Proposition 1.13.3, we get

$$\log^+|f(0)| = \int_0^{2\pi} \log^+|f(re^{i\theta})| \frac{d\theta}{2\pi} + N(f,\infty,r) - \int_0^{2\pi} N(f,e^{i\varphi},r)\frac{d\varphi}{2\pi}.$$

In case that $f(0) = \infty$, we get $\log|\mathrm{ilc}(f,0)|$ on the left hand side instead, and we have proved the first equality.

For the second inequality, by the First Main Theorem, Theorem 1.3.1, for each φ, we have

$$\left| m(f,e^{i\varphi},r) + N(f,e^{i\varphi},r) - T(f,\infty,r) + \log|\mathrm{ilc}(f - e^{i\varphi},0)| \right| \le \log 2.$$

We now integrate with respect to φ. From the first equality in the statement of the theorem, we have that

$$T(f,\infty,r) - \int_0^{2\pi} N(f,e^{i\varphi},r)\frac{d\varphi}{2\pi} = C.$$

If $f(0) = \infty$, then $\mathrm{ilc}(f - e^{i\varphi},0) = \mathrm{ilc}(f,0)$ for all φ, and so

$$\int_0^{2\pi} \log|\mathrm{ilc}(f - e^{i\varphi},0)| \frac{d\varphi}{2\pi} = \log|\mathrm{ilc}(f,0)| = C.$$

If, on the other hand, $f(0) \neq \infty$, then for all φ such that $f(0) \neq e^{i\varphi}$, we have $\mathrm{ilc}(f - e^{i\varphi},0) = f(0) - e^{i\varphi}$, and so

$$\int_0^{2\pi} \log|\mathrm{ilc}(f - e^{i\varphi},0)| \frac{d\varphi}{2\pi} = \int_0^{2\pi} \log|f(0) - e^{i\varphi}| \frac{d\varphi}{2\pi} = C,$$

where the second inequality follows from Proposition 1.13.3. This cancellation leaves us with the desired inequality. $\square$

We now give another variation on this theme.

Theorem 1.13.4 *If $f(z)$ is meromorphic in $\mathbf{D}(R)$ and $0 < r < R$, then*

$$\mathring{T}(f,r) = \int_{a\in\mathbf{P}^1} N(f,a,r)\,\omega, \quad \textit{and} \quad \int_{a\in\mathbf{P}^1} m(f,a,r)\,\omega = \frac{1}{2},$$

where ω is the spherical area form defined in section §1.11.

Before proceeding to the proof, we state a couple of Propositions.

Proposition 1.13.5 *For any* $z \in \mathbf{C}$,

$$\int_{a\in\mathbf{P}^1} \log|z-a|\,\omega = \frac{1}{2}\log(1+|z|^2).$$

Proof. From the definition of ω and from Proposition 1.13.3, we have

$$\begin{aligned}\int_{a\in\mathbf{P}^1} \log|z-a|\,\omega &= \int_0^\infty \int_0^{2\pi} \log|z-re^{i\varphi}|\frac{d\varphi}{2\pi}\frac{2rdr}{(1+r^2)^2}\\ &= \int_0^\infty \max\{\log|z|,\log r\}\frac{2rdr}{(1+r^2)^2}\\ &= \frac{1}{2}\log(1+|z|^2). \quad \square\end{aligned}$$

Proposition 1.13.6

$$\int_{a\in\mathbf{P}^1} c_{\text{fmt}}(f,a)\,\omega = -\frac{1}{2}.$$

Proof. For almost all a in $\mathbf{P}^1$,

$$c_{\text{fmt}}(f,a) = \log|f(0)-a| - \frac{1}{2}\log(1+|f(0)|^2) - \frac{1}{2}\log(1+|a|^2).$$

Integrating over a in $\mathbf{P}^1$ and applying Proposition 1.13.5, we find

$$\int_{a\in\mathbf{P}^1} c_{\text{fmt}}(f,a) = -\frac{1}{2}\int_{a\in\mathbf{P}^1} \log(1+|a|^2)\,\omega.$$

The integral on the right is a straightforward computation:

$$-\frac{1}{2}\int_{a\in\mathbf{P}^1} \log(1+|a|^2)\,\omega = -\frac{1}{2}\int_0^{2\pi}\int_0^\infty \frac{\log(1+r^2)}{(1+r^2)^2}\cdot 2rdr\frac{d\theta}{2\pi} = -\frac{1}{2}. \quad \square$$

Proof of Theorem 1.13.4. By replacing f by $1/f$ if necessary, we may assume without loss of generality that $f(0) \neq \infty$. For all $a \neq f(0)$, we have again by Corollary 1.2.1 that

$$\log|f(0)-a| = \int_0^{2\pi} \log|f(re^{i\theta})-a|\frac{d\theta}{2\pi} + N(f,\infty,r) - N(f,a,r).$$

Integrating over $a \in \mathbf{P}^1$ and making use of Proposition 1.13.5, we get

$$\begin{aligned}\frac{1}{2}\log(1+|f(0)|^2) = \int_0^{2\pi} \frac{1}{2}\log(1+|f(re^{i\theta})|^2)\frac{d\theta}{2\pi} &+ N(f,\infty,r)\\ &- \int_{a\in\mathbf{P}^1} N(f,a,r)\,\omega.\end{aligned}$$

The term on the left is nothing other than $-c_{\text{fmt}}(f,\infty)$ and the first integral on the right is the definition of $\mathring{m}(f,\infty,r)$. Hence

$$\int_{a\in\mathbf{P}^1} N(f,a,r)\,\omega = \mathring{m}(f,\infty,r) + N(f,\infty,r) + c_{\text{fmt}}(f,\infty) = \mathring{T}(f,r).$$

This together with the First Main Theorem (Theorem 1.3.1), and Proposition 1.13.6 implies that

$$\int_{a\in\mathbf{P}^1} \mathring{m}(f,a,r)\,\omega = -\int_{a\in\mathbf{P}^1} c_{\text{fmt}}(f,a)\,\omega = \frac{1}{2}. \quad \square$$

In [Ahlf 1935a], Ahlfors extended the averaging technique presented in this section. Rather than integrating against just the spherical area form, Ahlfors considered integrals of the form

$$\int_{a\in\mathbf{P}^1} N(f,a,r)\,\rho(a)\omega,$$

where ρ is an almost everywhere continuous and positive probability distribution on $\mathbf{P}^1$, meaning in particular that

$$\int_{a\in\mathbf{P}^1} \rho(a)\omega = 1.$$

Choosing ρ judiciously and using some of the techniques we will explain in Chapter 2, Ahlfors gave a particularly clear and easy proof of Nevanlinna's Second Main Theorem that illuminates the underlying geometry behind the Second Main Theorem.

1.14 Jensen's Inequality

We conclude this chapter with an important lemma of a different character. Recall that a real-valued function χ on the interval (a,b) (where we allow $a=-\infty$ and $b=+\infty$) is called **concave** (or "concave down") if for every x and y in the interval (a,b), and for every t in the interval $[0,1]$, we have

$$\chi(tx+(1-t)y) \geq t\chi(x)+(1-t)\chi(y).$$

Note that this is just the familiar fact from calculus that a function is concave down if and only if all the secant lines lie below the graph of the function.

Theorem 1.14.1 (Jensen's inequality) *Let (X,μ) be a measure space such that $\mu(X)=1$. Let χ be a concave function on the interval (a,b) (where we allow $a=-\infty$ and $b=+\infty$). Then, for any integrable real valued function f with $a<f(x)<b$ for almost all x in X, we have*

$$\int_X \chi(f(x))d\mu(x) \le \chi\left(\int_X f(x)d\mu(x)\right).$$

Remark. The concavity of χ implies that χ is continuous, and hence measurable, so the integral on the left makes sense. As we will only apply Theorem 1.14.1 to specific functions like log and $\log^+$ which are obviously continuous, we leave this fact as an exercise. We will typically apply Theorem 1.14.1 in the following context:

$$\int_0^{2\pi} \log f(\theta)\frac{d\theta}{2\pi} \le \log\left(\int_0^{2\pi} f(\theta)\frac{d\theta}{2\pi}\right).$$

Proof of Theorem 1.14.1. Let

$$c=\int_X f(x)d\mu(x).$$

By the concavity of χ, for any s and t such that $a<s<c<t<b$, we have

$$\frac{\chi(c)-\chi(s)}{c-s} \ge \frac{\chi(t)-\chi(c)}{t-c}.$$

Hence, if we let

$$m=\sup_{c<t<b}\frac{\chi(t)-\chi(c)}{t-c},$$

then for $a<s<c<t<b$,

$$\frac{\chi(c)-\chi(s)}{c-s} \ge m \ge \frac{\chi(t)-\chi(c)}{t-c}.$$

This implies that

$$\chi(f(x))-\chi(c)\le m(f(x)-c)$$

for almost all x because $a<f(x)<b$. Integrating this last inequality and making use of the fact that $\mu(X)=1$, we see that

$$\int_X \chi(f(x))d\mu(x)-\chi(c)\le m\left[\int_X f(x)d\mu(x)-c\right].$$

But,

$$c=\int_X f(x)\,d\mu(x) \quad\text{and so}\quad \int_X \chi(f(x))d\mu(x)\le\chi\left(\int_X f(x)d\mu(x)\right),$$

as desired. □

2 The Second Main Theorem via Negative Curvature

Shortly after R. Nevanlinna's first proof of the Second Main Theorem, Nevanlinna's brother, F. Nevanlinna, gave a "geometric" proof of the Second Main Theorem. In this chapter, we give a geometric proof of the Second Main Theorem based on "negative curvature," which broadly speaking, has the same overall structure as F. Nevanlinna's proof, although in terms of details, the proof we present here bears much greater resemblance to the work of Ahlfors [Ahlf 1941]. In Chapter 4, we will give another proof of the Second Main Theorem that is closer to R. Nevanlinna's original proof. Of course, neither the Nevanlinna brothers nor Ahlfors were interested in the exact structure of the error term. The error term we will present here is essentially due to P.-M. Wong [Wong 1989].

We have decided to begin with a negative curvature proof because obtaining a good error term using the geometric method requires less preparation than does the logarithmic derivative approach used in R. Nevanlinna's original proof. The geometric proof also has the virtue that the explicit constants occurring in the error terms have natural geometric interpretations. Another advantage of the geometric method is that one can use it to obtain the best possible error term for what's known as the "Ramification theorem," something which has yet to be accomplished using logarithmic derivatives. Moreover, in higher dimensions, the geometric approach has been successfully applied to a greater variety of situations than has the logarithmic derivative approach. On the downside, the structure of the error term here initially appears to be somewhat more complicated than the error term we will obtain in Chapter 4.

2.1 Khinchin Functions and Exceptional Sets

Before proceeding directly to the Second Main Theorem, we introduce some notation and terminology that will help us in our discussion of "error terms."

We begin with a definition. A function ψ will be called a **Khinchin function** if ψ is a real-valued, continuous, nondecreasing, and ≥ 1 on the interval $[e,\infty)$, and ψ satisfies the following convergence condition (known as the **Khinchin convergence condition**):

$$\int_e^\infty \frac{1}{x\psi(x)}\,dx<\infty.$$

In fact, since we will be interested in the value of the above integral, we will give it a name. Given a Khinchin function ψ, define

$$k_0 = k_0(\psi) = \int_e^\infty \frac{1}{x\psi(x)}\,dx < \infty.$$

We will explain in §6.2 why we have chosen the names "Khinchin function" and "Khinchin convergence condition." We note that in the original work of Nevanlinna, $\psi(x)$ was taken to be $(\log x)^{1+\varepsilon}$ for positive ε. One might find it helpful to keep this example in mind. Note that our choice of e as the start of the interval of definition of ψ could be replaced with an arbitrary positive number, but because we have the prototypical example of $\psi(x) = (\log x)^{1+\epsilon}$ in mind, and because we want $\psi(x) \geq 1$, the number e is a convenient choice.

One technical element that exists in all proofs of Nevanlinna's Second Main Theorem is the use of what is known as a "growth lemma." Historically, the growth lemmas were stated and proved for arbitrary functions ψ satisfying the Khinchin convergence condition, but then before using the growth lemmas to prove theorems in value distribution theory, the function ψ was immediately specialized to be $(\log r)^{1+\epsilon}$. S. Lang suggested keeping ψ arbitrary throughout. We will also explain the number theoretic motivation for keeping ψ arbitrary in §6.2.

Nevanlinna's Second Main Theorem will assert that a certain inequality holds for all radii r outside of a certain set E known as the **exceptional set** (of radii). The Second Main Theorem will have non-trivial content because the "size," in a sense we will make more precise in the next paragraph, of the exceptional set will be bounded in terms of our choice of Khinchin function ψ, or more precisely in terms of $k_0(\psi)$. Thus, the larger our function ψ, the smaller $k_0(\psi)$ will be, and therefore the smaller our exceptional set. However, there will also be a tradeoff, in that the Khinchin function ψ will also appear in the various inequalities themselves in such a way that the larger the function ψ, the weaker the inequalities will be.

Included in the statements the Second Main Theorem, the Khinchin function ψ will always have a companion function which we will always denote by ϕ. As we mentioned above, the Second Main Theorem will tell us that a certain inequality is true for all radii outside an exceptional set whose "size" is bounded in terms of $k_0(\psi)$. The function ϕ is what specifies what we mean by "size." More precisely, our exceptional sets E will be such that the integral

$$\int_E \frac{dr}{\phi(r)}$$

can be bounded in terms of $k_0(\psi)$. For example, if we take $\phi(r) \equiv 1$, then our exceptional sets will have finite Lebesgue measure, and in fact, Lebesgue measure bounded in terms of $k_0(\psi)$. If we take $\phi(r) = r$, then our exceptional sets will have finite "logarithmic measure," meaning that if E is such an exceptional set, then

$$\int_E \frac{dr}{r} < \infty.$$

In applications of the Second Main Theorem, as in Chapter 5, the idea is to choose ψ and ϕ so that the exceptional set will be "small" and yet ψ will not be too "large." When one studies meromorphic functions defined (except for poles) on the whole plane, one often wants to choose ϕ so that for every r_1,

$$\int_{r_1}^\infty \frac{dr}{\phi(r)} = \infty.$$

Then, since inequalities coming from the growth lemmas will hold true for all r outside a set E with

$$\int_E \frac{dr}{\phi(r)} < \infty,$$

we conclude that the exceptional set E cannot contain any intervals of the form (r_1, ∞), and so there must be a sequence of r outside E which tends toward infinity. In other words, whatever inequality is of interest to us will hold for a sequence of r tending toward infinity. For example, one often chooses $\phi(r) \equiv 1$ or $\phi(r) \equiv r$. Similarly, when one studies functions meromorphic in a disc of radius $R < \infty$, then one often chooses, for example, $\phi(r) = R - r$ so that for every $r_1 < R$,

$$\int_{r_1}^R \frac{dr}{\phi(r)} = \infty,$$

and again we can find a sequence of r tending to R outside any exceptional set E whose measure with respect to ϕ is bounded.

If one chooses a larger function ϕ, then the exceptional set E can grow larger. However, it is also often true that the larger the function ϕ, the better the inequality that will hold. Thus, the advantage of keeping ϕ arbitrary is that one can trade a larger exceptional set for an improvement in an inequality. This trade-off was recognized by the Nevanlinnas almost from the beginning. So, although the appearance of arbitrary Khinchin functions ψ in the statement of key theorems in value distribution appears only recently, the presence of arbitrary functions ϕ in these statements goes back almost to the beginning.

Finally, we would like to point out that although continuity is part of our definition of Khinchin function, it is not really necessary to require the continuity of ψ. With a little effort, one easily sees that the same qualitative results can be obtained without assuming the continuity of ψ.

2.2 The Nevanlinna Growth Lemma and the Height Transform

The main result of this section is a technical result, known as a "growth lemma." When one is interested in good explicit error terms, as we shall be, one must carefully formulate one's growth lemma because it affects the quality and structure both of the error terms and of the exceptional set. Although the statement and proofs of the results in this section can get a little "technical," the ideas involved are never deeper than those of freshman calculus, and therefore growth lemmas are often referred to as "calculus lemmas." A reader who is new to the subject of Nevanlinna theory might want to skip this section on a first reading, referring back to it as the various lemmas are used later on.

We now present the Nevanlinna growth lemma, an "elementary" growth lemma, which we will use in an iterated fashion (Lemma 2.2.3 below) in what we term the "curvature" approach to the proof of the Second Main Theorem. Because we iterate the growth lemma in this way, the structure of the error terms we will obtain via the curvature approach will be somewhat complicated.

Lemma 2.2.1 *Let $F(r)$ be a positive nondecreasing function defined and $\geq e$ for $r_0 \leq r < R \leq \infty$. Assume further that F has piecewise continuous derivative. Let $\xi(x)$ be a positive, nondecreasing, continuous function defined for $e \leq x < \infty$, and let $\phi(r)$ be any positive measurable function defined for $r_0 \leq r < R$. Let C be a positive constant and let E be the closed subset of $[r_0, R)$ defined by*

$$E = \left\{ r \in [r_0, R) : F'(r) \geq C F(r) \frac{\xi(F(r))}{\phi(r)} \right\}.$$

Then, for all ρ with $r_0 \leq \rho < R$,

$$\int_{E \cap [r_0, \rho]} \frac{dr}{\phi(r)} \leq \frac{1}{C} \int_e^{F(\rho)} \frac{dx}{x \xi(x)}.$$

Proof. From the definition of E and change of variables,

$$\int_{E \cap [r_0, \rho]} \frac{dr}{\phi(r)} \leq \frac{1}{C} \int_{r_0}^{\rho} \frac{F'(r) dr}{F(r)\xi(F(r))} \leq \frac{1}{C} \int_{F(r_0)}^{F(\rho)} \frac{dx}{x\xi(x)}.$$

The lemma follows since $F(r_0) \geq e$. □

Lemma 2.2.2 (Nevanlinna) *Let R, $F(r)$, and ϕ be as in Lemma 2.2.1, and let ψ be a Khinchin function. Then, there exists a measurable set of radii E with*

$$\int_E \frac{dr}{\phi(r)} \leq k_0(\psi) = \int_e^{\infty} \frac{dx}{x\psi(x)}.$$

such that for all $r_0 \leq r < R$ and not in E, the inequality

$$F'(r) < F(r) \frac{\psi(F(r))}{\phi(r)}$$

is valid.

Remark. Intuitively, Lemma 2.2.2 simply says that the derivative F' of a nondecreasing function F cannot be big unless F itself is big most of the time.

Proof. Let $C = 1$, let $\xi(x) = \phi(x)$, and choose E as in Lemma 2.2.1. Because ψ satisfies the Khinchin convergence condition, for all $\rho < R$, we have

$$\int_e^{F(\rho)} \frac{dx}{x\psi(x)} < \int_e^{\infty} \frac{dx}{x\psi(x)} = k_0(\psi).$$

The proof is completed by letting $\rho \to R$. □

In the Second Main Theorem we will want to bound a second derivative of the characteristic function $\mathring{T}$ in terms of $\mathring{T}$ itself. Hence, we iterate Lemma 2.2.2 to get the following complicated looking lemma which simply says the second derivative of a nondecreasing function F can be bounded in terms of F outside an "exceptional set."

Lemma 2.2.3 *Let $F(r)$ be a C^2 function defined for $r_0 \leq r < R \leq \infty$. Assume further that both $F(r)$ and $rF'(r)$ are positive nondecreasing functions of r, and that $F(r_0) \geq e$. Let b_1 be any number such that $b_1 r F'(r) \geq e$ for all $r \geq r_0$. (Such a number trivially exists since we have assumed $rF'(r)$ is positive and nondecreasing.) Let ψ be a Khinchin function and let ϕ be a positive measurable function defined for $r_0 \leq r < R \leq \infty$. Then, there exists a measurable set of radii E (called the "exceptional set") such that*

$$\int_E \frac{dr}{\phi(r)} \leq 2k_0(\psi),$$

such that for all $r_0 \leq r < R$ and not in E, we have

$$b_1 r F(r) \frac{\psi(F(r))}{\phi(r)} \geq b_1 r F'(r) \geq e,$$

and such that for all $r_0 \leq r < R$ and not in E, the inequality

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dF}{dr}\right] < F(r)\frac{\psi(F(r))}{\phi(r)} \frac{\psi\left(b_1 r F(r)\dfrac{\psi(F(r))}{\phi(r)}\right)}{\phi(r)}$$

is valid.

Proof. Apply Lemma 2.2.2 twice, first to $F(r)$, and then to $b_1 r F'(r)$. □

Remark. Lemma 2.2.3 was stated and proved in this generality by R. Nevanlinna. However, Nevanlinna and those who followed immediately specialized the function ψ to $\psi(x) = (\log x)^{1+\epsilon}$. S. Lang [Lang 1990] suggested keeping the function ψ arbitrary to better see its connection to Diophantine approximation.

We now describe the main context in which the functions to which we will apply Lemma 2.2.3 arise. Let u be a function defined on $\mathbf{D}(R)$, $R \leq \infty$ satisfying the following three hypotheses:

HT 1. u is continuous and > 0 except at a discrete set of points

HT 2. For all $0 < r < R$, the integral

$$\int_0^{2\pi} u(re^{i\theta})\frac{d\theta}{2\pi}$$

is absolutely convergent and the function

$$r \mapsto \int_0^{2\pi} u(re^{i\theta})\frac{d\theta}{2\pi}$$

is continuous.

HT 3. There exists r_0 such that

$$\int_0^{r_0}\frac{dt}{t}\int_{\mathbf{D}(t)} u\Phi \geq e.$$

Here Φ is the Euclidean area form on $\mathbf{C}$, defined in §1.11. Given a function u satisfying **HT 1-3**, define the **height transform**, which we will denote by $\mathring{\mathcal{H}}(u,r)$, by

$$\mathring{\mathcal{H}}(u,r) = \int_0^r \frac{dt}{t}\int_{\mathbf{D}(t)} u\Phi.$$

Lemma 2.2.4 *If u satisfies* **HT 1-3**, *$\mathring{\mathcal{H}}(u,r)$ is a C^2 function of r and*

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d\mathring{\mathcal{H}}(u,r)}{dr}\right] = 2\int_0^{2\pi} u(re^{i\theta})\frac{d\theta}{2\pi}.$$

Proof. Recalling that in polar coordinates $\Phi = 2r\,dr\,d\theta/2\pi$, we see that

$$r\frac{dF}{dr} = 2\int_0^{2\pi}\int_0^r u(te^{i\theta})t\,dt\frac{d\theta}{2\pi}.$$

Differentiating once more and dividing by r gives us the formula. □

Combining Lemma 2.2.3 and Lemma 2.2.4, we immediately get the following fundamental estimate for the height transform.

Lemma 2.2.5 *Let u be a function on $\mathbf{D}(R), R \leq \infty$ satisfying* **HT 1-3**, *let r_0 be such that $\mathring{\mathcal{H}}(u,r_0) \geq e$, and let b_1 be any number such that*

$$b_1\int_{\mathbf{D}(r_0)} u\Phi \geq e.$$

Let ψ be a Khinchin function, and let ϕ be a positive measurable function defined for $r_0 \leq r < R$. Then, there exists a set E of radii with

$$\int_E \frac{dr}{\phi(r)} \leq 2k_0(\psi)$$

and such that for all $r \geq r_0$ and not in E, the inequality

$$\log\int_0^{2\pi} u(re^{i\theta})\frac{d\theta}{2\pi}$$

$$< \log\mathring{\mathcal{H}}(u,r) + \log\left\{\frac{\psi(\mathring{\mathcal{H}}(u,r))}{\phi(r)}\right\} + \log\left\{\frac{\psi\left(b_1 r\mathring{\mathcal{H}}(u,r)\dfrac{\psi(\mathring{\mathcal{H}}(u,r))}{\phi(r)}\right)}{\phi(r)}\right\}$$

is valid.

Remark. We could have subtracted $\log 2$ from the right hand side, but we've chosen to throw this out.

The right hand side of Lemma 2.2.5 is a little messy, and we will use the following lemma to simplify some results after applying Lemma 2.2.5.

Lemma 2.2.6 *Let $\psi(r)$ be a Khinchin function. Let $p \geq 1$ be a real number. Set*

$$\psi_*(r) = \min\{\psi(\max\{e, r^{1/p}\}), r^{1/p}\}.$$

Then $\psi_*(r)$ is also a Khinchin function with

$$k_0(\psi_*) \leq 2p - 1 + pk_0(\psi).$$

Moreover

$$\psi_*(r) \leq r^{1/p}, \quad \psi_*(r^p) \leq \psi(r), \quad \textit{and} \quad \psi_*(r) \leq \psi(\max\{e, r^{1/p}\}),$$

for all $r \geq e$.

Proof. Clearly ψ_* is continuous, nondecreasing, and greater than 1. Moreover,

$$\int_e^\infty \frac{dr}{r\psi_*(r)} = \int_{\{r\geq e:\,\psi_*(r)\geq r^{1/p}\}} \frac{dr}{r\psi_*(r)} + \int_{\{r\geq e:\,\psi_*(r)< r^{1/p}\}} \frac{dr}{r\psi_*(r)}$$

$$\leq \int_e^\infty \frac{dr}{rr^{1/p}} + \int_e^\infty \frac{dr}{r\psi(\max\{e, r^{1/p}\})}.$$

Clearly,

$$\int_e^\infty \frac{dr}{r^{1+1/p}} = \frac{p}{e^{1/p}} \leq p.$$

Setting $s = r^{1/p}$, we get

$$\int_e^\infty \frac{dr}{r\psi(\max\{e, r^{1/p}\})} = \int_{e^{1/p}}^\infty \frac{pds}{s\psi(\max\{e,s\})}$$

$$\leq \int_{e^{1/p}}^e \frac{pds}{s} + \int_e^\infty \frac{pds}{s\psi(s)} = p - 1 + pk_0(\psi). \quad \square$$

2.3 Definitions and Notation

Throughout the remainder of this chapter, we maintain the following notational conventions:

R will be a positive real number or ∞.

$\mathbf{D}(R)$ will be the disc of radius R (or $\mathbf{C}$ when $R = \infty$).

$f = f_1/f_0$ will be a meromorphic function on $\mathbf{D}(R)$, and f_0 and f_1 will be holomorphic without common zeros.

$W = f_0 f_1' - f_1 f_0'$ will be the Wronskian of f.

$N_{\mathrm{ram}}(f, r) = N(W, 0, r)$.

q will be an integer ≥ 0.

$a_1, \ldots, a_q$ will be distinct points in $\mathbf{P}^1$.

$\mathring{\mathcal{D}}(a_1, \ldots, a_q) = -\log \min_j \prod_{i \neq j} \|a_i, a_j\| + (q-1)\log 2.$

r_0 will be a positive real number so that $\mathring{T}(f, r_0) \geq e$.

ψ will be a Khinchin function.

$\phi(r)$ will be a positive measurable function on $[r_0, R)$.

$k_0(\psi) = \displaystyle\int_e^\infty \frac{dx}{x\psi(x)} < \infty.$

b_1 will be any positive number such that

$$b_1 r_0 \frac{d}{dr}\bigg|_{r_0} \mathring{T}(f, r) \geq e.$$

$b_2(f) = \log|\mathrm{ilc}(W, 0)| - \log(|f_0(0)|^2 + |f_1(0)|^2)$

$b_3(q) = 12q^2 + 2q^3 \log 2.$

$b_4(f, a_1, \ldots, a_q)$

$$= \begin{cases} 0 & \text{if } f(0) \neq a_j \text{ for all } j \\ \log^+|\mathrm{ilc}(f - a_j, 0)| & \text{if for some } j, \\ \quad +(\mathrm{ord}_0(f - a_j))\log^+(1/r_0) & \quad f(0) = a_j \neq \infty \\ \log^+(1/|\mathrm{ilc}(f, 0)|) & \text{if for some } j, \\ \quad +(\mathrm{ord}_0(1/f))\log^+(1/r_0) & \quad f(0) = a_j = \infty. \end{cases}$$

$\Phi = \dfrac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}$ will be the Euclidean form on $\mathbf{C}$.

$\omega = \dfrac{\sqrt{-1}}{2\pi} \dfrac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}$ will be the Fubini-Study form on $\mathbf{P}^1$.

λ will be a positive constant between 0 and 1.

$\Lambda(r)$ will be a nonincreasing function of r with values between 0 and 1.

$$\gamma_\Lambda(z) = \left(\prod_{j=1}^q \|f(z), a_j\|^{-2(1-\Lambda(|z|))}\right)(f^\sharp(z))^2.$$

$$\alpha_\Lambda(z) = \left(\sum_{j=1}^q \|f(z), a_j\|^{-2(1-\Lambda(|z|))}\right)(f^\sharp(z))^2.$$

$$\Omega(f, \{a_1, \ldots, a_q\}, \Lambda) = \left(\sum_{j=1}^q \|f, a_j\|^{-2(1-\Lambda)}\right) f^*\omega = \alpha_\Lambda \Phi.$$

Remarks. When $f^\sharp(0) \neq 0$, then $b_2(f) = \log f^\sharp(0)$. We will see later that if $f^\sharp(0) \neq 0$, then we can estimate r_0 and b_1 only in terms of $f^\sharp(0)$.

2.4 The Ramification Theorem

One of the consequences of Nevanlinna's Second Main Theorem is a bound on how "ramified" a meromorphic function can be, in terms of its growth. This bound is known as the "Ramification Theorem," and obtaining the Ramification Theorem by using negative curvature is somewhat easier than obtaining the general Second Main Theorem. Hence, we will begin our discussion of the Second Main Theorem with the Ramification Theorem. Before stating the Ramification Theorem, we state a proposition expressing the Ahlfors-Shimizu height or characteristic function as a height transform, as in §2.2.

Proposition 2.4.1 *Let f be a meromorphic function on $\mathbf{D}(R)$. Then for all $0 < r < R$, we have $\mathring{T}(f, r) = \mathring{\mathcal{H}}((f^\sharp)^2, r)$.*

Proof. This follows directly from Theorem 1.11.3 and the definitions. □

Proposition 2.4.2 *Let f be a meromorphic function on $\mathbf{D}(R)$, and let $b_2(f)$ be defined as in §2.3. Then, for all $r < R$, we have*

$$N_{\mathrm{ram}}(f, r) - 2\mathring{T}(f, r) = \frac{1}{2}\int_0^{2\pi} \log(f^\sharp)^2 \frac{d\theta}{2\pi} - b_2(f).$$

If $f^\sharp(0) \neq 0$, then $b_2(f) = \log f^\sharp(0)$, and this simplifies to

$$N_{\mathrm{ram}}(f, r) - 2\mathring{T}(f, r) = \frac{1}{2}\int_0^{2\pi} \log(f^\sharp)^2 \frac{d\theta}{2\pi} - \log f^\sharp(0).$$

Proof. Let $s = n_{\mathrm{ram}}(f,0)$ and let

$$u = \log(f^\sharp)^2 - s\log|z|^2.$$

Note that because we have corrected for the ramification of f at 0, the function u is C^∞ at 0, and satisfies the conditions of Theorem 1.10.3 by Proposition 1.11.2. The formula stated in the proposition is a consequence of the Green-Jensen formula as follows.

To compute the interior term in Stokes's Theorem, note that away from singularities, $dd^c \log|z|^2 = 0$, and so by Proposition 1.11.1 and Theorem 1.11.3, we have

$$\int_0^r \frac{dt}{t} \int_{\mathbf{D}(t)} dd^c u = \int_0^r \frac{dt}{t} \int_{\mathbf{D}(t)} dd^c \log(f^\sharp)^2$$
$$= \int_0^r \frac{dt}{t} \int_{\mathbf{D}(t)} -dd^c \log(|f_0|^2 + |f_1|^2)^2 = -2\dot{T}(f,r).$$

For the singular term, note that

$$\int_0^r \frac{dt}{t} \lim_{\epsilon\to 0} \int_{S(Z,\epsilon,t)} d^c u = \int_0^r \frac{dt}{t} \lim_{\epsilon\to 0} \int_{S(Z,\epsilon,t)} d^c \log\left|\frac{W}{z^s}\right|^2.$$

Keeping in mind that we are canceling out the ramification at zero, we now have from Proposition 1.11.2, and the fact that the zeros of W count the ramification, that the singular term is

$$N_{\mathrm{ram}}(f,r) - s\log r.$$

Putting the singular and interior terms together and applying Green-Jensen (Theorem 1.10.3) we get that

$$N_{\mathrm{ram}}(f,r) - s\log r - 2\dot{T}(f,r)$$
$$= \int_0^r \frac{dt}{t} \lim_{\epsilon\to 0} \int_{S(Z,\epsilon,t)} d^c u + \int_0^r \frac{dt}{t} \int_{\mathbf{D}(t)} dd^c u = \frac{1}{2}\int_0^{2\pi} u(re^{i\theta})\frac{d\theta}{2\pi} - \frac{1}{2}u(0).$$

Now,

$$\frac{1}{2}\int_0^{2\pi} u(re^{i\theta})\frac{d\theta}{2\pi} = \frac{1}{2}\int_0^{2\pi} \log(f^\sharp(re^{i\theta}))^2 \frac{d\theta}{2\pi} - s\log r.$$

Canceling the $s\log r$ terms on both sides and computing $u(0)$ gives us what we want. □

Theorem 2.4.3 (Ramification Theorem) *Let f be a non-constant meromorphic function on $\mathbf{D}(R)$. Let ψ, ϕ, r_0, k_0, b_1, and b_2 be defined as in §2.3. Then, there exists a measurable set E of radii, called "the exceptional set," with*

$$\int_E \frac{dr}{\phi(r)} \le 2k_0(\psi)$$

such that for all $r \ge r_0$ and outside E, the following inequality holds:

$$N_{\mathrm{ram}}(f,r) - 2\dot{T}(f,r)$$
$$\le \frac{1}{2}\left[\log\dot{T}(f,r) + \log\left\{\frac{\psi(\dot{T}(f,r))}{\phi(r)}\right\}\right.$$
$$\left. + \log\left\{\frac{\psi\left(b_1 r T(f,r)\frac{\psi(\dot{T}(f,r))}{\phi(r)}\right)}{\phi(r)}\right\}\right] - b_2(f).$$

Proof. From Proposition 2.4.2, we have that

$$N_{\mathrm{ram}}(f,r) - 2\dot{T}(f,r) = \frac{1}{2}\int_0^{2\pi} \log(f^\sharp)^2\frac{d\theta}{2\pi} - b_2(f).$$

We use the concavity of the logarithm, Theorem 1.14.1, to pull the logarithm out of the integral

$$\frac{1}{2}\int_0^{2\pi}\log(f^\sharp)^2\frac{d\theta}{2\pi} \le \frac{1}{2}\log\int_0^{2\pi}(f^\sharp)^2\frac{d\theta}{2\pi}.$$

Noting that $\dot{T}(f,r) = \mathcal{H}((f^\sharp)^2, r)$ by Proposition 2.4.1, and applying the growth lemma, namely Lemma 2.2.5, we get a set E as in the statement of the theorem such that for r not in E,

$$\frac{1}{2}\log\int_0^{2\pi}(f^\sharp(re^{i\theta}))^2\frac{d\theta}{2\pi}$$
$$\le \frac{1}{2}\left[\log\dot{T}(f,r) + \log\left\{\frac{\psi(\dot{T}(f,r))}{\phi(r)}\right\} + \log\left\{\frac{\psi\left(b_1 r\dot{T}(f,r)\frac{\psi(\dot{T}(f,r))}{\phi(r)}\right)}{\phi(r)}\right\}\right],$$

and hence the theorem follows. □

The Ramification Theorem gives an upper bound on the growth of the ramification divisor of a meromorphic function f. Namely, it says that except for a small error term, the ramification divisor can grow at most twice as fast as $\dot{T}(f,r)$. Compare this to the fact that the ramification divisor of a rational function has degree at most twice that of the function itself.

Notice that f ramifies precisely when $f^\sharp = 0$. Recall also that the pseudo area form $f^*\omega$ is "pseudo" precisely when $f^\sharp = 0$. In some sense, the ramification of f is adding negative curvature to the pseudo area form $f^*\omega$, and thus what is happening in this "curvature" proof of Theorem 2.4.3 is that we are bounding the growth of the ramification divisor of f by bounding the amount of negative curvature a pseudo area form on $\mathbf{D}(R)$ can have.

The right hand side of the inequality in Theorem 2.4.3 is called the "error term." It is called an error term because in many applications it is insignificant compared to the terms on the left, and thus the error term can often be more or less ignored. This is one of the reasons that historically people were not very interested in the precise structure of the error term. We emphasize though that the structure of the error term is important for some applications, and we have chosen some of our applications in Chapter 5 to illustrate this.

The Ramification Theorem, with this error term, is due to Lang [Lang 1988]. It is interesting that we can get the $1/2$ out front here. That the coefficient $1/2$ is best possible was shown by Z. Ye [Ye 1991].

Notice also that although the exceptional set E in the Ramification Theorem can and does depend on the particular function f, the size of E is bounded independent of f. Of course, the inequality only holds for $r \geq r_0(f)$, and so there is some dependence on f. However, this is a very explicit dependence, and in applications to certain families of functions, one can often easily estimate r_0 uniformly over the family. See Corollary 2.8.2 where such a uniform bound is established.

2.5 The Second Main Theorem

We now state the Second Main Theorem that can be obtained by the method of this chapter.

Theorem 2.5.1 (Second Main Theorem) *Let $a_1, \ldots, a_q$ be a finite set of distinct points in $\mathbf{P}^1$, and let f be a non-constant meromorphic function on $\mathbf{D}(R)$. Let ψ, ϕ, $\mathcal{D}(a_1, \ldots, a_q)$, r_0, k_0, b_1, b_2, b_3, and b_4 be defined as in §2.3. Then, there exists a measurable set E of radii with*

$$\int_E \frac{dr}{\phi(r)} \leq 2k_0(\psi)$$

such that for all $r \geq r_0$ and outside E, the following inequality holds:

$$\begin{aligned}(q-2)\mathring{T}(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{ram}}(f,r) \leq \frac{1}{2}\Bigg[&\log\left\{b_3(q)\mathring{T}^2(f,r)\right\} + \log\left\{\frac{\psi(b_3(q)\mathring{T}^2(f,r))}{\phi(r)}\right\} \\ &+ \log\left\{\frac{\psi\left(b_1 r b_3(q)\mathring{T}^2(f,r)\dfrac{\psi(b_3(q)\mathring{T}^2(f,r))}{\phi(r)}\right)}{\phi(r)}\right\}\Bigg] \\ &-b_2(f) + \mathring{\mathcal{D}}(a_1,\ldots,a_q) + b_4(f,a_1,\ldots,a_q) + 1.\end{aligned}$$

Remark. Recall that $b_4 = 0$ if $f(0) \neq a_j$ for all j, and $b_2(f) = \log f^\sharp(0)$ if $f^\sharp(0) \neq 0$.

When $q > 2$, Theorem 2.5.1 complements the upper bound on the counting functions provided by the First Main Theorem by saying that the sum of q counting functions is bounded below by $(q-2)$ times the characteristic function, again up to the error term. Thus, for most values a, the characteristic function gives both an upper and lower bound on the number of times the function f takes on the value a, and the Second Main Theorem gives a precise limit on how much the lower bound can fail for all a's taken together.

Before proceeding toward the proof, we make some remarks about the error term. First, notice that in Theorem 2.5.1, we have $\mathring{T}^2(f,r)$ on the right hand side, whereas in Theorem 2.4.3, we only had $\mathring{T}(f,r)$. After combining $\log \mathring{T}^2(f,r)$ with the $1/2$ out front, we get $\log \mathring{T}(f,r)$ as the dominant term in the error term in Theorem 2.5.1. Theorem 2.5.1 was proved by R. Nevanlinna with $O(\log r\mathring{T}(f,r))$ for an error term. Theorem 2.5.1 with the error term given is due to P.-M. Wong [Wong 1989]. S.-S. Chern, in [Chern 1960], obtained an error term with dominant term $(1/2+\varepsilon)\log \mathring{T}(f,r)$, but Chern's paper contains an error that went unnoticed until S. Lang [Lang 1988] found it. We will give examples of functions, due to Z. Ye [Ye 1990] (also [Ye 1991]) in §7.1 that show one cannot obtain an error term for the Second Main Theorem with $(1/2+\varepsilon)\log \mathring{T}(f,r)$ as the dominant term.

The overall structure of the proof of Theorem 2.5.1 using curvature is similar to the proof of Theorem 2.4.3. Of course, the added complication here is the fact that we now also have to deal with the points $a_1, \ldots, a_q$ via what is called a "curvature computation." The use of curvature in value distribution theory was initiated by F. Nevanlinna (cf. [Nev(R) 1970]), although the way curvature appears in this chapter is closer to Ahlfors [Ahlf 1941], and was often used by Stoll (cf. [Stoll 1981]). The result we present here in its precise form is due to P.-M. Wong [Wong 1989], although a weaker version of it appears in Stoll [Stoll 1981, Theorem 10.3] as an "Ahlfors estimate."

To describe what we mean by a "curvature computation," we begin by introducing some notation. We let $\Lambda(r)$ denote a (for now arbitrary) nonincreasing function of r, and we recall the following functions and forms on $\mathbf{D}(R)$, which we defined in §2.3:

$$\gamma_\Lambda(z) = \left(\prod_{j=1}^{q} \|f(z), a_j\|^{-2(1-\Lambda(|z|))}\right)(f^\sharp(z))^2,$$

$$\alpha_\Lambda(z) = \left(\sum_{j=1}^{q} \|f(z), a_j\|^{-2(1-\Lambda(|z|))}\right)(f^\sharp(z))^2,$$

$$\Omega(f, \{a_1, \ldots, a_q\}, \Lambda) = \left(\sum_{j=1}^{q} \|f, a_j\|^{-2(1-\Lambda)}\right) f^*\omega = \alpha_\Lambda \Phi.$$

These functions and forms appear in the various works of Ahlfors, Stoll, and Wong. The form $\Omega(f, \{a_1, \ldots, a_q\}, \Lambda)$ is known as a "singular area form," and note that this is not quite an area form because it can be zero at those points z where $f^\sharp(z)$ vanishes. Moreover, $\Omega(f, \{a_1, \ldots, a_q\}, \Lambda)$ is not C^∞ at those points z where $f(z) = a_j$, and this is the reason that it is called a "singular" area form. The "curvature" we will compute is the curvature of the singular metric coming from the singular area form $\Omega(f, \{a_1, \ldots, a_q\}, \Lambda)$.

In some sense, we would really like to consider $\Lambda \equiv 0$, but in that case our curvature estimates "blow up." The way to get around this is to choose Λ allowing $\Lambda \to 0$ as $r \to R$. The error in Chern's work [Chern 1960] that was mentioned above was caused by Chern's attempt to simplify Ahlfors's work by considering only constant λ, rather than functions $\Lambda(r)$ of the radius r. In the end, Chern needs to take a limit as λ tends to zero (actually $\lambda \to 1$ in Chern's notation), but as this happens, the size of Chern's exceptional set might *a priori* grow and become unbounded.

We present Wong's curvature computation in a sequence of lemmas. We begin with a lemma collecting useful derivative formulas involving the expression $\|f, a\|^{2\lambda}$.

