R.E. Edwards

Fourier Series

A Modern Introduction

Volume 1

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PREFACE TO REVISED EDITION (OF VOLUME I)

There are a multitude of minor corrections. In addition there are a few substantial changes in and supplements to the exposition, including some proofs. (Such is the case, for example, with Sections 7.1–7.3.)

Professor Katznelson's book [Kz] is recommended as a companion text.

Many references to *Mathematical Reviews* have been inserted. None of these is essential to an understanding of the main text, and all may be ignored on a first reading. There is an already very large and rapidly increasing literature, and a preliminary glance at a review (often more rapidly accessible than the corresponding original paper) may help more ambitious readers to decide which research papers they wish to study. The list of such references is not claimed to be complete.

I am grateful to Professor Goes for correspondence which has led to a number of additions and improvements. My warmest thanks go to my friend and colleague Dr. Jeff Sanders for his help with the revision.

Finally, my wife earns my gratitude for her help in preparing the revised typescript.

R. E. E.

CANBERRA, January 1979

PREFACE

The principal aim in writing this book has been to provide an introduction, barely more, to some aspects of Fourier series and related topics in which a liberal use is made of modern techniques and which guides the reader toward some of the problems of current interest in harmonic analysis generally. The use of modern concepts and techniques is, in fact, as widespread as is deemed to be compatible with the desire that the book shall be useful to senior undergraduates and beginning graduate students, for whom it may perhaps serve as preparation for Rudin's *Harmonic Analysis on Groups* and the promised second volume of Hewitt and Ross's *Abstract Harmonic Analysis*.

The emphasis on modern techniques and outlook has affected not only the type of arguments favored, but also to a considerable extent the choice of material. Above all, it has led to a minimal treatment of pointwise convergence and summability: as is argued in Chapter 1, Fourier series are not necessarily seen in their best or most natural role through pointwise-tinted spectacles. Moreover, the famous treatises by Zygmund and by Bary on trigonometric series cover these aspects in great detail, while leaving some gaps in the presentation of the modern viewpoint; the same is true of the more elementary account given by Tolstov. Likewise, and again for reasons discussed in Chapter 1, trigonometric series in general form no part of the program attempted.

A considerable amount of space has been devoted to matters that cannot in a book of this size and scope receive detailed treatment. Among such material, much of which appears in small print, appear comments on diverse specialized topics (capacity, spectral synthesis sets, Helson sets, and so forth), as well as remarks on extensions of results to more general groups. The object in including such material is, in the first case, to say enough for the reader to gain some idea of the meaning and significance of the problems involved, and to provide a guide to further reading; and in the second case, to provide some sort of "cultural" background stressing a unity that underlies apparently diverse fields. It cannot be over-emphasized that the book is perforce introductory in all such matters.

The demands made in terms of the reader's active cooperation increase fairly steadily with the chapter numbers, and although the book is surely best regarded as a whole, Volume I is self-contained, is easier than Volume II, and might be used as the basis of a short course. In such a short course, it would be feasible to omit Chapter 9 and Section 10.6.

As to specific requirements made of the reader, the primary and essential item is a fair degree of familiarity with Lebesgue integration to at least the extent described in Williamson's introductory book *Lebesgue Integration*. Occasionally somewhat more is needed, in which case reference is made to Appendix C, to Hewitt and Stromberg's *Real and Abstract Analysis*, or to Asplund and Bungart's *A First Course in Integration*. In addition, the reader needs to know what metric spaces and normed linear spaces are, and to have some knowledge of the rudiments of point-set topology. The remaining results in functional analysis (category arguments, uniform boundedness principles, the closed graph, open mapping, and Hahn-Banach theorems) are dealt with in Appendixes A and B. The basic terminology of linear algebra is used, but no result of any depth is assumed.

Exercises appear at the end of each chapter, the more difficult ones being provided with hints to their solutions.

The bibliography, which refers to both book and periodical literature, contains many suggestions for further reading in almost all relevant directions and also a sample of relevant research papers that have appeared since the publication of the works by Zygmund, Bary, and Rudin already cited. Occasionally, the text contains references to reviews of periodical literature.

My first acknowledgment is to thank Professors Hanna Neumann and Edwin Hewitt for encouragement to begin the book, the former also for the opportunity to try out early drafts of Volume I on undergraduate students in the School of General Studies of the Australian National University, and the latter also for continued encouragement and advice. My thanks are due also to the aforesaid students for corrections to the early drafts.

In respect to the technical side of composition, I am extremely grateful to my colleague, Dr. Garth Gaudry, who read the entire typescript (apart from last-minute changes) with meticulous care, made innumerable valuable suggestions and vital corrections, and frequently dragged me from the brink of disaster. Beside this, the compilation of Sections 13.7 and 13.8 and Subsection 13.9.1 is due entirely to him. Since, however, we did not always agree on minor points of presentation, I alone must take the blame for shortcomings of this nature. To him I extend my warmest thanks.

My thanks are offered to Mrs. Avis Debnam, Mrs. K. Sumeghy, and Mrs. Gail Liddell for their joint labors on the typescript.

Finally, I am deeply in debt to my wife for all her help with the proofreading and her unfailing encouragement.

R. E. E.

CANBERRA, 1967

CONTENTS

CHAPTER 1

Trigonometric Series and Fourier Series

1.1 The Genesis of Trigonometric Series and Fourier Series

1.1.1. The Beginnings. D. Bernoulli, D'Alembert, Lagrange, and Euler, from about 1740 onward, were led by problems in mathematical physics to consider and discuss heatedly the possibility of representing a more or less arbitrary function f with period 2π as the sum of a *trigonometric series* of the form

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{1.1.1}$$

or of the formally equivalent series in its so-called "complex" form

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{1.1.1*}$$

in which, on writing $b_0 = 0$, the coefficients c_n are given by the formulae

$$c_n = \tfrac{1}{2}(a_n - ib_n), \quad c_{-n} = \tfrac{1}{2}(a_n + ib_n) \qquad (n = 0, 1, 2, \cdots).$$

This discussion sparked off one of the crises in the development of analysis.

Fourier announced his belief in the possibility of such a representation in 1811. His book *Théorie Analytique de la Chaleur*, which was published in 1822, contains many particular instances of such representations and makes widespread heuristic use of trigonometric expansions. As a result, Fourier's name is customarily attached to the following prescription for the coefficients a_n, b_n, and c_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \tag{1.1.2}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx, \tag{1.1.2*}$$

the a_n and b_n being now universally known as the "real," and the c_n as the "complex," *Fourier coefficients* of the function f (which is tacitly assumed to be integrable over $(-\pi, \pi)$). The formulae (1.1.2) were, however, known earlier to Euler and Lagrange.

The grounds for adopting Fourier's prescription, which assigns a definite trigonometric series to each function f that is integrable over $(-\pi, \pi)$, will be scrutinized more closely in 1.2.3. The series (1.1.1) and (1.1.1*), with the coefficients prescribed by (1.1.2) and (1.1.2*), respectively, thereby assigned to f are termed the "real" and "complex" *Fourier series* of f, respectively.

During the period 1823–1827, both Poisson and Cauchy constructed proofs of the representation of restricted types of functions f by their Fourier series, but they imposed conditions which were soon shown to be unnecessarily stringent.

It seems fair to credit Dirichlet with the beginning of the rigorous study of Fourier series in 1829, and with the closely related concept of function in 1837. Both topics have been pursued with great vigor ever since, in spite of more than one crisis no less serious than that which engaged the attentions of Bernoulli, Euler, d'Alembert, and others and which related to the prevailing concept of functions and their representation by trigonometric series. (Cantor's work in set theory, which led ultimately to another major crisis, had its origins in the study of trigonometric series.)

1.1.2. The rigorous developments just mentioned showed in due course that there are subtle differences between trigonometric series which converge at all points and Fourier series of functions which are integrable over $(-\pi, \pi)$, even though there may be no obvious clue to this difference. For example, the trigonometric series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{\log n}$$

converges everywhere; but, as will be seen in Exercise 7.7 and again in 10.1.6, it is not the Fourier series of any function that is (Lebesgue-)integrable over $(-\pi, \pi)$.

The theory of trigonometric series in general has come to involve itself with many questions that simply do not arise for Fourier series. For the express purpose of attacking such questions, many techniques have been evolved which are largely irrelevant to the study of Fourier series. It thus comes about that Fourier series may in fact be studied quite effectively without reference to general trigonometric series, and this is the course to be adopted in this book.

The remaining sections of this chapter are devoted to showing that, while Fourier series have their limitations, general trigonometric series have others no less serious; and that there are well-defined senses and contexts in which Fourier series are the natural and distinguished tools for representing functions in useful ways. Any reader who is prepared to accept without question the restriction of attention to Fourier series can pass from 1.1.3 to the exercises at the end of this chapter.

1.1.3. The Orthogonality Relations. Before embarking upon the discussion promised in the last paragraph, it is necessary to record some facts that provide the heuristic basis for the Fourier formulae (1.1.2) and (1.1.2*) and for whatever grounds there are for according a special role to Fourier series.

These facts, which result from straightforward and elementary calculations, are expressed in the following so-called *orthogonality relations* satisfied by the circular and complex exponential functions:

$$\left.\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos mx\cos nx\,dx = \begin{cases}0 & (m \neq n, m \geqslant 0, n \geqslant 0)\\ 1/2 & (m = n > 0),\\ 1 & (m = n = 0)\end{cases}\\
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin mx\sin nx\,dx = \begin{cases}0 & (m \neq n, m \geqslant 0, n \geqslant 0)\\ 1/2 & (m = n > 0),\\ 0 & (m = n = 0)\end{cases}\\
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos mx\sin nx\,dx = 0,\\
&\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{imx}\,e^{-inx}\,dx = \begin{cases}0 & (m \neq n)\\ 1 & (m = n);\end{cases}
\end{aligned}\right\} \quad (1.1.3)$$

in these relations m and n denote integers, and the interval $[-\pi, \pi]$ may be replaced by any other interval of length 2π.

1.2 Pointwise Representation of Functions by Trigonometric Series

1.2.1. Pointwise Representation. The general theory of trigonometric series was inaugurated by Riemann in 1854, since when it has been pursued with vigor and to the great enrichment of analysis as a whole. For modern accounts of the general theory, see [Z_1], Chapter IX and [$Ba_{1,2}$], Chapters XII–XV.

From the beginning a basic problem was that of representing a more or less arbitrary given function f defined on a period-interval I (say the interval $[-\pi, \pi]$) as the sum of at least one trigonometric series (1.1.1), together with a discussion of the uniqueness of this representation.

A moment's thought will make it clear that the content of this problem depends on the interpretation assigned to the verb "to represent" or, what comes to much the same thing, to the term "sum" as applied to an infinite series. Initially, the verb was taken to mean the pointwise convergence of the series at all points of the period interval to the given function f. With the passage of time this interpretation underwent modification in at least two ways. In the first place, the demand for convergence of the series to f at *all* points of the period-interval I was relaxed to convergence at *almost all* points of that interval. In the second place, convergence of the series to f at all or almost all points was weakened to the demand that the series be *summable* to f by one of several possible methods, again at all or almost all points. For the purposes of the present discussion it will suffice to speak of just one such summability method, that known after Cesàro, which consists of replacing the partial sums

$$\begin{aligned} s_0(x) &= \tfrac{1}{2}a_0, \\ s_N(x) &= \tfrac{1}{2}a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \qquad (N = 1, 2, \cdots) \end{aligned} \tag{1.2.1}$$

of the series (1.1.1) by their arithmetic means

$$\sigma_N = \frac{s_0 + \cdots + s_N}{N+1} \qquad (N = 0, 1, 2, \cdots). \tag{1.2.2}$$

Thus we shall say that the series (1.1.1) is summable at a point x to the function f if and only if

$$\lim_{N\to\infty} \sigma_N(x) = f(x).$$

It will be convenient to group all these interpretations of the verb "to represent" under the heading of *pointwise representation* (everywhere or almost everywhere, by convergence or by summability, as the case may be) of the function f by the series (1.1.1).

In terms of these admittedly rather crude definitions we can essay a bird's-eye view of the state of affairs in the realm of pointwise representation, and in particular we can attempt to describe the place occupied by Fourier series in the general picture.

1.2.2. Limitations of Pointwise Representation. Although it is undeniably of great intrinsic interest to know that a certain function, or each member of a given class of functions, admits a pointwise representation by some trigonometric series, it must be pointed out without delay that this type of representation leaves much to be desired on the grounds of *utility*. A mode of representation can be judged to be successful or otherwise useful as a tool in subsequent investigations by estimating what standard analytical operations applied to the represented function can, via the representation, be expressed with reasonable simplicity in terms of the expansion coefficients a_n and b_n. This is, after all, one of the main reasons for seeking a representation in series form. Now it is a sad fact that pointwise representations are in themselves not very useful in this sense; they are simply too weak to justify the termwise application of standard analytical procedures.

Another inherent defect is that a pointwise representation at almost all points of I is never unique. This is so because, as was established by Men'shov in 1916, there exist trigonometric series which converge to zero almost everywhere and which nevertheless have at least one nonvanishing coefficient; see 12.12.8. (That this can happen came as a considerable surprise to the mathematical community.)

1.2.3. **The Role of the Orthogonality Relations.** The a priori grounds for expecting the Fourier series of an integrable function f to effect a pointwise representation of f (or, indeed, to effect a representation in any reasonable sense) rest on the orthogonality relations (1.1.3). It is indeed a simple consequence of these relations that, if there exists *any* trigonometric series (1.1.1) which represents f in the pointwise sense, and if furthermore the s_N (or the σ_N) converge dominatedly (see [W], p. 60) to f, then the series (1.1.1) must be the Fourier series of f. However, the second conditional clause prevents any very wide-sweeping conclusions being drawn at the outset.

As will be seen in due course, the requirements expressed by the second conditional clause are fulfilled by the Fourier series of sufficiently smooth functions f (for instance, for those functions f that are continuous and of bounded variation). But, alas, the desired extra condition simply does not obtain for more general functions of types we wish to consider in this book. True, a greater degree of success results if convergence is replaced by summability (see 1.2.4). But in either case the investigation of this extra condition itself carries one well into Fourier-series lore. This means that this would-be simple and satisfying explanation for according a dominating role to Fourier series can scarcely be maintained at the *outset* for functions of the type we aim to study.

1.2.4. **Fourier Series and Pointwise Representations.** What has been said in 1.2.3 indicates that Fourier series can be expected to have but limited success in the pointwise representation problem. Let us tabulate a little specific evidence.

The Fourier series of a periodic function f which is continuous and of bounded variation converges boundedly at all points to that function. The Fourier series of a periodic continuous function may, on the contrary, diverge at infinitely many points; even the pointwise convergence almost everywhere of the Fourier series of a general continuous function remained in doubt until 1966 (see 10.4.5), although it had been established much earlier and much more simply that certain fixed subsequences of the sequence of partial sums of the Fourier series of any such function is almost everywhere convergent to that function (the details will appear in Section 8.6). The Fourier series of an integrable function may diverge at all points.

If ordinary convergence be replaced by summability, the situation improves. The Fourier series of a periodic continuous function is uniformly summable to that function. The Fourier series of any periodic integrable function is summable at almost all points to that function, but in this case neither the s_N nor the σ_N need be dominated.

1.2.5. Trigonometric Series and Pointwise Representations. Having reviewed a few of the limitations of Fourier series vis-à-vis the problem of pointwise representation, we should indicate what success is attainable by using trigonometric series in general.

In 1915 both Lusin and Privalov established the existence of a pointwise representation almost everywhere by summability methods of any function f which is measurable and finite almost everywhere. For 25 years doubts lingered as to whether summability could here be replaced by ordinary convergence, the question being resolved affirmatively by Men'shov in 1940. This result was sharpened in 1952 by Bary, who showed that, if the function f is measurable and finite almost everywhere on the interval I, there exists a continuous function F such that $F'(x) = f(x)$ at almost all points of I, and such that the series obtained by termwise differentiation of the Fourier series of F converges at almost all points x of I to $f(x)$. Meanwhile Men'shov had in 1950 shown also that to any measurable f (which may be infinite on a set of positive measure) corresponds at least one trigonometric series (1.1.1) whose partial sums s_N have the property that $\lim_{N\to\infty} s_N = f$ in *measure* on I. This means that one can write $s_N = u_N + v_N$, where u_N and v_N are finite-valued almost everywhere, $\lim_{N\to\infty} u_N(x) = f(x)$ at almost all points x of I, and where, for any fixed $\varepsilon > 0$, the set of points x of I for which $|v_N(x)| > \varepsilon$ has a measure which tends to zero as $N \to \infty$. (The stated condition on the v_N is equivalent to the demand that

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \frac{|v_N|\,dx}{1 + |v_N|} = 0;$$

and the circuitous phrasing is necessary because f may take infinite values on a set of positive measure.) This sense of representation is weaker than pointwise representation. For more details see [Ba_2], Chapter XV.

These theorems of Men'shov and Bary lie very deep and represent enormous achievements. However, as has been indicated at the end of 1.2.2, the representations whose existence they postulate are by no means unique.

Cantor succeeded in showing that a representation at *all* points by a convergent trigonometric series is necessarily unique, if it exists at all. Unfortunately, only relatively few functions f admit such a representation: for instance, there are continuous periodic functions f that admit no such representation. (This follows on combining a theorem due to du Bois-Reymond and Lebesgue, which appears on p. 202 of [Ba_1], with results about Fourier series dealt with in Chapter 10 of this book.) It is indeed the case that, in a sense, "most" continuous functions admit no representation of this sort.

1.2.6. **Summary.** It can thus be said in summary that pointwise representations are subject to inherent limitations as analytical tools, and that Fourier series can be accorded a distinguished role in respect of this type of representation only for functions of a type more restricted than one might hope to handle.

This being so, it is natural to experiment by varying the meaning assigned to the verb "to represent" in the hope of finding a more operationally effective meaning and of installing Fourier series in a more dominating role.

Before embarking on this program, it is perhaps of interest to add that a similar choice prevails in the interpretation of differentiation (which in fact has connections with the representation problem). The pointwise everywhere or almost everywhere interpretation of the derivative, if deprived of any further qualification, is also not entirely effective operationally. A new interpretation is possible and leads to distributional concepts; Chapter 12 is devoted to this topic.

1.3 New Ideas about Representation

1.3.1. **Plan of Action.** In the preceding section we have recounted some of the difficulties in the way of according a unique position to Fourier series on the grounds of their behavior in relation to the traditionally phrased problem of representing functions by trigonometric series. We have also indicated the shortcomings of this type of representation.

To this it may be added that in cases where the mathematical model of a physical problem suggests the use of expansions in trigonometric series, pointwise representations frequently do not correspond very closely to the physical realities.

Faced with all this, we propose to consider new meanings for the verb "to represent" that are in complete accord with modern trends, and which will in due course be seen to justify fully a concentration on Fourier series as a representational device.

1.3.2. **Different Senses of Convergence and Representation.** In recent times analysts have become accustomed to, and adept at working in diverse fields with, other meanings for the verb "to represent," most of which (and all of which we shall have occasion to consider) are tantamount to novel ways in which a series of functions may be said to converge. Such ideas are indeed the concrete beginnings of general topology and the theory of topological linear spaces.

Thus encouraged, we contemplate some possible relationships between an integrable function f on $(-\pi, \pi)$ and a trigonometric series (1.1.1) or (1.1.1*) expressed by each of equations (A) to (D) below.

For this purpose we write again

$$s_0(x) = \tfrac{1}{2}a_0, \quad s_N(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx),$$

so that

$$s_N(x) = \sum_{|n| \leqslant N} c_n e^{inx}, \tag{1.3.1}$$

and also

$$\sigma_N(x) = \frac{s_0(x) + \cdots + s_N(x)}{N+1}.$$

The relationships referred to are (compare 6.1.1, 6.2.6, 12.5.3, and 12.10.1):

(A) $$\lim_{N \to \infty} \int_{-\pi}^{\pi} |f(x) - \sigma_N(x)|\, dx = 0;$$

(B) $$\lim_{N \to \infty} \int_{-\pi}^{\pi} |f(x) - s_N(x)|^p\, dx = 0;$$

(C) $$\lim_{N \to \infty} \sup_x |f(x) - \sigma_N(x)| = 0;$$

(D) $$\lim_{N \to \infty} \int_{-\pi}^{\pi} u(x) s_N(x)\, dx = \int_{-\pi}^{\pi} u(x) f(x)\, dx$$

for each indefinitely differentiable periodic function u.

If any one of these relations holds for a given f and a given trigonometric series, one may say that the trigonometric series represents f in the corresponding sense: in case (A) it would be usual to say that the trigonometric series is *Cesàro-summable in mean with exponent* (or *index*) 1 *to* f; in case (B) that the trigonometric series is *convergent in mean with exponent* (or *index*) p *to* f; in case (C) that the trigonometric series is *uniformly Cesàro-summable to* f; and in case (D) that the trigonometric series is *distributionally convergent to* f.

1.3.3. The Role of Fourier Series. It is genuinely simple to verify that, given f, there is *at most one* trigonometric series for which any one of relations (A) to (D) is true, and that this only contender is the Fourier series of f (see the argument in 6.1.3). Moreover, it is true that the relations *do* hold if the trigonometric series is the Fourier series of f, provided in case (B) that *either* $1 < p < \infty$ and $f \in \mathbf{L}^p$ *or* $p = 1$ and $f \log^+ |f| \in \mathbf{L}^1$ (see 8.2.1, 12.10.1, and 12.10.2); and in case (C) that f is continuous and periodic. (The symbols $\mathbf{L}^1$ and $\mathbf{L}^p$ here denote the sets of measurable functions f on $(-\pi, \pi)$ such that $|f|$ and $|f|^p$, respectively, are Lebesgue-integrable over $(-\pi, \pi)$. A tiny modification to this definition is explained in detail in 2.2.4 and will be adopted thereafter in this book.)

Each of the relations (A) to (D) can, therefore, be used to characterize the Fourier series of f under the stated conditions, and each provides some justification for singling out the Fourier series for study. (There are, by the way, numerous other relationships that might be added to the list.)

It turns out that the weakest relationship (D) is suggestive of fruitful generalizations of the concept of Fourier series of such a type that the distinction between Fourier series and trigonometric series largely disappears. It suggests in fact the introduction of so-called *distributions* or *generalized functions* in the manner first done by L. Schwartz [$S_{1,2}$]. It will then appear that any trigonometric series in which $c_n = O(|n|^k)$ for some k may be regarded as the Fourier series of a distribution, to which this series is distributionally convergent. These matters will be dealt with in Chapter 12.

1.3.4. **Summary.** The substance of Section 1.2 and 1.3.3 summarizes the justification for subsequent concentration of attention on Fourier series in particular, at least insofar as reference is restricted to harmonic analysis in its classical setting. We shall soon embark on a program that will include at appropriate points a verification of each of the unproved statements upon which this justification is based. As for trigonometric series in general, we shall do no more than pause occasionally to mention a few of the simpler results that demand no special techniques.

A bird's-eye view of many of the topics to be discussed at some length in this book is provided by the survey article G. Weiss [1].

1.3.5. **Fourier Series and General Groups.** There are still other reasons in favor of the chosen policy which are based upon recent trends in analysis. Harmonic analysis has not remained tied to the study of Fourier series of periodic functions of a real variable; in particular it is now quite clear that Fourier-series theory has its analogue for functions defined on compact Abelian groups (and even, to some extent, on still more general groups); see, for example, [HR], [Re], [E_1]. While the level at which this book is written precludes a detailed treatment of such extensions, we shall make frequent reference to modern developments. However regrettable it may seem, it is a fact that these developments cluster around the extension of precisely those portions of the classical theory which do not depend upon the deeper properties of pointwise convergence and summability, and that a detailed treatment of the analogue for compact groups of the theory of general trigonometric series appears to lie in the future. Moreover, the portions of the classical theory that have so far been extended appear to be those most natural for handling those problems which are currently the center of attention in general harmonic analysis. Of course, these prevailing features may well change with the passage of time. While they prevail, however, they add support to the view that it is reasonable to accord some autonomy to a theory in which the modes of representation mentioned in 1.3.2 take precedence over that of pointwise representation.

EXERCISES

1.1. Establish the formulae

$$D_N(x) = \sum_{|n| \leqslant N} e^{inx} = \frac{\sin (N + \frac{1}{2})x}{\sin \frac{1}{2}x},$$
$$F_N(x) = (N + 1)^{-1} [D_0(x) + \cdots + D_N(x)]$$
$$= (N + 1)^{-1} \left[\frac{\sin \frac{1}{2}(N + 1)x}{\sin \frac{1}{2}x}\right]^2$$

for $N \geqslant 0$ an integer and $x \not\equiv 0$ modulo 2π, where the equality signs immediately following $D_N(x)$ and $F_N(x)$ are intended as definitions for all real x.

1.2. Prove that if p and q are integers and $p < q$, and if $x \not\equiv 0$ modulo 2π, then

$$\Big|\sum_{p \leqslant n \leqslant q} e^{inx}\Big| \leqslant |\operatorname{cosec} \tfrac{1}{2}x|.$$

By using partial summation (see 7.1.2 and [H], p. 97 ff.) deduce that if $c_p \geqslant c_{p+1} \geqslant \cdots \geqslant c_q \geqslant 0$, then, for $x \not\equiv 0$ modulo 2π,

$$\Big|\sum_{p \leqslant n \leqslant q} c_n e^{inx}\Big| \leqslant c_p\, |\operatorname{cosec} \tfrac{1}{2}x|.$$

1.3. Assume that $c_n \geqslant c_{n+1}$ and $\lim_{n \to \infty} c_n = 0$. Show that the series

$$\sum_{n=0}^{\infty} c_n e^{inx}$$

is convergent for $x \not\equiv 0$ modulo 2π, and that the convergence is uniform on any compact set of real numbers x which contains no number $\equiv 0$ modulo 2π.

1.4. Assume that $c_n \geqslant c_{n+1} \geqslant 0$ and $nc_n \leqslant A$. Show that

$$\Big|\sum_{n=1}^{N} c_n \sin nx\Big| \leqslant A(\pi + 1).$$

Hints: One may assume $0 < x < \pi$. Put $m = \min (N, [\pi/x])$ and split the sum into $\sum_1^m + \sum_{m+1}^N$, an empty sum being counted zero. Estimate the partial sums separately, using Exercise 1.2 for $\sum_{m+1}^N$.

1.5. Assume that the c_n are as in Exercise 1.4. Show that the series $\sum_{n=1}^{\infty} c_n \sin nx$ is boundedly convergent, and that the sum function is continuous, except perhaps at the points $x \equiv 0$ modulo 2π. (More general results will appear in Chapter 7.)

1.6. Compute the complex Fourier coefficients of the following functions, each defined by the prescribed formula over $[-\pi, \pi)$ and defined elsewhere so as to have period 2π:

(1) $f(x) = x$;

(2) $f(x) = |\sin x|$;

(3) $f(x) = x$ for $-\pi \leqslant x \leqslant 0, f(x) = 0$ for $0 < x < \pi$.

1.7. By a *trigonometric polynomial* is meant a function f admitting at least one expression of the form

$$f(x) = \sum_{|n| \leqslant N} c_n e^{inx},$$

where the c_n are f-dependent complex numbers.

(1) Use the orthogonality relations to show that, if f is a trigonometric polynomial, then

$$\hat{f}(n) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

vanishes for all but a finite number of integers n and that $f(x) = \sum_{n \in Z} \hat{f}(n) e^{inx}$. Show also that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx = \sum_{n \in Z} |\hat{f}(n)|^2$$

whenever f is a trigonometric polynomial. (This is a special case of Parseval's formula, to which we shall return in Chapter 8 and Section 10.5; see also Remark 6.2.7.)

A trigonometric polynomial f such that $\hat{f}(n) = 0$ for $|n| > N$ is said to be *of degree at most N*.

(2) Verify that the set $\mathbf{T}_N$ of trigonometric polynomials of degree at most N forms a complex linear space of dimension $2N + 1$ with respect to pointwise operations, and that if $f \in \mathbf{T}_N$, then also $\operatorname{Re} f \in \mathbf{T}_N$ and $\operatorname{Im} f \in \mathbf{T}_N$.

(3) Show that if $f \in \mathbf{T}_N$, $f \neq 0$, then f admits at most $2N$ zeros (counted according to multiplicity) in the interval $[0, 2\pi)$ (or in any interval congruent modulo 2π to this one).

1.8. (Stečkin's lemma) Suppose $f \in \mathbf{T}_N$ is real-valued, and that

$$\|f\|_\infty \equiv \sup_x |f(x)| = M = f(x_0).$$

Prove that

$$f(x_0 + y) \geqslant M \cos Ny \qquad \text{for } |y| \leqslant \frac{\pi}{N}.$$

Hints: Put $g(y) = f(x_0 + y) - M \cos Ny$. Assuming the assertion false, we choose y_0 so that $|y_0| < \pi/N$ and $g(y_0) < 0$. We assume $0 < y_0 < \pi/N$; otherwise the subsequent argument proceeds with the interval $[-2\pi, 0)$ in place of $[0, 2\pi)$. By examining closely the signs of g at the points $k\pi/N$ ($k = 0, 1, 2, \cdots, 2N$), show that g admits at least $2N + 1$ zeros in $[0, 2\pi)$. A contradiction results from Exercise 1.7.

1.9. (Bernstein's inequality) Prove that if $f \in \mathbf{T}_N$, then $\|f'\|_\infty \leqslant N \|f\|_\infty$ (the notation being as in the preceding exercise).

Hints: It suffices, by Exercises 1.7 and 1.10, to prove the inequality for real-valued $f \in \mathbf{T}_N$. If $f'(x_0) = m \equiv \|f'\|_\infty$ (which can be arranged by changing f into $-f$ if necessary) and $M = \|f\|_\infty$, Exercise 1.8 gives $f'(x_0 + y) \geqslant m \cos Ny$ for $|y| \leqslant \pi/N$. Integrate this inequality.

Notes: Many other proofs are known; the above, due to Stečkin, is perhaps the simplest. For a proof based upon interpolation methods, see [Z_2], p. 11. More general results, also due to Bernstein, apply to entire functions of order one and exponential type; see [Z_2], p. 277.

See also the approach in [Kz], p. 17; W. R. Bloom [1], [2]; MR **51** # 1239; **52** ## 6288, 11446; **53** # 11289; **54** # 829.

The inequality has also been extended in an entirely different way by Privalov, who showed that if $I = (a', b')$ and $J = (a, b)$ are any two subintervals of $[-\pi, \pi]$ satisfying $a < a' < b' < b$, then there exists a number $c(I, J)$ such that

$$\sup_{x \in I} |f'(x)| \leqslant c(I, J)N \cdot \sup_{x \in J} |f(x)|$$

for any $f \in \mathbf{T}_N$. It is furthermore established that similarly (but perhaps with a different value for $c(I, J)$) one has

$$\Big\{\int_I |f'(x)|^p \, dx\Big\}^{1/p} \leqslant c(I, J)N \cdot \Big\{\int_J |f(x)|^p \, dx\Big\}^{1/p}$$

for any $f \in \mathbf{T}_N$ and any p satisfying $1 \leqslant p < \infty$. Both inequalities are also valid when $I = J = [-\pi, \pi]$ and $c(I, J) = 1$, the first reducing to that of Bernstein and the second being in this case due to Zygmund. For more details, see [Ba_2], pp. 458–462. See also [L_2], Chapter 3.

1.10. Suppose that $\mathbf{E}$ is a complex linear space of complex-valued functions on a given set (pointwise operations), that $\mathbf{E} = \mathbf{E}_0 + i\mathbf{E}_0$ where $\mathbf{E}_0$ is the set of real-valued functions in $\mathbf{E}$, that l is a complex-linear functional on $\mathbf{E}$ which is real-valued on $\mathbf{E}_0$, and that p is a seminorm on $\mathbf{E}$ (see Appendix B.1.2). Suppose finally that $p(x) \leqslant p(y)$ whenever $x, y \in \mathbf{E}$ and $|x| \leqslant |y|$, and that $|l(x)| \leqslant p(x)$ for $x \in \mathbf{E}_0$. Prove that $|l(x)| \leqslant p(x)$ for $x \in \mathbf{E}$.

Hints: Write $x = a + ib$ with $a, b \in \mathbf{E}_0$ and $l(x) = r(\alpha + i\beta)$ with $r \geqslant 0$, α and β real, and $\alpha^2 + \beta^2 = 1$. Then

$$|l(x)| = r = (\alpha - i\beta)l(x) = l[(\alpha - i\beta)(a + ib)];$$

expanding and taking real parts: $|l(x)| = l(\alpha a + \beta b) \leqslant p(\alpha a + \beta b)$, and so forth.

1.11. Prove that, if a trigonometric polynomial f is real-valued and nonnegative, then $f = |g|^2$ for some trigonometric polynomial g (Fejér and F. Riesz).

Hints: Suppose $f(x) \equiv \sum_{|n| \leqslant N} c_n e^{inx}$ and consider first the case in which $f(x) > 0$ for all x. Assume (without loss of generality) that $c_{-N} \neq 0$ and examine the polynomial $P(z) = z^N \sum_{|n| \leqslant N} c_n z^n$. Observe that $P(z) = z^{2N}\overline{P(\bar{z}^{-1})}$ and $f(x) = e^{-iNx}P(e^{ix})$. Verify that the zeros of P are of the form $a_1, a_2, \cdots$, and $\bar{a}_1^{-1}, \bar{a}_2^{-1}, \cdots$, where $0 < |a_r| < 1$, and factorize P accordingly.

In case one knows merely that $f \geqslant 0$, apply the above to the $f_k = f + 1/k$ $(k = 1, 2, \cdots)$ and use a limiting argument.

Remarks. The theorem does not extend in the expected way to other groups; see [R], 8.4.5.

CHAPTER 2

Group Structure and Fourier Series

The aim of the first two sections of this chapter is to show how and to what extent the topological group structure of the set R of real numbers, and of some of its subgroups and quotient groups, lead to the study of periodic functions, the complex exponential functions, and the problem of expansions in trigonometric series in general and Fourier expansions in particular. In the remaining sections of this chapter we shall begin the study of Fourier coefficients in some detail.

In pursuing the aims of Sections 2.1 and 2.2 we are led to refer to fairly general topological groups and to constructs related to them. It is hoped thus to convey a very rough idea of how the classical theory of Fourier series fits into contemporary developments in parts of analysis, and to prepare the reader for a later perception of genuine structural unity underlying obvious similarities. It is of course not expected, nor is it necessary for an understanding of subsequent developments in this book, that the reader should at this stage stop to gather the details concerning topological groups and the related concepts to be spoken of (duality, invariant integrals, and so on); this writer will indeed venture the opinion that the return to a detailed study of generalizations is best made *after* some familiarity with special cases has been attained. On the other hand the reader will, it is hoped, gain from the realization that the classical theory is tributary to a broader stream, and will in due course want to try his hand at exploring the latter with the help of the references cited in this chapter.

2.1 Periodic Functions

For any reasonable interpretation of the term "represent" (see Chapter 1), any function of a real variable which is to be globally representable by a trigonometric series must admit 2π as a period, or must do this after suitable correction on a null set. Insofar as such correction does not alter the Fourier series of the function, we may and will assume that all functions of a real variable have period 2π. (Representation over a restricted range by so-called "half-range series" does not in any way conflict with this convention.)

2.1.1. **The Groups R and T.** The set R of real numbers, taken with addition as its law of composition and with its usual metric topology, is an example of an Abelian topological group. This means that it is first an Abelian group, and second a topological space, and that moreover the algebra and the topology are so related that the mapping $(x, y) \to x - y$ is continuous from $R \times R$ into R. If one drops the demand that the group structure be Abelian, one has here the concept of a *topological group* in general; see [B], pp. 98 ff., and/or [HR], Chapter II. Hereinafter the term "group" will always mean "locally compact group whose topology satisfies the Hausdorff separation axiom." This particular topological group R is locally compact but noncompact. We wish to focus attention, not so much on R, as on quotient groups thereof.

It is a simple matter to show that the only closed subgroups of R, other than $\{0\}$ and R itself, are those consisting of all integer multiples of some nonzero positive number (see Exercise 2.1). Which of these is selected is largely immaterial: we choose that one which is formed of all integer multiples of 2π and which is hereinafter denoted by $2\pi Z$ (Z denoting the additive group of integers).

Let us form the quotient group $R/2\pi Z = T$ and denote by p the natural projection of R onto T, which assigns to $x \in R$ the coset $\dot{x} = x + 2\pi Z$ containing x. The group T is made into a topological group by endowing it with the so-called *quotient topology*. In concrete terms, this means that the open sets in T are precisely the sets $p(U)$ where U is open in R. Even more concretely put, the quotient topology on T is that defined by the metric $d(\dot{x}, \dot{y}) = \inf\{|x - y + 2n\pi| : n \in Z\}$.

Another way of looking at T is to recognize that the mapping $\dot{x} \to \exp(ix)$ is an isomorphism of T onto the multiplicative group of complex numbers having unit absolute value. In this isomorphism, the quotient topology corresponds to that induced on the unit circumference in the complex plane by the usual metric topology on the latter. In view of this, the group T is often referred to as the *circle group* or the *one-dimensional torus group*.

Perhaps the most essential difference between R and T is that the latter is compact. Were we to attempt to apply to R the subsequent considerations concerning T, we should be led to Fourier integrals in place of Fourier series; almost all the additional difficulties thereby encountered would stem from the fact that R is noncompact.

2.1.2. **Periodic Functions.** If f is a periodic function on R (by "periodic" we shall always mean "with period 2π"), there is just one function $\dot{f}$ on T such that $f = \dot{f} \circ p$. (Notice that we shall *never* speak of so-called "many-valued functions.") Conversely, every periodic function f on R is expressible in this way. Moreover, in this one-to-one correspondence $f \leftrightarrow \dot{f}$, continuous

f's correspond to continuous $\dot{f}$'s. It will in fact be the case that this correspondence preserves every structure relevant to our purpose, and we shall before long ask the reader to make a mental identification of f and $\dot{f}$.

It is also perfectly legitimate to regard functions on the circle group as functions of the complex variable $z = e^{ix}$ having unit absolute value, but we shall make no systematic use of this notation.

2.1.3. **Role of the Group Structure.** As we shall see in Section 2.2, the topological group structure of T is inextricably bound up with the genesis and study of Fourier series. Indeed, it will slowly emerge that many of the most fundamental aspects of this study depend almost exclusively on the fact that T is a compact Abelian group. It will be seen, too, that the Lebesgue integral itself is determined (up to a nonzero constant factor of proportionality) by the topological group structure.

To this basic ingredient may be added, for the sake of richness and refinement, more specialized structures and concepts—the concepts of bounded variation and differentiability for functions, for example. In line with the remarks in 2.1.2, we say that a function f on T is of class $\mathbf{C}^k$ ($=k$ times continuously differentiable, or indefinitely differentiable if $k = \infty$), or is of bounded variation, if and only if the function $f \circ p$ on R has the corresponding property on some one (and therefore every) interval in R of length 2π.

2.2 Translates of Functions. Characters and Exponentials. The Invariant Integral

2.2.1. **Translates and Characters.** We pose the question: What are the fundamental reasons for considering expansions in terms of cosines and sines $\cos \lambda x$ and $\sin \lambda x$, or, equivalently, in terms of the complex exponentials $e^{i\lambda x}$?

The historical answer, which is also the one based on applications, might be that these functions are the eigenfunctions of particularly simple linear differential operators. The restriction of the continuous parameter λ to the discrete range $2\pi Z$ reflects periodic boundary conditions.

There is, however, another and even more fundamental explanation, which hinges only on the topological group structure of R and T. Let us look into this.

The simplest and most obvious way in which the group structure can be used in handling functions is via the *translation operators* T_a (a = a group element) acting on functions according to the rule

$$T_a f(x) = f(x - a).$$

Attention paid to the T_a is justified in retrospect, because most of the linear operators featuring in harmonic analysis prove to be limits in some sense of linear combinations of translation operators (see, for example, 3.1.9 and Chapter 16).

To fix ideas, we visualize the T_a as acting on the linear space $\mathbf{C} = \mathbf{C}(R)$ or $\mathbf{C}(T)$ of continuous, complex-valued functions on R or T as the case may be. (Almost all we have to say would remain true on replacing $\mathbf{C} = \mathbf{C}(R)$ or $\mathbf{C}(T)$ by various other function spaces over R or T.)

If $f \in \mathbf{C}$, then $T_a f \in \mathbf{C}$. Each T_a is indeed an automorphism of the linear space $\mathbf{C}$. To this we add for future reference the relations

$$T_0 = I, \qquad T_{a+b} = T_a T_b, \qquad T_{-a} = T_a^{-1} \tag{2.2.1}$$

where I denotes the identity automorphism of $\mathbf{C}$.

In general, and certainly for the groups R and T here considered, the space $\mathbf{C}$ is infinite-dimensional and the problem of analyzing the behavior of the operators T_a on $\mathbf{C}$ is a complicated one. However, elementary linear algebra (and, even more so, suitable forms of the simultaneous spectral resolution theorem) encourage one to hope for simplification if one can "reduce" the problem by finding linear subspaces $\mathbf{V}$ of $\mathbf{C}$ which are invariant in the sense that $T_a(\mathbf{V}) \subset \mathbf{V}$ for all group elements a. For brevity we term such a $\mathbf{V}$ an *invariant subspace.* The hope would lie in decomposing $\mathbf{C}$ into some sort of (possibly infinite) direct sum of invariant subspaces $\mathbf{V}_1, \mathbf{V}_2, \cdots$, each $\mathbf{V}_i$ being as small as possible. The T_a could then be examined on each $\mathbf{V}_i$ separately.

In this way one is led to consider the existence of minimal invariant subspaces $\mathbf{V}$ of $\mathbf{C}$, "minimal" meaning that $\mathbf{V}$ contains properly no invariant subspace other than $\{0\}$. Now it is evident that a one-dimensional invariant subspace $\mathbf{V}$ (if any such there be) is certainly minimal; and that such a subspace $\mathbf{V}$ is generated or spanned by a function f which is a simultaneous eigenvector of the T_a (if any such functions exist). So, without more ado, we seek such functions. (For non-Abelian groups in general there would not exist any one-dimensional invariant subspaces—one would have to be content with seeking finite-dimensional ones, which in fact exist in abundance for compact groups; for noncompact, non-Abelian groups, the situation is even more complicated.)

Given $f \in \mathbf{C}$, denote by $\mathbf{V}_f$ the smallest invariant subspace containing f, that is, the set of all finite linear combinations of translates $T_a f$ of f. We seek functions f such that $\dim \mathbf{V}_f = 1$. Clearly, therefore, $f \neq 0$ and to each group element a corresponds a complex scalar $\chi(-a)$ such that

$$T_a f = \chi(-a) f.$$

This signifies that

$$f(x - a) = \chi(-a) f(x) \tag{2.2.2}$$

for all pairs (a, x). If $x = 0, f(-a) = f(0)\,\chi(-a)$, which shows in particular that χ is continuous and that $\chi \neq 0$. On the other hand, (2.2.1) and (2.2.2) yield the functional equation

$$\chi(a + b) = \chi(a)\,\chi(b). \tag{2.2.3}$$

As a matter of definition, a complex-valued function $\chi \neq 0$ that satisfies (2.2.3) is termed a *character* of the group in question. It follows at once that χ is nonvanishing, $\chi(0) = 1$, and $\chi(-a) = \chi(a)^{-1}$. We shall have occasion to consider only characters that are continuous. If a character χ is bounded, then (2.2.3) shows that $|\chi(a)| = 1$ for all group elements a, so that χ defines a homomorphism of the group into the multiplicative group of complex numbers of absolute value 1.

Returning to (2.2.2), we may say that the function χ appearing there is a continuous character. Moreover, since (2.2.2) gives $f(-a) = \chi(-a)f(0)$ for all a, it follows that $c = f(0)$ is nonzero, and $f = c\chi$ is thus a nonzero scalar multiple of the character χ.

Let us next determine explicitly all the continuous characters of R and of T. Concerning characters that are not assumed to be continuous, see Exercise 3.19.

Supposing that χ is a continuous character of R, we integrate the relation (2.2.3) with respect to b, over an interval $(0, h)$, to get

$$\int_0^h \chi(a + b)\,db = \chi(a) \cdot \int_0^h \chi(b)\,db.$$

Since χ is continuous and $\chi(0) = 1$, h may be chosen and fixed so that the factor

$$\int_0^h \chi(b)\,db$$

is nonzero. Moreover,

$$\int_0^h \chi(a + b)\,db = \int_a^{a+h} \chi(c)\,dc.$$

Again since χ is continuous, this last expression is a differentiable function of a. It follows that χ is differentiable. Knowing this, we find that (2.2.3) yields

$$\begin{aligned}\chi'(a) &= \lim_{h\to 0} \frac{\chi(a + h) - \chi(a)}{h} \\ &= \lim_{h\to 0} \frac{\chi(h) - \chi(0)}{h} \cdot \chi(a),\end{aligned}$$

so that χ satisfies the differential equation

$$\chi' = ik\,\chi, \tag{2.2.4}$$

where $k = -i\chi'(0)$. The only solution of (2.2.4) taking the value 1 at the origin is

$$\chi(x) = e^{ikx}. \tag{2.2.5}$$

Evidently, whatever the complex number k, (2.2.5) defines a continuous character of R. This character is bounded, if and only if k is real.

To determine the continuous characters of T, it is merely necessary to add the demand that χ have period 2π. This signifies that $k \in Z$.

To sum up, we find that

(1) The continuous (and so necessarily bounded) characters of T are in one-to-one correspondence with Z, the character corresponding to $n \in Z$ being (derived by passage to the quotient from) the function

$$e_n(x) = e^{inx}. \tag{2.2.6}$$

Corresponding to $n = 0$ is the character e_0, which is the constant function 1; this is usually termed the *principal character*.

(2) The one-dimensional invariant subspaces of $\mathbf{C}(T)$ are precisely the subspaces $\mathbf{V}_n = \{\lambda e_n : \lambda \text{ a scalar}\}$, where n ranges over Z.

(3) The problem of harmonic analysis in respect to $\mathbf{C}(T)$ (and similarly in respect of other function spaces) may be suggestively but perhaps oversimply described as that of expressing $\mathbf{C}(T)$ as some sort of direct sum of the subspaces V_n ($n \in Z$). This task falls into two parts:

(a) Given $f \in \mathbf{C}(T)$, it is required to determine the corresponding "components" of f lying in the various subspaces $\mathbf{V}_n$. This is, strictly speaking, the problem of *harmonic* (or *spectral*) *analysis* and is, in the case of compact Abelian groups anyway, relatively simple. The said components are just the functions $\hat{f}(n)e_n$, where

$$\hat{f}(n) = \frac{1}{2\pi} \int f(x)\overline{e_n(x)}\, dx.$$

It will appear in Chapter 11 that the component $\hat{f}(n)e_n$ is nonzero, if and only if $\mathbf{V}_n \cap \mathbf{V} \neq \{0\}$, where $\mathbf{V}$ is the closed invariant subspace generated by f.