Lemma 2.5.2 *Let f be a meromorphic function, let a be a point in $\mathbf{P}^1$, let λ be a real number > 0, and let $u = \|f, a\|^2$. Then,*

(i) $dd^c \log u^\lambda = -\lambda (f^\sharp)^2 \Phi,$

(ii) $du \wedge d^c u = u(1-u)(f^\sharp)^2 \Phi,$

(iii) $dd^c u^\lambda = \{\lambda^2 u^{-(1-\lambda)} - \lambda(\lambda+1)u^\lambda\}(f^\sharp)^2 \Phi,$

(iv) $dd^c \log(1 + u^\lambda) = \dfrac{\lambda^2 u^{-(1-\lambda)} - \lambda(\lambda+1)u^\lambda - \lambda u^{2\lambda}}{(1+u^\lambda)^2}(f^\sharp)^2 \Phi.$

Proof. We recall that

$$\|f, a\|^2 = \frac{(f-a)(\bar f - \bar a)}{(1 + f\bar f)(1 + a\bar a)}.$$

For equation (i), note that taking log converts the product into a sum, and the only term that contributes to dd^c is therefore the $(1 + f\bar f)$ term in the denominator.

For equation (ii), note that for any C^2 function ξ, $d\xi \wedge d^c\xi$ differs from $\partial\xi \wedge \bar\partial\xi$ by a factor of $\sqrt{-1}/2\pi$. By direct computation, we see that

$$\partial\|f, a\|^2 = \frac{\bar f - \bar a}{1 + a\bar a} \cdot \frac{1}{(1 + f\bar f)^2} \cdot (1 + a\bar f)\partial f.$$

Since $\|f, a\|^2$ is real valued, we get the $\bar\partial$ derivative just by conjugating the above. Wedging them together gives us

$$\partial\|f, a\|^2 \wedge \bar\partial\|f, a\|^2 = \frac{(f-a)(\bar f - \bar a)(1 + a\bar f + \bar a f + a\bar a f\bar f)}{(1 + a\bar a)^2(1 + f\bar f)^4}\partial f \wedge \bar\partial \bar f.$$

The definition of $\|f, a\|$, switching back to d and d^c, and noting that

$$\frac{\sqrt{-1}}{2\pi}\partial f \wedge \bar\partial \bar f = |f'|^2 \Phi$$

completes the proof of (ii).

For the proof of (iii), first note that

$$dd^c u^\lambda = \lambda(\lambda - 1)u^{\lambda-2} du \wedge d^c u + \lambda u^{\lambda-1} dd^c u.$$

We can use equation (ii) to express $du \wedge d^c u$ in terms of $(f^\sharp)^2\Phi$, and formula $\mathbf{dd^c1}$ from Proposition 1.9.2 gives us that

$$dd^c u = u\, dd^c \log u + \frac{du \wedge d^c u}{u}.$$

Using (i) for the $dd^c \log u$ term and (ii) for the $du \wedge d^c u$ term gives us another expression in $(f^\sharp)^2\Phi$. Combining all the $(f^\sharp)^2\Phi$ factors results in the expression on the right hand side of (iii).

For (iv), we can use $\mathbf{dd^c3}$ from Proposition 1.9.2 to express $dd^c \log(1 + u^\lambda)$ in terms of $dd^c u^\lambda$ and $dd^c \log u^\lambda$. We then use equations (iii) and (i) to convert $dd^c u^\lambda$ and $dd^c \log u^\lambda$ into expressions involving $(f^\sharp)^2\Phi$. So doing, gives us (iv). □

Next we give the curvature estimate for constant λ.

Lemma 2.5.3 *Let f be a meromorphic function, let a be a point in $\mathbf{P}^1$, and let λ be a real number between 0 and 1. Then*

$$\lambda^2 \|f, a\|^{-2(1-\lambda)}(f^\sharp)^2\Phi \le 4 dd^c \log(1 + \|f, a\|^{2\lambda}) + 12\lambda (f^\sharp)^2 \Phi.$$

Proof. We start with equation (iv) of Lemma 2.5.2, and then note that since $0 \le \|f,a\|^{2\lambda} \le 1$, we get

$$dd^c \log(1+\|f,a\|^{2\lambda}) \ge \left\{\frac{1}{4}\lambda^2\|f,a\|^{-2(1-\lambda)} - \lambda(\lambda+1) - \lambda\right\}(f^\sharp)^2\Phi.$$

Now, $\lambda + 1 \le 2$, and so

$$dd^c \log(1+\|f,a\|^{2\lambda}) \ge \left\{\frac{\lambda^2}{4}\|f,a\|^{-2(1-\lambda)} - 3\lambda\right\}(f^\sharp)^2\Phi,$$

which, after rewriting, is what was to be shown. □

Now we check that we can apply the Green-Jensen formula.

Lemma 2.5.4 *Let f be a meromorphic function on $\mathbf{D}(R)$, let a be a point in $\mathbf{P}^1$, let λ be a real number > 0, and let $u = \|f,a\|^2$. Then, for all $r < R$,*

$$\int_0^r \frac{dt}{t}\int_{\mathbf{D}(t)} dd^c \log(1+u^\lambda) = \frac{1}{2}\int_0^{2\pi} \log(1+u(re^{i\theta})^\lambda)\frac{d\theta}{2\pi} - \frac{1}{2}\log(1+u(0)^\lambda).$$

Proof. In what follows, one needs to make obvious adjustments for the case $a=\infty$. We leave these adjustments as an exercise for the reader.

Note that $v = \log(1+u^\lambda)$ is continuous, and so in particular it is continuous at 0, integrable on $\partial\mathbf{D}(r)$, and satisfies condition **GJ 3**.

The singularities of v occur at points z_0 where $f(z_0) = a$. Let w be a local coordinate around such a point z_0 such that $w=0$ corresponds to z_0. Then,

$$|f(w)-a|^2 = |w|^{2m}\beta(w),$$

where m is a positive integer, and $\beta(w)$ is non-zero and C^∞ in a neighborhood of $w=0$. Let $w = \rho e^{i\varphi}$ be polar coordinates in a neighborhood of z_0.

To see that v satisfies **GJ 4**, note that by equation (iv) of Lemma 2.5.2, the discontinuity of $dd^c v$ comes from the term

$$u^{-(1-\lambda)}(f^\sharp)^2\Phi.$$

To check that this is integrable reduces to checking that

$$\frac{|f'(w)|^2}{|f(w)-a|^{2(1-\lambda)}}|dw\wedge d\bar{w}|$$

is integrable in a neighborhood of $w=0$. Up to invertible factors, $|f'(w)|^2$ looks like $|w|^{2(m-1)}$ and $|f(w)-a|^2$ looks like $|w|^{2m}$. Thus, the question of integrability boils down to whether

$$\frac{|w|^{2(m-1)}}{|w|^{2m(1-\lambda)}}|dw\wedge d\bar{w}|$$

is integrable. Switching to polar coordinates, this is the same as

$$\rho^{2[(m-1)-m(1-\lambda)]}\rho d\rho d\varphi = \rho^{-1+m\lambda}d\rho d\varphi.$$

Since $m\lambda > 0$, this is indeed integrable.

To check **GJ 2**, note that

$$dv = \frac{\lambda u^{\lambda-1}du}{1+u^\lambda}.$$

In a neighborhood of $w=0$,

$$|u^{\lambda-1}| \le M|w|^{2m(\lambda-1)},$$

for some constant M. By direct computation,

$$du = m|w|^{2(m-1)}\tilde{\beta}(w)(\bar{w}dw + wd\bar{w}) + |w|^{2m}d\tilde{\beta}(w),$$

where $\tilde{\beta}$ is a C^∞ function. Thus, in a neighborhood of $w=0$,

$$|u^{\lambda-1}du| \le M'|w|^{2m\lambda-1},$$

for some constant M'. If $z_0 \ne 0$, then $d\theta/2\pi$ is smooth in a neighborhood of $w=0$, and so

$$\left|dv\wedge\frac{d\theta}{2\pi}\right| \le M''|w|^{2m\lambda-1}|dw\wedge d\bar{w}| = M''\rho^{2m\lambda-1}\rho d\rho d\varphi,$$

for a constant M''. Thus,

$$dv\wedge\frac{d\theta}{2\pi}$$

is integrable (and even continuous) at z_0. If $z_0 = 0$, then

$$\left|\frac{d\theta}{2\pi}\right| \le \frac{1}{|w|},$$

and so we only lose one power of ρ in the above estimate for

$$dv\wedge\frac{d\theta}{2\pi},$$

which is therefore still integrable.

It remains to check that the singular term in Stokes's Theorem makes no contribution. Using the polar coordinate expression for d^c, equation (1.9.1), this amounts to verifying that

$$\lim_{\varepsilon\to 0}\int_{\rho=\varepsilon} \rho\frac{\partial}{\partial\rho}\log\left(1+\rho^{2m\lambda}\beta_1(\rho e^{i\varphi})\right)\frac{d\varphi}{2\pi} = 0,$$

where β_1 is C^∞ and non-zero in a neighborhood of $w=0$. But, by direct computation,

$$\rho\frac{\partial}{\partial\rho}\log\left(1+\rho^{2m\lambda}\beta_1(\rho e^{i\varphi})\right)$$
$$= \frac{\rho 2m\lambda\rho^{2m\lambda-1}\beta_1(\rho e^{i\varphi})+\rho^{2m\lambda+1}\partial\beta_1/\partial\rho}{1+\rho^{2m\lambda}\beta_1(\rho e^{i\varphi})},$$

and this clearly tends to 0 as $\rho\to 0$. □

We now state the key estimate.

Lemma 2.5.5 *Let $\Lambda(r)$ be a nonincreasing function of r with $0<\Lambda<1$. Let f be a non-constant meromorphic function. Given q distinct points $a_1,\ldots,a_q$ in $\mathbf{P}^1$, we have for all $r<R$,*

$$\mathring{\mathcal{H}}(\alpha_\Lambda, r) \le \frac{2q\log 2}{\Lambda^2(r)} + \frac{12q\mathring{T}(f,r)}{\Lambda(r)}.$$

Proof. Because α_Λ, and hence also $\mathring{\mathcal{H}}(\alpha_\Lambda,r)$, are additive in the a_j, it suffices to verify the inequality when $q=1$. Thus, we may assume

$$\alpha_\Lambda(z) = \|f(z),a\|^{-2(1-\Lambda(|z|))}(f^\sharp(z))^2,$$

for some point a in $\mathbf{P}^1$. Let $\lambda=\Lambda(r)$, and let

$$\alpha_\lambda(z) = \|f(z),a\|^{-2(1-\lambda)}(f^\sharp(z))^2.$$

Since Λ is nonincreasing and between 0 and 1,

$$\|f(z),a\|^{-2(1-\Lambda(|z|))} \le \|f(z),a\|^{-2(1-\Lambda(r))} \quad \text{for all } |z|\le r.$$

Hence,

$$\mathring{\mathcal{H}}(\alpha_\Lambda,r) \le \mathring{\mathcal{H}}(\alpha_\lambda,r).$$

Then, from Lemma 2.5.3, we get

$$\mathring{\mathcal{H}}(\alpha_\lambda,r) \stackrel{\text{def}}{=} \int_0^r\frac{dt}{t}\int_{\mathbf{D}(t)}\|f,a\|^{-2(1-\lambda)}(f^\sharp)^2\Phi$$
$$\le \frac{4}{\lambda^2}\int_0^r\frac{dt}{t}\int_{\mathbf{D}(t)}dd^c\log(1+\|f,a\|^{2\lambda}) + \frac{12}{\lambda}\int_0^r\frac{dt}{t}\int_{\mathbf{D}(t)}(f^\sharp)^2\Phi$$
$$= \frac{4}{\lambda^2}\int_0^r\frac{dt}{t}\int_{\mathbf{D}(t)}dd^c\log(1+\|f,a\|^{2\lambda}) + \frac{12}{\lambda}\mathring{T}(f,r).$$

Now, applying the Green-Jensen formula in the form of Lemma 2.5.4, we get

$$\int_0^r\frac{dt}{t}\int_{\mathbf{D}(t)}dd^c\log(1+\|f,a\|^{2\lambda})$$
$$= \frac{1}{2}\int_0^{2\pi}\log(1+\|f(re^{i\theta}),a\|^{2\lambda})\frac{d\theta}{2\pi} - \frac{1}{2}\log(1+\|f(0),a\|^{2\lambda}).$$

Since $\|f(z),a\|\le 1$, we get

$$\mathring{\mathcal{H}}(\alpha_\lambda,r) \le \frac{2\log 2}{\lambda^2} + \frac{12}{\lambda}\mathring{T}(f,r). \quad \square$$

In the above curvature estimate we used the additivity of α_Λ in the a_j. However, it is γ_Λ that we will need in our proof of the Second Main Theorem. Thus, we need a way to go from γ_Λ to α_Λ. We do this by what is known as a "product into sum" estimate, which we now state.

Proposition 2.5.6 (Product into Sum) *Let $a_1,\ldots,a_q$ be q distinct points in $\mathbf{P}^1$. Then, for all $w\ne a_j$ in $\mathbf{P}^1$, and for all λ with $0<\lambda<1$, we have*

$$\prod_{j=1}^q\|w,a_j\|^{-2(1-\lambda)} \le e^{2\mathring{\mathcal{D}}(a_1,\ldots,a_q)}\sum_{j=1}^q\|w,a_j\|^{-2(1-\lambda)},$$

where

$$\mathring{\mathcal{D}}(a_1,\ldots,a_q) = -\log\min_j\prod_{i\ne j}\|a_i,a_j\| + (q-1)\log 2,$$

just as in Proposition 1.2.3.

Proof. Let

$$s = \left(\frac{1}{2}\right)^{q-1}\min_j\prod_{i\ne j}\|a_i,a_j\|,$$

and note that by the triangle inequality, *i.e.* Proposition 1.2.3,

$$\prod_{j=1}^q\|w,a_j\|^{-2(1-\lambda)} \le s^{-2(1-\lambda)}\max_j\|w,a_j\|^{-2(1-\lambda)}$$
$$\le s^{-2(1-\lambda)}\sum_{j=1}^q\|w,a_j\|^{-2(1-\lambda)}.$$

Since $s,\lambda<1$, we have $s^{-2(1-\lambda)}\le s^{-2}$, and the proposition follows from the definition of $\mathcal{D}(a_1,\ldots,a_q)$. □

We have as an immediate corollary:

Corollary 2.5.7 $\log \gamma_\Lambda \leq \alpha_\Lambda + 2\dot{\mathcal{D}}(a_1, \ldots, a_q)$.

We now have all the tools we need to give the negative curvature proof of the Second Main Theorem.

Proof of Theorem 2.5.1. Directly from the definition,

$$\dot{m}(f, a_j, r) = \frac{1}{2}\int_0^{2\pi} \log \|f(re^{i\theta}), a_j\|^{-2} \frac{d\theta}{2\pi}.$$

Thus, for any constant λ,

$$(1-\lambda)\dot{m}(f, a_j, r) = \frac{1}{2}\int_0^{2\pi} \log \|f(re^{i\theta}), a_j\|^{-2(1-\lambda)} \frac{d\theta}{2\pi}.$$

Combining this with the Proposition 2.4.2, we get that

$$\sum_{j=1}^{q}(1-\lambda)\dot{m}(f, a_j, r) - 2\dot{T}(f, r) + N_{\mathrm{ram}}(f, r)$$
$$= \sum_{j=1}^{q}\frac{1}{2}\int_0^{2\pi} \log \|f(re^{i\theta}), a_j\|^{-2(1-\lambda)} \frac{d\theta}{2\pi} + \frac{1}{2}\int_0^{2\pi} \log(f^\natural(re^{i\theta}))^2 \frac{d\theta}{2\pi} - b_2(f).$$

So, if we let

$$\gamma_\lambda(z) = \left(\prod_{j=1}^{q} \|f(z), a_j\|^{-2(1-\lambda)}\right)(f^\natural(z))^2,$$

we have that

$$\sum_{j=1}^{q}(1-\lambda)\dot{m}(f, a_j, r) - 2\dot{T}(f, r) + N_{\mathrm{ram}}(f, r)$$
$$= \frac{1}{2}\int_0^{2\pi} \log \gamma_\lambda(re^{i\theta}) \frac{d\theta}{2\pi} - b_2(f).$$

Note that r remains constant in the integral on the right, so if we let

$$\Lambda(r) = \frac{1}{q\dot{T}(f, r)}.$$

and let $\lambda = \Lambda(r)$, we get

$$\sum_{j=1}^{q}(1-\Lambda(r))\dot{m}(f, a_j, r) - 2\dot{T}(f, r) + N_{\mathrm{ram}}(f, r)$$
$$= \frac{1}{2}\int_0^{2\pi} \log \gamma_\Lambda(re^{i\theta}) \frac{d\theta}{2\pi} - b_2(f).$$

By concavity of the logarithm, Theorem 1.14.1,

$$\int_0^{2\pi} \log \gamma_\Lambda(re^{i\theta}) \frac{d\theta}{2\pi} \leq \log \int_0^{2\pi} \gamma_\Lambda(re^{i\theta}) \frac{d\theta}{2\pi}.$$

Note that by the definition of r_0, $\Lambda(r) \leq 1$ for all $r \geq r_0$. Therefore, by the product into sum estimate, Lemma 2.5.7,

$$\log \int_0^{2\pi} \gamma_\Lambda(re^{i\theta}) \frac{d\theta}{2\pi} \leq \log \int_0^{2\pi} \alpha_\Lambda(re^{i\theta}) \frac{d\theta}{2\pi} + 2\dot{\mathcal{D}}(a_1, \ldots, a_q),$$

for all $r \geq r_0$.

Because $\alpha_\Lambda \geq (f^\natural)^2$, we have

$$\dot{\mathcal{H}}(\alpha_\Lambda, r) \geq 1 \text{ for } r \geq r_0 \quad \text{and} \quad b_1 \int_{\mathbf{D}(r_0)} \alpha_\Lambda \Phi \geq 1.$$

Therefore we can apply the growth lemma, Lemma 2.2.5, to get the existence of a measurable set of radii E such that

$$\int_E \frac{dr}{\phi(r)} \leq 2k_0(\psi).$$

and such that for all $r \geq r_0$,

$$\log \int_0^{2\pi} \alpha_\Lambda(re^{i\theta}) \frac{d\theta}{2\pi}$$
$$\leq \log F(r) + \log\left\{\frac{\psi(F(r))}{\phi(r)}\right\} + \log\left\{\frac{\psi\left(b_1 r F(r)\frac{\psi(F(r))}{\phi(r)}\right)}{\phi(r)}\right\},$$

where $F(r) = \dot{\mathcal{H}}(\alpha_\Lambda, r)$. From the curvature computation, Lemma 2.5.5, we get that

$$F(r) \leq \frac{2q\log 2}{\Lambda^2(r)} + \frac{12q\dot{T}(f, r)}{\Lambda(r)}.$$

Substituting $1/q\dot{T}(f, r)$ for Λ, we get

$$F(r) \leq (2q^3\log 2 + 12q^2)\dot{T}^2(f, r) = b_3(q)\dot{T}^2(f, r).$$

At this point, we have

$$(1-\Lambda(r))\sum_{j=1}^{q}\dot{m}(f, a_j, r) - 2\dot{T}(f, r) + N_{\mathrm{ram}}(f, r)$$

$$\leq \frac{1}{2}\left[\log b_3\dot{T}^2(f,r) + \log\left\{\frac{\psi(b_3\dot{T}^2(f,r))}{\phi(r)}\right\}\right.$$

$$\left. + \log\left\{\frac{\psi\left(b_1 r b_3\dot{T}^2(f,r)\dfrac{\psi(b_3\dot{T}^2(f,r))}{\phi(r)}\right)}{\phi(r)}\right\}\right]$$

$$-b_2(f) + \dot{D}(a_1,\dots,a_q).$$

By the definition of $\dot{T}(f,r)$, we have that if $f(0) \neq a_j$, then

$$\dot{T}(f,r) - N(f,a_j,r) = \dot{m}(f,a_j,r) + \log\|f(0),a_j\| \leq \dot{m}(f,a_j,r).$$

If $f(0) = a_j$ for some j, then for that particular j,

$$\dot{T}(f,r) - N(f,a_j,r) \leq \dot{m}(f,a_j,r) + C,$$

where

$$C = \begin{cases}\max\{0,\log|\mathrm{ilc}(f-a_j,0)|\} & \text{if } a_j \neq \infty\\ \max\{0,-\log|\mathrm{ilc}(f,0)|\} & \text{if } a_j = \infty.\end{cases}$$

Hence, keeping in mind that $0 < \Lambda(r) < 1$,

$$(1-\Lambda(r))\left(q\dot{T}(f,r) - \sum_{j=1}^{q} N(f,a_j,r)\right) \leq (1-\Lambda(r))\sum_{j=1}^{q} m(f,a_j,r) + C.$$

Thus,

$$\begin{aligned}(1-\Lambda(r))q\dot{T}(f,r) &- \sum_{j=1}^{q} N(f,a_j,r)\\ &\leq (1-\Lambda(r))\sum_{j=1}^{q} m(f,a_j,r) + C - \Lambda(r)\sum_{j=1}^{q} N(f,a_j,r)\\ &\leq (1-\Lambda(r))\sum_{j=1}^{q} m(f,a_j,r) + C + \sum_{j=1}^{q}\max\{0,-N(f,a_j,r)\}\\ &\leq (1-\Lambda(r))\sum_{j=1}^{q} m(f,a_j,r) + b_4,\end{aligned}$$

by our choice of constant b_4. Recalling that

$$\Lambda(r) = \frac{1}{q\dot{T}(f,r)},$$

leaves us with

$$q\dot{T}(f,r) - \sum_{j=1}^{q} N(f,a_j,r) \leq (1-\Lambda(r))\sum_{j=1}^{q} m(f,a_j,r) + b_4 + 1,$$

and completes the proof of the theorem. □

2.6 A Simpler Error Term

As was done by Z. Ye [Ye 1991], we now use a calculus lemma (Lemma 2.2.6) to show that the error terms in the theorems in this chapter can be simplified at the cost of enlarging the exceptional set by a little bit.

Theorem 2.6.1 (Ramification Theorem) *Let f be a non-constant meromorphic function on $\mathbf{D}(R)$. Let ψ, ϕ, k_0, b_1, and b_2 be defined as in §2.3. Then, there exists a measurable set E of radii with*

$$\int_E \frac{dr}{\phi(r)} \leq 8k_0(\psi)$$

such that for all r outside E and so large that

$$\dot{T}(f,r) \geq \max\{e, b_1(f)\},$$

the following inequality holds:

$$\begin{aligned}&N_{\mathrm{ram}}(f,r) - 2\dot{T}(f,r)\\ &\leq \frac{1}{2}\log\dot{T}(f,r) + \log\frac{\psi(\dot{T}(f,r))}{\phi(r)} + \frac{1}{2}\log\psi\left(\max\left\{e,\frac{r}{\phi(r)}\right\}\right) - b_2(f).\end{aligned}$$

Proof. Define ψ_* as in Lemma 2.2.6 with $p = 4$. Apply Theorem 2.4.3 with ψ_*, which provides an exceptional set such that

$$\int_E \frac{dr}{\phi(r)} \leq 8k_0(\psi).$$

We now examine the terms on the right-hand side of the inequality of Theorem 2.4.3 one by one. We leave the term $(1/2)\log\dot{T}(f,r)$ alone. For the next term, we have

$$\psi_*(t) \leq \psi_*(t^4) \leq \psi(t), \quad \text{for } t \geq 1$$

by construction, and so

$$\frac{1}{2}\log\frac{\psi_*(\dot{T}(f,r))}{\phi(r)} \leq \frac{1}{2}\log\frac{\psi(\dot{T}(f,r))}{\phi(r)}.$$

We now come to the term involving

$$\psi_*\left(b_1 r\dot{T}(f,r)\frac{\psi_*(\dot{T}(f,r))}{\phi(r)}\right).$$

By construction,

$$\psi_*(\dot{T}(f,r)) \leq \dot{T}^{1/4}(f,r) \leq \dot{T}(f,r).$$

Moreover, we assume that $\dot{T}(f,r) \geq b_1$. Thus,

$$b_1 r\dot{T}(f,r)\frac{\psi_*(\dot{T}(f,r))}{\phi(r)} \leq \frac{r}{\phi(r)}\dot{T}^3(f,r).$$

If $T(f,r) \geq r/\phi(r)$, then the above expression is $\leq \dot{T}^4(f,r)$. If, on the other hand, $T(f,r) \leq r/\phi(r)$, then the above expression is $\leq (\max\{e, r/\phi(r)\})^4$. By construction, we have that $\psi_*(t^4) \leq \psi(t)$, and so we have

$$\log\psi_*\left(b_1 r\dot{T}(f,r)\frac{\psi_*(\dot{T}(f,r))}{\phi(r)}\right) \leq \log\psi(\dot{T}(f,r)) + \log\psi(\max\{e, r/\phi(r)\}).$$

The proof is completed by combining these estimates. □

As was mentioned earlier, we know from the work of Z. Ye [Ye 1991] that the coefficient 1/2 in front of the term $\log\dot{T}(f,r)$, the dominant term in the error term, is best possible, and the term $\log\psi(\dot{T}(f,r))/\phi(r)$ is also necessary. It is not known if the term

$$\frac{1}{2}\log\psi(\max\{e, r/\phi(r)\})$$

is extraneous, and it remains an open question whether one can prove

$$N_{\mathrm{ram}}(f,r) - 2\dot{T}(f,r) \leq \frac{1}{2}\log\dot{T}(f,r) + \log\frac{\psi(\dot{T}(f,r))}{\phi(r)} + O(1)$$

for all r sufficiently large and outside a set whose measure with respect to ϕ depends only on ψ.

We can use the same trick to simplify the error term for the Second Main Theorem. This time Lemma 2.2.6 is applied with $p = 5$, and we leave the details as an exercise for the reader.

Theorem 2.6.2 (Second Main Theorem) *Let $a_1, \ldots, a_q$ be a finite set of distinct points in $\mathbf{P}^1$, and let f be a non-constant meromorphic function on $\mathbf{D}(R)$. Let ψ, ϕ, $\dot{D}(a_1, \ldots, a_q)$, k_0, b_1, b_2, b_3, and b_4 be defined as in §2.3. Then, there exists a measurable set E of radii with*

$$\int_E \frac{dr}{\phi(r)} \leq 10k_0(\psi)$$

such that for all r and outside E and so large that

$$\dot{T}(f,r) \geq \max\{e, b_1(f)b_3(q)\},$$

the following inequality holds:

$$(q-2)\dot{T}(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{ram}}(f,r)$$
$$\leq \log\dot{T}(f,r) + \log\frac{\psi(\dot{T}(f,r))}{\phi(r)} + \frac{1}{2}\log\psi\left(\max\left\{e, \frac{r}{\phi(r)}\right\}\right)$$
$$+\frac{1}{2}\log b_3(q) - b_2(f) + \check{D}(a_1, \ldots, a_q) + b_4(f, a_1, \ldots, a_q) + 1.$$

In particular, if $\phi(r) \geq r$, then for all r as above,

$$(q-2)\dot{T}(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{ram}}(f,r)$$
$$\leq \log\dot{T}(f,r) + \log\frac{\psi(\dot{T}(f,r))}{\phi(r)} + O(1).$$

2.7 The Unintegrated Second Main Theorem

In §1.12, we briefly discussed the unintegrated First Main Theorem, which said that if f is a non-constant meromorphic function on $\mathbf{C}$, then for all a, there exists a sequence of $r \to \infty$ such that

$$n(f,a,r) \leq A(f,r) + o(1), \qquad \text{where} \qquad A(f,r) = \int_{\mathbf{D}(r)} f^*\omega.$$

By "differentiating" the Second Main Theorem, we get what is known as the "unintegrated Second Main Theorem."

Theorem 2.7.1 (Unintegrated Second Main Theorem) *If f is a meromorphic function on $\mathbf{C}$ and $a_1, \ldots, a_q$ are q distinct points in $\mathbf{P}^1$, then*

$$(q-2)A(f,r) - \sum_{j=1}^{q} n(f,a_j,r) + n_{\mathrm{ram}}(f,r) \leq o(A(f,r))$$

for a sequence of $r \to \infty$.

In [Ahlf 1935b], in work that won Ahlfors one of the first Fields medals, Ahlfors developed a theory of "covering surfaces" to interpret Nevanlinna's theorems as geometric properties of the covering surface determined by the meromorphic function f. In particular, Ahlfors proved the following.

Theorem 2.7.2 *Let f be a non-constant meromorphic function on $\mathbf{C}$ and let $a_1, \ldots, a_q$ be a finite set of distinct points in $\mathbf{P}^1$. Then,*

$$\sum_{j=1}^{q} n^{(1)}(f, a_j, r) \geq (q-2)A(f,r) - kL(f,r),$$

where k is a positive constant, $L(f,r)$ is the length of the boundary of the covering surface determined by f restricted to $\mathbf{D}(r)$, and moreover,

$$\liminf_{r \to \infty} \frac{L(f,r)}{A(f,r)} = 0.$$

Ignoring the geometric interpretation of $L(f,r)$, the statement of Theorem 2.7.2 is weaker than Theorem 2.7.1, which was a trivial consequence of the Second Main Theorem, but the significance of Theorem 2.7.2 is its geometric character and proof. Ahlfors's method allowed him to generalize his theory to both quasiconformal mappings and to replace the "points" a_j by simply connected domains D_j. We will not attempt to give a detailed account of Ahlfors's theory here. We simply remark that Theorem 2.7.2 can be viewed as the natural generalization of the Riemann-Hurwitz Theorem to meromorphic functions. The geometric basis for Value Distribution theory developed by Ahlfors in [Ahlf 1935b] and the explicit connection he made between the Second Main Theorem and the "Gauss-Bonnet Formula" in [Ahlf 1937] were profoundly important in extending Nevanlinna's theory to quasiconformal mappings and to higher dimensional holomorphic mappings. For a detailed discussion of Ahlfors's theory, we refer the reader to [Ahlf 1935b], [Nev(R) 1970], or [Hay 1964].

In the case of entire functions, some may consider it more natural to compare the counting functions $n(f, a_j, r)$ to $\log M(f,r)$. In [Berg 1997], W. Bergweiler gives a lower bound on the unintegrated counting functions associated to an entire function in terms of $\log M(f,r)$ that is analogous to the Second Main Theorem. This is more subtle than Theorem 2.7.1 because one must be able to compare the counting functions to $A(f,r)$ and $A(f,r)$ to $\log M(f,r)$ using the same sequence of $r \to \infty$.

2.8 A Uniform Second Main Theorem

In Theorem 2.5.1, the constants r_0, b_1, b_2, and b_4 depend on the function f. For some of our applications, we wish to bound these constants uniformly in f. Recall

that if $f(0) \neq a_1, \ldots, a_q$, then the constant b_4 is zero, so for convenience we now assume that $f(0) \neq a_j$ for all j. We wish to bound the remaining constants uniformly in f. In particular, if we normalize our function so that $f'(0) = 1$, then we would like to find explicit bounds for these constants, independent of f given this normalization. Of course, once we normalize $f'(0)$ to be 1, then $b_2 = 0$ directly from the definition. Thus, we are faced with bounding r_0 and b_1.

We begin with an estimate for r_0, which follows easily from the First Main Theorem. The inequality in Theorem 2.5.1 is valid for all r (outside an exceptional set) greater than r_0, where r_0 is such that $\mathring{T}(f, r_0) \geq e$. We are thus looking for a universal constant r_0 such that if $f'(0) = 1$, then $\mathring{T}(f, r_0) \geq e$ for all such f. In fact, we can take r_0 to be the number e^e, which is an easy consequence of the following proposition.

Proposition 2.8.1 *Let f be a meromorphic function on $\mathbf{D}(R)$ with $f'(0) \neq 0$. Then for all $r < R$, we have*

$$\mathring{T}(f,r) \geq \log r + \log f'(0).$$

Proof. Because replacing f by $1/f$ affects neither $f'(0)$ nor $\mathring{T}(f,r)$, by Theorem 1.11.3, we assume that $f(0) \neq \infty$. In that case, the First Main Theorem and equation (1.2.4), which defines $\mathring{T}$, give us

$$\begin{aligned} \mathring{T}(f,r) &= \mathring{T}(f, f(0), r) \\ &= \mathring{m}(f, f(0), r) + N(f, f(0), r) \\ &\qquad + \log|\mathrm{ilc}(f - f(0), 0)| + 2\log \|f(0), \infty\|. \end{aligned}$$

Now, $\mathring{m}$ is always positive, and directly from the definition, since f is assumed unramified at 0,

$$N(f, f(0), r) \geq \log r.$$

Since $f'(0) \neq 0$, the other two terms combine together to give $\log f'(0)$. □

Corollary 2.8.2 *If $f'(0) = 1$, then $\mathring{T}(f,r) \geq e$ for all $r \geq e^e$, or in other words, $r_0(f) \leq e^e$.*

Remark. The function $f(z) = z$ shows that the constant e^e cannot be improved.

The estimate for b_1 is more involved than the very elementary estimate we just gave for r_0. Recall that b_1 is a constant such that

$$b_1 r_0 \frac{d}{dr}\bigg|_{r_0} \mathring{T}(f,r) \geq e,$$

or what is the same thing,

$$\int_{\mathbf{D}(r_0)} (f')^2 \Phi \geq \frac{e}{b_1}.$$

In the case we are interested in we will take $r_0 = e^e$, so the estimate for b_1 amounts to finding a lower bound for

$$A(f, e^e) = \int_{\mathbf{D}(e^e)} (f^\sharp)^2 \Phi.$$

Geometrically this integral measures how much area on the Riemann sphere is covered by the function f. Thus giving an upper bound on b_1 amounts to giving a lower bound on the spherical area covered by f. The constant b_1 is then also a natural geometric constant.

We base our bound on b_1 on the following theorem of Dufresnoy [Duf 1941].

Theorem 2.8.3 (Dufresnoy) *Suppose f is meromorphic on $\overline{\mathbf{D}(R)}$ and suppose $A(f, R) < 1$. Then, for all $0 < r < R$, we have*

$$\frac{1}{r^2} \frac{A(f,r)}{1 - A(f,r)} \le \frac{1}{R^2} \frac{A(f,R)}{1 - A(f,R)}.$$

Moreover,

$$[f^\sharp(0)]^2 \le \frac{1}{R^2} \frac{A(f,R)}{1 - A(f,R)}.$$

Proof. To prove the theorem we need a spherical isoperimetric inequality, which we will prove in §2.9. Theorem 2.9.3 will tell us that because $A(f, R) < 1$, we have

$$A(f,r)[1 - A(f,r)] \le [L(f,r)]^2,$$

for all $r \le R$, where

$$L(f,r) = \int_0^{2\pi} f^\sharp(re^{i\theta}) \frac{d\theta}{2\pi}.$$

We apply the Cauchy-Schwarz Inequality to the L^2 term,

$$[L(f,r)]^2 \le \left[\int_0^{2\pi} 1 \frac{d\theta}{2\pi}\right] \left[\int_0^{2\pi} [f^\sharp(re^{i\theta})]^2 \frac{d\theta}{2\pi}\right].$$

We recognize the right hand side as

$$\int_0^{2\pi} [f^\sharp(re^{i\theta})]^2 \frac{d\theta}{2\pi} = \frac{1}{2r} \frac{dA(f,r)}{dr} = \frac{1}{2} \frac{dA(f,r)}{d \log r}.$$

and hence we have

$$2A(f,r)[1 - A(f,r)] \le \frac{dA(f,r)}{d \log r}.$$

Thus,

$$2 \int_r^R d \log r \le \int_r^R \frac{dA(f,r)}{A(f,r)[1 - A(f,r)]},$$

and we get

$$\log \frac{R^2}{r^2} \le \log \frac{A(f,R)/[1 - A(f,R)]}{A(f,r)/[1 - A(f,r)]}.$$

Exponentiating results in

$$\frac{1}{r^2} \frac{A(f,r)}{1 - A(f,r)} \le \frac{1}{R^2} \frac{A(r,R)}{1 - A(f,R)}.$$

Now, for r near 0, we see from the definition of $A(f,r)$ that $A(f,r)$ is like $[f^\sharp(0)]^2 r^2$. Thus, taking the limit as $r \to 0$ in the above inequality results in

$$[f^\sharp(0)]^2 \le \frac{1}{R^2} \frac{A(f,R)}{1 - A(f,R)}. \quad \square$$

Corollary 2.8.4 *Set*

$$\tilde{b}_1 = e(1 + e^{-2e})$$

If f is a meromorphic function with $f^\sharp(0) = 1$, then

$$\tilde{b}_1 r_0 \frac{d}{dr}\bigg|_{r_0} \mathring{T}(f,r) \ge e,$$

for all $r_0 \ge e^e$.

Remark. Again the function $f(z) = z$ shows the constant $\tilde{b}_1$ cannot be improved

Proof. It suffices to assume $A(f, e^e) < 1$, in which case we can apply Theorem 2.8.3 to conclude that

$$1 = [f^\sharp(0)]^2 \le \frac{1}{e^{2e}} \frac{A(f,e^e)}{1 - A(f,e^e)}.$$

Isolating $A(f, e^e)$ gives us that

$$A(f, e^e) \ge \frac{1}{1 + e^{-2e}}.$$

Taking the reciprocal and multiplying by e gives us our bound. $\square$

Theorem 2.8.5 (Uniform Second Main Theorem) *Let $a_1, \ldots, a_q$ be a finite set of distinct points in $\mathbf{P}^1$, and let f be a non-constant meromorphic function on $\mathbf{D}(R)$. Assume further that $f(0) \neq a_j$ for all j and that $f'(0) = 1$. Let ψ, ϕ, k_0, b_3, and $\dot{D}(a_1, \ldots, a_q)$ be defined as in §2.3 (and note that none of the constants depends on f). Let $\tilde{b}_1 \leq 2.74$ be the constant defined in Corollary 2.8.4. Then, there exists a measurable set E of radii with*

$$\int_E \frac{dr}{\phi(r)} \leq 2k_0(\psi)$$

such that for all $r \geq e^e$ and outside E, the following inequality holds:

$$\begin{aligned}(q-2)\dot{T}(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{ram}}(f,r) \\ \leq \frac{1}{2}\Bigg[\log b_3\dot{T}^2(f,r) + \log\left\{\frac{\psi(b_3\dot{T}^2(f,r))}{\phi(r)}\right\} \\ + \log\left\{\frac{\psi\left(\tilde{b}_1 r b_3\dot{T}^2(f,r)\dfrac{\psi(b_3\dot{T}^2(f,r))}{\phi(r)}\right)}{\phi(r)}\right\}\Bigg] \\ + \dot{D}(a_1,\ldots,a_q) + 1.\end{aligned}$$

Proof. Given Corollary 2.8.2 and Corollary 2.8.4, this "theorem" is an immediate corollary of Theorem 2.5.1. □

2.9 The Spherical Isoperimetric Inequality

In this section we introduce the spherical isoperimetric inequality which we needed to prove Theorem 2.8.3, which we used in §2.8 to give the best possible estimate on $\tilde{b}_1$. Our presentation here is potential theoretic and based on a paper of A. Huber [Hub 1954], with some ideas from [Ran 1995]. As such, this section is of a rather different character from the rest of the book. We work toward our goal of our specific case of the isoperimetric inequality as quickly as possible and do not attempt to give a detailed introduction to potential theory or isoperimetric inequalities in general. To try to keep things as simple as possible for those reader unfamiliar with potential theory and subharmonic functions, we work only with the specific subharmonic functions we have already met in this chapter. Those already familiar with the theory will recognize the general principles behind what we do here. Those readers interested only in Nevanlinna's theory may skip this section without impairing their understanding of the rest of the book.

We begin with the Ahlfors-Beurling Inequality [AhlBeu 1950].

Theorem 2.9.1 (Ahlfors-Beurling Inequality) *Let K be a compact subset of $\mathbf{C}$ and let $A = \int_K \Phi$, where Φ is the Euclidean form defined in §1.9. Then, for any z in $\mathbf{C}$,*

$$\left|\int_K \frac{1}{w-z}\Phi(w)\right| \leq \sqrt{A}.$$

Proof. The inequality is clear if $A = 0$, so we may assume A positive. By translating and rotating K, we may assume that $z = 0$ and that $\int_K w^{-1}\Phi(w)$ is positive. Let

$$\Delta = \left\{w : \mathrm{Re}\left(\frac{1}{w}\right) > \frac{1}{2\sqrt{A}}\right\},$$

which is a disc of radius $\sqrt{A}$. Then, since $\int_K w^{-1}\Phi(w)$ is real,

$$\int_K \frac{1}{w}\Phi(w) = \int_K \mathrm{Re}\left(\frac{1}{w}\right)\Phi(w).$$

By the definition of Δ, and since

$$\int_\Delta \Phi = A = \int_K \Phi,$$

we have

$$\begin{aligned}\int_K \mathrm{Re}\left(\frac{1}{w}\right)\Phi(w) &\leq \int_{K\cap\Delta} \mathrm{Re}\left(\frac{1}{w}\right)\Phi(w) + \int_{K\setminus(K\cap\Delta)} \frac{1}{2a}\Phi(w) \\ &= \int_{K\cap\Delta} \mathrm{Re}\left(\frac{1}{w}\right)\Phi(w) + \int_{\Delta\setminus(K\cap\Delta)} \frac{1}{2a}\Phi(w).\end{aligned}$$

Thus,

$$\begin{aligned}\int_K \frac{1}{w}\Phi(w) &\leq \int_\Delta \mathrm{Re}\left(\frac{1}{w}\right)\Phi(w) \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\sqrt{A}\cos\theta} \frac{\cos\theta}{r}\frac{r\,dr\,d\theta}{\pi} = \sqrt{A}. \quad \square\end{aligned}$$

This leads immediately to the following version of the planar isoperimetric inequality.

Theorem 2.9.2 (Isoperimetric Inequality) *Let F be analytic on $\overline{\mathbf{D}(r)}$. Then,*

$$\int_{\mathbf{D}(r)} |F|^2\Phi \leq \left[\int_0^{2\pi} |F(re^{i\theta})|\frac{d\theta}{2\pi}\right]^2.$$

Remark. This form of the isoperimetric inequality is due to Carleman [Carl 1921].

Proof. Let G be analytic on $\overline{\mathbf{D}(r)}$ such that $G' = F$. Define

$$A \stackrel{\text{def}}{=} \int_{\mathbf{D}(r)} |F|^2\Phi = \int_{\mathbf{D}(r)} G^*\Phi,$$

$$\text{and} \quad L \stackrel{\text{def}}{=} \int_0^{2\pi} |F(re^{i\theta})|\frac{2\theta}{2\pi} = \int_0^{2\pi} |G'(re^{i\theta})|\frac{2\theta}{2\pi}.$$

Then, by a change of variables formula that counts multiplicity,

$$\begin{aligned}
A &= \int_{\mathbf{D}(r)} G^*\Phi = \int_{G(\mathbf{D}(r))} \left[\int_0^{2\pi} \frac{G'(z)}{G(z)-w}\frac{dz}{2\pi\sqrt{-1}}\right]\Phi(w) \\
\text{[Fubini]} \quad &= \int_0^{2\pi}\int_{G(\mathbf{D}(r))} \frac{1}{G(z)-w}\,\Phi(w)\,\frac{G'(z)dz}{2\pi\sqrt{-1}} \\
&\leq \int_0^{2\pi}\left|\int_{G(\mathbf{D}(r))} \frac{1}{G(z)-w}\,\Phi(w)\right| |G'(re^{i\theta})|\,\frac{d\theta}{2\pi} \\
&= \left|\int_{G(\mathbf{D}(r))} \frac{1}{G(z)-w}\,\Phi(w)\right| \int_0^{2\pi} |G'(re^{i\theta})|\,\frac{d\theta}{2\pi} \\
\text{[Ahlfors-Beurling]} \quad &\leq \sqrt{\int_{G(\mathbf{D}(r))} \Phi(w)\cdot L} \\
\text{[Change of Variables]} \quad &\leq \sqrt{\int_{\mathbf{D}(r)} G^*\Phi}\cdot L = L\sqrt{A}. \quad \square
\end{aligned}$$

Remark. One sees the geometric content of the theorem most clearly when one thinks of $F = G'$ with G a conformal homeomorphism of $\mathbf{D}(R)$. Then, Theorem 2.9.2 says that of all curves of a fixed length, none encloses more area than a circle. In fact, one can also show that if equality holds, then the boundary curve must be a circle.

Now we will work towards a spherical isoperimetric inequality. For the remainder of this section we will let f be a meromorphic function on $\mathbf{D}(R)$, and will will have $0 < r < R$. We also define

$$\boxed{\begin{aligned} L(f,r) &= \int_0^{2\pi} f^\sharp(re^{i\theta})\,\frac{d\theta}{2\pi} \\ A(f,r) &= \int_{\mathbf{D}(r)} f^*\omega = \int_{\mathbf{D}(r)} (f^\sharp)^2\,\Phi. \end{aligned}}$$

Theorem 2.9.3 (Spherical Isoperimetric Inequality) *With notation as above,*

$$A(f,r)[1 - A(f,r)] \leq [L(f,r)]^2.$$

Remark. Note that the spherical isoperimetric inequality only has non-trivial content when $A(f,r) < 1$. This should not be a surprise because if $A(f,r)$ is more than one, then f can map $\mathbf{D}(r)$ onto the whole sphere.

Before proceeding to prove Theorem 2.9.3, we need to recall the notion of "subharmonic function." A function ϕ is said to be **subharmonic** if for every point z_0 and for every ρ such that the disc $|z - z_0| < \rho$ is in the domain of ϕ, we have

$$\phi(z_0) \leq \int_0^{2\pi} \phi(z_0 + \rho e^{i\theta})\frac{d\theta}{2\pi}.$$

This inequality is known as the **sub mean value property**. Compare this with the mean value property of harmonic functions, where equality holds. Also note that if a harmonic function is added to or subtracted from a subharmonic function, the result is still subharmonic.

As in the previous sections we choose holomorphic functions f_0 and f_1 without common zeros so that $f = f_1/f_0$, and we let W be the Wronskian of f_0 and f_1.

Proposition 2.9.4 *The functions*

$$z \mapsto \log|W(z)| \quad \textit{and} \quad z \mapsto \log(|f_0(z)|^2 + |f_1(z)|^2)$$

are subharmonic on $\mathbf{D}(R)$.

Proof. Fix z_0 in $\mathbf{D}(R)$ and let $\rho < R - |z_0|$. We need to show

$$\log|W(z_0)| \leq \int_0^{2\pi} \log|W(z_0 + \rho e^{i\theta})|\,\frac{d\theta}{2\pi} \quad \text{and}$$

$$\log(|f_0(z_0)|^2 + |f_1(z_0)|^2) \leq \int_0^{2\pi} \log(|f_0(z_0 + \rho e^{i\theta})|^2 + |f_1(z_0 + \rho e^{i\theta})|^2)\,\frac{d\theta}{2\pi}.$$

If z_0 is a zero of W, then the inequality is clear, so we may assume $\log|W|$ is continuous at z_0. Thus, we have by continuity in both cases, that

$$\log|W(z_0)| = \lim_{\varepsilon\to 0}\int_0^{2\pi} \log|W(z_0+\varepsilon e^{i\theta})|\,\frac{d\theta}{2\pi} \qquad \text{and}$$

$$\log(|f_0(z_0)|^2+|f_1(z_0)|^2) = \lim_{\varepsilon\to 0}\int_0^{2\pi} \log(|f_0(z_0+\varepsilon e^{i\theta})|^2+|f_1(z_0+\varepsilon e^{i\theta})|^2)\,\frac{d\theta}{2\pi}.$$

In both cases, we use Stokes's Theorem (Theorem 1.10.1) to compare the integral along $|z_0-z|=\rho$ and $|z_0-z|=\varepsilon$. Noting that away from singularities $dd^c\log|W|=0$, we have

$$\int_0^{2\pi}\log|W(z_0+\rho e^{i\theta})|\,\frac{d\theta}{2\pi} \geq \int_0^{2\pi}\log|W(z_0+\varepsilon e^{i\theta})|\,\frac{d\theta}{2\pi}$$

by Proposition 1.11.2. On the other hand,

$$\int_0^{2\pi}\log(|f_0(z_0+\varepsilon e^{i\theta})|^2+|f_1(z_0+\varepsilon e^{i\theta})|^2)\,\frac{d\theta}{2\pi} \geq \int_0^{2\pi}\log(|f_0(z_0+\rho e^{i\theta})|^2+|f_1(z_0+\rho e^{i\theta})|^2)\,\frac{d\theta}{2\pi}$$

by Proposition 1.11.1. □

We now use Theorem 1.1.5 to create harmonic functions associated with the two subharmonic functions in Proposition 2.9.4. Henceforth we fix r so that the Wronskian W has no zeros on the circle $|z|=r$, and let $P(z,\zeta)$ be the Poisson kernel. Define

$$\boxed{\begin{aligned} u_1(z) &= \textstyle\int_0^{2\pi}\log|W(re^{i\theta})|P(z,re^{i\theta})\,\frac{d\theta}{2\pi}\\ u_2(z) &= \textstyle\int_0^{2\pi}\log(|f_0(re^{i\theta})|^2+|f_1(re^{i\theta})|^2)P(z,re^{i\theta})\,\frac{d\theta}{2\pi}.\end{aligned}}$$

Then, by the definition of P and Theorem 1.1.5, u_1 and u_2 are harmonic functions that agree with $\log|W|$ and $\log(|f_0|^2+|f_1|^2)$ respectively on the boundary circle $|z|=r$. Let

$$\Psi_1(z) = u_1(z)-\log|W(z)| \qquad \text{and} \qquad \Psi_2(z) = u_2(z)-\log(|f_0(z)|^2+|f_1(z)|^2).$$

Then, because $-\Psi_1$ and $-\Psi_2$ are subharmonic functions that are identically zero on the boundary circle $|z|=r$, we see from the sub mean value property for subharmonic functions that Ψ_1 and Ψ_2 are non-negative functions. We also define $\boxed{u=u_1-u_2.}$

It turns out it will be useful to have an integral representation for Ψ_2, so we briefly introduce here the concept of a "Green's function," which has a close relationship with the Poisson kernel. We define the **Green's function** $g(z,\zeta)$ for $\mathbf{D}(r)$ by

$$g(z,\zeta) = \log\left|\frac{r^2-\bar{\zeta}z}{r(z-\zeta)}\right|.$$

Notice g is harmonic in both variables except for the singularity at $z=\zeta$. Notice also that g is defined in terms of a Möbius automorphism of $\mathbf{D}(r)$. Those readers familiar with distribution theory will recognize this as a distribution that inverts the Laplacian. We make a connection here to the Poisson kernel.

Proposition 2.9.5 *Let $g(z,\zeta)$ be the Green's function for $\mathbf{D}(r)$ and let $P(z,\zeta)$ be the Poisson kernel. Then for $|\zeta|=r$ and all $z\in\mathbf{D}(r)$, we have*

$$-d^c_\zeta g(z,\zeta) = P(z,\zeta)d^c\log|\zeta|$$

Remark. The ζ in the subscript of d^c is to remind us that we are differentiating with respect to ζ and keeping z fixed.

Proof. Let

$$\zeta = T(w) = \frac{r^2(z-w)}{r^2-\bar{z}w},$$

as in Proposition 1.1.1. Then, Proposition 1.1.1 tells us that for $|w|=r$,

$$d^c\log|w| = P(z,\zeta)d^c\log|\zeta|.$$

But,

$$\begin{aligned}\log|w| &= \log\left|\frac{r^2(z-\zeta)}{r^2-\bar{z}\zeta}\right|\\ &= \log\left|\frac{r^2(z-\zeta)}{r^2-\bar{\zeta}z}\right|\\ &= -g(z,\zeta)+\log r \quad \square\end{aligned}$$

This allows us to find our integral representation for Ψ_2.

Proposition 2.9.6 *With the notation above,*

$$\Psi_2(z) = 2\int_{\zeta\in\mathbf{D}(r)} g(z,\zeta)f^*\omega(\zeta).$$

Proof. By Proposition 1.11.1, we have

$$\int_{\zeta\in\mathbf{D}(r)} g(z,\zeta)f^*\omega(\zeta) = \int_{\zeta\in\mathbf{D}(r)} g(z,\zeta)dd^c\log(|f_0(\zeta)|^2 + |f_1(\zeta)|^2).$$

Let α and β be C^2 functions. Then,

$$d(\alpha d^c\beta - \beta d^c\alpha) = d\alpha\wedge d^c\beta + \alpha dd^c\beta - d\beta\wedge d^c\alpha - \beta dd^c\alpha.$$

Since $d\alpha\wedge d^c\beta = d\beta\wedge d^c\alpha$, we find from Stokes's Theorem that,

$$\int_{\mathbf{D}(r)} \alpha dd^c\beta - \int_{\mathbf{D}(r)} \beta dd^c\alpha = \int_{|z|=r} \alpha d^c\beta - \int_{|z|=r} \beta d^c\alpha.$$

We will apply this with

$$\alpha(\zeta) = g(z,\zeta) \qquad \text{and} \qquad \beta(\zeta) = \log(|f_0(\zeta)|^2 + |f_1(\zeta)|^2),$$

although we have to remember that α is singular at z. In fact, except for this singularity, α is harmonic, and hence we can use Proposition 1.11.2 to calculate that the singular term in Stokes's Theorem will be precisely

$$-\frac{1}{2}\log(|f_0(z)|^2 + |f_1(z)|^2).$$

Because $g(z,\zeta) = 0$ for $|\zeta| = r$, we only have to compute the boundary term

$$\begin{aligned} -\int_0^{2\pi} &\log(|f_0(re^{i\theta})|^2 + |f_1(re^{i\theta})|^2)d^c g(z,re^{i\theta}) \\ &= \frac{1}{2}\int_0^{2\pi} \log(|f_0(re^{i\theta})|^2 + |f_1(re^{i\theta})|^2)P(z,re^{i\theta})\frac{d\theta}{2\pi} \\ &= \frac{1}{2}u_2(z) \end{aligned}$$

from Proposition 2.9.5 and the definition of u_2. Putting this together we find

$$2\int_{\zeta\in\mathbf{D}(r)} g(z,\zeta)f^*\omega(\zeta) = u_2(z) - \log(|f_0(z)|^2 + |f_1(z)|^2) = \Psi_2(z). \qquad \square$$

We now begin putting together the proof of Theorem 2.9.3. We being by applying the planar isoperimetric inequality as follows.

Lemma 2.9.7 *If u is the harmonic function defined earlier in this section, then*

$$\int_{\mathbf{D}(r)} e^{2u}\Phi \le [L(f,r)]^2.$$

Proof. Let u^* be a conjugate harmonic function for u on $\mathbf{D}(r)$. Then

$$F = e^{u+iu^*}$$

is holomorphic on $\mathbf{D}(r)$ with $|F| = e^u$. Thus, the planar isoperimetric inequality (Theorem 2.9.2) tells us

$$\int_{\mathbf{D}(r)} e^{2u}\Phi \le \left[\int_0^{2\pi} e^u(re^{i\theta})\frac{d\theta}{2\pi}\right]^2.$$

Because Ψ_1 and Ψ_2 vanish on $|z| = r$, the integral on the right is just

$$\begin{aligned} \int_0^{2\pi} &\exp[\log|W(re^{i\theta})| - \log(|f_0(re^{i\theta})|^2 + |f_1(re^{i\theta})|^2)]\frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} f^\sharp(re^{i\theta})\frac{d\theta}{2\pi} = L(f,r). \qquad \square \end{aligned}$$

Beginning of the proof of Theorem 2.9.3. Let $A(f,r)$, u, Ψ_1, and Ψ_2 be defined as earlier in the section. Then,

$$\begin{aligned} A(f,r) &= \int_{\mathbf{D}(r)} f^*\omega \\ &= \int_{\mathbf{D}(r)} (f^\sharp)^2\Phi \\ &= \int_{\mathbf{D}(r)} \exp(2[\log|W| - \log(|f_0|^2 + |f_1|^2)])\Phi \\ &= \int_{\mathbf{D}(r)} e^{2u-2\Psi_1+2\Psi_2}\Phi \\ &\le \int_{\mathbf{D}(r)} e^{2u+2\Psi_2}\Phi \end{aligned}$$

since Ψ_1 is non-negative.

Let

$$\eta(z) = \int_0^z \exp(u + iu^*),$$

where again u^* is a conjugate harmonic function for u on $\mathbf{D}(r)$. Clearly η always has non-zero derivative. Therefore, we can find a simply connected Riemann surface

X spread over a planer domain D so that X is conformally homeomorphic to $\mathbf{D}(r)$. Then, let $\mathcal{F}$ be the isometric mapping from X down onto D. By the change of variables formula,

$$\int_{\mathbf{D}(r)} e^{2u+2\Psi_2}\Phi = \int_{\mathbf{D}(r)} e^{2\Psi_2}|\eta'|^2\Phi = \int_X e^{2\Psi_2\circ\eta^{-1}\circ\mathcal{F}}\mathcal{F}^*\Phi.$$

We now take a break from the proof of Theorem 2.9.3 in order to make some estimates on the surface X. Our first lemma comes from [PoSz 1945, IV.89].

Lemma 2.9.8 *Let D be a planar domain and let $\mathcal{F} : X \to D$ be an analytic map with non-vanishing derivative from a simply connected Riemann surface X. Assume*

$$\int_X \mathcal{F}^*\Phi < \infty.$$

Let $\mathcal{G} : X \to \mathbf{D}(\rho)$ be a conformal homeomorphism of X onto a disc, and choose ρ so that

$$\rho^2 = \int_X \mathcal{F}^*\Phi.$$

Then for all $\rho' \leq \rho$,

$$\int_{\mathcal{G}^{-1}(\mathbf{D}(\rho'))} \mathcal{F}^*\Phi \leq (\rho')^2.$$

Proof. Write

$$\mathcal{F}\circ\mathcal{G}^{-1}(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Then,

$$A \stackrel{\text{def}}{=} \int_X \mathcal{F}^*\Phi = \int_0^\rho \int_0^{2\pi} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} nmc_n\bar{c}_m t^{m+n-2}e^{i(n-m)\theta}\,\frac{d\theta}{2\pi}\,2tdt.$$

By periodicity, the θ integrals remove all terms from the sum where $m \neq n$, and we are left with

$$\int_X \mathcal{F}^*\Phi = \int_0^\rho \sum_{n=1}^{\infty} n^2|c_n|^2 t^{2n-2}\,2tdt = \sum_{n=1}^{\infty} n|c_n|^2\rho^{2n}.$$

Similarly,

$$a \stackrel{\text{def}}{=} \int_{\mathcal{G}^{-1}(\mathbf{D}(\rho'))} \mathcal{F}^*\Phi = \sum_{n=1}^{\infty} n|c_n|^2(\rho')^{2n}.$$

Thus,

$$\frac{A}{a} - \frac{\rho^2}{(\rho')^2} = \frac{(\rho')^2\rho^2\sum_{n=2}^{\infty} n|c_n|^2(\rho^{2n-2} - (\rho')^{2n-2})}{\sum_{n=1}^{\infty} n|c_n|^2(\rho')^{2n-2}} \geq 0.$$

The lemma follows from the assumption that $A = \rho^2$. □

Corollary 2.9.9 *Let X be the Riemann surface for the proof of Theorem 2.9.3. Choose ρ so that*

$$\rho^2 = \int_X \mathcal{F}^*\Phi.$$

Let x_0 in X and let $\mathcal{G}$ be a conformal homeomorphism from X onto $\mathbf{D}(\rho)$ such that $\mathcal{G}(x_0) = 0$. Note that such a $\mathcal{G}$ exists by the Riemann mapping theorem. Assume $A(f,r) < 1$. Let $g(z,0) = \log\rho - \log|z|$ be the Green's function on $\mathbf{D}(\rho)$. Then,

$$\int_X \exp[2A(f,r)g(\mathcal{G}(z),0)]\mathcal{F}^*\Phi \leq \frac{[L(f,r)]^2}{1 - A(f,r)}.$$

Proof. Because the level lines of the function $g(z,0)$ are concentric circles centered at the origin in $\mathbf{D}(\rho)$, we conclude from Lemma 2.9.8 that

$$\begin{aligned}\int_X \exp[2A(f,r)g(\mathcal{G}(z),0)]\mathcal{F}^*\Phi &\leq \int_{\mathbf{D}(\rho)} \exp[2A(f,r)g(z,0)]\Phi \\ &= \int_0^{2\pi}\int_0^\rho \exp[2A(f,r)(\log\rho - \log t)\,2tdt\,\frac{d\theta}{2\pi} \\ &= \frac{\rho^2}{1 - A(f,r)}.\end{aligned}$$

The corollary then follows from

$$\rho^2 = \int_X \mathcal{F}^*\Phi = \int \mathbf{D}(r)e^{2u}\Phi \leq [L(f,r)]^2,$$

with the last inequality being Lemma 2.9.7. □

Lemma 2.9.10 *Let X, $\mathcal{G}$, $\mathcal{F}$, and u be as above. Assume that $A(f,r) < 1$. Then*

$$\int_X \exp[2\Psi_2\circ\eta^{-1}\circ\mathcal{F}]\mathcal{F}^*\Phi \leq \frac{[L(f,r)]^2}{1 - A(f,r)}.$$

Proof. By Lemma 2.9.6, we can compute Ψ_2 as an integral representation involving the Green's functions. Pulling this up to X, that means

$$\int_X \exp[2\Psi_2 \circ \eta^{-1} \circ \mathcal{F}]\mathcal{F}^*\Phi = \int_{z\in X} \exp[4 \int_{w\in X} g(\mathcal{G}(z), \mathcal{G}(w))d\mu(w)]\mathcal{F}^*\Phi,$$

where $d\mu$ is the Borel measure on X equal to $(f \circ \eta^{-1} \circ \mathcal{F})^*\omega$. For N a positive constant, let

$$g_N(z,w) = \min\{N, g(\mathcal{G}(z), \mathcal{G}(w))\}$$

and let

$$J_N = \int_{z\in X} \exp[2 \int_{w\in X} g_N(z,w)d\mu(w)]\mathcal{F}^*\Phi.$$

The point is that by cutting off the Green's function, the integral in the exponent is well behaved and we can now approximate the Borel measure μ by a discrete collection of point masses. Indeed, suppose that the points $w_1, \ldots, w_m$ with weights $q_1, \ldots, q_m$ approximate $d\mu$, which of course implies that

$$\sum_{j=1}^m q_j = \int_X d\mu = A(f,r).$$

For convenience, set $p_j = q_j/A(f,r)$. Then,

$$\begin{aligned} J_N &= \int_{z\in X} \exp[2\sum q_j g_N(z,w_j)]\mathcal{F}^*\Phi \\ &= \int_{z\in X} \prod_{j=1}^m [\exp[2A(f,r)g_N(z,w_j)]]^{p_j} \mathcal{F}^*\Phi. \end{aligned}$$

Now we apply the Hölder inequality to get that the integral on the right is bounded by

$$\prod_{j=1}^m \left[\int_{z\in X} \exp[2A(f,r)g_N(z,w_j)]\mathcal{F}^*\Phi \right]^{p_j}.$$

Now that the second integration is gone, we are in the situation of Corollary 2.9.9. The proof is completed by taking a limit as $N \to \infty$. □

Completion of the proof of Theorem 2.9.3. Lemma 2.9.10 essentially completes the proof of Theorem 2.9.3, except for the technical detail that we often assumed that the Wronskian W did not have any zeros on $|z| = r$. This is not a serious problem since these r values are discrete and the integrals in the statement of the theorem are well-behaved for all values of r. Thus, we can conclude that the theorem holds for all values of r by, for example, Lebesgue dominated convergence. □

3 Logarithmic Derivatives

For obvious reasons, a meromorphic function g is said to be a **logarithmic derivative** if

$$g = \frac{f'}{f}$$

for some meromorphic function f. Not any meromorphic function can be expressed as a logarithmic derivative, and thus one might ask what special properties functions which are logarithmic derivatives posses. For example, if $g = f'/f$ is a logarithmic derivative, then all the poles of g must be simple poles. In terms of what to expect from a value distribution point of view, we again look at polynomials for a clue. If P is a polynomial, then P' has degree smaller than P, and so P'/P stays small as z goes to infinity. Of course, one cannot expect exactly this behavior in general. For example, if $f = e^{z^2}$, then $f'/f = 2z$, and so $|f'/f| \to \infty$ as $|z| \to \infty$. However, we see here that the rate at which $\log|f'/f|$ approaches infinity is very slow by comparison to $T(f,r)$. The main result of this chapter is what is known as the Lemma on the Logarithhmic Derivative (Theorem 3.4.1), which essentially says that if f is a meromorphic function, then the integral means of $\log^+|f'/f|$ over large circles cannot approach infinity quickly compared with the rate at which $T(f,r)$ tends to infinity.

The Lemma on the Logarithmic Derivative was first proved by R. Nevanlinna and was the basis for his first proof of the Second Main Theorem. The presentation we give in this chapter is closer to that given in [GoOs 1970]. However, we incorporate various refinements from the work of A. Gol'dberg and A. Grinshtein [GoGr 1976], as in the work of J. Miles [Miles 1992]. In fact, it is often the Gol'dberg-Grinshtein result, either in the form of Lemma 3.2.1 or in the form of Theorem 3.2.2, that is most convenient to apply in applications, particularly if one is after the best error terms.

3.1 Inequalities of Smirnov and Kolokolnikov

Before stating any results about logarithmic derivatives, we derive some preliminary estimates. Our first lemma is due to V. Smirnov [Smir 1928].

Lemma 3.1.1 (Smirnov's inequality) *Let $R < \infty$, and let $F(z)$ be analytic in the disc $\mathbf{D}(R)$. For $0 \le \theta \le 2\pi$, define $u(\theta)$ by*

$$u(\theta) = \liminf_{\substack{z \to Re^{i\theta}, \\ |z|<R}} |F(z)|.$$

If either of the functions $\operatorname{Re} F(z)$ or $\operatorname{Im} F(z)$ has constant sign, then for any α with $0 < \alpha < 1$, we have

$$\int_0^{2\pi} u^\alpha(\theta)\frac{d\theta}{2\pi} \le \sec(\pi\alpha/2)\,|F(0)|^\alpha.$$

Proof. Assume, without loss of generality, that $\operatorname{Re} F(z) > 0$ for $|z| < R$. The function F is non-zero in $\mathbf{D}(R)$, and we can fix a choice of $\arg F(z)$ so that $|\arg F(z)| < \pi/2$ for $|z| < R$. Thus the function

$$F^\alpha(z) = |F(z)|^\alpha e^{i\alpha \arg F(z)}$$

is analytic in $\mathbf{D}(R)$, and it follows that $\operatorname{Re}\{F^\alpha(z)\}$ is harmonic. Since

$$\operatorname{Re} F^\alpha(z) = |F(z)|^\alpha \cos(\alpha \arg F(z)) \ge |F(z)|^\alpha \cos(\alpha\pi/2),$$

we have

$$\int_0^{2\pi} |F(re^{i\theta})|^\alpha \frac{d\theta}{2\pi} \le \sec(\alpha\pi/2) \int_0^{2\pi} \operatorname{Re} F^\alpha(re^{i\theta})\frac{d\theta}{2\pi} = \sec(\alpha\pi/2)\operatorname{Re} F^\alpha(0),$$

and so

$$\int_0^{2\pi} |F(re^{i\theta})|^\alpha \frac{d\theta}{2\pi} \le \sec(\alpha\pi/2)\,|F(0)|^\alpha, \quad \text{for any } r < R.$$

Finally, we conclude the proof of the lemma by passing to the limit as $r \to R$ and making use of Fatou's Lemma. □

We will apply Smirnov's inequality in the form of the following inequality due to A. Kolokolnikov, who also studied logarithmic derivatives [Kol 1974].

Lemma 3.1.2 (Kolokolnikov's inequality) *Let $R < \infty$, and let $\{c_k\}$ be a finite sequence of complex numbers in $\mathbf{D}(R)$. For $\delta_k = \pm 1$, define*

$$H(z) = \sum_k \frac{\delta_k}{z - c_k}.$$

Then, for any α with $0 < \alpha < 1$, and for $0 < r < R$,

$$\int_0^{2\pi} |H(re^{i\theta})|^\alpha \frac{d\theta}{2\pi} \le (2 + 2^{2-\alpha})\sec(\alpha\pi/2)\left(\frac{n(H,\infty,R)}{r}\right)^\alpha.$$

Proof. Let $\varphi_k = \arg c_k$, $p_k = \delta_k \cos\varphi_k$, and $q_k = \delta_k \sin\varphi_k$. We can then write the function $H(z)$ in the form

$$\begin{aligned} H(z) &= \sum_{\substack{|c_k|>r \\ p_k>0}} \frac{e^{i\varphi_k} p_k}{z - c_k} + \sum_{\substack{|c_k|>r \\ p_k<0}} \frac{e^{i\varphi_k} p_k}{z - c_k} \\ &\quad + \sqrt{-1}\sum_{\substack{|c_k|>r \\ q_k>0}} \frac{-e^{i\varphi_k} q_k}{z - c_k} + \sqrt{-1}\sum_{\substack{|c_k|>r \\ q_k<0}} \frac{-e^{i\varphi_k} q_k}{z - c_k} \\ &\quad + \sum_{\substack{|c_k|\le r \\ \delta_k=1}} \frac{\delta_k}{z - c_k} + \sum_{\substack{|c_k|\le r \\ \delta_k=-1}} \frac{\delta_k}{z - c_k} \\ &= F_1(z) + F_2(z) + \ldots + F_6(z). \end{aligned}$$

Since for $|c_k| > r$ and $|z| < r$, we have that $\operatorname{Re}\{e^{i\varphi_k}/(z - c_k)\} < 0$, we know the functions $F_1, \ldots, F_4$ satisfy the assumptions of Lemma 3.1.1. Therefore, for $j = 1, \ldots, 4$,

$$\begin{aligned} \int_0^{2\pi} |F_j(re^{i\theta})|^\alpha \frac{d\theta}{2\pi} &\le \sec(\alpha\pi/2)\,|F_j(0)|^\alpha \\ &\le \sec(\alpha\pi/2)\left(\sum_{|c_k|>r} \frac{1}{|c_k|}\right)^\alpha \\ &\le \sec(\alpha\pi/2)\left(\frac{n(H,\infty,R) - n(H,\infty,r)}{r}\right)^\alpha. \end{aligned}$$

When $|z| = r$, we have

$$\left|\sum_{|c_k|\le r} \frac{1}{z - c_k}\right| = \left|\sum_{|c_k|\le r} \frac{r}{r^2 - \bar{c}_k z}\right|.$$

Therefore, if we define

$$\tilde{F}_5(z) = \sum_{\substack{|c_k|\le r \\ \delta_k=1}} \frac{r}{r^2 - \bar{c}_k z},$$

then because when $|c_k| \le r$ and $|z| < r$ we have $\operatorname{Re}(r^2 - \bar{c}_k z) > 0$, we can apply Lemma 3.1.1 to

$$|F_5(re^{i\theta})| = |\tilde{F}_5(re^{i\theta})| = \left|\sum_{\substack{|c_k|\le r \\ \delta_k=1}} \frac{r}{r^2 - \bar{c}_k re^{i\theta}}\right|,$$

and we get

$$\int_0^{2\pi} |F_5(re^{i\theta})|^\alpha \frac{d\theta}{2\pi} \le \sec(\alpha\pi/2)\,|\tilde{F}_5(0)|^\alpha \le \sec(\alpha\pi/2)\left(\frac{n(H,\infty,r)}{r}\right)^\alpha.$$

Similarly, we obtain

$$\int_0^{2\pi} |F_6(re^{i\theta})|^\alpha \frac{d\theta}{2\pi} \le \sec\frac{\alpha\pi}{2}\left(\frac{n(H,\infty,r)}{r}\right)^\alpha.$$

We remark that if d_j are non-negative real numbers and $0 < \alpha < 1$, then

$$\left(\sum_j d_j\right)^\alpha \le \sum_j d_j^\alpha,$$

and we use this remark to conclude that

$$\int_0^{2\pi} |H(re^{i\theta})|^\alpha \frac{d\theta}{2\pi} \le \sum_{j=1}^{6} \int_0^{2\pi} |F_j(re^{i\theta})|^\alpha \frac{d\theta}{2\pi}$$

$$\le 2\sec(\alpha\pi/2)\left(\frac{n(H,\infty,R)}{r}\right)^\alpha\left[2\left(1-\frac{n(H,\infty,r)}{n(H,\infty,R)}\right)^\alpha + \left(\frac{n(H,\infty,r)}{n(H,\infty,R)}\right)^\alpha\right].$$

The proof is completed by noting that if $0 < d \le 1$, then

$$(1-d)^\alpha + d^\alpha \le 2^{1-\alpha}. \qquad \square$$

3.2 The Gol'dberg-Grinshtein Estimate

The fundamental estimate of A. Gol'dberg and A. Grinshtein [GoGr 1976] is essentially the following lemma bounding the integral of $|f'/f|^\alpha$ over the circle of radius r. The bound is in terms of $m(f,[0]+[\infty],s)$, and $n(f,[0]+[\infty],s)$, where s is any number $> r$, and by definition

$$m(f,[0]+[\infty],s) = m(f,0,s) + m(f,\infty,s),$$
$$\text{and} \quad n(f,[0]+[\infty],s) = n(f,0,s) + n(f,\infty,s).$$

We similarly define $N(f,[0]+[\infty],s)$.

Lemma 3.2.1 (Gol'dberg-Grinshtein) *Let f be a meromorphic function on $\mathbf{D}(R)$ $(0 < R \le \infty)$, and let $0 < \alpha < 1$. Then, for $r < s < R$, we have*

$$\int_0^{2\pi} \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^\alpha \frac{d\theta}{2\pi} \le \left(\frac{s}{r(s-r)} m(f,[0]+[\infty],s)\right)^\alpha + (2+2^{3-\alpha})\sec(\alpha\pi/2)\left(\frac{n(f,[0]+[\infty],s)}{r}\right)^\alpha.$$

Proof. Let $a_1,\dots,a_p$ (resp. $b_1,\dots b_q$) denote the zeros (resp. poles) of f in $\mathbf{D}(s)$, repeated according to multiplicity. From the Poisson-Jensen formula (Corollary 1.1.7), we have that for any z in $\mathbf{D}(s)$ which is not a zero or pole of f,

$$\frac{f'(z)}{f(z)} = \int_0^{2\pi} \frac{2se^{i\varphi}}{(se^{i\varphi}-z)^2} \log|f(se^{i\varphi})| \frac{d\varphi}{2\pi} + \sum_{j=1}^{p}\left(\frac{\bar{a}_j}{s^2-\bar{a}_j z} + \frac{1}{z-a_j}\right) - \sum_{j=1}^{q}\left(\frac{\bar{b}_j}{s^2-\bar{b}_j z} + \frac{1}{z-b_j}\right).$$

Consequently,

$$\left|\frac{f'(z)}{f(z)}\right| \le \int_0^{2\pi} \frac{2s}{|se^{i\varphi}-z|^2}\left|\log|f(se^{i\varphi})|\right| \frac{d\varphi}{2\pi} + \left|\sum_{j=1}^{p}\frac{\bar{a}_j}{s^2-\bar{a}_j z} - \sum_{j=1}^{q}\frac{\bar{b}_j}{s^2-\bar{b}_j z}\right| + \left|\sum_{j=1}^{p}\frac{1}{z-a_j} - \sum_{j=1}^{q}\frac{1}{z-b_j}\right|.$$

For $0 < \alpha < 1$ and for positive real numbers d_j, we know that $(\sum d_j)^\alpha \le \sum d_j^\alpha$. Applying this to the above inequality, we get

$$\left|\frac{f'(z)}{f(z)}\right|^\alpha \le \left(\int_0^{2\pi} \frac{2s}{|se^{i\varphi}-z|^2}\left|\log|f(se^{i\varphi})|\right| \frac{d\varphi}{2\pi}\right)^\alpha + \left|\sum_{j=1}^{p}\frac{\bar{a}_j}{s^2-\bar{a}_j z} - \sum_{j=1}^{q}\frac{\bar{b}_j}{s^2-\bar{b}_j z}\right|^\alpha + \left|\sum_{j=1}^{p}\frac{1}{z-a_j} - \sum_{j=1}^{q}\frac{1}{z-b_j}\right|^\alpha.$$

Now, set $z = re^{i\theta}$ and integrate with respect to θ to get

$$\int_0^{2\pi} \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^\alpha \frac{d\theta}{2\pi} \le \int_0^{2\pi}\left(\int_0^{2\pi} \frac{2s}{|se^{i\varphi}-re^{i\theta}|^2}\left|\log|f(se^{i\varphi})|\right| \frac{d\varphi}{2\pi}\right)^\alpha \frac{d\theta}{2\pi}$$
$$+ \int_0^{2\pi}\left|\sum_{j=1}^{p}\frac{\bar{a}_j}{s^2-\bar{a}_j re^{i\theta}} - \sum_{j=1}^{q}\frac{\bar{b}_j}{s^2-\bar{b}_j re^{i\theta}}\right|^\alpha \frac{d\theta}{2\pi}$$
$$+ \int_0^{2\pi}\left|\sum_{j=1}^{p}\frac{1}{re^{i\theta}-a_j} - \sum_{j=1}^{q}\frac{1}{re^{i\theta}-b_j}\right|^\alpha \frac{d\theta}{2\pi}$$
$$= I_1 + I_2 + I_3.$$

By applying the Hölder inequality and interchanging the order of integration, we see that

$$I_1^{1/\alpha} \le \int_0^{2\pi}\left(\int_0^{2\pi} \frac{2s}{|se^{i\varphi}-re^{i\theta}|^2}\left|\log|f(se^{i\theta})|\right| \frac{d\varphi}{2\pi}\right) \frac{d\theta}{2\pi}$$
$$= 2s\int_0^{2\pi}\left(\int_0^{2\pi} \frac{1}{|se^{i\varphi}-re^{i\theta}|^2} \frac{d\theta}{2\pi}\right)\left|\log|f(se^{i\varphi})|\right| \frac{d\varphi}{2\pi}.$$

By the Poisson Formula (Theorem 1.1.2), taking $u \equiv 1$, we see that the inner integral above is equal to

$$\frac{1}{s^2 - r^2}.$$

Moreover,

$$\int_0^{2\pi} \left|\log|f(se^{i\varphi})|\right| \frac{d\varphi}{2\pi} = \int_0^{2\pi} \log^+|f(se^{i\varphi})| \frac{d\varphi}{2\pi} + \int_0^{2\pi} \log^+\left|\frac{1}{f(se^{i\varphi})}\right| \frac{d\varphi}{2\pi}$$
$$= m(f,\infty,s) + m(f,0,s) = m(f,[0]+[\infty],s).$$

Thus,

$$I_1 \leq \left(\frac{2s}{s^2-r^2} m(f,[0]+[\infty],s)\right)^\alpha \leq \left(\frac{s}{r(s-r)} m(f,[0]+[\infty],s)\right)^\alpha,$$

where the last equality follows directly from $s + r \geq 2r$.

Now we estimate I_2. For each zero a_j (resp. each pole b_j), let $\xi_{1j} = \operatorname{Re} a_j$ (resp. $\eta_{1j} = \operatorname{Re} b_j$), and let $\xi_{2j} = \operatorname{Im} a_j$ (resp. $\eta_{2j} = \operatorname{Im} b_j$). Write

$$\sum_{j=1}^{p} \frac{\bar{a}_j}{s^2 - \bar{a}_j z} - \sum_{j=1}^{q} \frac{\bar{b}_j}{s^2 - \bar{b}_j z} = \left(\sum_{\xi_{1j}>0} \frac{\xi_{1j}}{s^2 - \bar{a}_j z} - \sum_{\eta_{1j}<0} \frac{\eta_{1j}}{s^2 - \bar{b}_j z}\right)$$
$$- \left(\sum_{\xi_{1j}\leq 0} \frac{-\xi_{1j}}{s^2 - \bar{a}_j z} - \sum_{\eta_{1j}\geq 0} \frac{-\eta_{1j}}{s^2 - \bar{b}_j z}\right)$$
$$-\sqrt{-1}\left(\sum_{\xi_{2j}>0} \frac{\xi_{2j}}{s^2 - \bar{a}_j z} - \sum_{\eta_{2j}<0} \frac{\eta_{2j}}{s^2 - \bar{b}_j z}\right)$$
$$+\sqrt{-1}\left(\sum_{\xi_{2j}\leq 0} \frac{-\xi_{2j}}{s^2 - \bar{a}_j z} - \sum_{\eta_{2j}\geq 0} \frac{-\eta_{2j}}{s^2 - \bar{b}_j z}\right)$$
$$= h_1(z) - h_2(z) - \sqrt{-1}h_3(z) + \sqrt{-1}h_4(z).$$

Note that we have $\bar{a}_j$ and $\bar{b}_j$ in the numerator, which accounts for the minus signs in the imaginary parts. So,

$$I_2 \leq \sum_{j=1}^{4} \int_0^{2\pi} |h_j(re^{i\theta})|^\alpha \frac{d\theta}{2\pi}.$$

Because $\operatorname{Re}(s^2 - \bar{a}_j re^{i\theta})$ and $\operatorname{Re}(s^2 - \bar{b}_j re^{i\theta})$ are positive, we can now apply Lemma 3.1.1 to get

$$\frac{s^{2\alpha}}{\sec(\alpha\pi/2)} I_2 \leq s^{2\alpha} \sum_{j=1}^{4} |h_j(0)|^\alpha$$
$$\leq \sum_{k=1}^{2}\left[\left(\sum_{\xi_{kj}>0} |\xi_{kj}| + \sum_{\eta_{kj}<0} |\eta_{kj}|\right)^\alpha + \left(\sum_{\xi_{kj}\leq 0} |\xi_{kj}| + \sum_{\eta_{kj}\geq 0} |\eta_{kj}|\right)^\alpha\right].$$

Note that $|\xi_{kj}|$ and $|\eta_{kj}|$ are $\leq s$, so the last sum is

$$\leq s^\alpha \sum_{k=1}^{2}\left[\left(\sum_{\xi_{kj}>0} 1 + \sum_{\eta_{kj}<0} 1\right)^\alpha + \left(\sum_{\xi_{kj}\leq 0} 1 + \sum_{\eta_{kj}\geq 0} 1\right)^\alpha\right].$$

Let

$$\gamma_k = \frac{\sum_{\xi_{kj}>0} 1 + \sum_{\eta_{kj}<0} 1}{n(f,[0]+[\infty],s)} \leq 1.$$

So, in other words,

$$s^\alpha \sum_{k=1}^{2}\left[\left(\sum_{\xi_{kj}>0} 1 + \sum_{\eta_{kj}<0} 1\right)^\alpha + \left(\sum_{\xi_{kj}\leq 0} 1 + \sum_{\eta_{kj}\geq 0} 1\right)^\alpha\right]$$
$$= s^\alpha \sum_{k=1}^{2}\left[\left(\gamma_k n(f,[0]+[\infty],s)\right)^\alpha + \left((1-\gamma_k) n(f,[0]+[\infty],s)\right)^\alpha\right],$$

and hence

$$\frac{s^{2\alpha}}{\sec(\alpha\pi/2)} I_2 \leq (sn(f,[0]+[\infty],s))^\alpha \sum_{k=1}^{2}[\gamma_k^\alpha + (1-\gamma_k)^\alpha].$$

We will now estimate the second factor on the right in the last inequality. Note that for any γ between 0 and 1, and for any α between 0 and 1,

$$\gamma^\alpha + (1-\gamma)^\alpha \leq 2^{1-\alpha} \quad \text{and so} \quad \sum_{k=1}^{2}[\gamma_k^\alpha + (1-\gamma_k)^\alpha] \leq 2^{2-\alpha}.$$

Thus,

$$I_2 \leq \sec(\alpha\pi/2) s^{-2\alpha} \cdot s^\alpha \cdot (n(f,[0]+[\infty],s))^\alpha \cdot 2^{2-\alpha}$$
$$= 2^{2-\alpha} \sec(\alpha\pi/2) \left(\frac{n(f,[0]+[\infty],s)}{s}\right)^\alpha$$
$$\leq 2^{2-\alpha} \sec(\alpha\pi/2) \left(\frac{n(f,[0]+[\infty],s)}{r}\right)^\alpha,$$

where the last inequality is simply $s > r$.

Finally, we estimate I_3 by applying Lemma 3.1.2 on $\mathbf{D}(s)$, whereby

$$I_3 \leq (2 + 2^{2-\alpha}) \sec(\alpha\pi/2) \left(\frac{n(f,[0]+[\infty],s)}{r}\right)^\alpha.$$

The lemma follows by combining the estimates for I_1, I_2 and I_3. $\square$

An important feature of Lemma 3.2.1 is that the constants involved are independent of the function f. One often wants to bound the integral on the left in the Lemma in terms of the characteristic function T. Such a bound is an easy consequence of the lemma and the First Main Theorem, as in the following theorem, but because of the use of the First Main Theorem, one must either formulate the bound under the normalizing assumptions $f(0) = 1$, or one must allow the constant on the right hand side to depend on f. Our precise formulation of Theorem 3.2.2 differs a little bit from what is in [GoGr 1976] because Gol'dberg and Grinshtein chose to normalize f so that $f(0) = 1$, whereas we prefer to treat the general case.

Theorem 3.2.2 (Gol'dberg-Grinshtein) *Let f be a meromorphic function in $\mathbf{D}(R)$ $(0 < R \le \infty)$, and let $0<\alpha<1$, then, for $r_0<r<\rho<R$, we have*

$$\int_0^{2\pi} \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^\alpha \frac{d\theta}{2\pi} \le C_{gg}(\alpha)\left(\frac{\rho}{r(\rho-r)}\right)^\alpha (2T(f,\rho)+\beta_1)^\alpha,$$

where

$$\beta_1 = \beta_1(f, r_0) = |\mathrm{ord}_0 f| \log^+ \frac{1}{r_0} + \Big|\log|\mathrm{ilc}(f,0)|\Big| + \log 2$$

and

$$C_{gg}(\alpha) = 2^\alpha + (8 + 2\cdot 2^\alpha)\sec\frac{\alpha\pi}{2}.$$

Remark. It turns out that we will want to apply Theorem 3.2.2 with an α such that

$$c_{gg}(\alpha) \stackrel{\text{def}}{=} \frac{1}{\alpha}\log C_{gg}(\alpha)$$

is small. Thus, we let

$$\boxed{c_{gg} = \inf_\alpha c_{gg}(\alpha).}$$

By doing some numerical calculation, we see that

$$\boxed{c_{gg} \le c_{gg}(0.7866) \le 4.5743.}$$

Proof. Let $s = (r+\rho)/2$. Then, Lemma 3.2.1 tells us

$$\int_0^{2\pi} \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^\alpha \frac{d\theta}{2\pi} \le \left(\frac{s}{r(s-r)} m(f,[0]+[\infty],s)\right)^\alpha + (2+2^{3-\alpha})\sec(\alpha\pi/2)\left(\frac{n(f,[0]+[\infty],s)}{r}\right)^\alpha. \qquad (*)$$

Note that $s - r = \rho - s = (\rho - r)/2$, and so

$$\frac{s}{r(s-r)} = \frac{r+\rho}{r(\rho-r)} \le \frac{2\rho}{r(\rho-r)}.$$

Now, using the definitions of the Nevanlinna functions together with the First Main Theorem (Theorem 1.3.1), we see that

$$\begin{aligned} m(f,0,s) &= T(f,0,s) - N(f,0,s) \\ &\le T(f,0,s) + \max\{0, \mathrm{ord}_0 f\}\log^+\frac{1}{r_0} \\ &\le T(f,\infty,s) + \max\{0,\mathrm{ord}_0 f\}\log^+\frac{1}{r_0} + \Big|\log|\mathrm{ilc}(f,0)|\Big| + \log 2. \end{aligned}$$

Similarly, $m(f,\infty,s) \le T(f,\infty,s) + \max\{0, -\mathrm{ord}_0 f\}\log^+(1/r_0)$, and so we have

$$m(f,[0]+[\infty],s) = m(f,0,s) + m(f,\infty,s) \le 2T(f,s) + \beta_1.$$

Moreover, because $\rho > s$, we know that $T(f,s) < T(f,\rho)$. Thus, the first term on the right in $(*)$ is less than or equal to

$$2^\alpha\left(\frac{\rho}{r(\rho-r)}\right)^\alpha (2T(f,\rho)+\beta_1)^\alpha.$$

Now we estimate the second term on the right in $(*)$ in terms of $T(f,\rho)$. We have

$$\begin{aligned} \frac{\rho}{\rho-s}\Big(N(f,[0]+[\infty],\rho) &+ |\mathrm{ord}_0 f|\log^+\frac{1}{r_0}\Big) \\ &\ge \frac{\rho}{\rho-s}\int_s^\rho n(f,[0]+[\infty],t)\frac{dt}{t} \\ &\ge \frac{\rho}{\rho-s}\int_s^\rho n(f,[0]+[\infty],s)\frac{dt}{\rho} \\ &= n(f,[0]+[\infty],s). \end{aligned}$$

Hence,

$$\begin{aligned} n(f,[0]+[\infty],s) &\le \frac{\rho}{\rho-s}\left(N(f,[0]+[\infty],\rho) + |\mathrm{ord}_0 f|\log^+\frac{1}{r_0}\right) \\ &\le \frac{2\rho}{\rho-r}(2T(f,\rho)+\beta_1), \end{aligned}$$

where the last inequality follows from the First Main Theorem. Thus, the second term on the right in $(*)$ is

$$\le (8+2\cdot 2^\alpha)\sec(\alpha\pi/2)\left(\frac{\rho}{r(\rho-r)}\right)^\alpha (2T(f,\rho)+\beta_1)^\alpha. \quad \square$$

Choosing an α lets us bound $m(f'/f,\infty,r)$.