(b) The study of the formula

$$f = \sum_{n \in Z} \hat{f}(n)e_n,$$

which it is hoped will reconstitute f from its harmonic components. This may be described as the problem of the *harmonic* (or *spectral*) *synthesis* of f. It presents what is by far the more difficult part of the program and embraces, of course, the question of representing f by a trigonometric series. It must be stressed that such a series representation is indeed generally impossible in $\mathbf{C}$, if one demands pointwise convergence. The study of the sense in which the synthesis is valid (which will vary from one function space to another) is an essential part of the problem before us; see the remarks in 10.3.6 and Section 16.8.

In connection with (1) above, it is interesting to observe that the group structure of Z corresponds, when Z is used to label the characters e_n, to pointwise multiplication of characters. Moreover, the corresponding "dual topology" on Z is that for which the relation $n \to n_0$ signifies that

$$e_n(x) \to e_{n_0}(x)$$

uniformly for $x \in T$, and turns out to be just the discrete topology on Z (having a base of neighborhoods of $0 \in Z$ comprising the one set $\{0\}$). This is a general feature: the bounded continuous characters of any given group may themselves be formed into a group under pointwise multiplication, termed the *dual* or *character group* of the given group, and topologized in such a way that (speaking informally) a sequence or net (χ_i) of characters converges to the character χ if and only if $\lim_i \chi_i(x) = \chi(x)$ uniformly for $x \in K$, and this for each compact subset K of the original group. Up to this point, everything is largely a matter of observation and definition. The interesting and decidedly nontrivial fact is that, by way of justification of the term "duality," the character group of the character group turns out to be (isomorphic with) the original group. This duality is profound and is fundamental in general harmonic analysis, but to develop the ideas in any generality would take us much too far afield. Suffice it to say that locally compact Abelian groups run around in mutually dual pairs—such as (R, R) and (T, Z)—either member of such a pair being isomorphic with the dual of the other: this is the so-called Pontryagin duality law, for more details of which the reader is referred to [B], Chapter 11, and [HR], Chapter VI. Our main concern will always be harmonic analysis on the group T, but we shall from time to time cast fleeting glances at the dual problems concerning harmonic analysis on the group Z, which is always to be thought of as being endowed with its discrete topology. To the reader we issue a standing invitation to reflect on the possible analogues for Z of results established in the text for T. As a start, he might verify that, in conformity with the Pontryagin duality law, the character group of Z can be identified with T in the manner suggested by (2.2.6); that is, to each bounded (necessarily continuous) character ζ of Z corresponds exactly one $x \in T$ such that $\zeta(n) = e_n(x)$ for $n \in Z$, and that the initial topology on T corresponds exactly to its dual topology under the association $x \leftrightarrow \zeta_x$ (see Exercise 2.3).

Studies of harmonic analysis on each of the groups T and Z form, when taken together, a useful forerunner to that of general harmonic analysis. This is partly because they illustrate separately a number of the difficulties that one encounters in an intermingled state when one moves along to harmonic analysis on general groups. Actually, the next degree of complexity is represented by the group R (the additive group of real numbers with its usual topology). In T, R, and Z one has, so to speak, the natural building bricks from which quite general locally compact Abelian groups may be built up. It is known, for example, that any such compactly generated group is isomorphic with a product $R^a \times Z^b \times F$, where a and b are nonnegative integers and F is a compact Abelian group (see [HR], Theorem (9.8)); moreover, F is isomorphic to a closed subgroup of a possibly infinite product of copies of T; and there exist arbitrarily small closed subgroups H of F such that F/H is isomorphic with $T^c \times F_0$, where c is a nonnegative integer and F_0 is a finite group (see [HR], Theorem (24.7)). These facts are quoted merely in order that the reader may get some idea of how the limited program attempted by this book fits into the scheme of general harmonic analysis; they will never be used hereafter in this book.

The reader would do well to peruse the survey article G. Weiss [1], which deals with both classical and modern aspects of the subject.

Before temporarily leaving the present topics we should indicate that in Chapter 11 the theory of Fourier series obtained up to that point will be used to classify all the closed invariant subspaces of $\mathbf{C}(T)$ (and of certain other function spaces). The theory will also show to what extent $\mathbf{C}(T)$ (and these other function spaces) can be decomposed into a direct sum of one-dimensional (and therefore minimal) invariant subspaces.

2.2.2. **The Invariant Integral.** Let us momentarily broaden the outlook by considering a locally compact topological group G (see 2.1.1); for the moment we do not assume that G is Abelian. Owing to this we must be careful to specify that our concern will lie with the left translation operators T_a defined by $T_a f(x) = f(-a + x)$. If G is Abelian, this agrees with the notation introduced in 2.2.1; in the contrary case one must distinguish these T_a from the right translation operators $f(x) \to f(x - a)$.

Denote by $\mathbf{C}_c(G)$ the linear space of complex-valued continuous functions f on G, each of which vanishes outside some f-dependent compact subset of G. Evidently, $\mathbf{C}_c(G)$ is a linear subspace of $\mathbf{C}(G)$. If G is compact (for example, if $G = R/2\pi Z$), $\mathbf{C}_c(G)$ and $\mathbf{C}(G)$ are identical.

A fundamental and cardinal fact underlying all general harmonic analysis is the existence of a linear functional I on $\mathbf{C}_c(G)$ which is

(1) positive, in the sense that $I(f) > 0$ if $f \neq 0$ is a nonnegative real-valued function in $\mathbf{C}_c(G)$; and

(2) left (translation) invariant, in the sense that

$$I(T_a f) = I(f)$$

for all $f \in \mathbf{C}_c(G)$ and all $a \in G$.

It is also a fact that, apart from a positive factor of proportionality, there is only one such functional. Any such functional I is called a *left invariant* (or *left Haar*) *integral* on G. (Similar remarks apply to *right Haar integrals*.)

It is known that the left invariant integral can in all cases be extended to more general functions in such a way as to preserve the basic, crucial, and pleasant properties of the Lebesgue integral of functions of one real variable. The details of this extension are to be found in any one of several references, for example: [HR], Chapters III and IV; [HS], Chapter 3; [B], Chapters 8–10; [E], Chapter 4. However, an intelligent reading of the present book will demand no more than a knowledge of the results of this extension for functions of one real variable; it is of little importance which of several possible approaches to the Lebesgue integral has been followed. More details about what we shall need to assume appear in 2.2.4.

The choice of $\mathbf{C}_c(G)$ in place of $\mathbf{C}(G)$ as the initial domain of definition of I comes about in the following way. It is quite easy to see at the outset that, whenever G is noncompact, there cannot be any invariant integral I for which $I(f)$ is finite-valued for all nonnegative real-valued $f \in \mathbf{C}(G)$ (or even for all nonnegative real-valued $f \in \mathbf{C}(G)$ which tend to zero at infinity). In other words, the "integrability" of a function will demand quite severe restrictions on the "average smallness" of the function at infinity. One very simple and, as it turns out, entirely effective way of imposing a priori such a restriction on f is to demand that it shall vanish outside some compact subset of G. (Of course, it turns out ultimately that this condition is not *necessary* for integrability.)

It is not too much to say that the inauguration of modern harmonic analysis on groups had to await the discovery, by Haar in 1933, of the existence of a left invariant integral on any locally compact group G satisfying the second countability axiom. Subsequent developments, including the removal of all countability restrictions on G, have been due to Weil, Kakutani, H. Cartan, von Neumann, and many others. The interested reader may also wish to consult a recent note by Bredon [1]. See also MR **39** # 7066.

On considering some familiar groups, old friends appear in a new light. For example, if $G = R^n$, the characteristic properties (1) and (2) show that an invariant integral is

$$I(f) = \int \cdots \int f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n,$$

a Lebesgue (or Riemann) integral extended over any hypercube outside which f vanishes.

Again, if G is $R/2\pi Z$, an invariant integral is

$$I(f) = \frac{1}{2\pi} \int f \circ p(x)\, dx, \tag{2.2.7}$$

a Lebesgue (or Riemann) integral extended over any interval of R of length 2π. Here we have chosen the disposable proportionality factor so as to arrange that $I(1) = 1$ (a choice that is possible for compact groups and for those only).

The essential uniqueness of the invariant integral for the circle group T (and likewise for R^n) can be established by quite simple and down-to-earth arguments, as follows. We handle functions on T as if they were periodic functions on R (see 2.1.2). Let us first note that any invariant integral I has the property that

$$|I(f)| \leqslant I(1) \cdot \sup |f|; \tag{2.2.8}$$

this follows from property (1) and the linearity of I. Now, if f is continuously differentiable,

$$\lim_{a \to 0} \frac{T_a f - f}{a} = -f'$$

holds uniformly. Consequently (2.2.8) and property (2) combine to show that $I(f') = 0$ for any continuously differentiable f. Next, if g is continuous and periodic and satisfies

$$\int_0^{2\pi} g(x)\, dx = 0,$$

then $g = f'$, where

$$f(x) = \int_0^x g(t)\, dt$$

is continuously differentiable and periodic. Thus $I(g) = 0$ for such g. Finally, choose any nonnegative continuous periodic h_0 such that

$$\int_0^{2\pi} h_0(x)\, dx = 1.$$

Given any continuous periodic h, we apply what precedes to the function g defined by

$$g(x) = h(x) - h_0(x) \int_0^{2\pi} h(t)\, dt,$$

which is continuous and periodic, and satisfies

$$\int_0^{2\pi} g(x)\, dx = 0.$$

The result of this application, namely, the conclusion $I(g) = 0$, signifies exactly that

$$I(h) = I(h_0) \cdot \int_0^{2\pi} h(x)\, dx,$$

showing that I differs from the expression appearing on the right-hand side of (2.2.7) by the constant factor $2\pi I(h_0) > 0$. This completes the verification of the essential uniqueness of the invariant integral on T.

Armed with this uniqueness property, it is simple to deduce other invariance properties of the integral. The elementary properties of the Riemann integral show that, if f is a continuous periodic function on R, and if $k \in Z$ and $k \neq 0$, then

$$\frac{1}{2\pi} \int_0^{2\pi} f(kx)\, dx = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx. \tag{2.2.9}$$

This can be established by using the identification afforded by (2.2.7) and invoking the uniqueness property of the invariant integral. In doing this, we may replace T by any compact group G and the mapping $x \to kx$ by any continuous group homomorphism t of G onto itself. Let I be the invariant integral on G, normalized so that $I(1) = 1$. We will show that

$$I(f \circ t) = I(f) \tag{2.2.10}$$

for all continuous functions f on G, from which (2.2.9) will follow by specialization. It is to be observed that, since $k \in Z$ and $k \neq 0$, the mapping $t: \dot{x} \to (kx)^{\cdot}$ is a continuous homomorphism of T onto itself.

To prove (2.2.10), we consider the new functional I' defined by

$$I'(f) = I(f \circ t); \tag{2.2.11}$$

this definition is effective since, owing to the continuity of t, $f \circ t$ is continuous whenever f is continuous. Since t maps G onto G, it is clear that I' enjoys property (1) of invariant integrals. Since also

$$\begin{aligned} T_a(f \circ t)(x) &= f \circ t(x - a) = f[t(x - a)] = f[t(x) - t(a)] \\ &= T_{t(a)}f[t(x)] = (T_{t(a)}f) \circ t(x), \end{aligned}$$

owing to the fact that t is a group homomorphism, we have $T_a(f \circ t) = (T_{t(a)}f) \circ t$. We again use the assumption that t maps G onto G; then the translation invariance of I shows that I' is also translation-invariant. By uniqueness, therefore, there is a number c such that $I'(f) = c \cdot I(f)$ for all continuous functions f. Choosing $f = 1$, we find that (2.2.11) gives $I'(1) = I(1 \circ t) = I(1) = 1$. Hence $c = 1$ and I' is identical with I. This is just what (2.2.10) asserts.

The dual situation. Let us turn momentarily from the circle group T to the dual group Z. There is no lasting mystery about the invariant integral on Z; apart from a disposable constant of proportionality, it must be expressed by summation:

$$I(\phi) = \sum_{n \in Z} \phi(n), \tag{2.2.7*}$$

at least for those functions ϕ on Z whose support $\{n \in Z : \phi(n) \neq 0\}$ is finite. (The compact subsets of a discrete space, such as Z, are exactly the finite subsets thereof.)

The linear space of functions ϕ on Z having finite support is, however, too narrow to accommodate fully effective operation, and it is desirable that the invariant integral be extended to other functions. No problem arises for those functions ϕ for which the series in (2.2.7*) is absolutely convergent: this is the space usually denoted by $\ell^1(Z)$ and is the exact analogue, for the group Z, of the space $\mathbf{L}^1$ of Lebesgue integrable functions on T.

To go still further, it will be necessary to interpret the right-hand side of (2.2.7*) according to one of a number of conventions. For example, the conditional convergence of the series will by convention always mean the existence of a finite limit for the sequence of symmetric partial sums

$$\sum_{|n| \leq N} \phi(n)$$

when $N \to \infty$. A yet more general interpretation which will play a fundamental role in the sequel lies in interpreting the right-hand side of (2.2.7*) as the limit as $N \to \infty$, when it exists, of the arithmetic means of the first $N + 1$ symmetric partial sums. This arithmetic mean is expressible in the form

$$\sum_{|n| \leq N} \left(1 - \frac{|n|}{N + 1}\right) \phi(n),$$

and this process of attaching a generalized sum to the series in (2.2.7*) is known as *summation by Cesàro means of the first order*. As applied to Fourier series, the method will be studied in some detail in Chapters 5 and 6. Yet other summability methods are known to be useful and effective, though we shall not dwell on them to any length in this book (see Section 6.6).

2.2.3. **The Orthogonality Relations.** It is interesting to note at this point that the orthogonality relations (1.1.3), which have been seen to be at the basis of the formation of Fourier series, flow inevitably from the defining properties of the invariant integral.

Suppose here that G is any compact topological group and that I is that left invariant integral on G for which $I(1) = 1$. Consider any nonprincipal continuous (and therefore bounded) character χ of G and choose any $a \in G$ such that $\chi(-a) \neq 1$. Then, by (2.2.3) and property (2) of 2.2.2,

$$I(\chi) = I(T_a\chi) = I[\chi(-a) \cdot \chi] = \chi(-a) \cdot I(\chi)$$

showing that $I(\chi) = 0$. Applying this to the product $\chi = \chi_1 \cdot \bar{\chi}_2$ of two continuous characters χ_1 and $\bar{\chi}_2$, we obtain the orthogonality relations

$$I(\chi_1\bar{\chi}_2) = \begin{cases} 1 & \text{if } \chi_1 = \chi_2 \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.12}$$

In view of (2.2.6) and (2.2.7), these relations reduce, when $G = T$, to the relations (1.1.3), which are now seen in their true relationship to the group structure of T.

There are other orthogonality relations pertaining to irreducible unitary representations of compact topological groups that reduce to (2.2.12) when the representations are one-dimensional (see 2.2.1); to discuss these would take us too far afield, and is in any case irrelevant to our main theme.

The dual orthogonality relations. In view of (2.2.7*), any would-be orthogonality relations for the discrete group Z would read somewhat as follows:

$$\sum_{n \in Z} e^{inx} \cdot \overline{e^{iny}} = 1 \quad \text{or} \quad 0$$

according as the real numbers x and y are, or are not, congruent modulo 2π. There is, however, no way of making sense of this relation which is based upon applying a summability method to the series on the left for individual values of x and y. On using concepts to be introduced in Chapter 12, it is nevertheless the case that, for a fixed y, the series converges distributionally to a certain distribution (or generalized function) known as the *Dirac measure placed at the point* y. This latter entity does, in a sense, vanish on the open set of points $x \not\equiv y$ modulo 2π, but there is no reasonable way of attaching to it a numerical value at points $x \equiv y$ modulo 2π.

There is therefore a residual and irreducible asymmetry separating the mutually dual situations; this is, in the last analysis, because of the profound topological differences between the "smooth" compact group G and the discrete noncompact group Z.

2.2.4. **L^p and Other Function Spaces.** It has been remarked in 2.2.2 that the invariant integral can in all cases be extended to functions more general than those in $\mathbf{C}_c(G)$. For $G = T$, in which case the invariant integral has been identified in (2.2.7), the extension involved is that from the Riemann to the Lebesgue integral; for the dual group Z, several stages in the extension have already been mentioned at the end of 2.2.2. It is essential for a smooth and satisfactory development of Fourier theory that advantage be taken of this extension. Broadly and figuratively speaking, the Lebesgue theory of integration is that which is necessary and sufficient for the major portion of contemporary analysis; integration theories for functions on more general sets and spaces almost invariably share the characteristic basic properties of the Lebesgue theory. However, in certain special connections involving functions of a real variable, more elaborate theories have proved useful. We shall have neither occasion nor space for more than a passing reference in 12.8.2(3) to some such theories. (Others, mainly designed to handle integration strictly as an antiderivation process, will receive no mention at all in this book.)

We shall therefore assume that the reader is familiar with the definition and basic properties of the Lebesgue integral of a function of one real variable. With but relatively few exceptions, some of which are dealt with in Appendix C, all the results we shall need will be found in the brief account in [W]. For the exceptional points the reader is referred to [HS], [AB], or [E], Chapter 4, or to any one of the several excellent accounts of integration theory now available. In making use of these sources of results about the Lebesgue integral of functions of one real variable, it will be agreed that a function f on T is measurable (or integrable) if the associated periodic function $f \circ p$ is Lebesgue measurable (or integrable) over some one—and hence every—interval of length 2π.

Having reached this stage we shall drop the notational distinction between f and $f \circ p$—in other words, we shall not distinguish between a periodic function on R and the corresponding function on the circle group.

It will be convenient to introduce some notations for the function spaces that will appear constantly in the following pages.

If k is an integer, $k \geqslant 0$, $\mathbf{C}^k = \mathbf{C}^k(T)$ will denote the set of complex-valued functions with period 2π and with k continuous derivatives, and $\mathbf{C}^\infty = \mathbf{C}^\infty(T) = \bigcap \{\mathbf{C}^k : k = 1, 2, \cdots\}$. For brevity, $\mathbf{C}$ is written in place of $\mathbf{C}^0$.

For any real number $p > 0$ we denote by $\mathbf{L}^p = \mathbf{L}^p(T)$ the set of periodic complex-valued measurable functions f such that

$$\|f\|_p \equiv [\frac{1}{2\pi} \int |f(x)|^p \, dx]^{1/p} \tag{2.2.13}$$

is finite, the integral being extended over any interval of length 2π; compare [W], p. 68, [AB], p. 215, or [HS], p. 188. In addition, $\mathbf{L}^\infty = \mathbf{L}^\infty(T)$ denotes the set of essentially bounded periodic complex-valued measurable functions, that is, of periodic complex-valued measurable functions f for which

$$\|f\|_\infty \equiv \text{ess . sup } |f(x)| \tag{2.2.14}$$

is finite, the essential supremum being taken relative to any interval of x-values of length 2π.

To be perfectly accurate, we shall frequently use $\mathbf{L}^p$ $(0 < p \leqslant \infty)$ to denote the set of *equivalence classes* of the appropriate type, two functions going into the same class if and only if they agree almost everywhere (a.e.). Since we shall not always signal which viewpoint is being adopted, the reader is warned to be on his guard and to be prepared to devote a little thought to deciding which interpretation is appropriate. The Fourier series of a function depends only on the class determined by that function.

Each of $\mathbf{C}^k$ (k an integer $\geqslant 0$, or ∞) and $\mathbf{L}^p$ $(0 < p \leqslant \infty)$ is a linear space; in view of preceding remarks, the reader should check the truth of this when $\mathbf{L}^p$ is regarded as a set of equivalence classes.

When $1 \leqslant p \leqslant \infty$, $\| \cdot \|_p$ is a norm on $\mathbf{L}^p$ if the latter is considered as a set of equivalence classes of functions (but only a seminorm if $\mathbf{L}^p$ is viewed as a set of individual functions); see Appendix B.1.2 for an explanation of the terminology. This statement is virtually the content of Minkowski's inequality, which asserts that $f + g \in \mathbf{L}^p$ and

$$\|f + g\|_p \leqslant \|f\|_p + \|g\|_p \tag{2.2.15}$$

whenever $1 \leqslant p \leqslant \infty$ and $f, g \in \mathbf{L}^p$. For a proof of Minkowski's inequality, see p. 68 of [W], or p. 146 of [HLP], or Section 4.11 of [E], or [AB], p. 218, or

[HS], pp. 191–192. The assertion is false if $0 < p < 1$ (see [HLP], *loc. cit.*), but it is then true that $\|f - g\|_p^p$ is a metric on $\mathbf{L}^p$ qua set of equivalence classes (or a semimetric if $\mathbf{L}^p$ is considered as a set of individual functions).

For $0 < p \leqslant \infty$, $\mathbf{L}^p$ is complete for the metric $\|f - g\|_p$ if $p \geqslant 1$, or for the metric $\|f - g\|_p^p$ if $0 < p < 1$; the former case is dealt with in [W], Theorem 4.5a, and the same argument adapts readily to the case $0 < p < 1$; alternatively, see [HS], p. 192, or [AB], p. 220.

To complete the picture, on $\mathbf{C}^k$ (k an integer $\geqslant 0$) we introduce the norm

$$\|f\|_{(k)} = \sup_{0 \leqslant h \leqslant k} \|D^h f\|_\infty, \tag{2.2.16}$$

here and subsequently D is the symbol of derivation.

On $\mathbf{C}^\infty$ we introduce the metric $\|f - g\|_{(\infty)}$, where

$$\|f\|_{(\infty)} = \sum_{k=0}^{\infty} \frac{2^{-k}\|f\|_{(k)}}{1 + \|f\|_{(k)}}; \tag{2.2.17}$$

Despite the notation, $\|f\|_{(\infty)}$ is *not* a norm. Then $\mathbf{C}^k$ is complete for the metric $\|f - g\|_{(k)}$ whenever $k = 0, 1, 2, \cdots, \infty$ (the reader should supply a proof of this).

With their appropriate metric topologies, all these spaces are topological linear spaces (see Appendix B.1.1), that is (compare 2.1.1 in relation to topological groups), the linear space operations $(f, g) \to f - g$ and $(\lambda, f) \to \lambda \cdot f$ (λ a complex scalar) are continuous. Further details concerning $\mathbf{C}^\infty$ appear in Section 12.1.

There will be constant use for one or more links in the chain of inclusion relations

$$\mathbf{C}^\infty \subset \cdots \subset \mathbf{C}^{k+1} \subset \mathbf{C}^k \subset \cdots \subset \mathbf{C}^0 = \mathbf{C} \subset \mathbf{L}^\infty \subset \mathbf{L}^p \subset \mathbf{L}^q, \tag{2.2.18}$$

where k is an integer $\geqslant 0$ and where $\infty > p > q > 0$. What is more, each inclusion map of one term of this sequence into any other lying to its right is continuous. The only nontrivial portion of this last assertion depends on the inequality

$$\|f\|_q \leqslant \|f\|_p \qquad 0 < q < p \leqslant \infty, \tag{2.2.19}$$

The estimate (2.2.19) is itself a consequence of Hölder's inequality, which asserts that if $1 \leqslant p \leqslant \infty$, and if p' denotes the *conjugate exponent* (or *index*) defined by $1/p + 1/p' = 1$ (supplemented by the convention that $p' = \infty$ if $p = 1$ and $p' = 1$ if $p = \infty$), then $f \cdot g \in \mathbf{L}^1$ and

$$\|f \cdot g\|_1 \leqslant \|f\|_p \cdot \|g\|_{p'} \tag{2.2.20}$$

whenever $f \in \mathbf{L}^p$ and $g \in \mathbf{L}^{p'}$. A proof of Hölder's inequality will be found on pp. 72–73 of [Ka], the assumption there made concerning continuity of f and g being unnecessary; see also Section 4.11 of [E], or [HS], pp. 190–191, or [AB], p. 217. An extended discussion of both the Minkowski and Hölder inequalities is undertaken in Chapter VI of [HLP], but this is unnecessarily elaborate for our purposes.

Each of the spaces $\mathbf{L}^p$ $(0 < p \leqslant \infty)$ and $\mathbf{C}^k$ $(k = 0, 1, 2, \cdots, \infty)$ is translation-invariant, as also are the appropriate metrics or norms. If $\mathbf{E}$ denotes any one of these topological linear spaces other than $\mathbf{L}^\infty$, and if $f \in \mathbf{E}$, the mapping $a \to T_a f$ is continuous from R (or from $R/2\pi Z$) into $\mathbf{E}$. (For the case $\mathbf{E} = \mathbf{L}^1$ a proof appears on p. 67 of [W]; this proof is readily adaptable to the case $\mathbf{E} = \mathbf{L}^p$ whenever $0 < p < \infty$. In the remaining admissible cases the result is almost evident in view of the well-known result that a continuous complex-valued function on a compact metric space is uniformly continuous. Regarding the excluded case $p = \infty$, see Exercise 3.5.) In all cases the mapping $f \to T_a f$ is, for any fixed $a \in R$, a continuous endomorphism of $\mathbf{E}$; moreover

$$\|T_a f\|_p = \|f\|_p$$

if $f \in \mathbf{L}^p$ and $0 < p \leqslant \infty$, and

$$\|T_a f\|_{(k)} = \|f\|_{(k)}$$

if $f \in \mathbf{C}^k$ and $k = 0, 1, 2, \cdots, \infty$.

Convergence in the sense of the metric on $\mathbf{L}^p$ $(0 < p \leqslant \infty)$ will be termed *convergence in* $\mathbf{L}^p$ or *convergence in mean with index* (or *exponent*) p. We note also that convergence in $\mathbf{C}$ in the sense of the norm $\|\cdot\|_\infty$ is equivalent to uniform convergence.

2.2.5. **The Dual Concepts.** In view of (2.2.7*), the natural analogues, for the group Z, of the spaces $\mathbf{L}^p$ introduced above, are the spaces $\ell^p = \ell^p(Z)$ of complex-valued functions ϕ on Z such that

$$\|\phi\|_p \equiv \{\sum_{n\in Z} |\phi(n)|^p\}^{1/p} \qquad \text{if } 0 < p < \infty,$$

or

$$\|\phi\|_\infty \equiv \sup_{n\in Z} |\phi(n)| \qquad \text{if } p = \infty,$$

is finite.

In addition to these, we occasionally wish to refer to the subspace $\mathbf{c}_0 = \mathbf{c}_0(Z)$ of $\ell^\infty(Z)$ formed of those ϕ for which

$$\lim_{|n|\to\infty} \phi(n) = 0.$$

Each of $\mathbf{c}_0$ and ℓ^p $(1 \leqslant p \leqslant \infty)$ is a Banach space; if $0 < p < 1$, ℓ^p is a complete metric space.

In lieu of (2.2.18) and (2.2.19) one has the relations

$$\ell^q \subset \ell^p \subset \mathbf{c}_0 \subset \ell^\infty \tag{2.2.18*}$$

and

$$\|\phi\|_\infty \leqslant \|\phi\|_p \leqslant \|\phi\|_q \tag{2.2.19*}$$

for $0 < q < p < \infty$. (Notice that $\|\phi\|_q \leqslant 1$ implies $|\phi(n)| \leqslant 1$ for all $n \in Z$, hence $|\phi(n)|^p \leqslant |\phi(n)|^q$ for all $n \in Z$, hence $\sum |\phi|^p \leqslant \sum |\phi|^q \leqslant 1$.)

The Hölder and Minkowski inequalities suffer no change in form other than the obvious replacement of integrals by the appropriate sums. Proofs for the case of finite sums appear on pp. 67–72 of [Ka]; for our purposes, which involve infinite sums, transparent limiting processes constitute the final step; see [HS], p. 194. A much more elaborate account appears in Chapter II of [HLP].

Concerning notation, we shall sometimes denote a function ϕ on Z in the sequential form: $(\phi_n)_{n\in Z}$; this is sometimes a convenience and is in any case in accord with tradition. There is, however, a nonvanishing chance of confusion with the convention according to which $(\phi_n)_{n\in Z}$ might also denote a two-way infinite sequence of functions on Z. The context will in all cases dispel initial doubts on this score.

2.3 Fourier Coefficients and Their Elementary Properties

Except in the discussion of certain specific examples, we shall use systematically the so-called "complex" Fourier coefficients. Indeed, the substance of Sections 2.1 and 2.2 constitutes ample indication that the exponentials e^{inx} play a much more fundamental role than do their real and imaginary parts separately. Not until Chapter 12 is reached shall we consider the Fourier coefficients and Fourier series of anything more general than integrable functions.

For $f \in \mathbf{L}^1$ we adopt in general work the systematic notation

$$\hat{f}(n) = \frac{1}{2\pi}\int f(x)e^{-inx}\,dx \qquad \text{for all } n \in Z \tag{2.3.1}$$

for the (complex) Fourier coefficients of f. The integral in (2.3.1) extends over any interval of length 2π. The symbol $\hat{f}$ naturally denotes the function $n \to \hat{f}(n)$ defined on Z; it is a two-way infinite sequence. Throughout this section we shall establish some of the simplest properties of the Fourier transformation $f \to \hat{f}$. In order to avoid confusion with the Fourier series of measures and distributions introduced in Chapter 12, a series of the type $\sum_{n\in Z}\hat{f}(n)e^{inx}$ with $f \in \mathbf{L}^1$ will be termed a *Fourier-Lebesgue series.*

The reader will notice that (2.3.1) makes no sense for a general $f \in \mathbf{L}^p$, if $0 < p < 1$. At no time shall we contemplate in detail any such extension, though the methods of Chapter 12 would permit us to make one sort of extension to restricted nonintegrable functions; see the example in 12.5.8.

Before beginning to display the elementary properties of the Fourier transformation, we introduce the following notations:

The symbol D will be that of derivation, as applied to functions of a real variable. There will be no occasion, until we reach Chapter 12, to apply D to any functions that are not absolutely continuous; for absolutely continuous functions, the new interpretation of D introduced in Chapter 12 is in agreement with the traditional one referred to here. For any complex-valued function f, $\bar{f}$ denotes the complex-conjugate function. For any function f defined on any group (on R, T, or Z in particular), $\check{f}$ denotes the function $t \to f(-t)$, and f^* the function $t \to \overline{f(-t)}$; thus $f^* = (\hat{f})^{\vee} = (\bar{f})^{-}$. Accordingly D and $f \to \check{f}$ are linear, whereas $f \to \bar{f}$ and $f \to f^*$ are conjugate-linear.

2.3.1. The mapping $f \to \hat{f}$ is linear. Moreover, $(\bar{f})^{\wedge} = (\hat{f})^*$ and $(f^*)^{\wedge} = (\hat{f})^{-}$.

Proof. The first statement is clear, integration being a linear process on the integrand. Of the remaining two assertions it will suffice to indicate the proof of the first, thus

$$\begin{aligned}(\bar{f})^{\wedge}(n) &= \frac{1}{2\pi}\int \overline{f(x)e}^{-inx}\,dx \\ &= \left[\frac{1}{2\pi}\int f(x)e^{inx}\,dx\right]^{-} = [\hat{f}(-n)]^{-} = (\hat{f})^*(n)\end{aligned}$$

for all $n \in Z$.

Note: In 3.1.1 and 4.1.2 we shall add some most important complements to the first assertion in 2.3.1.

2.3.2. For each $f \in \mathbf{L}^1$ and $n \in Z$, $|\hat{f}(n)| \leqslant \|f\|_1$.

Proof. By [W], Theorem 3.4c, we have

$$\begin{aligned}|\hat{f}(n)| \leqslant \frac{1}{2\pi}\int |f(x)e^{-inx}|\,dx &= \frac{1}{2\pi}\int |f(x)|\,dx \\ &\equiv \|f\|_1.\end{aligned}$$

Note: If we write $\|\hat{f}\|_\infty = \sup\{|\hat{f}(n)| : n \in Z\}$, 2.3.2 is equivalent to the inequality $\|\hat{f}\|_\infty \leqslant \|f\|_1$. This estimate, although well-nigh trivial, is the best possible in the sense that $\hat{f}(0) = \|f\|_1$ whenever f is real and nonnegative. On the other hand, for general real- or complex-valued functions f the relationship between $\|\hat{f}\|_\infty$ and $\|f\|_1$ is complicated; see Exercise 8.8 and Subsections 11.3.1 and 11.4.14. In particular, there exist functions $f \in \mathbf{L}^1$ for which $\|f\|_1 > 0$ and the ratio $\|\hat{f}\|_\infty/\|f\|_1$ is arbitrarily small. Hosts of examples of this phenomenon can be constructed by using the results of Chapter 15. A simpler example is provided by the so-called *Dirichlet kernel*

$$D_N(x) = \sum_{|n| \leqslant N} e^{inx} = \frac{\sin (N + 1/2)x}{\sin x/2};$$

if $f = D_N$, $\|\hat{f}\|_\infty = 1$ and yet $\|f\|_1$ is (as will be seen in 5.1.1) asymptotic to $(4/\pi^2) \log N$ as $N \to \infty$, so that the said ratio is in this case $\pi^2/(4 \log N)$ and can be made as small as we wish by choosing N sufficiently large. Moreover, it has been shown (D. J. Newman [2]) that for each positive integer N there exists a trigonometric polynomial

$$f(x) = \sum_{n=0}^{N} c_n e^{inx}$$

such that $|c_n| = 1$ $(n = 0, 1, \cdots, N)$ (and therefore $\|\hat{f}\|_\infty = 1$) and $\|f\|_1 > N^{1/2} - c$, where c is a suitable absolute constant.

Were this type of phenomenon absent, the theory of Fourier series would be much simpler and much less intriguing than it in fact is.

2.3.3. $(T_a f)^\wedge(n) = e^{-ina}\hat{f}(n)$ for $n \in Z$ and $f \in \mathbf{L}^1$.

Proof. It is easy to verify that, if $g(x) = f(x)e^{-inx}$, then

$$T_a g(x) = e^{ina} \cdot T_a f(x) \cdot e^{-inx}.$$

Integrating this relation and using translation-invariance of the integral, we obtain

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi}\int g(x)\,dx = \frac{1}{2\pi}\int T_a g(x)\,dx \\ &= e^{ina} \cdot \frac{1}{2\pi}\int T_a f(x) \cdot e^{-inx}\,dx \\ &= e^{ina} \cdot (T_a f)^\wedge(n),\end{aligned}$$

which is equivalent to the stated result.

Note: On being asked for a proof of 2.3.3, the reader's first reaction might be to apply the usual formula for change of variable in the integrals involved: this procedure is, of course, perfectly legitimate. But we prefer to phrase the device in terms of the characteristic invariance property of the integral (see 2.2.2).

2.3.4. Suppose that f is absolutely continuous, and let Df denote any integrable function equal almost everywhere to the derivative of f. Then

$$(Df)^\wedge(n) = in \cdot \hat{f}(n) \qquad \text{for all } n \in Z.$$

Proof. That the derivative of f exists almost everywhere and is integrable follows from [W], Section 6.3, Exercises 15 and 16 on p. 111, and Theorem 5.2g. The formula for partial integration ([W], Theorem 5.4a) then yields

$$\begin{aligned}(Df)^\wedge(n) &= \frac{1}{2\pi}\int_0^{2\pi} Df(x) \cdot e^{-inx}\,dx \\ &= \frac{1}{2\pi}[f(x) \cdot e^{-inx}]_0^{2\pi} + \frac{1}{2\pi}\int_0^{2\pi} f(x) \cdot e^{-inx} \cdot in \cdot dx \\ &= \hat{f}(n) \cdot in,\end{aligned}$$

which completes the proof.

Remarks. All that is required of Df is that the partial integration formula

$$\int_0^{2\pi} Df \cdot u \cdot dx = -\int_0^{2\pi} f \cdot Du \cdot dx$$

shall hold for all periodic, indefinitely differentiable functions u. This means that the preceding interpretation of Df for absolutely continuous functions f will accord with the generalized concept of derivation introduced in Chapter 12. The reader is reminded, however, that the result is not generally true for functions f possessing almost everywhere an integrable derivative: it is in addition necessary that f be equal to the indefinite integral of this derivative, which is ensured by (and indeed equivalent to) absolute continuity.

2.3.5. Suppose that f is absolutely continuous, and that its derivative Df is equal almost everywhere to an absolutely continuous function. Then $\hat{f}(n) = O(1/n^2)$ as $|n| \to \infty$, so that the Fourier series of f is absolutely and uniformly convergent.

Proof. The result expressed in 2.3.4 may now be applied to Df in place of f, showing that

$$\hat{f}(n) = (in)^{-2}(D^2f)\hat{\ }(n)$$

for $n \neq 0$. The desired majorization follows upon using 2.3.2.

Remarks. (1) Much stronger results will be noted in Section 10.6.

(2) On using 2.3.8 (which could quite well be established immediately following 2.3.4), the O appearing in 2.3.5 could be replaced by o. Notice that the hypotheses of 2.3.5 are amply fulfilled whenever $f \in \mathbf{C}^2$.

(3) The hypotheses of 2.3.5 ensure that the Fourier series of f is indeed convergent to $f(x)$ at all points, though we are not in a position to prove this just yet; see 2.4.3.

(4) In a similar way, 2.3.4 shows that $(n)\ \hat{f} = O(1/|n|)$ (and, with 2.3.8, that $\hat{f}(n) = o(1/|n|)$) whenever f is absolutely continuous. The next result asserts that the former majorization is in fact true for any f of bounded variation.

For a periodic function f we define the *total variation* $V(f)$ to be the supremum of all sums

$$\sum_{k=1}^{m} |f(x_k) - f(x_{k-1})|$$

with respect to all sequences $(x_k)_{k=0}^{m}$ such that $x_0 < x_1 < \cdots < x_m \leqslant x_0 + 2\pi$. Then f is of *bounded variation* if and only if $V(f) < \infty$ in which case we shall write $f \in \mathbf{BV}$; compare [W], p. 105; [HS], p. 266; and [AB], p. 256. Evidently, in taking the supremum above one may assume that the x_k fall into any preassigned interval of length 2π.

2.3.6. If f is of bounded variation, then

$$|n \cdot \hat{f}(n)| \leqslant \frac{1}{2\pi} V(f) \qquad \text{for all } n \in Z.$$

Proof. Granted a knowledge of Riemann-Stieltjes integrals ([HS], Section 8; [AB], Chapter 8), one may write

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, d\left[\frac{e^{-inx}}{(-in)}\right]$$

for $n \neq 0$, and apply partial integration for such integrals. Since we do not wish to make explicit use of properties of Riemann-Stieltjes integrals, we shall adopt a more pedestrian approach.

Suppose first that f is continuous. Put, for $n \neq 0$, $g(x) = e^{-inx}/(-in)$. It is then easy to verify that, given $\varepsilon > 0$, one has for $n \neq 0$

$$\left|\hat{f}(n) - \frac{1}{2\pi}\sum_{k=1}^{m} f(x_k)[g(x_k) - g(x_{k-1})]\right| \leqslant \varepsilon$$

for all sufficiently fine partitions $0 = x_0 < x_1 < \cdots < x_m = 2\pi$ of the interval $[0, 2\pi]$. Denoting by $\sum$ the sum appearing above, and applying partial summation, we obtain

$$\sum = [f(2\pi) - f(x_1)]g(0) - \sum_{k=1}^{m-1}[f(x_{k+1}) - f(x_k)]g(x_k).$$

By continuity (and periodicity) of f, the first summand on the right will not exceed ε in absolute value, provided the partition is sufficiently fine. Thus

$$\begin{aligned}|\hat{f}(n)| &\leqslant \varepsilon + \frac{\varepsilon}{2\pi} + \frac{1}{2\pi}\sum_{k=1}^{m-1}|f(x_{k+1}) - f(x_k)|\cdot|g(x_k)| \\ &\leqslant \left(1 + \frac{1}{2\pi}\right)\varepsilon + \frac{1}{2\pi}\cdot V(f)\cdot\frac{1}{|n|},\end{aligned}$$

since $|g(x)| \leqslant 1/|n|$. Letting $\varepsilon \to 0$, we obtain

$$|\hat{f}(n)| \leqslant \frac{V(f)}{2\pi|n|}$$

for $n \neq 0$, which is equivalent to the stated result.

Suppose finally that f is merely of bounded variation. We shall obtain the desired result in this case by approximating f by a suitable sequence of continuous functions f_r of bounded variation. Perhaps the simplest choice is

$$f_r(x) = r\int_x^{x+(1/r)} f(t)\, dt = r\int_0^{1/r} f(x+t)\, dt.$$

Whatever the increasing finite sequence (x_k) of points of $[0, 2\pi]$, one has

$$\sum_k |f_r(x_k) - f_r(x_{k-1})| \leqslant r\int_0^{1/r}\sum_k |f(x_k + t) - f(x_{k-1} + t)|\, dt,$$

which, since the integrand never exceeds $V(f)$, is majorized by $V(f)$. Thus $V(f_r) \leqslant V(f)$ for all r. By what is already established, therefore, we have

$$|\hat{f}_r(n)| \leqslant \frac{V(f_r)}{2\pi|n|} \leqslant \frac{V(f)}{2\pi|n|} \tag{2.3.2}$$

for $n \neq 0$ and all r.

Beside this, a simple computation shows that

$$\hat{f}_r(n) = \exp\left(\frac{in}{2r}\right) \cdot \hat{f}(n) \cdot \frac{\sin(n/2r)}{n/2r}$$

so that

$$\lim_{r\to\infty} |\hat{f}_r(n)| = |\hat{f}(n)|.$$

Combining this with (2.3.2), we are led to

$$|\hat{f}(n)| \leqslant \frac{V(f)}{2\pi|n|}$$

for $n \neq 0$, and the proof is complete.

Remarks. (1) The converse of 2.3.6 is false: there exist continuous functions f for which $\hat{f}(n) = o(1/|n|)$ as $|n| \to \infty$ and yet $f \notin \mathbf{BV}$.

(2) The estimate in 2.3.6, namely, that $\hat{f}(n) = O(1/|n|)$ as $|n| \to \infty$, cannot be improved, even if it be assumed that f is continuous as well as being of bounded variation. In other words, there exist continuous functions f of bounded variation such that $\hat{f}(n) \neq o(1/|n|)$ as $|n| \to \infty$. For a proof, see [Ba$_1$], pp. 210–211; or Exercise 12.44. In view of 2.3.4 and 2.3.8, any such function f fails to be absolutely continuous.

Incidentally, it is known (after Wiener) that a function f of bounded variation is continuous if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{|n|\leqslant N} |n\hat{f}(n)| = 0;$$

see Exercise 8.13.

(3) At the expense of replacing the factor $(2\pi)^{-1}$ by 1 in 2.3.6, there is a very neat proof due to Taibleson [1]. Thus, if $n \in Z$ and $n \neq 0$, write $a_k = 2k\pi|n|^{-1}$ for $k \in \{0, 1, 2, \cdots, |n|\}$. Denote by g the step function which is equal to $f(a_k)$ on (a_{k-1}, a_k) for $k \in \{1, \cdots, |n|\}$. Then, since

$$\int_{2k\pi|n|^{-1}}^{2\pi(k+1)|n|^{-1}} e^{-inx}\, dx = 0,$$

it follows that

$$\begin{aligned} 2\pi|\hat{f}(n)| &= \left| \int_0^{2\pi} (f(x) - g(x))\, e^{-inx}\, dx \right| \\ &\leqslant \sum_{k=1}^{|n|} \int_{a_{k-1}}^{a_k} |f(x) - f(a_k)|\, dx \\ &\leqslant \sum_{k=1}^{|n|} \int_{a_{k-1}}^{a_k} V_k\, dx = \sum_{k=1}^{|n|} V_k(a_k - a_{k-1}), \end{aligned}$$

where V_n is the total variation of f on the interval $[a_{k-1}, a_k]$. Since $V_1 + \cdots + V_n \leqslant V(f)$ and $a_k - a_{k-1} = 2\pi|n|^{-1}$,

$$2\pi|\hat{f}(n)| \leqslant V(f)\cdot 2\pi|n|^{-1}$$

and hence

$$|n\hat{f}(n)| \leqslant V(f).$$

See also M. and S.-I. Izumi [1].

2.3.7. Define the *mean modulus of continuity of f with exponent* (or *index*) 1 by

$$\omega_1 f(a) = \|T_a f - f\|_1 = \omega_1 f(-a).$$

Then, if $f \in \mathbf{L}^1$,

$$|\hat{f}(n)| \leqslant \frac{1}{2}\,\omega_1 f\left(\frac{\pi}{n}\right) \qquad (n \in Z,\ n \neq 0).$$

Proof. By definition

$$\hat{f}(n) = \frac{1}{2\pi}\int f(x)e^{-inx}\,dx,$$

and by 2.3.3

$$-\hat{f}(n) = \frac{1}{2\pi}\int f\left(x + \frac{\pi}{n}\right)e^{-inx}\,dx.$$

Subtracting and dividing by two, we obtain

$$\hat{f}(n) = \tfrac{1}{2}(f - T_{-\pi/n}f)^\wedge(n),$$

whence the result follows on applying 2.3.2.

2.3.8. (Riemann-Lebesgue lemma) For any integrable f one has

$$\lim_{|n|\to\infty} \hat{f}(n) = 0.$$

Proof. This follows immediately from 2.3.7 and the fact ([W], Theorem 4.3c) that $\omega_1 f(a) \to 0$ as $a \to 0$.

Remarks. (1) The Riemann-Lebesgue lemma is so fundamental that it is worth pointing out another method of proof (which indeed lies behind the proof of Williamson's Theorem 4.3c just cited and used). Suppose we denote by $\mathbf{E}$ the set of integrable functions f for which the statement of the lemma holds. Then 2.3.2 shows that $\mathbf{E}$ is a closed subset of $\mathbf{L}^1$ (relative to the topology defined by the norm $\|\cdot\|_1$). It is otherwise evident that $\mathbf{E}$ is a linear subspace of $\mathbf{L}^1$. To prove the lemma it therefore suffices to show that $\mathbf{E}$ contains a set of functions, say $\mathbf{S}$, the finite linear combinations of which are dense in $\mathbf{L}^1$. There are many such sets $\mathbf{S}$ which may be indicated. Examples are: (i) the set of characteristic functions of intervals $[a, b]$ $(0 < a < b < 2\pi)$, extended by periodicity. The finite linear combinations of these are dense in $\mathbf{L}^1$ (as is shown in [W], Theorem 4.3a); and each such function is directly verifiable to have a Fourier transform satisfying the lemma (assurance on this point also comes from 2.3.6). (ii) the set $\mathbf{C}^\infty$; see [W], Theorem 4.3b and 2.3.4.

(2) It is worth pointing out that 2.3.4 to 2.3.8 are all essentially concerned with restrictions on the rate of decay of $\hat{f}(n)$ as $|n| \to \infty$. The indications are clearly that the smoother the function f, the more rapid this decay. This conclusion will receive further reinforcement as we progress; some extreme instances are covered by Exercises 2.7 and 2.8.

2.3.9. **Introduction of $\mathbf{A}(Z)$.** The preceding results and remarks might raise hopes that the membership of f to various function spaces (such as $\mathbf{C}$ or $\mathbf{L}^p$ for various values of p) might be decidable solely by inspection of the rate of decay of $\hat{f}$, or at any rate by examining $|\hat{f}|$. However, while there are many criteria of this sort that are either sufficient or necessary, with the sole exception of the case of $\mathbf{L}^2$ (dealt with in Chapter 8), there are no known necessary and sufficient conditions of this type. Moreover, it will appear in Chapters 12 and 14 that there definitely cannot be any such complete characterization involving only the values of $|\hat{f}|$. The few necessary and sufficient conditions that are known are of a much more complicated sort and are unfortunately extremely difficult to apply in specific instances; see 2.3.10. Much remains to be discovered in this direction.