Corollary 3.2.3 (Gol'dberg-Grinshtein) *Let f be a meromorphic function on $\mathbf{D}(R)$ $(0 < R \le \infty)$. Then, for any r and ρ with $r_0 < r < \rho < R$,*

$$\begin{aligned} m(f'/f,\infty,r) &\le \log^+ \frac{\rho(2T(f,\rho)+\beta_1)}{r(\rho-r)} + c_{gg} \\ &\le \log^+ T(f,\rho) + \log^+ \frac{\rho}{r(\rho-r)} + \log^+\beta_1 + c_{gg} + 2\log 2, \end{aligned}$$

where, as in the theorem,

$$\beta_1 = \beta_1(f,r_0) = |\mathrm{ord}_0 f| \log^+ \frac{1}{r_0} + \Big|\log|\mathrm{ilc}(f,0)|\Big| + \log 2,$$

and c_{gg} is the constant defined in the remark following the theorem.

Remark. In [GoGr 1976], Gol'dberg and Grinshtein consider only the case when $f(0) = 1$, but in that case they obtain

$$m(f'/f,r) < \log^+ T(f,\rho) + \log^+ \frac{\rho}{r(\rho-r)} + 5.8501,$$

where the 5.8501 is a somewhat better constant than what we obtain if we specialize the inequality in Corollary 3.2.3 to the case $f(0) = 1$.

Proof. Using the concavity of $\log^+$ (Theorem 1.14.1) to pull the $\log^+$ outside the integral, and then using Theorem 3.2.2, we get

$$\begin{aligned} m(f'/f,\infty,r) &= \frac{1}{\alpha}\int_0^{2\pi} \log^+ \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^\alpha \frac{d\theta}{2\pi} \le \frac{1}{\alpha}\log^+ \int_0^{2\pi} \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^\alpha \frac{d\theta}{2\pi} \\ &\le \frac{1}{\alpha}\log^+ C_{gg}(\alpha) + \log^+ \frac{\rho(2T(f,\rho)+\beta_1)}{r(\rho-r)}. \end{aligned}$$

Using the identity

$$\log^+(x+y) \le \log^+ x + \log^+ y + \log 2,$$

we get,

$$\log^+ \frac{\rho(2T(f,\rho)+\beta_1)}{r(\rho-r)} \le \log^+ T(f,\rho) + \log^+ \frac{\rho}{r(\rho-r)} + \log^+\beta_1 + 2\log 2.$$

The proof is completed by replacing $\frac{1}{\alpha}\log^+ C_{gg}(\alpha)$ with c_{gg}. □

3.3 The Borel-Nevanlinna Growth Lemma

In this section we will prove another "growth" or "calculus" lemma which will be used in contexts similar to that of Lemma 2.2.3. If the reader has skipped Chapter 2, he or she may now want to go back and read Section 2.1 before proceeding with the rest of this chapter. The growth lemma we state and prove here stems from E. Borel [Bor 1897]. The idea was then further developed by R. Nevanlinna [Nev(R) 1931]. The form we present here is similar to the formulations in: [Hay 1964], [Hink 1992], and [Wang 1997].

Lemma 3.3.1 *Let $F(r)$ and $\phi(r)$ be positive, nondecreasing, continuous functions defined for $r_0 \le r < \infty$, and assume that $F(r) \ge e$ for $r \ge r_0$. Let $\xi(x)$ be a positive, nondecreasing, continuous function defined for $e \le x < \infty$. Let $C > 1$ be a constant, and let E be the closed subset of $[r_0,\infty)$ defined by*

$$E = \left\{ r \in [r_0,\infty) : F\left(r + \frac{\phi(r)}{\xi(F(r))}\right) \ge CF(r)\right\}.$$

Then, for all $R < \infty$,

$$\int_{E\cap[r_0,R]} \frac{dr}{\phi(r)} \le \frac{1}{\xi(e)} + \frac{1}{\log C}\int_e^{F(R)} \frac{dx}{x\xi(x)}.$$

Proof. To ease notation, set $h(r) = \phi(r)/\xi(F(r))$.

We may assume the set E is non-empty, otherwise the lemma is trivial. Since E is not empty, let r_1 be the smallest $r \in E$ with $r \ge r_0$.

Now assume we have found numbers $r_1,\dots,r_n,s_1,\dots,s_{n-1}$. We describe here how to inductively extend this set, and we continue this process as long as possible. If there is no number s with $F(s) \ge CF(r_n)$, then stop here. Otherwise, by the continuity of F, there exists an s with $F(s) = CF(r_n)$. Let s_n be the smallest such s. Then, if there is an $r \in E$ with $r \ge s_n$, let r_{n+1} be the smallest such r. Otherwise, stop here.

For each pair r_j, s_j, clearly $s_j > r_j$, and since $r_j \in E$,

$$F(r_j + h(r_j)) \ge CF(r_j) = F(s_j).$$

Since F is nondecreasing, this implies $r_j + h(r_j) \ge s_j$, and so $s_j - r_j \le h(r_j)$. Moreover, $F(r_{j+1}) \ge F(s_j) = CF(r_j)$ since $r_{j+1} \ge s_j$. Hence,

$$F(r_{n+1}) \ge CF(r_n) \ge C^2F(r_{n-1}) \ge \cdots \ge C^nF(r_1).$$

It follows that either we could only find finitely many r_n or else the sequence r_n goes to ∞ as n goes to ∞.

Let $N < \infty$ denote the largest n such that $r_n \le R$. If $s_n > R$ or could not be found, then replace s_n by R. The set $E \cap [r_0,R]$ is clearly contained in the union of intervals

$$\bigcup_{n=1}^N [r_n,s_n],$$

and so

$$\int_{E\cap[r_0,R]} \frac{dr}{\phi(r)} \le \sum_{n=1}^N \int_{r_n}^{s_n} \frac{dr}{\phi(r)}.$$

Because ϕ is nondecreasing, and since $s_n - r_n \le h(r_n)$,

$$\int_{r_n}^{s_n} \frac{dr}{\phi(r)} \le \frac{s_n - r_n}{\phi(r_n)} \le \frac{1}{\xi(F(r_n))}.$$

Since ξ and F are nondecreasing,

$$\int_{E\cap[r_0,R]} \frac{dr}{\phi(r)} \le \sum_{n=1}^{N} \frac{1}{\xi(F(r_n))} \le \sum_{n=1}^{N} \frac{1}{\xi(C^{n-1}F(r_1))}$$

$$\le \frac{1}{\psi(F(r_1))} + \frac{1}{\log C}\int_{F(r_1)}^{F(R)} \frac{du}{u\xi(u)}.$$

The lemma follows since $F(r_1) \ge e$ and $\xi(x) \ge \xi(e)$. □

We will apply Lemma 3.3.1 in the following two contexts, depending on whether we are working with functions defined on the whole plane or only in a disc of finite radius.

Lemma 3.3.2 (Plane Growth Lemma) *Let $F(r)$ be a positive, nondecreasing, continuous function defined and $\ge e$ for $e \le r_0 \le r < \infty$. Let ψ be a Khinchin function, and let $\phi(r)$ be a positive, nondecreasing, continuous function defined for $r_0 \le r < \infty$. Set $\rho = r + \phi(r)/\psi(F(r))$. If $\phi(r) \le r$ for all $r \ge r_0$, then there exists a closed set $E \subset [r_0, \infty)$ (called the "exceptional set") such that*

$$\int_E \frac{dr}{\phi(r)} \le 1 + k_0(\psi) < \infty,$$

and such that for all $r \ge r_0$ and not in E, we have

$$\log F(\rho) < \log F(r) + 1,$$

and

$$\log \frac{\rho}{r(\rho - r)} \le \log \frac{\psi(F(r))}{\phi(r)} + \log 2.$$

If, on the other hand, $\phi(r) \not\le r$, then there exists a closed set $E \subset [r_0, \infty)$ (called the "exceptional set") such that

$$\int_E \frac{dr}{\phi(r)} \le 1 + 2k_0(\psi) < \infty,$$

and such that for all $r \ge r_0$ and not in E, we have

$$\log F(\rho) < \log F(r) + 1,$$

and

$$\log^+ \frac{\rho}{r(\rho - r)} \le \log^+ \frac{\psi(F(r))}{\phi(r)} + \log 2.$$

Proof. Apply Lemma 3.3.1 with $\xi = \psi$, $C = e$, and $R \to \infty$ to conclude there exists a set E_1 such that

$$\int_{E_1} \frac{dr}{\phi(r)} \le 1 + \lim_{R\to\infty}\int_e^{F(R)} \frac{dx}{x\psi(x)} \le 1 + \int_e^{\infty} \frac{dx}{x\psi(x)} = 1 + k_0(\psi)$$

and such that

$$F(\rho) = F\left(r + \frac{\phi(r)}{\psi(F(r))}\right) < eF(r).$$

for all $r \ge r_0$ and not in E_1. Thus, for r not in E_1,

$$\log F(\rho) < \log F(r) + 1.$$

If $\phi(r) \le r$, then note that

$$\frac{\rho}{r(\rho - r)} = \frac{\psi(F(r))}{\phi(r)} + \frac{1}{r} \le \frac{\psi(F(r)) + 1}{\phi(r)} \le \frac{2\psi(F(r))}{\phi(r)},$$

and so the lemma follows with $E = E_1$.

If $\phi(r) \not\le r$, then let

$$E_2 = \{r \ge r_0 : \phi(r) \ge r\psi(r)\},$$

and observe that

$$\int_{E_2} \frac{dr}{\phi(r)} \le \int_e^{\infty} \frac{1}{r\psi(r)} = k_0(\psi).$$

We now work with the term

$$\log \frac{\rho}{r(\rho - r)}$$

and consider various cases. For those r not in E_2 such that $F(r) \ge r \ge r_0$, we have

$$\frac{\rho}{r(\rho - r)} = \frac{1}{\rho - r}\left(1 + \frac{\rho - r}{r}\right) = \frac{\psi(F(r))}{\phi(r)}\left(1 + \frac{\phi(r)}{r\psi(F(r))}\right)$$

$$\le \frac{\psi(F(r))}{\phi(r)}\left(1 + \frac{\phi(r)}{r\psi(r)}\right)$$

$$\le 2\frac{\psi(F(r))}{\phi(r)}.$$

If, on the other hand, $F(r) \le r$, and in addition $\rho \le 2r$, then

$$\frac{\rho}{r(\rho - r)} \le \frac{2}{\rho - r} = 2\frac{\psi(F(r))}{\phi(r)}.$$

Therefore, in either of these two cases,

$$\log^+ \frac{\rho}{r(\rho - r)} \le \log^+ \frac{\psi(F(r))}{\phi(r)} + \log 2.$$

The last case to consider is the case that $F(r) \le r$, and $\rho \ge 2r$. Note that

$$\frac{d}{d\rho}\left[\frac{\rho}{r(\rho - r)}\right] = \frac{-r^2}{r^2(\rho - r)^2} < 0,$$

and so

$$\rho \mapsto \frac{\rho}{r(\rho - r)}$$

is decreasing in ρ. Since $2r \ge \rho$ by assumption, if we replace ρ by $2r$, we get

$$\frac{\rho}{r(\rho - r)} \le \frac{2r}{r(2r - r)} = \frac{2}{r} \le 2,$$

where the last inequality follows since $r \ge F(r) \ge e \ge 1$. Hence in this case,

$$\log^+ \frac{\rho}{r(\rho - r)} \le \log 2.$$

The proof of the lemma is completed by taking the E to be $E_1 \cup E_2$. □

Lemma 3.3.3 (Disc Growth Lemma) *Let $F(r)$ be a positive, nondecreasing, continuous function defined and $\ge e$ for $0 < r_0 \le r < R < \infty$. Let ψ be a Khinchin function, and let $\phi(r)$ be a positive continuous function defined for $r_0 \le r < R$, and such that the function $r \mapsto (R - r)^{-2}\phi(r)$ is nondecreasing. Then, there exists a closed set $E \subset [r_0, R)$ (called the "exceptional set") such that*

$$\int_E \frac{dr}{\phi(r)} \le 1 + k_0(\psi) < \infty,$$

and such that if we set

$$\rho = R - \left[\frac{1}{R - r} + \frac{(R - r)^{-2}\phi(r)}{\psi(F(r))}\right]^{-1},$$

then for all $r \ge r_0$ and not in E, we have

$$\log F(\rho) \le \log F(r) + 1,$$

and

$$\log^+ \frac{\rho}{r(\rho - r)} \le \log^+ \frac{\psi(F(r))}{\phi(r)} + \log^+ \frac{1}{R - r} + \log \frac{R}{r} + \log 2.$$

Moreover, if $\phi(r) \le R - r$, then

$$\log \frac{\rho}{r(\rho - r)} \le \log \frac{\psi(F(r))}{\phi(r)} + \log \frac{R}{r} + \log 2.$$

Proof. Consider the function

$$s = s(r) = \frac{1}{R - r}$$

whose inverse is given by

$$r = r(s) = R - \frac{1}{s}.$$

Clearly the functions s and r are increasing, and the function s is a bijection from the interval $[r_0, R)$ to the interval $[1/(R - r_0), \infty)$. Let

$$s_0 = s(r_0) = \frac{1}{R - r_0}.$$

Set $\tilde{F}(s) = F(r(s))$, and observe that $\tilde{F}(s)$ is a continuous, positive, nondecreasing function $\ge e$ on the interval $[s_0, \infty)$. Set

$$\tilde{\phi}(s) = s^2\phi(r(s)),$$

which by assumption is positive, continuous, and nondecreasing on $[s_0, \infty)$. Thus we may apply Lemma 3.3.1 (with $\xi = \psi$, $C = e$, and $R \to \infty$) to get a closed set $\tilde{E}$ in $[s_0, \infty)$ such that

$$\int_{\tilde{E}} \frac{ds}{\tilde{\phi}(s)} \le 1 + k_0(\psi),$$

and such that for all $s \ge s_0$ and not in $\tilde{E}$, we have

$$\tilde{F}\left(s + \frac{\tilde{\phi}(s)}{\psi(\tilde{F}(s))}\right) \le e\tilde{F}(s).$$

This can be rewritten as

$$\begin{aligned} F\left(R - \left[\frac{1}{R - r} + \frac{(R - r)^{-2}\phi(r)}{\psi(F(r))}\right]^{-1}\right) &= F\left(R - \left[s + \frac{s^2\phi(r(s))}{\psi(F(r(s)))}\right]^{-1}\right) \\ &= \tilde{F}\left(s + \frac{\tilde{\phi}(s)}{\psi(\tilde{F}(s))}\right) \\ &\le e\tilde{F}(s) = eF(r), \end{aligned}$$

and the inequality holds provided r is not in the set $E \stackrel{\text{def}}{=} r(\tilde{E})$. Let ρ be as in the statement of the lemma, and note that since

$$s(\rho) = s(r) + \frac{\tilde{\phi}(s)}{\psi(\tilde{F}(s))} > s(r),$$

we have $r < \rho < R$. After some manipulation, one finds that

$$\frac{1}{\rho - r} = \frac{\psi(F(r))}{\phi(r)} + \frac{1}{R-r}.$$

If $\phi(r) \le R - r$, then

$$\frac{1}{\rho - r} \le \frac{\psi(F(r)) + 1}{\phi(r)} \le 2\frac{\psi(F(r))}{\phi(r)}.$$

In general,

$$\log^+ \frac{1}{\rho - r} \le \log^+ \frac{\psi(F(r))}{\phi(r)} + \log^+ \frac{1}{R-r} + \log 2.$$

The proof of the theorem is completed by noting that for r outside E,

$$\log F(\rho) \le \log F(r) + 1,$$

and that by the change of variables formula,

$$\int_E \frac{dr}{\phi(r)} = \int_{s(E)} \frac{ds}{s^2\phi(s)} = \int_{\tilde{E}} \frac{ds}{\tilde{\phi}(s)} \le \frac{1}{\psi(F(r))} + k_0(\psi). \qquad \square$$

3.4 The Logarithmic Derivative Lemma

Theorem 3.4.1 (Lemma on the Logarithmic Derivative) *Let c_{ss} be the constant from Corollary 3.2.3, and let f be a non-constant meromorphic function on $\mathbf{D}(R)$, $R \le \infty$. Assume that $T(f, r_0) \ge e$ for some r_0, and that in the case $R = \infty$ that $r_0 \ge e$. Let*

$$\beta_1 = \beta_1(f, r_0) = |\mathrm{ord}_0 f| \log^+ \frac{1}{r_0} + \left|\log|\mathrm{lc}(f, 0)|\right| + \log 2,$$

and let ψ be a Khinchin function.

(1) In the case that $R = \infty$, let $\phi(r)$ be a positive, nondecreasing, continuous function defined for $r_0 \le r < \infty$. Then, there exists a closed set E of radii such that

$$\int_E \frac{dr}{\phi(r)} \le 2k_0(\psi) + 1,$$

and such that for all $r \ge r_0$ outside of E, we have

$$m(f'/f, \infty, r) \le \log T(f, r) + \log^+ \frac{\psi(T(f,r))}{\phi(r)} + c_{ss} + \log^+ \beta_1 + 3\log 2 + 1.$$

Moreover, if $\phi(r) \le r$, then

$$\int_E \frac{dr}{\phi(r)} \le k_0(\psi) + 1,$$

and for all $r \ge r_0$ outside of E, we have

$$m(f'/f, \infty, r) \le \log T(f, r) + \log \frac{\psi(T(f,r))}{\phi(r)} + c_{ss} + \log^+ \beta_1 + 3\log 2 + 1.$$

(2) In the case that $R < \infty$, let $\phi(r)$ be a positive continuous function defined for $r_0 \le r < R$, such that $r \mapsto (R-r)^{-2}\phi(r)$ is a nondecreasing function. Then, there exists a closed set E of radii such that

$$\int_E \frac{dr}{\phi(r)} \le k_0(\psi) + \frac{1}{\psi(T(f, r_0))} \le k_0(\psi) + 1,$$

and such that for all $r_0 \le r < R$ and outside of E, we have

$$m(f'/f, \infty, r) \le \log T(f, r) + \log^+ \frac{\psi(T(f,r))}{\phi(r)} + \log^+ \frac{1}{R-r} + \log \frac{R}{r} + \log^+ \beta_1 + c_{ss} + 3\log 2 + 1.$$

Moreover, if $\phi(r) \le R - r$, then

$$m(f'/f, \infty, r) \le \log T(f, r) + \log \frac{\psi(T(f,r))}{\phi(r)} + \log \frac{R}{r} + \log^+ \beta_1 + c_{ss} + 3\log 2 + 1.$$

Remarks. Theorem 3.4.1 is known as the "Lemma on the Logarithmic Derivative," but it is a significant result and of independent interest, so we have decided to call it a "theorem."

So that the reader can appreciate the form in which we have stated the Lemma on the Logarithmic Derivative, we briefly recall our discussion of exceptional sets at the beginning of Chapter 2. Namely, the idea is to take $\psi(x)$ small subject to the Khinchin convergence condition, for example $\psi(x) = (\log x)^{1+\epsilon}$. Similarly, one often wants to choose ϕ so that for every r_1,

$$\int_{r_1}^{R} \frac{dr}{\phi(r)} = \infty,$$

and thus the exceptional set E cannot contain any intervals of the form (r_1, R), and so there must be a sequence of r tending toward R for which the key inequality holds. Thus, the Lemma on the Logarithmic Derivative says that a logarithmic derivative poorly approximates infinity when $|z|$ is large and outside of a small exceptional set.

Proof. The theorem follows immediately from Corollary 3.2.3 and the Borel-Nevanlinna growth lemma in the form of Lemma 3.3.2 or Lemma 3.3.3 as appropriate. □

3.5 Functions of Finite Order

In the case of meromorphic functions of finite order, the error term in the Lemma on the Logarithmic Derivative can be further simplified.

Theorem 3.5.1 (Ngoan-Ostrovskiĭ) *If f is a non-constant meromorphic function on $\mathbf{C}$ of finite order p, then for all $\varepsilon > 0$, and for all r sufficiently large (depending on ε),*

$$m(f'/f,r) \le \max\{0,(p-1+\varepsilon)\log r\} + O(1).$$

Remark. Note how the order of the function appears as a coefficient in the error term in a natural way. Theorem 3.5.1 says that the mean-proximity function of the logarithmic derivative of a meromorphic function of order p grows essentially at most like the characteristic function of a polynomial of degree $p-1$. We emphasize that the inequality in Theorem 3.5.1 is true for *all* sufficiently large r, and that there is no need for an exceptional set of radii. Theorem 3.5.1 was first proved by V. Ngoan and I. V. Ostrovskiĭ in [NgoOst 1965].

Proof. From Corollary 3.2.3, we have for $\rho > r$,

$$m(f'/f,r) \le \log^+ \frac{\rho(2T(f,\rho)+O(1))}{r(\rho-r)} + O(1).$$

Taking $\rho = 2r$, the right hand side simplifies to

$$\log^+ \frac{4T(f,2r)+O(1)}{r} + O(1).$$

For r sufficiently large, $4T(f,2r)+O(1) \le 5T(f,2r)$. Thus, for r sufficiently large,

$$m(f'/f,r) \le \max\{0, \log 5 + \log T(f,2r) - \log r\} + O(1).$$

Since f has order p, we have $\log T(f,2r) \le (p+\varepsilon)\log r$ for r sufficiently large, and hence the theorem follows. □

4 The Second Main Theorem via Logarithmic Derivatives

In this chapter we give a logarithmic derivative based proof of the Second Main Theorem. R. Nevanlinna's original proof of this theorem was along these lines, and the proof we give here is generally speaking similar to the proof given in Hayman's book [Hay 1964]. Neither Nevanlinna nor Hayman were interested in the precise structure of the error term, and they did not use the refined logarithmic derivative estimates of Gol'dberg and Grinshtein, as we shall here. A sharp error term in the Second Main Theorem using this type of method was first obtained by Hinkkanen [Hink 1992]. The exact method of proof given here makes use of ideas from Ye's refinement of Cartan's method [Cart 1933], as in [Ye 1995].

4.1 Definitions and Notation

Throughout this chapter, unless explicitly stated otherwise, we maintain the following notational conventions:

R will be a positive real number or ∞.

$\mathbf{D}(R)$ will be the disc of radius R (or $\mathbf{C}$ if $R = \infty$).

f will be a meromorphic function on $\mathbf{D}(R)$.

$N_{\mathrm{ram}}(f,r) = N(f',0,r) + 2N(f,\infty,r) - N(f',\infty,r)$ (see Proposition 1.4.1).

q will be an integer ≥ 2.

$a_1,\dots,a_q$ will be distinct points in $\mathbf{P}^1$, at least two of which are finite. Without loss of generality, we will assume that if one of the points a_j is infinity, then it is $a_q = \infty$.

$q' \ge 2$ will equal the number of finite a_j.

$$\mathcal{D}(a_1,\dots,a_q) = (q'-1)\left[\log^+ \max_{1\le j\le q'} |a_j| + \log^+\left(\frac{2}{\min\limits_{1\le i<j\le q'} |a_i - a_j|}\right)\right].$$

r_0 will be a positive real number so that $T(f,r_0) \ge e$. Moreover, if $R = \infty$, then we assume $r_0 \ge e$.

ψ will be a Khinchin function.

$$k_0(\psi) = \int_e^\infty \frac{dx}{x\psi(x)} < \infty.$$

$$\beta_1(f - a_j, r_0) = |\mathrm{ord}_0(f - a_j)|\log^+ \frac{1}{r_0} + \left|\log|\mathrm{ilc}(f - a_j, 0)|\right| + \log 2.$$

$$\beta_2(f, a_1, \ldots, a_q) = \sum_{j=1}^{q'} \log|\mathrm{ilc}(f - a_j, 0)| - \log|\mathrm{ilc}(f', 0)|.$$

$$\beta_3(f, a_1, \ldots, a_q, r_0) = \log^+ \max_{1\le j\le q'} \{2\log^+|a_j| + \beta_1(f - a_j, r_0)\}.$$

Remark. The constant $\mathcal{D}(a_1, \ldots, a_q)$ is a geometric constant that more or less measures how close the a_j are to one another on the Riemann sphere. By this we mean that $\mathcal{D}(a_1, \ldots, a_q)$ gets big in one of two ways. Either one of the points gets close to ∞, or two of the finite points get close to one another. Compare this with the term $\mathring{\mathcal{D}}(a_1, \ldots, a_q)$ defined in §1.2 in connection with Proposition 1.2.3.

4.2 The Second Main Theorem

Theorem 4.2.1 (Second Main Theorem) *Let $a_1, \ldots, a_q$ be a finite set of distinct points in $\mathbf{P}^1$ such that at least two of them are finite. Let f be a nonconstant meromorphic function on $\mathbf{D}(R)$, and let r_0, β_2, β_3, and $\mathcal{D}(a_1, \ldots, a_q)$ be defined as at the beginning of the section. Then,*

(1) We have for all $r_0 < r < \rho < R$,

$$\begin{aligned}(q-2)T(f,r) - \sum_{j=1}^{q} N(f, a_j, r) + N_{\mathrm{ram}}(f, r) &\le \log T(f,\rho) + \log\frac{\rho}{r(\rho - r)} + \mathcal{D}(a_1, \ldots, a_q)\\ &\quad + \beta_2(f, a_1, \ldots, a_q) + \beta_3(f, a_1, \ldots, a_q, r_0)\\ &\quad + \frac{4}{3}\log q + 5 + (q+1)\log 2\\ &= \log T(f,\rho) + \log\frac{\rho}{r(\rho - r)} + O(1).\end{aligned}$$

(2) If $R = \infty$, and $\phi(r)$ is a positive, nondecreasing, continuous function on $[r_0, \infty)$, then there exists a closed set of radii E such that

$$\int_E \frac{dr}{\phi(r)} \le \frac{1}{\psi(T(f, r_0))} + 2k_0(\psi) \le 1 + 2k_0(\psi) < \infty,$$

and such that for all $r \ge r_0$ and outside of E,

$$\begin{aligned}(q-2)T(f,r) - \sum_{j=1}^{q} N(f, a_j, r) + N_{\mathrm{ram}}(f, r) &\le \log T(f,r) + \log^+\frac{\psi(T(f,r))}{\phi(r)} + \mathcal{D}(a_1, \ldots, a_q)\\ &\quad + \beta_2(f, a_1, \ldots, a_q) + \beta_3(f, a_1, \ldots, a_q, r_0)\\ &\quad + \frac{4}{3}\log q + 6 + (q+2)\log 2\\ &= \log T(f,r) + \log^+\frac{\psi(T(f,r))}{\phi(r)} + O(1).\end{aligned}$$

If, in addition, $\phi(r) \le r$, then

$$\int_E \frac{dr}{\phi(r)} \le \frac{1}{\psi(T(f, r_0))} + k_0(\psi) \le 1 + k_0(\psi) < \infty,$$

and for all $r \ge r_0$ and outside of E,

$$\begin{aligned}(q-2)T(f,r) - \sum_{j=1}^{q} N(f, a_j, r) + N_{\mathrm{ram}}(f, r) &\le \log T(f,r) + \log\frac{\psi(T(f,r))}{\phi(r)} + \mathcal{D}(a_1, \ldots, a_q)\\ &\quad + \beta_2(f, a_1, \ldots, a_q) + \beta_3(f, a_1, \ldots, a_q, r_0)\\ &\quad + \frac{4}{3}\log q + 6 + (q+2)\log 2\\ &= \log T(f,r) + \log\frac{\psi(T(f,r))}{\phi(r)} + O(1).\end{aligned}$$

(3) If $R < \infty$, and $\phi(r)$ is a positive continuous function defined on $[r_0, R)$ such that $r \mapsto (R - r)^{-2}\phi(r)$ is nondecreasing, then there exists a closed set of radii E such that

$$\int_E \frac{dr}{\phi(r)} \le \frac{1}{\psi(T(f, r_0))} + k_0(\psi) \le 1 + k_0(\psi) < \infty,$$

and such that for all $r_0 \le r < R$ and outside E,

$$\begin{aligned}(q-2)T(f,r) - \sum_{j=1}^{q} N(f, a_j, r) + N_{\mathrm{ram}}(f, r) &\le \log T(f,r) + \log^+\frac{\psi(T(f,r))}{\phi(r)}\\ &\quad + \log^+\frac{1}{R-r} + \log\frac{R}{r} + \mathcal{D}(a_1, \ldots, a_q)\\ &\quad + \beta_2(f, a_1, \ldots, a_q) + \beta_3(f, a_1, \ldots, a_q, r_0)\\ &\quad + \frac{4}{3}\log q + 6 + (q+2)\log 2\\ &= \log T(f,r) + \log^+\frac{\psi(T(f,r))}{\phi(r)}\\ &\quad + \log^+\frac{1}{R-r} + O(1).\end{aligned}$$

If, in addition, $\phi(r) \le R - r$, then for all $r_0 \le r < R$ and outside E,

$$\begin{aligned}(q-2)T(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{ram}}(f,r) &\le \log T(f,r) + \log\frac{\psi(T(f,r))}{\phi(r)} \\ &\quad + \log\frac{R}{r} + \mathcal{D}(a_1,\dots,a_q) \\ &\quad + \beta_2(f,a_1,\dots,a_q) + \beta_3(f,a_1,\dots,a_q,r_0) \\ &\quad + \frac{4}{3}\log q + 6 + (q+2)\log 2 \\ &= \log T(f,r) + \log\frac{\psi(T(f,r))}{\phi(r)} + O(1).\end{aligned}$$

Before we actually give a proof of the Second Main Theorem, we make a few comments and state some corollaries.

If one chooses ϕ and ψ appropriately, one sees that the error term in inequality (2) in the theorem is of order $O(\log T(f,r))$, and the error term in inequality (3) is of order

$$O\left(\log T(f,r) + \log^+\frac{1}{R-r}\right).$$

With an error term of order $O(\log rT(f,r))$ in the $R = \infty$ case, and of order

$$O\left(\log(rT(f,r)) + \log^+\frac{1}{R-r}\right),$$

in the case of finite R, the Second Main Theorem is due to R. Nevanlinna. An error term of the form

$$\log T(f,r) + \log^+\frac{\psi(T(f,r))}{\phi(r)} + O(1)$$

in the case $R = \infty$, or of the form

$$\log T(f,r) + \log^+\frac{\psi(T(f,r))}{\phi(r)} + \log^+\frac{1}{R-r} + O(1)$$

in the case of finite R was first obtained by A. Hinkkanen [Hink 1992]. We will see in §7.1 that the coefficient 1 in front of the $\log T(f,r)$ term in the error term is best possible. Hinkkanen also gives explicit $O(1)$ constants. Since the details of our proof differ from Hinkkanen's proof, the explicit $O(1)$ constants we get are different from those that Hinkkanen gets, though not in any important way. However, notice that in inequality (1) in Theorem 4.2.1, we have

$$\log[\rho/(r(\rho - r))]$$

on the right-hand side, whereas Hinkkanen has $\log^+$ in his corresponding term. We will take advantage of this seemingly small improvement when we discuss functions of finite order. Moreover, as we will show in Theorem 4.2.4, by combining Theorem 4.2.1 with Theorem 2.6.2, one can avoid the use of $\log^+$ entirely in the $R = \infty$ case.

We would also like to point out that by the First Main Theorem,

$$(q-2)T(f,r) - \sum_{j=1}^{q} N(f,a_j,r) = \sum_{j=1}^{q} m(f,a_j,r) - 2T(f,r) + O(1),$$

and thus the left-hand side of the inequalities in Theorem 4.2.1 (which is always the same) can be replaced with

$$\sum_{j=1}^{q} m(f,a_j,r) - 2T(f,r) + N_{\mathrm{ram}}(f,r),$$

provided that the $O(1)$ term on the right is modified according to the constant terms that appear in the First Main Theorem. Moreover, since the mean-proximity functions are all positive, for any $q_1 \le q$, we have

$$N_{\mathrm{ram}}(f,r) \le \sum_{j=1}^{q_1} m(f,a_j,r) + N_{\mathrm{ram}}(f,r) \le \sum_{j=1}^{q} m(f,a_j,r) + N_{\mathrm{ram}}(f,r).$$

Thus, we can immediately conclude the following corollaries.

Corollary 4.2.2 (Ramification Theorem) *Let f be a non-constant meromorphic function on $\mathbf{D}(R)$, and let r_0, ψ and ϕ be as in the theorem. Then*

(1) If $R = \infty$, then for $r \ge r_0$, outside an exceptional set as in the theorem,

$$-2T(f,r) + N_{\mathrm{ram}}(f,r) \le \log T(f,r) + \log^+\frac{\psi(T(f,r))}{\phi(r)} + O(1).$$

(2) If $R < \infty$, then for $r \ge r_0$, outside an exceptional set as in the theorem,

$$\begin{aligned}-2T(f,r)&+N_{\mathrm{ram}}(f,r) \\ &\le \log T(f,r) + \log^+\frac{\psi(T(f,r))}{\phi(r)} + \log^+\frac{1}{R-r} + O(1).\end{aligned}$$

Corollary 4.2.3 *Let $a_1,\dots,a_q$ be arbitrary distinct points in $\mathbf{P}^1$, without any restriction on the number. Let f be a non-constant meromorphic function on $\mathbf{D}(R)$, and let r_0, ψ and ϕ be as in the theorem. Then*

(1) If $R = \infty$, then for $r \geq r_0$, outside an exceptional set as in the theorem,

$$(q-2)T(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{ram}}(f,r) \leq \log T(f,r) + \log^+ \frac{\psi(T(f,r))}{\phi(r)} + O(1).$$

(2) If $R < \infty$, then for $r \geq r_0$, outside an exceptional set as in the theorem,

$$(q-2)T(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{ram}}(f,r) \leq \log T(f,r) + \log^+ \frac{\psi(T(f,r))}{\phi(r)} + \log^+ \frac{1}{R-r} + O(1).$$

Remark. If one further restricts ϕ, as in Theorem 4.2.1, then one can change $\log^+ \psi(T(f,r))/\phi(r)$ to $\log \psi(T(f,r))/\phi(r)$, as in Theorem 4.2.1.

Note that the difference between Corollary 4.2.3 and the theorem itself is that in the corollary we allow $q < 2$ and we do not require that two of the a_j be finite. One could of course work out the precise constants $O(1)$ in these corollaries, as we did in the theorem itself. Indeed, one simply needs to add points a_j until the hypotheses of Theorem 4.2.1 are satisfied, apply the First Main Theorem to convert to mean-proximity functions, and then adjust the explicit constants given in the theorem accordingly. However, this is quite tedious, and so we have decided to omit this computation. Recall that we gave explicit $O(1)$ constants for the Second Main Theorem, without any conditions on the number of points a_j, in Chapter 2.

We remark that here in Corollary 4.2.2 the coefficient in front of the $\log T(f,r)$ term in the error term is 1, whereas it was $1/2$ in Theorem 2.6.1 (the Ramification Theorem proven by way of negative curvature). On the other hand, the error term in Theorem 2.6.1 had a term of the form $\log \psi(\max\{e, \phi(r)/r\})$, and no such term is present here. To either prove or find a counterexample to the Ramification Theorem with error term of the form

$$\frac{1}{2}\log T(f,r) + \log^+ \frac{\psi(T(f,r))}{\phi(r)} + O(1)$$

is still an open problem.

Proof of Theorem 4.2.1. Just as in the Lemma on the Logarithmic Derivative (Theorem 3.4.1), inequalities (2) and (3) follow from inequality (1) and the Borel-Nevanlinna Growth Lemma in the form of Lemma 3.3.2 or Lemma 3.3.3 as appropriate. Thus we prove (1).

Recall that we assume that if one of the a_j's is infinite, then it is $a_q = \infty$. Recall also that $q' = q - 1$ in that case, and otherwise $q' = q$.

Let

$$s = \min_{1 \leq i < j \leq q'} |a_i - a_j|.$$

As in Proposition 1.2.3, for each z, $f(z)$ can be close to at most one point a_j. More precisely, by the triangle inequality, for each z which is not a pole of f, there is at most one index $j_0 \leq q'$ such that $|f(z) - a_{j_0}| < s/2$. For now we will regard z as a fixed point such that: (i) z is not a pole of f; (ii) $f(z) \neq a_j$ for any j; and (iii) $f'(z) \neq 0$. We let j_0 be an index such that

$$|f(z) - a_{j_0}| \leq |f(z) - a_j| \quad \text{for all } j \leq q',$$

and so in particular, by what was just said,

$$|f(z) - a_j| \geq \frac{s}{2} \quad \text{for all } j \leq q' \text{ and } \neq j_0.$$

Thus, for all $j \leq q'$ and not equal to j_0,

$$\begin{aligned} \log^+|f(z)| &\leq \log^+|f(z) - a_j| + \log^+|a_j| + \log 2 \\ &\leq \log|f(z) - a_j| + \log^+ \frac{2}{s} + \log^+|a_j| + \log 2. \end{aligned}$$

Therefore,

$$\begin{aligned} &(q'-1)\log^+|f(z)| \\ &\quad \leq \sum_{\substack{j=1 \\ j \neq j_0}}^{q'} \log|f(z) - a_j| + \sum_{\substack{j=1 \\ j \neq j_0}}^{q'} \log^+|a_j| + (q'-1)\left(\log^+ \frac{2}{s} + \log 2\right) \\ &\quad \leq \sum_{\substack{j=1 \\ j \neq j_0}}^{q'} \log|f(z) - a_j| + \sum_{\substack{j=1 \\ j \neq j_0}}^{q'} \log^+|a_j| + (q'-1)\left(\log^+ \frac{2}{s} + \log 2\right) \\ &\quad \leq \sum_{\substack{j=1 \\ j \neq j_0}}^{q'} \log|f(z) - a_j| \ + \ \mathcal{D}(a_1, \ldots, a_q) + (q-1)\log 2. \end{aligned}$$

Now,

$$\begin{aligned} &\sum_{\substack{j=1 \\ j \neq j_0}}^{q'} \log|f(z) - a_j| \\ &\quad = \sum_{j=1}^{q'} \log|f(z) - a_j| - \log|f'(z)| + \log\left|\frac{f'(z)}{f(z) - a_{j_0}}\right| \\ &\quad \leq \sum_{j=1}^{q'} \log|f(z) - a_j| - \log|f'(z)| + \log\left(\sum_{j=1}^{q'}\left|\frac{f'(z)}{f(z) - a_j}\right|\right). \end{aligned}$$

This last expression does not involve j_0, and therefore we no longer need to regard z as fixed. Note that f' is the derivative of $f - a_j$, and so the last term on the right above involves logarithmic derivatives. We have thus established a bound for $(q'-1)\log^+|f(z)|$ that involves logarithmic derivatives.

The strategy for the rest of the proof will be to use the techniques of the previous section to bound this logarithmic derivative term, and to use Jensen's Formula to identify the other terms with terms in the statement of the theorem.

By integrating $\log^+|f(z)|$ around the circle and making use of the bound we established above, we find

$$\begin{aligned}(q'-1)m(f,\infty,r) &= \int_0^{2\pi}(q'-1)\log^+|f(re^{i\theta})|\frac{d\theta}{2\pi}\\ &\le \sum_{j=1}^{q'}\int_0^{2\pi}\log|f(re^{i\theta})-a_j|\frac{d\theta}{2\pi} - \int_0^{2\pi}\log|f'(re^{i\theta})|\frac{d\theta}{2\pi}\\ &\quad + \int_0^{2\pi}\log\left(\sum_{j=1}^{q'}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})-a_j}\right|\right)\frac{d\theta}{2\pi} + \mathcal{D}(a_1,\dots,a_q)\\ &\quad +(q-1)\log 2.\end{aligned}$$

From Jensen's Formula (Corollary 1.2.1), we have

$$\int_0^{2\pi}\log|f(re^{i\theta})-a_j|\frac{d\theta}{2\pi} = N(f,a_j,r) - N(f,\infty,r) + \log|\mathrm{ilc}(f-a_j,0)|,$$

and
$$\int_0^{2\pi}\log|f'(re^{i\theta})|\frac{d\theta}{2\pi} = N(f',0,r) - N(f',\infty,r) + \log|\mathrm{ilc}(f',0)|.$$

Thus,

$$\begin{aligned}&(q'-1)m(f,\infty,r)\\ &\quad -\sum_{j=1}^{q'}N(f,a_j,r) + q'N(f,\infty,r) + N(f',0,r) - N(f',\infty,r)\\ &\le \int_0^{2\pi}\log\left(\sum_{j=1}^{q'}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})-a_j}\right|\right)\frac{d\theta}{2\pi}\\ &\quad + \left(\sum_{j=1}^{q'}\log|\mathrm{ilc}(f-a_j,0)|\right) - \log|\mathrm{ilc}(f',0)|\\ &\quad +\mathcal{D}(a_1,\dots,a_q) + (q-1)\log 2\end{aligned}$$

$$\begin{aligned}&= \int_0^{2\pi}\log\left(\sum_{j=1}^{q'}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})-a_j}\right|\right)\frac{d\theta}{2\pi}\\ &\quad +\beta_2(f,a_1,\dots,a_q) + \mathcal{D}(a_1,\dots,a_q) + (q-1)\log 2.\end{aligned}$$

The left-hand side of this last inequality is nothing other than

$$(q'-1)T(f,r) - \sum_{j=1}^{q'}N(f,a_j,r) - N(f,\infty,r) + N_{\mathrm{ram}}(f,r).$$

If $a_q = \infty$, then note that this is exactly

$$(q-2)T(f,r) - \sum_{j=1}^{q}N(f,a_j,r) + N_{\mathrm{ram}}(f,r),$$

which is what appears on the left-hand side of the inequality in the statement of the theorem. If $a_q \ne \infty$, then since $-T(f,r) \le -N(f,\infty,r)$, we still have the expression on the left in the theorem bounded by

$$\begin{aligned}&\int_0^{2\pi}\log\left(\sum_{j=1}^{q'}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})-a_j}\right|\right)\frac{d\theta}{2\pi} +\beta_2(f,a_1,\dots,a_q) + \mathcal{D}(a_1,\dots,a_q)\\ &\qquad +(q-1)\log 2.\end{aligned}$$

To complete the proof of the theorem, we still need to bound the integral term involving logarithmic derivatives. Let α be a real number between 0 and 1, so

$$\int_0^{2\pi}\log\left(\sum_{j=1}^{q'}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})-a_j}\right|\right)\frac{d\theta}{2\pi} \le \frac{1}{\alpha}\int_0^{2\pi}\log\left(\sum_{j=1}^{q'}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})-a_j}\right|\right)^{\alpha}\frac{d\theta}{2\pi}.$$

Since $\alpha < 1$, we can bring the α inside the sum to get the above

$$\le \frac{1}{\alpha}\int_0^{2\pi}\left(\log\sum_{j=1}^{q'}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})-a_j}\right|^{\alpha}\right)\frac{d\theta}{2\pi}.$$

Next, apply concavity of the logarithm in the form of Jensen's Inequality (Theorem 1.14.1) and interchange the integral with the sum to get this

$$\le \frac{1}{\alpha}\log\left(\sum_{j=1}^{q'}\int_0^{2\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})-a_j}\right|^{\alpha}\frac{d\theta}{2\pi}\right).$$

Now we apply the Goldberg-Grinshtein estimate (Theorem 3.2.2) to get that the above expression is

$$\leq c_{gg}(\alpha) + \log \frac{\rho}{r(\rho - r)} + \frac{1}{\alpha} \log \left(\sum_{j=1}^{q'} (2T(f - a_j, \rho) + \beta_1(f - a_j, r_0))^\alpha \right),$$

where $c_{gg}(\alpha)$ is as in Theorem 3.2.2. We will eventually take $\alpha = 3/4$, and then one can check numerically that $c_{gg}(3/4) \leq 5$. We thus concern ourselves with the term involving the $T(f - a_j, \rho)$. Now

$$\begin{aligned} \frac{1}{\alpha} \log \sum_{j=1}^{q'} (2T(f - a_j, \rho) &+ \beta_1(f - a_j, r_0))^\alpha \\ &\leq \frac{1}{\alpha} \log q' \max_{j \leq q'} \{(2T(f - a_j, \rho) + \beta_1(f - a_j, r_0))^\alpha\} \\ &= \frac{1}{\alpha} \log q' + \log \max_{j \leq q'} \{2T(f - a_j, \rho) + \beta_1(f - a_j, r_0)\}. \end{aligned}$$

Using Proposition 1.5.1, we see

$$\begin{aligned} \log \max_{j \leq q'} \{2T(f - a_j, \rho) &+ \beta_1(f - a_j, r_0)\} \\ &\leq \log \max_{j \leq q'} \{2T(f, \rho) + 2\log^+|a_j| + \beta_1(f - a_j, r_0)\}. \end{aligned}$$

Finally, using $\log^+(x + y) \leq \log^+ x + \log^+ y + \log 2$, we get

$$\begin{aligned} \log \max_{j \leq q'} \{2T(f, \rho) &+ 2\log^+|a_j| + \beta_1(f - a_j, r_0)\} \\ &\leq \log T(f, \rho) + \beta_3(f, a_1, \ldots, a_q, r_0) + 2\log 2. \end{aligned}$$

Combining these various estimates completes the proof of the theorem. □

If we combine the Second Main Theorem in this Chapter with the Second Main Theorem in Chapter 2, then we obtain the following Second Main Theorem with the simplest error term possible:

Theorem 4.2.4 *Let $a_1, \ldots, a_q$ be a finite set of distinct points in $\mathbf{P}^1$, let f be a non-constant meromorphic function on $\mathbf{C}$, let ψ be a Khinchin function, and let ϕ be a positive, non-decreasing, continuous function on $[r_0, \infty)$. Then for all r sufficiently large and outside an exceptional set E of radii with*

$$\int_E \frac{dr}{\phi(r)} < \infty,$$

we have

$$\begin{aligned} (q-2)T(f, r) - \sum_{j=1}^{q} N(f, a_j, r) &+ N_{\text{ram}}(f, r) \\ &\leq \log T(f, r) + \log \frac{\psi(T(f, r))}{\phi(r)} + O(1). \end{aligned}$$

Proof. Combine Theorems 4.2.1 and Theorem 2.6.2. Use the inequality in Theorem 4.2.1 for those r where $\phi(r) < r$, and use the inequality in Theorem 2.6.2 when $\phi(r) > r$. □

That one could combine Theorems 4.2.1 and Theorem 2.6.2 to obtain an inequality without any $\log^+$'s in the error term, as in Theorem 4.2.4 was first observed by H. Chen and Z. Ye [ChYe 2000].

4.3 Functions of Finite Order

We will now see how the error term in the Second Main Theorem can be simplified in the event that the meromorphic function f has finite order.

Theorem 4.3.1 *Let f be a non-constant meromorphic function on $\mathbf{C}$ of finite order ρ, let $a_1, \ldots, a_q$ be $q \geq 0$ distinct points in $\mathbf{P}^1$, and let $\varepsilon > 0$. Then, for all r sufficiently large,*

$$(q-2)T(f, r) - \sum_{j=1}^{q} N(f, a_j, r) + N_{\text{ram}}(f, r) \leq (p - 1 + \varepsilon) \log r + O(1).$$

Remark. Note how the order of the function appears as a coefficient in the error term in a natural way, and note that we again get $(p - 1 + \varepsilon)$ as in Theorem 3.5.1. However, unlike in Theorem 3.5.1, the error term in Theorem 4.3.1 actually tends to $-\infty$ in the case that f has order smaller than 1, and in particular if f is rational. We emphasize, as we did in the logarithmic derivative case, that the inequality in Theorem 4.3.1 is true for *all* sufficiently large r, and that there is no need for an exceptional set. The error term of the precise form $(p - 1 + \varepsilon) \log r + O(1)$ for the Second Main Theorem is due to Z. Ye [Ye 1996a].

Proof. As in the discussion preceding Corollary 4.2.2, since we make no statement about the value of the implicit constant in the $O(1)$ notation, we may, without loss of generality, assume $q \geq 3$. If we then take $\rho = 2r$ in statement (1) of Theorem 4.2.1, we get

$$(q-2)T(f, r) - \sum_{j=1}^{q} N(f, a_j, r) + N_{\text{ram}}(f, r) \leq \log T(f, 2r) - \log r + O(1).$$

Note that to get the $-\log r$ term, it is important that we have written statement (1) of Theorem 4.2.1 with $\log$ on the right hand side and not with $\log^+$. Now, since f has order p, and since ε is regarded as fixed, we then have that for all r sufficiently large (depending on ε and f), that $\log T(f, 2r) \leq (p + \varepsilon) \log r$, and hence the theorem is proved. □

5 Some Applications

The emphasis of this book is Nevanlinna's Second Main Theorem, and the precise structure of its error term; applications of the Second Main Theorem, or of Nevanlinna theory more generally, are not our focus. However, for the student just learning Nevanlinna theory, we feel we must provide some sort of survey of what Nevanlinna theory is "good for" so to speak, and this is what we aim to do in the present chapter. Our goal here is to describe a variety of applications in as little space as possible, not to present the most current or refined application of any given type. Indeed, many of the applications we give here are due to R. Nevanlinna himself, and Nevanlinna's own 1929 monograph [Nev(R) 1929] is one place the reader can find more refined versions and more detailed exploration of some of the applications we present here. We have, however, also made some limited effort to direct the interested reader to some of the current literature related to the applications discussed here.

We have chosen the topics in this chapter according to whether we feel they meet one of two criteria: (1) the problem to which the Second Main Theorem is applied can be easily described to someone who is not familiar with Nevanlinna theory, or (2) the application requires knowledge about the structure of the error term in the Second Main Theorem. Examples of applications meeting the first criterion are our sharing values and iteration applications. Examples such as these were chosen because we feel they have the greatest impact in convincing someone new to Nevanlinna theory of its utility. Examples, like our radius bound application, that fit the second criterion were chosen because they illustrate the value of studying the error terms, which is what distinguishes our book from most others on the subject. In fact, a few of the applications we discuss do not really fit either of these criterion, but have been included because they will help us discuss another application which does. We do not mention a number of celebrated applications of the Second Main Theorem, either because the result does not fit either of our criteria or because we could not treat the application in what we felt to be a reasonable amount of space, given the emphasis of our book. We therefore apologize to the expert in Nevanlinna theory who does not find his or her favorite type of application of the Second Main Theorem in this chapter.

We remark that many of the applications that we chose because they fit our first criterion are applications of the Second Main Theorem that do not require any knowledge about the error term except that it is of order $o(T(f,r))$, and this is one of the reasons error terms were ignored for so long. We should also point out that many of the results in this chapter can be proved without making use of the Second Main Theorem at all, and that we occasionally give a brief indication of these alternate approaches.

5.1 Infinite Products

We begin by discussing an application that does not even require the Second Main Theorem, but which further emphasizes the connection Nevanlinna theory makes between entire (meromorphic) functions and polynomials (rational functions). A consequence of the Fundamental Theorem of Algebra is that any polynomial $P(z)$ can be written

$$P(z) = cz^p \prod_{j=p+1}^{n} \left(1 - \frac{z}{z_j}\right),$$

where the z_j are the non-zero zeros of P, repeated with multiplicity. Of course, given any finite set of zeros, we can also use such a product to make a polynomial with precisely those zeros.

We will now see how this can be generalized so that meromorphic functions of finite order can be written as infinite products of functions specifying the zeros and poles. We define, for any non-negative integer q, the **Weierstrass primary factor**

$$E_q(z) = (1-z)e^{z+z^2/2+\dots+z^q/q},$$

where for $q = 0$ we take $E_0(z) = 1 - z$.

Theorem 5.1.1 *Let f be a meromorphic function on $\mathbf{C}$ with zeros a_j and poles b_j repeated according to multiplicity. Suppose there exists an integer q such that either*

$$(a) \quad \liminf_{r\to\infty} \frac{T(f,r)}{r^{q+1}} = 0 \qquad or \qquad (b) \quad \lim_{r\to\infty} \frac{T(f,r)}{r^{q+1}} = 0.$$

Then, there exists an integer p and a polynomial of degree at most q such that

$$f(z) = z^p e^{P_q(z)} \lim_{r\to\infty} \frac{\prod_{|a_j|<r} E_q(z/a_j)}{\prod_{|b_j|<r} E_q(z/b_j)}$$

converges uniformly on compact subsets of the plane, where in case (a) the limit takes place over some sequence of $r \to \infty$, whereas in case (b), the limit takes place over all $r \to \infty$.

Remark. Theorem 5.1.1 does *not* assert that the products in the numerator and denominator converge separately. Note also that if f is a meromorphic function of finite order (respectively finite lower order), then f automatically satisfies hypothesis (b) (respectively hypthesis (a)) of the theorem for sufficently large q.

Corollary 5.1.2 *If f is a meromorphic function on $\mathbf{C}$ then f is a rational function if and only if $T(f,r) = O(\log r)$ as $r \to \infty$. Moreover, if f is not rational, then*

$$\lim_{r\to\infty} \frac{T(f,r)}{\log r} = +\infty.$$

Proof of Corollary 5.1.2. We already saw in §1.6 that rational functions satisfy $T(f,r) = O(\log r)$. For the converse, note that if

$$\liminf_{r\to\infty} \frac{T(f,r)}{\log r} = C < \infty,$$

then because, for any r,

$$\begin{aligned} n(f,a,r) &\le \liminf_{\rho\to\infty} \frac{N(f,a,\rho) - N(f,a,r)}{\log\rho - \log r} \\ &\le \liminf_{\rho\to\infty} \frac{T(f,\rho) + O(1) - N(f,a,r)}{\log\rho - \log r} = C < \infty, \end{aligned}$$

f can only take on each value a at most C times. In particular, f has at most a finite number of zeros and poles. The corollary follows by noting that we may take $q = 0$ in the theorem, and hence the theorem tells us that f is the quotient of two polynomials. □

Proof of Theorem 5.1.1. Because of the factor z^p out front, we can always divide out by z^p and assume $f(0) \ne 0, \infty$. We wish to compute

$$\frac{\partial^{q+1}}{\partial z^{q+1}} \log f(z).$$

From the Poisson-Jensen Formula (Theorem 1.1.6), as long as z is not a zero or pole of f, we can write

$$\begin{aligned} \log|f(z)| = &\int_0^{2\pi} \log|(f(re^{i\theta})| \operatorname{Re} \frac{re^{i\theta}+z}{re^{i\theta}-z}\frac{d\theta}{2\pi} \\ &- \sum_{|a_j|<r} \log\left|\frac{r^2-\bar{a}_j z}{r(z-a_j)}\right| + \sum_{|b_j|<r} \left|\frac{r^2-\bar{b}_j z}{r(z-b_j)}\right|. \end{aligned}$$

As in Corollary 1.1.7, from Cauchy-Riemann, multiplying by 2, and moving the derivatives inside the integral, we get

$$\frac{\partial^{q+1}}{\partial z^{q+1}} \log f(z) = \sum_{|a_j|<r} \frac{(-1)^q q!}{(z-a_j)^{q+1}} - \sum_{|b_j|<r} \frac{(-1)^q q!}{(z-b_j)^{q+1}} + S_r(z) + I_r(z).$$

where

$$S_r(z) = q! \sum_{|a_j|<r} \left(\frac{\bar a_j}{r^2 - \bar a_j z}\right)^{q+1} - q! \sum_{|b_j|<r} \left(\frac{\bar b_j}{r^2 - \bar b_j z}\right)^{q+1},$$

$$\text{and} \quad I_r(z) = (q+1)! \int_0^{2\pi} \log|f(re^{i\theta})| \frac{2re^{i\theta}}{(re^{i\theta} - z)^{q+2}} \frac{d\theta}{2\pi}.$$

The idea is to show that S_r and I_r tend to zero uniformly on compact subsets as $r \to \infty$, possibly through a particular sequence in case (a). Once we know that, we can write

$$\frac{\partial^{q+1}}{\partial z^{q+1}} \log f(z) = (-1)^{q+1} q! \lim_{r\to\infty} \left\{ \sum_{|b_j|<r} \frac{1}{(z - b_j)^{q+1}} - \sum_{|a_j|<r} \frac{1}{(z - a_j)^{q+1}} \right\}.$$

Since the sums converge uniformly on compact subsets, we can integrate term by term $q+1$ times and exponentiate to get the statement of the theorem. Note that $P_q(z)$ is the polynomial of degree (at most) q such that

$$\left.\frac{\partial^k}{\partial z^k}\right|_{z=0} P_q(z) = \left.\frac{\partial^k}{\partial z^k}\right|_{z=0} \log f(z), \quad k = 0, \ldots, q.$$

Thus, we just need to show that S_r and I_r vanish uniformly on compact sets.

For S_r, note that for $|z| \le t < r$, we have

$$\left|\frac{\bar a_j}{r^2 - \bar a_j z}\right| \le \frac{|a_j|}{r^2 - |a_j| r} < \frac{1}{r - t}.$$

Therefore,

$$\left| \sum_{|a_j|<r} \left(\frac{\bar a_j}{r^2 - \bar a_j z}\right)^{q+1} \right| \le \frac{n(f,0,r)}{(r-t)^{q+1}} = \frac{n(f,0,r)}{r^{q+1}(1 - t/r)^{q+1}}.$$

Note that

$$\begin{aligned} n(f,0,r) &= n(f,0,r) \int_r^{er} \frac{dt}{t} \\ &\le \int_r^{er} n(f,0,t) \frac{dt}{t} \\ &\le N(f,er,0) \le T(f,er) + O(1). \end{aligned}$$

Therefore, since

$$\liminf_{r\to\infty} \frac{T(f,r)}{r^{q+1}} = 0 \quad \text{or} \quad \lim_{r\to\infty} \frac{T(f,r)}{r^{q+1}} = 0,$$

the same is true for $n(f,0,r)/r^{q+1}$, and hence the sum converges uniformly for $|z| \le t$. A similar estimates shows the convergence for the sum involving the poles b_j, and hence S_r tends to zero uniformly on compact sets as $r \to \infty$, through an appropirate sequence in case (a).

We now estimate I_r. Again, let $|z| \le t$. Then,

$$\begin{aligned} |I_r(z)| &\le \frac{(q+1)! 2r}{(r-t)^{q+2}} \int_0^{2\pi} \left|\log|f(re^{i\theta})|\right| \frac{d\theta}{2\pi} \\ &\le \frac{2(q+1)! [m(f,0,r) + m(f,\infty,r) + O(1)]}{(1 - t/r)^{q+2} r^{q+1}} \\ &\le \frac{4(q+1)!}{(1-t/r)^{q+2}} \frac{T(f,r) + O(1)}{r^{q+1}}, \end{aligned}$$

and hence I_r also tends to zero uniformly on compact subsets. □

5.2 Defect Relations

Given a meromorphic function f, and $a_1, \ldots, a_q$ distinct points in $\mathbf{P}^1$, denote by

$$\begin{aligned} S(f, \{a_j\}_{j=1}^q, r) &= (q-2)T(f,r) - \sum_{j=1}^q N(f, a_j, r) + N_{\mathrm{ram}}(f,r) \\ &= (q-2)\mathring{T}(f,r) - \sum_{j=1}^q N(f, a_j, r) + N_{\mathrm{ram}}(f,r) + O(1). \end{aligned}$$

The following weak formulation of the Second Main Theorem is all that is needed for many of the applications discussed in this chapter.

Theorem 5.2.1 (Weak Second Main Theorem) *Let $a_1, \ldots, a_q$ be distinct points in $\mathbf{P}^1$. If f is a non-constant meromorphic function defined on $\mathbf{C}$, then*

$$\liminf_{r\to\infty} \frac{S(f, \{a_j\}_{j=1}^q, r)}{T(f,r)} \le 0.$$

If f is a meromorphic function defined on $\mathbf{D}(R)$, $R < \infty$, such that

$$\mu = \liminf_{r\to R} \frac{T(f,r)}{\log(1/(R-r))} > 0,$$

then

$$\liminf_{r\to R} \frac{S(f, \{a_j\}_{j=1}^q, r)}{T(f,r)} \le \frac{1}{\mu}.$$

Proof. If f is defined on all of $\mathbf{C}$, then by taking $\phi(r) \equiv 1$ and by taking $\psi(x) = (\log x)^2$ in the Second Main Theorem, in the form of either Corollary 4.2.3 or Theorem 2.5.1, we have that

$$S(f, \{a_j\}_{j=1}^q, r) \leq O(\log T(f,r)),$$

for r sufficiently large and outside of a set of finite Lebesgue measure.

If f is defined on $\mathbf{D}(R), R < \infty$, then instead choose $\phi(r) = R - r$ and apply the Second Main Theorem in the form of Theorem 2.5.1 to conclude that

$$S(f, \{a_j\}_{j=1}^q, r) \leq \log \frac{1}{R-r} + O(\log T(f,r))$$

for all r sufficiently close to R and outside an exceptional set that cannot contain any intervals of the form $(R - \epsilon, R)$ for $\epsilon > 0$. Thus,

$$\liminf_{r\to R} \frac{S(f, \{a_j\}_{j=1}^q, r)}{T(f,r)} \leq \limsup_{r\to R} \frac{\log(1/(R-r))}{T(f,r)} = \frac{1}{\mu}$$

by the definition of μ. Note that the lim sup is required on the right because we do not know where the exceptional set lies. □

We pause to remark that in the disc case $R < \infty$, the precise bound of $1/\mu$ cannot be obtained by Theorem 4.2.1 or its corollaries; we must use the negative curvature version of the theorem, as we have done in the proof. This is because the error term in Theorem 4.2.1 has an extra term of the form $\log 1/(R-r)$ which, when combined with the $\log 1/\phi(r)$ term, makes the bound of $1/\mu$ unobtainable by this method.

Our first application of the Second Main Theorem will be an application to the study of what are known as "defects." In all honesty, this application is more of a weaker re-formulation of the Second Main Theorem than a true application of it. We introduce this topic first because we derive some of our other applications as a consequence of the so-called "defect relation" rather than of the full Second Main Theorem.

We begin by defining defects and introducing notation for them. Although defects are most often discussed in the context of meromorphic functions defined on all of $\mathbf{C}$, we will also define them for merormorphic functions on a disc, $\mathbf{D}(R)$, so let f be a meromorphic function on $\mathbf{D}(R), R \leq \infty$. We will only be able to make meaningful statements about defects in the event that

$$\lim_{r\to R} T(f,r) = \infty,$$

so we henceforth assume this, and we remark that if $R = \infty$, then this assumption is nothing more than assuming f to be non-constant. Given a point a in $\mathbf{P}^1$, the **Nevanlinna defect** $\delta(f,a)$ is defined by

$$\delta(f,a) = \liminf_{r\to R} \left\{1 - \frac{N(f,a,r)}{T(f,r)}\right\}.$$

We remark that by the First Main Theorem,

$$\delta(f,a) = \liminf_{r\to R} \frac{m(f,a,r)}{T(f,r)}.$$

We also point out that one gets the same thing if one uses $\mathring{T}(f,r)$ in place of $T(f,r)$ (or $\mathring{m}(f,a,r)$ in place of $m(f,a,r)$).

Recall that we defined the truncated counting function $N^{(1)}$ in §1.2, and that this function counts how often a meromorphic function takes on a particular value without regard for multiplicity. We warn the reader not to confuse what we have denoted by $N^{(1)}$ with what R. Nevanlinna and subsequent authors have denoted by N_1, which they use to denote the counting function associated to the ramification divisor. One of the reasons we have broken with tradition and use N_{ram} to denote the ramification counting function is so that it will not be confused with the counting function truncated to order 1.

We define the **ramification defect**

$$\theta(f,a) = \liminf_{r\to R} \left\{\frac{N(f,a,r) - N^{(1)}(f,a,r)}{T(f,r)}\right\}.$$

The quantity

$$\delta^{(1)}(f,a) = \delta(f,a) + \theta(f,a) = \liminf_{r\to\infty} \left\{1 - \frac{N^{(1)}(f,a,r)}{T(f,r)}\right\}$$

is known as the **truncated defect** (to order 1) since the truncated counting function is used in its definition. Calling the defect "truncated" is a bit of a misnomer though in the sense that $\delta^{(1)} \geq \delta$. We also point out that what we have denoted by $\delta^{(1)}$ is often denoted by Θ. We prefer the $\delta^{(1)}$ notation because in higher dimensions one wants to discuss defects truncated to order j, which is convenient to denote by $\delta^{(j)}$.

As a direct consequence of the First Main Theorem, we have

Proposition 5.2.2 *For every a in $\mathbf{P}^1$,*

$$0 \leq \delta(f,a), \quad 0 \leq \theta(f,a), \quad \textit{and} \quad \delta^{(1)}(f,a) = \delta(f,a) + \theta(f,a) \leq 1.$$

A value a is called **deficient** for f if $\delta(f,a) > 0$. The value a is said to be **maximally deficient** if $\delta(f,a) = 1$. Note that any omitted value is maximally deficient.

Theorem 5.2.3 *Let f be a non-constant meromorphic function on $\mathbf{C}$. Then, the set of values a such that $\delta(f,a) > 0$ or $\theta(f,a) > 0$ is countable, and moreover,*

$$\sum_{a\in\mathbf{P}^1} \delta^{(1)}(f,a) = \sum_{a\in\mathbf{P}^1} (\delta(f,a) + \theta(f,a)) \leq 2.$$

Proof. Let $a_1, \dots, a_q$ be a finite number of distinct points in $\mathbf{P}^1$. Then,

$$\sum_{j=1}^{q}(\delta(f,a_j)+\theta(f,a_j))$$

$$= \liminf_{r\to\infty} \frac{qT(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + \sum_{j=1}^{q} N(f,a_j,r) - \sum_{j=1}^{q} N^{(1)}(f,a_j,r)}{T(f,r)}.$$

Observe that

$$N(f,a,r) - N^{(1)}(f,a,r)$$

counts the number of times $f = a$ with multiplicity greater than 1, and so

$$\sum_{j=1}^{q} N(f,a_j,r) - \sum_{j=1}^{q} N^{(1)}(f,a_j,r) \le N_{\mathrm{ram}}(f,r) + n_{\mathrm{ram}}(f,0)\log^+\frac{1}{r}.$$

Note that the $\log^+(1/r)$ term is zero for $r \ge 1$, and in any case bounded as $r \to \infty$. Thus,

$$\sum_{j=1}^{q}(\delta(f,a_j)+\theta(f,a_j)) \le \liminf_{r\to\infty} \frac{qT(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{ram}}(f,r)}{T(f,r)}$$

$$= 2 + \liminf_{r\to\infty} \frac{(q-2)T(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{ram}}(f,r)}{T(f,r)}$$

$$= 2 + \liminf_{r\to\infty} \frac{S(f,\{a_j\}_{j=1}^{q},r)}{T(f,r)} \le 2,$$

the last inequality being Theorem 5.2.1.

Thus, for any finite collection of distinct points, the sum of the defects is at most two. In particular, for every positive integer k, there are at most finitely many values a with $\delta^{(1)}(f,a) \ge 1/k$. Since

$$\{a : \delta^{(1)}(f,a) > 0\} = \bigcup_{k=1}^{\infty}\{a : \delta^{(1)}(f,a) \ge 1/k\},$$

there are at most countably many a such that $\delta^{(1)}(f,a) > 0$. If there are only finitely many such a, we are done. If there are countably many such a, then enumerate them by a_j and note that

$$\sum_{j=1}^{\infty}\delta^{(1)}(f,a_j) = \lim_{q\to\infty}\sum_{j=1}^{q}\delta^{(1)}(f,a_j) \le 2,$$

since each finite sum is ≤ 2 by what was shown above. □

Corollary 5.2.4 *If f is an entire function, then*

$$\sum_{a\in\mathbf{C}}\delta^{(1)}(f,a) = \sum_{a\in\mathbf{C}}(\delta(f,a)+\theta(f,a)) \le 1.$$

Proof. If f is entire, then $\delta^{(1)}(f,\infty) = \delta(f,\infty) = 1$. □

These so-called defect relations are our long sought after generalizations to the Fundamental Theorem of Algebra. The defects measure how far the Fundamental Theorem of Algebra is from holding for f at the value a (*i.e.* $\delta(f,a)$ measures to what extent f does not take on the value a with the expected frequency), and the defect relation says in some sense that the analogue of the Fundamental Theorem of Algebra fails by a finite amount, and provides a quantitative bound for that failure.

The following question was posed by Nevanlinna and is known as the Nevanlinna Inverse Problem.

Question 5.2.5 (Nevanlinna Inverse Problem) *For $1 \le i < N \le \infty$, let $\{\delta_i\}$ and $\{\theta_i\}$ be sequences of non-negative real numbers such that*

$$0 < \delta_i + \theta_i \le 1 \qquad \text{and} \qquad \sum_i(\delta_i+\theta_i) \le 2.$$

Let $a_i, 1 \le i < N$ be distinct points in $\mathbf{P}^1$. Does there exist a meromorphic function f on $\mathbf{C}$ such that

$$\delta(f,a_i) = \delta_i \quad \theta(f,a_i) = \theta_i \quad 1 \le i < N$$

and so that $\delta(f,a) = \theta(f,a) = 0$ for all $a \ne \{a_i\}$?

Nevanlinna solved this "inverse problem" in some special cases, and the problem was completely solved by D. Drasin in [Dras 1976].

We conclude this topic by stating a disc version of the defect relation.

Theorem 5.2.6 *Let f be a meromorphic function on $\mathbf{D}(R), R < \infty$. If*

$$\mu = \liminf_{r\to R}\frac{T(f,r)}{\log(1/(R-r))} > 0,$$

then the set of values a in $\mathbf{P}^1$ such that $\delta(f,a) > 0$ or $\theta(f,a) > 0$ is countable, and moreover,

$$\sum_{a\in\mathbf{P}^1}\delta^{(1)}(f,a) = \sum_{a\in\mathbf{P}^1}(\delta(f,a)+\theta(f,a)) \le 2 + \frac{1}{\mu}.$$

Proof. The proof is exactly the same as the proof of Theorem 5.2.3. □

We mention without proof that if

$$f : \mathbf{D}(R) \to \mathbf{P}^1 \setminus \{a_1, \ldots, a_q\}$$

is a universal covering map with $q \geq 3$, then

$$\liminf_{r \to R} \frac{T(f,r)}{\log(1/(R-r))} = \frac{1}{q-2}$$

and f clearly has q maximally deficient values. Thus, the bound in Theorem 5.2.6 is sharp. We refer the reader to [Nev(R) 1929] for a further discussion.

For a more thorough treatment of defects as well as the discussion of some illuminating examples we refer the reader to Chapter X of [Nev(R) 1970]. Chapter 4 of [Hay 1964] consists of a much broader study of defects than we have presented here, and includes, among other things, a partial solution to the Nevanlinna Inverse Problem as well as some additional conditions that defects of functions of finite order must satisfy.

5.3 Picard's Theorem

The defect relation we just derived has as immediate consequence:

Theorem 5.3.1 (Picard's Theorem) *Let f be a meromorphic function on $\mathbf{C}$. If f omits three distinct values in $\mathbf{P}^1$, then f must be constant.*

Proof. If f omits three distinct values, then there are three distinct points a with $\delta(f,a) = 1$. This clearly contradicts Theorem 5.2.3. □

We take a moment to remind the reader of Picard's original proof [Pic 1879]. If f omits three values, then by post-composition with a fractional linear transformation, one may assume the values are 0, 1, and ∞. Then, in present day language, the Picard modular function gives a universal covering map from the unit disc to $\mathbf{P}^1 \setminus \{0, 1, \infty\}$. Thus any meromorphic function on $\mathbf{C}$ that omits 0, 1, and ∞ can be lifted to the universal cover, namely the disc. This lifted map is bounded (since it maps into the disc) and therefore must be constant by Liouville's Theorem.

In some sense Picard's proof is simpler than the proof we present based on the Second Main Theorem. On the other hand, Picard's proof relies on knowing that the thrice punctured sphere is uniformized by the disc, and this is in some sense "deeper" (though certainly not more complicated) than the proof of the Second Main Theorem, which is really only Stokes's Theorem and a lot of calculus. Therefore, some people regard the Second Main Theorem proof of Picard's Theorem as a more "elementary," if more complicated, proof of Picard's Theorem. An advantage of the proof being "elementary" is that this proof generalizes to prove higher dimensional analogs of Picard's Theorem in cases that Picard's "uniformization" proof does not.

5.4 Totally Ramified Values

We call a point a in $\mathbf{P}^1$ a **totally ramified value** for a meromorphic function f if for every z such that $f(z) = a$, we have $\mathrm{ord}_z(f-a) > 1$ if a is finite, or $\mathrm{ord}_z f < -1$ if $a = \infty$. Note that an omitted value trivially satisfies this definition, but that it will be useful to distinguish between omitted values and non-omitted totally ramified values.

Theorem 5.4.1 (R. Nevanlinna) *If f is a non-constant meromorphic function defined on all of $\mathbf{C}$, then twice the number of omitted values plus the number of non-omitted totally ramified values cannot exceed 4. In particular, a meromorphic function f has at most four totally ramified values, a holomorphic function has at most two finite totally ramified values, and a unit (i.e. a holomorphic function that is never zero) can have no totally ramified values (other than 0 and ∞ which are omitted).*

Proof. If a is an omitted value, then a is totally ramified and

$$\delta^{(1)}(f,a) = \delta(f,a) = 1.$$

If a is totally ramified, but not omitted, then since every value is taken with at least multiplicity 2, we have by the First Main Theorem that

$$\delta^{(1)}(f,a) = \liminf_{r\to\infty}\left\{1 - \frac{N^{(1)}(f,a,r)}{T(f,a,r)}\right\} \geq \liminf_{r\to\infty}\left\{1 - \frac{N^{(1)}(f,a,r)}{N(f,a,r)}\right\} \geq \frac{1}{2}.$$

Thus, the defect relation, Theorem 5.2.3, implies that the number of omitted values, plus half the number of totally ramified but not omitted values, can be no more than two. □

We remark that $\cos z$ is an entire function with 1 and -1 as totally ramified values, and thus an entire function can have two totally ramified values. We leave it as an exercise for the interested reader to check that the Weierstrass $\wp$ function has four totally ramified (and not omitted) values, and thus Theorem 5.4.1 is sharp.

5.5 Meromorphic Solutions to Differential Equations

Value distribution theory has proved to be an indispensable tool in the study of entire or meromorphic solutions to differential equations. Here we give only one modest example of a theorem which can be proven using Nevanlinna theory.

Theorem 5.5.1 (Malmquist's Theorem) *Denote by $\mathbf{C}(z)$ the field of rational functions in the variable z with complex coefficients. Denote by $\mathbf{C}(z)[X]$ the one variable polynomial ring with coefficients in $\mathbf{C}(z)$. Let $P(X)$ and $Q(X)$ be relatively prime elements of $\mathbf{C}(z)[X]$. If the differential equation*

$$f' = \frac{P(f)}{Q(f)}$$

has a transcendental meromorphic solution f, then Q has degree zero in X and P has degree at most 2 in X.

Remarks. A differential equation of the form

$$f' = a_0(z) + a_1(z)f + a_2(z)f^2,$$

is called a Riccati equation. Thus, Malmquist's Theorem says that the only differential equations of the form $f' = P(f)/Q(f)$ that admit transcendental meromorphic solutions are the Riccati equations.

Malmquist's proof [Malm 1913] of Theorem 5.5.1 was published in 1913, well before Nevanlinna invented his theory. K. Yosida gave the first Nevanlinna theory based proof when he generalized Malmquist's Theorem in [Yos 1933]. Since that time, a whole range of theorems in differential equations can be classified as "Malmquist-Yosida" theorems.

We break the proof of Malmquist's Theorem up into pieces. First we prove a proposition relating the characteristic function of $P(f)/Q(f)$ to the characteristic function of f.

Proposition 5.5.2 *Let P and Q be as in Theorem 5.5.1, and let f be a transcendental meromorphic function. Set $g = P(f)/Q(f)$. Then*

$$T(g,r) = \max\{\deg P, \deg Q\}T(f,r) + o(T(f,r)).$$

Proof. Write

$$P(X) = \sum_{j=0}^{p} a_j(z)X^j \quad \text{and} \quad Q(X) = \sum_{j=0}^{q} b_j(z)X^j,$$

where $a_j(z)$ and $b_j(z)$ are rational functions of z. Note that by Corollary 5.1.2, $T(a_j,r) = O(\log r) = o(T(f,r))$, and similarly for the b_j.

We first treat the polynomial case, that is the case when $q = \deg Q = 0$, in which case we may assume $Q \equiv 1$.

We show $T(g,r) = T(P(f),r) \leq pT(f,r) + o(T(f,r))$ by induction on p. When $p = 0$ this inequality is trivial. In general, note that

$$P(f) = f\left(\sum_{j=1}^{p} a_j f^{j-1}\right) + a_0.$$

Thus, by Proposition 1.5.1, we have

$$T(P(f),r) \leq T(f,r) + T\left(\sum_{j=1}^{p} a_j f^{j-1}, r\right) + T(a_0,r) + \log 2,$$

and hence the induction step.

To show $T(P(f),r) \geq pT(f,r) + o(T(f,r))$, we consider the counting and mean-proximity functions separately.

The counting function is the easier of the two. There are only finitely many points where one of the rational functions a_j can have zeros or poles. If z_0 is not one of these finitely many points and if f has a pole of order k at z_0, then f^p has a pole of order pk at z_0, and hence so does $P(f)$. Thus, if it weren't for the zeros and poles of the coefficient functions a_j, we would have $N(P(f),\infty,r) = pN(f,\infty,r)$. Taking the finitely many zeros and poles of the coefficients into account gives us

$$N(P(f),\infty,r) \geq pN(f,\infty,r) - O(\log r).$$

Because the coefficient functions a_j are rational functions, there exists an integer d such that for $r \gg 0$ and for $j = 0,\ldots,p-1$,

$$\frac{|a_j(re^{i\theta})|}{|a_p(re^{i\theta})|} \leq r^d.$$

Define

$$S_r = \{\theta \in [0,2\pi] : |f(re^{i\theta})| \geq r^{d+1}\},$$

and denote by S_r^c the complement of S_r in the interval $[0,2\pi]$. Then, for $z = re^{i\theta}$ with $\theta \in S_r$ and $r \gg 0$, we see

$$\begin{aligned} |P(f(z))| &= |a_p(z)||f(z)|^p \left|1 + \sum_{j=0}^{p-1} \frac{a_j(z)}{a_p(z)[f(z)]^{p-j}}\right| \\ &\geq |a_p(z)||f(z)|^p\left(1 - \frac{1}{r}\right) \geq \frac{1}{2}|a_p(z)||f(z)|^p, \end{aligned}$$

and thus for these z and $r \gg 0$,

$$|f(z)| \leq |f(z)|^p \leq \frac{2|P(f(z))|}{|a_p(z)|}.$$

On the other hand, for $z = re^{i\theta}$ with $\theta \notin S_r$, we have $|f(z)| \leq r^{d+1}$. Thus, for all sufficiently large r,

$$\begin{aligned} pm(f,\infty,r) &= \int_{S_r} \log^+|f(re^{i\theta})|^p \frac{d\theta}{2\pi} + \int_{S_r^c} \log^+|f(re^{i\theta})|^p \frac{d\theta}{2\pi} \\ &\leq \int_{S_r} \log^+ 2\frac{|P(f(re^{i\theta}))|}{|a_p(re^{i\theta})|} \frac{d\theta}{2\pi} + \int_{S_r^c} \log^+ r^{p(d+1)} \frac{d\theta}{2\pi} \\ &\leq \int_0^{2\pi} \log^+|P(f(re^{i\theta}))| \frac{d\theta}{2\pi} + O(\log r), \end{aligned}$$

and hence $m(P(f),\infty,r) \geq pm(f,\infty,r) - O(\log r)$. Combining this with our earlier estimate for the counting functions, we have

$$T(P(f),r) \geq pT(f,r) + o(T(f,r)).$$

We now show

$$T(g,r) \leq \max\{p,q\}T(f,r) + o(T(f,r))$$

in the general case $g = P(f)/Q(f)$ when $\deg Q \geq 1$. By the First Main Theorem (Theorem 1.3.1), $T(g,r) = T(1/g,r) + O(1)$, and so without loss of generality we may assume $p \geq q$, and we proceed by induction on q. By the division algorithm, we can write $P/Q = A + B/Q$, were $A, B \in \mathbf{C}(z)[X]$, with the degree of A (in X) equal to $p - q$, and the degree of B strictly less than the degree of Q. Thus, from Proposition 1.5.1,

$$T(g,r) \leq T(A(f),r) + T(B(f)/Q(f),r) + O(1).$$

We have already proven the inequality for polynomials, so

$$T(A(f),r) \leq (p-q)T(f,r) + o(T(f,r)).$$

Using the First Main Theorem (Theorem 1.3.1) and the induction hypothesis, we have

$$T(B(f)/Q(f),r) = T(Q(f)/B(f),r) + O(1) \leq qT(f,r) + o(T(f,r)),$$

and hence $T(g,r) \leq pT(f,r) + o(T(f,r))$ as was to be shown.

It remains to show $T(g,r) \geq \max\{p,q\}T(f,r) + O(1)$, and here we need to use the assumption that P and Q are relatively prime. Since P and Q are relatively prime, we have by the Chinese Remainder Theorem, $1 = CP + DQ$, for some $C, D \in \mathbf{C}(z)[X]$. Dividing through by CQ, we have

$$\frac{1}{CQ} = \frac{P}{Q} + \frac{D}{C}.$$

Again by the First Main Theorem, we may assume $q \geq p$, in which case

$$c = \deg C \geq \deg D.$$

Because CQ is a polynomial (in X), and we have already proved the proposition for polynomials,

$$T(C(f)Q(f),r) = (c+q)T(f,r) + o(T(f,r)).$$

On the other hand, by Proposition 1.5.1 and the inequality in the other direction

$$\begin{aligned} T\left(\frac{1}{C(f)Q(f)},r\right) &\leq T\left(\frac{P(f)}{Q(f)},r\right) + T\left(\frac{B(f)}{Q(f)},r\right) + \log 2 \\ &\leq T\left(\frac{P(f)}{Q(f)},r\right) + cT(f,r) + o(T(f,r)). \end{aligned}$$

Making use of the First Main Theorem one last time,

$$\begin{aligned} (c+q)T(f,r) + o(T(f,r)) &= T\left(\frac{1}{C(f)Q(f)},r\right) \\ &\leq T\left(\frac{P(f)}{Q(f)},r\right) + cT(f,r) + o(T(f,r)), \end{aligned}$$

and hence the proposition. □

We now state a weaker version of Malmquist's Theorem as a lemma.

Lemma 5.5.3 *Let $P(X)$ and $Q(x)$ be relatively prime polynomials in $\mathbf{C}(z)[X]$ and let f be a transcendental meromorphic solution to*

$$f' = \frac{P(f)}{Q(f)}.$$

Then, $\deg P \leq 2$ *and* $\deg Q \leq 2$.

Proof. Let $d = \max\{\deg P, \deg Q\}$. Since $f' = P(f)/Q(f)$, by Proposition 5.5.2 we have

$$\begin{aligned} dT(f,r) &= T(P(f)/Q(f),r) + o(T(f,r)) \\ &= T(f',r) + o(T(f,r)) \;=\; m(f',\infty,r) + N(f',\infty,r) + o(T(f,r)). \end{aligned}$$

Now, $N(f',\infty,r) \leq 2N(f,\infty,r)$ for $r \geq 1$, and by Proposition 1.5.1,

$$m(f',\infty,r) \leq m\left(\frac{f'}{f},\infty,r\right) + m(f,\infty,r).$$

Clearly, $m(f,\infty,r) + 2N(f,\infty,r) \leq 2T(f,r)$, and so we conclude from the Logarithmic Derivative Lemma (Theorem 3.4.1) that

$$dT(f,r) \leq 2T(f,r) + o(T(f,r)),$$

for a sequence of $r \to \infty$, or in other words that $d \leq 2$. □

Lemma 5.5.3 reduces the proof of Theorem 5.5.1 to the case when P and Q are quadratic. The proof is essentially completed by a the following elementary proposition about quadratic polynomials.

Proposition 5.5.4 *Let*

$$P(X) = a_2X^2 + a_1X + a_0 \qquad \text{and} \qquad Q(X) = b_2X^2 + b_1X + b_0,$$

where a_0, a_1, a_2, b_0, b_1, and b_2 are in $\mathbf{C}(z)$. Let

$$\tilde{P}(X) = a_0X^2 + a_1X + a_2 \qquad \text{and} \qquad \tilde{Q}(X) = b_0X^2 + b_1X + b_2.$$

Then, P and Q are relatively prime in $\mathbf{C}(z)[X]$ if and only if $\tilde{P}$ and $\tilde{Q}$ are relatively prime in $\mathbf{C}(z)[X]$.