To make things more specific, let us consider $\mathbf{L}^1$ itself. If we denote by $c_0(Z)$ the linear space of complex-valued functions (two-way infinite sequences) ϕ on Z for which $\lim_{|n| \to \infty} \phi(n) = 0$ and equip it with the norm

$$\|\phi\|_\infty = \sup\{|\phi(n)| : n \in Z\} \tag{2.3.3}$$

(see 2.2.5), we have learned so far that $f \to \hat{f}$ is a continuous linear mapping of $\mathbf{L}^1$ into $c_0(Z)$. Denote by $\mathbf{A}(Z)$ the range of this mapping. The question is: Given in advance a $\phi \in c_0(Z)$, how can one determine whether or not $\phi \in \mathbf{A}(Z)$? No effective and general method is known for doing this.

Although we know that $\hat{f}$ tends to zero at infinity for each $f \in \mathbf{L}^1$, the rate of decay can be arbitrarily slow. For example, given any $\phi \in c_0(Z)$, one may choose positive integers $N_1 < N_2 < \cdots$ so that

$$|\phi(n)| \leqslant k^{-2} \qquad \text{for} \qquad |n| \geqslant N_k.$$

Then

$$f(x) = \sum_{k=1}^{\infty} 2\phi(N_k) \cos N_k x$$

is a continuous function for which

$$\hat{f}(n) = \phi(N_k)$$

for $n = \pm N_k$ ($k = 1, 2, \cdots$). Furthermore we shall see in Section 7.4 that (again for any assigned $\phi \in c_0(Z)$) a function $f \in \mathbf{L}^1$ can be chosen so that

$$\hat{f}(n) \geqslant |\phi(n)|$$

for all $n \in Z$.

Again, although the sequence ϕ defined by

$$\phi(n) = \begin{cases} \dfrac{1}{\log |n|} & \text{for } |n| \geqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

belongs to $\mathbf{A}(Z)$, the sequence ϕ_1 defined by

$$\phi_1(n) = \begin{cases} \dfrac{\operatorname{sgn} n}{\log |n|} & \text{for } |n| \geqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

has not this property (see Exercise 7.7 and 10.1.6). This shows that an orderly, and therefore seemingly harmless, change of sign can destroy membership of $\mathbf{A}(Z)$.

This (or any other similar) example shows incidentally that $\mathbf{A}(Z)$ is a proper subset of $\mathbf{c}_0(Z)$; it also shows that $|\phi|$ may belong to $\mathbf{A}(Z)$ while ϕ fails to do so. There is an entirely different, and typically modern, approach to the proper inclusion relation $\mathbf{A}(Z) \subsetneqq \mathbf{c}_0(Z)$ which shows a little more, namely, that $\mathbf{A}(Z)$ is in fact a meager (that is, first category; see Appendix A.1) subset of $\mathbf{c}_0(Z)$.

To see this, we must observe that $\mathbf{L}^1$ and $\mathbf{c}_0(Z)$ are Banach spaces when endowed with the norms defined in (2.2.13) and (2.3.3), respectively, and that $T: f \to \hat{f}$ is a continuous linear operator mapping $\mathbf{L}^1$ into $\mathbf{c}_0(Z)$ whose range is $\mathbf{A}(Z)$ (see 2.3.1 and 2.3.2). If, contrary to our assertion, $\mathbf{A}(Z)$ were nonmeager in $\mathbf{c}_0(Z)$, the open mapping theorem (Appendix B.3.2) would entail that T is an *open* map of $\mathbf{L}^1$ *onto* $\mathbf{c}_0(Z)$. Assuming for the moment the uniqueness theorem (2.4.1), this would imply the existence of a number $B > 0$ such that

$$\|f\|_1 \leqslant B \cdot \|\hat{f}\|_\infty \tag{2.3.4}$$

for each $f \in \mathbf{L}^1$. However, (2.3.4) can be negatived, the resulting contradiction thus establishing our assertion. For example, if $f = D_N$, as in Exercise 1.1, a direct computation, which will be carried out in detail in 5.1.1, shows that

$$\|f\|_1 \equiv \|D_N\|_1 \sim \frac{4}{\pi^2} \log N$$

for large N. Since in this case $\|\hat{f}\|_\infty = 1$, it is plain that a contradiction of (2.3.4) results whenever N is sufficiently large.

At this point see also Exercise 9.8.

In addition to the linear space structure of $\mathbf{c}_0(Z)$, one may consider its structure as an algebra under pointwise operations. It is then natural to ask whether $\mathbf{A}(Z)$ is a subalgebra (as well as a linear subspace) of $\mathbf{c}_0(Z)$. This leads us to seek a way of combining integrable functions—a sort of multiplication—which corresponds to pointwise multiplication of their Fourier transforms. We shall consider this question and its ramifications in Chapter 3, returning in Chapter 4 to the consideration of $\mathbf{A}(Z)$ and the Fourier transformation in this enriched setting. Further results will appear in 11.4.13 and 11.4.16.

2.3.10. Criteria for Membership of A(Z). Simply as a matter of interest (for we shall make no subsequent use of these results), we sample a few of the known criteria for a given sequence $\phi = (\phi_n)_{n\in Z}$ to be the sequence of Fourier coefficients of a function belonging to certain prescribed function spaces. Further results of this sort will appear in Section 8.7, 10.6.3(1), and (2), 12.7.5, 12.7.6, 12.7.9(2), and Exercise 12.50. If the reader will ponder these conditions, he will soon be convinced of the difficulty of applying them in specific instances.

(1) In order that ϕ shall belong to A(Z), it is necessary and sufficient that, having chosen any p satisfying $0 < p < \infty$, one has

$$\lim_{r\to\infty} \sum_{n\in Z} \phi_n \hat{u}_r(n) = 0$$

for any sequence $(u_r)_{r=1}^{\infty}$ of trigonometric polynomials satisfying

$$\|u_r\|_\infty \leqslant 1, \qquad \lim_{r\to\infty} \|u_r\|_p = 0.$$

An equivalent condition is that to each $\varepsilon > 0$ shall correspond a number $k(\varepsilon) \geqslant 0$ such that

$$\Big|\sum_{n\in Z} \phi_n \hat{u}(n)\Big| \leqslant \varepsilon \cdot \|u\|_\infty + k(\varepsilon)\cdot\|u\|_p \tag{2.3.5}$$

for all trigonometric polynomials u; see R. E. Edwards [1] and Ryan [1].

The case $p = 2$ of this result is due to Salem; see [Ba_1], pp. 239–240.

(2) In order that ϕ shall be the sequence of Fourier coefficients of a continuous function, it is necessary and sufficient that to each $\varepsilon > 0$ shall correspond a number $k(\varepsilon) \geqslant 0$ and a finite subset F_ε of Z such that

$$\Big|\sum_{n\in Z} \phi_n \hat{u}(n)\Big| \leqslant \varepsilon \cdot \|u\|_1 + k(\varepsilon)\cdot \sup_{n\in F_\varepsilon} |\hat{u}(n)| \tag{2.3.6}$$

for all trigonometric polynomials u; see R. E. Edwards [1].

(3) It will appear in 13.5.1 that, if $\phi = \hat{f}$ for some $f \in \mathbf{L}^p$, where $1 < p \leqslant 2$, then $\phi \in \ell^{p'}$, that is,

$$\sum_{n\in Z} |\phi_n|^{p'} < \infty, \tag{2.3.7}$$

p' being defined by $1/p + 1/p' = 1$. It is known (Rooney [1]) that if ϕ satisfies (2.3.7), then it is the sequence of Fourier coefficients of some function in $\mathbf{L}^p(1 < p \leqslant 2)$ if and only if

$$\sup_\nu (\nu + 1)^{p-1} \sum_{m=0}^{\nu} |M_{\nu,m}(\phi)|^p < \infty, \tag{2.3.8}$$

the supremum being taken as ν ranges over all nonnegative integers, where

$$M_{\nu,m}(\phi) = \sum_{n \in Z} a_{n,\nu,m}\phi_n,$$

and

$$a_{n,\nu,m} = {}_\nu\mathrm{C}_m \int_0^1 t^m(1 - t)^{\nu-m} e^{\pi i n t}\, dt$$

for $n \in Z$, $\nu \in Z$, $m \in Z$, $\nu \geqslant 0$, and $0 \leqslant m \leqslant \nu$.

(4) A further sufficient condition for membership of $\mathbf{A}(Z)$ will be discussed at some length in Section 8.7.

(5) The behavior of $\mathbf{A}(Z)$ under permutations of Z is discussed by Kahane [4].

2.4 The Uniqueness Theorem and the Density of Trigonometric Polynomials

In this section we shall establish the uniqueness theorem, which asserts that a function is determined almost everywhere by its Fourier transform, and certain consequences thereof concerning approximation by trigonometric polynomials.

2.4.1. (1) If $f \in \mathbf{C}$ and $\hat{f} = 0$, then $f = 0$.

(2) If $f \in \mathbf{L}^1$ and $\hat{f} = 0$, then $f = 0$ a.e.

Proof. Statement (1) is, of course, a special case of (2). We shall prove it first and deduce statement (2) from it.

By 2.3.1, we may in all cases assume that f is real-valued. Moreover, since $\hat{f} = 0$ entails $(T_a f)^\wedge = 0$ for all a (by 2.3.3), it will suffice to show that if $f \in \mathbf{C}$ and

$$\frac{1}{2\pi} \int ft\, dx = 0 \qquad \text{for all trigonometric polynomials } t, \tag{2.4.1}$$

then

$$f(0) = 0. \tag{2.4.2}$$

We will in fact show that the negation of (2.4.2) implies the negation of (2.4.1).

If (2.4.2) is false, we may (by changing f into $-f$ if necessary) assume that $f(0) = c > 0$ and then choose $\delta > 0$ so that

$$f(x) \geqslant \tfrac{1}{2}c \qquad \text{for } |x| \leqslant \delta. \tag{2.4.3}$$

To construct a trigonometric polynomial t violating (2.4.1), write

$$t_0(x) = 1 + \cos x - \cos \delta$$

and then $t = t_0{}^N$, where the large positive integer N will be chosen later. It is plain that t is indeed a trigonometric polynomial. It is also clear that

$$\left.\begin{array}{l} |t(x)| \leqslant 1 \quad \text{for } \delta \leqslant |x| \leqslant \pi, \quad t(x) \geqslant 0 \quad \text{for } |x| \leqslant \delta, \\ \qquad t(x) \geqslant q^N \quad \text{for } |x| \leqslant \tfrac{1}{2}\delta, \end{array}\right\} \tag{2.4.4}$$

where $q = 1 + \cos \frac{1}{2}\delta - \cos \delta > 1$. By (2.4.3) and (2.4.4) we have

$$\begin{aligned} \frac{1}{2\pi}\int ft\,dx &\geqslant \frac{1}{2\pi}\int_{|x|\leqslant\delta} ft\,dx - \frac{1}{2\pi}\|f\|_\infty (2\pi - 2\delta) \\ &\geqslant \frac{\frac{1}{2}c}{2\pi}\int_{|x|\leqslant\delta} t\,dx - \|f\|_\infty \\ &\geqslant \frac{c}{4\pi}\int_{|x|\leqslant\frac{1}{2}\delta} t\,dx - \|f\|_\infty \\ &\geqslant \frac{c}{4\pi}\cdot\delta\cdot q^N - \|f\|_\infty, \end{aligned}$$

which is positive provided we choose N larger than $\log(4\pi\|f\|_\infty/c\delta)/\log q$, thus negating (2.4.1). This proves statement (1).

Now assume that f is as in statement (2). Define

$$F(x) = c + \int_0^x f(y)\,dy,$$

where the number c is chosen so as to make $\hat{F}(0)$ vanish. Since $\hat{f}(0) = 0$, F is periodic. Now F is absolutely continuous and $DF = f$ a.e. ([W], Theorem 5.2g). By 2.3.4, the choice of c, and the main hypothesis on f, it follows that $\hat{F} = 0$. Thus statement (1) shows that $F = 0$, and so $f = 0$ a.e., as alleged.

Remark. The uniqueness theorem for *trigonometric polynomials* is a direct consequence of the orthogonality relations (1.1.3) and is covered by Exercise 1.7(1).

We proceed to deduce from 2.4.1 two rather special and purely provisional results concerning the recapture of a function from its Fourier series; more satisfactory results of this nature will appear in Chapters 6 and 10. As was pointed out in 2.2.1, these results concern harmonic synthesis on the circle group.

2.4.2. If $f \in \mathbf{L}^1$ has a Fourier series that is dominatedly convergent almost everywhere, then

$$f(x) = \sum_{n\in Z} \hat{f}(n)e^{inx} \qquad \text{a.e.}$$

Proof. Let g be defined almost everywhere to be the sum of the Fourier series of f wherever the latter converges, and to be, say, 0 elsewhere. By dominated convergence, $g \in \mathbf{L}^1$. By the same token, one has for any $m \in Z$,

$$\hat{g}(m) = \frac{1}{2\pi}\int g(x)e^{-imx}\,dx = \sum_{n\in Z} \hat{f}(n)\cdot\frac{1}{2\pi}\int e^{inx}\cdot e^{-imx}\,dx$$
$$= \hat{f}(m),$$

the last step by the orthogonality relations. By the uniqueness theorem (2.4.1), $f = g$ a.e.

2.4.3. If f is a continuous function with a uniformly convergent Fourier series, then

$$f(x) = \sum_{n\in Z} \hat{f}(n)e^{inx}$$

everywhere.

Proof. Using the notation of the proof of 2.4.2, the sum function g is now everywhere defined and is continuous thanks to uniform convergence of the series. Also, since the range of integration involved is a bounded interval, uniform convergence entails dominated convergence. So 2.4.2 entails that f and g agree almost everywhere. But, since both are continuous, this in turn implies agreement everywhere.

A further and very important deduction from 2.4.1 is the following density theorem; it too will be refined later.

2.4.4. The set $\mathbf{T}$ of all trigonometric polynomials is everywhere dense in each of the Banach spaces $\mathbf{C}$, $\mathbf{L}^p$ $(1 \leqslant p < \infty)$, that is, given $f \in \mathbf{C}$ (respectively $f \in \mathbf{L}^p$) and $\varepsilon > 0$, there exists $t \in \mathbf{T}$ such that

$$\|f - t\|_\infty \leqslant \varepsilon \qquad (\text{respectively } \|f - t\|_p \leqslant \varepsilon). \tag{2.4.5}$$

Proof. (1) First take the case of $\mathbf{C}$. Given $\varepsilon > 0$, first choose $g \in \mathbf{C}^2$ such that

$$\|f - g\|_\infty \leqslant \tfrac{1}{2}\varepsilon. \tag{2.4.6}$$

This may be done by choosing a sufficiently small positive a and setting

$$g_1(x) = a^{-1}\int_x^{x+a} f(y)\,dy, \qquad g(x) = a^{-1}\int_x^{x+a} g_1(y)\,dy.$$

By 2.3.5 and 2.4.3,

$$g = \sum_{n\in Z} \hat{g}(n)e_n,$$

the series being convergent in $\mathbf{C}$ (that is, uniformly convergent). One may therefore choose N so large that

$$\|g - \sum_{|n|\leqslant N} \hat{g}(n)e_n\|_\infty \leqslant \tfrac{1}{2}\varepsilon. \tag{2.4.7}$$

Then (2.4.6) and (2.4.7) combine to yield (2.4.5) with

$$t = \sum_{|n| \leqslant N} \hat{g}(n) e_n .$$

(2) This case follows from (1), the fact that **C** is everywhere dense in $\mathbf{L}^p$ (compare [W], Theorem 4.4e), and the inequality $\|h\|_p \leqslant \|h\|_\infty$. Thus, given $f \in \mathbf{L}^p$ and $\varepsilon > 0$, first choose $g \in \mathbf{C}$ so that $\|f - g\|_p \leqslant \frac{1}{2}\varepsilon$, and then [by (1)] a $t \in \mathbf{T}$ so that $\|g - t\|_\infty \leqslant \frac{1}{2}\varepsilon$. Then, a fortiori, $\|g - t\|_p \leqslant \frac{1}{2}\varepsilon$ and so

$$\|f - t\|_p \leqslant \|f - g\|_p + \|g - t\|_p \leqslant \varepsilon .$$

Remarks. (1) The assertion in 2.4.4 is false for $p = \infty$. (Why?)

(2) It is possible to deduce 2.4.4 from 2.4.1 via the Hahn-Banach theorem (see Appendix B.5) and results about the topological duals of the spaces **C** and $\mathbf{L}^p$ given in Chapter 12 and Appendix C, respectively. One of the consequences of the Hahn-Banach theorem is, in fact, that uniqueness theorems and density theorems consort in "dual pairs," so to speak.

(3) When combined with 2.3.2, 2.4.4 leads to an independent proof of 2.3.8; see Exercise 2.9.

(4) Assuming that part of 2.4.4 which refers to the space **C**, one can derive 2.4.1; see Exercise 2.10. Thus, 2.4.1 and 2.4.4 are equivalent, a fact which illustrates the substance of Remark (2) immediately above.

(5) Other proofs of 2.4.1 and 2.4.4 will appear in 5.1.2 and 6.1.1, where more refined versions of 2.4.4 are considered; see also Section 6.2, where some applications are mentioned.

2.5 Remarks on the Dual Problems

2.5.1. Definition of the Fourier Transform. If we are given a function ϕ on Z, it is natural to attempt to define its Fourier transform $\hat{\phi}$ as the function on T given by

$$\hat{\phi}(x) = \sum_{n \in Z} \phi(n) e^{inx} ; \tag{2.5.1}$$

in comparing this with (2.3.1), the reader will observe a change from e^{-inx} to e^{inx}, which is made purely on the grounds of subsequent convenience. Although (2.5.1) makes excellent sense whenever $\phi \in \ell^1(Z)$, in which case $\hat{\phi}$ is evidently a continuous function on T satisfying

$$\|\hat{\phi}\|_\infty \leqslant \|\phi\|_1 \tag{2.5.2}$$

(compare 2.3.2), it is plain that complications arise if, for example, ϕ is known merely to belong to $\ell^p(Z)$ for some $p > 1$. (In the case of the group T no analogous complications appeared, because of the compactness of T.) One has in fact to contemplate conditional convergence and summability, perhaps merely for almost all x, as was heralded by the remarks at the end of 2.2.2 concerning the interpretation of the invariant integral on Z.

As a matter of fact, and as will appear in the course of Chapters 8, 12, and 13, it is often more effective to replace considerations of pointwise convergence (everywhere or almost everywhere) of the series on the right of (2.5.1) by that of convergence, either in one of the spaces $\mathbf{L}^q$ or distributionally, of the symmetric partial sums

$$\hat{\phi}_N(x) \equiv \sum_{|n| \leqslant N} \phi(n)e^{inx}.$$

On the other hand, for functions ϕ which are structurally special, the results of Chapter 7 yield pointwise convergence (at least almost everywhere) of the series defining $\hat{\phi}$, although even here there is no assurance that the function $\hat{\phi}$ so defined almost everywhere will belong to $\mathbf{L}^1$. Further very special results of this sort, applying to cases in which ϕ is known to be of the form $\hat{f}$ for some $f \in \mathbf{L}^1$, are contained in Chapters 5, 6, and 10.

2.5.2. **The Uniqueness Theorem.** Related difficulties arise in connection with the appropriate uniqueness theorem, at least if pointwise convergence or summability is envisaged and unless severe a priori restrictions are imposed upon ϕ. [For instance, there are no difficulties if we assume that the series on the right of (2.5.1) is dominatedly convergent almost everywhere; but this can only be the case if ϕ belongs to $\mathbf{A}(Z)$, a criterion extremely difficult to verify at the outset.] If convergence is meant in the distributional sense, the uniqueness theorem presents no difficulties and is implicit in results to be obtained in Chapter 12. Dogged insistence on the pointwise interpretation of convergence leads right to the heart of the Riemann-inspired theory of general trigonometric series, including a number of problems (such as those concerning the characterization of the so-called sets of multiplicity and sets of uniqueness) of great difficulty and whose delicacy is such that they "fall through the mesh" imposed by requirements of distributional convergence or convergence in some space $\mathbf{L}^q$. Two relatively very simple results concerning pointwise convergence appear in Exercises 2.13 and 2.14.

2.5.3. **The Space A.** Dual to $\mathbf{A}(Z)$ is $\mathbf{A} = \mathbf{A}(T)$: this is the linear space of functions on T of the form $\hat{\phi}$ obtained when ϕ ranges over $\ell^1(Z)$. Equivalently, $\mathbf{A}$ consists precisely of those continuous functions f on T such that $\hat{f} \in \ell^1$. As in the case of $\mathbf{A}(Z)$, so with $\mathbf{A}$: there is no complete solution to the problem of characterizing directly in terms of their functional values the elements of $\mathbf{A}$. We shall return to the consideration of $\mathbf{A}$ in Sections 10.6 and 12.11, where partial results will be obtained and used.

2.5.4. **The Dual of 2.4.4: Almost Periodicity.** In view of the closing remarks in 2.2.1 and Exercise 2.3, it is natural to attach the label *trigonometric polynomial on* Z to each function on Z which is a finite linear combination of characters $e_a: n \to e^{ian}$ of Z, the parameter a ranging over T (or, equivalently, over R).

This being so, the dual of 2.4.4 is concerned with the characterization of those complex-valued functions on Z that are uniform limits of trigonometric polynomials on Z. The investigation of this problem cannot be undertaken here. The functions which are approximable in the stated fashion are precisely the so-called *almost periodic functions* on Z.

Almost periodicity is a concept applying to functions on arbitrary groups and has behind it a vast literature. For the case of the group R, see [Bes]; for more general groups, see [Mk]; [HR], Section 18; [We], Chapitre VII; [Lo], Chapter VIII.

On a compact group, such as T, all continuous functions are almost periodic; this fact explains the form taken by 2.4.4.

On the noncompact group Z, the only member of $\mathbf{c}_0(Z)$ that is almost periodic is the zero element.

EXERCISES

2.1. Let S be a closed subgroup of R distinct from $\{0\}$ and R. Show that there exists a number $d > 0$ such that S consists precisely of all integer multiples of d.

Hint: Consider the infimum of all positive members of S.

2.2. Let x be a real number such that x/π is irrational. Show that the set $\{e^{inx} : n \in Z\}$ is everywhere dense in the unit circumference in the complex plane.

Remarks. A stronger result will appear in Exercise 2.15. The stated result is a special case of Kronecker's theorem, for which see [HW], Chapter XXIII, especially p. 370. There is a general group-theoretic formulation of this theorem which is discussed in [HR], pp. 431–432, 435–436, and which asserts in particular that any character χ of $R/2\pi Z$, continuous or not, can be approximated arbitrarily closely on any preassigned finite subset of $R/2\pi Z$ by a suitably chosen continuous character e_n.

Hint: Show that $e^{imx} \neq e^{inx}$ if $m, n \in Z$ and $m \neq n$. Deduce that the said set has 1 as a limiting point.

2.3. Let ζ be a bounded character of the group Z. Show that there exists exactly one $x \in T$ such that $\zeta(n) = e^{inx} \equiv \zeta_x(n)$ for all $n \in Z$.

Assume that Z is endowed with its discrete topology; verify that the dual topology on T (when it is regarded as the dual of Z) is identical with its initial topology (as defined in 2.1.1).

2.4. Determine the continuous linear functionals I on $\mathbf{C} = \mathbf{C}(T)$ which are "relatively invariant" in the sense that there exists a function Δ on T such that

$$I(T_a f) = \Delta(a) \cdot I(f)$$

for all $a \in T$ and all $f \in \mathbf{C}$.

Hint: Show first that, if $I \neq 0$, then $\Delta(a) = e^{ina}$ for some $n \in Z$. Then consider the functional J defined by $J(f) = I(e_{-n} \cdot f)$.

2.5. Consider the finite group $Z_m = Z/mZ$, where m is a positive integer, taken with the discrete topology. What are the invariant integrals on Z_m? What are the characters of Z_m? Discuss the Fourier theory for this group.

2.6. Let t be a continuous homomorphism of $R/2\pi Z$ onto itself. Show that there exists $k \in Z$ such that t maps the coset $x + 2\pi Z$ into the coset $kx + 2\pi Z$.

Hint: Consider the character of T which carries $x + 2\pi Z$ into e^{inx}, so getting a homomorphism s of Z into itself for which $e^{int(x+2\pi Z)} = e^{is(n)(x+2\pi Z)}$. Then examine s.

2.7. Suppose that $f \in \mathbf{L}^1$. Show that

$$\hat{f}(n) = O(e^{-\varepsilon|n|})$$

for some $\varepsilon > 0$ if and only if f is equal almost everywhere to a function which is analytic in a horizontal strip $|\operatorname{Im} z| < \delta$ for some $\delta > 0$.

Hint: For the "if" part, apply Cauchy's theorem for a suitable rectangle to the integral defining $\hat{f}(n)$.

2.8. Suppose $f \in \mathbf{C}^\infty$ and put

$$M_k = \|D^k f\|_1 \qquad (k = 1, 2, \cdots).$$

Show that for $n \neq 0$

$$|\hat{f}(n)| \leqslant M_k |n|^{-k}.$$

Show also that if

$$M_k \leqslant \text{const}\ R^k \Gamma(\alpha k + 1) \qquad (k = 1, 2, \cdots) \tag{1}$$

for some $R > 0$, $\alpha > 0$, then

$$|\hat{f}(n)| \leqslant \text{const}\ |n|^{1/2\alpha} \cdot \exp\left[-\left(\frac{|n|}{R}\right)^{1/\alpha}\right] \qquad (n \in Z, n \neq 0); \tag{2}$$

and that if (2) holds, then

$$M_k \leqslant \text{const}\ R^k \Gamma(\alpha k + 1)(1 + k^{\alpha - 1/2}) \qquad (k = 1, 2, \cdots). \tag{3}$$

Hint: Make use of Stirling's formula describing the behavior of $\Gamma(z)$ for large positive values of z.

Remarks. Functions f for which an inequality of the type (1) holds (the constant possibly depending upon f) form the simplest types of what are termed *quasi-analytic classes* of functions; the case $\alpha = 1$ corresponds to the analytic functions. The relationships between such classes and those defined by means of inequalities involving the Fourier coefficients—like (2), for example—

have been studied in great detail. See, for example, [M], especially pp. 78–79, 138–139.

2.9. Give a proof of 2.3.8 based solely on 2.4.4 and 2.3.2.

2.10. Derive 2.4.1 directly from that part of 2.4.4 which refers to the space **C**.

2.11. Deduce from 2.4.1 the following uniqueness theorem: If f is defined and integrable over $(-\pi, \pi)$, and if

$$\int_{-\pi}^{\pi} f(x)x^N\,dx = 0 \qquad (N = 0, 1, 2, \cdots),$$

then $f = 0$ a.e. on $(-\pi, \pi)$.

Note: On the basis of the Hahn-Banach theorem (see Appendix B.5), this assertion implies results about approximation by ordinary polynomials akin to Weierstrass' theorem (see 6.2.2).

2.12. Show that if f is defined and integrable over $(-\pi, \pi)$, and if

$$\int_{-\pi}^{\pi} f(x)e^{i\alpha_N x}\,dx = 0 \qquad (N = 1, 2, \cdots),$$

where $(\alpha_N)_{N=1}^{\infty}$ is a sequence of complex numbers having at least one (finite) limiting point, then $f = 0$ a.e. on $(-\pi, \pi)$.

Hint: We may assume without loss of generality that $\alpha_N \neq 0$, $\alpha_N \to 0$. Consider $\int_{-\pi}^{\pi} f(x)e^{izx}\,dx$ as a function of the complex variable z.

2.13. (Lusin-Denjoy theorem) Write the trigonometric series $\sum_{n\in Z} c_n e^{inx}$ in its real form $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, and suppose that

$$\sum_{n=0}^{\infty} |c_n e^{inx} + c_{-n}e^{-inx}| \equiv |a_0| + \sum_{n=1}^{\infty} |a_n \cos nx + b_n \sin nx|$$

converges for $x \in E$, where E is measurable and the Lebesgue measure $m(E)$ of E is positive. Prove that

$$\sum_{n\in Z} |c_n| < \infty,$$

that is, that

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty.$$

Hints: Assume without loss of generality that the a_n and b_n are real-valued and put $a_n = r_n \cos \theta_n$, $b_n = r_n \sin \theta_n$, where $r_n \geqslant 0$ and θ_n is real. Use Egorov's theorem to justify termwise integration of the series

$$\sum_{n=1}^{\infty} r_n |\cos (nx - \theta_n)|$$

over some set E_0 with $m(E_0) > 0$, and observe that

$$\begin{aligned}\int_{E_0} |\cos (nx - \theta_n)|\,dx &\geqslant \int_{E_0} \cos^2 (nx - \theta_n)\,dx \\ &= \tfrac{1}{2}m(E_0) + o(1).\end{aligned}$$

Note: For a simple generalization of this result, see [KS], p. 84, Théorème II. The Lusin-Denjoy theorem has prompted numerous more elaborate investigations of the absolute convergence of trigonometric series: see [Z_1], Chapter VI; [Ba_2], Chapter IX; [KS], Chapitre VII.

2.14. (Cantor-Lebesgue theorem) As indicated at the end of 2.2.2, a trigonometric series $\sum_{n\in Z} c_n e^{inx}$ is said to converge for a particular value of x if and only if

$$\lim_{N\to\infty} \sum_{|n|\leqslant N} c_n e^{inx}$$

exists finitely for that value of x. Show that if this is true for each point x belonging to a measurable set E having positive Lebesgue measure, then $\lim_{|n|\to\infty} c_n = 0$.

Hints: As in the hints for the preceding exercise, reduce the problem to the case in which $\sum r_n \cos(nx - \theta_n)$ is uniformly convergent for $x \in E_0$, where $m(E_0) > 0$. Were the assertion to be false, there would exist integers $n_1 < n_2 < \cdots$ so that

$$\cos(n_k x - \theta_{n_k}) \to 0$$

uniformly for $x \in E_0$. Consider the integrals

$$\int_{E_0} \cos^2(n_k x - \theta_{n_k})\, dx.$$

Notes: Cantor considered the case in which E is a nondegenerate interval. Steinhaus produced examples of series $\sum_{n\in Z} c_n e^{inx}$ for which $c_n \to 0$ and yet the series diverges everywhere. One such example is

$$\sum_{n=3}^{\infty} (\log n)^{-1} \cdot \cos n(x - \log\log n);$$

see [Ba_1], p. 176.

2.15. (Equidistributed sequences) Let x_0 be real and such that x_0/π is irrational. Suppose g is a periodic function with the property that to each $\varepsilon > 0$ correspond continuous periodic functions u and v such that $u \leqslant g \leqslant v$ and

$$\frac{1}{2\pi}\int (v - u)\, dx < \varepsilon.$$

Show that

$$\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} g(nx_0) = \hat{g}(0). \tag{1}$$

Deduce that if I is a subinterval of $[0, 2\pi)$, then N^{-1} times the number of points $x_0, 2x_0, \cdots, Nx_0$ which belong modulo 2π to I converges, as $N \to \infty$, to $(2\pi)^{-1}$ times the length of I, that is, that the points nx_0 $(n = 1, 2, \cdots)$ are equidistributed modulo 2π. Observe that this result implies that of Exercise 2.2. See also [Ba_2], p. 473.

Hints: First prove (1) for continuous periodic g by using 2.4.4.

Notes: The idea of equidistributed sequences is due to Weyl. Several results will be found in [PS], Band I, pp. 70–74.

For extensions to general groups, see [HR], pp. 432, 437–438 and Rubel [1].

2.16. (Fejér's lemma) Suppose that $1 \leqslant p \leqslant \infty$, that $f \in \mathbf{L}^p$, and that $g \in \mathbf{L}^{p'}$, where $1/p + 1/p' = 1$. Prove that

$$\frac{1}{2\pi} \int f(x) g(nx)\, dx \to \hat{f}(0)\hat{g}(0)$$

for $n \in Z$ and $|n| \to \infty$.

Hints: Assume first that $p > 1$ and use 2.4.4 to approximate g in $\mathbf{L}^{p'}$ by trigonometric polynomials; then use 2.3.8. If $p = 1$, so that $p' = \infty$, approximate f in $\mathbf{L}^1$ by continuous functions.

Note: The result actually remains true if the restriction $n \in Z$ is replaced by $n \in R$.

2.17. Let $(\varepsilon)_{n=1}^{\infty}$ be any sequence of positive numbers converging to zero. By extracting a suitable subsequence (ε_{n_k}) and considering the series

$$\sum_{k=1}^{\infty} \varepsilon_{n_k} \exp{(in_k x)},$$

show that there exist continuous functions f such that

$$\limsup_{n \to \infty} \frac{\omega_1 f(\pi/n)}{\varepsilon_n} > 0.$$

Hint: Use 2.3.7.

2.18. Prove that any nonnegative continuous function f is the uniform limit of functions $|g|^2$, where g denotes a trigonometric polynomial.

Formulate and prove an analogous result for functions f in $\mathbf{L}^p$ $(1 \leqslant p < \infty)$.

Hint: See Exercise 1.11.

2.19. Prove that, for any finite set $F \subset Z$ and any $\varepsilon > 0$, a trigonometric polynomial f exists such that

$$0 \leqslant \hat{f}(n) \leqslant 1 \quad \text{for all } n \in Z, \qquad \hat{f}(n) = 1 \quad \text{for all } n \in F,$$
$$\|f\|_1 \leqslant 1 + \varepsilon.$$

For which sets F does the result remain true when $\varepsilon = 0$?

Hints: Suppose r is a positive integer such that $F \subset [-r, r]$ and choose a large positive integer N. Consider

$$\begin{aligned} f(x) &= \Big[(2N+1)^{-1} \sum_{|n| \leqslant N} e^{inx}\Big] \cdot \Big[\sum_{|n| \leqslant N+r} e^{inx}\Big] \\ &= u(x) \cdot v(x) \end{aligned}$$

and use Exercise 1.7(1) and the Cauchy-Schwarz inequality.

CHAPTER 3

Convolutions of Functions

3.1 Definition and First Properties of Convolution

At the end of 2.3.9 we posed the problem of finding a binary operation on integrable functions that would correspond to pointwise multiplication of their Fourier transforms. To attempt directness by trying to *define* the result, say $f * g$, of applying this operation to functions $f, g \in \mathbf{L}^1$ by requiring that $(f * g)^\wedge = \hat{f} \cdot \hat{g}$ is not very effective, because we do not know how to characterize $\mathbf{A}(Z)$ in such a way that it is clear that it is closed under pointwise multiplication. A more useful clue is provided by the orthogonality relations combined with the special properties of characters.

Suppose that we write e_n for the function $x \to e^{inx}$ $(n \in Z)$. For m and n in Z, the orthogonality relations (1.1.3) show that

$$\frac{1}{2\pi} \int e_m(x - y) e_n(y)\, dy = \begin{cases} e_n(x) & \text{if } m = n \\ 0 & \text{otherwise}. \end{cases}$$

Accordingly, if we define $f * g$ by

$$f * g(x) = \frac{1}{2\pi} \int f(x - y) g(y)\, dy, \tag{3.1.1}$$

then it appears that $e_m * e_n$ has as its Fourier transform the pointwise product of the Fourier transforms $\hat{e}_m$ and $\hat{e}_n$. Since each of $f * g$ and $\hat{f} \cdot \hat{g}$ is evidently bilinear in the pair (f, g), the desired relation will obtain for functions f and g which are trigonometric polynomials, that is, finite linear combinations of the e_n. It thus appears that (3.1.1) constitutes a hopeful starting point. We proceed to the details forthwith.

Suppose that f and g belong to $\mathbf{L}^1$. Then the Fubini-Tonelli theorem is applicable (see [W], Theorems 4.2b, 4.2c, and 4.2d; [HS], pp. 384–386, 396; [AB], pp. 154–155) and shows that the integrand appearing on the right of (3.1.1) is, for almost all x, an integrable function of y, so that (3.1.1) effectively defines $f * g(x)$ for almost all x; moreover the function so defined almost everywhere is measurable and

$$\|f * g\|_1 \leqslant \|f\|_1 \cdot \|g\|_1. \tag{3.1.2}$$

In particular, $f * g \in \mathbf{L}^1$.

From these arguments it also appears that

$$|f * g(x)| \leqslant |f| * |g|(x) \qquad \text{a.e.} \tag{3.1.3}$$

As a consequence of invariance of the integral it appears that at any point x for which $f * g(x)$ exists, $g * f(x)$ exists and has the same value. Thus

$$f * g = g * f. \tag{3.1.4}$$

Let us now compute the Fourier coefficients of $f * g$, using en route the Fubini-Tonelli theorem and the relation $e^{-inx} = e^{-in(x-y)} \cdot e^{-iny}$:

$$\begin{aligned}
(f * g)^\wedge(n) &= \frac{1}{2\pi}\int f * g(x) e^{-inx}\,dx \\
&= \frac{1}{2\pi}\int e^{-inx}\left\{\frac{1}{2\pi}\int f(x-y)g(y)\,dy\right\}dx \\
&= \left(\frac{1}{2\pi}\right)^2 \iint e^{-inx} f(x-y)g(y)\,d(x,y) \\
&= \left(\frac{1}{2\pi}\right)^2 \int\left\{\int f(x-y)e^{-in(x-y)}\,g(y)e^{-iny}\,dx\right\}dy
\end{aligned}$$

by two appeals to the Fubini-Tonelli theorem

$$\begin{aligned}
&= \frac{1}{2\pi}\int g(y)e^{-iny}\left\{\frac{1}{2\pi}\int f(x-y)e^{-in(x-y)}\,dx\right\}dy \\
&= \frac{1}{2\pi}\int g(y)e^{-iny}\,\{\hat{f}(n)\}\,dy
\end{aligned}$$

by translation invariance of the inner integral

$$= \hat{f}(n)\hat{g}(n).$$

Thus we have the desired relationship:

$$(f * g)^\wedge(n) = \hat{f}(n)\cdot\hat{g}(n) \qquad \text{for all } n \in Z. \tag{3.1.5}$$

Convolution is associative, that is, $(f * g) * h = f * (g * h)$ for $f, g, h \in \mathbf{L}^1$. A direct verification is possible, using the Fubini-Tonelli theorem. Alternatively, one may appeal to (3.1.5), to the uniqueness theorem of 2.4.1, and to the evident associativity of pointwise multiplication of the transforms $\hat{f}$, $\hat{g}$, and $\hat{h}$.

Remarks. The definition and the above properties of convolution may be formulated and established in another way, thereby making appeal to no more than the most primitive form of the Fubini theorem applying to continuous integrands (which somebody aptly christened "Fubinito").

One begins by defining $f * g$ for $f, g \in \mathbf{C}$ by (3.1.1). Fubinito then yields (3.1.2) to (3.1.5) for such f and g. It is furthermore evident that $(f, g) \to f * g$ is bilinear from $\mathbf{C} \times \mathbf{C}$ into $\mathbf{C} \subset \mathbf{L}^1$.

The inequality (3.1.2), which expresses the continuity of the bilinear map

$(f, g) \to f * g$ from $\mathbf{C} \times \mathbf{C}$ (with the topology induced by that on $\mathbf{L}^1 \times \mathbf{L}^1$) into $\mathbf{L}^1$, ensures that this mapping can be uniquely extended continuously into a bilinear mapping from $\mathbf{L}^1 \times \mathbf{L}^1$ into $\mathbf{L}^1$. Specifically, if $f, g \in \mathbf{L}^1$, $f * g$ will be the $\mathbf{L}^1$-limit of $(f_i * g_i)$, where (f_i) and (g_i) are sequences extracted from $\mathbf{C}$ and converging to f and g in $\mathbf{L}^1$, respectively. [The inequality (3.1.2) ensures that $(f_i * g_i)$ is Cauchy, and therefore convergent, in $\mathbf{L}^1$; and that the limit does not depend on the chosen sequences but only on f and g.] This mode of extension ensures that (3.1.2), (3.1.4), and (3.1.5) continue to hold, the last by virtue of 2.3.2.

It remains to verify that (3.1.1) holds almost everywhere for general f and g in $\mathbf{L}^1$. Since both sides of this inequality are bilinear in (f, g), and since (as the reader should pause to prove) any real-valued $h \in \mathbf{L}^1$ is equal a.e. to the difference of two nonnegative integrable functions h_1 and h_2, each equal to the limit of a monotone increasing sequence of nonnegative continuous functions, it may and will be assumed that sequences (f_i) and (g_i) may be chosen from $\mathbf{C}$ so that $0 \leqslant f_i \uparrow f$ a.e. and $0 \leqslant g_i \uparrow g$ a.e. Then the monotone convergence theorem shows that

$$f_i * g_i(x) \uparrow \frac{1}{2\pi} \int f(x - y) g(y) \, dy .$$

Since $(f_i * g_i)$ converges in $\mathbf{L}^1$ to $f * g$, it follows that (3.1.1) holds for almost all x. This shows in particular that $y \to f(x - y)g(y)$ is integrable for almost all x whenever f and g are nonnegative functions in $\mathbf{L}^1$. If f and g are replaced by $|f|$ and $|g|$, it is seen that the same is true whenever f and g belong to $\mathbf{L}^1$. Once (3.1.1) is established for general $f, g \in \mathbf{L}^1$, (3.1.3) results immediately.

[The reader should note that it is *not* the case that every nonnegative integrable function h is equal a.e. to the limit of an $\uparrow$ sequence (h_n) of continuous nonnegative functions. As a counterexample, take h to be defined on $[0, 2\pi]$ as the characteristic function of the complement, relative to $[0, 2\pi]$, of a closed, nowhere dense set $K \subseteq [0, 2\pi]$ having positive measure.]

Similar techniques are applicable in connection with 3.1.4 to 3.1.6.

At this point it is convenient to summarize what little we do know so far about convolution and to mention a few questions that arise, which will guide some of the subsequent developments.

3.1.1. **Some Problems.** The convolution $f * g$ of two functions f and g chosen from $\mathbf{L}^1$ being defined by (3.1.1), the mapping $(f, g) \to f * g$ is an associative and commutative bilinear mapping of $\mathbf{L}^1 \times \mathbf{L}^1$ into $\mathbf{L}^1$; this mapping is continuous by virtue of (3.1.2). In current terminology (which will be explained at greater length in Section 11.4), $\mathbf{L}^1$ forms a commutative complex Banach algebra under convolution. As will be seen subsequently in (c), however, $\mathbf{L}^1$ possesses no identity (or unit) element relative to convolution.

By 2.3.1, 2.3.2, and (3.1.5), the Fourier transformation $f \to \hat{f}$ is a continuous homomorphism of the convolution algebra $\mathbf{L}^1$ into the algebra $c_0(Z)$, the latter being taken with pointwise operations as in 2.3.9. In particular, for each $n \in Z$ the mapping

$$\gamma_n : f \rightarrow \hat{f}(n) \tag{3.1.6}$$

is a continuous homomorphism of $\mathbf{L}^1$ onto the complex field (the latter being regarded as an algebra over itself).

A few questions with an algebraic flavor arise quite naturally at this point. The following selection is typical and significant.

(a) Is the mapping $f \rightarrow \hat{f}$ actually an isomorphism of $\mathbf{L}^1$ into $\mathbf{c}_0(Z)$? This is equivalent to asking whether the relations $\hat{f}(n) = 0$ $(n \in Z)$ entail that $f = 0$ a.e., and has thus been answered affirmatively in 2.4.1.

(b) Are there any continuous homomorphisms of $\mathbf{L}^1$ onto the complex numbers distinct from the γ_n $(n \in Z)$? In 4.1.2 we shall see that the response is negative, thereby providing a very satisfactory explanation of the fundamental nature of the Fourier transformation.

(c) From 2.3.8 and (3.1.5) it appears at once that $\mathbf{L}^1$ contains no identity for convolution, that is, no element e such that $e * f = f$ for all $f \in \mathbf{L}^1$. [Were such an element e to exist, one would infer from (3.1.5) that $\hat{e}(n) = 1$ for all $n \in Z$, a possibility which is ruled out by 2.3.8.] In view of this, one is prompted to ask whether every $f \in \mathbf{L}^1$ can be factored into a convolution product $f_1 * f_2$ with f_1 and f_2 in $\mathbf{L}^1$. An affirmative answer was published relatively recently (Walter Rudin and P. J. Cohen), though the result was known to Salem and Zygmund somewhat earlier; see [Z_1], p. 378. Cohen's method is a most elegant one, applying to a general class of Banach algebras. We shall return to this and similar problems in Section 7.5; see also 11.4.18(6).

(d) Which elements e of $\mathbf{L}^1$ are idempotent, that is, satisfy $e * e = e$? Plainly, every trigonometric polynomial of the form

$$e = \sum e_n \qquad \text{(a finite sum)}$$

is idempotent. From 2.3.8 and 2.4.1 it appears that these are indeed the only idempotents in $\mathbf{L}^1$.

(e) Since $e_m * e_n = 0$ if $m \neq n$, it is clear that $\mathbf{L}^1$ is not an integral domain (that is, that it possesses an abundance of zero divisors). Which subalgebras of $\mathbf{L}^1$ possess no zero divisors? Some light will be shed upon this in 11.3.9.

(f) Can one classify or describe the closed subalgebras of $\mathbf{L}^1$? This appears to be an extremely difficult problem. It is easy to see that, if one takes a sequence $(S_k)_{k=1}^{\infty}$ of pairwise disjoint and finite subsets of Z, then the set of $f \in \mathbf{L}^1$, such that $\hat{f}$ takes an (f-dependent) constant value on S_k $(k = 1, 2, \cdots)$ and $\hat{f} = 0$ on $Z \backslash \bigcup_{k=1}^{\infty} S_k$, is a closed subalgebra of $\mathbf{L}^1$. It was natural to hope that all closed subalgebras of $\mathbf{L}^1$ would prove to be of this type. However, J.-P. Kahane has recently disproved this conjecture. His results, as well as some of the simpler aspects of the study of closed subalgebras, will be discussed in Section 11.3.

(g) Conforming with normal algebraic terminology, an ideal in $\mathbf{L}^1$ will mean a linear subspace $\mathbf{I}$ of $\mathbf{L}^1$ having the property that $f * g \in \mathbf{I}$ whenever $f \in \mathbf{I}$ and $g \in \mathbf{L}^1$; a closed ideal will mean a subset that is both an ideal and a closed subset of $\mathbf{L}^1$.

This being so, can one effectively describe all the closed ideals in $\mathbf{L}^1$?

In Section 11.2 we shall uncover an affirmative answer to this and a number of similar questions for the case in which the underlying group is T, noticing at the same time, however, that for many groups of interest the analogous questions have no known answer that is completely satisfactory. Meanwhile, see Exercises 3.2 to 3.5.

The reader may be struck by the fact that the questions in (f) and (g) are not phrased in purely algebraic terms, inasmuch as we speak of *closed* subalgebras and ideals. This topological restriction is customary when one is dealing with infinite dimensional algebras. To seek to classify all (not necessarily closed) subalgebras or ideals is both rather unnatural and overambitious. Topological restrictions compensate in some measure for the infinite dimensionality and are natural just because of this feature.