Proof. Let c_0 and c_1 be elements of $\mathbf{C}(z)$. Then, $(c_1X + c_0)$ is a common factor of P and Q if and only if $(c_0X + c_1)$ is a common factor of $\tilde{P}$ and $\tilde{Q}$. □

Proof of Theorem 5.5.1. Let f be a transcendental meromorphic solution to the differential equation

$$f' = \frac{P(f)}{Q(f)}.$$

By Lemma 5.5.3,

$$P(X) = a_2X^2 + a_1X + a_0 \qquad \text{and} \qquad Q(X) = b_2X^2 + b_1X + b_0,$$

where a_0, a_1, a_2, b_0, b_1, and b_2 are in $\mathbf{C}(z)$. We need to show that

$$b_1(z) \equiv b_2(z) \equiv 0.$$

Let $g = 1/f$ Then,

$$\begin{aligned} g' = \frac{-f'}{f^2} = -\frac{a_0 + a_1f + a_2f^2}{f^2[b_0 + b_1f + b_2f^2]} &= \frac{g^2[a_0 + a_1/g + a_2/g^2]}{b_0 + b_1/g^2 + b_2/g^2} \\ &= \frac{g^2[a_0g^2 + a_1g + a_2]}{b_0g^2 + b_1g + b_2} \end{aligned}$$

Thus, g is a transcendental meromorphic solution to

$$g' = \frac{g^2\tilde{P}(g)}{\tilde{Q}(g)},$$

where $\tilde{P}(X)$ and $\tilde{Q}(X)$ are defined as in Proposition 5.5.4. Since $X^2\tilde{P}(X)$ has degree $4 \geq 2$, Lemma 5.5.3 implies that $X^2\tilde{P}(X)$ and $\tilde{Q}(X)$ have a quadratic factor in common. On the other hand, $\tilde{P}$ and $\tilde{Q}$ are relatively prime by Proposition 5.5.4. Thus, X^2 divides $\tilde{Q}(X)$. □

For a comprehensive introduction to the role of Nevanlinna theory in the study of meromorphic solutions to differential equations, see [Laine 1993].

5.6 Functions Sharing Values

Perhaps some of the most striking applications of Nevanlinna theory are results that state if two functions share values then they must be identical (or differ by a constant, or differ by a fractional linear transformation, *etc.*). We content ourself here with giving just one such example due to Nevanlinna himself.

Theorem 5.6.1 (Nevanlinna Five Values Theorem) *Let f_1 and f_2 be meromorphic functions on $\mathbf{C}$. For a point a in $\mathbf{P}^1$, and for $\ell = 1, 2$, denote by*

$$G_\ell(a) = \{z \in \mathbf{C} : f_\ell(z) = a\}.$$

If $G_1(a_j) = G_2(a_j)$ for five distinct values $a_1, \ldots, a_5$, then either $f_1 \equiv f_2$ or f_1 and f_2 are both constant.

Remark. When $G_1(a) = G_2(a)$, the functions f_1 and f_2 are said to share the value a (ignoring multiplicity). Note that e^z and e^{-z} share the four values 0, 1, -1, and ∞, so the 5 in the statement of the theorem cannot be replaced by 4. We emphasize that the sets G_ℓ do not account for multiplicity. If one takes the multiplicities into account, meaning that $f_1 = a$ and $f_2 = a$ occur at the same points and with the same multiplicities, then Nevanlinna [Nev(R) 1929] also showed that apart from certain explicit exceptions (like the example of e^z and e^{-z} mentioned above), three or four shared values are sufficient to determine f.

Proof. Suppose that f_1 and f_2 are neither both constant, nor identical. If either f_1 or f_2 is constant, then our assumption implies the other omits at least four values, and so is constant also by Picard's Theorem. Thus, we may assume neither f_ℓ is constant.

Let $g(z) = (f_1(z) - f_2(z))^{-1}$, and note that g is not identically ∞ since f_1 and f_2 are assumed not to be identical. From the First Main Theorem (Theorem 1.3.1) and Proposition 1.5.1, we know that

$$T(g,r) = T(f_1 - f_2, r) + O(1) \leq T(f_1, r) + T(f_2, r) + O(1).$$

Let $a_1, \ldots, a_5$ be five values such that $G_1(a_j) = G_2(a_j)$ for each $j = 1, \ldots, 5$. Without loss of generality, we may assume none of the a_j are infinite. Define

$$N^{(1)}(r) = \sum_{j=1}^{5} N^{(1)}(f_1, a_j, r) = \sum_{j=1}^{5} N^{(1)}(f_2, a_j, r).$$

By construction, for $r \geq 1$,

$$N^{(1)}(r) \leq N^{(1)}(g, \infty, r) \leq N(g, \infty, r) \leq T(g, r).$$

On the other hand, for $\ell = 1, 2$, and $r \geq 1$, we have

$$N^{(1)}(r) \geq \sum_{j=1}^{5} N(f_\ell, a_j, r) - N_{\text{ram}}(f, r) = 3T(f_\ell, r) - S(f_\ell, \{a_j\}_{j=1}^{5}, r).$$

Therefore,

$$\begin{aligned}
3[T(f_1,r)+T(f_2,r)] &\le 2N^{(1)}(r) + \sum_{\ell=1}^{2} S(f_\ell,\{a_j\}_{j=1}^5,r) \\
&\le 2T(g,r) + \sum_{\ell=1}^{2} S(f_\ell,\{a_j\}_{j=1}^5,r) \\
&\le 2[T(f_1,r)+T(f_2,r)] + \sum_{\ell=1}^{2} S(f_\ell,\{a_j\}_{j=1}^5,r) + O(1).
\end{aligned}$$

As the f_ℓ are non-constant by assumption, this contradicts the weak Second Main Theorem (Theorem 5.2.1). □

We now describe a variation of this idea. Given a subset S of $\mathbf{P}^1$ and a meromorphic function f on $\mathbf{C}$, define

$$G_{\mathrm{IM}}(f,S) = \{z \in \mathbf{C} : f(z) = a \text{ for some } a \in S\}$$

and $G_{\mathrm{CM}}(f,S) =$

$$\{(z,m) \in \mathbf{C} \times \mathbf{Z}_{>0} : f(z) = a \text{ with multiplicity } m \text{ for some } a \in S\}.$$

Here "IM" and "CM" stand for ignoring and counting multiplicity. If a subset S of $\mathbf{P}^1$ is such that whenever $G_{\mathrm{IM}}(f_1,S) = G_{\mathrm{IM}}(f_2,S)$ for two non-constant meromorphic functions f_1 and f_2, then f_1 must be identically equal to f_2, then S is called a **unique range set** ignoring multiplicity. We similarly define the notion of a unique range set counting multiplicity. A very interesting and active topic of current research in Nevanlinna theory is to provide necessary and sufficient conditions for a subset of $\mathbf{P}^1$ to be a unique range set, and another problem is to find the minimal cardinality of a unique range set (counting or ignoring multiplicity). As things are not settled yet, and there is rapid progress in this area, we prefer not to state any results, but simply reference the recent work of E. Mues and M. Reinders [MuRe 1995], the work of G. Frank and M. Reinders [FrRe 1998], and the work of P. Li and C.C. Yang: [LiYa 1996] and [LiYa 1995].

5.7 Bounding Radii of Discs

We saw above in the discussion of Picard's Theorem that a meromorphic function omitting three values in the Riemann sphere must be constant. We now turn to a finite version of this question. That is to say suppose that a function meromorphic in a disc omits at least three values, then is there a bound on how big the radius of that disc can be? Of course for this question to make sense, one must normalize the derivative of the meromorphic function at the origin. Thus, we will now examine the following question:

Question 5.7.1 *Let $q \ge 3$, and let $a_1,\dots,a_q$ be q distinct points in $\mathbf{P}^1$. Let f be a function meromorphic on $\mathbf{D}(R)$, omitting the values $a_1,\dots,a_q$, and normalized so that $f^\sharp(0) = 1$. Can we find an explicit upper bound on R (independent of f)?*

We remark that any such bound must certainly depend on the omitted values a_j because as the omitted values cluster together (*i.e.* as the distance between the a_j decreases) one can obviously map larger and larger discs into the sphere omitting the a_j.

It turns out that finding an explicit bound on R in Question 5.7.1 is a direct consequence on our uniform Second Main Theorem (Theorem 2.8.5), although because of the generality with which we have stated the fundamental inequality in Theorem 2.8.5, we need to do some routine manipulations to get such a bound out of the inequality. We also need to make several arbitrary choices as we proceed. We have tried to make these choices in a relatively straightforward, yet efficient, manner; one can make different such choices to get better bounds.

We begin with a proposition that chooses specific functions for the functions ψ and ϕ in Theorem 2.8.5 and then separates terms containing the characteristic function $\mathring{T}$ from the other terms.

Proposition 5.7.2 *Let $a_1,\dots,a_q$ be $q \ge 3$ distinct points in $\mathbf{P}^1$. Let $b_3(q)$, $\tilde{b}_1$, and $\mathring{\mathcal{D}}(a_1,\dots,a_q)$ be as in Theorem 2.8.5. Let f be a meromorphic function on $\mathbf{D}(R)$ which omits the a_j. Assume that f is normalized so that $f^\sharp(0) = 1$. Then, for all r such that $(\log r)^2 \ge e^e$ and outside a set of Lebesgue measure at most 2,*

$$\begin{aligned}
(q-2)\mathring{T}(f,r) - \log \mathring{T}(f,r) - 2\log\log \mathring{T}(f,r) - \log\log\log \mathring{T}^2(f,r) \\
\le \log\log r + \mathring{\mathcal{D}}(a_1,\dots,a_q) + \frac{1}{2}\log b_3(q) + \log\log b_3(q) \\
+ \log\log\log b_3(q) + \log\log \tilde{b}_1 b_3(q) + 4\log 2 + 1.
\end{aligned}$$

Remark. The right hand side of the above inequality is independent of f, and the only term on the right that depends on r is the $\log\log r$ term.

Proof. Start with the inequality in Theorem 2.8.5, and note that the $N(f,a_j,r)$ terms are all zero since the a_j are all omitted values. The function f is unramified at the origin, and in any case $r \ge 1$, so the $N_{\mathrm{ram}}(f,r)$ term is positive and can be ignored. Choose

$$\phi(r) \equiv 1 \quad \text{and} \quad \psi(x) = (\log x)^2.$$

By freshman calculus, $k_0(\psi) = 1$, and so the Lebesgue measure (since $\phi \equiv 1$) of the exceptional set is bounded by 2. By the First Main Theorem, or more precisely by Proposition 2.8.1, since we have assumed $(\log r)^2 \ge e^e$, we know that

$$2\log\log \mathring{T}^2(f,r) \ge 2\log\log((\log r)^2) \ge 2.$$

Similarly, $\log \mathring{T}(f,r)$, $\log b_3$, $2\log\log b_3$ and $\log \tilde{b}_1 b_3$ are all ≥ 2, and so we can repeatedly use the inequality

$$\log(x+y) \leq \log x + \log y \quad x, y \geq 2$$

to simplify the inequality in Theorem 2.8.5. Collecting all the terms involving $T(f,r)$ on the left then gives the desired inequality. □

To simplify our estimate, we now make the left had side of the inequality in the previous proposition linear in $\mathring{T}$.

Proposition 5.7.3 *The left hand side of the inequality in Proposition 5.7.2 can be replaced by* $(q-5/2)\mathring{T}(f,r)$.

Proof. We assume $\mathring{T}^2(f,r) \geq e^e \geq 9$, and by elementary calculus one easily sees that for all $x \geq 9$,

$$\frac{1}{2}x \leq x - \log x - 2\log\log x - \log\log\log x^2. \quad \square$$

Next we again use that $\mathring{T}(f,r) \geq \log r$ to eliminate the $\mathring{T}(f,r)$ terms, and we will be left with a bound on r. Namely,

Theorem 5.7.4 *Let $a_1, \ldots, a_q$ be $q \geq 3$ distinct points in $\mathbf{P}^1$. Let $\tilde{b}_1$, $b_3(q)$ and $\mathring{\mathcal{D}}(a_1, \ldots, a_q)$ be defined as in Theorem 2.8.5. Let f be a meromorphic function in $\mathbf{D}(R)$ omitting the values a_j and normalized so that $f^\sharp(0) = 1$. Then, we have that*

$$\log R \leq (q - 13/5)^{-1}\Big[\mathring{\mathcal{D}}(a_1, \ldots, a_q) + \frac{1}{2}\log b_3 + \log\log b_3$$
$$+ \log\log\log b_3 + \log\log \tilde{b}_1 b_3 + 4\log 2 + 1\Big] + \log 2.$$

Proof. Since $\mathring{T}(f,r) \geq \log r$ by Proposition 2.8.1, we can replace the term $(q-5/2)\mathring{T}(f,r)$ with $(q-5/2)\log r$. Then we gather the terms depending on r and use the fact that

$$\frac{2}{5}\log r \leq \frac{1}{2}\log r - \log\log r, \quad (\log r)^2 \geq e^e,$$

which can be verified by elementary calculus, to conclude that

$$(q - 13/5)\log r \leq \mathring{\mathcal{D}}(a_1, \ldots, a_q) + \frac{1}{2}\log b_3 + \log\log b_3 + \log\log\log b_3$$
$$+ \log\log \tilde{b}_1 b_3 + 4\log 2 + 1.$$

Solving for $\log r$ and adding $\log 2$ to account for the possibility that the exceptional set could be the interval $[R-2, R)$ gives us the upper bound for R. □

We would like to point out that one can prove a theorem like Theorem 5.7.4 without actually using the Second Main Theorem. One can construct a negatively curved singular area form (in a similar manner as in the curvature proof of the Second Main Theorem) and then apply an Ahlfors-Schwarz type lemma. We also point out that one cannot prove a theorem like Theorem 5.7.4 by using the Second Main Theorem as we have if one only knows that the error term in the Second Main Theorem is of order $O(\log T(f,r))$, because the error term (and the size of the exceptional set for that matter) could *a priori* depend on the function f in some complicated way. To get any kind of radius bound, one must have some sort of uniform estimate on the error term as we provided in Theorem 2.8.5, and if one wants an explicit bound as we have given here, then the constants in the error term must be estimated explicitly as well.

We conclude this topic with a discussion of the sharpness (or lack thereof) of the bound in Theorem 5.7.4. For special cases, Theorem 5.7.4 is very far from sharp. We leave it as an exercise to the reader to check that if $q = 3$ and if $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, then the bound given to us by Theorem 5.7.4 is on the order of $R \leq e^{30}$, which, although explicit, is much much larger than the sharp bound. M. Bonk and W. Cherry [BoCh 1996] showed that the sharp bound in this case is

$$R \leq \frac{\Gamma(1/4)^4}{4\pi^2} < 4.4 \ll e^{30}.$$

On the other hand, thinking of q as fixed, the only part of the bound in Theorem 5.7.4 that depends on the points $a_1, \ldots, a_q$ is $\mathring{\mathcal{D}}(a_1, \ldots, a_q)$. If we set

$$s = \min_{i \neq j} \|a_i, a_j\|,$$

then we easily see that as $s \to 0$,

$$e^{\mathring{\mathcal{D}}(a_1, \ldots, a_q)} = O\left(\left(\frac{1}{s}\right)^{q-1}\right).$$

Thus the bound in Theorem 5.7.4 is of the type

$$R \leq O\left(\left(\frac{1}{s}\right)^{\frac{q-1}{q-13/5}}\right)$$

as $s \to 0$, where the constants implicit in the O notation depend on q. Actually, one easily sees that the same techniques we used to derive Theorem 5.7.4 from Theorem 5.7.2, can be used to show that given any $\varepsilon > 0$,

$$R \leq O\left(\left(\frac{1}{s}\right)^{\frac{q-1}{q-2-\varepsilon}}\right),$$

where the implicit constants now depend on both q and ε. In any case, the point we wish to make here is that in either case, the exponent of $1/s$ tends to 1 as $q \to \infty$.

To see the significance of this, specialize to the case that

$$a_j = Me^{2\pi j\sqrt{-1}/q}, \quad j = 1, \ldots, q,$$

and M large. Using stereographic projection to compute the minimal chordal distance between the points a_j, one easily sees that

$$\frac{1}{s} = O(M)$$

as $M \to \infty$, where again the implicit constants depend on q. Thus, when the a_j are chosen in this manner,

$$R \leq O\left(M^{\frac{q-1}{q-13/5}}\right).$$

On the other hand, the map $f(z) = z$ satisfies $f^\sharp(0) = 1$ and maps the disc of radius M into $\mathbf{C} \setminus \{a_1, \ldots, a_q\}$, and so we see that Theorem 5.7.4 is in some sense sharp as both $s \to 0$ and $q \to \infty$ at the same time. This leaves us with the following question:

Question 5.7.5 *Given a fixed $q \geq 3$, what is the smallest possible exponent $p(q)$ such that if $a_1, \ldots, a_q$ are distinct points with $\|a_i, a_j\| \geq s$ for all $i \neq j$, and if f is a meromorphic function on $\mathbf{D}(R)$ omitting the values a_j and normalized so that $f^\sharp(0) = 1$, then*

$$R \leq O\left(\left(\frac{1}{s}\right)^{p(q)}\right),$$

where the constants implicit in the big O notation may depend on q, but not on f or the a_j.

Remark. The discussion above shows that

$$1 \leq p(q) \leq \frac{q-1}{q-2} + \varepsilon,$$

for any positive ε. Note that this does not imply $p(q) \leq (q-1)/(q-2)$, since the constants implicit in the big O notation could *a priori* blow up as $\varepsilon \to 0$.

5.8 Theorems of Landau and Schottky Type

Theorem 5.7.4 gives us a bound on the radius of a map given a normalization of its derivative at the origin. Obviously, one can turn this around and normalize the radius, and then get a bound on the derivative at the origin. When such a theorem is stated this way, it is known as a **theorem of Landau type** after E. Landau's Theorem [Land 1904]. Composing with a disc automorphism allows one to bound the derivative at any point, and it is this result we now state as a corollary of Theorem 5.7.4.

Corollary 5.8.1 (Landau's Theorem) *Let $a_1, \ldots, a_q$ be $q \geq 3$ distinct values in $\mathbf{P}^1$. Let $\tilde{b}_1$, $b_3(q)$ and $\mathring{\mathcal{D}}(a_1, \ldots, a_q)$ be as in Theorem 2.8.5. Let*

$$C = \exp\left\{(q - 13/5)^{-1}\left[\mathring{\mathcal{D}}(a_1, \ldots, a_q) + \frac{1}{2}\log b_3 + \log\log b_3 \right.\right.$$
$$\left.\left. + \log\log\log b_3 + \log\log \tilde{b}_1 b_3 + 4\log 2 + 1\right] + \log 2\right\}.$$

Let f be a meromorphic function on $\mathbf{D}(R)$, $R < \infty$ omitting the a_j. Then, for all z in $\mathbf{D}(R)$,

$$f^\sharp(z) \leq \frac{C}{R\left(1 - \left|\frac{z}{R}\right|^2\right)}.$$

Proof. Assume the conclusion of the theorem is false for some z_0 in $\mathbf{D}(R)$. Since we may replace f by $1/f$ and the a_j by $1/a_j$ without changing the statement of the corollary, we may assume without loss of generality that z_0 is not a pole of f. Let

$$M = Rf^\sharp(z_0)\left(1 - \left|\frac{z_0}{R}\right|^2\right) > C.$$

Let h be the automorphism of $\mathbf{D}(R)$ given by

$$h(z) = \frac{R^2(z - z_0)}{R^2 - \bar{z}_0 z},$$

and let

$$g(z) = f \circ h\left(\frac{R}{M}z\right),$$

which is holomorphic in the disc of radius M. We see that $g(0) = f(z_0)$ and that

$$|g'(0)| = \frac{R}{M}\left(1 - \left|\frac{z_0}{R}\right|^2\right)|f'(z_0)| = (1 + |f(z_0|^2)$$

by construction. Hence, $g^\sharp(0) = 1$. Theorem 5.7.4 then tells us that

$$\log M \le \log C. \quad \square$$

We remark that this theorem gives an effective upper bound on $|f'(0)|$ in terms of $|f(0)|$ and the minimal distance between the a_j. We point out that the version of Landau's Theorem in [Land 1904] follows easily from the Schwarz Lemma and the fact that $\mathbf{P}^1$ minus the omitted points is covered by the unit disc. This observation, first made by Carathéodory [Cara 1905], bounds $|f'(0)|$ in terms of $|f(0)|$ and the universal covering map, which of course is not the same form of the theorem we have given here. The bound we have given here is of the form

$$|f'(0)| \le \text{Const.} \times (1 + |f(0)|^2).$$

On the other hand, if the omitted values are $0, 1$ and ∞, one knows (see Hempel [Hemp 1979] and Jenkins [Jenk 1981]) that

$$|f'(0)| \le 2|f(0)| \left(\Big|\log |f(0)|\Big| + \frac{\Gamma(1/4)^4}{4\pi^2} \right),$$

and this is clearly a better bound, and in fact best possible, as $|f(0)|$ goes to zero or infinity. Although, see [BoCh 1996] for a further discussion of the quadratic bound, including the computation of the best possible constant.

Integrating Corollary 5.8.1 results in an effective version of Schottky's Theorem [Scho 1904].

Corollary 5.8.2 *Let $a_1, \ldots, a_q$ be $q \ge 3$ distinct values in $\mathbf{P}^1$, and let C be as in Corollary 5.8.1. Let f be a meromorphic function on $\mathbf{D}(R)$, $R < \infty$ omitting the a_j. Then, for all $r < R$,*

$$\mathring{T}(f, r) \le \frac{C^2}{2} \log \frac{R^2}{R^2 - r^2}.$$

Proof. Using Theorem 1.11.3 expressing $\mathring{T}$ as an integral of $(f^\sharp)^2$ and the bound on $f^\sharp$ given in Corollary 5.8.1, we get

$$\begin{aligned} T(f, r) &= \int_0^r \frac{dt}{t} \int_{\mathbf{D}(t)} (f^\sharp)^2 \Phi \\ &\le \int_0^r \frac{dt}{t} \int_0^{2\pi} \int_0^t \frac{C^2}{R^2(1 - s^2/R^2)^2} 2s ds \frac{d\theta}{2\pi} \\ &= C^2 \int_0^r \left[\frac{1}{(1 - t^2/R^2)} - 1 \right] \frac{dt}{t} \\ &= \frac{C^2}{2} \log \frac{R^2}{R^2 - r^2}. \quad \square \end{aligned}$$

If the function f is holomorphic, rather than meromorphic, we can use Proposition 1.7.1 to restate this last corollary in terms of the maximum modulus of the function, or in other words give an upper bound on $|f(z)|$ in terms of $|z|$. Such a theorem is called a **theorem of Schottky type**.

Corollary 5.8.3 (Schottky's Theorem) *Let $a_1, \ldots, a_q$ be $q \ge 3$ distinct values in $\mathbf{P}^1$, and let C be as in Corollary 5.8.1. Let f be a holomorphic function on $\mathbf{D}(R)$, $R < \infty$ omitting the a_j. Then, for all $r < R$,*

$$\log |f(z)| < \frac{4R}{R - |z|} \left[\frac{C^2}{2} \log \frac{4R^2}{R^2 - |z|^2} - \log \|f(0), \infty\| \right].$$

Proof. Let z be in $\mathbf{D}(R)$ and let $r = |z|$. From Proposition 1.7.1 we have for any ρ between r and R,

$$\log |f(z)| \le \frac{\rho + r}{\rho - r} T(f, \rho).$$

By taking $\rho = (R + r)/2$, we get

$$\log |f(z)| < \frac{4R}{R - r} T(f, (R + r)/2).$$

Using equation (1.2.4) and Proposition 1.2.2 to relate T and $\mathring{T}$, we have

$$\log |f(z)| < \frac{4R}{R - r} \left[\mathring{T}(f, (R + r)/2) - \log \|f(0), \infty\| \right].$$

Finally, we use the estimate in Corollary 5.8.2 and the fact that

$$\frac{1}{R^2 - \left(\dfrac{R + r}{2}\right)^2} < \frac{4}{R^2 - r^2}$$

to conclude

$$\log |f(z)| < \frac{4R}{R - r} \left[\frac{C^2}{2} \log \frac{4R^2}{R^2 - r^2} - \log \|f(0), \infty\| \right]. \quad \square$$

We again point out that the version of Schottky's Theorem we have given here is effective, but we again stress that for typical cases, such as the complement of $0, 1, \infty$, the constant we give is very far from best possible.

5.9 Slowly Moving Targets

Borrowing the language of number theory (see Chapter 6), the Second Main Theorem gives us information about how well a meromorphic function f approximates a finite number of points on the Riemann sphere. (If the function has large deficiency for the value a, then it stays close to the value a often and so "well-approximates" the value a. The Second Main Theorem says that a non-constant function cannot well-approximate too many values.) One could also ask, "how well does f approximate other functions?" To be more precise, we say that a meromorphic function $a(z)$ on $\mathbf{D}(R)$ is **slowly moving** with respect to a meromorphic function f (also defined on $\mathbf{D}(R)$) if

$$\lim_{r \to R} \frac{T(a,r)}{T(f,r)} = 0.$$

One also calls a slowly moving function a **small function**. One can then define the various Nevanlinna functions in this context of **moving targets**; for example,

$$m(f,a,r) = m(f-a,0,r), \quad \text{and} \quad N^{(1)}(f,a,r) = N^{(1)}(f-a,0,r).$$

Nevanlinna showed that the Second Main Theorem for fixed targets implies the following partial second main theorem for moving targets.

Theorem 5.9.1 *Let f, a_1, a_2 and a_3 be meromorphic functions on $\mathbf{C}$. Assume that the a_j are slowly moving with respect to f, and that no two of the a_j are identically equal. Then, for all r sufficiently large and outside of a set of finite Lebesgue measure,*

$$T(f,r) - \sum_{j=1}^{3} N^{(1)}(f,a_j,r) \le o(T(f,r)).$$

Actually, Nevanlinna had an extra $O(\log r)$ term on the right because he had this term in his error term for the Second Main Theorem. Since we do not have this term in our error terms, we do not get it here either.

Because the main emphasis of this book is the precise structure of the error term in Nevanlinna theory, and because the fine structure of the error term in the case of moving targets has yet to be seriously studied, we have decided not to include any discussion of the modern treatment of the Second Main Theorem with moving targets. We content ourselves here with giving a proof for the theorem of Nevanlinna stated above, which we will need for our discussion of iteration later in this section, and we simply direct the readers attention to the work of Osgood, Steinmetz, Ru, and Stoll: [Osg 1985], [Stei 1986], [RuSt 1991a] [RuSt 1991b], and [Ru 1997] for more detailed treatments of moving targets. However, we remark that the three moving target theorem of Nevanlinna that we shall prove presently includes a "ramification term" in that we have stated the theorem using the truncated counting functions. This sort of ramification term seems elusive in the more general slowly moving

target theorems cited above, and finding a proof for a moving target second main theorem that works with truncated counting functions is an important open problem.

Proof of Theorem 5.9.1. Let

$$g = \frac{(f-a_1)(a_2-a_3)}{(f-a_3)(a_2-a_1)}.$$

If we let

$$S(r) = T(g,r) - N^{(1)}(g,\infty,r) - N^{(1)}(g,0,r) - N^{(1)}(g,1,r),$$

then we know from the Second Main Theorem for fixed targets that

$$S(r) \le O(\log T(g,r))$$

for all large r outside a set of finite Lebesgue measure. Using Proposition 1.5.1, the First Main Theorem (Theorem 1.3.1), and the fact that the a_j are slowly moving with respect to f, we see

$$\begin{aligned} T(f,r) &= T(f,\infty,r) \le T(f-a_3,r) + T(a_3,r) + O(1) \\ &\le T(f-a_3,0,r) + o(T(f,r)) \\ &\le T((a_3-a_1)/(f-a_3),\infty,r) + o(T(f,r)) \\ &\le T\left(1 + \frac{a_3-a_1}{f-a_3}, r\right) + o(T(f,r)) \\ &= T((f-a_1)/(f-a_3),r) + o(T(f,r)). \end{aligned}$$

Similarly,

$$T((a_2-a_1)/(a_2-a_3),r) = o(T(f,r)),$$

and so $T(f,r) \le T(g,r) + o(T(f,r))$. Therefore,

$$T(f,r) - \sum_{j=1}^{3} N^{(1)}(f,a_j,r) \le S(r) + o(T(f,r)) \le O(\log T(g,r)) + o(T(f,r))$$

for all r sufficiently large and outside a set of finite measure. But, again by Proposition 1.5.1,

$$T(g,r) \le T(f,r) + o(T(f,r)),$$

and so the theorem is proven. □

5.10 Fixed Points and Iteration

The study of the dynamics of rational functions is a rich and diverse branch of mathematics. One can also study the dynamics of transcendental holomorphic functions; namely if f is holomorphic, then consider

$$f^n(z) = f \circ \ldots \circ f(z),$$

where f is composed with itself n times. What happens as n becomes large? How is the behavior of f^n related to properties of f? These are some of the sorts of questions one studies in complex dynamics. We will not discuss dynamics in any detail, but we simply state that one theme that occurs frequently in dynamics is the study of fixed points. A point z_0 is called a **fixed point** for a holomorphic map f if $f(z_0) = z_0$. Generalizing this idea, we call a point z_0 a **fixed point of order** n, if $f^n(z_0) = z_0$. If z_0 is a fixed point of order n and not a fixed point of any lower order, then z_0 is called a fixed point of **exact order** n. We remark, that if $f^n(z_0) = z_0$, then z_0 is a zero of the function $f^n(z) - z$, and thus that Nevanlinna theory should be a useful tool in the study of fixed points, or complex dynamics more generally, really comes as no surprise.

We give here just one example of how Nevanlinna theory can be used in the study of dynamics of transcendental holomorphic functions. Our discussion here is taken essentially from Hayman [Hay 1964]. The result we discuss here depends on a Schottky type theorem, and the only real difference between our treatment and that of Hayman, is that we use Corollary 5.8.3, which was a direct consequence of the Second Main Theorem, whereas Hayman uses a Schottky type theorem derived by other means.

The example we have decided to give in this section is the following theorem of Baker [Bakr 1960].

Theorem 5.10.1 (Baker) *If f is a transcendental (i.e. non-polynomial) entire function, then except for at most one positive integer n, the function f possesses infinitely many fixed points of exact order n.*

We postpone the proof of Theorem 5.10.1 for now so that we may state and prove some results we shall need for the proof of Theorem 5.10.1; these results are also of idependent interest.

First we give a theorem of Bohr [Bohr 1923].

Theorem 5.10.2 *Let f be a holomorphic function in the unit disc such that $f(0) = 0$ and such that $M(f, 1/2) \geq 1$. Let*

$$r_1 = \left[1 + 48\left(\frac{16}{3}\right)^{4C^2}\right]^{-1},$$

where C is as in Corollary 5.8.1. Then, there exists some $r \geq r_0$ such that f takes on every value w on the circle $|w| = r$ at least once inside the unit disc $|z| < 1$.

Remark. Hayman [Hay 1951] has shown, by different methods, that Theorem 5.10.2 remains true with $r_1 = 1/8$, which is much better than what we have done here.

Proof. Let $r_2 > 0$ be such that for every $r \geq r_2$, there exists a point w_r with $|w_r| = r$, and such that w_r is omitted by f. We need to show that $r_2 > r_1$. For $j = 1, 2$, choose points w_j so that $|w_j| = jr_2$ and such that $f(z) \neq w_j$ for all $|z| < 1$. Then, let

$$g(z) = \frac{f(z) - w_1}{w_2 - w_1},$$

and note that g is analytic on the unit disc and omits the values 0 and 1. Note also that

$$|g(0)| = \left|\frac{-w_1}{w_2 - w_1}\right| \leq 1.$$

Then, Schottky's Theorem (Corollary 5.8.3) tells us that

$$\log|g(z)| < \tilde{C} = 4\left(C^2 \log\frac{16}{3} + \log 2\right) \quad \text{for } |z| \leq \frac{1}{2}.$$

Thus,

$$|f(z)| < |w_1| + |w_2 - w_1|\,|g(z)| \leq r_2(1 + 3e^{\tilde{C}})$$

On the other hand, we've assumed the maximum modulus on $|z| = 1/2$ to be at least one, and thus

$$r_2 > \frac{1}{1 + 3e^{\tilde{C}}} = r_1. \quad \square$$

Next we will state and prove a lemma which tells us that the characteristic function of the composite of two entire transcendental functions grows faster than the characteristic function of the outside function.

Lemma 5.10.3 *Let g and h be transcendental (i.e. non-polynomial) entire functions, and let $f(z) = g \circ h(z)$. Then,*

$$\lim_{r\to\infty} \frac{T(g, r)}{T(f, r)} = 0.$$

Proof. Let r be a positive number and let

$$\tilde{h}_r(z) = \frac{h(rz/2) - h(0)}{M(h - h(0), r/4)}.$$

Note that $\tilde{h}_r(0) = 0$ and $M(\tilde{h}_r, 1/2) = 1$. Let r_1 be as in Theorem 5.10.2 and apply Theorem 5.10.2 to $\tilde{h}_r$ to conclude that there exists some $\tilde{r} \geq r_1$ such that $\tilde{h}_r$

assumes all values $|w| = \tilde{r}$ at least once in the unit disc. In other words, h assumes in $|z| \le r/2$, all values on some circle

$$\{w : |w - h(0)| = R_1\},$$

with

$$R_1 \ge r_1 M(h - h(0), r/4) \ge r_1(M(h, r/4) - |h(0)|).$$

Choose θ_0 so that

$$g(h(0) + R_1 e^{i\theta_0}) = \max_{|w-h(0)|=R_1} |g(w)|,$$

and choose z_0 so that $|z_0| \le r/2$, and $f(z_0) = h(0) + R_1 e^{i\theta_0}$. Then, by the maximum modulus principle,

$$\begin{aligned} |f(z_0)| &= |g(h(z_0))| = |g(h(0) + R_1 e^{i\theta_0})| \\ &= \max_{|w-h(0)|=R_1} |g(w)| \\ &\ge \max_{|w|=R_1-|h(0)|} |g(w)| = M(g, R_1 - |h(0)|). \end{aligned}$$

In other words, $M(f, r/2) \ge M(g, R_2)$, where $R_2 = R_1 - |h(0)|$. Thus, by using Proposition 1.7.1 to relate the Nevanlinna characteristic function to the maximum modulus, we obtain

$$T(f,r) \ge \frac{1}{3}\log M(f, r/2) \ge \frac{1}{3}\log M(g, R_2) \ge \frac{1}{3}T(g, R_2).$$

Note that $R_2 \ge r_1 M(h, r/4) - (r_1 + 1)|h(0)|$. Because h is transcendental, for any fixed positive integer k and for all r sufficiently large, we have

$$r_1 M(f, r/4) \ge r^{k+1} + (r_1 + 1)|h(0)|,$$

and so

$$T(f,r) \ge \frac{1}{3}T(g, r^{k+1}).$$

By Corollary 1.13.2, $T(g,r)$ is a convex function of $\log r$ and thus

$$\frac{T(g,r) - T(g,r_2)}{\log r - \log r_1}$$

is non-decreasing as $r \to \infty$ and r_2 remains fixed. Thus,

$$T(g, r^{k+1}) \ge (k+1)T(g,r) + O(1)$$

for r sufficiently large, where the $O(1)$ term does not depend on r or k. Thus, for r sufficiently large (depending on k),

$$T(f,r) \ge \frac{k+1}{3}T(g,r) + O(1),$$

and hence the theorem is proved. □

Proof of Theorem 5.10.1. Suppose that f has only a finite number of fixed points of exact order n, and call them $\zeta_1, \ldots, \zeta_q$. It suffices to prove that f has infinitely many fixed points of exact order k for all $k > n$. So, let $k > n$, and let z_0 be a solution to the equation $f^k(z) = f^{k-n}(z)$. Then

$$f^n(f^{k-n}(z_0)) = f^{k-n}(z_0),$$

and so $\zeta = f^{k-n}(z_0)$ is a fixed point of order n. Thus, either ζ is one of the ζ_j, or else ζ is a fixed point of exact order strictly less than n.

If ζ is not among the ζ_j, then note that

$$f^{k-n+m}(z_0) = f^{k-n}(z_0),$$

where $m \le n - 1$ is the exact order of the fixed point ζ. Thus, any solution to $f^k(z) = f^{k-n}(z)$ is either a solution to $f^{k-n}(z) = \zeta_j$ for some j, or a solution to $f^{k-n+m}(z) = f^{k-n}(z)$ for some $1 \le m \le n-1$. Thus,

$$\begin{aligned} &N^{(1)}(f^k - f^{k-n}, 0, r) \\ &\quad \le \sum_{m=1}^{n-1} N^{(1)}(f^{k-n+m} - f^{k-n}, 0, r) + \sum_{j=1}^{q} N^{(1)}(f^{k-n}, \zeta_j, r) \\ &\quad \le O\left(\sum_{\ell=1}^{k-1} T(f^\ell, r)\right), \end{aligned}$$

where the last inequality follows from the First Main Theorem. Clearly,

$$f^k = f^\ell \circ f^{k-\ell}, \quad 1 \le \ell \le k-1,$$

and f^ℓ and $f^{k-\ell}$ are transcendental. Hence we know that $T(f^\ell, r) = o(T(f^k, r))$ by Lemma 5.10.3. Thus the functions

$$a_1(z) = z, \quad a_2(z) = f^{k-n}(z), \quad \text{and} \quad a_3(z) = \infty$$

are all slowly moving with respect to f^k, and therefore by Theorem 5.9.1, we have a sequence of radii $r_i \to \infty$ such that

$$T(f^k, r_i) \le \sum_{j=1}^{3} N^{(1)}(f^k, a_j, r_i) + o(T(f^k, r_i)).$$

However, f^k has no poles since it is entire, and we just saw that

$$N^{(1)}(f^k, f^{k-n}, r) = o(T(f^k, r)).$$

Thus, $T(f^k, r_i) \leq N^{(1)}(f^k, a_1, r_i) + o(T(f^k, r_i))$. Now, again by the First Main Theorem and Lemma 5.10.3, the number of fixed points of order strictly less than k is $o(T(f^k, r))$, and so f has infinitely many fixed points of exact order k for all $k > n$. □

Remark. In the proof, we used Lemma 5.10.3 simply to compare $T(f^k, r)$ with $T(f^\ell, r)$. We chose to include Lemma 5.10.3 even though its proof, making use of Theorem 5.10.2, is somewhat complicated because it is a nice application of Schottky's Theorem (Corollary 5.8.3). However, if Theorem 5.10.1 were our only goal, it would have sufficed to use the following lemma, which, given the Second Main Theorem, is much easier to prove.

Lemma 5.10.4 *Let g and h be transcendental (i.e. non-polynomial) entire functions, and let $f(z) = g \circ h(z)$. Then, there exists an exceptional set E with finite Lebesgue measure such that*

$$\lim_{\substack{r\to\infty \\ r\notin E}} \frac{T(h,r)}{T(f,r)} = 0.$$

Proof. Because g is transcendental, there exists a point $b \in \mathbf{C}$ such that $g(z) = b$ has infinitely many solutions, $a_1, a_2, \ldots$. To see this, suppose no such b existed. Then, we conclude from Theorem 5.1.2 that every value b is deficient for g. This contradicts Theorem 5.2.3. Then, from the First Main Theorem, for each q,

$$T(f,r) + O(1) \geq N(f,b,r) \geq \sum_{j=1}^{q} N(h,a_j,r).$$

For each q we can use the Second Main Theorem to find an exceptional set E_q with Lebesgue measure $\leq 2^{-q}$ such that

$$\sum_{j=1}^{q} N(h,a_j,r) \geq (q-2)T(h,r) - o(T(h,r))$$

for all $r \geq r_0$ outside E_q. Thus,

$$\lim_{\substack{r\to\infty \\ r\notin E_q}} \frac{T(h,r)}{T(f,r)} \leq \frac{1}{q-2}.$$

The proof is completed by taking E to the union of all the E_q. □

One can remove the exceptional set from Lemma 5.10.4. See, for instance, [Clun 1970].

We conclude by remarking without proof that W. Bergweiler [Berg 1991] has improved Baker's Theorem.

Theorem 5.10.5 (Bergweiler) *If f is a transcendental entire function and n is an integer ≥ 2, then f has infinitely many fixed points of exact order n.*

For a general introduction to they dynamics of entire and meromorphic functions, see [Berg 1993], [ChuYa1990], [Gro 1972], or [HuaYa 1998].

6 A Further Digression into Number Theory: Theorems of Roth and Khinchin

This chapter is our second digression into number theory. We saw in §1.8 that Jensen's Formula (or the First Main Theorem) can be viewed as an analogue of the Artin-Whaples product formula in number theory. In the present chapter, we discuss a celebrated number theory theorem known as Roth's Theorem, and we explain how this is analogous to a weak form of the Second Main Theorem. In fact, it was the formal similarity between Nevanlinna's Second Main Theorem and Roth's Theorem that led C. F. Osgood (see [Osg 1981] and [Osg 1985]) to the discovery of an analogy between Nevanlinna theory and Diophantine approximations. As we said previously, such an analogy was later, but independently, explored in greater depth by P. Vojta. In this chapter we will discuss, without proof, some of the key points in Vojta's monograph [Vojt 1987], and we include Vojta's so called "dictionary" relating Nevanlinna theory and number theory. This section is intended for the analytically inclined and is only intended to provide the most basic insight into the beautiful analogy between Nevanlinna theory and Diophantine approximation theory. By omitting proofs, we have tried to make this section less demanding on the reader than [Vojt 1987]. However, a true appreciation for Vojta's analogy cannot be obtained without also studying the proofs of the number theoretic analogues in their full generality. Any reader that is seriously interested in the connection between Nevanlinna theory and Diophantine approximation is highly encouraged to carefully read Vojta's monograph [Vojt 1987].

We will also discuss in this section some of the ways that the analogy we explore here has affected the study of Nevanlinna theory, and in particular, we will discuss how it led Lang [Lang 1988], and then others, to the more detailed study of the error term in Nevanlinna's Second Main Theorem, which is the main emphasis of this book.

6.1 Roth's Theorem and Vojta's Dictionary

We begin by stating Roth's theorem over the rational numbers, as in Roth's original paper [Roth 1955]. We recall that a real or complex number is called an **algebraic number** if it is the zero of a polynomial with rational coefficients.

Theorem 6.1.1 (Roth's Theorem) *Let α be an algebraic number and let ε be a real number > 0. Then, for all but finitely many pairs of relatively prime integers m and n ($n > 0$), the following inequality holds:*

$$\left|\alpha - \frac{m}{n}\right| \geq \frac{1}{n^{2+\varepsilon}}.$$

Roth's theorem is a quantitative statement, but qualitatively what it is saying is that an algebraic number cannot be too closely approximated by infinitely many rational numbers. The appearance of the 2 in the exponent in the inequality in Roth's theorem, and the appearance of the 2 in the Second Main Theorem (as in $(q-2)T(f,r)$, or as in the sum of the defects is less than 2) is what, in part, led Osgood to find an analogy between Nevanlinna theory and number theory. We point out here that the Euler characteristic of the Riemann sphere is also 2, and as one sees if one studies higher dimensional Nevanlinna theory and higher dimensional versions of Roth's Theorem, this is no coincidence.

Before explaining in more detail how Roth's Theorem connects to Nevanlinna theory, we first introduce some notation and state a more general version of Roth's theorem. This more general version of Roth's theorem demands some patience from the reader unfamiliar with number theory and its notation, but we feel this general version of Roth's theorem is essential to understanding the analogy. We will state the general Roth theorem over an arbitrary number field, but we encourage the reader who has never seen these ideas before to continually think of what happens in the concrete case of the rational number field $\mathbf{Q}$.

We will let F denote a number field, recalling that a number field is a finite extension of the rationals $\mathbf{Q}$. We let $[F:\mathbf{Q}]$ denote the degree of F over $\mathbf{Q}$, or in other words the dimension of F as a $\mathbf{Q}$-vector space. Now, recall that in §1.8, we discussed the Artin-Whaples product formula (equation (1.8.1)), which says, when written multiplicatively, that we can choose absolute values $|\ |_v$ in the various equivalence classes v of absolute values on F such that for every x in $F\setminus\{0\}$, we have $|x|_v \neq 1$ for at most finitely many v, and moreover

$$\prod_v |x|_v = 1.$$

We will now describe the absolute values $|\ |_v$ in more detail, recalling that these absolute values can be divided into Archimedean and non-Archimedean absolute values.

The Archimedean absolute values arise in the following way. We note that any number field has only a finite number of embeddings $\sigma: F \to \mathbf{C}$. For each such embedding σ, and for x in F, we define the absolute value

$$|x|_\sigma = |\sigma(x)|,$$

where the absolute value on the right is the standard absolute value on $\mathbf{C}$. We remark that if $\sigma(F)$ is not contained in the real numbers $\mathbf{R}$, then σ has a conjugate

embedding $\bar{\sigma}$ coming from complex conjugation, and that in this case $|x|_\sigma = |x|_{\bar{\sigma}}$ for all x. In such cases, we consider both $|\ |_\sigma$ and $|\ |_{\bar{\sigma}}$, even though they are really the same. Alternatively, one can consider conjugate pairs of embeddings to be one embedding, but counted with multiplicity two.

The non-Archimedean absolute values on F arise in much the same way as they do on $\mathbf{Q}$, although to get the product formula to work out, they have to be normalized appropriately. More precisely, recall that an algebraic number x is called an **algebraic integer** if x is a root of a *monic* polynomial with coefficients in $\mathbf{Z}$. Recall that a polynomial with integer coefficients is said to be **monic** if the leading coefficient is one. The elements of F which are also algebraic integers form a ring R, called the **ring of integers** in F. Many of the nice properties we take for granted about the ring $\mathbf{Z}$ need not apply to R. In particular, one may not be able to uniquely factor elements of R into primes, nor need R be a principal ideal domain. A key advance in number theory came about when one realized that rather than look at the factorization of elements of R, one should look at the factorization of *ideals* in R. Let x be an element of R, and let $\wp$ be a prime ideal in R. Although we may not be able to uniquely factor the element x into a product of prime or irreducible elements, the principal ideal in R generated by x does factor uniquely into a product of prime ideals. We denote by $\mathrm{ord}_\wp x$ the number of times the prime ideal $\wp$ appears in this ideal factorization. Every prime ideal $\wp$ lies above some everyday prime p in $\mathbf{Q}$, namely the characteristic of the finite field $R/\wp$. Let $\mathbf{Q}_p$ be the completion of $\mathbf{Q}$ with respect to the p-adic absolute value $|\ |_p$ on $\mathbf{Q}$. Note that this completion is formed in the standard way by looking at equivalence classes of Cauchy sequences, two Cauchy sequences being equivalent if their difference is a sequence converging to 0. Write $F = \mathbf{Q}(x_1, \ldots, x_q)$ so that the x_j generate F. Then, let $F_\wp = \mathbf{Q}_p(x_1, \ldots, x_q)$. Note that $F \hookrightarrow F_\wp$, although this embedding is not unique. We remark that some elements of F which are not in $\mathbf{Q}$ might already be in $\mathbf{Q}_p$, and so $[F_\wp:\mathbf{Q}_p]$ might be strictly less than $[F:\mathbf{Q}]$. For elements x in R, we now define

$$|x|_\wp = p^{-[F_\wp:\mathbf{Q}_p]\mathrm{ord}_\wp x}.$$

Note here the similarity with the absolute value $|\ |_p$ on $\mathbf{Q}$. The factors $[F_\wp:\mathbf{Q}_p]$ are necessary for the product formula to work out. The absolute value $|\ |_\wp$ and the order function $\mathrm{ord}_\wp$ extend to F in the natural way by writing any x in F as the quotient of two elements in R.

We remark that the factors $[F_\wp:\mathbf{Q}_p]$ used in the normalization of $|\ |_\wp$ can be thought of as the "multiplicities" with which we count an absolute value in that equivalence class. In this way, these factors correspond to the "multiplicities" $[\mathbf{R}:\mathbf{R}] = 1$ and $[\mathbf{C}:\mathbf{R}] = 2$ that arise in the Archimedean case depending on whether for a complex embedding σ, we have $\sigma(F)$ contained in $\mathbf{R}$ or not. If we insist on sticking with absolute values, we cannot include the Archimedean multiplicities in the normalization, as we have in the non-Archimedean case, because $|\ |_\sigma^2$ does not satisfy the triangle inequality if $|\ |_\sigma$ is an Archimedean absolute value.

The product formula (equation (1.8.1)) can then be written as

$$1 = \left(\prod_{\sigma: F\hookrightarrow \mathbf{C}} |x|_\sigma\right) \cdot \left(\prod_{\wp} |x|_\wp\right),$$

where if $x \neq 0$, then for all but finitely many prime ideals $\wp$, $|x|_\wp = 1$.

Let x be an element of F, and define the **relative height** $H(F,x)$, by

$$H(F,x) = \left(\prod_{\sigma: F\hookrightarrow \mathbf{C}} \max\{1, |x|_\sigma\}\right) \cdot \left(\prod_{\wp} \max\{1, |x|_\wp\}\right).$$

The height $H(F,x)$ measures the "size" or "complexity" of a number. To get an intuitive feel for the height, the reader should right now verify that if F is the rational number field $\mathbf{Q}$, then

$$H(\mathbf{Q},x) = \max\{|m|,|n|\},$$

where $x = m/n$, and m and n are relatively prime integers. Thus, the height of a number is large if either the numerator or the denominator is large. For the reader unfamiliar with number theory, it is useful to keep this special case in mind.

The height $H(F,x)$ is called "relative" because it is relative to the number field F. Note however that if E is a finite extension of F and if x is in F, then

$$H(E,x) = H(F,x)^{[E:F]}. \tag{6.1.2}$$

Also note that:

Given a positive number M, only a finite number of elements x in F have $H(F,x) \leq M$.

Given a in $\mathbf{P}^1$, let λ_a be any Weil function. For definiteness, one can take

$$\lambda_a(z) = \begin{cases} \log^+ \dfrac{1}{|z-a|} & a \neq \infty,\ z \neq a, \infty \\ \log^+ |z| & a = \infty,\ z \neq \infty \end{cases}$$

We are now ready to state the more general version of Roth's theorem.

Theorem 6.1.3 (General Roth Theorem) *Let $a_1, \ldots, a_q$ be (not necessarily distinct) elements of $F \cup \{\infty\}$, where F is a number field. Then, given $\varepsilon > 0$, there are only finitely many x in F that do not satisfy the inequality*

$$\sum_\sigma \max_j \lambda_{\sigma(a_j)}(\sigma(x)) \leq (2+\varepsilon)\log H(F,x),$$

where the sum is taken over all embeddings $\sigma: F \hookrightarrow \mathbf{C}$, and where by convention we let $\sigma(\infty) = \infty$.

Remarks. The inequality does not make sense for $x = a_j$, and so the a_j are among the finitely many exceptions.

We leave it as an exercise for the reader to prove that Theorem 6.1.3 actually implies Theorem 6.1.1. Note that in Theorem 6.1.1 we do not require α to be rational, but in Theorem 6.1.3 we require the a_j to be in F. By way of a hint, we remark that the one should consider for F the splitting field for the irreducible polynomial of which α is a root, and then take $\sigma^{-1}(\alpha)$ for the a_j as σ runs through the embeddings $\sigma: F \hookrightarrow \mathbf{C}$ (*i.e.* the other roots of the irreducible polynomial). We also point out that since to prove Theorem 6.1.1, one only needs to consider rational numbers close to α, and so one need only consider rational numbers $x = \pm m/n$ where m and n are relatively prime positive integers, and $H(\mathbf{Q},x) \leq Cn$ for some constant C.

If we introduce more notation, we will see how this is the analogue of Nevanlinna's Second Main Theorem. First, we modify our notion of height a little. Given a non-zero algebraic number x contained in a number field F, we define the **absolute logarithmic height** by

$$h(x) = \frac{1}{[F:\mathbf{Q}]}\log H(F,x).$$

We note that by equation (6.1.2) this height does not depend on the number field F, and hence the term absolute; we also thus omit F from the notation. This height $h(x)$ will be the Diophantine analogue to Nevanlinna's characteristic function $T(f,r)$.

Now, let's see what the analogues to the other Nevanlinna functions should be. To do this, let's think back to our discussion of the product formula in §1.8. In the notation of §1.8, for a in $\mathbf{C}$,

$$m(f,a,r) = \int_0^{2\pi} \log^+ \frac{1}{|f-a|_{\theta,r}} \frac{d\theta}{2\pi}.$$

Now, recall that we thought of the "absolute values" $|f|_{\theta,r}$ for θ between 0 and 2π as the analogues of the Archimedean absolute values on a number field, *i.e.* the absolute values $|\ |_\sigma$ coming from embeddings σ of F into $\mathbf{C}$. By this analogy then, given a number field F, and two distinct points a and x in F, define

$$m(F,a,x) = \frac{1}{[F:\mathbf{Q}]}\sum_{\sigma: F\hookrightarrow\mathbf{C}} \log^+ \frac{1}{|x-a|_\sigma} = \frac{1}{[F:\mathbf{Q}]}\sum_{\sigma: F\hookrightarrow\mathbf{C}} \lambda_{\sigma(a_j)}(\sigma(x)),$$

and similarly,

$$m(F,\infty,x) = \frac{1}{[F:\mathbf{Q}]}\sum_{\sigma: F\hookrightarrow\mathbf{C}} \log^+ |x|_\sigma = \frac{1}{[F:\mathbf{Q}]}\sum_{\sigma: F\hookrightarrow\mathbf{C}} \lambda_\infty(\sigma(x)),$$

where we have used λ to denote the Weil function formed with $\log^+$. Of course, one can define such "mean proximity" functions using any Weil function.

Now that we have all this additional notation, we can see the resemblance to the Second Main Theorem by stating Roth's Theorem as follows.

Theorem 6.1.4 *Let F be a number field, and let $a_1, \dots, a_q$ be q distinct elements of $F \cup \{\infty\}$. Then, given $\varepsilon > 0$, there are only finitely many x in F that do not satisfy the inequality*

$$\sum_{j=1}^{q} m(F, a_j, x) \leq (2+\varepsilon)h(x).$$

Note to see that Theorem 6.1.4 is equivalent to Theorem 6.1.3 one must use the fact that only finitely many elements of F have bounded height, and also Proposition 1.2.3 to go between the sum over j in Theorem 6.1.4 and the maximum over j in Theorem 6.1.3.

To further explore the analogy, we introduce even more notation. Recall that in Nevanlinna theory, we saw that when $f(0) \neq a$,

$$N(f,a,r) = \sum_{z \in \mathbf{D}(r)} (\mathrm{ord}_z^+(f-a)) \log \frac{r}{|z|}.$$

Now, let $\mathfrak{p}$ be a prime in F, and let $F_{\mathfrak{p}}$ be as above. As usual, we define

$$\mathrm{ord}_{\mathfrak{p}}^+ x = \max\{0, \mathrm{ord}_{\mathfrak{p}} x\}.$$

We will replace the $\log(r/|z|)$ in the definition of $N(f,a,r)$ with $[F_{\mathfrak{p}} : \mathbf{Q}_p] \log p$ to define the number theoretic analogue of the counting function. Namely, given two distinct elements x and a in a number field F, we define

$$N(F,a,x) = \frac{1}{[F:\mathbf{Q}]} \sum_{\mathfrak{p} \in F} (\mathrm{ord}_{\mathfrak{p}}^+(x-a))[F_{\mathfrak{p}} : \mathbf{Q}_p] \log p,$$

and similarly we define

$$N(F,\infty,x) = \frac{1}{[F:\mathbf{Q}]} \sum_{\mathfrak{p} \in F} (\mathrm{ord}_{\mathfrak{p}}^+ \frac{1}{x})[F_{\mathfrak{p}} : \mathbf{Q}_p] \log p,$$

taking $N(F,\infty,0) = 0$.

Let's now come back to the Artin-Whaples Product Formula. When put into the present notation, this says that for non-zero x,

$$m(F,0,x) + N(F,0,x) = h(x) = m(f,\infty,x) + N(f,\infty,x).$$

The reader should right now verify this assertion in the case that F is $\mathbf{Q}$. Moreover, one easily sees that

$$h(x-a) = h(x) + O(1),$$

thinking of x as variable and a as fixed. Thus,

$$h(x) = m(F,a,x) + N(F,a,x) + O(1),$$

where the $O(1)$ term depends on a, and x is considered the variable. In other words, we have the analogue of Nevanlinna's First Main Theorem, and so we can again restate Roth's Theorem as

Table 6.1: Vojta's Dictionary

Value Distribution	**Diophantine Approximation**
non-constant $f: \mathbf{C} \to \mathbf{P}^1$	infinite $\{x\}$ in a number field F
A radius r	An element x of F
A finite measure set E of radii	A finite subset of $\{x\}$
An angle θ	An embedding $\sigma: F \to \mathbf{C}$.
$\lvert f(re^{i\theta})\rvert$	$\lvert x\rvert_\sigma$
$(\mathrm{ord}_z f) \log \frac{r}{\lvert z\rvert}$	$(\mathrm{ord}_{\mathfrak{p}} x)[F_{\mathfrak{p}} : \mathbf{Q}_p] \log p$
Characteristic function: $T(f,r) = \int_0^{2\pi} \log^+ \lvert f(re^{i\theta})\rvert \frac{d\theta}{2\pi} + N(f,\infty,r)$	Logarithmic height: $h(x) = \frac{1}{[F:\mathbf{Q}]} \sum_{\sigma: F \hookrightarrow \mathbf{C}} \log^+ \lvert x\rvert_\sigma + N(F,\infty,r)$
Mean-proximity function: $m(f,a,r) = \int_0^{2\pi} \log^+ \left\lvert \frac{1}{f(re^{i\theta}) - a} \right\rvert \frac{d\theta}{2\pi}$	Mean-proximity function: $m(F,a,x) = \sum_{\sigma: F \hookrightarrow \mathbf{C}} \log^+ \left\lvert \frac{1}{x-a} \right\rvert_\sigma$
Counting function: $N(f,a,r) = \sum_{\lvert z\rvert < r} (\mathrm{ord}_z^+(f-a)) \log \frac{r}{\lvert z\rvert}$	Counting function: $N(F,a,x) = \frac{1}{[F:\mathbf{Q}]} \sum_{\mathfrak{p} \in F} (\mathrm{ord}_{\mathfrak{p}}^+(x-a))[F_{\mathfrak{p}} : \mathbf{Q}_p] \log p$
First Main Theorem: $N(f,a,r) + m(f,a,r) = T(f,r) + O(1)$	Height property $N(F,a,x) + m(F,a,x) = h(x) + O(1)$
Second Main Theorem: $(q-2)T(f,r) - \sum_{j=1}^{q} N^{(1)}(f,a_j,r) \leq \log(T(f,r)\psi(T(f,r))) + O(1)$	Roth type conjecture: $(q-2)h(x) - \sum_{j=1}^{q} N^{(1)}(F,a_j,x) \leq \log(h(x)\psi(h(x))) + O(1)$
Weaker Second Main Theorem: $(q-2)T(f,r) - \sum_{j=1}^{q} N(f,a_j,r) \leq \varepsilon T(f,r)$	Roth's Theorem: $(q-2)h(x) - \sum_{j=1}^{q} N(F,a_j,x) \leq \varepsilon h(x)$
Jensen's Formula: $\int_0^{2\pi} \log \lvert f(re^{i\theta})\rvert \frac{d\theta}{2\pi} = N(f,0,r) - N(f,\infty,r) + O(1)$	Artin-Whaples Product Formula: $\sum_{\sigma \in S_\infty} \log \lvert x\rvert_\sigma = N(F,0,x) - N(F,\infty,x)$

Theorem 6.1.5 *Let F be a number field. Let $a_1, \ldots, a_q$ be q distinct elements in $F \cup \{\infty\}$. Then, given $\varepsilon > 0$, there are only finitely many x in F that do not satisfy the inequality*

$$(q-2)h(x) - \sum_{j=1}^{q} N(F, a_j, x) \leq \varepsilon h(x).$$

In Vojta's dictionary, a non-constant meromorphic function in Nevanlinna theory is the analogue of an infinite set of numbers $\{x\}$ in a number field F. We emphasize that the meromorphic function in Nevanlinna theory does *not* correspond to an individual number. Rather, a theorem in Diophantine approximation which holds for all but finitely many numbers x corresponds to a theorem in Nevanlinna theory that holds for all *radii* r outside a set of finite measure. Thus, in some sense, the individual numbers x correspond to radii r.

As an illustration of how Roth's Theorem can be applied in ways similar to the Second Main Theorem, we look at the following analogue of Picard's Theorem, the proof of which we leave as an exercise to the reader.

Theorem 6.1.6 *Let F be a number field, let a_1, a_2, and a_3 be three distinct elements in $F \cup \{\infty\}$. Then, there are only finitely many elements x in F such that $1/(x - a_j)$ (or x itself if $a_j = \infty$) is an algebraic integer for all $j = 1, 2, 3$.*

Now that we have introduced this analogy and pointed out some similarities between the Second Main Theorem and Roth's Theorem, let's make some observations about the differences.

The most significant difference between Roth's Theorem and the Second Main Theorem is that none of the versions of Roth's Theorem we have stated here contains a term analogous to the ramification term $N_{ram}(f, r)$ in Nevanlinna's Second Main Theorem. Proving a version of Roth's Theorem with some sort of ramification term, for example truncated counting functions, would have dramatic consequences in number theory. In fact, one could convincingly argue that finding a proof for a Roth type theorem that includes a ramification term is the most important open problem in Diophantine approximation theory. For a more detailed discussion of conjectural number theory analogs to the ramification term and their importance, see Chapter 5 of Vojta's monograph [Vojt 1987].

6.2 The Khinchin Convergence Condition

Another difference between the versions of Roth's Theorem stated here and the Second Main Theorem is that the error term in the above Roth theorems is $\varepsilon h(x)$. This is not as strong as the error term we have given for the Nevanlinna Second Main Theorems. Given the error terms that have been obtained in Nevanlinna theory, one could ask whether the following stronger refinement of Roth's Theorem is true.

Conjecture 6.2.1 *Let F be a number field and let $a_1, \ldots, a_q$ be distinct elements of $F \cup \{\infty\}$. Let ψ be a Khinchin function. Then, for all but finitely many x in F,*

$$\sum_{j=1}^{q} m(F, a_j, x) - 2h(x) \leq \log(h(x)\psi(h(x))) + O(1).$$

In fact, that Roth's Theorem could be improved along the lines of Conjecture 6.2.1 is a conjecture of Lang dating back to the sixties, see [Lang 1965] and [Lang 1971]. Similar conjectures were also made by Bryuno [Bry 1964], and by Richtmyer, Devaney, and Metropolis [RDM 1962]. Lang's motivation for Conjecture 6.2.1 was the following theorem of Khinchin (see [Khin 1964]).

Theorem 6.2.2 (Khinchin's Theorem) *Let $\tilde{\psi}$ be a positive increasing function such that*

$$\sum_{n=1}^{\infty} \frac{1}{n\tilde{\psi}(n)} < \infty.$$

Then, all real numbers α outside a set of measure 0 (depending only on $\tilde{\psi}$) are such that for all but finitely many pairs (m, n) of relatively prime integers, $n > 0$, the inequality

$$-\log\left|\alpha - \frac{m}{n}\right| - 2\log n \leq \log\tilde{\psi}(n)$$

is valid, the finitely many exceptional pairs depending on $\tilde{\psi}$ and α.

Note the similarity with Roth's Theorem. To compare Theorem 6.2.2 with Conjecture 6.2.1, set

$$\tilde{\psi}(n) = (\log n)\psi(\log n),$$

and note that because ψ is a Khinchin function, $\tilde{\psi}$ satisfies the conditions of Theorem 6.2.2. In fact, this is why we call our functions ψ "Khinchin functions." Thus, Conjecture 6.2.1 says that algebraic numbers behave like almost all numbers. After Lang became aware of the analogy between the Second Main Theorem and Roth's Theorem, he conjectured that the best possible error term for the Second Main Theorem would have the same structure as his conjectured refinement of Roth's Theorem. This is what led people to compute the exact form of the error term, as in this book, and why Lang introduced the arbitrary Khinchin functions ψ into the theory.

Lang (see [Lang 1966a] and [Lang 1966b]) defined a real number α to be of type $\tilde{\psi}$ if all but finitely many pairs of relatively prime integers (m, n) with $n > 0$ satisfy the following inequality:

$$-\log\left|\alpha - \frac{m}{n}\right| - 2\log n \leq \log\tilde{\psi}(n).$$

Lang then posed the problem of computing the types of algebraic numbers, as well as "classical" numbers like e, π and so forth. Lang also transposed this problem to

Nevanlinna theory, namely to compute the precise form of the error term in the Second Main Theorem for the classical special functions. We will take up a discussion of this in §7.3.

Z. Ye [Ye 1996b] addressed the question of finding a Nevanlinna theory analog of Theorem 6.2.2.

7 More on the Error Term

In this final chapter, we further explore the error term in the Second Main Theorem and the Logarithmic Derivative Lemma. In §7.1, we construct examples showing that in the general case, the error terms we have given in the previous chapters are essentially the best possible. In §7.2, we explain how by putting rather modest growth restrictions on the functions under consideration and by enlarging the exceptional set, the lower order terms in the error term can be further improved. Finally, in §7.3, we compute the precise form of the error term for some of the more familiar special functions.

7.1 Sharpness of the Second Main Theorem and the Logarithmic Derivative Lemma

In this section we will discuss the sharpness of the error terms in the Second Main Theorem and in the Lemma on the Logarithmic Derivative. We do this by constructing specific holomorphic and meromorphic functions for which we can explicitly estimate the growth of these error terms. We show that the coefficient 1 in front of the $\log T(f,r)$ term in the error terms in the Second Main Theorem and Lemma on the Logarithmic Derivative cannot be reduced, and we show that some sort of term involving a Khinchin function ψ is necessary. More precisely,

Theorem 7.1.1 (Sharpness of SMT and LDL) *Let $r_0 < R \leq \infty$, and let $h(r)$ be a positive continuous function on $[r_0, R)$. Let $\xi(x)$ be any positive non-decreasing function defined on (e, ∞) such that*

$$\int_e^\infty \frac{dx}{\xi(x)} = \infty. \tag{7.1.2}$$

Then, there exists a closed set E in $[r_0, R)$ such that in the case that $R = \infty$,

$$\int_E dr < \infty,$$

and in the case that $R < \infty$,

$$\int_E \frac{dr}{(R-r)^2} < \infty;$$

and such that moreover,

(1) *There exists a holomorphic function f on $\mathbf{D}(R)$ such that for all $r \geq r_0$ and outside E,*

$$m(f'/f, \infty, r) \geq \log T(f,r) + \log \xi(T(f,r)) + h(r),$$

and

(2) *Given $a_1, \ldots, a_q$ $(q \geq 1)$ distinct points in $\mathbf{P}^1$, there exists a meromorphic function f such that for all $r \geq r_0$ and outside E,*

$$(q-2)T(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{ram}}(f,r) \geq \log T(f,r) + \log \xi(T(f,r)) + h(r).$$

Before proving the theorem, we make various remarks.

Because $\xi(x) = 1$ satisfies equation 7.1.2, we see that for any function $h(r)$, we can find a meromorphic function whose error term in the Logarithmic Derivative Lemma or Second Main Theorem is $\geq \log T(f,r) + h(r)$. Thus, the coefficient 1 in front of the $\log T(f,r)$ in the error terms for the Second Main Theorem and Logarithmic Derivative Lemma cannot be reduced.

Compare the role of the function ξ in Theorem 7.1.1 with the role of Khinchin functions ψ in the Lemma on the Logarithmic Derivative and the Second Main Theorem. Equation 7.1.2 says precisely that the functions ξ in Theorem 7.1.1 do not satisfy the Khinchin convergence condition. Thus, Theorem 7.1.1 shows that some sort of term involving Khinchin functions is necessary in the error terms of the Second Main Theorem and Logarithmic Derivative Lemma.

Notice that the role of the exceptional set E here in Theorem 7.1.1 is sort of dual to the role of the exceptional set in the Second Main Theorem and Logarithmic Derivative Lemmas.

The results in this section that we use to prove Theorem 7.1.1 are taken from the work of J. Miles [Miles 1992], Z. Ye [Ye 1991], and A. Hinkkanen [Hink 1992]. The type of construction use here has been used in various contexts in value distribution theory before, see for example [EdFu 1965]. Miles [Miles 1992] was the first to use this construction to demonstrate that the error term in the Lemma on the Logarithmic Derivative is sharp. For the case $R = \infty$, Miles obtained statement 1 in Theorem 7.1.1 in the special case that $h(r) = -\log r$, except that he only showed the inequality for a set of r outside a set E such that

$$\int_E \frac{dr}{r} < \infty.$$

His treatment of the case $R < \infty$ was for the case $h(r) = -\log(R-r)$, and outside a set E such that

$$\int_E \frac{dr}{(R-r)} < \infty.$$

At around the same time, Ye [Ye 1991] improved Miles's result in the case $R = \infty$ to the form presented here, and he observed that this same construction also shows the sharpness of the error term in the Second Main Theorem. A. Hinkkanen [Hink 1992] also constructed, in the same manner as we do here, examples showing sharpness of the Logarithmic Derivative Lemma and the Second Main Theorem, and Hinkkanen improved on Miles's treatment of the disc case (*i.e.* $R < \infty$), which was not addressed in [Ye 1991].

In [Ye 1991], Ye also showed that the coefficient $1/2$ in front of the $\log T(f,r)$ term in the error term in the Ramification Theorem (*i.e.* Theorem 2.4.3) is best possible. We have decided not to include the details of that computation here becuase it is more involved than the other examples.

We further postpone the proof of Theorem 7.1.1 until after we prove some preliminary results. We will use the given data, namely the functions ξ and h, to construct a holomorphic function g on $\mathbf{D}(R)$ and the exceptional set E. We will then prove various properties about this function g, and we will finally see that elementary modifications of the function g will give us the functions f needed in the theorem.

Given a sequence $r_j, j \geq j_0$, we will consider functions defined by infinite products of the following form:

$$\boxed{g(z) = \prod_{j=j_0}^{\infty} \left(1 + \left(\frac{z}{r_j}\right)^{2^j}\right).} \qquad (7.1.3)$$

Notice that the multiplicity of the zeros of g grow exponentially with j, and that therefore the function g has rather large growth.

Proposition 7.1.4 *Let r_j for $j \geq j_0$ be a nondecreasing sequence of real numbers in $[r_0, R)$ such that $r_j \to R$ as $j \to \infty$. Then the function g defined by equation* (7.1.3) *is holomorphic on $\mathbf{D}(R)$.*

Proof. Let $|z| \leq t < R$. Then, because $r_j \to R$, we have for all j sufficently large that

$$\left|\frac{t}{r_j}\right|^{2^j} \leq \frac{1}{2}.$$

We thus have that for all j sufficiently large,

$$\left|\log\left(1 + \left(\frac{z}{r_j}\right)^{2^j}\right)\right| \leq 2\left|\frac{z}{r_j}\right|^{2^j},$$

and thus for $|z| \leq t < R$, the sum

$$\sum_{j=j_0}^{\infty} \left|\log\left(1 + \left(\frac{z}{r_j}\right)^{2^j}\right)\right|$$

converges uniformly. □

We now turn to the question of how to choose the r_j. We begin by remarking that we can make the following simplifying assumptions. First, without loss of generality, we may assume that h is increasing and that $h(r) \to \infty$ as $r \to R$. This is because replacing h by a bigger function only makes the statement of the theorem stronger. Second, we can assume without loss of generality that ξ and h are both C^∞. Indeed, it is easy to see that one can smooth h in such a way that h only gets larger. It is only slightly harder to see that ξ can be smoothed in such a way that ξ only increases, but not so much that the divergence of the integral in equation (7.1.2) is affected.

Proposition 7.1.5 *Let $\xi(x)$ be as in the statement of Theorem 7.1.1, and assume the ξ is C^∞. Let $s_0 > 0$, let $H(s)$ be a positive C^∞ function on $[s_0, \infty)$ such that $H(s)$ is strictly increasing, and assume that $H(s) \to \infty$ as $s \to \infty$. Then, the function*

$$U(t) = \int_e^t \frac{dx}{x\xi(x)}$$

is a C^∞ strictly increasing function taking $[e, \infty)$ onto $[0, \infty)$. Moreover, if we then define

$$V(s) = U^{-1}\left(\int_{s_0}^s H(t)dt\right),$$

then V is a strictly increasing C^∞ function on $[s_0, \infty)$, and

$$V'(s) = H(s)V(s)\xi(V(s)).$$

Proof. Freshman calculus. □

Proposition 7.1.6 *Let r_0, R, ξ, and h be as in the statement of Theorem 7.1.1. Assume ξ and h are C^∞, that $h(r)$ is strictly increasing, and that $h(r) \to \infty$ as $r \to R$. Let*

$$s(r) = \begin{cases} r & \text{if } R = \infty \\ \dfrac{1}{R - r} & \text{if } R < \infty, \end{cases}$$

and let

$$r(s) = \begin{cases} s & \text{if } R = \infty \\ R - \dfrac{1}{s} & \text{if } R < \infty \end{cases}$$

be the inverse function of $s(r)$. Let $s_0 = s(r_0)$, and define $H(s)$ on $[s_0, \infty)$ by

$$H(s) = e^{h(r(s))+s}.$$

Let $V(s)$ be defined as in Proposition 7.1.5. Finally, define $v(r)$ by

$$v(r) = rH(s(r))V(s(r))\xi(V(s(r)))s'(r).$$

Then, there exists j_0 such that for each integer $j \geq j_0$, the equation

$$2^{j+1} = v(r_j) \tag{7.1.7}$$

has a unique solution r_j in $[r_0, R)$, and moreover the r_j are strictly increasing with $r_j \to R$ as $j \to \infty$.

Proof. The function $H(s)$ is increasing by assumption, the function $s(r)$ is increasing by definition, the function ξ is increasing by assumption, and the function $V(s)$ is increasing by Proposition 7.1.5. Clearly $s'(r)$ is nondecreasing, and so v is therefore strictly increasing. Since we have assumed $H(s) \to \infty$ as $s \to \infty$, we clearly have $v(r) \to \infty$ as $r \to R$. The rest easily follows. □

We now state the key lemma.

Lemma 7.1.8 *Let r_0, R, ξ, and h be as in the statement of Theorem 7.1.1, with ξ and h also assumed C^∞, h increasing, and $h(r) \to \infty$ as $r \to R$. Let $V(s)$ be as in Proposition 7.1.5, and let $s(r)$, $r(s)$, $H(r)$, and $v(r)$ be as in Proposition 7.1.6. Let $r_j, j \geq j_0$ be defined as in equation (7.1.7), and let $g(z)$ be defined by equation (7.1.3) using these r_j. Then, g is a holomorphic function on $\mathbf{D}(R)$ such that*

$$N(g, 0, r) \leq V(s(r)) + O(1).$$

Moreover, there exists a closed set E in $[r_0, R)$ such that

$$\int_E dr < \infty \quad \text{if } R = \infty \qquad \text{or} \qquad \int_E \frac{dr}{(R - r)^2} < \infty \quad \text{if } R < \infty,$$

such that for all $r \geq r_0$ and not in E, we have

(1) $$T(g, r) = N(g, 0, r) + O(1),$$

and such that for all z with $|z| = r$, and $r \geq r_0$ and not in E, we have

(2) $$\log\left|\frac{g'(z)}{g(z)}\right| \geq \log T(g, r) + \log\xi(T(g, r)) + h(r) + s(r) - O(1).$$

Proof. In as much as possible we try to uniformly treat the cases $R < \infty$ and $R = \infty$. Whenever we say, "for r sufficiently large," this will be mean for r sufficiently close to R in the case that $R < \infty$. We also point out we can add any interval of the form $[r_0, t]$ for any $t < R$ to the set E without destroying the finite measure condition required by the theorem. Thus, whenever we say for r sufficiently large, we add such an interval to E, and provided we only do this a finite number of times, we will be OK.

Now then, we already know that g is holomorphic on $\mathbf{D}(R)$ by Proposition 7.1.4, so the first thing we need to show is:

$$N(g, 0, r) \leq V(s(r)) + O(1).$$

Given r sufficiently large, choose j such that $r_{j-1} \le r < r_j$. Then,

$$n(g,0,r) = \sum_{k=j_0}^{j} 2^k = 2^j - 2^{j_0} \le v(r) - 2^{j_0}.$$

By the definition of v and Proposition 7.1.5,

$$\frac{v(r)}{r} = H(s(r))V(s(r))\xi(V(s(r)))s'(r) = V'(s(r))s'(r).$$

Thus

$$\begin{aligned} N(g,0,r) &= \int_{r_0}^{r} \frac{n(g,0,t)}{t}\,dt + O(1) \\ &\le \int_{r_0}^{r} \frac{v(t)}{t}dt - 2^{j_0}\log r + O(1) \\ &\le \int_{s(r_0)}^{s(r)} V'(s)ds + O(1) \le V(s(r)) + O(1), \end{aligned}$$

as was to be shown.

We now describe the set E. For each $j \ge j_0$, let

$$r_j' = r(s(r_j) - j^{-2}), \quad r_j'' = r(s(r_j) + j^{-2}), \quad \text{and} \quad E_j = [r_j', r_j''].$$

In addition to any small r that we want to exclude as per our comment in the first paragraph of this proof, we will take

$$E \supseteq \bigcup_{j=j_0}^{\infty} E_j.$$

If $R = \infty$, then clearly

$$\int_E dr \le O(1) + \sum_{j=j_0} \frac{2}{j^2} < \infty.$$

If, on the other hand, $R < \infty$, then

$$\int_E \frac{dr}{(R-r)^2} = \int_{s(E)} ds \le O(1) + \sum_{j=j_0}^{\infty} \frac{2}{j^2} < \infty.$$

Note that although the E_j need not be disjoint, E has the property that for any r not in E, there exists a unique integer k such that

$$r > r_j'' \text{ for } j = j_0, \dots, k \quad \text{and} \quad r < r_j' \text{ for } j = k+1, \dots.$$

Note that if $|z| = r$, and r is not in E, then if we choose k as above, we have

$$\begin{aligned} \log|g(z)| = \sum_{j=1}^{k} 2^j \log\frac{r}{r_j} &+ \sum_{j=1}^{k} \log\left|\left(\frac{r_j}{z}\right)^{2^j} + 1\right| \\ &+ \sum_{j=k+1}^{\infty} \log\left|1 + \left(\frac{z}{r_j}\right)^{2^j}\right| \qquad (*) \\ &= N(g,0,r) + \Sigma_1 + \Sigma_2. \end{aligned}$$

We will now show that $|\Sigma_1|$ and $|\Sigma_2|$ are $O(1)$ as $r \to R$, from which (1) in the statement of the lemma follows. In order to do these estimates, we remark that if $R = \infty$, then for large r,

$$v(r) > e^{h(r)+r} > e^r$$

and so $r_j < j+1$ for large j. On the other hand, if $R < \infty$, then for large j,

$$r_j < R < j+1.$$

Therefore, if $r \ge r_j''$, and j is sufficiently large, then

$$\log\frac{r}{r_j} = \log\left(1 + \frac{r - r_j}{r_j}\right) \ge \log\left(1 + \frac{j^{-2}}{r_j}\right) \ge \frac{1}{2}\cdot\frac{j^{-2}}{r_j} \ge \frac{j^{-2}}{2(j+1)}.$$

Thus, for those r, we have

$$\left(\frac{r}{r_j}\right)^{2^j} \ge \exp\left(\frac{2^j}{2j^2(j+1)}\right). \qquad (**)$$

Similarly, if $r < r_j'$ and j is sufficiently large, then

$$\log\frac{r}{r_j} = \log\left(1 - \frac{r_j - r}{r_j}\right) \le \log\left(1 - \frac{j^{-2}}{r_j}\right) \le -\frac{j^{-2}}{r_j} \le \frac{-1}{j^2(j+1)}.$$

So we have for these r,

$$\left(\frac{r}{r_j}\right)^{2^j} \le \exp\left(-\frac{2^j}{j^2(j+1)}\right). \qquad (***)$$

Now, from $(**)$, there exists ℓ such that if $j \ge \ell$, and $r \ge r_j$ and not in E, we have

$$\left|\frac{r_j}{r}\right|^{2^j} \le \frac{1}{2}.$$

Thus, for r sufficiently large, again using (**),

$$\begin{aligned}|\Sigma_1| &\le \sum_{j=1}^{k}\left|\log\left|\left(\frac{r_j}{z}\right)^{2^j}+1\right|\right| \\ &\le \sum_{j=1}^{\ell-1}\left|\log\left|\left(\frac{r_j}{z}\right)^{2^j}+1\right|\right| + \sum_{j=\ell}^{k}2\left(\frac{r_j}{r}\right)^{2^j} \\ &\le O(1)+2\sum_{j=\ell}^{\infty}\exp\left(-\frac{2^j}{2j^2(j+1)}\right) \le O(1).\end{aligned}$$

Similarly, using (***), for r sufficiently large, we have

$$|\Sigma_2| \le 2\sum_{j=k+1}^{\infty}\left(\frac{r}{r_j}\right)^{2^j} \le 2\sum_{j=1}^{\infty}\exp\left(-\frac{2^j}{j^2(j+1)}\right) \le O(1).$$

Hence, we have proven (1).

We now proceed to prove (2) in the statment of the lemma. Differentiating (*) and multiplying by z we get

$$\begin{aligned}\frac{zg'(z)}{g(z)} &= \sum_{j=1}^{k}2^j + \sum_{j=1}^{k}\frac{-2^j}{1+(z/r_j)^{2^j}} + \sum_{j=k+1}^{\infty}\frac{2^j(z/r_j)^{2^j}}{1+(z/r_j)^{2^j}} \\ &= n(g,0,r)+\Sigma_3+\Sigma_4.\end{aligned}$$

From (**) and (***), we find

$$|\Sigma_3| \le 2\sum_{j=1}^{k}\frac{2^j}{(r/r_j)^{2^j}} \le 2\sum_{j=1}^{\infty}2^j\exp\left(-\frac{2^j}{2j^2(j+1)}\right)+O(1) \le O(1),$$

and

$$|\Sigma_4| \le \sum_{j=k+1}^{\infty}\frac{2^j(r/r_j)^{2^j}}{1-(r/r_j)^{2^j}} \le 2\sum_{j=1}^{\infty}2^j\exp\left(-\frac{2^j}{j^2(j+1)}\right)+O(1) \le O(1).$$

Thus, when $|z| = r$ and r is not in E, we have

$$\left|\frac{f'(z)}{f(z)}\right| = \frac{n(g,0,r)}{r}+O(1).$$

Choose j so that $r_{j-1} \le r < r_j$, and so then,

$$n(g,0,r) = 2^j - 2^{j_0} = \frac{1}{2}v(r_j) - 2^{j_0} \ge \frac{1}{2}v(r) - O(1).$$

Thus, using the definition of v and H,

$$\begin{aligned}\log\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| &\ge \log\frac{n(g,0,r)}{r} - O(1) \\ &\ge \log\frac{v(r)}{r} - O(1) \\ &\ge \log H(s(r)) + \log(V(s(r))\xi(V(s(r)))) + \log s'(r) - O(1) \\ &\ge h(r) + s(r) + \log V(s(r)) + \log\xi(V(s(r))) - O(1).\end{aligned}$$

Now, we have already shown that for r not in E,

$$T(g,0,r) \le N(g,0,r) + O(1) \le V(s(r)) + O(1),$$

and hence we have shown (2). □

Now that we have Lemma 7.1.8, it is very easy to complete the proof of Theorem 7.1.1.

Proof of Theorem 7.1.1. Let g and E be as in Lemma 7.1.8. Let $s(r)$ be as in Proposition 7.1.6. Note that from (2) in Lemma 7.1.8, we immediatly have for all $r \ge r_0$ and not in E, that

$$\begin{aligned}m(g'/g,\infty,r) &\ge \int_E \log\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|\frac{d\theta}{2\pi} \\ &\ge \log T(g,r) + \log\xi(T(g,r)) + h(r) + s(r) - O(1).\end{aligned}$$

Now note that $s(r) \to \infty$ as $r \to R$, so by enlarging E if necessary we can assume the $s(r)$ term cancels the $-O(1)$ term and we have proven statement (1) in Theorem 7.1.1 by taking $f = g$.

For statement (2) in the theorem, recall that we have assumed $q \ge 1$. In the case that $a_1 = \infty$, then again take $f = g$. Otherwise, take

$$f = a_1 + \frac{1}{g}.$$

In either case, note that

$$N_{\mathrm{ram}}(f,r) = N(g',0,r).$$

Also note that

$$m(f,a_1,r) = m(g,\infty,r),$$

and of course $T(f,r) = T(g,r) + O(1)$. Because g is holomorphic, we also have $m(g,\infty,r) = T(g,r)$, and so we have

$$\begin{aligned}(q-2)T(f,r) - \sum_{j=1}^{q}N(f,a_j,r) + N_{\mathrm{ram}}(f,r) &\ge m(f,a_1,r) + N_{\mathrm{ram}}(f,r) - 2T(f,r) - O(1) \\ &= N(g',0,r) - T(g,r) - O(1).\end{aligned}$$

An easy computation reveals (again using that g is holomorphic),

$$N(g',0,r) = N(g,0,r) - N(g'/g,\infty,r) + N(g'/g,0,r).$$

Now, the First Main Theorem (Theorem 1.3.1) tells us

$$N(g'/g,0,r) - N(g'/g,\infty,r) = m(g'/g,\infty,r) - m(g'/g,0,r) + O(1).$$

Thus,

$$N(g',0,r) - T(g,r) = N(g,0,r) + m(g'/g,\infty,r) - m(g'/g,0,r) - T(g,r).$$

Using (1) in Lemma 7.1.8, we have

$$N(g,0,r) - T(g,r) = O(1)$$

for r outside E. Thus, for r outside E,

$$\begin{aligned}(q-2)T(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{ram}}(f,r) &\geq N(g',0,r) - T(g,r) - O(1)\\ &\geq m(g'/g,\infty,r) - m(g'/g,0,r) - O(1)\\ &= \int_0^{2\pi} \log\left|\frac{g'(re^{i\theta})}{g(re^{i\theta})}\right| \frac{d\theta}{2\pi} - O(1).\end{aligned}$$

The proof is completed by applying (2) of Lemma 7.1.8 and enlarging E as necessary so that $s(r)$ absorbs any $-O(1)$ term. □

7.2 Better Error Terms for Functions with Controlled Growth

The error terms in the versions of the Logarithmic Derivative Lemma and Second Main Theorem presented in this book have always had terms of the form

$$\log \frac{\psi(T(f,r))}{\phi(r)},$$

where ψ satisfies the Khinchin convergence condition. Such inequalities are valid for radii outside a set E of finite measure. The measure of the set E is bounded by the integral in the Khinchin convergence condition, and the type of measure (Lebesgue, logarithmic, *etc.*) is specified by the function ϕ. As we have emphasized before, taking smaller functions ψ and larger functions ϕ improves the

$\log\psi(T(f,r))/\phi(r)$ term in the error terms at the cost of enlarging the exceptional set. It is then a natural question whether this kind of trade off can be further exploited. That is, can we get even further improvements in the error term by enlarging the exceptional set even further?