As has been indicated, problem (f) is not yet solved completely. Beside this we shall in 12.7.4 encounter problems (b), (d), and (g) in a new setting, the algebra $\mathbf{L}^1$ being replaced by a larger one; and in Section 16.8 problem (d) will appear for still larger convolution algebras. The analogues of (b) and (g) for some of these enlarged algebras have not yet received complete solutions. It may also be added that, for underlying groups of types markedly different from T, the $\mathbf{L}^1$ version of problem (d) is a good deal more difficult; see Chapter 3 of [R] and also Rudin [3], Rudin and Schneider [1], Rider [1].

With these traces of ignorance left showing, we turn to some simple analytical properties of the convolution process that will play a fundamental role in subsequent developments.

A start is made with two properties that stem directly from the invariance of the integral; the proofs are left for the reader to provide.

3.1.2. $$T_a(f * g) = T_a f * g = f * T_a g.$$

3.1.3. $$T_a f * T_b g = T_{a+b} f * g.$$

3.1.4. Suppose that $1 \leqslant p \leqslant \infty$ and that p' is the conjugate exponent (or index), defined by $1/p + 1/p' = 1$ ($p' = 1$ if $p = \infty$ and $p' = \infty$ if $p = 1$). If $f \in \mathbf{L}^p$ and $g \in \mathbf{L}^{p'}$, then $f * g$ is defined everywhere, is continuous, and

$$\|f * g\|_\infty \leqslant \|f\|_p \cdot \|g\|_{p'}.$$

Proof. Hölder's inequality shows that the function $y \to f(x - y)g(y)$ is in this case integrable for each x, so that $f * g(x)$ is defined for all x, and that

$$\begin{aligned} |f * g(x)| &= \left| \frac{1}{2\pi} \int f(x - y) g(y)\, dy \right| \\ &\leqslant \|f(x - y)\|_p \cdot \|g\|_{p'}, \end{aligned}$$

where in the first factor $f(x - y)$ is regarded as a function of y. By translation-invariance, $\|f(x - y)\|_p = \|f\|_p$; whence the stated inequality. To show that $f * g$ is continuous, we may by symmetry suppose that $p < \infty$ and then use 3.1.1, 3.1.2, and the inequality just established to obtain

$$\begin{aligned}\|T_a(f * g) - f * g\|_\infty &= \|T_a f * g - f * g\|_\infty \\ &= \|(T_a f - f) * g\|_\infty \\ &\leqslant \|T_a f - f\|_p \cdot \|g\|_{p'}.\end{aligned}$$

Finally, $\|T_a f - f\|_p \to 0$ as $a \to 0$ whenever $f \in \mathbf{L}^p$ and $p < \infty$ (see 2.2.4).

Remarks. (1) The preceding result contains the first hint that convolution is a smoothing process. The next two results develop this theme by showing that, if $f \in \mathbf{L}^1$, then $f * g$ shares with g a number of smoothness properties. Further developments along these lines must be deferred until Chapter 12; see especially 12.6.2, 12.7.2, and 12.7.3.

(2) There are valid converses of 3.1.4 which are in essence contained in 12.8.4 and 16.3.5.

3.1.5. (1) If $f \in \mathbf{L}^1$ and if $g \in \mathbf{C}^k$, or is of bounded variation, or is absolutely continuous, then $f * g$ has the same property. Moreover, in the first case one has

$$D^m(f * g) = f * D^m g \tag{3.1.7}$$

for any integer $m \geqslant 0$ not exceeding k.

(2) The formula (3.1.7) holds for $m = 1$ whenever $f \in \mathbf{L}^1$ and g is absolutely continuous.

Proof. (1) We shall deal with the assertion involving $\mathbf{C}^k$, leaving the remaining (similar) arguments to be provided by the reader. In dealing with this selected case, it will suffice to show that $f * g \in \mathbf{C}^1$ if $g \in \mathbf{C}^1$ and that $D(f * g) = f * Dg$; the rest will follow by induction on m.

Now if $a \neq 0$

$$a^{-1}[f * g(x + a) - f * g(x)] = \frac{1}{2\pi}\int f(y)\,\frac{g(x + a - y) - g(x - y)}{a}\,dy.$$

Since $g \in \mathbf{C}^1$, the cofactor of $f(y)$ in the integrand tends, as $a \to 0$, to $Dg(x - y)$; and, as the mean value theorem shows, the convergence is uniform with respect to y (and to x). It follows from general convergence theorems ([W], Theorem 4.1b, for example) that $f * g$ has a derivative equal to $f * Dg$, this last function being continuous by virtue of the case $p = 1$ of 3.1.4.

(2) Suppose that $f \in \mathbf{L}^1$ and that g is absolutely continuous. We will first show that $f * g$ is absolutely continuous.

For any two real numbers a and b we have

$$|f * g(b) - f * g(a)| \leqslant \frac{1}{2\pi}\int |f(y)|\,|g(b - y) - g(a - y)|\,dy. \tag{3.1.8}$$

Since g is absolutely continuous, to any $\varepsilon > 0$ corresponds a number $\delta = \delta(\varepsilon) > 0$ such that

$$\sum_{k=1}^{r} |g(b_k) - g(a_k)| \leqslant \varepsilon$$

for any sequence $([a_k, b_k])_{k=1}^{r}$ of nonoverlapping intervals $[a_k, b_k]$ for which $\sum_{k=1}^{r} (b_k - a_k) \leqslant \delta$. But then, under the same conditions on these intervals,

$$\sum_{k=1}^{r} |g(b_k - y) - g(a_k - y)| \leqslant \varepsilon$$

for all y and so (3.1.8) shows that

$$\sum_{k=1}^{r} |f * g(b_k) - f * g(a_k)| \leqslant \|f\|_1 \cdot \varepsilon.$$

This shows that $f * g$ is absolutely continuous.

That (3.1.7) holds with $m = 1$ is now most easily seen by applying 2.3.4 to each side and referring to (3.1.5).

3.1.6. If $f \in \mathbf{L}^1$ and $g \in \mathbf{L}^p$ $(1 \leqslant p \leqslant \infty)$, then $f * g \in \mathbf{L}^p$ and

$$\|f * g\|_p \leqslant \|f\|_1 \cdot \|g\|_p.$$

Proof. For any $h \in \mathbf{L}^{p'}$ the Fubini-Tonelli theorem gives

$$\begin{aligned} \left|\frac{1}{2\pi} \int f * g(x)h(x)\, dx\right| &\leqslant \frac{1}{2\pi} \int |h(x)| \left\{\frac{1}{2\pi} \int |f(y)g(x - y)|\, dy\right\} dx \\ &= \frac{1}{2\pi} \int |f(y)| \left\{\frac{1}{2\pi} \int |h(x)g(x - y)|\, dx\right\} dy. \end{aligned}$$

By Hölder's inequality the inner integral is majorized by $\|h\|_{p'} \cdot \|g\|_p$. Hence

$$\left|\frac{1}{2\pi} \int f * g(x)h(x)\, dx\right| \leqslant \|f\|_1 \cdot \|h\|_{p'} \cdot \|g\|_p.$$

The converse of Hölder's inequality (see Exercise 3.6) now goes to show that $f * g \in \mathbf{L}^p$ and $\|f * g\|_p \leqslant \|f\|_1 \cdot \|g\|_p$, as alleged.

Remarks. (1) The argument can be made less sophisticated by assuming first that f and g are continuous. One may then assume that h, too, is continuous. The required versions of the Fubini-Tonelli theorem and the Hölder inequality and its converse then become simpler. This leads to the stated result for f and g continuous. In general, we may assume that $p < \infty$, since otherwise the result is contained in 3.1.4, and then approximate f and g in $\mathbf{L}^1$ and in $\mathbf{L}^p$, respectively, by continuous functions f_n and g_n $(n = 1, 2, \cdots)$. By (3.1.2), $f_n * g_n \to f * g$ in $\mathbf{L}^1$, and so a subsequence converges almost everywhere. By applying the result already established for continuous f and g to the terms of such a subsequence, and making use of Fatou's lemma ([W], Theorem 4.1d) on the way, the desired result appears.

(2) The assertion in 3.1.6 can be improved in several ways; see 12.7.3 and Exercise 13.5. A result that combines and extends 3.1.4 and 3.1.6 will be obtained in Section 13.6.

3.1.7. **Other Convolution Algebras.** From 3.1.5 it appears in particular that each $\mathbf{C}^k$ is an associative and commutative complex algebra under $*$, and that the same is true of the set of functions of bounded variation, and the set of absolutely continuous functions. Each of these, save $\mathbf{C}^\infty$, can in fact be made into a commutative complex Banach algebra under $*$. ($\mathbf{C}^\infty$ is not normable, but it is a good complete, metrisable, associative, and commutative complex topological algebra under $*$.) Similarly, by 3.1.6, $\mathbf{L}^p$ ($1 \leqslant p \leqslant \infty$) is an associative and commutative complex Banach algebra. Being subalgebras of $\mathbf{L}^1$ (in the purely algebraic sense) which contain all trigonometric polynomials, none of these algebras possess an identity element; see 3.1.1(c).

Beside this, 3.1.6 shows that $\mathbf{L}^p$ ($1 \leqslant p \leqslant \infty$) can be regarded as a module over the ring $\mathbf{L}^1$ ($*$ being both the ring product in $\mathbf{L}^1$ and the module product between elements of $\mathbf{L}^1$ and elements of $\mathbf{L}^p$).

3.1.8. **Convolution and Translation.** Both 3.1.2 and 3.1.3 hint at close connections between translation operators T_a and convolution. This will be borne out as we progress (see especially Sections 16.2 and 16.3). Meanwhile here is a basic result in this direction.

3.1.9. Let $f \in \mathbf{L}^1$ and let $\mathbf{E}$ denote any one of the normed spaces $\mathbf{C}$ or $\mathbf{L}^p$ ($1 \leqslant p < \infty$). If $g \in \mathbf{E}$, then $f * g$ is the limit in $\mathbf{E}$ of finite linear combinations of translates of g.

Proof. Let $g \in \mathbf{E}$ be given. Denote by $\mathbf{V}_g$ the closed linear subspace of $\mathbf{E}$ generated by the translates $T_a g$ of g, that is, the closure in $\mathbf{E}$ of the set of all finite linear combinations of elements $T_a g$. Denote also by $\mathbf{S}$ the set of $f \in \mathbf{L}^1$ such that $f * g \in \mathbf{V}_g$. It has to be shown that $\mathbf{S} = \mathbf{L}^1$. Now it is evident that $\mathbf{S}$ is a linear subspace of $\mathbf{L}^1$; and from 3.1.6 it follows that $\mathbf{S}$ is closed in $\mathbf{L}^1$. It will therefore suffice to show that $\mathbf{S}$ contains a subset $\mathbf{S}_0$ such that the finite linear combinations of elements of $\mathbf{S}_0$ are dense in $\mathbf{L}^1$.

If $\mathbf{E} = \mathbf{C}$, a convenient choice of $\mathbf{S}_0$ is $\mathbf{C}$ (see [W], Theorem 4.3b). We will leave to the reader the task of showing that in fact $f * g$ is the uniform limit of finite linear combinations of translates of g, whenever f and g are continuous. (*Hint:* Approximate the integral defining $f * g$ by Riemann sums, using uniform continuity of the functions involved.)

We pass on to the remaining cases.

Suppose, then, that $\mathbf{E} = \mathbf{L}^p$ ($1 \leqslant p < \infty$). In this case a convenient choice of $\mathbf{S}_0$ is the set of functions f which coincide on $[0, 2\pi]$ with the characteristic function of an interval $I = [a, b]$, where $0 < a < b < 2\pi$, and which are defined elsewhere by periodicity (compare [W], Theorem 4.3a). In this case we partition I by a finite number of subintervals I_k whose lengths $|I_k|$ are majorized by a number δ to be chosen shortly. Choose and fix a point a_k in each I_k. We then have

$$\begin{aligned} f * g(x) &- \frac{1}{2\pi}\sum_k |I_k| g(x - a_k) \\ &= \frac{1}{2\pi}\sum_k \int_{I_k} [g(x - y) - g(x - a_k)]\, dy \\ &= \frac{1}{2\pi}\sum_k h_k(x), \end{aligned}$$

say, and so Minkowski's inequality yields

$$\|f * g - \frac{1}{2\pi}\sum_k |I_k| \cdot T_{a_k} g\|_p \leqslant \frac{1}{2\pi}\sum \|h_k\|_p. \tag{3.1.9}$$

Next, using Hölder's inequality and the Fubini-Tonelli theorem,

$$\begin{aligned} \|h_k\|_p^p &= \frac{1}{2\pi}\int \{|\int_{I_k} [g(x - y) - g(x - a_k)]\, dy|^p\}\, dx \\ &\leqslant \frac{1}{2\pi}\int \{|I_k|^{p/p'} \cdot \int_{I_k} |g(x - y) - g(x - a_k)|^p\, dy\}\, dx \\ &= |I_k|^{p/p'} \int_{I_k} \{\left(\frac{1}{2\pi}\right)\int |g(x - y) - g(x - a_k)|^p\, dx\}\, dy \\ &= |I_k|^{p/p'} \cdot \int_{I_k} \|T_y g - T_{a_k} g\|_p^p\, dy. \end{aligned} \tag{3.1.10}$$

Now, given $\varepsilon > 0$, we can choose $\delta > 0$ so small that

$$\|T_y g - T_{a_k} g\|_p^p \leqslant \varepsilon^p$$

for all $y \in I_k$ (see 2.2.4). Then (3.1.10) shows that

$$\|h_k\|_p^p \leqslant |I_k|^{p/p'} \cdot |I_k| \cdot \varepsilon^p = |I_k|^p \cdot \varepsilon^p,$$

since $p/p' + 1 = p$, and therefore

$$\|h_k\|_p \leqslant |I_k| \cdot \varepsilon. \tag{3.1.11}$$

Combining (3.1.9) with (3.1.11), we obtain

$$\begin{aligned} \|f * g - \frac{1}{2\pi}\sum_k |I_k| \cdot T_{a_k} g\|_p &\leqslant \frac{1}{2\pi}\sum_k |I_k| \cdot \varepsilon \\ &= \frac{1}{2\pi} |I| \cdot \varepsilon. \end{aligned}$$

Since

$$\frac{1}{2\pi}\sum_k |I_k| \cdot T_{a_k} g$$

is a finite linear combination of translates of g, this shows that $f * g \in \mathbf{V}_g$ and the proof is complete.

Remarks. For a complement to 3.1.9, see 3.2.3. See also Exercise 3.7. An alternative proof of 3.1.9 will appear in part (2) of the proof of 11.1.2.

3.1.10. **Characterization of Convolution.** Several of the results appearing in this section, taken singly or in combination, have converses which are interesting in that they virtually characterize convolution as a linear or bilinear process in terms of such basic concepts as the invariant integral, the standard function spaces, and the translation operators.

For example, 3.1.2 and 3.1.4 combine to show that, if $f \in \mathbf{L}^{p'}$ is given, then the mapping $U: g \to f * g$ is a continuous linear operator from $\mathbf{L}^p$ into $\mathbf{C}$ which commutes with translations (that is, $T_a U = U T_a$ for all $a \in R/2\pi Z$). As will be seen in 16.3.5, the converse is also true.

Again, in Subsection 16.3.11 we shall comment on converses of the result appearing in Subsection 3.1.9.

To take a third example, it has appeared that the mapping B: $(f, g) \to f * g$ is bilinear, has various continuity and positivity properties, and is related to translation in such a way that

$$B(T_a f, g) = T_a B(f, g) = B(f, T_a g).$$

It will appear in Subsection 16.3.12 that these properties go a long way toward characterizing convolution as a bilinear operator.

3.2 Approximate Identities for Convolution

In 3.1.1(c) it has been remarked that $\mathbf{L}^1$ contains no identity element for $*$. The same is true of the smaller $*$-algebras, $\mathbf{C}^k$ and $\mathbf{L}^p$ $(1 \leqslant p \leqslant \infty)$. This being so, we are going to consider and seek the next best thing, namely, a so-called "approximate identity."

3.2.1. By an *approximate identity* (for convolution) we shall mean a sequence $(K_n)_{n=1}^{\infty}$ of elements of $\mathbf{L}^1$ such that

$$\sup_n \|K_n\|_1 < \infty, \tag{3.2.1}$$

$$\lim_{n\to\infty} \frac{1}{2\pi} \int K_n(x)\, dx = 1, \tag{3.2.2}$$

and

$$\lim_{n\to\infty} \int_{\delta \leqslant |x| \leqslant \pi} |K_n(x)|\, dx = 0 \tag{3.2.3}$$

for any fixed δ satisfying $0 < \delta < \pi$.

It will be seen in Chapter 5, and it is fundamental to our work, that there exist approximate identities $(K_n)_{n=1}^{\infty}$ in which each K_n is a trigonometric polynomial.

More immediate examples stem from the observation that any sequence (K_n) of nonnegative integrable functions satisfying (3.2.2) and

$$\lim_{n\to\infty} \int_{\delta \leqslant |x| \leqslant \pi} K_n(x)\,dx = 0$$

for each fixed δ satisfying $0 < \delta < \pi$, constitutes an approximate identity. Thus, we might take for K_n $(n = 1, 2, \cdots)$ the function which is defined on $[-\pi, \pi)$ to be πn times the characteristic function of the interval $[-1/n, 1/n]$, and defined elsewhere by periodicity.

The name "approximate identity" is justified by the following result.

3.2.2. Let $(K_n)_{n=1}^{\infty}$ be an approximate identity.
Then

$$\lim_{n\to\infty} \|K_n * f - f\|_\infty = 0 \qquad (f \in \mathbf{C});$$

$$\lim_{n\to\infty} \|D^m(K_n * f) - D^m f\|_\infty = 0 \qquad (f \in \mathbf{C}^k)$$

provided m is an integer $\geqslant 0$ not exceeding k; and

$$\lim_{n\to\infty} \|K_n * f - f\|_p = 0 \qquad (f \in \mathbf{L}^p),$$

provided $1 \leqslant p < \infty$.

Proof. Inasmuch as $D^m(K_n * f) = K_n * D^m f$ whenever $K_n \in \mathbf{L}^1$ and $f \in \mathbf{C}^m$ (see 3.1.5), the second statement will follow from the first. The first follows from the uniform continuity of f in the following way.

We have

$$K_n * f(x) - f(x)\frac{1}{2\pi}\int K_n(y)\,dy = \frac{1}{2\pi}\int K_n(y)[f(x-y) - f(x)]\,dy.$$

Putting

$$\alpha_n = \frac{1}{2\pi}\int K_n(y)\,dy,$$

gives

$$\|K_n * f - \alpha_n f\|_\infty \leqslant \frac{1}{2\pi}\int |K_n(y)| \cdot \|T_y f - f\|_\infty \cdot dy \equiv I, \qquad (3.2.4)$$

say. Being assigned any $\varepsilon > 0$, choose and fix δ satisfying $0 < \delta \leqslant \pi$, so that $\|T_y f - f\|_\infty \leqslant \varepsilon$ for $|y| \leqslant \delta$.

Then

$$I \equiv \frac{1}{2\pi}\int |K_n(y)| \cdot \|T_y f - f\|_\infty \cdot dy$$
$$= \frac{1}{2\pi}\int_{|y|<\delta} + \frac{1}{2\pi}\int_{\delta \leqslant |y| \leqslant \pi}. \qquad (3.2.5)$$

The first integral is, by virtue of (3.2.1), majorized by

$$\varepsilon \cdot \frac{1}{2\pi} \int |K_n| \, dy \leqslant M\varepsilon, \tag{3.2.6}$$

where M is independent of n. Since also $\|T_y f - f\|_\infty \leqslant 2\|f\|_\infty$ for all y, the second integral is majorized by

$$2\|f\|_\infty \cdot \frac{1}{2\pi} \int_{\delta \leqslant |y| \leqslant \pi} |K_n(y)| \, dy. \tag{3.2.7}$$

Keeping ε and δ fixed, we find that (3.2.3) to (3.2.7) show that

$$\limsup_{n \to \infty} \|K_n * f - \alpha_n f\|_\infty \leqslant M\varepsilon.$$

Since ε is an arbitrary positive number, and since $\lim \alpha_n = 1$ by (3.2.2), it follows that

$$\lim_{n \to \infty} \|K_n * f - f\|_\infty = 0$$

whenever f is continuous.

To prove the third statement, given $f \in \mathbf{L}^p$ and $\varepsilon > 0$, first choose $f^\dagger \in \mathbf{C}$ such that $\|f - f^\dagger\|_p \leqslant \varepsilon$. By 3.1.6 and (3.2.1),

$$\|K_n * f - K_n * f^\dagger\|_p \leqslant \|K_n\|_1 \cdot \varepsilon \leqslant M\varepsilon, \tag{3.2.8}$$

where M is independent of n. By what has been established, there exists $n_0 = n_0(\varepsilon)$ so that

$$\|K_n * f^\dagger - f^\dagger\|_\infty \leqslant \varepsilon \qquad \text{for } n > n_0.$$

A fortiori, then,

$$\|K_n * f^\dagger - f^\dagger\|_p \leqslant \varepsilon \qquad \text{for } n > n_0,$$

and therefore

$$\|K_n * f^\dagger - f\|_p \leqslant 2\varepsilon \qquad \text{for } n > n_0. \tag{3.2.9}$$

Hence, by (3.2.8) and (3.2.9),

$$\|K_n * f - f\|_p \leqslant M\varepsilon + 2\varepsilon \qquad \text{for } n > n_0,$$

which proves the third statement and completes the proof.

3.2.3. Let $\mathbf{E}$ denote any one of the normed spaces $\mathbf{C}$ or $\mathbf{L}^p$ $(1 \leqslant p < \infty)$. Since each T_a is a continuous endomorphism of $\mathbf{E}$, 3.2.2 shows that $T_a(K_n * f) \to T_a f$ for each $f \in \mathbf{E}$. Also, by 3.1.2, $T_a(K_n * f) = T_a K_n * f$. It thus appears that $T_a f$ is the limit in $\mathbf{E}$ of convolutions $k * f$ with $k \in \mathbf{L}^1$. This complements 3.1.9.

The two results taken together show that, given $f \in \mathbf{E}$, the closed linear subspace of $\mathbf{E}$ generated by all translates of f is identical with the closure in $\mathbf{E}$ of the set of all convolutions $k * f$ with k ranging over $\mathbf{L}^1$; if $\mathbf{E}$ is regarded as a $*$-module over $\mathbf{L}^1$, this latter set is just the closed submodule of $\mathbf{E}$ generated by f. Further results of this type will appear in Section 11.1.

3.2.4. Approximate Identities and the Dirac δ-Function. The first part of 3.2.2 shows in particular that

$$\lim_{n\to\infty} K_n * f(0) = f(0),$$

or, what is equivalent after replacing f by $\check{f}$,

$$\lim_{n\to\infty} \frac{1}{2\pi} \int K_n(x) f(x)\, dx = f(0)$$

for each continuous f. (Actually, scrutiny of the proof would show that this holds for any $f \in \mathbf{L}^\infty$ such that f is continuous at 0.) This means that the sequence (K_n) is of the type which is often said to converge (in some unspecified sense) to the so-called *Dirac δ-function*. Complete precision will be attained in terms of the ideas to be studied in Chapter 12; see especially 12.2.3 and 12.3.2(3).

3.2.5. Approximate Identities and Summation Factors. Insofar as the study of Fourier series is concerned, one of the main effects of using approximate identities is the insertion of "summation factors" into series which, in their unadorned state, will in general diverge. The summation factors are, indeed, just the Fourier transforms $\hat{K}_n$, which, as the results of 3.2.4 show, have the property that

$$\lim_{n\to\infty} \hat{K}_n(m) = 1 \qquad (m \in Z).$$

Specific examples appear in Sections 5.1 and 6.6.

3.3 The Group Algebra Concept

3.3.1. The Classical Concept. In the classical and purely algebraic theory of a finite group G, additional flexibility was sought by introducing the so-called *group algebra* (or *group ring*) $\mathfrak{A}$ of G. This was defined, after choosing a field K of scalars, as the set of all formal (finite) linear combinations

$$f = \sum_{x\in G} f(x)\cdot x$$

of group elements $x \in G$ with coefficients $f(x) \in K$. The algebraic operations are as follows:

$$\alpha f = \sum_{x\in G} [\alpha f(x)]\cdot x \qquad \alpha \in K,$$

$$f + g = \sum_{x\in G} [f(x) + g(x)]\cdot x,$$

$$fg = \sum_{x\in G} [\sum_{y\in G} f(x-y) g(y)]\cdot x.$$

Taking stock of the fact that

$$I(f) = \sum_{x \in G} f(x)$$

is an invariant integral on G (see 2.2.2 and Exercise 2.5), a little thought will show that the group algebra may very well be pictured as the algebra of K-valued functions on G, the linear space operations being point-wise and the product being convolution; the sum

$$\sum_{y \in G} f(x - y)g(y)$$

corresponds exactly to the integral

$$\frac{1}{2\pi} \int f(x - y)g(y)\, dy$$

used in Section 3.1 to define the convolution of two functions on the group T.

It is no part of our purpose to carry forward the study of the group algebra of a finite group (see, for example, [Bo] and [vW]): the concept is mentioned merely because it is the forerunner of one that holds an important place in the modern developments in harmonic analysis (see 3.3.2). For purposes of subsequent comparison it is to be noted that the study of the group algebra of a finite group leads ultimately to a good deal of information about the structure of the underlying group, albeit only when combined with the study of representations of the group.

3.3.2. The Modern Concept. On turning to infinite groups, and specifically to locally compact topological groups G, there are various ways in which the group algebra concept can be extended. It is customary in these conditions to assume that the ground field K is the complex field. Nonetheless, considerable freedom of choice remains, especially when G is compact. For $G = T$, for instance, one might attach the term "group algebra" to any one of $\mathbf{L}^p$ $(1 \leqslant p \leqslant \infty)$, or $\mathbf{C}^k$ $(0 \leqslant k \leqslant \infty)$, or again the measure algebra $\mathbf{M}$ introduced in Chapter 12. The favorite contender for the title is $\mathbf{L}^1$, mainly perhaps because $\mathbf{L}^1$ remains a convolution algebra even when the underlying group is noncompact. (This last property is shared by $\mathbf{M}$, but $\mathbf{M}$ is generally far more mysterious than is $\mathbf{L}^1$; see 12.7.4.)

There are, of course, important differences between the group algebra $\mathfrak{A}$ of a finite group and any one of $\mathbf{L}^p$ or $\mathbf{C}^k$. Thus (1) $\mathfrak{A}$ possesses an identity element (to wit, the function taking the value 1 at $x = 0$ and the value 0 elsewhere; compare 3.1.1(c)); (2) $\mathfrak{A}$ is a finite-dimensional linear space, which is evidently not the case with $\mathbf{L}^p$ or $\mathbf{C}^k$. The difference pointed out in (2) means that the study of the group algebra in its modern guise is as much one of analysis as of algebra.

Again because of these differences, it is much more difficult to establish clear-cut relations between the properties of a group algebra and those of the underlying group; see the remarks in 4.2.7. The fact is that the modern approach lays more emphasis on the structure of function spaces carried by a group, and less on the underlying group itself.

3.4 The Dual Concepts

There is no trouble involved in framing the definition of the convolution $\phi * \psi$ of two functions on Z, provided that these functions are suitably restricted in their behavior at infinity; see Exercise 3.15.

The same is not true of the analogue of (3.1.5), however. The analogue reads

$$(\phi * \psi)^{\wedge} = \hat{\phi} \cdot \hat{\psi} \tag{3.4.1}$$

and the remarks in Section 2.5 suffice to point up some of the difficulties encountered in establishing (3.4.1). The only simple case is that in which both ϕ and ψ belong to ℓ^1, in which case $\phi * \psi$ belongs to ℓ^1, too [compare (3.1.2)].

Our more immediate concern in the sequel will lie with cases of (3.4.1) in which ϕ and ψ have the forms $\hat{f}$ and $\hat{g}$, respectively, where f and g are suitably restricted functions on $G = T$: in this case the formula (3.4.1) appears essentially in the disguise of versions of the Parseval formula to be discussed in Chapters 8 and 10. See also 12.6.9 and 12.11.3.

EXERCISES

3.1. For $f \in \mathbf{L}^1$ let

$$s_N f = \sum_{|n| \leqslant N} \hat{f}(n) e_n,$$

and suppose that D_N is defined as in Exercise 1.1. Verify that $s_N f = D_N * f$ and that

$$s_N f * g = s_N(f * g) = f * s_N g$$

for $f, g \in \mathbf{L}^1$. Deduce that $s_N f$ is the limit in $\mathbf{L}^1$ of linear combinations of translates of f.

3.2. Suppose that $1 \leqslant p \leqslant \infty$ and define

$$\mathbf{I} = \{f \in \mathbf{L}^p : \|s_N f\|_p = O(1) \text{ as } N \to \infty\},$$
$$\mathbf{J} = \{f \in \mathbf{L}^p : \lim_{N \to \infty} \|f - s_N f\|_p = 0\}.$$

Verify that $\mathbf{I}$ and $\mathbf{J}$ are submodules of $\mathbf{L}^p$ (considered as a $*$-module over $\mathbf{L}^1$) and that $\mathbf{J} \subset \mathbf{I}$. Show that $\mathbf{I}$ and $\mathbf{J}$ are everywhere dense in $\mathbf{L}^p$ if $p < \infty$. If $p = \infty$, is $\mathbf{J}$ dense in $\mathbf{L}^\infty$? Give reasons for your answer.

3.3. Take $p = 1$ in the preceding exercise. Show that $\mathbf{I}$ and $\mathbf{J}$ are non-closed in $\mathbf{L}^1$.

Hint: Assume the result of the computations in 5.1.1 and use the uniform boundedness principle as stated in Appendix B.2.1 or in B.2.2.

3.4. (1) Let $\mathbf{E}$ be the set of $f \in \mathbf{L}^1$ such that $f(x) = f(-x)$ a.e. Is $\mathbf{E}$ an ideal in $\mathbf{L}^1$? Is it a subalgebra of $\mathbf{L}^1$? Is $\mathbf{E}$ closed in $\mathbf{L}^1$? Give reasons for your answers.

(2) Prove that $\omega_1(f * g) \leqslant \|f\|_1 \cdot \omega_1 g$ for $f, g \in \mathbf{L}^1$. Deduce that the set

$$\mathbf{I} = \{f \in \mathbf{L}^1 \colon \omega_1 f(a) = O(|a|^{1/2}) \text{ as } a \to 0\}$$

is an ideal in $\mathbf{L}^1$. Is $\mathbf{I}$ closed in $\mathbf{L}^1$? Give reasons.

Hint for (2): Observe that $\mathbf{I}$ is everywhere dense in $\mathbf{L}^1$.

3.5. Suppose that $f \in \mathbf{L}^\infty$ is such that the function $a \to T_a f$ is continuous from R into $\mathbf{L}^\infty$ for the normed topology on the latter space (that is, that $\|T_a f - f\|_\infty \to 0$ as $a \to 0$). Prove that f is equal almost everywhere to a continuous function. (The converse is true and almost trivial.)

Hints: Take an approximate identity $(K_n)_{n=1}^\infty$ in $\mathbf{L}^1$ and consider the functions $f_n = K_n * f$. Show that the f_n are equicontinuous and uniformly bounded. Let $(x_i)_{i=1}^\infty$ be a sequence that is everywhere dense in $(0, 2\pi)$ and pick strictly increasing sequences of natural numbers $(n_k^{(i)})_{k=1}^\infty$ so that $(n_k^{(i+1)})_{k=1}^\infty$ is a subsequence of $(n_k^{(i)})_{k=1}^\infty$ and $\lim_{k\to\infty} f_{n_k^{(i)}}(x_i)$ exists finitely for each i. Deduce that there exists $g \in \mathbf{C}$ such that, if $n_k = n_k^{(k)}$ (the "diagonal subsequence"), then $f_{n_k} \to g$ uniformly. Use 3.2.2 to compare f and g.

> ***Remarks.*** This is the special case, for the group $R/2\pi Z$, of a result due to D. A. Edwards [1] for general groups. An analogous and older result for Radon measures (see Chapter 12) is the work of Plessner and Raikov. Both types of result are treated in R. E. Edwards [2]. See also Exercises 11.22 and 12.23. The existence of a uniformly convergent subsequence of (f_n) is a special case of Ascoli's theorem; see [E], Section 0.4.

3.6. (Converse of Hölder's inequality) Suppose that $1 \leqslant p \leqslant \infty$ and that $f \in \mathbf{L}^1$ is such that

$$\left|\frac{1}{2\pi}\int fg\,dx\right| \leqslant m \cdot \|g\|_p \tag{1}$$

for each $g \in \mathbf{C}$, m being a number independent of g. Prove that $f \in \mathbf{L}^{p'}$, where $1/p + 1/p' = 1$, and that

$$\|f\|_{p'} \leqslant m.$$

Hint: Show that (1) continues to hold for $g \in \mathbf{L}^\infty$.

Remark. It is even enough to assume that (1) holds for $g \in \mathbf{C}^\infty$. For another variant, see 13.1.5.

3.7. Does 3.1.9 remain valid if therein one takes $\mathbf{E} = \mathbf{C}^k$? Justify your answer.

3.8. Let $(K_n)_{n=1}^{\infty}$ be any approximate identity. Show that, if $p > 1$, then

$$\lim_{n \to \infty} \|K_n\|_p = \infty;$$

compare Exercise 7.5.

3.9. Let $\mathbf{H}^p$ denote the set of $f \in \mathbf{L}^p$ such that $\hat{f}(n) = 0$ for $n < 0$ (here $1 \leqslant p \leqslant \infty$). Show that $\mathbf{H}^p$ is a closed ideal in $\mathbf{L}^p$.

Remarks. The study of the so-called *Hardy spaces* $\mathbf{H}^p$ is an elaborate procedure having close connections with complex analytic function theory, the connection being explained by the remark that each $f \in \mathbf{H}^p$ may be regarded as the boundary values (for $|z| = 1$ and in a sense that can, and must, be made precise) of the function $f\#$ defined for complex z satisfying $|z| < 1$ by the power series $\sum_{n \geqslant 0} \hat{f}(n) z^n$. (In terms of the functions $f\#$, the definition of $\mathbf{H}^p$ can be extended to cases where $0 < p < 1$.) This book contains no attempt to discuss the subject systematically, although sidelong glances are thrown in that direction in Exercises 6.15, 8.15, 11.8, 11.10 and in Section 12.9 and 12.10.3. For detailed accounts of the subject, see [Ho]; [Hel]; [dBR]; [R], Chapter 8; [Z_1], Chapter VII; [Kz], pp. 81 ff.; and [Ba_2], pp. 70–93. For a survey of the abstract theory, see Srinivasan and Wang [1] and the references cited there and also MR 37 # 1982; 55 ## 989, 990.

3.10. Let $(K_N)_{N=1}^{\infty}$ be a sequence of nonnegative integrable functions such that $\lim_{N \to \infty} \hat{K}_N(n) = 1$ $(n \in Z)$. Show that

$$\lim_{N \to \infty} K_N * f = f$$

uniformly for each continuous f. Deduce that $(K_N)_{N=1}^{\infty}$ is an approximate identity (see 3.2.1).

3.11. Assume that a is a real number such that a/π is irrational and that f is a measurable complex-valued function such that $T_a f = f$ a.e. Show that $f =$ const a.e. (Recall that all functions considered have period 2π.)

3.12. Suppose $1 \leqslant p \leqslant \infty$. Show that the convolution algebra $\mathbf{L}^p$ has no nonzero *generalized* (or *topological*) *nilpotents*, that is, elements f such that

$$\inf_k \|f^{*k}\|_p^{1/k} = 0,$$

where $f^{*1} = f$ and $f^{*(k+1)} = f * f^{*k}$ $(k = 1, 2, \cdots)$. See 11.4.18(1).

3.13. Assume that $K \in \mathbf{L}^1$ and $\|K\|_1 < 1$. Given $g \in \mathbf{L}^p$ (where $1 \leqslant p \leqslant \infty$), show that the equation

$$f - K * f = g$$

has the unique solution

$$f = g + \sum_{n=1}^{\infty} K^{*n} * g,$$

and that this solution satisfies

$$\|f\|_p \leqslant (1 - \|K\|_1)^{-1} \cdot \|g\|_p.$$

Remarks. An obvious *necessary* condition in order that the given equation be soluble for each $g \in \mathbf{L}^p$ is that $\hat{K}(n) \neq 1$ $(n \in Z)$. That this condition is *sufficient* lies somewhat deeper. It is a corollary of 11.4.13.

We remark in passing that convolution *inequalities* of the form $f - K * f \geqslant 0$ have come to play a role in certain parts of modern potential theory and harmonic analysis. In this connection K, which has hitherto denoted an integrable function, frequently denotes a measure (see 12.2.3). Such inequalities will play no role in this book, but the interested reader should consult Essén [1] and the references cited there.

3.14. Suppose that $(c_n)_{n \in Z}$ is a sequence such that

$$\sum_{n \in Z} |c_n \hat{f}(n)| < \infty$$

for each $f \in \mathbf{L}^1$. Show that $\sum_{n \in Z} |c_n| < \infty$.

Hint: Use the uniform boundedness principle in Appendix B.2.1.

Note: The hypothesis should *not* be confused with the demand that $\sum c_n \hat{f}(n)$ be merely convergent for each $f \in \mathbf{L}^1$; see 10.5.1.

Similar arguments will establish a more general lemma of Bosanquet and Kestelman, which asserts that if the functions u_k $(k = 1, 2, \cdots)$ are such that $fu_k \in \mathbf{L}^1$ and $\sum_{k=1}^{\infty} |\int fu_k \, dx| < \infty$ whenever $f \in \mathbf{L}^1$, then $\sum_{k=1}^{\infty} |u_k| \in \mathbf{L}^{\infty}$.

3.15. Consider sequences (= complex-valued functions on Z) $\phi, \psi, \cdots$. Frame a definition of the convolution $\phi * \psi$ which will be such that $(fg)^\wedge = \hat{f} * \hat{g}$ for trigonometric polynomials f and g.

Using the notations introduced in 2.2.5, discuss the dual aspects of the results in Section 3.1.

Note: Further discussion of the relation $(fg)^\wedge = \hat{f} * \hat{g}$ appears later in the guise of the Parseval formula; see Chapters 8 [especially (8.2.5)] and 10. The change of the underlying group from T to Z has led to some interesting problems concerning convolution over noncompact groups, which have been studied in a sequence of papers by Rajagopalan and Zelazko, a useful summary of which appears in Math. Rev. **32** # 2506. See also Exercise 4.6 below; [HR], (38.26) and (38.27); MR **34** # 1868, 8213; **35** # 7136; **37** # 4509; Gaudet and Gamlen [1].

3.16. Let S be a set of real numbers whose *interior measure*

$$m_*(S) \equiv \sup \{m(F) : F \quad \text{closed}, F \subset S\}$$

is positive. Show that the set of differences $S^\dagger \equiv S - S \equiv \{x - y : x \in S, y \in S\}$ contains a neighborhood of 0.

Hints: By taking a suitable closed subset of some translate of S, it may be assumed that S is measurable, is contained in $(0, \frac{1}{2}\pi)$, and that $m(S) > 0$.

Let f be the periodic function that coincides on $[-\pi, \pi)$ with the characteristic function of S. Apply 3.1.4 to $f * \check{f}$.

Notes: The result is due to Steinhaus (Sur les distances des points des ensembles de mesure positive. *Fund. Math.* **1** (1920), 93–104). It could be formulated entirely in terms of the group T; in fact, the result is valid for any locally compact group whatsoever, the proof being a simple adaptation of that which is proposed above. A corollary is that a subgroup is either open or has zero interior measure.

For some generalizations, see Ray and Lahiri [1], Mueller [1], and the references cited there.

There is an analogous result due to Banach, Kuratowski, and Pettis applying to topological groups that are not necessarily locally compact, measure-theoretic concepts being replaced by category; see [K], p. 211 and (for special cases) [Z_1], p. 250, Example 2. For a converse, see MR **51** # 6286.

3.17. Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of real numbers and suppose that $\lim_{n\to\infty} \exp(i\alpha_n x)$ exists for each x belonging to a set of real numbers having positive interior measure (see the preceding exercise). Prove that $\lim_{n\to\infty} \alpha_n$ exists finitely.

Hints: The set S of points of convergence of the sequence $(\exp(i\alpha_n x))$ is evidently a subgroup of R. Use Exercise 3.16 to conclude that $S = R$, so that $g(x) = \lim_{n\to\infty} \exp(i\alpha_n x)$ exists for all real x. By integration theory, therefore,

$$\lim_{n\to\infty} \int_R f(x) \exp(i\alpha_n x)\, dx = \int_R f(x) g(x)\, dx$$

for every function f which is Lebesgue-integrable over R. Deduce first that $(\alpha_n)_{n=1}^{\infty}$ is bounded (an adaptation of 2.3.8 will be needed here), and then (by choosing f suitably, or by a compactness argument) that this sequence is convergent.

3.18. Let $(c_n)_{n=1}^{\infty}$ be a sequence of complex numbers and $(\alpha_n)_{n=1}^{\infty}$ a sequence of real numbers. Suppose that $\lim_{n\to\infty} c_n \cdot \exp(i\alpha_n x)$ exists for each x belonging to a set of real numbers having positive interior measure (see Exercise 3.16). Show that (i) (c_n) is convergent to some complex number and (ii) if $\lim_{n\to\infty} c_n \neq 0$, then (α_n) is convergent in R.

3.19. Let χ be a measurable character of T. Prove that χ is continuous.

Hints: Use Exercise 3.16 to show that χ is bounded. Then establish continuity of χ by adapting the reasoning used in 2.2.1 to show that a continuous character is differentiable.

Alternatively, see [HR], p. 346, where the result is stated and proved in a more general form.

Remark. Despite the stated result, there exist characters of T which are both bounded and nonmeasurable.

CHAPTER 4

Homomorphisms of Convolution Algebras

In this brief chapter we introduce the reader to two problems typical of the current outlook on harmonic analysis. The first problem, which will be solved in detail in Section 4.1, arises on choosing any one of the convolution algebras $\mathbf{E}$ mentioned in Subsection 3.3.2 and seeking to determine all the homomorphisms γ of $\mathbf{E}$ into the complex field. The answer highlights the fundamental importance of the Fourier transformation in relation to group structure.

The second problem is concerned with the (self-) homomorphisms of $\mathbf{E}$ (that is, the homomorphisms of $\mathbf{E}$ into itself). Of the available choices of $\mathbf{E}$, only the cases $\mathbf{L}^2$ and $\mathbf{L}^1$ are fully solved. The former case is easy and of relatively little interest (compare Exercise 8.1). The case $\mathbf{E} = \mathbf{L}^1$ is, on the contrary, comparatively very complex, and we shall be able only to indicate how the solution of the complex homomorphism problem allows a useful reduction to be made, and to indicate the solution for this case.

An incomplete account of the homomorphism problem is inserted at this early stage because it has been learned in 3.1.9 and 3.1.10 that convolution is related in a very basic way to the group structure, and because, granted the ensuing fundamental role of convolution, the homomorphism problem begs for recognition without delay. This problem has, in fact, proved to be one focus of interest in contemporary work.

4.1 Complex Homomorphisms and Fourier Coefficients

4.1.1. We have to consider nontrivial complex homomorphisms γ of the convolution algebra $\mathbf{L}^1$ into the complex field. In other words, γ is a linear functional on the complex linear space $\mathbf{L}^1$ which is not the zero functional and which satisfies

$$\gamma(f * g) = \gamma(f) \cdot \gamma(g) \tag{4.1.1}$$

for $f, g \in \mathbf{L}^1$.

It will appear in 11.4.9 and 11.4.12 that such a homomorphism γ is necessarily continuous on $\mathbf{L}^1$, that is, that

$$|\gamma(f)| \leqslant \text{const } \|f\|_1, \tag{4.1.2}$$

but we shall temporarily assume explicitly that γ is continuous.

4.1.2. Let γ be any nontrivial continuous complex homomorphism of $\mathbf{L}^1$. Then there exists a unique $n \in Z$ such that $\gamma = \gamma_n$, where [see (3.1.6)]

$$\gamma_n(f) = \hat{f}(n)$$

for $f \in \mathbf{L}^1$.

Proof. The uniqueness of n is clear.

We shall offer two proofs of the existence of n, the relative merits of which will be weighed in 4.1.3.

First Proof. Define $c_n = \gamma(e_n)$ for $n \in Z$. Since γ is continuous and nontrivial, the density theorem of 2.4.4 entails that $c_n \neq 0$ for at least one integer n. Since also $e_n * e_{n'} = e_n$ or 0 according as n' is, or is not, equal to n, an application of γ shows that $c_n \cdot c_{n'} = c_n$ or 0 according as n' is, or is not, equal to n. Thus $c_{n'}$ is equal to 1 or to 0 according as n' is, or is not, equal to n. Linearity of γ shows then that $\gamma(f) = \hat{f}(n)$ for all trigonometric polynomials f. Continuity of γ, together with the density theorem, accordingly show that $\gamma(f) = \hat{f}(n)$ for all $f \in \mathbf{L}^1$.

Second Proof. In view of the continuity of γ and the fact that $\mathbf{C}$ is everywhere dense in $\mathbf{L}^1$, it will suffice to show that, for some $n \in Z$, the formula

$$\gamma(f) = \hat{f}(n)$$

holds for each continuous f.

Now, again since γ is continuous, we can choose and fix a continuous f_0 such that $\gamma(f_0)$ is nonzero. Consider the function χ defined on $R/2\pi Z$ by

$$\chi(x) = \frac{\gamma(T_x f_0)}{\gamma(f_0)}. \tag{4.1.3}$$

Evidently, $\chi(0) = 1$. By (4.1.2) and the fact that

$$\|T_x f_0 - T_y f_0\|_1 = \|T_{x-y} f_0 - f_0\|_1,$$

which tends to zero with $x - y$, we see that χ is continuous. Moreover, 3.1.3 and (4.1.3) combine to show that

$$\chi(x + y) = \chi(x)\chi(y).$$

Consequently (see 2.2.1) there exists $n \in Z$ such that

$$\chi(x) = e^{-inx}. \tag{4.1.4}$$

Now take any continuous f. Then (compare the proof of 3.1.9), given any $\varepsilon > 0$, $f_0 * f$ is uniformly approximated to within ε by any sum of the form

$$\frac{1}{2\pi} \sum_{k=1}^{m} f(x_k) \cdot T_{x_k} f_0 \cdot (x_k - x_{k-1})$$

where $0 = x_0 < x_1 < \cdots < x_m = 2\pi$ is any partition of $[0, 2\pi]$ whose "mesh" $\max_k(x_k - x_{k-1})$ is sufficiently small (depending upon ε, f_0, and f). A fortiori,

$$\|f_0 * f - \frac{1}{2\pi}\sum_k f(x_k) \cdot T_{x_k} f_0 \cdot (x_k - x_{k-1})\|_1 \leqslant \varepsilon$$

for any such partition. Therefore, if we apply γ and use (4.1.2), we conclude that

$$|\gamma(f_0 * f) - \frac{1}{2\pi}\sum_k f(x_k) \cdot \gamma(T_{x_k} f_0) \cdot (x_k - x_{k-1})| \leqslant \text{const}\ \varepsilon.$$

Using (4.1.1), dividing through by $\gamma(f_0)$, and using (4.1.3) and (4.1.4), we find that

$$|\gamma(f) - \frac{1}{2\pi}\sum_k f(x_k) \cdot e^{-inx_k}(x_k - x_{k-1})| \leqslant \text{const}\ \frac{\varepsilon}{|\gamma(f_0)|} \tag{4.1.5}$$

for all partitions of sufficiently small mesh. But, as the mesh of the partition tends to zero, the sum appearing on the left of (4.1.5) converges (since f is continuous) to the integral

$$\frac{1}{2\pi}\int f(x) e^{-inx}\, dx = \hat{f}(n).$$

It follows that $\gamma(f) = \hat{f}(n)$, and the proof is finished.