We recall here the notion of the (upper) density of a set. Let ϕ be a positive function on (r_0, R), and let E be a measureable subset of (r_0, R). Then, we define the **upper density** of the set E with respect to the function ϕ to be

$$\limsup_{r\to R} \frac{\displaystyle\int_{E\cap(r_0,r)} \frac{dt}{\phi(t)}}{\displaystyle\int_{r_0}^{r} \frac{dt}{\phi(t)}}.$$

Thus, the upper density of a set is a real number between 0 and 1 and measures what percentage of the interval (r_0, R) the set E fills up, where percentage is measured with respect to ϕ. If ϕ is chosen so that

$$\int_{r_0}^{R} \frac{dt}{\phi(t)} = \infty,$$

then any set E with finite measure with respect to ϕ, *i.e.* such that

$$\int_E \frac{dt}{\phi(t)} < \infty,$$

will clearly have upper density 0.

A natural question is "can the error term in the Second Main Theorem or the Logarithmic Derivative Lemma be improved if we ask only that the inequality hold outside a set of density zero or outside a set of density less than 1?" Recall that for many applications, all that is needed to apply the Second Main Theorem is the existence of a sequence of $r \to R$ such that the inequality holds. If the Second Main Theorem holds outside an exceptional set with upper density less than one, then the existence of such r would of course follow.

The examples constructed in the last section show that, in general, no such improvement is possible. That is, Theorem 7.1.1 says that for ξ not satisfying the Khinchin convergence condition, there exist meromorphic functions f with error terms greater than

$$\log T(f,r) + \log\xi(T(f,r))$$

for all r outside a set E of finite measure (at least for certain ϕ), and so in particular the error terms are large outside a set of density zero.

On the other hand, we also remarked that the functions constructed in the previous section grow very rapidly. M. Jankowski [Jan 1994] recognized that by restricting the growth of the functions under consideration and applying Borel-Nevanlinna growth Lemma (Lemma 3.3.1) to functions that do not necessarily satisfy the Khinchin convergence condition, the error term in the Lemma on the Logarithmic Derivative could be improved outside an exceptional set of upper density less than one. This idea was carried out in complete generality based on the Borel-Nevanlinna growth lemma (Lemma 3.3.1) by Y. Wang [Wang 1997] and based on the Nevanlinna growth lemma (Lemma 2.2.1) by H. Chen and Z. Ye [ChYe 2000]. In this section, we briefly describe this work for functions meromorphic on the whole plane. Analogous results are true for functions meromorphic in a disc, but we leave the precise formulations in the disc case to the reader.

Wang's key observation was that the Borel-Nevanlinna growth lemma in the form of Lemma 3.3.1 immediately implies the following:

Lemma 7.2.1 *Let F and ϕ be a positive, non-decreasing, continuous functions on (r_0,∞), and let ξ be a positive, non-decreasing, continuous function defined on (e,∞). Suppose further that*

$$\limsup_{r\to\infty}\frac{\displaystyle\int_e^{F(r)}\frac{dx}{x\xi(x)}}{\displaystyle\int_e^r\frac{dx}{\phi(x)}}\le\tau<\infty,$$

and that

$$\int_{r_0}^\infty\frac{dr}{\phi(r)}=\infty.$$

Then, the upper density (with respect to ϕ) of the set

$$E=\left\{r\in[r_0,\infty):F\left(r+\frac{\phi(r)}{\xi(F(r))}\right)\ge CF(r)\right\}$$

is $\le\tau(\log C)^{-1}$, where C is a positive constant.

Note of course that Lemma 7.2.1 has non-trivial content only when $\log C>\tau$.

Theorem 7.2.2 (Wang) *Let ϕ be a positive, nondecreasing, continuous function defined for $0<r_0\le r<\infty$ such that*

$$\int_{r_0}^\infty\frac{dr}{\phi(r)}=\infty.$$

Let ξ be a positive, nondecreasing, continuous function defined on (e,∞). Let f be a meromorphic function on $\mathbf{C}$ such that

$$\alpha=\limsup_{r\to\infty}\frac{\displaystyle\int_1^{T(f,r)}\frac{dx}{x\xi(x)}}{\displaystyle\int_1^r\frac{dx}{\phi(x)}}<\infty.$$

If $\alpha>0$, let $0<\beta<1$. If $\alpha=0$, then let $\beta=0$. Then

$$m(f'/f,r)\le\log T(f,r)+\log^+\frac{\xi(T(f,r))}{\phi(r)}+O_{\alpha,\beta}(1)\tag{1}$$

outside an exceptional set of radii of upper density (with respect to ϕ) $\le\beta$. Moreover, for any finite set of distinct points $a_1,\ldots,a_q$ in $\mathbf{P}^1$,

$$\begin{aligned}(q-2)T(f,r)+&\textstyle\sum_{j=1}^q N(f,a_j,r)+N_{\mathrm{ram}}(f,r)\\&\le\log T(f,r)+\log^+\frac{\xi(T(f,r))}{\phi(r)}+O_{\alpha,\beta}(1)\end{aligned}\tag{2}$$

outside an exceptional set of radii of upper density (with respect to ϕ) $\le\beta$.

Remark. Theorem 7.2.2 is only an improvement over the theorems in previous chapters in the event that

$$\int_e^\infty\frac{dx}{x\xi(x)}=\infty.$$

Proof. The theorem follows immediately from Corollary 3.2.3, statement (1) of Theorem 4.2.1, and Lemma 7.2.1 in a manner that the reader should by now find routine. □

Chen and Ye recognized that an argument similar to Wang's, but applied to Lemma 2.2.1 instead of Lemma 3.3.1, would result in the following.

Lemma 7.2.3 *Let $F(r)$ be a C^2 function defined for $r_0\le r<\infty$. Assume further that both $F(r)$ and $rF'(r)$ are positive nondecreasing functions of r and that $F(r_0)\ge e$. Let b_1 be any number such that $b_1rF'(r)\ge e$ for all $r\ge r_0$. Let ϕ be a positive measureable function defined for $r_0\le r<\infty$. Let ξ be a positive nondecreasing continuous function defined on $(e,\infty$.) Suppose further that*

$$\limsup_{r\to\infty}\frac{\displaystyle\int_e^{F(r)}\frac{dx}{x\xi(x)}}{\displaystyle\int_e^r\frac{dx}{\phi(x)}}\le\tau<\infty,$$

and that

$$\int_{r_0}^\infty\frac{dr}{\phi(r)}=\infty.$$

Let C be a positive constant. Then, there exists a set E of upper density (with respect to ϕ) $\le 2\tau/C$ such that for all $r\ge r_0$ and outside E,

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dF}{dr}\right]<C^2F(r)\frac{\xi(F(r))}{\phi(r)}\frac{\xi\left(Cb_1rF(r)\frac{\xi(F(r))}{\phi(r)}\right)}{\phi(r)}.$$

Proof. Let

$$E_1 = \left\{ r : \frac{1}{r}\frac{d}{dr}[rF'(r)] \geq CF'(r)\frac{\xi(b_1 rF'(r))}{\phi(r)} \right\}.$$

Then, applying Lemma 2.2.1 to the function $b_1 rF'(r)$ and using

$$\limsup_{r\to\infty} \frac{\displaystyle\int_e^{F(r)} \frac{dx}{x\xi(x)}}{\displaystyle\int_e^r \frac{dx}{\phi(x)}} \leq \tau,$$

we bound the upper density of E_1 by τ/C. Outside E_1,

$$\frac{1}{r}\frac{d}{dr}[rF'(r)] < CF'(r)\frac{\xi(b_1 rF'(r))}{\phi(r)}.$$

Now, set

$$E_2 = \left\{ r : F'(r) \geq CF(r)\frac{\xi(F(r))}{\phi(r)} \right\}$$

and apply Lemma 2.2.1 to $F(r)$. Then, E_2 has upper density $\leq \tau/C$. Because ξ is nondecreasing, we have the desired inequality for all r outside $E_1 \cup E_2$, which has density $\leq 2\tau/C$. □

Theorem 7.2.4 (Chen-Ye) *Let ϕ be a positive, nondecreasing, continuous function on $0 < r_0 \leq r < \infty$ such that*

$$\int_{r_0}^{\infty} \frac{dr}{\phi(r)} = \infty.$$

Let ξ be a positive, nondecreasing, continuous function defined on (e, ∞). Let f be a meromorphic function on $\mathbf{C}$ such that

$$\alpha = \limsup_{r\to\infty} \frac{\displaystyle\int_e^{T(f,r)} \frac{dx}{x\xi(x)}}{\displaystyle\int_e^r \frac{dx}{\phi(x)}} < \infty.$$

If $\alpha > 0$, let $0 < \beta < 1$. If $\alpha = 0$, then let $\beta = 0$. Then, for any finite set of distinct points $a_1, \ldots, a_q$ in $\mathbf{P}^1$,

$$(q-2)T(f,r) + \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{ram}}(f,r)$$
$$\leq \log T(f,r) + \log\frac{\xi(T(f,r))}{\phi(r)} + O_{\alpha,\beta}(1)$$

outside an exceptional set of radii of upper density (with respect to ϕ) $\leq \beta$.

Proof. We break the proof into two cases. If $\phi(r) \leq r$, then choose C large in Lemma 7.2.1 and use statement (1) of Theorem 4.2.1 and an argument as in Lemma 3.3.2 to get the desired in equality.

If, on the other hand, $\phi(r) > r$, then we proceed as follows. As in Lemma 2.2.6, we let

$$\xi_*(x) = \min\{\xi(\max\{e, x^{1/5}\}), x^{1/5}\}.$$

Notice that, using the same computation as in the proof of Lemma 2.2.6, we see

$$\limsup_{r\to\infty} \frac{\displaystyle\int_e^{T(f,r)} \frac{dx}{x\xi_*(x)}}{\displaystyle\int_e^r \frac{dx}{\phi(x)}} \leq \limsup_{r\to\infty} \frac{9 + 5\displaystyle\int_e^{T(f,r)} \frac{dx}{x\xi(x)}}{\displaystyle\int_e^r \frac{dx}{\phi(x)}} \leq 5\alpha < \infty.$$

Moreover,

$$\xi_*(x^5) \leq \xi(x) \qquad \text{and} \qquad \xi_*(x) \leq x^{1/5}. \tag{$*$}$$

Then, we replace the use of Lemma 2.2.3 in the proof of Theorem 2.5.1 with Lemma 7.2.3 using ξ_* in place of ξ and choosing a large constant C. This gives us a set E of upper density $\leq 10\alpha/C$ such that for all $r \geq r_0$ and outside E, we have

$$(q-2)\mathring{T}(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{ram}}(f,r)$$
$$\leq \frac{1}{2}\left[\log\left\{C^2 b_3 \mathring{T}^2(f,r)\right\} + \log\left\{\frac{\xi_*(b_3\mathring{T}^2(f,r))}{\phi(r)}\right\}\right.$$
$$\left. + \log\left\{\frac{\xi_*\left(b_1 rCb_3\mathring{T}^2(f,r)\dfrac{\xi_*(b_3\mathring{T}^2(f,r))}{\phi(r)}\right)}{\phi(r)}\right\}\right] + O(1).$$

Here b_1 and b_3 are as in §2.3. We fix C so large that $10\alpha/C \leq \beta$. Then, the first term on the right in the above inequality becomes

$$\log\mathring{T}(f,r) + O_{\alpha,\beta}(1).$$

Because we can add any finite interval to E without changing its upper density, we may assume

$$\mathring{T}(f,r) \geq \max\{b_1 Cb_3, Cb_3\}.$$

Hence

$$\log\left\{\frac{\xi_*(b_3\mathring{T}^2(f,r))}{\phi(r)}\right\} \le \log\frac{\xi(T(f,r))}{\phi(r)}$$

by $(*)$. For the last term,

$$\xi_*(b_3\mathring{T}^2) \le [b_3\mathring{T}^2]^{1/5} \le \mathring{T}$$

by the construction of ξ_* and since $\mathring{T}(f,r) \ge b_3$. Moreover,

$$b_1rCb_3\mathring{T}^2(f,r)\frac{\xi_*(b_3\mathring{T}^2(f,r))}{\phi(r)} \le \frac{r}{\phi(r)}\mathring{T}^4(f,r).$$

For each fixed r, either $\mathring{T}(f,r) \ge r/\phi(r)$ or $\mathring{T}(f,r) < r/\phi(r)$, and hence

$$\begin{aligned}\log&\frac{\xi_*\left(b_1rCb_3\mathring{T}^2(f,r)\dfrac{\xi_*(b_3\mathring{T}^2(f,r))}{\phi(r)}\right)}{\phi(r)}\\ &\le \log\frac{\xi(\mathring{T}(f,r))}{\phi(r)} + \log\xi(\max\{e, r/\phi(r)\}),\end{aligned}$$

again by $(*)$. We are then done because we have assumed $\phi(r) > r$, so the second term on the right is bounded. □

Remark. Note that the proof of Theorem 7.2.4 shows that the theorem remains true if ϕ is assumed only to be positive and measureable, provided one adds the term

$$\frac{1}{2}\log\xi(\max\{e, r/\phi(r)\})$$

to the error term in the inequality.

Corollary 7.2.5 *Let f be meromorphic on $\mathbf{C}$ such that $\log T(f,r) = O(r)$, and let $a_1,\dots,a_q$ be distinct points in $\mathbf{P}^1$. Then, outside an exceptional set of Lebesgue upper density < 1,*

$$m(f'/f,r) \le \log T(f,r) + O(1)$$

and

$$(q-2)T(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{ram}}(f,r) \le \log T(f,r) + O(1).$$

Proof. Apply Theorem 7.2.4 with $\xi \equiv 1$ and $\phi \equiv 1$. □

We leave it as an exercise for the reader to check that the inequality in Theorem 4.3.1 is recovered by applying Theorem 7.2.2 or Theorem 7.2.4 with $\xi \equiv 1$ and $\phi(r) = r$, but only outside an exceptional set of logarithmic density < 1.

Given natural growth restrictions on f it is usually easy to see that there exist functions ξ satisfying the hypotheses of either Theorem 7.2.2 or Theorem 7.2.4 for f such that

$$\int_e^\infty \frac{dx}{x\xi(x)} = \infty.$$

In fact in [Wang 1997], Wang shows that for any meromorphic function f, there exists some function ξ satisfying the hypotheses of Theorem 7.2.2 and such that

$$\int_e^\infty \frac{dx}{x\xi(x)} = \infty.$$

Moreover, one may always find such divergent ξ so that the α in the theorems is 0.

7.3 Error Terms for Some Classical Special Functions

Recall that in §6.2 we discussed Khinchin's Theorem (Theorem 6.2.2), which said that almost all real numbers α have the property that for all but finitely many relatively prime pairs of integers (m,n),

$$-\log\left|\alpha - \frac{m}{n}\right| - 2\log n \le \log\tilde{\psi}(n),$$

where $\tilde{\psi}$ is analogous to a Khinchin function. Recall that we also mentioned that S. Lang defined the type of a number α to be the best (or smallest) function $\tilde{\psi}$ that could be taken so that the above inequality is satisfied for the specific number α. Lang then posed the problem of finding the "types" of the well-known numbers like e, π, γ, and so forth. After the analogy between Nevanlinna theory and Diophantine approximation was established, Lang [Lang 1990] transposed this problem of finding the type of well-known numbers to finding the type of well-known functions. Lang's problem of determining the type of certain well-known functions was taken up by L. Sons and Z. Ye in [SoYe 1993], where they compute the type of Euler's gamma function and the various Weierstrass functions from the theory of elliptic functions. In this chapter we will compute the types of rational functions, the exponential function, some trigonometric functions, and Euler's gamma function, which is the easiest of the functions treated by Sons and Ye.

We now make a precise definition of what we mean by the "type" of a function, which following Sons and Ye [SoYe 1993], we will call the "Lang type." Let ξ be a positive real-valued function on the interval (e,∞). A function f meromorphic on $\mathbf{C}$ is said to be of **Lang type less than or equal** to ξ if

LT $\leq$. Given *any* finite set of distinct complex numbers $a_1, a_2, \cdots, a_q$,

$$(q-2)\mathring{T}(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{ram}}(f,r) \leq \log \xi(\mathring{T}(f,r)) + O(1),$$

for all r outside a set of finite Lebesgue measure.

In the other direction, a function f meromorphic on $\mathbf{C}$ is said to be of **Lang type greater than or equal** to ξ if

LT $\geq$. There exists *some* (possibly empty) set of distinct complex numbers a_1, $a_2, \cdots, a_q$ such that

$$(q-2)\mathring{T}(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{ram}}(f,r) \geq \log \xi(\mathring{T}(f,\ r)) + O(1),$$

for all r outside a set of finite Lebesgue measure.

If conditions **LT** $\leq$ and **LT** $\geq$ are satisfied at the same time for the same function ξ then we say that the meromorphic function f has **precise Lang type** ξ. This agrees with what Sons and Ye termed simply "Lang type."

If ξ can be taken to be a constant function in **LT** $\leq$ or **LT** $\geq$, then we say f has Lang type less than or equal to 1, Lang type greater than or equal to 1, or precise Lang type 1, accordingly.

If $\xi(x) = x\psi(x)$ for any Khinchin function ψ, then the Second Main Theorem exactly tells us that any non-constant meromorphic function has Lang type $\leq \xi$. On the other hand, if we take $q = 0$ in condition **LT** $\geq$, we see that for all $r \geq 1$,

$$-2\mathring{T}(f,r) + N_{\mathrm{ram}}(f,r) \geq -2\mathring{T}(f,r) = \log e^{-2\mathring{T}(f,r)},$$

and so all non-constant meromorphic functions have Lang type $\geq e^{-2x}$.

What interests us here is the determination of the precise Lang type for various special functions. We begin with rational functions.

Rational Functions

Proposition 7.3.1 *If f is a rational function of degree $d > 0$, then f has precise Lang type $e^{-x/d}$.*

Proof. Note that we determined in §1.6 that

$$\mathring{T}(f,r) = T(f,r) + O(1) = d \log r + O(1).$$

Thus,

$$\log e^{-\mathring{T}(f,r)/d} = -\log r + O(1),$$

and so to prove the proposition, we must show on the one hand that for every finite set of distinct points $a_1, \ldots, a_q$, we have

$$(q-2)\mathring{T}(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{ram}}(f,r) \leq -\log r + O(1),$$

while on the other hand for some choice of $a_1, \ldots, a_q$,

$$(q-2)\mathring{T}(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{ram}}(f,r) \geq -\log r + O(1).$$

Because f is rational, it extends to a function on all of $\mathbf{P}^1$. Let $c = f(\infty)$, and let m be the multiplicity with which f takes on the value c at ∞. Then, for $a \neq c$ and for all large r,

$$N(f,a,r) = d \log r + O(1) = \mathring{T}(f,r) + O(1).$$

On the other hand, for the value c,

$$N(f,c,r) = (d-m)\log r + O(1),$$

for all large r. For all large r, we also have

$$N_{\mathrm{ram}}(f,r) = 2(d-1)\log r - (m-1)\log r.$$

Let $a_1 = c$ and let a_2 be any other point in $\mathbf{P}^1$ (so $q = 2$). Then,

$$\begin{aligned}(q-2)\mathring{T}(f,r) &- \sum_{j=1}^{2} N(f,a_j,r) + N_{\mathrm{ram}}(f,r) \\ &= 0 - [(d-m)\log r + d\log r] + [2(d-1)\log r - (m-1)\log r] + O(1) \\ &= -\log r + O(1),\end{aligned}$$

establishing that f has Lang type $\geq e^{-x/d}$. On the other hand, adding more points a_j does not change the left hand side, except by a bounded term, since the N and $\mathring{T}$ terms cancel. Taking away the points a_1 or a_2 makes the left hand side smaller. Thus f also has Lang type $\leq e^{-x/d}$. □

Remark. Since rational functions have order 0, Theorem 4.3.1 implies the weaker statement that rational functions of degree $d > 0$ have Lang type less than or equal to $e^{(\varepsilon-1)x/d}$ for all positive ε.

Exponential and Trigonometric Functions

Now we turn our attention to a transcendental function, namely e^z.

Proposition 7.3.2 *The exponential function e^z has precise Lang type 1.*

Proof. Letting $a_1 = 0$ and $a_2 = \infty$, the left hand side of the key inequality in the Second Main Theorem is equal to 0, so e^z is clearly of Lang type ≥ 1. The reverse inequality is a consequence of Proposition 1.6.1. □

We leave it as an exercise for the reader to show that $\sin z$ and $\cos z$ also have precise Lang type 1. The reader will need to show that for these functions

$$N_{\mathrm{ram}}(f,r) = T(f,r) + O(1).$$

See the discussion of $\sin z$ and $\cos z$ in §1.6 following Proposition 1.6.1.

Euler's Gamma Function

Recall that **Euler's gamma function** $\Gamma(z)$ is a meromorphic function with the following properties:

- Γ has simple poles at the non-positive integers: $0, -1, -2, \ldots$, and these are its only poles.
- $\Gamma(z+1) = z\Gamma(z)$ for all z which are not poles.
- $\Gamma(1) = 1$.

From this it follows that for all strictly positive integers n, $\Gamma(n) = (n-1)!$, and so Γ is a meromorphic function extending the factorial function on the positive integers. We will take for our definition of Γ the following product development:

$$\boxed{\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}, \quad \text{where } \gamma = \lim_{n\to\infty}\left(-\log n + \sum_{j=1}^{n}\frac{1}{j}\right).}$$

The constant γ is is called **Euler's constant**. For those unfamiliar with the properties of the gamma function, see Section 2.4 of Chapter 5 of [Ahlf 1966].

As was said before, Γ has simple poles at the non-positive integers, and it is clear from the product development of $1/\Gamma$ given above that it has no zeros.

We will now discuss some of the well-known properties of Γ', the derivative of the gamma function.

Proposition 7.3.3 *The derivative of the gamma function Γ' has at least one zero on the positive real-axis, and for each integer $k \geq 1$, there exists a negative real number x_k with $-k < x_k < -k+1$ such that $\Gamma'(x_k) = 0$.*

Remark. In fact, all the zeros of Γ' are real, there is precisely one zero between every two non-positive integers, and there is exactly one positive real zero. This will follow from what we will do a bit later in this chapter. See Corollary 7.3.8.

Proof. We consider $\Gamma(x)$ as a function of a real-variable x. Then Γ is real-valued and smooth away from the non-positive integers. Since Γ interpolates the factorial function, we know $\Gamma(1) = 0! = 1 = 1! = \Gamma(2)$, and so by calculus, Γ' has a zero between 1 and 2. Similarly, Γ has poles at the non-positive integers, it is never zero, and therefore must have local extrema between every non-positive integer. □

We now prove a well-known (see for example formula 6.3.20 of [Dav 1965]) estimate giving more precise information on the location of the negative real zeros of Γ'.

Proposition 7.3.4 *Let x_k with $-k < x_k < -k+1$ be a negative real zero of Γ'. Then,*

$$x_k = -k + \frac{1}{\log k} \pm O\left(\frac{1}{(\log k)^2}\right).$$

Proof. We find the zeros of Γ' by finding the zeros of

$$\frac{d}{dz}\log\Gamma(z) = -\gamma - \frac{1}{z} + \sum_{j=1}^{\infty}\left\{\frac{1}{j} - \frac{1}{z+j}\right\},$$

where this formula comes from logarithmically differentiating the product expansion for $1/\Gamma$. We assume x_k is a zero between $-k$ and $-k+1$ and plug in to get

$$0 = -\gamma - \frac{1}{x_k} + \sum_{j=1}^{\infty}\left\{\frac{1}{j} - \frac{1}{x_k+j}\right\}.$$

Now we separate the sum

$$\begin{aligned}\sum_{j=1}^{\infty}&\left\{\frac{1}{j} - \frac{1}{x_k+j}\right\}\\ &= \frac{1}{k} - \frac{1}{x_k+k} + \sum_{\substack{j=1\\ j\neq k}}^{\infty}\left\{\frac{1}{j} - \frac{1}{j-k}\right\} + \sum_{\substack{j=1\\ j\neq k}}^{\infty}\left\{\frac{1}{j-k} - \frac{1}{x_k+j}\right\}.\end{aligned}$$

The first sum on the right telescopes to

$$\sum_{j=1}^{k-1}\frac{1}{j},$$

and this differs from $\log k$ by a bounded amount. The second sum is easily seen to be bounded. Finally, since x_k and k are bounded away from zero, $1/x_k$ and $1/k$ are bounded. Thus,

$$\left|\frac{1}{x_k + k} - \log k\right| \le O(1).$$

From this the proposition follows. □

At this point, we know everything about the zeros and poles of Γ, and we know quite a bit about the real zeros of its derivative. This is enough to give a lower bound on the Lang type of Γ.

Lemma 7.3.5 *Euler's gamma function has Lang type $\ge \log x$.*

Proof. Because $N(\Gamma, 0, r) = 0$ for all r, to prove the lemma, it suffices to show

$$N_{\text{ram}}(\Gamma, r) - N(\Gamma, \infty, r) \ge \log\log r - O(1)$$

for all sufficiently large r. We know all the poles of Γ. Thus, if K is the largest integer $\le r$,

$$N(\Gamma, \infty, r) = \log r + \sum_{k=1}^{K} \log\frac{r}{k}.$$

All the poles of Γ are simple poles, so none of the poles are ramification points for Γ. Thus $N_{\text{ram}}(\Gamma, r) = N(\Gamma', 0, r)$. By Proposition 7.3.3, we know that for every pole of Γ, there is a zero of Γ' to the right of the pole on the real-axis. Thus, for $r \ge 2$, Γ' has at least as many zeros in $\mathbf{D}(r)$ as Γ has poles. Let x_0 denote the zero of Γ' on the positive real-axis, and let x_k, $k \ge 1$ denote the negative zeros of Γ' as in Proposition 7.3.4. Then, if $r \ge 2$ and K is as before,

$$N_{\text{ram}}(\Gamma, r) \ge \log\frac{r}{x_0} + \sum_{k=1}^{K} \log\frac{r}{-x_k},$$

and so

$$N_{\text{ram}}(\Gamma, r) - N(\Gamma, \infty, r) \ge \sum_{k=1}^{K} \log\frac{k}{-x_k} + O(1).$$

Note that $\log k/(-x_k) > 0$ since $k - 1 < -x_k < k$. From Proposition 7.3.4,

$$\begin{aligned}\sum_{k\gg 0}^{K} \log\frac{k}{-x_k} &= \sum_{k\gg 0}^{K} \frac{k}{k - 1/(\log k) \pm O(1/(\log k)^2)} \\ &= \sum_{k\gg 0}^{K} \frac{1}{1 - 1/(k\log k) \pm O(1/(k(\log k)^2))} \\ &= \sum_{k\gg 0}^{K} \log\left(1 + \frac{1}{k\log k} \pm O\left(\frac{1}{k(\log k)^2}\right)\right).\end{aligned}$$

We now use the Taylor expansion for log in a neighborhood of 1 to conclude

$$\sum_{k\gg 0}^{K} \log\frac{k}{-x_k} = \sum_{k\gg 0}^{K} \frac{1}{k\log k} \pm \sum_{k\gg 0}^{K} O\left(\frac{1}{k(\log k)^2}\right) \ge \log\log r \pm o(\log\log r).$$

From this the lemma follows. □

Remark. From the fact that Γ is zero-free, Proposition 2.4.2, the First Main Theorem, and the definition of the proximity functions $m(\Gamma, 0, r)$ and $m(\Gamma, \infty, r)$, we have

$$N_{\text{ram}}(\Gamma, r) - N(\Gamma, \infty, r) = \int_0^{2\pi} \log\left|\frac{\Gamma'(re^{i\theta})}{\Gamma(re^{i\theta})}\right| \frac{d\theta}{2\pi} + O(1).$$

In [SoYe 1993], Sons and Ye use a Stirling's formula type estimate for this integral to prove Lemma 7.3.5 for all r outside a set of finite Lebesgue measure. M. Bonk suggested the method of proof used here based on the precise locations of the zeros of Γ'.

We now directly estimate an upper bound for the Lang type of Γ. Our approach here is the same as in [SoYe 1993].

Proposition 7.3.6 $T(\Gamma, r) = (1 + o(1))\frac{r}{\pi}\log r.$

Proof. By the First Main Theorem, Theorem 1.3.1,

$$T(\Gamma, r) = m(\Gamma, 0, r) + N(\Gamma, 0, r) + O(1),$$

and since Γ is zero-free, we need only estimate

$$m(\Gamma, 0, r) = \int_0^{2\pi} \log^+\left|\frac{1}{\Gamma(re^{i\theta})}\right| \frac{d\theta}{2\pi}.$$

We do this using the product development for $1/\Gamma$. Write $\log|1/\Gamma|$ as

$$\log\left|\frac{1}{\Gamma(z)}\right| = \log|z| + \log|h_{r,1}(z)| + \log|h_{r,2}(z)| + \log|h_{r,3}(z)|,$$

where for fixed $r \ge 1$, the $h_{r,i}$ are the holomorphic functions defined by

$$h_{r,1}(z) = \exp\left(\gamma z - \sum_{n<2r} \frac{z}{n}\right)$$

$$h_{r,2}(z) = \prod_{n=1}^{n<2r} \left(1 + \frac{z}{n}\right)$$

$$h_{r,3}(z) = \prod_{n\ge 2r} \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

Note that

$$\log|h_{r,1}(re^{i\theta})| = \left(\gamma - \sum_{n<2r}\frac{1}{n}\right) r\cos\theta.$$

We see from the definition of γ that

$$\left(\gamma - \sum_{n<2r}\frac{1}{n}\right) = -(1+o(1))\log r,$$

where the $o(1)$ term tends to zero as $r\to\infty$. Thus,

$$\log^+|h_{r,1}(re^{i\theta})| = (1+o(1))r\log r\max\{0,-\cos\theta\},$$

and so

$$\int_0^{2\pi}\log^+|h_{r,1}(re^{i\theta})|\frac{d\theta}{2\pi} = (1+o(1))r\log r\int_{\pi/2}^{3\pi/2}-\cos\theta\frac{d\theta}{2\pi} = (1+o(1))\frac{r}{\pi}\log r.$$

If we show that $h_{r,1}$ is the dominant term and the other terms can be neglected, then we will have proved the proposition.

We make two observations about the function $h_{r,2}$. First, $h_{r,2}$ is largest on the positive real-axis, and so

$$\log|h_{r,2}(re^{i\theta})| \le \log|h_{r,2}(r)| \le \sum_{n=1}^{n<2r}\log(1+r/n) \le O(r)$$

by comparison with the integral

$$\int_1^{2r}\log(1+r/x)\,dx.$$

Second, since $h_{r,2}$ has no poles and since $N(h_{r,2},0,r)$ is non-negative for $r\ge 1$, we have from the Jensen Formula (Corollary 1.2.1) that

$$\int_0^{2\pi}\log^+\left|\frac{1}{h_{r,2}(re^{i\theta})}\right|\frac{d\theta}{2\pi} \le \int_0^{2\pi}\log^+|h_{r,2}(re^{i\theta})|\frac{d\theta}{2\pi} - N(f,0,r) \le O(r)$$

by our upper bound on $\log|h_{r,2}|$ above.

The function $h_{r,3}$ is even easier to estimate. Namely,

$$|\log|h_{r,3}(z)|| = \left|\sum_{n\ge 2r}\log\left(1+\frac{z}{n}\right) - \sum_{n\ge 2r}\frac{z}{n}\right|.$$

Because $|z/n|\le 1/2$ here, we can use the Taylor expansion for $\log x$ around 1, which is an alternating series, to conclude

$$|\log|h_{r,3}(re^{i\theta})|| \le O\left(\sum_{n\ge 2r}\left(\frac{r}{n}\right)^2\right) = O(r),$$

where the last equality follows by comparison with the integral

$$\int_{2r}^{\infty}\frac{dx}{x^2}.$$

Therefore, we clearly have

$$\begin{aligned} m(\Gamma,0,r) &\le m(h_{r,1},\infty,r) + m(h_{r,2},\infty,r) + m(h_{r,3},\infty,r) \\ &= (1+o(1))\frac{r}{\pi}\log r + o(r\log r). \end{aligned}$$

For the inequality in the other direction,

$$\begin{aligned} m(\Gamma,0,r) &= \int_0^{2\pi}\log^+|re^{i\theta}h_{r,1}(re^{i\theta})h_{r,2}(re^{i\theta})h_{r,3}(re^{i\theta})|\frac{d\theta}{2\pi} \\ &\ge \int_0^{2\pi}\log^+|h_{r,1}(re^{i\theta})|\frac{d\theta}{2\pi} - \int_0^{2\pi}\log^+\left|\frac{1}{h_{r,2}(re^{i\theta})}\right|\frac{d\theta}{2\pi} \\ &\quad - \int_0^{2\pi}\log^+\left|\frac{1}{h_{r,3}(re^{i\theta})}\right|\frac{d\theta}{2\pi} \\ &\ge (1+o(1))\frac{r}{\pi}\log r - O(r) - O(r) = (1+o(1))\frac{r}{\pi}\log r \end{aligned}$$

by our estimates above. □

Theorem 7.3.7 *Euler's gamma function has precise Lang type* $\log x$.

Proof. We have already shown in Lemma 7.3.5 that Γ has Lang type $\ge \log x$. We show here that Γ has Lang type $\le \log x$. Indeed, we have by the Second Main Theorem, in the form of statement (1) of Theorem 4.2.1 (taking $\rho = 2r$), that for r sufficiently large and for any finite collection of distinct points $a_1,\dots,a_q$,

$$(q-2)T(\Gamma,r) - \sum_{j=1}^{q}N(\Gamma,a_j,r) + N_{\mathrm{ram}}(\Gamma,r) \le \log\frac{T(\Gamma,2r)}{r} + O(1).$$

Then, using Proposition 7.3.6, we have

$$\log\frac{T(\Gamma,2r)}{r} = \log\log r + O(1). \quad □$$

Corollary 7.3.8 *The zeros of Γ' are all real, each occurring with multiplicity one. There is exactly one positive real zero, and for each integer $k \geq 1$, there is exactly one zero x_k with $-k < x_k < -k+1$.*

Proof. We have already seen that the zeros specified in the corollary exist. Were there any other zeros, we would have a term in $N_{\mathrm{ram}}(\Gamma, r)$ that would not be canceled by a term in $N(\Gamma, \infty, r)$, and thus we would have

$$N_{\mathrm{ram}}(\Gamma, r) - N(\Gamma, \infty, r) \geq \log r \pm o(\log r).$$

This contradicts the theorem. □

Remark. Corollary 7.3.8 follows from "Laguerre's Theorem on Separation of Zeros." This is because $1/\Gamma$ is contained in what is known as the "Polya-Laguerre class," which consists of entire functions that are real on the real axis, have only real zeros, and have "genus" 0 or 1. See [Rub 1996, Chapter 15] and [Lev 1980, Chapter VIII].

Sons and Ye did not compute the Lang type of the Riemann zeta function in their paper [SoYe 1993]. Ye has since made some computations of the Nevanlinna functions for the zeta function [Ye 1999]. Nevertheless, determining the Lang type for the Riemann zeta function remains an open problem. In Corollary 7.3.8 we used our precise determination of the error term for Γ to conclude that Γ' has no zeros, other than the ones we know must exist for obvious reasons. Thus, Corollary 7.3.8 is an example of how the precise error term for a special function can be used to deduce a strong statement about the value distribution of that function (or a closely related function). Because precise error terms can sometimes be used to make such conclusions, it would be very interesting to compute the Lang type of the Riemann zeta function. Because determining the zero distribution of the Riemann zeta function is known to be a hard problem, it is perhaps no surprise that the Lang type has also yet to be determined.

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Glossary of Notation

$f^\#$	spherical derivative of f, 36
(a,b)	open interval, ix
$(a,b]$	half-open interval, ix
$[a,b)$	half-open interval, ix
$[a,b]$	closed interval, ix
$\wedge$	wedge product, 31
$\lvert\ \rvert$	absolute value, 29
$\lvert\ \rvert_0$	trivial absolute value, 29
$\lvert\ \rvert_\infty$	Archimedean absolute value, 29
$\lvert\ \rvert_p$	p-adic absolute value, 29
$\lvert\ \rvert_\wp$	$\wp$-adic absolute value, 153
$\lvert\ \rvert_{\theta,r}$	Archimedean "absolute value" of meromorphic function, 30
$\lvert\ \rvert_{z,r}$	"z-adic" absolute value of meromorphic function, 30
$\ll$	much less than, ix
$\gg$	much greater than, ix
$\overline{X}$	closure of the set X, ix
$\lVert\,.\,\rVert$	chordal distance, 15
$\bar{z}$	complex conjugate of the complex number z, ix
$\equiv$	identically equal, ix
$\sqrt{-1}$	imaginary unit, ix
A	spherical area covered by a meromorphic function, 41, 81
a_j	q distinct points in $\mathbf{P}^1$, 56, 107
α_Λ	expression useful in proof of the Second Main Theorem, 57
b_1	chosen so $b_1 r_0 \frac{d}{dr}\Big\rvert_{r_0} \mathring{T}(f,r) \geq e$, 53, 56
$\tilde{b}_1$	b_1 chosen uniformly for all f with $f^\#(0) = 1.$, 77
b_2	$b_2(f) = \log\lvert\mathrm{ilc}(W,0)\rvert - \log(\lvert f_0(0)\rvert^2 + \lvert f_1(0)\rvert^2)$, 56
b_3	$b_3(q) = 12q^2 + 2q^3\log 2$, 56
b_4	a constant in the error term of the Second Main Theorem, 56
β_1	$\beta_1(f - a_j, r_0) = \lvert\mathrm{ord}_0(f - a_j)\rvert \log^+\frac{1}{r_0} + \Big\lvert \log\lvert\mathrm{ilc}(f - a_j, 0)\rvert \Big\rvert + \log 2$, 96, 104, 108
β_2	$\beta_2(f, a_1, \ldots, a_q) = \sum_{j=1}^{q'} \log\lvert\mathrm{ilc}(f - a_j, 0)\rvert - \log\lvert\mathrm{ilc}(f', 0)\rvert$, 108

β_3	$\beta_3(f,a_1,\dots,a_q,r_0)=\log^+\max\limits_{1\le j\le q'}\{2\log^+\lvert a_j\rvert+\beta_1(f-a_j,r_0)\}$. 108
C	an unspecified constant. x
$\mathbf{C}$	complex numbers or plane. ix, 6
$\overline{\mathbf{C}}$	Riemann sphere (or projective line). 6
$\mathbf{C}(z)$	field of rational functions in z with complex coefficients. 129
$\mathbf{C}(z)[X]$	one variable polynomial ring with coefficients in $\mathbf{C}(z)$. 129
C_{fmt}	First Main Theorem constant. 17
C_{gg}	Gol'dberg-Grinshtein constant. 96
c_{gg}	Gol'dberg-Grinshtein constant. 96
C^∞	infinitely differentiable. ix
C^k	k-differentiable. ix
$\mathbf{C}^n$	complex n-space. ix
d	exterior derivative $=\partial+\bar\partial$. 31
$\mathcal{D}$	constant depending on the distances between the points a_j. 107, 108
$\mathcal{D}$	constant from Proposition 1.2.3. 16, 56
$\mathbf{D}(\infty)$	a disc of infinite radius, or in other words the complex plane. 6
$\mathbf{D}(r)$	open disc of radius r in $\mathbf{C}$ centered at 0. 6, 56, 107
d^c	$=\frac{\sqrt{-1}}{4\pi}(\bar\partial-\partial)$. 31
d^c	polar coordinate form of d^c. 31
dd^c	complex Laplacian. 31
$dd^c\log$	first Chern operator. 32
∂	∂-operator. 31
$\bar\partial$	$\bar\partial$-operator. 31
$\partial/\partial z$	Wirtinger derivative with respect to z. 31
$\partial/\partial\bar z$	Wirtinger derivative with respect to $\bar z$. 31
δ	Nevanlinna defect. 124
$d\theta$	1-form $d\theta$. 33
E	exceptional set. 50
e^z	exponential function. 5
f	a meromorphic function on $\mathbf{D}(R)$., 12, 56, 107
f_0	analytic "denominator" of f, *i.e.* $f=f_1/f_0$. 21, 56
f_1	analytic "numerator" of f, *i.e.* $f=f_1/f_0$. 21, 56
$F_{\mathfrak{p}}$	field extension of $\mathbf{Q}_p$. 153
g	Green's function. 83
Γ	gamma function. 180
γ	Euler's constant. 180
H	relative height of an element of a field F. 154
h	absolute logarithmic height of an algebraic number. 155
$\hat{\mathcal{H}}$	height transform. 54
i	$\sqrt{-1}$. ix
$\mathrm{ilc}(f,a)$	initial Laurent coefficient of the meromorphic function f at the point a. 9

Im	imaginary part. ix
J	Jacobian. 32
k_0	$k_0(\psi)=\int_e^\infty 1/(x\psi(x))\,dx$. 50, 56, 107
L	lenth. 81
$\Lambda(r)$	nonincreasing function with values between 0 and 1. 57, 61
λ	lower (growth) order. 27
λ	positive real number between 0 and 1. 56
λ_a	Weil function with singularity at a. 13
λ_a	Weil function with singularity at a (Nevanlinna's). 14
$\mathring{\lambda}_a$	Weil function with singularity at a (geometric). 15
$\log^+$	$\log^+x=\max\{0,\log x\}$. 14
M	maximum modulus function. 28
m	mean proximity function (Nevanlinna's). 2, 14
m	number theoretic analog of mean proximity. 155
$\mathring{m}$	mean proximity function (geometric). 15
N	counting function (integrated). 1, 12
N	number theoretic analog of counting function. 156
n	counting function (unintegrated). 12
N_1	Nevanlinna's original notation for N_{ram}. 22, 125
$N^{(k)}$	truncated counting function. 13
$n^{(k)}$	truncated counting function (unintegrated). 13
N_{ram}	counting function associated to ramification divisor. 2, 22, 56, 107
n_{ram}	counting function (unintegrated) associated to ramification divisor. 22
O	big "oh". ix
o	little "oh". ix
O_ϵ	big "oh" with implicit constant depending on ϵ. x
Ω	singular area form. 57
ω	Fubini-Study form. 35, 56
$\mathrm{ord}_a f$	order of the function f at the point a. 9
$\mathrm{ord}_a^+ f$	$=\max\{0,\mathrm{ord}_a f\}$. 13
P	Poisson kernel. 6
$\wp$	Weierstrass $\wp$-function. 129
$\mathfrak{p}$	prime ideal. 153
$\mathbf{P}^1$	projective line (or Riemann sphere). 6
Φ	Euclidean form. 36, 56
ϕ	companion to Khinchin function. 50, 56
ψ	Khinchin function. 49, 56, 107
$\mathbf{Q}$	rational numbers. 29
q	An integer – the number of points a_j. 56, 107
q'	An integer – the number of finite points a_j. 107

$\mathbf{Q}_p$	field of p-adic numbers,	153
R	an upper bound on the lengths of all radii under consideration,	6, 56, 107
R	ring of integers,	153
r	the length of a radius, usually $< R$,	6
$\mathbf{R}$	real numbers,	ix
r_0	positive real number such that $\mathring{T}(f, r_0) \geq e$ or $T(f, r_0) \geq e$,	56, 75, 107
Re	real part,	ix
ρ	(growth) order,	27
$\mathbf{R}^n$	real n-space,	ix
S	$S(f, \{a_1, \ldots, a_q\}, r) = (q-2)T(f,r) - \sum_{j=1}^{q} N(f, a_j, r) + N_{\mathrm{ram}}(f,r)$,	123
$S(Z, \epsilon)$	formal sum of small circles around Z,	33
σ	embedding of a field in $\mathbf{C}$,	152
T	characteristic (or height) function (Nevanlinna's),	2, 17, 19
$\mathring{T}$	characteristic (or height) function (geometric),	17, 19, 39
Θ	traditional notation for the truncated defect $\delta^{(1)}$,	125
θ	ramification defect,	125
W	Wronskian,	21, 56
$\mathbf{Z}$	integers,	ix
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