4.1.3. Comments on the Preceding Proofs. (1) It is very easy to see that the first proof of 4.1.2 adapts readily to the case in which $\mathbf{L}^1$ is replaced by any subset $\mathbf{E}$ of $\mathbf{L}^1$ fulfilling the following four conditions:

(a) $\mathbf{E}$ is an algebra under convolution;
(b) $\mathbf{E}$ is a topological space;
(c) a set S of integers exists such that $e_n \in \mathbf{E}$ for $n \in S$, while the linear combinations of these e_n are everywhere dense in $\mathbf{E}$;
(d) for each $n \in S$, the function $f \to \hat{f}(n)$ is continuous on $\mathbf{E}$.

The conclusion is then that each nontrivial continuous complex homomorphism γ of $\mathbf{E}$ is of the form $\gamma(f) = \hat{f}(n)$ for all $f \in \mathbf{E}$ and some γ-dependent $n \in S$.

Moreover, when the substance of Chapter 12 has been absorbed the reader will see that in the above it is unnecessary to assume that the elements of $\mathbf{E}$ are integrable functions: they may be permitted to be distributions. Use will be made of this remark in Section 16.6.

(2) If the reader scrutinizes carefully the second proof, he will see that it also stands with only verbal changes when $\mathbf{L}^1$ is replaced by $\mathbf{C}^k$ or by $\mathbf{L}^p$ $(1 \leqslant p < \infty)$.

In the case of $\mathbf{L}^\infty$, one change is necessary owing to the fact that $\mathbf{C}$ is not dense in $\mathbf{L}^\infty$. However, (4.1.1) and the fact that $f * g \in \mathbf{C}$ whenever $f \in \mathbf{L}^\infty$ and $g \in \mathbf{L}^\infty$, ensures that γ, being by hypothesis not identically zero on $\mathbf{L}^\infty$, cannot vanish identically on $\mathbf{C}$. Hence f_0 can be chosen as before. The preceding argument shows that $\gamma(f) = \hat{f}(n) = \gamma_n(f)$ for all $f \in \mathbf{C}$. But then, if $g \in \mathbf{L}^\infty$, we have (since $f_0 * g \in \mathbf{C}$)

$$\gamma(f_0 * g) = \gamma_n(f_0 * g),$$

that is,

$$\gamma(f_0)\gamma(g) = \gamma_n(f_0)\gamma_n(g).$$

Since $\gamma(f_0) = \gamma_n(f_0) \neq 0$, so $\gamma(g) = \gamma_n(g)$ for all $g \in \mathbf{L}^\infty$. The conclusion is thus valid for $\mathbf{L}^\infty$ also.

The second proof is clearly more complicated and laborious than the first. It has been included, because it can be adapted to cases in which the underlying group is noncompact (the groups Z and R, for example). In such cases the first proof breaks down completely because $\mathbf{L}^1$ then contains no continuous characters at all. Moreover, the second proof bypasses the density theorem and relies on fewer facts concerning harmonic analysis.

4.1.4. An analogous and much more difficult problem arises when $\mathbf{L}^1$ is replaced by the measure algebra $\mathbf{M}$ introduced in Chapter 12; see especially the remarks in 12.7.4.

4.1.5. A full exploitation of the results obtained in 4.1.2 and 4.1.3 depends on the Gelfand theory of complex commutative Banach algebras. Were we seeking to develop harmonic analysis on a general group, it would at this point be advantageous to embark on the Gelfand theory and reap the fruits of its application. As it is, however, we shall defer this sowing and harvesting until Section 11.4.

4.2 Homomorphisms of the Group Algebra

4.2.1. **Statement of the Problem.** In Section 3.3 we have remarked that each of $\mathbf{L}^p$ $(1 \leqslant p \leq \infty)$ and $\mathbf{C}^k$ $(0 \leqslant k < \infty)$ is a possible analogue of the group algebra of a finite group, and that $\mathbf{L}^1$ is the favored contender for this title. Exhibiting no prejudice for the moment, we let $\mathbf{E}$ denote any one of these group algebras.

A problem exerting a natural appeal and depending for its solution (as far as this is known at present) on harmonic analysis, is that of determining as explicitly as possible the homomorphisms of $\mathbf{E}$ (into itself). By such a homomorphism we shall mean a continuous linear mapping T of $\mathbf{E}$ into itself with the property that

$$T(f * g) = Tf * Tg \tag{4.2.1}$$

for $f, g \in \mathbf{E}$. (By using the closed graph theorem as indicated in 4.2.3, together with the necessary continuity of complex homomorphisms of $\mathbf{E}$, one could show that any homomorphism of $\mathbf{E}$ into itself is necessarily continuous.)

In Exercises 4.2, 4.3, and 4.8, the reader's attention is directed to the simplest and most obvious such homomorphisms, namely those of the type

$$Tf(x) = e^{inx} f(kx),$$

where n, $k \in Z$ and $k \neq 0$. Each of these particular mappings T defines a homomorphism of each of the group algebras $\mathbf{E}$ envisaged. This feature is nontypical inasmuch as there are homomorphisms T of $\mathbf{C}^\infty$ (or of $\mathbf{L}^2$) which are not extendible into homomorphisms of $\mathbf{L}^1$; see Exercise 8.1.

This homomorphism problem has been posed for general groups. For the case in which $\mathbf{E}$ is either $\mathbf{L}^1$ or the measure algebra $\mathbf{M}$ (see Chapter 12), most of what is currently known is presented in detail in [R], Chapter 4. All the results of this nature are relatively recent.

Our aim is confined to indicating how a knowledge of the complex homomorphisms of $\mathbf{E}$, discussed in Section 4.1, permits a small step forward in the shape of representing the problem in a different and more tractable form, and to stating the solution.

4.2.2. **Reformulation of the Problem.** Let T be a homomorphism of $\mathbf{E}$. For each $n \in Z$ the mapping

$$f \to (Tf)\hat{}(n) \tag{4.2.2}$$

is a continuous complex homomorphism of $\mathbf{E}$. So, in accordance with 4.1.2 and 4.1.3, this mapping (4.2.2) is either trivial (that is, identically zero) or is of the type $\gamma_{n'}$ for some $n' \in Z$.

Denote by Y the set of $n \in Z$ for which the mapping (4.2.2) is nontrivial on $\mathbf{E}$. We then have

$$(Tf)\hat{}(n) = \hat{f}(n')$$

for $n \in Y$ and $f \in \mathbf{E}$. This relationship entails that n' is uniquely determined by $n \in Y$, so that a mapping

$$\alpha\colon Y \to Z$$

is obtained for which $\alpha(n) = n'$. Thus

$$(Tf)\hat{}(n) = \hat{f} \circ \alpha(n) \tag{4.2.3}$$

for $n \in Y$ and $f \in \mathbf{E}$.

For $n \in Z \backslash Y$, $(Tf)\hat{}(n) = 0$ for all $f \in \mathbf{E}$. The Riemann-Lebesgue lemma 2.3.8 suggests that we write accordingly $\alpha(n) = \infty$ for $n \in Z\backslash Y$, interpreting $\hat{f}(\infty)$ as 0. If this be done, α may be regarded as a mapping of Z into $Z \cup \{\infty\}$ and (4.2.3) holds for $n \in Z$ and $f \in \mathbf{E}$.

The mapping α must have the property that

$$\sum_{n \in Z} \hat{f}(\alpha(n))e_n = \sum_{n \in Y} \hat{f}(\alpha(n))e_n \in \mathbf{E} \tag{4.2.4}$$

whenever $f \in \mathbf{E}$. The reader will notice that (4.2.4) is intended to mean simply that there exists $g \in \mathbf{E}$ (unique by 2.4.1) such that $\hat{g} = \hat{f} \circ \alpha$; no statement is intended concerning the convergence of the series in (4.2.4). (The series do in fact converge distributionally, as will appear in Chapter 12; no use is made of this fact.)

4.2.3. The next step is to show that, if α is a mapping of Z into $Z \cup \{\infty\}$ having the property expressed in (4.2.4), then the mapping T of $\mathbf{E}$ into itself defined by

$$Tf = \sum_{n \in Z} \hat{f}(\alpha(n))e_n \tag{4.2.5}$$

is a continuous homomorphism of $\mathbf{E}$ into itself. The uniqueness theorem 2.4.1 is being invoked once again in order to reach assurance that there is precisely one $g \in \mathbf{E}$ for which $\hat{g} = \hat{f} \circ \alpha$, this g being Tf.

To begin with, this same uniqueness theorem shows at once that T is linear and satisfies (4.2.1) as a consequence of the relations $(\lambda\hat{f}) \circ \alpha = \lambda(\hat{f} \circ \alpha)$, $(\hat{f}_1 + \hat{f}_2) \circ \alpha = (\hat{f}_1 \circ \alpha) + (\hat{f}_2 \circ \alpha)$, and $(\hat{f}_1\hat{f}_2) \circ \alpha = (\hat{f}_1 \circ \alpha) \cdot (\hat{f}_2 \circ \alpha)$ holding for any scalar λ and any $f_1, f_2 \in \mathbf{E}$. It therefore remains to show that T is continuous, to achieve which end we shall invoke the closed graph theorem (see Appendix B.3.3). This invocation is permissible, since $\mathbf{E}$ is in all cases either a Banach space or a Fréchet space (see 2.2.4).

In order to prove continuity of T it is sufficient (according to the said closed graph theorem) to show that the assumptions

$$\lim_{k \to \infty} f_k = 0 \quad \text{in } \mathbf{E}, \qquad \lim_{k \to \infty} Tf_k = g \quad \text{in } \mathbf{E} \tag{4.2.6}$$

imply the conclusion

$$g = 0. \tag{4.2.7}$$

But, in all cases here envisaged, $\lim_{k \to \infty} f_k = 0$ in $\mathbf{E}$ entails that $\lim_{k \to \infty} \hat{f}_k = 0$ pointwise on Z. Similarly the second clause of (4.2.6) entails that $\lim_{k \to \infty}(Tf_k)^\wedge = \hat{g}$ pointwise on Z. Since $(Tf_k)^\wedge = \hat{f}_k \circ \alpha$, $\alpha(Z) \subset Z \cup \{\infty\}$, and $\hat{f}_k(\infty) = 0$, it follows that $\hat{g} = 0$ and so (by the uniqueness theorem once again) the conclusion (4.2.7) is obtained. This completes the proof that T is continuous.

4.2.4. To sum up, we find that the homomorphisms T of $\mathbf{E}$ into itself correspond to the mappings $\alpha : Z \to Z \cup \{\infty\}$ having the property expressed in (4.2.4), the correspondence $T \leftrightarrow \alpha$ being specified by (4.2.5).

At this point the reader may care to try Exercises 4.4 and 4.5.

4.2.5. The determination of the mappings α of Z into $Z \cup \{\infty\}$ which have the property expressed in (4.2.4) is (except when $\mathbf{E} = \mathbf{L}^2$; compare Exercise 8.1) lengthy and difficult. For the case in which $\mathbf{E} = \mathbf{L}^1$, the solution was given by Rudin in 1956 and is described in 4.2.6; the solution of the dual problem, in which the group $R/2\pi Z$ is replaced by Z, will be mentioned in 10.6.3(3). The analogous problem for general groups (hence, in particular, the dual problem for the group Z) and homomorphisms of $\mathbf{L}^1$ into the measure algebra $\mathbf{M}$ was solved in generality by P. J. Cohen [2] in 1960, although partial solutions had been discovered earlier by various writers (see the remarks in 12.7.4). Details and more references appear in [R], Chapter 4, and [Kah], Capítulos IV–VII. See also Exercises 12.49 and 13.6.

4.2.6. **Statement of the Solution.** Rudin's solution of the problem of homomorphisms of $\mathbf{L}^1$ into itself (for the group $R/2\pi Z$) is expressed in terms of the associated mappings α and is as follows.

In order that a mapping α of Z into $Z \cup \{\infty\}$ shall yield a homomorphism T of $\mathbf{L}^1$ into itself via the formula (4.2.5), it is necessary and sufficient that there exist an integer $q > 0$ and a mapping β of Z into itself with the following properties:

(1) If $A_1, \cdots, A_q$ denote the residue classes of Z modulo q, then $Y = \alpha^{-1}(Z)$ is of the form $S_1 \bigcup \cdots \bigcup S_r$, where each S_i is either finite or is contained in some A_j from which it differs by a finite set, and where the S_i are pairwise disjoint;

(2) $\beta(n + q) \neq \beta(n) \qquad (n \in Z)$;

(3) $\beta(n + q) + \beta(n - q) = 2\beta(n) \qquad (n \in Z)$;

(4) $\alpha(n) = \beta(n)$ for all save at most a finite number of $n \in Y$.

The conditions (1), (3), and (4) are necessary and sufficient in order that α shall yield a homomorphism of $\mathbf{L}^1$ into the measure algebra $\mathbf{M}$ introduced in Chapter 12.

The reader will experience no difficulty in verifying that condition (3) signifies the existence of integers u_h and v_h $(h = 0, 1, \cdots, q - 1)$ such that

$$\beta(kq + h) = u_h k + v_h \tag{4.2.8}$$

for $(k, h) \in Z \times \{0, 1, \cdots, q - 1\}$; and that the conjunction of conditions (2) and (3) signifies that (4.2.8) holds and that in addition $u_h \neq 0$ for $h = 0, 1, \cdots, q - 1$.

4.2.7. **Other Problems; Multipliers.** A specialized question which has received a good deal of attention is this: To what extent does the existence of an automorphism T of $\mathbf{L}^1$ entail the existence of an automorphism of the underlying group $R/2\pi Z$? Or, in a more general form: To what extent does the existence of an isomorphism T of $\mathbf{L}^1(G_1)$ onto $\mathbf{L}^1(G_2)$ entail the existence of an isomorphism of G_1 onto G_2? Partial solutions of these problems are known; see [R], Theorems 4.7.1, 4.7.2, and Section 4.7.7, and 16.7.1 below.

Analogous problems concerning homomorphisms and isomorphisms of $\mathbf{L}^p(G_1)$ onto $\mathbf{L}^p(G_2)$ have been posed. The Rudin-Cohen method does not appear to extend to values of $p \neq 1$. Another method of attack on the isomorphism problem rests upon a preliminary study of the so-called *multipliers* of an algebra $\mathbf{L}^p(G)$, that is, the continuous endomorphisms U of $\mathbf{L}^p(G)$ which commute with translations (so that $UT_a = T_aU$ for all $a \in G$). The connection between the two problems stems from the fact that, if T is an isomorphism of $\mathbf{L}^p(G_1)$ onto $\mathbf{L}^p(G_2)$, then the formula $U_1 = T^{-1}U_2T$ sets up a one-to-one correspondence between the multipliers of $\mathbf{L}^p(G_1)$ and those of $\mathbf{L}^p(G_2)$. From there on the hope is that connections between the multiplier algebras can be translated into ones between the underlying groups.

In Chapter 16 we shall study multipliers per se, turning aside in 16.7.1 in order to summarize how information about multipliers of special types bears upon isomorphism problems.

Regarding homomorphisms of $\mathbf{L}^p$, where $1 < p < \infty$, see also 15.3.6.

For a guide to further reading, see MR **41** # 4141; **53** # 8781; **54** ## 3296, 5746.

EXERCISES

4.1. Without using 4.1.2, show that if γ is a nontrivial complex homomorphism of $\mathbf{L}^1$, and if

$$M = \gamma^{-1}(\{0\}) \equiv \{f \in \mathbf{L}^1 : \gamma(f) = 0\},$$

then M is an ideal in $\mathbf{L}^1$ such that

(1) $M \neq \mathbf{L}^1$;

(2) M is maximal, that is, there exists no ideal I in $\mathbf{L}^1$, distinct from M and from $\mathbf{L}^1$, such that $M \subset I$;

(3) There is an identity modulo M, that is, an element e of $\mathbf{L}^1$ such that $e * f - f \in M$ for every $f \in \mathbf{L}^1$ (M is accordingly termed *regular* or *modular*).

Remark. It follows from 11.4.9 that the above relation between modular maximal ideals and complex homomorphisms is reversible and remains in force in a more general setting.

4.2. Let $m \in Z$, $k \in Z$, $k \neq 0$. Define

$$Tf(x) = e^{imx} \cdot f(kx).$$

Verify that T defines a continuous homomorphism of $\mathbf{E}$ ($= \mathbf{L}^p$ or $\mathbf{C}$) into itself. When is $T(\mathbf{E}) = \mathbf{E}$?

Hint: See 2.2.2.

4.3. T being as in the preceding exercise, what is the corresponding map α (as introduced in 4.2.2)?

4.4. Suppose that T and α are as in 4.2.1 and 4.2.2. Show that T is one-to-one and $T(\mathbf{E})$ contains all trigonometric polynomials if and only if $Y = Z$ and α maps Z one-to-one onto itself.

4.5. Suppose that T is a continuous homomorphism of $\mathbf{L}^p$ $(1 \leqslant p \leqslant \infty)$ into itself and that α is the corresponding map of Z into $Z \cup \{\infty\}$ (see 4.2.2). Show that

(1) For any $m \in Z$, $\alpha^{-1}(\{m\})$ is finite;

(2) A number $k \geqslant 0$ exists such that for all $m \in Z$

$$\Big\| \sum_{n \in \alpha^{-1}(\{m\})} e_n \Big\|_p \leqslant k.$$

Hint: Consider the T-image of e_m.

4.6. $\ell^p(Z)$ is defined for $1 \leqslant p \leqslant \infty$ as in 2.2.5. Verify that $\ell^1(Z)$ is an algebra under convolution. Has the algebra $\ell^1(Z)$ an identity element?

Is $\ell^p(Z)$ an algebra under convolution, if $p > 1$? (See the Note attached to Exercise 3.15.)

4.7. Find all the continuous complex homomorphisms of the algebra $\ell^1(Z)$. (See 11.4.17.)

4.8. Determine all complex-valued functions h on $R/2\pi Z$ having the property that the operator $T: f \to hf$ is a homomorphism of $\mathbf{T}$ into $\mathbf{L}^1$, each of the latter being considered as a convolution algebra.

4.9. In the general theory developed in Chapter 4 of [R], affine maps play a prominent role.

Let α be a map of Z into itself which is *affine* in the sense that

$$\alpha(n + n' - n'') = \alpha(n) + \alpha(n') - \alpha(n'')$$

whenever $n, n', n'' \in Z$. Show that α defines, via formula (4.2.3), a homomorphism T of $\mathbf{L}^1$ into itself, and find an explicit closed formula expressing Tf in terms of f.

CHAPTER 5

The Dirichlet and Fejér Kernels. Cesàro Summability

In this chapter we introduce the so-called Dirichlet and Fejér kernels and their elementary properties. These kernels are basic in the study of pointwise convergence and summability, respectively, of Fourier series. From their properties we shall derive the localization principle, together with alternative proofs of the uniqueness and approximation theorems of 2.4.1 and 2.4.4.

Included in the text and in some of the attached exercises are a few properties of Cesàro summability as applied to general series, most of which will later be applied in the case of Fourier series.

5.1 The Dirichlet and Fejér Kernels

In this section we define these kernels and state a few of their basic properties that are crucial in the study of pointwise convergence and Cesàro summability of Fourier series.

A few words about terminology and notation are required. The functions D_N and F_N introduced in (5.1.2) and (5.1.7), and herein called the Dirichlet and Fejér kernels, respectively, are precisely twice the functions to which these names are customarily attached; compare, for example, [Ba$_1$], pp. 85 and 133–134 and [Z$_1$], pp. 49 and 88, where K_N is written in place of $\frac{1}{2} F_N$. This choice of nomenclature has been made in order that the convolution expressions in (5.1.1) and (5.1.6) shall be valid.

If N is a nonnegative integer, the Nth symmetric partial sum of the Fourier series of f is

$$s_N f(x) = \sum_{|n| \leqslant N} \hat{f}(n) e^{inx}.$$

When we speak of the convergence of the Fourier series of f we shall always mean the convergence of these symmetric partial sums; see the end of 2.2.2.

Inserting the integral expression for $\hat{f}(n)$ we find that

$$s_N f(x) = \frac{1}{2\pi} \int f(y) D_N(x - y)\, dy = D_N * f(x), \tag{5.1.1}$$

where (see Exercise 1.1) one has for $x \not\equiv 0 \pmod{2\pi}$

$$D_N(x) = \sum_{|n| \leqslant N} e^{inx} = \frac{\sin (N + \frac{1}{2})x}{\sin \frac{1}{2}x}; \tag{5.1.2}$$

if x is congruent to $0 \pmod{2\pi}$, $D_N(x)$ has the value $2N + 1$, which is the value obtained by continuous extension of the expression on the extreme right in (5.1.2). The function D_N, or sometimes the sequence $(D_N)_{N=0}^{\infty}$, will be spoken of as the *Dirichlet kernel.*

It is to be observed that D_N is a trigonometric polynomial of degree N which is even in x and satisfies

$$\frac{1}{2\pi} \int D_N(x)\, dx = \frac{1}{\pi} \int_0^{\pi} D_N(x)\, dx = 1. \tag{5.1.3}$$

Furthermore,

$$|D_N(x)| \leqslant \operatorname{cosec} \tfrac{1}{2}\delta \qquad (0 < \delta \leqslant |x| \leqslant \pi). \tag{5.1.4}$$

The Nth Cesàro sum (see the end of 2.2.2) of the Fourier series of f is the arithmetic mean of the first $N + 1$ terms of the sequence of symmetric partial sums thereof, namely,

$$\sigma_N f = \frac{s_0 f + \cdots + s_N f}{N + 1}. \tag{5.1.5}$$

These are also spoken of as the $(C, 1)$-means of the Fourier series of $f \cdots$ "C" for Cesàro and the "1" indicating first order arithmetic means. Using (5.1.1) and some elementary calculations, we find that

$$\begin{aligned} \sigma_N f(x) &= \frac{1}{2\pi} \int f(y) F_N(x - y)\, dy = F_N * f(x) \\ &= \sum_{|n| \leqslant N} \left(1 - \frac{|n|}{N + 1}\right) \hat{f}(n) e^{inx}, \end{aligned} \tag{5.1.6}$$

where the functions (see Exercise 1.1)

$$\begin{aligned} F_N(x) &= \frac{D_0(x) + \cdots + D_N(x)}{N + 1} \\ &= \sum_{|n| \leqslant N} \left(1 - \frac{|n|}{N + 1}\right) e^{inx} \\ &= \frac{[\sin \frac{1}{2}(N + 1)x / \sin \frac{1}{2}x]^2}{N + 1} \end{aligned} \tag{5.1.7}$$

constitute the so-called *Fejér kernel* (because Fejér was the first to consider systematically the Cesàro summability of Fourier series); when $x \equiv 0$ (mod 2π), the final expression (5.1.7) is to be interpreted as $N + 1$ by continuous extension.

Notice that F_N is an even trigonometric polynomial of degree N and that

$$\frac{1}{2\pi} \int F_N(x)\, dx = \frac{1}{\pi} \int_0^{\pi} F_N(x)\, dx = 1. \tag{5.1.8}$$

Moreover,

$$\left.\begin{aligned} 0 \leqslant F_N(x) &\leqslant \operatorname{cosec} \tfrac{1}{2}\delta \\ 0 \leqslant F_N(x) &\leqslant \frac{\operatorname{cosec}^2 \frac{1}{2}\delta}{N+1} \end{aligned}\right\} \qquad (0 < \delta \leqslant |x| \leqslant \pi). \tag{5.1.9}$$

5.1.1. Concerning $\|D_N\|_1$. Since $F_N \geqslant 0$, the relations (5.1.8) and (5.1.9) suffice to show that $(F_N)_{N=0}^{\infty}$ is an approximate identity (see 3.2.1). This is *not* true of the D_N, for which (3.2.1) fails (as will be seen forthwith). This failure is the root cause of most troubles concerning the convergence of Fourier series, and also the reason why summability is often effective when convergence is not.

We shall show now that (3.2.1) fails for $(D_N)_{N=0}^{\infty}$, since in fact

$$\begin{aligned} \|D_N\|_1 &= \frac{1}{2\pi}\int |D_N(x)|\,dx = \frac{1}{\pi}\int_0^{\pi} |D_N(x)|\,dx \\ &= \frac{4}{\pi^2}\log N + O(1) \end{aligned} \tag{5.1.10}$$

as $N \to \infty$. Indeed, (5.1.2) gives

$$\begin{aligned} \|D_N\|_1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\sin (N+\frac{1}{2})x}{\sin \frac{1}{2}x}\right| dx \\ &= \frac{2}{\pi}\int_0^{\pi/2} \left|\frac{\sin (2N+1)y}{\sin y}\right| dy \qquad (\text{putting } y = \tfrac{1}{2}x) \\ &= \frac{2}{\pi}\int_0^{\pi/2} \left|\frac{\sin(2N+1)y}{y}\right| dy + O(1), \end{aligned}$$

since $(\sin y)^{-1} - y^{-1}$ is bounded on $(0, \pi/2)$. Putting $t = (2N+1)y$, the remaining integral becomes

$$\begin{aligned} \frac{2}{\pi}\int_0^{\frac{1}{2}(2N+1)\pi} \frac{|\sin t|\,dt}{t} &= \frac{2}{\pi}\sum_{k=0}^{2N}\int_{\frac{1}{2}k\pi}^{\frac{1}{2}(k+1)\pi} \frac{|\sin t|\,dt}{t} \\ &= \frac{2}{\pi}\sum_{k=0}^{2N}\int_0^{\frac{1}{2}\pi} \frac{u_k(s)\,ds}{\frac{1}{2}k\pi + s} \end{aligned}$$

on putting $t = \frac{1}{2}k\pi + s$ and $u_k(s) = \sin s$ or $\cos s$ according as k is even or odd. Now, for $k = 1, 2, \cdots$ and $0 \leqslant s \leqslant \pi$,

$$0 \leqslant \frac{1}{\frac{1}{2}k\pi} - \frac{1}{\frac{1}{2}k\pi + s} \leqslant \frac{\pi}{(\frac{1}{2}k\pi)^2} = \frac{4}{\pi k^2},$$

and $\sum_{k=1}^{\infty} 1/k^2$ is convergent. So

$$\begin{aligned} \|D_N\|_1 &= \frac{2}{\pi}\sum_{k=1}^{2N} 1/(\tfrac{1}{2}k\pi)\int_0^{\pi/2} u_k(s)\,ds + O(1) \\ &= \frac{2}{\pi}\sum_{k=1}^{2N} \frac{2}{k\pi} + O(1). \end{aligned}$$

Since $\sum_{k=1}^{2N} 1/k = \log N + O(1)$, (5.1.10) is established.

5.1.2. **New Proofs of 2.4.1 and 2.4.4.** It is worth noting that 3.2.2 and the opening sentence of 5.1.1 combine to yield independent proofs of 2.4.1 and 2.4.4.

Consider, for example, 2.4.1. If $\hat{f} = 0$, then (5.1.6) shows that $F_N * f = 0$ for all N. Since, as has been remarked in 5.1.1, the F_N form an approximate identity in $\mathbf{L}^1$, 3.2.2 shows that f, the mean limit in $\mathbf{L}^1$ (and, if f is continuous, the uniform limit) of $F_N * f$ as $N \to \infty$, is zero almost everywhere (or everywhere).

The deduction of 2.4.4 follows similarly from (5.1.6) and 3.2.2, the former showing that $F_N * f$ is a trigonometric polynomial for all N and all $f \in \mathbf{L}^1$.

The same arguments and sources yield results that may be interpreted in terms of the Cesàro summability of Fourier series, but we defer this development until Chapter 6.

5.2 The Localization Principle

The formulae collected in 5.1 can be used to show that the convergence or summability at a point x of the Fourier series of a function f depends solely on the behavior of f in the immediate neighborhood of x. A generalized version of this so-called *localization principle* reads as follows.

5.2.1. If f and g are integrable functions and if, for a given point x, the function

$$y \to \frac{f(y) - g(y)}{y - x}$$

is integrable over some neighborhood of the point x, then

$$\lim_{N \to \infty} [s_N f(x) - s_N g(x)] = 0 \tag{5.2.1}$$

and

$$\lim_{N \to \infty} [\sigma_N f(x) - \sigma_N g(x)] = 0. \tag{5.2.2}$$

Proof. Since both $h \to s_N h$ and $h \to \sigma_N h$ are linear operators, we may assume without loss of generality that $g = 0$ throughout. Again, since (5.1.1), (5.1.6), and 3.1.2 show that

$$s_N f(x) = s_N T_{-x} f(0), \qquad \sigma_N f(x) = \sigma_N T_{-x} f(0),$$

we may take $x = 0$.

Now, by (5.1.1),

$$\begin{aligned} s_N f(0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) D_N(y)\, dy \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \frac{f(y)}{\sin \frac{1}{2} y} \right\} \cdot \sin (N + \tfrac{1}{2}) y \, dy. \end{aligned} \tag{5.2.3}$$

If $f(y)/y$ is integrable over some neighborhood of 0, it is easily seen that $f(y)/\sin \frac{1}{2}y$ is integrable over $[-\pi, \pi]$, since for any small $\alpha > 0$ the ratio $y/\sin \frac{1}{2}y$ is bounded for $0 < |y| \leqslant \alpha$ and $\operatorname{cosec} \frac{1}{2}y$ is bounded for $\alpha \leqslant |y| \leqslant \pi$. Equation (5.2.1) [with $g = x = 0$] then follows from the Riemann-Lebesgue lemma 2.3.8 applied to the right-hand side of (5.2.3).

The least laborious way of arriving at (5.2.2) is to infer it from (5.2.1) on the basis of the simple and general result in 5.3.1.

5.2.2. Evidently, the hypotheses of 5.2.1 hold whenever f and g agree throughout some neighborhood of x, which is the situation to which the localization principle refers.

5.2.3. Taking g to be a constant function s, in which case

$$s_N g = s \qquad (N = 0, 1, 2, \cdots),$$

we see that 5.2.1 already implies that

$$\lim_N s_N f(x) = \lim_N \sigma_N f(x) = s \tag{5.2.4}$$

whenever $[f(y) - s]/(y - x)$ is integrable over some neighborhood of $y = x$. This is notably the case whenever $f'(x)$ exists and $s = f(x)$.

This, the first of our results on pointwise convergence and summability of Fourier series, will be supplemented by numerous other criteria in Chapters 6 and 10; it should be compared with Dini's test in 10.2.3. Crude though it is, it suffices to cover many specific instances. It shows that the Fourier series of most functions which arise in problems of applied mathematics converge to these functions at most points.

5.2.4. **Remarks.** (i) One cannot in 5.2.2 replace "some neighborhood of x" by "some set of positive measure containing x"; see [Ba_1], p. 465.

(ii) The localization principle (5.2.1) breaks down badly for groups T^m ($m > 1$), even for continuous functions f and g. The weaker principle (5.2.2) retains validity, at least for bounded functions f and g. See [Z_2], pp. 304–305. For certain other groups, see MR **37** ## 1527, 5330.

5.3 Remarks concerning Summability

Although summability theory is a highly developed field of activity, our concern rests almost entirely and solely in the use of Cesàro's method in connection with Fourier series. Even here, moreover, its principal merit is simply that it succeeds, in situations where ordinary convergence fails, in recapturing at almost all points a function from its Fourier series. Our remarks about summability are therefore very few.

Consider a two-way infinite series $\sum_{n \in Z} c_n$ and define the partial sums

$$s_N = \sum_{|n| \leqslant N} c_n$$

and the *Cesàro* (or *arithmetic*) *means*

$$\sigma_N = \frac{s_0 + \cdots + s_N}{N + 1}.$$

We recall (see the end of 2.2.2) that the given series is said to be *Cesàro* (or $(C, 1)$-) *summable* to sum s if and only if

$$\lim_{N \to \infty} \sigma_N = s.$$

[There is a Cesàro method (C, α) for every $\alpha \neq -1, -2, \cdots$, but we shall make no use of this concept; see, for example [Z_1], p. 76.]

Consider as an example the series $\sum_{n \in Z} e^{inx}$. Equation (5.1.2) shows that this series converges (to a finite sum) for no real values of x. On the other hand, (5.1.7) shows that this same series is Cesàro-summable to 0 for all real $x \not\equiv 0 \pmod{2\pi}$. It will appear in Chapters 6 and 10 that this very special example is surprisingly significant in relation to the behavior of Fourier series in general.

Turning to generalities, we shall first verify that the Cesàro method of summability is stronger than, and consistent with, ordinary convergence.

5.3.1. If $s_N \to s$, then also $\sigma_N \to s$ (as $N \to \infty$ in each case).

Proof. Since one would expect the arithmetic means σ_N to behave more regularly than the s_N, this statement should occasion no surprise. In view of the identity

$$\frac{(s_0 - s) + \cdots + (s_N - s)}{N + 1} = \sigma_N - s,$$

we may in the proof assume that $s = 0$. Then, given $\varepsilon > 0$, determine $N_0 = N_0(\varepsilon)$ so that $|s_N| \leqslant \varepsilon$ for $N > N_0$. For $N > N_0$ one has accordingly

$$\sigma_N = \frac{s_0 + \cdots + s_{N_0}}{N + 1} + \frac{s_{N_0 + 1} + \cdots + s_N}{N + 1},$$

so that

$$|\sigma_N| \leqslant \frac{M(N_0 + 1)}{N + 1} + \varepsilon,$$

where $M = \sup_N |s_N| < \infty$. Letting $N \to \infty$, we infer that

$$\limsup |\sigma_N| \leqslant \varepsilon.$$

Since ε is arbitrarily small, the result follows.

5.3.2. We have in 5.3.1 tacitly assumed that the limit s is finite. However, if we look at real-valued sequences s_N, the result extends to properly divergent ones. For example, the reader will verify easily that the preceding proof adapts easily to show that, if $\lim s_N = \infty$, then also $\lim \sigma_N = \infty$. More generally, indeed, one has in all cases

$$-\infty \leqslant \liminf s_N \leqslant \liminf \sigma_N \leqslant \limsup \sigma_N \leqslant \limsup s_N \leqslant \infty.$$

5.3.3. **Converse Results.** The unrestricted converse of 5.3.1 is false. Thus (5.1.2) shows that the series

$$\sum_{n \in Z} e^{inx}$$

diverges properly to ∞ if $x \equiv 0 (\mathrm{mod}\ 2\pi)$, and is boundedly divergent for all other values of x. On the other hand (5.1.7) shows that the series is Cesàro-summable to 0 for any x not congruent to 0 mod 2π; if $x \equiv 0 (\mathrm{mod}\ 2\pi)$, $\sigma_N \to \infty$, in accord with the remarks in 5.3.2.

An easy partial converse is contained in the next result.

5.3.4. If the s_N are real and increasing (for example, if $c_n \geqslant 0$), then Cesàro summability implies convergence.

Proof. There is no loss of generality in assuming that $s_N \geqslant 0$. Then

$$\sigma_{2N} \geqslant \frac{s_N + \cdots + s_{2N}}{2N + 1} \geqslant \frac{(N + 1)s_N}{2N + 1} \geqslant \tfrac{1}{2} s_N .$$

Hence the s_N are bounded above and convergence follows.

The result also follows from 5.3.2.

5.3.5. **Tauberian Theorems.** There are more subtle partial converses of 5.3.1, both for Cesàro and for other summability methods of importance. In these the positivity of the c_n (assumed in 5.3.4) is replaced by other conditions. Since the first results of this type (applying to Abel summability) were established by Tauber, such theorems have come to be known as *Tauberian theorems*. A little more about one source of such theorems will be found in 11.2.4.

A simple such theorem (due to Hardy; see Exercise 5.8) states that if $c_n = O(1/|n|)$, then Cesàro summability implies convergence. This is worth mentioning here because it could be used in conjunction with 2.3.6 to infer the convergence of the Fourier series of functions of bounded variation from the Cesàro summability thereof (yet to be established in Chapter 6). Not much economy would result from this approach, however, and we shall give a direct proof of convergence in due course (see 10.1.4).

EXERCISES

5.1. Suppose that $f \in \mathbf{L}^1$ satisfies

$$\omega_1 f(a) = o(|a|) \qquad \text{as } a \to 0,$$

the notation being as in 2.3.7. Prove that f is equal almost everywhere to a constant.

Hint: Consider first the case where f is a trigonometric polynomial. Then reduce the general case to this by using a suitable approximate identity.

5.2. Suppose that $f \in \mathbf{L}^1$ satisfies

$$\omega_1 f(a) = O(|a|^\alpha) \qquad \text{as } a \to 0$$

for some $\alpha > 0$. Let N be any integer such that $N\alpha > 1$. Prove that f^{*N} is equal almost everywhere to a continuous function, where $f^{*N} = f * \cdots * f$ (N factors). (Compare with Exercise 8.4.)

Note: In the following exercises the notation is as in Section 5.3.

5.3. Show that if $\lim_{N\to\infty}\sigma_N$ exists finitely, then $s_N = o(N)$ as $N \to \infty$.

5.4. Give detailed proofs of the statements in 5.3.2.

5.5. Show that $\Delta_N \equiv s_N - \sigma_N$ is expressible as

$$\Delta_N = (N+1)^{-1} \sum_{|n| \leqslant N} |n| c_n.$$

Show that if

$$M \equiv \sum_{n \in Z} |n|^{p-1} |c_n|^p < \infty$$

for some $p \geqslant 1$, and if $\lim_{N\to\infty}\sigma_N = s$, then $\lim_{N\to\infty} s_N = s$.

Hint: Use Hölder's inequality for series to show that

$$|\Delta_N| \leqslant A M^{1/p},$$

where A is an absolute constant. Notice that $\limsup_{N\to\infty}|\Delta_N|$ is unaffected if, for any k, c_n is redefined to be 0 for $|n| \leqslant k$.

5.6. (1) Suppose that $c_n \to 0$ as $|n| \to \infty$, and that $c_n = 0$, except perhaps for $n = 0, \pm n_k$ ($k = 1, 2, \cdots$), where $1 \leqslant n_1 < n_2 < \cdots$ and $\inf_k n_{k+1}/n_k \equiv q > 1$. Show that if $\lim_{N\to\infty}\sigma_N = s$, then $\lim_{N\to\infty} s_N = s$.

(2) Show also that the same conclusion is valid whenever $c_n = o(1/n)$.

Hint: Define Δ_N as in Exercise 5.5, and write

$$u_k = \sup\left(|c_{n_k}|, |c_{-n_k}|\right);$$

then show that for $n_k \leqslant N < n_{k+1}$

$$|\Delta_N| \leqslant 2 \cdot \sum_{r=1}^{k} u_r q^{r-k}.$$

Show that this last expression tends to zero as $k \to \infty$.

Remark. The result (1) allows a slight generalization to the case in which the hypotheses $\inf n_{k+1}/n_k > 1$ is replaced by

$$\sum_{k>m} \frac{1}{n_k} = O\left(\frac{1}{n_m}\right);$$

see [Ba$_1$], pp. 178–181.

5.7. Define, for N, $k = 1, 2, \cdots$,

$$\sigma_{N,k} \equiv k^{-1}(s_N + s_{N+1} + \cdots + s_{N+k-1}).$$

Verify that

$$\sigma_{N,k} = \left(1 + \frac{N}{k}\right)\sigma_{N+k-1} - \left(\frac{N}{k}\right)\sigma_{N-1},$$

and that

$$\sigma_{N,k} = s_N + \sum_{N<|n|\leqslant N+k} \left(1 - \frac{|n| - N}{k}\right)c_n.$$

Deduce from the former relation that if $k = k_N \to \infty$ with N in such a way that N/k_N remains bounded, and if $\lim_{N\to\infty}\sigma_N = s$, then $\lim_{N\to\infty}\sigma_{N,k} = s$.

5.8. Suppose that $c_n = O(1/|n|)$ as $|n| \to \infty$ and that $\lim_{N\to\infty}\sigma_N = s$. Show that $\lim_{N\to\infty}s_N = s$. (This is Hardy's theorem referred to in 5.3.5.)

Hint: Use the second relation established in the preceding exercise to show that

$$|\sigma_{N,k} - s_N| \leqslant \frac{Ak}{N}$$

for a suitable constant A. Choose $k = k_N$ suitably and employ the final assertion in the preceding exercise.

5.9. Suppose that $f \in \mathbf{L}^1$ satisfies

$$|f(x) - f(x_0)| = O\left[\left(\log \frac{1}{|x - x_0|}\right)^{-1-\varepsilon}\right]$$

as $x \to x_0$, for some $\varepsilon > 0$. Prove that

$$\sum_{n\in Z} \hat{f}(n)e^{inx_0} = f(x_0).$$

Hint: Use 5.2.3.

CHAPTER 6

Cesàro Summability of Fourier Series and Its Consequences

6.1 Uniform and Mean Summability

From 3.2.2 and the fact that the Fejér kernels F_N form an approximate identity (see 5.1.1), we infer at once the following basic results about uniform and mean summability of Fourier series.

6.1.1. If $f \in \mathbf{C}^k$, and if m is a nonnegative integer not exceeding k, then

$$\lim_{N \to \infty} \|D^m(f - \sigma_N f)\|_\infty = 0.$$

If $f \in \mathbf{L}^p$, where $1 \leqslant p < \infty$, then

$$\lim_{N \to \infty} \|f - \sigma_N f\|_p = 0.$$

6.1.2. The case $k = 0$ of 6.1.1 is especially significant. It asserts that the Fourier series of any continuous function is uniformly Cesàro-summable to that function. This, together with several other results of a similar nature dealt with in this chapter, were given by Fejér in 1904. Since it was already known by then that the Fourier series of a continuous function may diverge at certain points, Fejér's result can be expected to have helped analysts to breathe more freely once again. If the Cesàro method did nothing more than this, it would amply justify its existence; as we shall see, it actually does a good deal more.

The proof of 6.1.1 works equally well for many function spaces $\mathbf{H}$ other than $\mathbf{C}^k$ and $\mathbf{L}^p$ ($1 \leqslant p < \infty$), including at least all so-called *homogeneous Banach spaces over* T. These are by definition the linear subspaces of $\mathbf{L}^1(T)$ which are endowed with a norm $\|\cdot\|_{\mathbf{H}}$ under which it is a Banach space and such that:

(i) $\|f\|_1 \leqslant \|f\|_{\mathbf{H}}$ for all $f \in \mathbf{H}$;

(ii) if $f \in H$ and $a \in T$, then $T_a f \in \mathbf{H}$ and $\|T_a f\|_{\mathbf{H}} = \|f\|_{\mathbf{H}}$;

(iii) $\lim_{a \to 0} \|T_a f - f\|_{\mathbf{H}} = 0$ for all $f \in \mathbf{H}$.

(See [Kz], Chapter I, 2.10 and 2.11.)

It follows from Exercises 3.1 and 3.2 that the second part of 6.1.1 is not true, if $\sigma_N f$ is replaced by $s_N f$; the first part also fails. In both cases the failure is a consequence of the fact that

$$\sup_N \|D_N\|_1 = \infty,$$

itself a corollary of (5.1.10). (For the case $k = 0$ of 6.1.1, see also 10.3 below.)

For a homogeneous Banach space $\mathbf{H}$ over T satisfying $\|e_n f\|_{\mathbf{H}} = \|f\|_{\mathbf{H}}$ for all $n \in Z$ and all $f \in \mathbf{H}$, it is known (see [Kz], p. 49) that

$$\lim_{N \to \infty} \|f - s_N f\|_{\mathbf{H}} = 0 \quad \text{for all } f \in \mathbf{H}$$

if and only if $\mathbf{H}$ admits conjugation; that is to say, if and only if $f \in \mathbf{H}$ implies $\tilde{f} \in \mathbf{H}$, where the conjugate function $\tilde{f}$ is defined as in 12.8.1. [This last is equivalent to demanding that, for every $f \in \mathbf{H}$, there is a function $\tilde{f} \in \mathbf{H}$ such that

$$\hat{\tilde{f}}(n) = -i \cdot \operatorname{sgn} n \cdot \hat{f}(n) \quad \text{for all } n \in Z.]$$

The space $\mathbf{L}^p$ $(1 < p < \infty)$ admits conjugation (see 12.9.3), but neither $\mathbf{C}$ nor $\mathbf{L}^1$ admits conjugation (see 12.8.3–12.8.5).

6.1.3. **Characterization of Fourier Series.** At this point we can characterize the Fourier series of a given $f \in \mathbf{L}^1$ among all trigonometric series, thereby fulfilling a prediction made in 1.3. Thus the case $p = 1$ of 6.1.1 shows that the $\sigma_N f$, the Cesàro means of the Fourier series of f, converge in mean in $\mathbf{L}^1$ to f. On the other hand, the Fourier series of f is the only trigonometric series with this property.

Suppose indeed that $\sum c_n e^{inx}$ is a trigonometric series whose Cesàro means

$$\sigma_N(x) = \sum_{|n| \leqslant N} c_n \left(1 - \frac{|n|}{N+1}\right) e^{inx}$$

converge in mean in $\mathbf{L}^1$ to f. Then, by 2.3.2, one has

$$\lim_{N \to \infty} \hat{\sigma}_N(n) = \hat{f}(n).$$

However,

$$\hat{\sigma}_N(n) = \left(1 - \frac{|n|}{N+1}\right) c_n$$

if $|n| \leqslant N$ and is zero otherwise, so that $\lim_{N \to \infty} \hat{\sigma}_N(n) = c_n$. Thus $c_n = \hat{f}(n)$ for all $n \in Z$, and the trigonometric series in question is the Fourier series of f.

6.1.4. **Behavior of $\|f - \sigma_N f\|_p$ as $N \to \infty$.** Although we know from 6.1.1 that, as $N \to \infty$,

$$\|f - \sigma_N f\|_p = o(1)$$

if $f \in \mathbf{L}^p$ and $1 \leqslant p < \infty$, and that

$$\|f - \sigma_N f\|_\infty = o(1)$$

if $f \in \mathbf{C}$, these assertions cannot be improved to the extent of replacing $o(1)$ by $O(\varepsilon_N)$ for any fixed sequence $\varepsilon_N \to 0$.

More precisely, it is impossible to find a sequence $(K_N)_{N=1}^\infty$ of integrable functions and a sequence $(\varepsilon_N)_{N=1}^\infty$ of positive numbers such that

(1) $$\liminf_{N \to \infty} \varepsilon_N = 0$$

and *either*

(2) $\|f - K_N * f\|_p = O(\varepsilon_N)$ for some p satisfying $1 \leqslant p < \infty$ and each $f \in \mathbf{L}^p$,

or

(2a) $\|f - K_N * f\|_\infty = O(\varepsilon_N)$ for each $f \in \mathbf{C}$.

On the other hand, it will be seen in Section 6.5 and Exercises 6.6 to 6.9 that integrable functions K_N and positive numbers $\varepsilon_N \to 0$ can be chosen so that (2a) is valid for functions f satisfying additional smoothness conditions.

We will exhibit a proof of the impossibility of satisfying (1) and (2); an exactly similar argument may be used to establish the impossibility of satisfying (1) and (2a). This proof shows, by using functional analytic techniques, that the assumption that (1) and (2) can be fulfilled leads to a contradiction of 2.3.8. Before reading the following proof, the reader is urged to look at Appendices A and B.1.

Suppose then that (1) and (2) are satisfied by some choice of $(K_N)_{N=1}^\infty$ and $(\varepsilon_N)_{N=1}^\infty$. We shall study the function q defined on $\mathbf{L}^p$ by the formula

$$q(f) = \sup\{\varepsilon_N^{-1} \|f - K_N * f\|_p : N = 1, 2, \cdots\}.$$

According to (2), q is finite-valued on $\mathbf{L}^p$, and from this it is almost evident that q is a seminorm on $\mathbf{L}^p$ (see Appendix B.1.1). For each N, an application of 3.1.6 shows that the function

$$f \to \varepsilon_N^{-1} \|f - K_N * f\|_p$$

is continuous on $\mathbf{L}^p$. Hence (see Appendix A.4) q is a lower semicontinuous seminorm on $\mathbf{L}^p$. The crucial step is to apply Appendix B.2.1(1), using therein $\mathbf{E} = \mathbf{L}^p$ and $p_k = q$ for $k = 1, 2, \cdots$; since $\mathbf{L}^p$ is complete (see 2.2.4), this application is justified and leads to the conclusion that q is continuous on $\mathbf{L}^p$. This signifies the existence of a constant c such that $q(f) \leqslant c \cdot \|f\|_p$ for all $f \in \mathbf{L}^p$. In other words, we have

$$\|f - K_N * f\|_p \leqslant c \cdot \varepsilon_N \cdot \|f\|_p \tag{6.1.1}$$

for $f \in \mathbf{L}^p$ and $N = 1, 2, \cdots$. If herein we choose to take $f = e_n$, we shall obtain thereby the relation

$$|\hat{K}_N(n) - 1| \leqslant c \cdot \varepsilon_N \tag{6.1.2}$$

for all $n \in Z$ and $N = 1, 2, \cdots$. Therefore, in view of (1), it will appear from (6.1.2) that

$$|\hat{K}_N(n) - 1| \leqslant \tfrac{1}{2}$$

for all $n \in Z$ and infinitely many N. For any such N, it thus appears that

$$\liminf_{|n| \to \infty} |\hat{K}_N(n)| \geqslant \tfrac{1}{2} > 0,$$

which contradicts 2.3.8. This contradiction terminates the proof.

Further illustrations of this type of proof will be discussed in detail in Section 10.3.

6.2 Applications and Corollaries of 6.1.1

6.2.1. **Comments on 6.1.1.** Before pressing on to some refinements of 6.1.1 contained in Sections 6.3 and 6.4, we shall make a number of deductions from 6.1.1, each of which is of considerable importance.

Since it is evident that 6.1.1 refines the trigonometric polynomial approximation theorem of 2.4.4 to the extent of specifying an algorithm for the construction of trigonometric polynomials approximating a given function, it may well suggest two lines of thought, namely:

(1) Real analysis contains another famous approximation theorem, to wit, that of Weierstrass. This refers to the approximation, uniformly on a compact real interval I, of continuous functions on I by ordinary polynomials. In 6.2.2 and 6.2.4 we shall show how this theorem is deducible from 6.1.1 and indicate a common source of both theorems.

(2) What can be said in general about approximation by trigonometric polynomials? For a given f, how good an approximation is afforded by $\sigma_N f$ in comparison with other trigonometric polynomials of degree at most N? We shall come round to a brief discussion of these matters in Section 6.5.

Meanwhile, 6.2.5 to 6.2.8 will be concerned with some deductions from 6.1.1 more directly concerned with Fourier series.

6.2.2. **The Weierstrass Polynomial Approximation Theorem.** This asserts that if f is a (not necessarily periodic) continuous function on a compact interval $[a, b]$ of the line, then f is uniformly approximable on $[a, b]$ by (ordinary) polynomial functions.

In proving this on the basis of 6.1.1, we may without loss of generality assume that $[a, b]$ is $[-\pi, \pi]$. Then a constant c may be chosen so that $f - cx$ takes the same value at $-\pi$ as at π, and can therefore be extended into a periodic continuous function. It is evidently sufficient to show that this modified function is uniformly approximable on $[-\pi, \pi]$ by polynomials. Thus we may assume from the outset that f is periodic and continuous.

Given any $\varepsilon > 0$ we choose N so large that

$$\|f - \sigma_N f\|_\infty \leqslant \tfrac{1}{2}\varepsilon.$$

The trigonometric polynomial

$$\sigma_N f(x) = \sum_{|n| \leqslant N} c_n e^{inx},$$

can in turn be uniformly approximated on $[-\pi, \pi]$ to within $\frac{1}{2}\varepsilon$ by ordinary polynomials, to do which it suffices to replace each exponential e^{inx} by a sufficiently large number of terms of its Taylor expansion about the origin, the latter series converging uniformly on any compact set. The result is a polynomial function P such that

$$|(\sigma_N f - P)(x)| \leqslant \tfrac{1}{2}\varepsilon$$

uniformly for $|x| \leqslant \pi$. But then

$$|(f - P)(x)| \leqslant \varepsilon$$

uniformly for $|x| \leqslant \pi$, and Weierstrass' theorem is established.

6.2.3. **Other Proofs of Weierstrass' Theorem.** There are many other proofs of Weierstrass' theorem, both "classical" and "abstract-modern" in flavor; the latter are certainly the more enlightening.

It was M. H. Stone who, in 1937, first undertook an abstract analysis of the status of Weierstrass' theorem and its close relatives. His work and subsequent developments laid bare the anatomy of the situation and have resulted in very general approximation theorems concerning closed subalgebras of the Banach algebra (with pointwise operations) of continuous functions on any compact Hausdorff space; these algebras will be encountered again in 11.4.1. Both 2.4.4 and Weierstrass' theorem are contained as very special cases within this scheme. For a recent survey, see Stone's article "A generalization of Weierstrass' approximation theorem" appearing on pp. 30–87 of [SMA]; see also [E], Section 4.10, [HS], pp. 94–98, and [L_2], Chapter 1.

6.2.4. **Bernstein Polynomials.** In just the same way that 6.1.1 includes and refines 2.4.4, a famous theorem of Bernstein includes and refines the Weierstrass theorem in 6.2.2. Bernstein's theorem asserts that, if f is a continuous function on [0, 1], then the associated so-called *Bernstein polynomials*

$$B_N f(x) = \sum_{n=0}^{N} f(n/N) \cdot {}_N C_n x^n (1 - x)^{N-n} \qquad (N = 0, 1, 2, \cdots)$$

converge to f uniformly on [0, 1].

There is a very extensive literature dealing with Bernstein polynomials; for a start, the interested reader should consult [Ka], pp. 52–59, [L_1], and [L_2], Chapter 1.

We now turn to some deductions from 6.1.1 more closely connected with Fourier series.

6.2.5. Suppose that $f, g \in \mathbf{L}^1$ and that the Fourier series of g is essentially boundedly convergent almost everywhere, that is, that $\sup_N \|s_N g\|_\infty < \infty$ and that $\lim_{N\to\infty} s_N g(x)$ exists for almost all x. Then

$$\frac{1}{2\pi}\int f(x)g(x)\,dx = \sum_{n\in Z} \hat{f}(n)\cdot\hat{g}(-n),$$

the series being convergent.

Proof. By hypothesis we have

$$|s_N g(x)| \leqslant M \qquad \text{a.e.},$$

M being a number independent of N and of x, and $\lim_{N\to\infty} s_N g(x)$ exists almost everywhere. By 5.3.1 and 6.1.1, $\lim_{N\to\infty} s_N g(x) = g(x)$ almost everywhere. Consequently,

$$\begin{aligned}\frac{1}{2\pi}\int f(x)g(x)\,dx &= \lim_{N\to\infty}\frac{1}{2\pi}\int f(x)s_N g(x)\,dx \\ &= \lim_{N\to\infty}\sum_{|n|\leqslant N} \hat{f}(-n)\cdot\hat{g}(n),\end{aligned}$$

passage to the limit under the integral sign being justified by Lebesgue's theorem ([W], Theorem 4.1b). From this the stated results follow.

6.2.6. **Remarks.** (1) By 2.3.5, the hypotheses on g are certainly fulfilled whenever $g \in \mathbf{C}^2$. Consequently, 6.2.5 justifies the characterization of the Fourier series of $f \in \mathbf{L}^1$ among all trigonometric series mentioned in (D) of 1.3.2.

(2) The conclusion of 6.2.5 may be derived from the mere assumption that $\sup_N \|s_N g\|_\infty < \infty$. Thus, Appendix B.4.1 may be used to show that there is at least one subsequence $(s_{N_k} g)$ which converges weakly in $\mathbf{L}^\infty$. Moreover, if $(s_{N_k} g)$ is any subsequence converging to h weakly in $\mathbf{L}^\infty$, and if $u \in T$, then

$$\frac{1}{2\pi}\int u(x)s_{N_k}g(x)\,dx = \frac{1}{2\pi}\int u(x)g(x)\,dx$$

for all large k; hence

$$\frac{1}{2\pi}\int u(x)h(x)\,dx = \frac{1}{2\pi}\int u(x)g(x)\,dx$$

and so $h = g$ a.e. It follows that $s_N g \to g$ weakly in $\mathbf{L}^\infty$. In view of Appendix C.1, this is equivalent to the desired conclusion.

6.2.7. **Remark.** The formula appearing in 6.2.5 is one variant of the so-called Parseval formula, a prototype version of which has appeared in Exercise 1.7, and to which we shall return in Sections 8.2 and 10.5 with different hypotheses on f and g.

It is to be observed that the series appearing in 6.2.5 is not convergent for all $f \in \mathbf{L}^1$ and all continuous g; see Exercise 10.7. On the other hand, 6.1.1 is easily seen to imply that the series is Cesàro-summable to $(1/2\pi)\int fg\,dx$ whenever $f \in \mathbf{L}^1$ and $g \in \mathbf{L}^\infty$; see Exercise 6.2.

One very special but important case of 6.2.5 demands closer examination, since it leads to the conclusion that, no matter how badly the Fourier series of $f \in \mathbf{L}^1$ may behave in respect of pointwise convergence (see 10.3.4), yet nevertheless one may always legitimately integrate termwise this Fourier series.

6.2.8. If $f \in \mathbf{L}^1$, then

$$\int_a^b f(x)\,dx = \sum_{n \in Z} \hat{f}(n)\frac{e^{inb} - e^{ina}}{in},$$

the term corresponding to $n = 0$ being understood to mean $\hat{f}(0)(b - a)$.

Proof. Owing to the meaning assigned to the term corresponding to $n = 0$, it is sufficient to establish the formula for the case in which $a = 0$ and $0 < b < 2\pi$. Let g be the function equal to 1 on the interval $[0, b)$, zero elsewhere on $[0, 2\pi)$, and extended so as to have period 2π. A direct computation shows that

$$\hat{g}(n) = \frac{e^{-inb} - 1}{-2\pi in} = \frac{e^{-\frac{1}{2}ibn}\sin\frac{1}{2}bn}{\pi n}, \tag{6.2.1}$$

the right-hand side being understood to mean $b/2\pi$ when $n = 0$. Consequently one finds after some reduction that

$$s_N g(x) = \frac{b}{2\pi} + \sum_{n=1}^{N} \frac{\sin n(x - b) - \sin nx}{\pi n}.$$

Reference to Exercise 1.5 confirms that this series is boundedly convergent for all x. Also, by 5.2.3, the limit of $s_N g(x)$ is $g(x)$ provided $0 < x < 2\pi$ and $x \neq b$, since g is constant on some neighborhood of each such point x. Hence ([W], Theorem 4.1b)

$$\begin{aligned}\int_0^b f(x)\,dx &= \int_0^{2\pi} f(x)g(x)\,dx = \lim_{N\to\infty} \int_0^{2\pi} f(x)\cdot s_N g(x)\,dx \\ &= \lim_{N\to\infty} \sum_{|n|\leqslant N} \hat{g}(n)\cdot\int_0^{2\pi} f(x)e^{inx}\,dx \\ &= \lim_{N\to\infty} \sum_{|n|\leqslant N} \hat{g}(n)\cdot 2\pi\hat{f}(-n) \\ &= \lim_{N\to\infty} \sum_{|n|\leqslant N} \hat{f}(n)\cdot 2\pi\hat{g}(-n).\end{aligned}$$

Using (6.2.1), the desired result appears.

Remark. This result (and even a little more) will be obtained on the basis of more general theorems in 10.1.5.

6.3 More about Pointwise Summability

We shall here deal with some refinements of the first half of 6.1.1.

It is convenient to introduce some notation. Given a point x and a number s, we write

$$f_s^*(y) \equiv f_s^*(y, x) = \tfrac{1}{2}[f(x+y) + f(x-y) - 2s], \tag{6.3.1}$$

in terms of which (5.1.6) and (5.1.8) lead to

$$\sigma_N f(x) - s = \frac{1}{\pi}\int_0^{\pi} f_s^*(y) F_N(y)\, dy. \tag{6.3.2}$$

It is our aim to give simple conditions sufficient to ensure that the expression (6.3.2) tends to zero as $N \to \infty$. In studying this we shall not assume from the outset that s is the "right" value, namely $f(x)$. Indeed, it is the behavior of f_s^* in the neighborhood of $y = 0$ (that is, of f in the neighborhood of x) which is significant, and the prime feature is not the value of f_s^* at 0 (that is, of f at x) but rather the limiting behavior of f_s^* near 0 (that is, of f near x). In this section we consider the simplest case, in which we assume outright the existence of the limit of $f_s^*(y)$ as $y \to +0$.

6.3.1. Suppose that $f \in \mathbf{L}^1$ and that

$$f(x+0) + f(x-0) \equiv \lim_{y \to +0} [f(x+y) + f(x-y)] \tag{6.3.3}$$

exists finitely. Then

$$\lim_{N \to \infty} \sigma_N f(x) = \tfrac{1}{2}[f(x+0) + f(x-0)]. \tag{6.3.4}$$

The limit in (6.3.4) is attained uniformly on any set on which the limit in (6.3.3) is attained uniformly.

Proof. The formula (6.3.2) will be applied, taking therein

$$s = \tfrac{1}{2}[f(x+0) + f(x-0)].$$

The hypotheses signify that $f_s^*(y) \to 0$ as $y \to +0$. Given any $\varepsilon > 0$, there exists therefore a number $\delta > 0$ such that $|f_s^*(y)| \leqslant \varepsilon$ for $0 \leqslant y \leqslant \delta$. This number δ will depend upon x and ε, but can be chosen uniformly with respect to x when the latter varies in any set on which the limit in (6.3.3) is attained uniformly. The integral in (6.3.2) is then expressed as a sum

$$\frac{1}{\pi}\int_0^{\delta} + \frac{1}{\pi}\int_{\delta}^{\pi} = I_1 + I_2,$$

say. Then

$$|I_1| \leqslant \frac{\varepsilon}{\pi}\int_0^{\delta} F_N\, dy \leqslant \frac{\varepsilon}{\pi}\int_0^{\pi} F_N\, dy = \varepsilon,$$

by (5.1.8). (Notice that we are here using positivity of F_N: this is not available for D_N and the substitution of $|D_N|$ for D_N would vitiate the argument irreparably.) On the other hand, the use of (5.1.9) leads to

$$\begin{aligned} |I_2| &\leqslant \frac{\operatorname{cosec}^2 \frac{1}{2}\delta}{N+1} \cdot \int_0^{\pi} |f_s^*(y)|\, dy \\ &\leqslant \frac{\operatorname{cosec}^2 \frac{1}{2}\delta}{N+1} \cdot \pi[\|f\|_1 + |s|]. \end{aligned}$$

If ε and δ are held fixed while N is allowed to tend to infinity, it is seen that

$$\limsup_{N\to\infty} |\sigma_N f(x) - s| \leqslant \varepsilon + \limsup_{N\to\infty} |I_2| = \varepsilon.$$

Since ε is a freely chosen positive number, the desired conclusion now follows.

6.4 Pointwise Summability Almost Everywhere

So far we have depended solely on the fact that the Fejér kernels form an approximate identity. By using somewhat more special properties, combined with a fundamental theorem of Lebesgue related to the differentiability of indefinite integrals, we can establish the pointwise summability almost everywhere of the Fourier series of any integrable function. Before proving this result, we shall review the auxiliary requirements.

6.4.1. Auxiliary Inequalities. Concerning F_N we observe two inequalities. Temporarily using A, B to denote absolute constants, the first inequality reads

$$0 \leqslant F_N(y) \leqslant F_N^*(y) \equiv \frac{AN}{1 + N^2y^2} \qquad (0 \leqslant y \leqslant \pi), \tag{6.4.1}$$

which is easily established by examining separately the ranges $0 \leqslant y \leqslant \pi/N$ and $\pi/N \leqslant y \leqslant \pi$; in the first interval, $F_N(y)$ is majorized by a multiple of N, and in the second by a multiple of $N^{-1}y^{-2}$. This inequality shows in turn that

$$0 \leqslant F_N(y) \leqslant F_N^*(y) \leqslant \frac{A}{2y} \qquad (0 < y \leqslant \pi). \tag{6.4.2}$$

We shall also need the following consequence of (6.4.1):

$$\begin{aligned} \int_0^{\pi} y|F_N^{*\prime}(y)|\, dy &= 2AN^3 \int_0^{\pi} \frac{y^2\, dy}{(1 + N^2y^2)^2} \\ &\leqslant 2A \int_0^{\infty} \frac{t^2\, dt}{(1 + t^2)^2} = B. \end{aligned} \tag{6.4.3}$$

6.4.2. Regarding the theorem of Lebesgue it suffices to recall that for any integrable f, periodic or not, and almost all x it is true that

$$\int_0^h |f(x+y) - f(x)|\, dy = o(h)$$
$$\int_0^h |f(x-y) - f(x)|\, dy = o(h)$$

as $h \to +0$. For a proof, see [HS], p. 276.

Using the notation introduced in Section 6.3, it follows that for almost all values of x one has

$$\int_0^h |f_s^*(y)|\, dy = o(h) \qquad (h \to +0), \tag{6.4.4}$$

when s is taken to be $f(x)$. Such points x are usually termed *Lebesgue points* of f and the set of such points the *Lebesgue set* of f.

Now we can state and prove the main theorems of this section.

6.4.3. If $f \in \mathbf{L}^1$, and if for a given x and s one has

$$\int_0^h |f_s^*(y)|\, dy = o(h) \qquad (h \to +0), \tag{6.4.4}$$

then

$$\lim_{N \to \infty} \sigma_N f(x) = s. \tag{6.4.5}$$

Proof. We start again from the formula (6.3.2), namely,

$$\sigma_N f(x) - s = \frac{1}{\pi} \int_0^\pi f_s^*(y) F_N(y)\, dy.$$

Assuming that (6.4.4) is satisfied, we suppose that $\varepsilon > 0$ is given and choose $\delta > 0$ so that

$$0 \leqslant I(h) \equiv \int_0^h |f_s^*(y)|\, dy \leqslant \varepsilon h \qquad (0 \leqslant h \leqslant \delta). \tag{6.4.6}$$

By (6.4.1),

$$\left|\frac{1}{\pi} \int_0^\delta f_s^*(y) F_N(y)\, dy\right| \leqslant \frac{1}{\pi} \int_0^\delta |f_s^*(y)| F_N^*(y)\, dy.$$

Partial integration and (6.4.6) show that this is majorized by

$$\begin{aligned}\pi^{-1} I(\delta) F_N^*(\delta) - \pi^{-1} \int_0^\delta I(y) F_N^{*\prime}(y)\, dy \\ &\leqslant \pi^{-1}\varepsilon\delta \cdot F_N^*(\delta) + \pi^{-1}\varepsilon \int_0^\delta y |F_N^{*\prime}(y)|\, dy \\ &\leqslant \tfrac{1}{2} A \pi^{-1} \varepsilon + B \pi^{-1} \varepsilon\end{aligned}$$

by (6.4.2) and (6.4.3). Thus

$$\left|\frac{1}{\pi}\int_0^\delta f_s^*(y)F_N(y)\,dy\right| \leqslant \tfrac{1}{2}A\pi^{-1}\varepsilon + B\pi^{-1}\varepsilon,$$

and therefore

$$|\sigma_N f(x) - s| \leqslant \pi^{-1}(\tfrac{1}{2}A + B)\varepsilon + \left|\frac{1}{\pi}\int_\delta^\pi f_s^*(y)F_N(y)\,dy\right|. \qquad (6.4.7)$$

Now, for any $\delta > 0$, the inequality (5.1.9) shows that the integral appearing in (6.4.7) tends to zero as $N \to \infty$. So, keeping ε and δ fixed, we infer from (6.4.7) that

$$\limsup_{N\to\infty} |\sigma_N f(x) - s| \leqslant \pi^{-1}(\tfrac{1}{2}A + B)\varepsilon.$$

Finally, letting $\varepsilon \to 0$, (6.4.5) follows.

6.4.4. If $f \in \mathbf{L}^1$, then

$$\lim_{N\to\infty} \sigma_N f(x) = f(x) \qquad (6.4.8)$$

holds for almost all x.

Proof. This is immediate on combining the closing remark in 6.4.2 with 6.4.3.

6.4.5. If a trigonometric series $\sum c_n e^{inx}$ is Cesàro-summable almost everywhere to a sum $f(x)$, and if this series is a Fourier-Lebesgue series, then $f \in \mathbf{L}^1$ and the series is the Fourier series of f.

Proof. By hypothesis the series is the Fourier series of a function $g \in \mathbf{L}^1$. By 6.4.4, the Cesàro means σ_N of the given series converge almost everywhere to g. Hence $g = f$ a.e., and the stated results follow.

6.4.6. **Remark.** In connection with 6.4.5 it must be remarked that a trigonometric series $\sum c_n e^{inx}$ may well be Cesàro-summable almost everywhere to an integrable sum and yet fail to be a Fourier-Lebesgue series. Thus the series $1 + 2\sum_{n=1}^{\infty} \cos nx$ is Cesàro-summable to 0 for every $x \not\equiv 0 \pmod{2\pi}$ [as appears from (7.1.1) and (7.1.2)], but 2.3.8 shows that it is not a Fourier-Lebesgue series. (It is, however, the Fourier-Stieltjes series of the Dirac measure ε; see 12.2.3 and 12.5.10.)

6.4.7. **The Majorant Function $\sigma^* f$.** It is worth observing that, if $g \in \mathbf{L}^\infty$, then (5.1.8) gives at once for all x the inequality

$$|\sigma_N g(x)| \leqslant \|g\|_\infty. \qquad (6.4.9)$$

In view of (6.4.9) it is interesting to consider the *majorant function*

$$\sigma^* f(x) \equiv \sup_N |\sigma_N f(x)| \qquad (\leqslant \infty). \qquad (6.4.10)$$

Naturally, $\sigma^* f$ is a nonnegative measurable (in fact, lower semicontinuous) function. By using special properties of the F_N it may be shown (see [Z_1], pp. 154–156; Edwards and Hewitt [1], Theorem 3.1) that this majorant satisfies the following integral inequalities:

$$\|\sigma^* f\|_p \leqslant A_p \|f\|_p \qquad \text{if } f \in \mathbf{L}^p \text{ and } p > 1, \tag{6.4.11}$$

$$\|\sigma^* f\|_p \leqslant A_p \|f\|_1 \qquad \text{if } f \in \mathbf{L}^1 \text{ and } 0 < p < 1, \tag{6.4.12}$$

$$\|\sigma^* f\|_1 \leqslant \frac{A}{2\pi} \int |f| \log^+ |f| \, dx + B \qquad \text{if } f \cdot \log^+ |f| \in \mathbf{L}^1. \tag{6.4.13}$$

In these inequalities A and B denote absolute constants, A_p depends upon p only, and $\log^+ t = \log (\max (1, t))$.

Concerning the analogous assertions applying to the majorant

$$s^* f(x) \equiv \sup_N |s_N f(x)| \tag{6.4.14}$$

of the ordinary partial sums of the Fourier series of f, see 10.3.5 and 10.4.5.

6.4.8. **The Estimate (6.4.9).** The relation (6.4.9), valid for $g \in \mathbf{L}^\infty$, is close to being the best possible. Indeed, if we are given any set E of measure zero, there exists a nonnegative function f belonging to $\mathbf{L}^p$ for every $p < \infty$ and such that

$$\lim_{N \to \infty} \sigma_N f(x) = \infty \qquad (x \in E). \tag{6.4.15}$$

To verify this we may assume that E lies in $(0, 2\pi)$ and choose numbers $\alpha_k > 0$ and $c_k \geqslant 0$ $(k = 1, 2, \cdots)$ so that

$$\lim_{k \to \infty} c_k = \infty, \qquad \sum_{k=1}^{\infty} c_k{}^p \alpha_k < \infty \tag{6.4.16}$$

for every $p < \infty$; for example, $c_k = \log k$, $\alpha_k = k^{-2}$. Since E has measure zero, we may choose open sets E_k such that $E \subset E_k \subset (0, 2\pi)$ and having measure $m(E_k) \leqslant \alpha_k$. Let f_k denote the characteristic function of E_k, extended by periodicity, and put

$$f = \sum_{k=1}^{\infty} c_k f_k.$$

Then $f \in \mathbf{L}^p$ for every $p < \infty$, as follows from the second clause of (6.4.16) and Beppo Levi's theorem ([W], Theorem 4.1e). Observe also that f is nonnegative and lower semicontinuous. According to 6.3.1,

$$\lim_{N \to \infty} \sigma_N f_k(x) = 1$$

for all $x \in E_k$ and, a fortiori, for $x \in E$. Since F_N is nonnegative, one has for all x and all k

$$\sigma_N f(x) \geqslant c_k \cdot \sigma_N f_k(x),$$

and so for all $x \in E$ and all k

$$\liminf_{N \to \infty} \sigma_N f(x) \geqslant c_k.$$

This, together with the first clause of (6.4.16), yields (6.4.15).
See also Exercise 6.18.

6.4.9. **Remarks on Convergence.** By combining Exercises 5.5 and 5.6 with 6.4.4 one can specify some simple conditions on $f \in \mathbf{L}^1$ sufficient to ensure that

$$\lim_{N \to \infty} s_N f(x) = f(x) \qquad \text{a.e.}; \tag{6.4.17}$$

see Exercises 6.12 and 6.13, and also 10.1.1, 10.1.4, 10.2.1, and 10.2.3. As will appear in 10.3.4, the relation (6.4.17) is hopelessly over-optimistic for general $f \in \mathbf{L}^1$.

6.5 Approximation by Trigonometric Polynomials

In this section we shall take a look at the issues enumerated in 6.2.1(2). This involves an excursion around the fringes of the general question of approximation by trigonometric polynomials; for major inroads into the general theory the reader must turn to the relevant portions of [L$_2$], [Z], [Ba], [Ti], and [BK].

6.5.1. **The Functionals ρ_N and E_N.** For definiteness we shall work within the Banach space $\mathbf{C}$ of continuous functions, but the reader will scarcely need to be told that each question posed in this setting has some sort of analogue for the case in which $\mathbf{C}$ is replaced by $\mathbf{L}^p$ (or, indeed, by any one of a number of other quite natural function spaces).

In order to examine the questions raised in vague terms in 6.2.1(2), we introduce two sequences of functionals ρ_N and E_N ($N = 0, 1, 2, \cdots$) defined on $\mathbf{C}$ in the following way:

$$\rho_N f = \|f - \sigma_N f\|_\infty, \qquad E_N f = \inf \{\|f - t\|_\infty : t \in \mathbf{T}_N\}, \tag{6.5.1}$$

where $\mathbf{T}_N$ denotes the set of trigonometric polynomials of degree at most N. Plainly, the relative magnitude of $\rho_N f$ and $E_N f$ provides a sensible measure of just how good $\sigma_N f$ is as an approximant to f, when compared with other elements of $\mathbf{T}_N$.

It is evident that

$$E_{N+1} f \leqslant E_N f \leqslant \rho_N f,$$

and that $\rho_N f = 0$ if and only if f is a constant function. Moreover (see Exercise 6.5), the infimum $E_N f$ is actually an assumed minimum; as a consequence it follows that $E_N f = 0$ if and only if $f \in \mathbf{T}_N$.

According to 6.1.1, $\rho_N f = o(1)$ as $N \to \infty$ for each $f \in \mathbf{C}$; yet, by 6.1.4, the relation $\rho_N f = O(\epsilon_N)$ is false for some $f \in \mathbf{C}$ (in fact, for a nonmeager set of

$f \in \mathbf{C}$) whenever $\varepsilon_N = o(1)$ is given. It is also easily shown (see Exercise 6.10) that the relation $\rho_N f = o(1/N)$ holds if and only if f is a constant. This simple result is noteworthy when it is compared with Exercises 6.8 and 6.9; it then shows conclusively that, for sufficiently smooth nonconstant continuous functions f, $\sigma_N f$ is far from being the optimal approximant to f among all elements of $\mathbf{T}_N$. If f is very smooth, $s_N f$ is a decidedly better approximant to f than is $\sigma_N f$. Crudely speaking, the great advantage of $\sigma_N f$ is to be seen for general continuous functions f.

After these preliminary remarks we now proceed to establish some improved estimates for $\rho_N f$ in case f satisfies certain *Lipschitz* (or *Hölder*) conditions. This latter type of condition is for our immediate purposes best expressed in terms of a modified modulus of continuity, namely,

$$\Omega_\infty f(\delta) \equiv \sup_{|a| \leqslant \delta} \|T_a f - f\|_\infty;$$

compare with (8.5.1) and the definition of $\omega_1 f$ in 2.3.7. Applying (6.3.1) and choosing $s = f(x)$, we see that

$$|f_s^*(y)| \leqslant \Omega_\infty f(|y|),$$

so that (6.3.2) yields

$$\rho_N f \leqslant \frac{1}{\pi} \int_0^\pi F_N(y) \Omega_\infty f(y)\, dy. \tag{6.5.2}$$

Since (6.4.1) shows that

$$0 \leqslant F_N(y) \leqslant AN, \qquad F_N(y) \leqslant \frac{A}{Ny^2}$$

for a suitable absolute constant A, (6.5.2) entails that

$$\rho_N f \leqslant \frac{AN}{\pi} \int_0^{1/N} \Omega_\infty f(y)\, dy + \frac{A}{\pi N} \int_{1/N}^\pi \frac{\Omega_\infty f(y)\, dy}{y^2}. \tag{6.5.3}$$

Further progress will be facilitated by appeal to the following simple result, the proof of which will be left to the reader.

6.5.2. Suppose that α and β are nonnegative functions defined on some interval $(0, c)$, where $c > 0$, and integrable over (c', c) for each c' satisfying $0 < c' < c$. Suppose further that $\alpha(y) = o[\beta(y)]$ as $y \to +0$, and that

$$\int_t^c \beta(y)\, dy \to \infty \qquad \text{as } t \to +0.$$

Then also

$$\int_t^c \alpha(y)\, dy = o\Big[\int_t^c \beta(y)\, dy\Big] \qquad \text{as } t \to +0.$$

Here now is the main result of this section.

6.5.3. Suppose that the function ω is defined, nonnegative, and increasing on some interval $(0, c)$, where $c > 0$, while $y^{-a}\omega(y)$ is decreasing on the same interval for some choice of a satisfying $0 < a < 1$. Suppose further that f is continuous and satisfies

$$\Omega_\infty f(y) = O[\omega(y)] \text{ (respectively, } o[\omega(y)]) \quad \text{as } y \to +0. \quad (6.5.4)$$

Then

$$\rho_N f = O\left[\omega\left(\frac{1}{N}\right)\right] \left(\text{respectively, } o\left[\omega\left(\frac{1}{N}\right)\right]\right) \quad \text{as } N \to \infty. \quad (6.5.5)$$

Proof. If $c < \pi$, and if we extend ω by setting $\omega(y) = \omega(c)$ for $c < y \leqslant \pi$, this extended function satisfies the required conditions on the interval $(0, \pi]$. Thus, we may as well assume from the outset that $c = \pi$.

Now, without appeal to 6.5.2. it follows from (6.5.4) that

$$\begin{aligned} N \cdot \int_0^{1/N} \Omega_\infty f(y)\,dy &= O \text{ (respectively, } o) \left(N \cdot \int_0^{1/N} \omega(y)\,dy\right) \\ &= O \text{ (respectively, } o) \left(N \cdot N^{-1} \cdot \omega\left(\frac{1}{N}\right)\right) \\ &= O \text{ (respectively, } o) \left(\omega\left(\frac{1}{N}\right)\right), \end{aligned} \quad (6.5.6)$$

the second step being justified since ω is increasing.

To handle the second term on the right of (6.5.3), we apply 6.5.2, taking $\alpha(y) = \Omega_\infty f(y)/y^2$ and $\beta(y) = \omega(y)/y^2$. Unless ω is identically vanishing (in which case f is constant and nothing remains to be proved), the decreasing character of $\omega(y)/y^a$ for some a satisfying $0 < a < 1$ ensures that β satisfies the hypotheses of 6.5.2. We thus obtain

$$\begin{aligned} N^{-1} \cdot \int_{1/N}^{\pi} \frac{\Omega_\infty f(y)\,dy}{y^2} &= N^{-1} \cdot O \text{ (respectively, } o) \left(\int_{1/N}^{\pi} \frac{\omega(y)\,dy}{y^2}\right) \\ &= O \text{ (respectively, } o) \left[N^{-1} \cdot \int_{1/N}^{\pi} y^{-a}\omega(y) \cdot y^{a-2}\,dy\right] \\ &= O \text{ (respectively, } o) \left[N^{-1} \cdot (N^{-1})^{-a}\, \omega\left(\frac{1}{N}\right) \right. \\ &\qquad \left. \cdot \int_{1/N}^{\pi} y^{a-2}\,dy\right]; \end{aligned}$$

the last step is justified since $y^{-a}\omega(y)$ is decreasing. Evaluating the remaining integral and simplifying, one obtains

$$N^{-1} \cdot \int_{1/N}^{\pi} \frac{\Omega_\infty f(y)\,dy}{y^2} = O \text{ (respectively, } o) \left[\omega\left(\frac{1}{N}\right)\right]. \quad (6.5.7)$$

It now remains but to combine (6.5.3), (6.5.6), and (6.5.7) in order to derive (6.5.5).

6.5.4. **Remarks** (1). The majorization given in 6.5.3 is, at least for certain natural choices of ω, the best possible (see [Ba$_1$], p. 206). Somewhat similar results are known for functions f which satisfy a mean Lipschitz condition of the type $\|T_af - f\|_1 = O(|a|^\alpha)$ as $|a| \to 0$; see [Z₁], p. 117. See also MR **53** # 6203.

(2) Inasmuch as $E_Nf \leqslant \rho_Nf$, Subsection 6.5.3 and its analogues yield majorants for E_Nf for restricted functions f. Superior results are obtainable by estimating, not

$$\rho_Nf = \|f - \sigma_Nf\|_\infty,$$

but rather $\|f - \tau_Nf\|_\infty$, where

$$\tau_Nf = 2\sigma_{2N-1}f - \sigma_{N-1}f \in \mathbf{T}_{2N-1};$$

some calculations of this nature are proposed in Exercises 6.6 to 6.9. More elaborate results appear in Timan [1]. One might also use to the same end the so-called *Jackson polynomials* $J_N * f$, where

$$J_N(x) = c_N\left(\frac{\sin \frac{1}{2}N'x}{\sin \frac{1}{2}x}\right)^4,$$

where $N' = [\frac{1}{2}N] + 1$ and where the number c_N is chosen to make $\|J_N\|_1 = 1$; see [L_2], pp. 55–56. See also W. R. Bloom [3], [4].

6.5.5. **Converse Results.** It should also be mentioned that many results in the reverse direction are known: given the possibility of approximating f with a given degree of accuracy by trigonometric polynomials of degree at most N, one can infer smoothness properties of f. The earliest such results appear to be the work of Bernstein (1912) and of de la Vallée Poussin (1919); since then the subject has been studied vigorously (see [L_2], Chapter 4, and [BK], pp. 45–59 and the references cited there). A very special instance of this type of result appears in Exercise 6.10. A crucial role in these investigations is Bernstein's inequality (see Exercise 1.9), the basic reason being that this inequality combines with the first mean value theorem to yield an estimate for the modulus of continuity, $\|T_at - t\|_\infty$, of a trigonometric polynomial t in terms of $\|t\|_\infty$ and the degree N of t.

A sample result asserts that if $f \in \mathbf{C}$ is such that $E_Nf = O(N^{-\alpha})$, where $0 < \alpha \leqslant 1$, then

$$\|T_af - f\|_\infty = \begin{cases} O(|a|^\alpha) & \text{if } \alpha < 1, \\ O(|a| \log |a|^{-1}) & \text{if } \alpha = 1; \end{cases}$$

this is very close to being a converse to a special case of 6.5.3. Other results infer the existence of several continuous derivatives of f, together with estimates of their iterated differences; see [L_2], *loc. cit.*, and [BK], pp. 45–57, 72–88. See also MR **54** ## 832, 13433; **55** # 960; and Zamansky [1], [2].

6.6 General Comments on Summability of Fourier Series

Cesàro summability has so far received all our attention, but we should mention in passing the *Abel* (or *Abel-Poisson*) method, which is an equal favorite. See also Exercise 6.14 for yet another method of great importance in the theory of trigonometric series.

The *Abel means* of the Fourier series of f are the functions

$$\begin{aligned} A_r f(x) &= \sum_{n \in Z} r^{|n|} \hat{f}(n) e^{inx} \\ &= P_r * f(x), \end{aligned} \tag{6.6.1}$$

where the continuous parameter r satisfies $0 \leqslant r < 1$ and where P_r is the so-called *Poisson kernel*

$$P_r(x) = \frac{1 - r^2}{1 - 2r \cos x + r^2}$$

encountered in the Poisson representation formula for harmonic functions. *Abel summability* of the Fourier series of f refers to the limiting behavior of the means $A_r f$ as $r \to 1$ (from below).

All the results proved in this chapter about Cesàro summability of Fourier series remain true in respect of their Abel summability. (The reader is urged to verify this statement as an extended exercise.) In a few regards (which will nowhere concern us), the Abel method is slightly to be preferred, partly because of its evidently closer connections with complex variable theory; for details, see [Z_1], Chapters III and VII; [Z_2], Chapter XIV. See also Exercise 6.16.

Even for quite general (locally compact Abelian or compact) groups, there is indeed an unlimited number of summability methods, each expressible in terms of a limiting process $\lim K_t * f$, and each just as effective as the Cesàro or Abel methods in respect of norm-convergence in $\mathbf{C}$ or in $\mathbf{L}^p (1 \leqslant p < \infty)$. In each case, (K_t) is a sequence or net of kernels, usually forming an approximate identity in $\mathbf{L}^1$ and composed of very well-behaved functions. The fun begins when one wishes to examine the associated problem of pointwise almost everywhere summability for discontinuous functions, concerning which surprisingly little is yet known (except for the circle group T, R, and their finite products). Some progress is reported by Stein [1] for the case of compact groups and spaces $\mathbf{L}^p$ with $1 \leqslant p \leqslant 2$; somewhat weaker results with a wider range of applicability are discussed by Edwards and Hewitt [1]. These results apply in fact to sequences of $\mathfrak{m}$-operators of type $(\mathbf{L}^p, \mathbf{L}^p)$ $(1 \leqslant p < \infty)$; see 16.2.7 and 16.2.8.

6.7 Remarks on the Dual Aspects

The results in this chapter and in Chapter 10 to follow can both be regarded as investigations bearing upon how one may interpret, in a pointwise sense, the "inversion formula"

$$f = (\hat{f})^\wedge, \tag{6.7.1}$$

where f is a given integrable function on $G = T$. In this chapter we have concentrated on the case in which the second Fourier transformation in

(6.7.1) is interpreted via pointwise summability methods; in Chapter 10 the interpretation is via pointwise convergence (see the discussion at the end of 2.2.2).

In Chapters 8, 12, and 13 it will appear that the same inversion formula can be interpreted by using mean and distributional convergence.

There remains still the dual inversion formula, to wit,

$$\phi = (\hat{\phi})^{\wedge}, \tag{6.7.2}$$

where now ϕ is a given function on Z. Here again, the difficulties concentrate around the definition of $\hat{\phi}$; see Section 2.5. As usual, the adoption of pointwise convergence or summability as the means of defining $\hat{\phi}$ raises problems (except in the transparent case in which $\phi \in \ell^1$). These thorny features largely evaporate if mean or distributional convergence is used, as will be done in 8.3.3, 12.5.4, and 13.5.1(2).

EXERCISES

6.1. Suppose that $f \in \mathbf{L}^p$ $(1 \leqslant p \leqslant \infty)$ and that $\sum c_n e^{inx}$ is a trigonometric series with partial sums s_N whose Cesàro means are σ_N. Prove that if

$$\lim_{N \to \infty} \|f - s_N\|_p = 0,$$

then

$$\lim_{N \to \infty} \|f - \sigma_N\|_p = 0.$$

This is an analogue, for mean convergence, of 5.3.1.

6.2. Suppose that $f \in \mathbf{L}^1$ and $g \in \mathbf{L}^\infty$. Show that

$$\frac{1}{2\pi} \int fg \, dx = \lim_{N \to \infty} \sum_{|n| \leqslant N} \left(1 - \frac{|n|}{N + 1}\right) \hat{f}(n) \hat{g}(-n).$$

6.3. Write, for $f \in \mathbf{L}^1$,

$$m(\delta) = \operatorname*{ess.\,sup}_{|x| \leqslant \delta} |f(x)|, \qquad m = \lim_{\delta \downarrow 0} m(\delta).$$

Prove that

$$\limsup_{N \to \infty} |\sigma_N f(0)| \leqslant m.$$

6.4. Is

$$\sum_{n \in Z} \frac{(-1)^n e^{inx}}{(1 + |n|) \log (2 + n^2)}$$

the Fourier series of a bounded (measurable) function? Give reasons for your answer.

Note: The next five exercises form a sequence concerned with approximation by trigonometric polynomials; they should be attempted in order of enumeration.

6.5. For $f \in \mathbf{C}$ write, as in (6.5.1),

$$E_N f = \inf \|f - t\|_\infty,$$

the infimum being taken with respect to all trigonometric polynomials t of order at most N. Prove that this infimum is assumed.

6.6. Assuming the formula

$$\sigma_{N-1} f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x + y) \cdot 2 \sin^2 \tfrac{1}{2} N y \cdot \frac{dy}{N y^2}$$

for $f \in \mathbf{C}$ (see [Z_1], p. 92), verify that for $N \geqslant 1$

$$\tau_N f \equiv 2\sigma_{2N-1} f - \sigma_{N-1} f$$

is given by

$$\tau_N f(x) = \frac{2}{\pi} \int_0^\infty \left[f\left(x + \frac{y}{N}\right) + f\left(x - \frac{y}{N}\right) \right] \frac{h(y)\, dy}{y^2}.$$

where

$$h(y) = \sin^2 y - \sin^2 \tfrac{1}{2} y = \tfrac{1}{2}(\cos y - \cos 2y).$$

6.7. The notation being as in Exercise 6.6, define

$$H_0(y) = h(y) y^{-2}, \qquad H_i(y) = \int_y^\infty H_{i-1}(t)\, dt \qquad (i = 1, 2, \cdots; y > 0).$$

Verify that

(1) $\displaystyle\int_0^\infty |H_i(y)|\, dy < \infty \qquad (i = 0, 1, 2, \cdots),$

(2) $H_i(0) = 0 \qquad$ if $i > 1$ is odd.

Hints: For (1), check that $H_i(y) = O(y^{-2})$ as $y \to \infty$. For (2), apply the results of Exercise 6.6 taking in turn $f(x) = 1$ and $f(x) = \cos x$. Taking $x = 0$ yields

$$\frac{4}{\pi} \int_0^\infty H_0(y)\, dy = 1, \qquad \frac{4}{\pi} \int_0^\infty \cos\left(\frac{y}{N}\right) \cdot H_0(y)\, dy = 1.$$

Integrate the second relation by parts and use the first to obtain

$$\int_0^\infty \cos\left(\frac{y}{N}\right) \cdot H_2(y)\, dy = 0.$$

Now let $N \to \infty$ to conclude that $H_3(0) = 0$. Similarly,

$$H_5(0) = \int_0^\infty H_4(y)\, dy = 0,$$

and so forth.

6.8. Suppose that $f \in \mathbf{C}^k$ (k an integer $\geqslant 0$) and that N is a positive integer. Show that

$$\|\tau_N f - f\|_\infty \leqslant A_k \cdot \|D^k f\|_\infty \cdot N^{-k},$$

and deduce that

$$E_N f \leqslant A'_k \cdot \|D^k f\|_\infty \cdot N^{-k},$$

where A_k and A'_k depend only on k.

Hints: From Exercise 6.6,

$$\tau_N f(x) - f(x) = \frac{2}{\pi}\int_0^\infty \left[f\left(x + \frac{y}{N}\right) + f\left(x - \frac{y}{N}\right) - 2f(x)\right]\frac{h(y)\,dy}{y^2}.$$

Apply partial integration k times and use the results of Exercise 6.7.

6.9. Suppose that $f \in \mathbf{C}^k$ (k an integer $\geqslant 0$) and show that

$$\|\tau_N f - f\|_\infty \leqslant B_k \cdot \Omega_\infty(D^k f)\left(\frac{2\pi}{N}\right) \cdot N^{-k}.$$

Deduce that

$$E_N f \leqslant B'_k \cdot \Omega_\infty(D^k f)\left(\frac{2\pi}{N}\right) \cdot N^{-k}.$$

Here B_k and B'_k depend only upon k, and

$$\Omega_p g(\delta) = \sup_{|a| \leqslant \delta} \|T_a g - g\|_p \qquad (1 \leqslant p \leqslant \infty).$$

Hints: Consider $f_\delta(x) = (2\delta)^{-1}\int_{-\delta}^{\delta} f(x + y)\,dy$ where $\delta > 0$. Show that $f_\delta \in \mathbf{C}^{k+1}$,

$$\|D^{k+1} f_\delta\|_\infty \leqslant (2\delta)^{-1}\,\Omega_\infty(D^k f)(2\delta) \leqslant \delta^{-1}\,\Omega_\infty(D^k f)(\delta),$$

and

$$\|D^k(f - f_\delta)\|_\infty \leqslant \Omega_\infty(D^k f)(\delta).$$

Since $\tau_N f = \tau_N f_\delta + \tau_N(f - f_\delta)$, the desired results follow on using Exercise 6.8 and taking $\delta = 2\pi/N$.

6.10. Suppose that $f \in \mathbf{L}^1$ and $\|f - \sigma_N f\|_1 = o(N^{-1})$ as $N \to \infty$. Show that f is equal almost everywhere to a constant.

Show also that if $f \in \mathbf{L}^1$ and

$$E_N^{(1)} f = \inf\{\|f - t\|_1 : t \in \mathbf{T}_N\},$$

then

$$|\hat{f}(n)| \leqslant E^{(1)}_{|n|-1} f \qquad (n = \pm 1, \pm 2, \cdots).$$

Remark. The first part of this exercise asserts that the sequence of operators $f \to \sigma_N f$ ($N = 1, 2, \cdots$) on $\mathbf{C}$ is "saturated by the function $\psi(N) = N^{-1}$"; for this concept, see [L₂], pp. 98–102. See also MR **36** # 5605.

6.11. Let $\sum_{n \in Z} c_n e^{inx}$ be a trigonometric series and

$$\sigma_N = \sum_{|n| \leqslant N} (1 - |n|/N + 1) c_n e^{inx}$$

its Nth Cesàro mean. Show that the given series is a Fourier-Lebesgue series if and only if

$$\lim_{N, N' \to \infty} \|\sigma_N - \sigma_{N'}\|_1 = 0.$$

(Compare the results in 12.7.5 and 12.7.6.)

6.12. (Fatou's theorem) Show that if $f \in \mathbf{L}^1$ and

$$N^{-1} \sum_{|n| \leqslant N} |n\hat{f}(n)| = o(1) \qquad \text{as } N \to \infty,$$

then $\lim_{N\to\infty} s_N f(x) = f(x)$ for almost all x; and that if, furthermore, f is continuous, then $\lim_{N\to\infty} s_N f = f$ uniformly.

Hint: Use Exercise 5.5 and the results appearing in 6.1.1 and 6.4.4.

Note: One can show, by using the same method in combination with 8.3.1, that if $c_n = o(1/|n|)$ as $|n| \to \infty$, then the trigonometric series $\sum_{n\in Z} c_n e^{inx}$ is convergent for almost all x.

6.13. Suppose that $f \in \mathbf{L}^1$ and that $\hat{f}(n) = 0$ save perhaps for $n = 0$, $\pm n_1, \pm n_2, \cdots$, where $0 < n_1 < n_2 < \cdots$ and $\inf n_{k+1}/n_k > 1$. Prove that $\lim_{N\to\infty} s_N f(x) = f(x)$ for almost all x; and that if, in addition, f is continuous, then $\lim_{N\to\infty} s_N f = f$ uniformly.

Hint: Use Exercise 5.6, and Subsections 6.1.1 and 6.4.4.

Note: These so-called *lacunary series* will receive further attention in Chapter 15; see especially the remarks in Section 15.6. The result, like that in Exercise 5.6, admits some generalization.

6.14. Let the function r_N ($N = 1, 2, \cdots$) be defined to be equal to πN in $[-1/N, 1/N]$, to 0 in $[-\pi, -1/N)$ and in $(1/N, \pi]$, and be defined elsewhere so as to be periodic. Put $R_N = r_N * r_N$. Verify that $(R_N)_{N=1}^{\infty}$ is an approximate identity. Compute $\hat{R}_N$ and deduce that

$$\lim_{N\to\infty} \sum_{n\in Z} \left[\frac{\sin (N^{-1}n)}{N^{-1}n}\right]^2 \hat{f}(n)e^{inx} = f(x)$$

uniformly for each continuous f, provided sin 0/0 is interpreted as 1.

Note: This is a case of *Riemann's method of summability*, which is of fundamental importance in the general theory of trigonometric series; see [Z_1], Chapter IX and [Ba_1], p. 192.

6.15. Show that a necessary and sufficient condition that a continuous function f be of the form $f = F \circ e_1$, where $e_1(x) = e^{ix}$ and F is defined and continuous on the closed unit disk in the complex plane and holomorphic in the interior of this disk, is that $\hat{f}(n) = 0$ for $n \in Z$ and $n < 0$ (compare Exercise 3.9).

6.16. Suppose that $f \in \mathbf{L}^1$ and define

$$F(x) = \int_{-\pi}^{x} f(y)\, dy.$$

Prove that, if x is a point for which the *symmetric derivative*

$$D_s F(x) \equiv \lim_{\varepsilon \downarrow 0} \frac{F(x + \varepsilon) - F(x - \varepsilon)}{2\varepsilon}$$

exists finitely, then

$$\lim_{r \to 1-0} A_r f(x) = D_s F(x),$$

$A_r f$ being defined as in Section 6.6.

Remark. The relation $D_s F(x) = f(x)$ holds at all points x of the Lebesgue set of f (see 6.4.2), but it may well hold at other points as well. Compare the result with 6.4.3.

Hints: Assume, without loss of generality, that $F(\pi) = 0$. Integrate by parts in the formula (6.6.1) to obtain

$$A_r f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (2 \sin y)^{-1} \{F(x + y) - F(x - y)\} Q_r(y)\, dy,$$

where $Q_r(y) = -\sin y \cdot P_r'(y)$. Verify that $Q_r \geqslant 0$, $\|Q_r\|_1 = r$, and that

$$\lim_{r \to 1-0} \sup_{\delta \leqslant |y| \leqslant \pi} Q_r(y) = 0$$

for any $\delta \in (0, \pi]$. Mimic the proof of 6.4.3.

6.17. Sometimes (see 6.4.7, 12.9.9, 13.8.3, and 13.10, Exercises 13.21 to 13.23; [Z_1], pp. 170–175; [Z_2], pp. 116–119, 158) one wishes to consider the set $\mathbf{L}^\Phi$ of measurable functions f such that $\Phi(|f|) \in \mathbf{L}^1$, Φ being a fixed nonnegative function of a suitable type, and one wants to know that the set $\mathbf{T}$ of trigonometric polynomials is contained and everywhere dense in $\mathbf{L}^\Phi$ in the sense that, given any $f \in \mathbf{L}^\Phi$ and $\varepsilon > 0$, there exists $t \in \mathbf{T}$ such that

$$N(|f - t|) \equiv \frac{1}{2\pi} \int \Phi(|f - t|)\, dx < \varepsilon. \tag{*}$$

Suppose that Φ is a nonnegative real-valued function defined on $[0, \infty)$ having the following properties:

(1) Φ is increasing and $\Phi(s) \to 0$ as $s \to +0$;

(2) $\Phi(s) \geqslant As$ for large $s > 0$;

(3) $\Phi(s + s') \leqslant B\{s + s' + \Phi(s) + \Phi(s')\}$ for $s, s' \geqslant 0$.

Here A and B denote positive numbers. Prove that $\mathbf{L}^\Phi$ is a linear subspace of $\mathbf{L}^1$, that $\mathbf{L}^\Phi \supset \mathbf{L}^\infty$, and that the approximation (*) above is always possible.

6.18. Let $(\gamma_N)_{N=1}^\infty$ be any sequence of positive numbers such that $\gamma_N = o(N)$ as $N \to \infty$. Use the uniform boundedness principle (Appendix B.2.1) in order to prove the existence of a nonnegative function $f \in \mathbf{L}^1$ such that

$$\limsup_{N \to \infty} \frac{\sigma_N f(0)}{\gamma_N} = \infty. \tag{1}$$

Is there any sense in which this result is the best possible?

Can you construct explicitly nonnegative functions $f \in \mathbf{L}^1$ that satisfy (1), for certain specific choices of $(\gamma_N) \cdots$ for example, when $\gamma_N = N/\{\log(N + 2)\}^\varepsilon$ and $\varepsilon > 0$?

CHAPTER 7

Some Special Series and Their Applications

In this chapter we assemble a few results about two special types of series, namely,

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx = \sum_{n\in Z} c_n e^{inx}, \tag{C}$$

where $c_n = \tfrac{1}{2}a_{|n|}$; and

$$\sum_{n=1}^{\infty} a_n \sin nx = \sum_{n\in Z} c_n e^{inx}, \tag{S}$$

where $c_n = (1/2i)\operatorname{sgn} n \cdot a_{|n|}$. We shall assume throughout that the a_n are real-valued, and write s_N and σ_N for the Nth partial sum and the Nth Cesàro mean, respectively, of whichever series happens to be under discussion.

The series (C) and (S) are examples of so-called conjugate series, a topic to which we return in Section 12.8.

A primary concern will be the determination of conditions under which these series are Fourier-Lebesgue series. While the results are rather special, inasmuch as they assume heavy restrictions on the sequence (a_n), they frequently play an important role in handling questions of general significance (as, for example, in Section 7.5).

For an extended study of more special series, see [Z_1], Chapter V.

The results we shall obtain can be easily recast into statements about the definition and nature of $\hat{\phi}$ (see Section 2.5) for rather special functions ϕ on Z.

7.1 Some Preliminaries

7.1.1. Modified Kernels. It will be helpful to list a few formulae, some of which have been encountered in Section 5.1.

$$\begin{aligned}\tfrac{1}{2} + \sum_{n=1}^{N} \cos nx &= \tfrac{1}{2}\sum_{|n|\leqslant N} e^{inx} = \tfrac{1}{2}D_N(x)\\ &= \frac{\sin(N+\tfrac{1}{2})x}{2\sin\tfrac{1}{2}x},\end{aligned} \tag{7.1.1}$$

$$F_N(x) = \frac{D_0(x) + \cdots + D_N(x)}{N+1}$$
$$= (N+1)^{-1}\left[\frac{\sin \tfrac{1}{2}(N+1)x}{\sin \tfrac{1}{2}x}\right]^2. \tag{7.1.2}$$

Running parallel to (7.1.1) is a formula serving to introduce the so-called *Dirichlet conjugate kernel*:

$$\tfrac{1}{2}\tilde{D}_N(x) = \sum_{n=1}^{N} \sin nx = \frac{\cos \tfrac{1}{2}x - \cos (N + \tfrac{1}{2})x}{2 \sin \tfrac{1}{2}x}; \tag{7.1.3}$$

the symbol $\tilde{}$ marks the passage to the so-called conjugate series, a topic dealt with in Section 12.8 in some generality.

The modified kernels defined by

$$\tfrac{1}{2}\, D_N^{\#}(x) = \tfrac{1}{2}\, D_N(x) - \tfrac{1}{2}\cos Nx = \frac{\sin Nx}{2 \tan \tfrac{1}{2}x} \tag{7.1.4}$$

and

$$\tfrac{1}{2}\, \tilde{D}_N^{\#}(x) = \tfrac{1}{2}\, \tilde{D}_N(x) - \tfrac{1}{2}\sin Nx = \frac{1 - \cos Nx}{2 \tan \tfrac{1}{2}x} \tag{7.1.5}$$

play useful, if transient, roles. Notice that $D_N^{\#}$ is even, and that $\tilde{D}_N^{\#}$ is odd. Moreover,

$$\tilde{D}_N^{\#}(x) \geqslant 0 \qquad (0 < x < \pi), \tag{7.1.6}$$

and

$$|\tilde{D}_N(x)| \leqslant \frac{2\pi}{x} \qquad (0 < x < \pi). \tag{7.1.7}$$

From 5.1.1 and the defining formulae above we infer that

$$\|D_N\|_1 \sim \frac{4}{\pi^2}\log N + O(1), \tag{7.1.8}$$

$$\|D_N^{\#}\|_1 \sim \frac{4}{\pi^2}\log N + O(1), \tag{7.1.9}$$

as $N \to \infty$. It may be shown similarly that

$$\|\tilde{D}_N\|_1 \sim \frac{2}{\pi}\log N + O(1), \tag{7.1.10}$$

$$\|\tilde{D}_N^{\#}\|_1 \sim \frac{2}{\pi}\log N + O(1) \tag{7.1.11}$$

as $N \to \infty$; see Exercise 12.20 and [Z_1], pp. 49, 67.

7.1.2. **Convex Sequences.** In the remainder of this section we deal with some matters concerning sequences (a_n) to play the role of coefficients in the series (C) and (S). We shall assume that the sequence is indexed from $n = 0$

[as is the case with (C)]; only minor changes are needed to take care of sequences indexed from $n = 1$ [as is the case with (S)].

For any real-valued sequence $(a_n)_{n=0}^{\infty}$ we define the sequence of differences $\Delta a_n = a_n - a_{n+1}$. In particular, then, (a_n) is decreasing (in the wide sense), if and only if $\Delta a_n \geqslant 0$. The sequence of second differences is defined by

$$\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}.$$

The sequence (a_n) is said to be *convex*, if and only if $\Delta^2 a_n \geqslant 0 \geqslant 0$; *quasiconvex*, if and only if

$$\sum_{n=0}^{\infty} (n + 1)|\Delta^2 a_n| < \infty;$$

of bounded variation (BV) if and only if

$$\sum_{n=0}^{\infty} |\Delta a_n| < \infty.$$

It is worth explaining at once that the intervention of the difference sequences is explained by frequent use of the technique of partial summation, which has already been used without special comment in earlier chapters; see also [H], pp. 97 ff. Given two sequences (a_n) and (b_n), the formula for partial summation reads

$$\sum_{p \leqslant n \leqslant q} a_n b_n = a_q B_q - a_p B_{p-1} + \sum_{p \leqslant n \leqslant q-1} \Delta a_n \cdot B_n; \tag{7.1.12}$$

in this formula it is understood that $p \leqslant q$, and that

$$B_n = \sum_{r \leqslant k \leqslant n} b_k,$$

where r is any fixed integer satisfying $r \leqslant p$ and such that a_n and b_n are defined for $n \geqslant r$; an empty sum (that is, a sum extending over a range that is empty) is always understood to have the value zero. Repetition of the technique introduces the second differences $\Delta^2 a_n$.

The following simple result about convex sequences helps to illuminate 7.3.1 to follow.

7.1.3. (1) If $(a_n)_{n=0}^{\infty}$ is convex and bounded, then it is decreasing,

$$\lim_{n \to \infty} n \cdot \Delta a_n = 0, \tag{7.1.13}$$

and

$$\sum_{n=0}^{\infty} (n + 1) \cdot \Delta^2 a_n = a_0 - \lim_{n \to \infty} a_n. \tag{7.1.14}$$

(2) If (a_n) is quasiconvex and bounded, then (a_n) is BV and $(n \cdot \Delta a_n)$ is bounded.

(3) If (a_n) is quasiconvex and convergent (to a finite limit), then (a_n) is BV and (7.1.13) is true.

Proof. (1) Convexity signifies that (Δa_n) is decreasing. If $\Delta a_m = c < 0$, then $\Delta a_n \leqslant c$ for $n \geqslant m$, in which case $a_n \to \infty$ as $n \to \infty$, contrary to the assumed boundedness of (a_n). So $\Delta a_n \geqslant 0$, (a_n) is decreasing, and therefore boundedness implies that $a = \lim a_n$ exists finitely. Now

$$a_0 - a_N = \Delta a_0 + \cdots + \Delta a_{N-1}, \tag{7.1.15}$$

which we now know to be a series of nonnegative terms converging to $a_0 - a$. Since also $\Delta(\Delta a_n) \geqslant 0$, it follows easily that $n \cdot \Delta a_n \to 0$, which is (7.1.13). Finally, (7.1.14) follows on applying partial summation to (7.1.15) and using (7.1.13).

(2) We have

$$a_m - a_{m+n+1} = \Delta^2 a_m + 2\Delta^2 a_{m+1} + \cdots + n\Delta^2 a_{m+n+1} + (n+1)\Delta a_{m+n}. \tag{7.1.16}$$

[This may be proved by induction on n, thus: It is trivially true for $n = 0$. If it is true for n, then

$$\begin{aligned}
a_m - a_{m+(n+1)+1} &= a_m - a_{m+n+1} + a_{m+n+1} - a_{m+n+2} \\
&= (a_m - a_{m+n+1}) + \Delta a_{m+n+1} \\
&= \Delta^2 a_m + 2\Delta^2 a_{m+1} + \cdots + n\Delta^2 a_{m+n-1} \\
&\quad + (n+1)\Delta a_{m+n} + \Delta a_{m+n+1} \\
&= \Delta^2 a_m + 2\Delta^2 a_{m+1} + \cdots + n\Delta^2 a_{m+n-1} \\
&\quad + (n+1)\{\Delta a_{m+n+1} + \Delta^2 a_{m+n}\} + \Delta a_{m+n+1} \\
&= (\Delta^2 a_m + 2\Delta^2 a_{m+1} + \cdots + (n+1)\Delta^2 a_{m+n}) \\
&\quad + (n+2)\Delta a_{m+n+1};
\end{aligned}$$

that is, the equality holds with n replaced throughout by $n + 1$. Induction does the rest.]

Taking $m = 0$ in (7.1.16), we infer that

$$|(n+1)\Delta a_n| \leqslant |a_0 - a_{n+1}| + |\Delta^2 a_0| + 2|\Delta^2 a_1| + \cdots + n|\Delta^2 a_{n-1}|, \tag{7.1.17}$$

which is bounded whenever (a_n) is bounded and quasiconvex.

Next, interchanging (a_n) and (b_n) in (7.1.12) and then taking $a_n = \operatorname{sgn} b_n$, it appears that

$$\sum_{n=0}^{q} |b_n| \leqslant (q+1)|b_q| + \sum_{n=0}^{q} n|\Delta b_n|.$$

In this, take $b_n = \Delta a_n$, to conclude that

$$\sum_{n=0}^{q} |\Delta a_n| \leqslant (q+1)|\Delta a_q| + \sum_{n=0}^{q} n|\Delta^2 a_n|,$$

which is bounded with respect to q [by (7.1.17)] whenever (a_n) is quasiconvex and bounded.

This completes the proof of (2).

(3) In view of (2), it remains only to show that (7.1.13) is true. But this follows in taking in turn $m = n$ and $m = n + 1$ in (7.1.16), leading to

$$\begin{aligned} |(n+1)\Delta a_{2n}| &\leqslant |a_n - a_{2n+1}| + |\Delta^2 a_n| + 2|\Delta^2 a_{n+1}| + \cdots + n|\Delta^2 a_{2n-1}| \\ &\leqslant |a_n - a_{2n+1}| + \sum_{k=1}^{2n-1} |(k+1)\Delta^2 a_k| \end{aligned}$$

and similarly

$$|(n+2)\Delta a_{2n+1}| \leqslant |a_{n+1} - a_{2n+2}| + \sum_{k=n+1}^{2n} (k+1)|\Delta^2 a_k|.$$

In Section 7.4 we shall need to know that there exist sequences (a_n) which are positive and convex, and which tend to zero arbitrarily slowly. How such sequences may be constructed will appear from the next two results.

7.1.4. Let $a > 1$ and suppose that $(N_k)_{k=1}^{\infty}$ is a strictly increasing sequence of positive integers such that

$$\begin{gathered} N_2 \geqslant [1 + \tfrac{1}{2}(a-1)^{-1}]N_1, \\ (k+1)N_{k+1} \geqslant 2kN_k - (k-1)N_{k-1} \qquad (k = 2, 3, \cdots). \end{gathered} \tag{7.1.18}$$

If (a_n) is the sequence defined so that $a_0 = a$, $a_n = 1/k$ for $n = N_k$ ($k = 1, 2, \cdots$), and so as to be linear for values of n satisfying $0 < n < N_1$ or $N_k < n < N_{k+1}$ ($k = 1, 2, \cdots$), then (a_n) is positive, decreases to zero, and is convex.

Proof. It is evident that (a_n) is positive and decreases to zero. Conditions (7.1.18) express the convexity of (a_n), which amounts to saying that the negatives of the slopes of the line segments, obtained by joining neighbors in the sequence of points of the plane having coordinates

$$(0, a), (N_1, 1), \cdots, \left(N_k, \frac{1}{k}\right), \cdots,$$

form a decreasing sequence.

7.1.5. Let $(c_n)_{n=0}^{\infty}$ be any complex sequence such that

$$\lim_{n\to\infty} c_n = 0.$$

There exists a sequence $(a_n)_{n=0}^{\infty}$ which is positive, convex, decreasing to zero, and such that

$$|c_n| \leqslant a_n \qquad (n = 0, 1, 2, \cdots). \tag{7.1.19}$$

Proof. First choose $N_1 > 0$ so that $|c_n| \leqslant \frac{1}{2}$ for $n \geqslant N_1$, and then $a > 1$ so large that the line segment in the plane joining the points $(0, a)$ and $(N_1, 1)$ lies above all the points

$$(0, |c_0|), \cdots, (N_1 - 1, |c_{N_1-1}|).$$

Then, as is easily seen, the integers $N_2, N_3, \cdots$ may be chosen inductively so as to increase strictly, to satisfy (7.1.18), and to be such that

$$|c_n| \leqslant \frac{1}{k+1} \qquad \text{for } n \geqslant N_k.$$

The sequence (a_n) constructed in 7.1.4 will then satisfy all requirements.

7.2 Pointwise Convergence of the Series (C) and (S)

7.2.1. Suppose that $a_n = O(1)$ and $\sum_{n=0}^{\infty} |\Delta a_n| < \infty$. Then

(1) (S) is convergent every where;

(2) (C) and (S) are uniformly convergent for $\delta \leqslant |x| \leqslant \pi$ for any $\delta > 0$.

Proof. Consider (S). By (7.1.12) and (7.1.3),

$$\sum_{p \leqslant n \leqslant q} a_n \sin nx = a_q \cdot \tfrac{1}{2}\tilde{D}_q(x) - a_p \cdot \tfrac{1}{2}\tilde{D}_{p-1}(x) + \sum_{p \leqslant n \leqslant q} \Delta a_n \cdot \tfrac{1}{2}\tilde{D}_n(x)$$

and so, if $0 < |x| \leqslant \pi$,

$$\Big|\sum_{p \leqslant n \leqslant q} a_n \sin nx\Big| \leqslant |a_q|/|\sin \tfrac{1}{2}x| + |a_p|/|\sin \tfrac{1}{2}x| + \sum_{p \leqslant n \leqslant q} |\Delta a_n|/|\sin \tfrac{1}{2}x|.$$

The statements concerning (S) follow from this and the hypotheses. Those about (C) are proved similarly, using (7.1.1) in place of (7.1.3).

Conditions for the uniform convergence of (S) are not so superficial.

7.2.2. Suppose that $a_n \downarrow 0$. Then

(1) the series (S) is uniformly convergent, if and only if $na_n = o(1)$;

(2) the series (S) is boundedly convergent if and only if $na_n = O(1)$;

(3) the series (S) is the Fourier series of a continuous function if and only if $na_n \to 0$;

(4) the series (S) is the Fourier series of a function in $\mathbf{L}^\infty$ if and only if $na_n = O(1)$.

Proof. (1) Suppose first that (S) is uniformly convergent. Putting $x = \pi/2N$ we have

$$\sum_{[\frac{1}{2}N]+1}^{N} a_n \sin nx \geqslant \sin\left(\frac{\pi}{4}\right) \cdot a_N \cdot \sum_{[\frac{1}{2}N]+1}^{N} 1,$$

whence uniform convergence is seen to imply that $na_n \to 0$.

Next suppose that, conversely, $na_n \to 0$. Write $b_k = \sup_{n \geqslant k} na_n$, so that $b_k \to 0$ as $k \to \infty$. Let $0 < x \leqslant \pi$ and define $N = N_x$ to be that integer for which

$$\frac{\pi}{N+1} < x \leqslant \frac{\pi}{N}.$$

Decompose the remainder:

$$\sum_{n \geqslant m} a_n \sin nx = \sum_{m \leqslant n < m+N} + \sum_{n \geqslant m+N} = R + R'.$$

Then

$$|R| = \Big|\sum_{m \leqslant n < m+N} a_n \sin nx\Big| \leqslant x \cdot \sum_{m \leqslant n < m+N} na_n$$
$$\leqslant xN \cdot b_m \leqslant \pi b_m. \tag{7.2.1}$$

On the other hand, using partial summation in conjunction with (7.1.7), we find that

$$|R'| = \Big|\sum_{n \geqslant m+N} \Delta a_n \cdot \tfrac{1}{2}\tilde{D}_n(x) - a_{m+N}\tfrac{1}{2}\tilde{D}_{m+N-1}(x)\Big|$$
$$\leqslant \frac{2a_{m+N}\pi}{x} \leqslant 2(N+1)a_{m+N} \leqslant 2b_m. \tag{7.2.2}$$

The combination of (7.2.1) and (7.2.2) yields

$$\Big|\sum_{n \geqslant m} a_n \sin nx\Big| \leqslant (\pi + 2)b_m,$$

and uniform convergence is seen to obtain.

(2) The proof is an obvious adaptation of that of (1), the sole difference being that now $b_k = O(1)$.

(3) If $na_n \to 0$, (1) asserts that the series (*S*) is uniformly convergent and is, therefore, the Fourier series of its continuous sum function.

Conversely suppose that (*S*) is the Fourier series of a continuous function f. Then, by 6.1.1, $\sigma_N \to f$ uniformly. This shows that $f(0) = 0$. So, by uniform convergence,

$$\sigma_N\left(\frac{\pi}{N}\right) \to 0. \tag{7.2.3}$$

Since $\sin t \geqslant 2t/\pi$ for $0 < t < \tfrac{1}{2}\pi$, we have

$$\sigma_N\left(\frac{\pi}{N}\right) \geqslant \sum_{n \leqslant \frac{1}{2}N} a_n\left(1 - \frac{n}{N+1}\right)\cdot\frac{2}{\pi}\cdot\frac{\pi n}{N}.$$

So (7.2.3) entails

$$N^{-1}\sum_{n \leqslant \frac{1}{2}N} na_n \to 0,$$

hence

$$N^{-1}a_{[\frac{1}{2}N]}\sum_{n \leqslant \frac{1}{2}N} n \to 0,$$

and so finally $Na_N \to 0$.

(4) The proof is entirely similar to that of (3), using (2) and the fact that $\|\sigma_N\|_\infty = O(1)$, if $\sum_{n=1}^{\infty} a_n \sin nx$ is the Fourier series of the bounded (measurable) function f; see (6.4.9).

Remark. In cases (3) and (4) the proofs show that any function, of which (S) is the Fourier series, is equal almost everywhere to the sum function of the series (S). This also follows from 6.4.5.

7.2.3. Define

$$\alpha_n = \sup\left\{k^{-1} \sum_{n \leqslant m < n+k} m|a_m| : \ k \in \{1, 2, \cdots\}\right\},$$
$$\beta_n = n \sum_{m \geqslant n} |\Delta a_m| .$$

Then

(1) if $\alpha_n = O(1)$ and $\beta_n = O(1)$, (S) is boundedly convergent;

(2) if $\alpha_n = o(1)$ and $\beta_n = o(1)$, (S) is uniformly convergent.

Proof. We may and will suppose that $0 < x \leqslant \pi$. Denote by $N = N_x$ the positive integer such that

$$\frac{\pi}{N+1} < x \leqslant \frac{\pi}{N}.$$

Then, if $m \in \{1, 2, \cdots\}$,

$$\sum_{n \geqslant m} a_n \sin nx = \sum_{m \leqslant n < m+N} + \sum_{m \geqslant m+N}$$
$$= R + R'.$$

Here,

$$\begin{aligned} |R| &\leqslant \sum_{m \leqslant n < m+n} |a_n| \cdot nx \\ &\leqslant x \sum_{m \leqslant n < m+N} n|a_n| = Nx \cdot N^{-1} \sum_{m \leqslant n < m+N} n|a_n| \\ &\leqslant Nx \cdot \alpha_m \leqslant \pi\alpha_m \end{aligned} \tag{7.2.4}$$

and [using partial summation and noting that necessarily $a_n = O(1)$]

$$\begin{aligned} |R'| &= \left| \sum_{n \geqslant m+n} \Delta a_n \cdot \tfrac{1}{2}\tilde{D}_n(x) - a_{m+N} \cdot \tfrac{1}{2}\tilde{D}_{m+N-1}(x) \right| \\ &\leqslant \sum_{n \geqslant m+N} |\Delta a_n| \cdot \pi x^{-1} + |a_{m+N}| \cdot \pi x^{-1} \\ &\leqslant \left(|a_{m+N}| + \sum_{n \geqslant m+N} |\Delta a_n| \right)(N+1). \end{aligned}$$

Since

$$\alpha_n \geqslant n|a_n| \quad \text{for all } n \in \{1, 2, \cdots\},$$

therefore

$$(N+1)|a_{m+N}| \leqslant (m+N)|a_{m+N}| \leqslant \alpha_{m+N}.$$

Hence

$$\begin{aligned} |R'| &\leqslant \alpha_{m+N} + (m+N) \sum_{n \geqslant m+N} |\Delta a_n| \\ &\leqslant \alpha_{m+N} + \beta_{m+N}. \end{aligned} \tag{7.2.5}$$

Both statements (1) and (2) follow on combining (7.2.4) and (7.2.5).

7.3 The Series (*C*) and (*S*) as Fourier Series

The aim is to establish analogues of cases (3) and (4) of 7.2.2 under weaker hypotheses on the coefficients a_n. We begin with the series (*C*).

7.3.1. Assume that

$$a_n \to 0, \tag{7.3.1}$$

$$\sum_{n=0}^{\infty} (n+1)|\Delta^2 a_n| < \infty. \tag{7.3.2}$$

Then, for the series (*C*),

$$s_N(x) \to f(x) = \sum_{n=0}^{\infty} (n+1)\Delta^2 a_n \cdot \tfrac{1}{2} F_n(x) \tag{7.3.3}$$

pointwise for $x \not\equiv 0 \pmod{2\pi}$, $f \in \mathbf{L}^1$ and (*C*) is the FS of f. The hypotheses are satisfied if $a_n \downarrow 0$ and (a_n) is convex (see 7.1.3) in which case $f \geqslant 0$.

Proof. Two applications of partial summation yield, via the formulas in 7.1.1, the equality

$$s_N(x) = \sum_{n=0}^{N-2} (n+1)\Delta^2 a_n \cdot \tfrac{1}{2} F_n(x) + \tfrac{1}{2} N \Delta a_{N-1} \cdot F_{N-1}(x) + \tfrac{1}{2} a_N D_N(x). \tag{7.3.4}$$

By 7.1.3(3), $n\Delta a_n \to 0$, and so pointwise convergence for $x \not\equiv 0 \pmod{2\pi}$ is clear from (7.1.1), (7.1.2) and (7.3.4). Since also $\|F_n\|_1 = 1$, the series

$$\sum_{n=0}^{\infty} (n+1)\Delta^2 a_n \cdot \tfrac{1}{2} F_n \tag{7.3.5}$$

converges in $\mathbf{L}^1$ to f. [We are not here asserting that $s_N f$ in $\mathbf{L}^1$; see 7.3.2(1).]

It remains to show that (*C*) is the FS of f. We may and will assume without loss of generality that $a_0 = 0$. Consider

$$g(x) = \sum_{n=1}^{\infty} n^{-1} a_n \sin nx = \sum_{n=1}^{\infty} a_n^* \sin na. \tag{7.3.6}$$

By 7.2.3, this series is uniformly convergent, since

$$\alpha_n^* = \sup_k \frac{1}{k} \sum_{n \leqslant m < n+k} m|a_m^*| = \sup_k \frac{1}{k} \sum_{n \leqslant m < n+k} |a_m|$$
$$\leqslant \sup_{m \geqslant n} |a_m| \to 0,$$

$$\Delta a_n^* = \frac{\Delta a_n}{n} + a_{n+1}\left(\frac{1}{n} - \frac{1}{n+1}\right),$$

and so

$$\beta_n^* = n \sum_{m \geqslant n} |\Delta a_m^*| \leqslant n \sum_{m \geqslant n} \left(\frac{|\Delta a_m|}{m} + |a_{m+1}|\left(\frac{1}{m} - \frac{1}{m+1}\right)\right)$$
$$= n \sum_{m \geqslant n} \frac{|\Delta a_m|}{m} + n \sum_{m \geqslant n} \frac{|a_{m+1}|}{m^2}$$
$$\to 0;$$

recall that $m\Delta a_m \to 0$ and $a_m \to 0$.

Now, if

$$g_N = \sum_{n=1}^{N} n^{-1} a_n \sin nx$$

we have $Dg_N = s_N$ and so, by (7.3.4),

$$g_N(x) = \int_0^x s_N(y)\, dy$$
$$= \int_0^x \sum_{n=0}^{N-2} (n+1)\Delta^2 a_n \cdot \tfrac{1}{2} F_n(y)\, dy + \tfrac{1}{2} N \Delta a_{N-1} \cdot \int_0^x F_{N-1}(y)\, dy$$
$$+ \tfrac{1}{2} a_N \int_0^x D_N(y)\, dy.$$

By convergence in $\mathbf{L}^1$ of (7.3.5), the first term converges to $\int_0^x f(y)\, dy$; since $n\Delta a_n \to 0$, the second term tends to 0; since $a_n \to 0$, the third term tends to 0. So

$$g(x) = \int_0^x f(y)\, dy.$$

Hence, by 2.3.4, $\hat{f}(n) = in\hat{g}(n)$ for all $n \in Z$. Also, by uniform convergence of the series in (7.3.6), $\hat{g}(n) = (2in)^{-1} a_{|n|}$ for all nonzero $n \in Z$. So $\hat{f}(n) = \tfrac{1}{2} a_{|n|}$ for nonzero $n \in Z$ and $\hat{f}(0) = 0$, showing that the Fourier series of f is indeed the series (C).

7.3.2. Supplements to 7.3.1.

(1) If (a_n) is quasiconvex, then $s_N \to f$ in $\mathbf{L}^1$ if and only if

$$a_n \cdot \log n = o(1). \tag{7.3.7}$$

Proof. Since (a_n) is quasiconvex, the series (7.3.5) is convergent in $\mathbf{L}^1$. Hence, by (7.3.4), $s_N \to f$ in $\mathbf{L}^1$ if and only if

$$\tfrac{1}{2}N \cdot \Delta a_{N-1} \cdot F_{N-1}(x) + \tfrac{1}{2}a_N \cdot D_N(x) \to 0 \tag{7.3.7'}$$

in $\mathbf{L}^1$.

If (7.3.7) is true, then $a_n = o(1)$. Then 7.1.3(3) shows that $n \cdot \Delta a_n = o(1)$. Hence (7.3.7′) follows from (7.1.8) and the equation $\|F_N\|_1 = 1$, valid for $N \in \{0, 1, 2, \cdots\}$.

Conversely, if (7.3.7′) is true, then

$$\tfrac{1}{2}N \cdot \Delta a_{N-1} \cdot \hat{F}_N(n) + \tfrac{1}{2}a_N \cdot \hat{D}_N(n) \to 0 \quad \text{as } N \to \infty,$$

uniformly for $n \in Z$. Taking first $n = 0$ and then $n = \tfrac{1}{2}N$ or $\tfrac{1}{2}(N-1)$ according as N is even or odd, it is easy to conclude that

$$N \cdot \Delta a_{N-1} \to 0 \quad \text{as } N \to \infty.$$

Then, since $\|F_{N-1}\|_1 = 1$ for all $N \in \{1, 2, \cdots\}$, (7.3.7′) entails $a_N \cdot D_N \to 0$ in $\mathbf{L}'$ and so, by (7.1.8), that (7.3.7) is true.

(2) If (a_n) is quasiconvex and

$$a_n \cdot \log n = O(1) \tag{7.3.8}$$

then (7.3.4), 7.1.3(3) and (7.1.8) combine to show that

$$\|s_N\|_1 = O(1).$$

From this one may conclude (see 12.5.2 and 12.7.5) that (C) is the Fourier (-Stieltjes) series of a measure.

See also Teljakowski [2]; MR **48** # 794; **52** # 14808a,b; **54** # 13436.

(3) It is easily shown (see Exercise 7.5) that, if $p > 1$,

$$\|F_n\|_p = O(n^{1/p'}), \qquad \|D_n\|_p = O(n^{1/p'}) \tag{7.3.9}$$

where $(1/p) + (1/p') = 1$. From this and (7.3.4) it appears that the conditions

$$n^{1-(1/p)}a_n = O(1), \tag{7.3.10}$$

$$n^{2-(1/p)}\,\Delta a_n = O(1) \tag{7.3.11}$$

$$\sum_{n=0}^{\infty} n^{2-(1/p)}|\Delta^2 a_n| < \infty \tag{7.3.12}$$

together suffice to ensure that $\|s_N\|_p$ is bounded with respect to N, in which case it may be shown (see 12.7.6) that (C) is the Fourier series of a function in $\mathbf{L}^p$.

For the case in which $a_n \downarrow 0$, a sharper result appears in 7.3.5(2).

(4) If (a_n) is quasiconvex and $a_n = O(1)$, (C) is the Fourier series of the function $f \in \mathbf{L}^1$; it converges in $\mathbf{L}^1$ to f, if and only if (7.3.7) holds; and

$$\sup_N \|s_N\|_1$$

if and only if (7.3.8) holds.

The first assertion here is included in 7.3.1; the second is covered by (1) above; and the third is proved almost exactly as was (1) above.

These results are due essentially to W. H. Young and A. N. Kolmogorov; see [Ba$_1$], Chapter I, §30; [Ba$_2$], Chapter X, §2; [Z], Chapter V, Theorem (1.12).

(5) If $a_n \downarrow 0$, 7.2.1 shows that each of (C) and (S) is convergent for $x \not\equiv 0$ (mod 2π). In either case, the series is a Fourier series, if and only if its sum function belongs to $\mathbf{L}^1$. (This requires proof; see, for example, [Ba$_2$], p. 199.)

(6) If $a_n \downarrow 0$, the proof of 7.3.1 can be modified in the following way (kindly suggested to me by Professor G. Goes). By 7.2.1(2), the series (C) converges uniformly for $\delta \leqslant |x| \leqslant \pi$, for any $\delta > 0$. The sum function f is thus continuous except perhaps at points $x \equiv 0$ (mod 2π). The equation (7.3.4) and the estimates [see (7.1.1) and (7.1.2)]

$$|\tfrac{1}{2}N\Delta a_{N-1}\cdot F_{N-1}(x)| \leqslant \tfrac{1}{2}|\Delta a_{N-1}|\cdot(\sin \tfrac{1}{2}x)^{-2},$$
$$|\tfrac{1}{2}a_N D_N(x)| \leqslant \tfrac{1}{2}a_N|\sin \tfrac{1}{2}x|^{-1}$$

combine with the assumption $a_n \downarrow 0$ to show that (7.3.13) holds for $0 < |x| \leqslant \pi$. Defining

$$g_N(x) = \sum_{n=1}^{N} n^{-1}a_n \sin nx,$$

one has for $0 < \delta \leqslant x \leqslant \pi$

$$g_N(x) - g_n(\delta) = \int_\delta^x s_N(y)\,dy;$$

and, by uniformity of the convergence of s_N to f on $[\delta, \pi]$, it follows that

$$g(x) - g(\delta) = \int_\delta^x f(y)\,dy$$

for $0 < \delta \leqslant x \leqslant \pi$. By (7.3.3) and (7.3.2), $f \in \mathbf{L}^1$. Since also g is continuous and $g(0) = 0$, it follows that

$$g(x) = \int_0^x f(y)\,dy$$

for $0 \leqslant x \leqslant \pi$. A similar argument yields the same equation for $-\pi \leqslant x \leqslant 0$. From this point on, the argument proceeds as before.

Moreover, if $a_n \downarrow 0$, one can even dispense to some extent with (7.3.2), though one cannot now conclude that $f \in \mathbf{L}^1$. However, it will still be true that

$$g(x) - g(x_0) = \int_{x_0}^{x} f(y)\,dy$$

for $-\pi \leqslant x_0 \leqslant x < 0$ and for $0 < x_0 \leqslant x \leqslant \pi$. Because of this one may integrate by parts in the formula

$$n^{-1}a_n = 2/\pi \int_0^{\pi} g(x)\sin nx\,dx$$

$$\lim_{\delta\to 0} 2/\pi \int_{\delta}^{\pi} g(x)\sin nx\,dx$$

to conclude that

$$a_n = \lim_{\delta\to 0} 2/\pi \int_{\delta}^{\pi} f(x)\cos nx\,dx;$$

recall that g is continuous and $g(0) = 0$, and that we are assuming (as before) that $a_0 = 0$. In other words, (*C*) is the series

$$\sum_{n\in Z} \hat{f}(n)e^{inx},$$

where now

$$\hat{f}(n) = \lim_{\delta\to 0} 1/2\pi \int_{\delta\leqslant|x|\leqslant\pi} f(x)e^{-inx}\,dx$$

is a Cauchy principal value. Thus (*C*) is, in this wider sense, still the Fourier series of f.

We next turn to a few analogous results for the sine series (*S*).

7.3.3. Suppose that $a_n \downarrow 0$ and write

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx,$$

the series being everywhere convergent by 7.2.1. Then $f \in \mathbf{L}^1$ if and only if

$$\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty, \tag{7.3.13}$$

in which case (*S*) is the Fourier series of f and

$$\lim_{N\to\infty} \|s_N - f\|_1 = 0. \tag{7.3.14}$$

Proof. The reader will verify easily that, under the stated conditions on (a_n), (7.3.13) is equivalent to

$$\sum_{n=1}^{\infty} \Delta a_n \cdot \log n < \infty. \tag{7.3.13'}$$

By partial summation,

$$\begin{aligned} s_N(x) &= \sum_{n=1}^{N-1} \Delta a_n \cdot \tfrac{1}{2} \tilde{D}_n(x) + \tfrac{1}{2} a_N \cdot \tilde{D}_N(x) \\ &\to \sum_{n=1}^{\infty} \Delta a_n \cdot \tfrac{1}{2} \tilde{D}_n(x) = f(x). \end{aligned} \tag{7.3.15}$$

If we replace herein $\tilde{D}_n$ by $\tilde{D}_n{}^{\#}$ we obtain a function

$$f^{\#}(x) = \sum_{n=1}^{\infty} \Delta a_n \cdot \tfrac{1}{2} \tilde{D}_n{}^{\#}(x) \tag{7.3.16}$$

which differs from f by

$$\sum_{n=1}^{\infty} \Delta a_n \cdot \tfrac{1}{2} \sin nx,$$

which is continuous since, by virtue of the relation $a_n \downarrow 0$, one has

$$\sum_{n=1}^{\infty} |\Delta a_n| < \infty.$$

Thus $f \in \mathbf{L}^1$ if and only if $f^{\#} \in \mathbf{L}^1$.

On the other hand, since $\tilde{D}_n{}^{\#}$ is odd and is nonnegative on $(0, \pi)$ [see (7.1.5)], $f^{\#} \in \mathbf{L}^1$ if and only if

$$\sum_{n=1}^{\infty} \Delta a_n \cdot \|\tilde{D}_n{}^{\#}\|_1 < \infty.$$

This requirement is equivalent to (7.3.13′), in view of (7.1.11).

Finally, assuming (7.3.13′) to hold, we have

$$a_n \log n = \log n \cdot \sum_{k \geqslant n} \Delta a_k \leqslant \sum_{k \geqslant n} \Delta a_k \cdot \log k,$$

which tends to zero as $n \to \infty$. Hence (7.1.10) and (7.3.15) show that $s_N \to f$ in mean in $\mathbf{L}^1$ as $N \to \infty$.

7.3.4. Suppose that $a_n \downarrow 0$ and that

$$\sum_{n=1}^{\infty} \frac{a_n}{n} = \infty. \tag{7.3.17}$$

Then $\sum_{n=1}^{\infty} a_n \sin nx$, although everywhere convergent, is not a Fourier-Lebesgue series.

Proof. This follows immediately on combining 5.3.1, 6.4.5, and 7.3.3.

7.3.5 Supplements.

(i) 7.3.3 can be generalized. For example, Teljakowskiĭ [1] has proved that, if (a_n) is quasiconvex and $a_n \to 0$, then (S) is a Fourier series, if and only if

$$\sum_{n=1}^{\infty} n^{-1}|a_n| < \infty.$$

Also, Kano and Uchiyama [1] have proved that, if (a_n) is quasiconvex and bounded, then (S) converges in $\mathbf{L}^1$, if and only if

$$\sum_{n=1}^{\infty} n^{-1}|a_n| < \infty \quad \text{and} \quad a_n \cdot \log n \to 0,$$

while

$$\sup \|s_N\|_1 < \infty$$

if and only if

$$\sum_{n=1}^{\infty} n^{-1}|a_n| < \infty \quad \text{and} \quad a_n \cdot \log n = O(1).$$

These writers show also that there exists a nonnegative quasiconvex sequence (a_n) such that $a_n \to 0$, both (C) and (S) are Fourier series, and yet both (C) and (S) diverge in $\mathbf{L}^1$.

See also Teljakowskiĭ [2] MR **52** ## 14805, 14819; MR **55** #8673.

(ii) In 7.3.2(3) we have indicated conditions sufficient to ensure that the sum function f of the cosine series (C) shall belong to $\mathbf{L}^p$. For the case in which $a_n \downarrow 0$, a sharper necessary and sufficient condition is known. Denoting by g the sum function of the sine series (S), and assuming that $1 < p < \infty$, it is known that the following conditions are equivalent:

(1) $f \in \mathbf{L}^p$;

(2) $g \in \mathbf{L}^p$;

(3) the sequence $(a_n)_{n=1}^{\infty}$, assumed to decrease monotonely to zero, satisfies the condition

$$\sum_{n=1}^{\infty} n^{p-2} a_n^p < \infty.$$

Despite appearances, this result is considerably deeper than that mentioned in 7.3.2(3). Its proof depends on Theorem 12.9.1 and an inequality of Hardy; see [Z_2], p. 129 or [Ba_2], p. 207.

For further results, see Aljančič [1] and the references cited there.

Hardy [1] proved that, if (S) is the Fourier series of some $f \in \mathbf{L}^p$ ($p \leqslant 1$), then so too is the series

$$\sum_{n=1}^{\infty} (Ta)_n \sin nx, \tag{7.3.18}$$

where

$$(Ta)_n = n^{-1}(a_1 + \cdots + a_n) \quad \text{for } n \in \{1, 2, \cdots\}.$$

On the other hand, G. and S. Goes [1] proved that if (a_n) is BV and $a_n \to 0$, then (7.3.18) is a Fourier series, if and only if

$$\sum_{n=1}^{\infty} n^{-1}|a_n| < \infty;$$

See also MR 55 # 8673.

(iii) See also [Boa] and Boas [2].

7.4 Application to A(Z)

We can now substantiate a statement made in 2.3.9 concerning the existence of integrable functions whose Fourier coefficients tend to zero arbitrarily slowly.

Let $\phi \in \mathbf{c}_0(Z)$ be given and define

$$\lambda_n = \max\,[|\phi(n)|, |\phi(-n)|] \qquad (n = 0, 1, 2, \cdots).$$

Then $\lambda_n \to 0$ as $n \to \infty$ and we may, as in 7.1.5, construct a sequence (a_n) which is convex and satisfies $a_n \downarrow 0$ and

$$2\lambda_n \leqslant a_n \qquad (n = 0, 1, 2, \cdots).$$

Consider the function f figuring in the proof of 7.3.1. Since (a_n) is convex, f is nonnegative and integrable. The Fourier series of f is $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$, so that $\hat{f}(n) = \frac{1}{2}a_{|n|}$ for all $n \in Z$. Hence

$$\hat{f}(n) \geqslant |\phi(n)| \qquad \text{for all } n \in Z.$$

See also [HR], (32.47).

7.5 Application to Factorization Problems

In this section it will be shown how 7.3.1 aids in the solution of a number of problems concerning the possibility of factorizations $f = g * h$ with f, g, and h in specified function spaces. Included in the discussion will be the Salem-Zygmund-Rudin-Cohen result mentioned in 3.1.1(c). The approach will be "classical" (as opposed to the Banach algebra-based technique introduced by Cohen [4] for certain such problems; see 11.4.18(6)), being a modified form of the arguments published by Rudin [1].

7.5.1. Let $\mathbf{E}$ denote any one of $\mathbf{L}^p$ $(1 \leqslant p < \infty)$ or $\mathbf{C}^k$ $(0 \leqslant k < \infty)$. Then $\mathbf{E} = \mathbf{L}^1 * \mathbf{E}$.

Proof. Since, by 3.1.5 and 3.1.6, $\mathbf{L}^1 * \mathbf{E} \subset \mathbf{E}$, it suffices to show that $\mathbf{E} \subset \mathbf{L}^1 * \mathbf{E}$. In doing this we shall use the simplest results about Cesàro summability; numerous other summability methods would serve equally well, however.

Suppose that $f \in \mathbf{E}$. The aim is to prove the existence of $g \in \mathbf{E}$ and $h \in \mathbf{L}^1$ such that

$$f = g * h. \tag{7.5.1}$$

The starting point in the construction of g and h is the remark that, by 6.1.1,

$$\lim_{N \to \infty} \|f - F_N * f\| = 0, \tag{7.5.2}$$

where $\|\cdot\|$ denotes the appropriate norm ($\|\cdot\|_p$ or $\|\cdot\|_{(k)}$) in $\mathbf{E}$. As a consequence of (7.5.2) one may, as it is easy to verify, choose nonnegative numbers α_N $(N = 1, 2, \cdots)$ so that

$$\sum_{N=1}^{\infty} \alpha_N = \infty, \tag{7.5.3}$$

$$\sum_{N=1}^{\infty} \alpha_N \|f - F_N * f\| < \infty, \tag{7.5.4}$$

and

$$\sum_{N=1}^{\infty} \frac{\alpha_N}{N} < \infty. \tag{7.5.5}$$

Now (7.5.4) and the completeness of $\mathbf{E}$ together ensure that

$$g = f + \sum_{N=1}^{\infty} \alpha_N (f - F_N * f) \tag{7.5.6}$$

belongs to $\mathbf{E}$, the series being convergent in $\mathbf{E}$. This in turn entails that

$$\hat{g} = \hat{f} \cdot \left[1 + \sum_{N=1}^{\infty} \alpha_N (1 - \hat{F}_N)\right]; \tag{7.5.7}$$

the reader will observe that, for a fixed $n \in Z$, $1 - \hat{F}_N(n) = O(1/N)$ for large N, so that (7.5.5) ensures the pointwise convergence on Z of the series appearing in (7.5.7).

In view of 2.4.1, (3.1.5), and (7.5.7), to establish (7.5.1) for some $h \in \mathbf{L}^1$, it suffices to show that the sequence $(a_n)_{n \in Z}$ defined by

$$a_n = \frac{1}{\left[1 + \sum_{N=1}^{\infty} \alpha_N (1 - \hat{F}_N(n))\right]} \equiv \frac{1}{b_n} \tag{7.5.8}$$

is a Fourier-Lebesgue sequence. It is at this point that 7.3.1 comes to our aid.

Noting that $a_n = a_{|n|}$, 7.3.1 shows that we have only to verify the following two points:

(1) $a_n \downarrow 0$ as $n \uparrow \infty$;

(2) $(a_n)_{n \geqslant 0}$ is convex.

As to (1), we have

$$\hat{F}_N(n) = \begin{cases} 1 - \dfrac{n}{N+1} & \text{if } 0 \leqslant n \leqslant N, \\ 0 & \text{if } n > N, \end{cases}$$

so that $0 \leqslant \hat{F}_N(n) \leqslant 1$ and $\hat{F}_N(n) = 0$ for $n > N$. Therefore, by (7.5.3),

$$b_n \geqslant 1 + \sum_{N<n} \alpha_N \to \infty \qquad \text{as } n \to \infty.$$

Moreover $b_{n+1} \geqslant b_n$, since $\alpha_N \geqslant 0$, and $\hat{F}_N(n+1) \leqslant \hat{F}_N(n)$. Thus (1) is satisfied.

Turning to (2), the convexity of $(a_n)_{n \geqslant 0}$ is equivalent to the condition

$$\frac{b_{n+1} - b_n}{b_n} \geqslant \frac{b_{n+2} - b_{n+1}}{b_{n+2}}. \tag{7.5.9}$$

Now it turns out that

$$b_{n+1} - b_n = \sum_{N \geqslant n} \frac{\alpha_N}{N+1},$$

and so, since $\alpha_N \geqslant 0$ and $b_n \leqslant b_{n+1} \leqslant b_{n+2}$, it is clear that (7.5.9) holds. Thus (2) is satisfied and the proof is complete.

7.5.2. **Remarks.** (1) The excluded case $k = \infty$ of 7.5.1 is easily dealt with separately: indeed, on using 12.1.1 it is easily seen that $\mathbf{C}^\infty = \mathbf{C}^\infty * \mathbf{C}^\infty$. On the other hand, the excluded case $p = \infty$ is a false assertion (as follows from 3.1.4).

As with 6.1.1, the method of proof may be adapted to apply to homogeneous Banach spaces over T; see [Kz], p. 61, Exercise 1.

(2) Extensions of parts of 7.5.1 (other than the case $\mathbf{E} = \mathbf{L}^1$) to more general groups have been examined by Hewitt [1] and by Curtis and Figà-Talamanca [1] jointly. (Rudin's original argument applies to locally Euclidean groups.) Hewitt exploits Cohen's method, working within a general Banach algebra setting; see 11.4.18(6).

7.5.3. Having now shown that $\mathbf{L}^1 = \mathbf{L}^1 * \mathbf{L}^1$, it is natural to ask whether $\mathbf{L}^p = \mathbf{L}^p * \mathbf{L}^p$ for $p > 1$. That the answer is negative will be established for $p = 2$ in Section 8.4, and for general $p > 1$ in Exercise 13.20 and again in 15.3.4.

7.5.4. Supplements to 7.5.1. (1) Concerning the functions g and h appearing in (7.5.1) and constructed in the proof of 7.5.1, two comments deserve to be recorded.

In the first place, the sum appearing in (7.5.4) may plainly be made inferior to any preassigned $\delta > 0$, in which case (7.5.6) shows that

$$\|g - f\| < \delta. \tag{7.5.10}$$

In the second place, having arranged that the sequence $(a_n)_{n \geqslant 0}$ decreases to zero and is convex, equation (7.3.4) shows that the function h is nonnegative, so that $\|h\|_1 = a_0$ and therefore

$$\|h\|_1 = 1. \tag{7.5.11}$$

(2) With but little further effort 7.5.1 may be sharpened to the following extent.

Suppose that $\mathbf{E}_1$ is a σ-compact subset of $\mathbf{E}$ (by which it is meant that $\mathbf{E}_1$ is contained in the union of a countable sequence A_i $(i = 1, 2, \cdots)$ of compact subsets of $\mathbf{E}$). Then there exists an $h \in \mathbf{L}^1$, which is nonnegative and satisfies (7.5.11), with the property that to each $f \in \mathbf{E}_1$ corresponds a $g \in \mathbf{E}$ such that (7.5.1) holds. (The choice of h may be made the same for all $f \in \mathbf{E}_1$.)

The basis of the proof of this extension is the remark that, since A_i is a compact subset of $\mathbf{E}$, the numbers

$$\beta_i(N) = \sup \{\|f - F_N * f\| : f \in A_i\}$$

tend to zero as $N \to \infty$ for each fixed i. (This in turn depends on the fact that, in any metric space, any compact set can be covered by a finite number of balls of arbitrarily small radius; this remark is used in conjunction with the observation that $\|f - F_N * f\| \leqslant 2\|f\|$ for all N and all $f \in \mathbf{E}$.) This being so, the numbers $\alpha_N \geqslant 0$ are chosen to satisfy (7.5.3) and (7.5.5), while in place of (7.5.4) we impose (as we may) the demand that

$$\sum_{N=1}^{\infty} \alpha_N \beta_i(N) < \infty \tag{7.5.4'}$$

for each $i = 1, 2, \cdots$. The proof then proceeds exactly as before.

(3) Other factorization theorems may be derived from 7.5.1. For example, starting from the relation $\mathbf{C} = \mathbf{C} * \mathbf{L}^1$, one may infer that

$$\mathbf{C}^1 = \mathbf{C} * \mathbf{AC}, \tag{7.5.12}$$

where $\mathbf{AC}$ denotes the space of (periodic) absolutely continuous functions; see Exercise 7.10.

7.5.5 For further reading, see 11.4.18(6) below; [HR], (32.14) ff.; [DW]; MR **34** ## 4817, 4818; MR **46** # 2355; MR **52** # 6327; MR **53** ## 8782, 8789; MR **54** ## 843, 8151.

EXERCISES

7.1. Show that if $\sum_{n=0}^{\infty} c_n$ is convergent, and if $\sum_{n=0}^{\infty} |\Delta\lambda_n| < \infty$, then $\sum_{n=0}^{\infty} \lambda_n c_n$ is convergent.

7.2. Prove the converse of the result in Exercise 7.1, that is, that if $\sum_{n=0}^{\infty} \lambda_n c_n$ converges whenever $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} |\Delta\lambda_n| < \infty$.

Hint: Show that the hypothesis means that $\sum_{n=0}^{\infty} \Delta\lambda_n \cdot s_n$ is convergent for any sequence $s_n \to 0$.

7.3. Prove that, if $\sum_{n=0}^{\infty} c_n/\log(2 + n)$ is convergent, then $\sum_{n=0}^{\infty} c_n/(1+n)^{\alpha}$ is convergent for any $\alpha > 0$.

7.4. Write $s_n = \sum_{k=0}^{n} c_k$ and $\sigma_k = (n + 1)^{-1}(s_0 + s_1 + \cdots + s_n)$. If we are given that (σ_n) is a bounded sequence and that $s_n = o(\log n)$ as $n \to \infty$, prove that $\sum_{n=0}^{\infty} c_n/\log(2 + n)$ and $\sum_{n=0}^{\infty} c_n/(1 + n)^{\alpha}$ are convergent for $\alpha > 0$.

Hint: Write $\lambda_n = 1/\log(2 + n)$ and apply partial summation twice in succession to the series $\sum_{n=0}^{\infty} \lambda_n c_n$.

7.5. Verify that, if $p > 1$,

$$\|F_N\|_p \sim A_p N^{1/p'}, \qquad \|D_N\|_p \sim B_p N^{1/p'}$$

as $N \to \infty$, where $1/p + 1/p' = 1$ and where A_p and B_p are positive and independent of N. [Compare (7.3.9).]

7.6. Suppose that $a(x)$ is defined and continuous for $x \geqslant 0$, and that the second derivative $a''(x)$ exists and is nonnegative for $x > 0$. Show that the sequence $(a(n))_{n=0}^{\infty}$ is convex.

7.7. Show that $\sum_{n=2}^{\infty} \cos nx/\log n$ is the Fourier series of an integrable function $f \geqslant 0$, but that $\sum_{n=2}^{\infty} \sin nx/\log n$ is not a Fourier-Lebesgue series at all. (This example will gain in significance in 12.8.3.)

7.8. Using 7.3.2(3), show that if $0 < \alpha \leqslant 1$, then $\sum_{n=1}^{\infty} \cos nx/n^{\alpha}$ is the Fourier series of a nonnegative function $f \in \mathbf{L}^p$ provided $1 \leqslant p < (1 - \alpha)^{-1}$, and that then

$$\lim_{N\to\infty} \left\| f - \sum_{n=1}^{N} \frac{\cos nx}{n^{\alpha}} \right\|_p = 0.$$

What can be said if $\alpha > 1$?

7.9. Using Exercise 7.8, show that, if $0 < \alpha \leqslant 1$, then $\sum_{n\in Z, n\neq 0} \hat{g}(n)/|n|^{\alpha}$ is convergent for each $g \in \mathbf{L}^q$ with $q > \alpha^{-1}$.

For further results in this direction, see 10.4.3, Exercises 10.4 to 10.6, and Exercise 13.1.

7.10. Construct a proof of (7.5.12), and deduce that

$$\mathbf{C}^m = \mathbf{C}^{m-1} * \mathbf{AC} = \mathbf{C} * \underbrace{\mathbf{AC} * \cdots * \mathbf{AC}}_{m \text{ factors}}$$

for $m = 1, 2, \cdots$.

7.11. Prove that [with the notation used in Remark (4) following 2.3.5 and 7.5.4(3)]

(1) $\mathbf{AC} = \mathbf{L}^1 * \mathbf{AC}$;

(2) $\mathbf{AC} = \mathbf{L}^1 * \mathbf{BV}$.

Note: From (2) we see that $\mathbf{L}^1 * \mathbf{BV}$ is a proper subset of $\mathbf{BV}$. In proving that $\mathbf{L}^1 * \mathbf{BV} \subset \mathbf{AC}$, it is useful to note that $V(f) = \|Df\|_1$ for each trigonometric polynomial f. (Actually, $V(f) = \|Df\|_1$ is true for each absolutely continuous f; see [Na], p. 259.)

CHAPTER 8

Fourier Series in $\mathbf{L}^2$

It will be shown in this chapter that the problem of *mean* convergence of Fourier series in $\mathbf{L}^2$ has a complete and simple solution. The abstract foundation for this situation lies in the fact that $\mathbf{L}^2$ is a Hilbert space with the inner (or scalar) product

$$(f, g) = \frac{1}{2\pi} \int f\bar{g}\, dx, \tag{8.1}$$

and that moreover the functions e_n defined by

$$e_n(x) = e^{inx} \qquad (n \in Z) \tag{8.2}$$

form an orthonormal base in $\mathbf{L}^2$. This last means that the family (e_n) is *orthonormal*, in the sense that

$$(e_m, e_n) = \delta_{mn} \qquad (m, n \in Z), \tag{8.3}$$

and that

$$f \in \mathbf{L}^2, (f, e_n) = 0 \ (n \in Z) \quad \Rightarrow \quad f = 0 \qquad \text{a.e.} \tag{8.4}$$

Indeed, (8.3) is simply a restatement of the orthogonality relations, and the implication (8.4) is a special case of the uniqueness theorem 2.4.1. As Hilbert space theory shows, these two facts imply that each $f \in \mathbf{L}^2$ has a convergent expansion

$$f = \sum_{n \in Z} (f, e_n) e_n; \tag{8.5}$$

see, for example, [E], Corollary 1.12.5, or [HS], pp. 245–246, or [AB], pp. 239–240.

Despite this ready-made solution to the problem before us, we shall not assume a knowledge of Hilbert space and will give all the necessary details pertaining to the present situation. For general orthogonal expansions, see [KSt], Kapitel III.

With the exception of Section 8.6, what little we have to say about pointwise convergence is included in Chapter 10.

8.1 A Minimal Property

We make a start by showing that, for a given function $f \in \mathbf{L}^2$, the sequence of partial sums $s_N f$ of the Fourier series of f possesses a minimal property which already serves to distinguish the Fourier series of f among all trigonometric series (see the discussion in Section 1.2).

Denote by $\mathbf{T}_N$ the linear space of all trigonometric polynomials of degree at most N, that is, all linear combinations

$$t = \sum_{|n| \leqslant N} \alpha_n e_n \tag{8.1.1}$$

of the functions e_n for which $|n| \leqslant N$; see Exercise 1.7.

8.1.1. For a given $f \in \mathbf{L}^2$ one has

$$\|f - t\|_2 > \|f - s_N f\|_2$$

for every $t \in \mathbf{T}_N$ different from $s_N f$.

Proof. A perfectly straightforward calculation, based upon the orthogonality relations (8.3), leads to the identity

$$\|f - t\|_2^2 = \|f\|_2^2 + \sum_{|n| \leqslant N} |\alpha_n - \hat{f}(n)|^2 - \sum_{|n| \leqslant N} |\hat{f}(n)|^2 \tag{8.1.2}$$

for an arbitrary $t \in \mathbf{T}_N$, given by (8.1.1). The right-hand side of (8.1.2) plainly has a strict minimum which is attained for the choice $\alpha_n = \hat{f}(n)$ ($|n| \leqslant N$) and for no other, and this minimum value is $\|f - s_N f\|_2^2$.

8.2 Mean Convergence of Fourier Series in L². Parseval's Formula

Using 8.1.1 and the results in Chapter 6, it is a simple matter to establish mean convergence of Fourier series in $\mathbf{L}^2$.

8.2.1. If $f \in \mathbf{L}^2$, then

$$\lim_{N \to \infty} \|f - s_N f\|_2 = \lim_{N \to \infty} \|f - \sigma_N f\|_2 = 0 \tag{8.2.1}$$

and

$$\frac{1}{2\pi} \int |f(x)|^2 \, dx \equiv \|f\|_2^2 = \sum_{n \in Z} |\hat{f}(n)|^2. \tag{8.2.2}$$

Furthermore, given any $\varepsilon > 0$ there exists a finite subset $F_0 = F_0(\varepsilon)$ of Z such that for every finite subset F of Z satisfying $F \supset F_0$ one has

$$\|f - \sum_{n \in F} \hat{f}(n) e_n\|_2 \leqslant \varepsilon. \tag{8.2.3}$$

Proof. Since $\sigma_N f \in \mathbf{T}_N$, 8.1.1 entails that

$$\|f - s_N f\|_2 \leqslant \|f - \sigma_N f\|_2.$$

On the other hand, the case $p = 2$ of 6.1.1 shows that

$$\lim_{N\to\infty} \|f - \sigma_N f\|_2 = 0,$$

and (8.2.1) is thereby established. Taking $t = s_N f$ in (8.1.2) yields

$$\|f - s_N f\|_2^2 = \|f\|_2^2 - \sum_{|n|\leqslant N} |\hat{f}(n)|^2;$$

if we let $N \to \infty$, (8.2.2) follows.

Finally, putting $s_F = \sum_{n\in F} \hat{f}(n)e_n$ for any finite set $F \subset Z$, and $g = f - s_F$, we have $g \in \mathbf{L}^2$ and

$$\hat{g}(n) = 0 \quad (n \in F), \qquad \hat{g}(n) = \hat{f}(n) \quad (n \in Z\backslash F).$$

An application of (8.2.2) to g yields

$$\|f - s_F\|_2^2 = \sum_{n\in Z\backslash F} |\hat{f}(n)|^2.$$

Since $\sum |\hat{f}(n)|^2$ is a convergent series of nonnegative terms, (8.2.3) follows at once.

Remarks. Equation (8.2.2) is one form of the *Parseval formula*. Valid extensions of (8.2.2) will appear in Sections 13.5 and 13.11. If one takes $f, g \in \mathbf{L}^2$, replaces f by $f + \lambda g$ in (8.2.2), and varies the scalar λ, one infers directly the so-called *polarized* version of the formula, which reads as follows.

8.2.2. If $f, g \in \mathbf{L}^2$, then

$$(f, g) \equiv \frac{1}{2\pi}\int f(x)\overline{g(x)}\,dx = \sum_{n\in Z} \hat{f}(n)\overline{\hat{g}(n)}, \tag{8.2.4}$$

the series being absolutely convergent. In view of 2.3.1, it is equivalent to assert that

$$\frac{1}{2\pi}\int f(x)g(x)\,dx = \sum_{n\in Z} \hat{f}(n)\hat{g}(-n), \tag{8.2.5}$$

the series being absolutely convergent, for $f, g \in \mathbf{L}^2$.

8.3 The Riesz-Fischer Theorem

In view of the fact that

$$\sum_{n\in Z} |\hat{f}(n)|^2 < \infty$$

for each $f \in \mathbf{L}^2$, it is especially satisfying to be able to state and prove an unqualified converse.

8.3.1. (Riesz-Fischer theorem) Let $(c_n)_{n\in Z}$ be any sequence of complex numbers such that

$$\sum_{n\in Z} |c_n|^2 < \infty. \tag{8.3.1}$$

Then there exists an $f \in \mathbf{L}^2$, uniquely determined almost everywhere, such that $\hat{f}(n) = c_n$ $(n \in Z)$.

Proof. If we define

$$s_N = \sum_{|n|\leqslant N} c_n e_n,$$

the orthogonality relations (8.3) yield for $M < N$ the equality

$$\|s_M - s_N\|_2{}^2 = \sum_{M<|n|\leqslant N} |c_n|^2$$

which, by (8.3.1), tends to zero as $M, N \to \infty$. By completeness of $\mathbf{L}^2$ ([W], Theorem 4.5a), there exists an essentially unique $f \in \mathbf{L}^2$ such that

$$\lim_{N\to\infty} \|f - s_N\|_2 = 0. \tag{8.3.2}$$

Moreover (8.3.2) entails (Cauchy-Schwarz inequality) that $\hat{f}(n) = \lim_{N\to\infty} \hat{s}_N(n)$ for each $n \in Z$. Since $\hat{s}_N(n)$ is c_n or 0 according as $|n| \leqslant N$ or $|n| > N$, it follows that $\hat{f}(n) = c_n$ for all $n \in Z$.

8.3.2. **Remarks.** The result 8.3.1 is known to be the best possible in the following sense: given any sequence $(c_n)_{n\in Z}$ for which

$$\sum_{n\in Z} |c_n|^2 = \infty,$$

it is possible to choose the $\pm$ signs in such a way that the series $\sum \pm c_n e^{inx}$ is not the Fourier series of any integrable function (nor even the Fourier-Stieltjes series of any measure, as defined in 12.5.2). Such questions will receive more attention in Chapter 14; see especially 14.3.5 and 14.3.6.

It can also be shown that the relation $\sum_{n\in Z} |\hat{f}(n)|^2 < \infty$, valid for $f \in \mathbf{L}^2$, cannot be much improved even for continuous functions f. For example (see Exercise 8.9), given any positive function ω on Z such that $\lim_{|n|\to\infty} \omega(n) = \infty$, there exist continuous functions f such that

$$\limsup_{n\to\infty} |\omega(n)\hat{f}(n)| = \infty \tag{8.3.3}$$

and

$$\sum_{n\in Z} \omega(n)|\hat{f}(n)|^2 = \infty. \tag{8.3.4}$$

Moreover, there exist (Carleman) continuous functions f such that

$$\sum_{n\in Z} |\hat{f}(n)|^{2-\varepsilon} = \infty \quad \text{for any given } \varepsilon > 0, \tag{8.3.5}$$

and even (Banach) such that

$$\sum_{n\in Z} |\hat{f}(n)|^{2-\varepsilon_n} = \infty$$

for suitable sequences $\varepsilon_n \to 0$ (see Exercise 15.13). A specific example of (8.3.5) is

$$f(x) = \sum_{n=2}^{\infty} \frac{e^{icn \log n}}{n^{1/2}(\log n)^{\beta}} e^{inx},$$

where $\beta > 1$ and $c > 0$; see [Z_1], p. 200, Theorem (4.11); this example is quite different in nature from those indicated in Exercise 15.13. See also [Ba_1], p. 337.

For some valid extensions of 8.3.1, see Sections 13.5 and 13.11. Regarding extensions of 8.2.1 and 8.3.1 to general groups, see 13.5.2.

8.3.3. **Dual Version of 8.3.1.** As has been hinted at in Section 2.5, 8.3.1 can be recast into the form of a dual result. It says in effect that, if $\phi \in \ell^2$, then the trigonometric polynomials

$$\hat{\phi}_N = \sum_{|n|\leqslant N} \phi(n)e_n$$

converge in mean in L^2 to a function $f = \hat{\phi}$ on T such that $\hat{f} = \phi$. This includes a possible interpretation of the inversion formula spoken of in Section 6.7, and in particular attaches a good meaning to the Fourier transform $\hat{\phi}$ whenever $\phi \in \ell^2$ (see Section 2.5).

See also Sections 13.5 and 13.11.

8.4 Factorization Problems Again

It has been stated in 3.1.1(c) and proved in 7.5.1 that there are no prime elements in the $*$-algebra L^1, that is, that every $f \in L^1$ can be factorized in at least one way as $f = f_1 * f_2$ with $f_1, f_2 \in L^1$. In 7.5.3 we alleged that the analogous assertion for L^p ($p > 1$) is false; see also Exercise 13.20. We can now verify this when $p = 2$.

Indeed, if $p = 2$ the Parseval formula (8.2.2) and the Riesz-Fischer theorem 8.3.1 combine to show that any $f \in L^2$ for which $\sum_{n\in Z} |\hat{f}(n)| = \infty$ is a prime element of L^2 and (see Exercise 8.2) that these are the only primes in L^2.

In view of this it is natural to ask whether it is true that any nonprime element of L^p ($p > 1$) is expressible as a finite convolution product of prime elements of L^p. An affirmative answer for $p = 2$ is given in Exercise 8.3.

8.5 More about Mean Moduli of Continuity

Extending the notation introduced in 2.3.7, we define the *mean modulus of continuity of f with exponent $p > 0$* by

$$\omega_p f(a) = \|T_a f - f\|_p. \tag{8.5.1}$$

This may be regarded as defined for any measurable f, setting $\omega_p f(a) = \infty$ if $T_a f - f$ does not belong to $\mathbf{L}^p$. Translation invariance of the integral shows that

$$\omega_p f(-a) = \omega_p f(a). \tag{8.5.2}$$

For $p \geqslant 1$ one has also by Minkowski's inequality

$$\begin{aligned}\omega_p f(a + b) &= \|T_{a+b} f - f\|_p \leqslant \|T_{a+b} f - T_b f\|_p + \|T_b f - f\|_p \\ &= \omega_p f(a) + \omega_p f(b),\end{aligned} \tag{8.5.3}$$

again by invariance of the integral. Also, by (2.2.19),

$$\omega_p f(a) \leqslant \omega_q f(a) \qquad (0 < p < q). \tag{8.5.4}$$

It has been seen in Exercises 5.1 and 5.2 that restrictions on the rate of decrease of $\omega_1 f(a)$ as $a \to 0$ bear upon smoothness properties of f. It will now be seen, by using the results of 8.3, that further results of this nature may be expressed in terms of $\omega_1 f$ and $\omega_2 f$. Similar and much more elaborate results will be mentioned at the end of 10.4.6.; see also 10.6.2.

8.5.1. If $f \in \mathbf{L}^1$ and

$$\sum_{N=1}^{\infty} \left[\omega_1 f\left(\frac{\pi}{N}\right)\right]^2 < \infty,$$

then $f \in \mathbf{L}^2$ and

$$\|f\|_2^2 \leqslant \frac{1}{2} \sum_{N=1}^{\infty} \left[\omega_1 f\left(\frac{\pi}{N}\right)\right]^2 + |\hat{f}(0)|^2. \tag{8.5.5}$$

Proof. By 2.3.7 and (8.5.2) we have

$$|\hat{f}(n)| \leqslant \tfrac{1}{2} \omega_1 f\left(\frac{\pi}{|n|}\right)$$

for $n \in Z$, $n \neq 0$. So (8.2.2) gives

$$\|f\|_2^2 \leqslant |\hat{f}(0)|^2 + 2 \sum_{N=1}^{\infty} \left[\tfrac{1}{2} \omega_1 f\left(\frac{\pi}{N}\right)\right]^2,$$

which is equivalent to (8.5.5).

8.5.2. Suppose that $f \in \mathbf{L}^2$ and that $a > 0$. Then

$$\frac{4}{\pi^2} \sum_{|n| \leqslant \pi/a} |n\hat{f}(n)|^2 \leqslant \left[\frac{\omega_2 f(a)}{a}\right]^2 \leqslant \sum_{n \in Z} |n\hat{f}(n)|^2. \tag{8.5.6}$$

Proof. By 2.3.3 and (8.2.2),

$$[\omega_2 f(a)]^2 = \sum |e^{ina} - 1|^2 \, |\hat{f}(n)|^2. \tag{8.5.7}$$

Furthermore,

$$|e^{ina} - 1| = 2|\sin \tfrac{1}{2} na|;$$

and

$$2|\sin \tfrac{1}{2} na| \geqslant 2 \frac{2}{\pi} |\tfrac{1}{2} na| \qquad \text{if } |\tfrac{1}{2} na| \leqslant \tfrac{1}{2}\pi,$$

while

$$2|\sin \tfrac{1}{2} na| \leqslant 2|\tfrac{1}{2} na| = |na|$$

for all n and all a. Insert these estimates into (8.5.7): (8.5.6) emerges after division by a^2.

8.5.3. **Concerning Absolute Continuity.** On the basis of 8.5.2 we can establish a number of interesting conditions, each necessary and sufficient in order that a given $\mathbf{L}^2$ function shall be equal almost everywhere to an absolutely continuous function whose derivative (existing pointwise almost everywhere) belongs to $\mathbf{L}^2$.

8.5.4. Suppose that $f \in \mathbf{L}^2$. Then the following four conditions are equivalent:

(1) after correction on a null set, f is absolutely continuous and $Df \in \mathbf{L}^2$;

(2) $\sum_{n \in Z} |n\hat{f}(n)|^2 < \infty$;

(3) $\lim_{a \to 0} a^{-1}(T_{-a}f - f)$ exists in mean in $\mathbf{L}^2$;

(4) $\omega_2 f(a)/a = O(1)$ as $a \to 0$.

If any one of these conditions is fulfilled, the limit mentioned in (3) is Df.

Proof. That (1) implies (2) follows directly from 2.3.4 and (8.2.2).

Assuming (2), 8.3.1 ensures that there exist $g \in \mathbf{L}^2$ such that $\hat{g}(n) = in\hat{f}(n)$ for all $n \in Z$. Then 6.2.8 shows that

$$\sum_{n \in Z}' \hat{f}(n)(e^{inb} - e^{ina}) = \int_a^b g(x)\, dx \tag{8.5.8}$$

for all a and b. Moreover (2) entails that $\sum_{n \in Z} |\hat{f}(n)| < \infty$, and 2.4.2 shows that, after correcting f on a null set,

$$f(x) = \sum_{n \in Z} \hat{f}(n) e^{inx}$$

for all x. So (8.5.8) and Lebesgue's theorem on the derivation of integrals ([W], Theorem 5.2g) combine to show that f is absolutely continuous and

$Df = g$ almost everywhere. Thus (1) is satisfied and the equivalence of (1) and (2) is established.

Again assuming (2), and using the above notation, Parseval's formula (8.2.2) gives

$$\begin{aligned}\|a^{-1}(T_{-a}f - f) - g\|_2^2 &= \sum_{n\in Z} |a^{-1}(e^{ina} - 1)\hat{f}(n) - in\hat{f}(n)|^2 \\ &= \sum_{n\in Z} |a^{-1}(e^{ina} - 1) - in|^2|\hat{f}(n)|^2. \qquad (8.5.9)\end{aligned}$$

Now

$$\lim_{a\to 0} a^{-1}(e^{ina} - 1) - in = 0 \qquad (n \in Z),$$

and (see the proof of 8.5.2)

$$|a^{-1}(e^{ina} - 1)| \leqslant \text{const}\, |n|.$$

These facts, combined with (2) and (8.5.9), show that

$$\|a^{-1}(T_{-a}f - f) - g\|_2^2 \to 0$$

as $a \to 0$. Thus (2) implies (3).

Since $\omega_2 f(a) = \|T_{-a}f - f\|_2$, it is evident that (3) implies (4).

The first inequality in 8.5.2 shows that (4) implies (2).

We now know that

$$(1) \Leftrightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (2),$$

so that the proof is complete.

8.5.5. **Remarks.** (1) By using the uniform boundedness principles (Appendix B.2.1 and B.2.2) it could be shown that a fifth equivalent condition is obtained on apparently weakening (3) to the demand that the said limit exists weakly in $\mathbf{L}^2$, that is, (see Appendix C.1), that

$$\lim_{a\to 0} \int a^{-1}(T_{-a}f - f) \cdot h\, dx$$

exists finitely for each $h \in \mathbf{L}^2$.

(2) Yet another equivalent condition is the existence of $g \in \mathbf{L}^2$ such that

$$\lim_{a\to 0} \int a^{-1}(T_{-a}f - f) \cdot u\, dx = \int gu\, dx$$

for each $u \in \mathbf{C}^\infty$; see Exercise 12.24.

8.6 Concerning Subsequences of $s_N f$

As will be described in 10.4.5, Carleson announced in 1966 a proof of the almost everywhere pointwise convergence of $s_N f$ for each $f \in \mathbf{L}^2$. Despite this, we shall here illustrate in detail the use of the Parseval formula in proving a much earlier result due to Kolmogorov (1925). This theorem of Kolmogorov, which asserts the existence of specifiable, rapidly increasing sequences $(N_k)_{k=1}^{\infty}$ of positive integers having the property that $\lim_{k\to\infty} s_{N_k} f = f$ pointwise almost everywhere for each $f \in \mathbf{L}^2$, prompted a number of similar investigations which are not superseded by Carleson's theorems; see 10.4.6.

We begin with a definition. A sequence $(N_k)_{k=1}^{\infty}$ of positive integers is termed a *Hadamard sequence* if

$$q \equiv \inf_k \frac{N_{k+1}}{N_k} > 1;$$

such sequences have appeared already in Exercises 5.6 and 6.13, and they will be encountered again in Chapter 15.

8.6.1. (Kolmogorov) Let $(N_k)_{k=1}^{\infty}$ be a Hadamard sequence of positive integers. If $f \in \mathbf{L}^2$, we have $s_{N_k} f(x) \to f(x)$ for almost all x.

Proof. Recall from 6.4.4 that $\sigma_N f \to f$ almost everywhere. Observe also that if $(g_r)_{r=1}^{\infty}$ is any sequence of nonnegative integrable functions, and if

$$\sum_{r=1}^{\infty} \int g_r < \infty,$$

then ([W], Theorem 4.1e) $\sum_{r=1}^{\infty} g_r$ is integrable, hence finite-valued almost everywhere, and so $g_r \to 0$ almost everywhere. In view of these two remarks, it will be sufficient to show that

$$S = \sum_{k=1}^{\infty} \|\sigma_{N_k} f - s_{N_k} f\|_2^2 < \infty. \tag{8.6.1}$$

Now, by (8.2.2),

$$\|\sigma_{N_k} f - s_{N_k} f\|_2^2 = (N_k + 1)^{-2} \sum_{|n| \leqslant N_k} n^2 |\hat{f}(n)|^2,$$

so that

$$\begin{aligned}\sum_{k \leqslant p} \|\sigma_{N_k} f - s_{N_k} f\|_2^2 &\leqslant \sum_{k \leqslant p} N_k^{-2} \sum_{|n| \leqslant N_k} n^2 |\hat{f}(n)|^2 \\ &\equiv S_p,\end{aligned} \tag{8.6.2}$$

say. Putting $u_n = n^2 |\hat{f}(n)|^2$, we may write

$$S_p = \sum_{j=1}^{p} \sum_{N_{j-1} < |k| \leqslant N_j} u_k \sum_{m=j}^{p} N_m^{-2}$$

with the understanding that N_0 be read as 0. On the other hand, since (N_k) is a Hadamard sequence, it is easily seen that

$$\sum_{m \geqslant j} N_m^{-2} \leqslant C \cdot N_j^{-2},$$

where C is a number depending only upon q. Hence

$$S_p \leqslant C \cdot \sum_{j=1}^{p} N_j^{-2} \sum_{N_{j-1} < |k| \leqslant N_j} u_k. \tag{8.6.3}$$

Moreover,

$$\sum_{N_{j-1} < |k| \leqslant N_j} u_k \leqslant N_j^2 \sum_{N_{j-1} < |k| \leqslant N_j} |\hat{f}(k)|^2.$$

So, by (8.6.3) and (8.2.2),

$$S_p \leqslant C \cdot \sum_{|k| \leqslant N_p} |f(k)|^2 \leqslant C \cdot \|f\|_2^2.$$

From this estimate and the obvious relation

$$S = \sup_p S_p,$$

(8.6.1) follows and the proof is complete. More general results of similar nature are proved in [Z_2], Chapter XV; see especially p. 231.

8.7 A(Z) Once Again

In this section we shall use 8.2.1 and 8.3.1 in a manner due to Hirschman so as to obtain a sufficient condition for a complex-valued function ϕ on Z to belong to A(Z); see 2.3.10(4). The result actually given by Hirschman ([1], Lemma 3a) appears as 8.7.3 and will be used in Section 16.4.

For $k \in Z$, $T_k\phi$ will denote the translated function $n \to \phi(n - k)$ on Z (compare the notation introduced in 2.2.1 for functions defined on groups).

8.7.1. Suppose that $\phi \in \mathbf{c}_0(Z)$ and that

$$S \equiv \sum_{m=0}^{\infty} 2^{-\frac{1}{2}m} \|T_{(2^m)}\phi - \phi\|_2 < \infty. \tag{8.7.1}$$

Then there exists a function $f \in \mathbf{L}^1$ such that $\hat{f} = \phi$ and

$$\|f\|_1 \leqslant \tfrac{1}{2}S. \tag{8.7.2}$$

Proof. Let m be a nonnegative integer and define $k = 2^m$. According to 8.3.1, (8.7.1) shows first that there exists a function $f_k \in \mathbf{L}^2$ such that $\hat{f}_k = T_k\phi - \phi$ and

$$\lim_{N \to \infty} s_{N,k} = f_k$$

in $\mathbf{L}^2$, where

$$s_{N,k} = \sum_{|n| \leq N} \{\phi(n - k) - \phi(n)\} e_n.$$

So (see the proof of Theorem 4.5a of [W]) one can choose sequences $(N_r^{(k)})_{r=1}^{\infty}$ ($k = 2, 4, \cdots$) of integers such that $(N_r^{(k+1)})_{r=1}^{\infty}$ is a subsequence of $(N_r^{(k)})_{r=1}^{\infty}$ and

$$\lim_{r \to \infty} s_{N_r^{(k)},k} = f_k$$

pointwise almost everywhere. On taking the "diagonal subsequence" $(N_r)_{r=1}^{\infty} = (N_r^{(r)})_{r=1}^{\infty}$, one will have

$$f_k(x) = \lim_{r\to\infty} \sum_{|n|\leqslant N_r} \{\phi(n-k) - \phi(n)\}e^{inx}$$

for almost all x and all $k = 1, 2, \cdots$. Since ϕ is known to belong to $\mathbf{c}_0(Z)$, this last relation may be written

$$f_k(x) = \lim_{r\to\infty} (e^{ikx} - 1) \sum_{|n|\leqslant N_r} \phi(n)e^{inx}$$

for almost all x and all $k = 2, 4, \cdots$. This relation makes it plain that there exists a measurable periodic function f such that

$$f_k(x) = (e^{ikx} - 1)f(x) \qquad \text{a.e.}, \tag{8.7.3}$$

and 8.2.1 shows that

$$\frac{1}{2\pi}\int |(e^{ikx} - 1)f(x)|^2\,dx = \|T_k\phi - \phi\|_2^{\,2}. \tag{8.7.4}$$

Let I_m denote the interval defined by the inequalities

$$\frac{2\pi}{4k} \leqslant |x| < \frac{2\pi}{2k},$$

so that $\bigcup_{m=0}^{\infty} I_m = (-\pi, \pi)$. For $x \in I_m$,

$$|e^{ikx} - 1| \geqslant 2^{1/2},$$

so that (8.7.4) entails that

$$\int_{I_m} |f(x)|^2\,dx \leqslant \pi\|T_k\phi - \phi\|_2^{\,2}. \tag{8.7.5}$$

From the Cauchy-Schwarz inequality and (8.7.5) it may be inferred that

$$\int_{I_m} |f(x)|\,dx \leqslant \pi 2^{-1/2 m}\|T_k\phi - \phi\|_2,$$

and, summing over m, (8.7.1) yields

$$\|f\|_1 \leqslant \tfrac{1}{2}S,$$

which is (8.7.2). In particular, $f \in \mathbf{L}^1$ as a consequence of (8.7.1).

Finally, for all $n \in Z$,

$$\phi(n-k) - \phi(n) = \frac{1}{2\pi}\int f_k(x)e^{-inx}\,dx,$$

which, by (8.7.3), is equal to $\hat{f}(n-k) - \hat{f}(n)$. If we let $m \to \infty$ and use 2.3.8 and the hypothesis that $\phi \in \mathbf{c}_0(Z)$, it appears that $\phi = \hat{f}$. The proof is complete.

8.7.2. The statement of Hirschman's lemma is made in terms of the expression

$$V_\beta(\phi) = \sup\{\sum_{k=0}^{r-1} |\phi(n_{k+1}) - \phi(n_k)|^\beta\}^{1/\beta},$$

where β is a positive real number and the supremum is taken with respect to all strictly increasing finite sequences $(n_k)_{k=0}^r$ of integers. The supremum may, of course, be ∞.

It is evident that $V_1(\phi) = \|\Delta\phi\|_1$, where the difference operator Δ is defined as in 7.1.2. Also, if ϕ is bounded,

$$V_\beta(\phi) \leqslant (2\|\phi\|_\infty)^{1-1/\beta}\|\Delta\phi\|_1{}^{1/\beta}$$

for any $\beta > 1$.

Less evident is Hirschman's estimate which follows.

8.7.3. Suppose that

$$|\phi(n)| \leqslant K(1 + |n|)^{-2/\alpha} \qquad (n \in Z) \tag{8.7.6}$$

for some $\alpha > 2$, and that $V_\beta(\phi) < \infty$ for some β satisfying $1 \leqslant \beta < 2$. Then, for $k = 1, 2, 3, \cdots$,

$$\|T_k\phi - \phi\|_2 \leqslant A_{\alpha,\beta}V_\beta(\phi)^{\beta(\alpha-2)/2(\alpha-\beta)}K^{\alpha(2-\beta)/2(\alpha-\beta)}k^{(\alpha-2)/2(\alpha-\beta)}. \tag{8.7.7}$$

In particular, there exists an $f \in \mathbf{L}^1$ such that $\hat{f} = \phi$ and

$$\|f\|_1 \leqslant A'_{\alpha,\beta}V_\beta(\phi)^{\beta(\alpha-2)/2(\alpha-\beta)}K^{\alpha(2-\beta)/2(\alpha-\beta)}.$$

Proof. Hölder's inequality for sums gives

$$\|T_k\phi - \phi\|_2{}^2 \leqslant \{\sum_{n\in Z} |\phi(n) - \phi(n+k)|^\beta\}^{(\alpha-2)/(\alpha-\beta)}\cdot \{\sum_{n\in Z} |\phi(n) - \phi(n+k)|^\alpha\}^{(2-\beta)/(\alpha-\beta)}. \tag{8.7.8}$$

Beside this,

$$\sum_{n\in Z} |\phi(n) - \phi(n+k)|^\beta = \sum_{j=0}^{k-1}\sum_{m\in Z} |\phi(mk+j) - \phi(mk+j-k)|^\beta \leqslant k\cdot V_\beta(\phi)^\beta, \tag{8.7.9}$$

and, by (8.7.6),

$$\sum_{n\in Z} |\phi(n) - \phi(n+k)|^\alpha \leqslant \sum_{n\in Z} A_\alpha\{|\phi(n)|^\alpha + |\phi(n+k)|^\alpha\} \leqslant 2A_\alpha K^\alpha \cdot \sum_{n\in Z}(1+|n|)^{-2}. \tag{8.7.10}$$

The estimate (8.7.7) follows at once on combining (8.7.8), (8.7.9), and (8.7.10).

The final statement is a consequence of (8.7.7) and 8.7.1.

EXERCISES

8.1. Let α be any permutation of Z. Define Tf for $f \in \mathbf{L}^2$ by

$$Tf = \sum_{n \in Z} \hat{f}(\alpha(n))e_n.$$

Verify that T is an isometric isomorphism of the convolution algebra $\mathbf{L}^2$ onto itself.

Construct permutations α of Z such that T is not the restriction to $\mathbf{L}^2$ of any homomorphism of the convolution algebra $\mathbf{L}^1$ into itself.

Hint: For the second part refer to conditions (3) and (4) of 4.2.6, and show how to construct permutations α of Z with the property that, for any integer $q > 0$, the relation,

$$\alpha(n + q) + \alpha(n - q) = 2\alpha(n)$$

is false for infinitely many $n \in Z$.

8.2. Show that an element f of $\mathbf{L}^2$ is a prime element of $\mathbf{L}^2$ if and only if $\sum_{n \in Z} |\hat{f}(n)| = \infty$. (See 8.4.)

8.3. Show that any nonprime element of $\mathbf{L}^2$, say f, is expressible as the product of two prime elements of $\mathbf{L}^2$.

Hint: Reduce the problem to showing that, if $\sum |c_n| < \infty$, then one can write $c_n = a_n b_n$, where $\sum |a_n|^2 < \infty$, $\sum |b_n|^2 < \infty$, $\sum |a_n| = \sum |b_n| = \infty$.

8.4. Suppose that $f \in \mathbf{L}^1$ and that $\omega_1 f(a) = O(|a|^\alpha)$ as $a \to 0$ for some $\alpha > 0$. Show that $f^{*N} \in \mathbf{L}^2$ for any integer N for which $N\alpha > \frac{1}{2}$. (Compare this result with that of Exercise 5.2.)

8.5. Let $k \in \mathbf{L}^1$ and let $Tf = k * f$. Show that T is a continuous endomorphism of $\mathbf{L}^2$ such that

$$\|T\| \equiv \sup \{\|Tf\|_2 : f \in \mathbf{L}^2, \|f\|_2 \leqslant 1\} = \|\hat{k}\|_\infty;$$

$\|\hat{k}\|_\infty$ is defined as in 2.2.5, $\hat{k}$ being a function on Z.

What are the eigenvalues and eigenvectors of T? Can it ever happen that $T(\mathbf{L}^2) = \mathbf{L}^2$? (give reasons for your answer). Under what conditions is $T(\mathbf{L}^2)$ everywhere dense in $\mathbf{L}^2$?

8.6. The notation being as in Exercise 8.5, consider the same questions when $T - \lambda I$ replaces T, λ being a complex number and I the identity endomorphism of $\mathbf{L}^2$.

8.7. Let $\alpha = (\alpha_n)_{n \in Z}$ be a sequence such that $\alpha \in \ell^p$ for some $p > 0$ (the notation being as in 2.2.5). Prove that

$$\lim_{k \to \infty} \|\alpha^k\|_1^{1/k} = \|\alpha\|_\infty,$$

where α^k is the pointwise product of k sequences each identical with α.

8.8. *The spectral radius formula for* $\mathbf{L}^2$ (see Exercise 3.12 and Subsection 11.4.14). Show that, if $f \in \mathbf{L}^2$, then

$$\lim_{k \to \infty} \|f^{*k}\|_2^{1/k} = \|\hat{f}\|_\infty .$$

Hint: Use Exercise 8.7 in conjunction with the Parseval formula.

8.9. Let ω be a positive function on Z such that $\lim_{|n| \to \infty} \omega(n) = \infty$. Prove the existence of continuous functions f satisfying (8.3.3) and (8.3.4).

Hint: Consider $\sum_{k=1}^{\infty} \omega(n_k)^{-1/2} e^{in_k x}$, where the integers n_k increase sufficiently rapidly.

8.10. The wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},$$

with boundary conditions $u(0, t) = u(2\pi, t)$ and initial conditions

$$u(x, 0) = f(x), \qquad \frac{\partial u}{\partial t}(x, 0) = g(x),$$

is to be considered in the following interpretation:

(1) for each $t > 0$, $u_t \in \mathbf{C}^1$, $u_t(0) = u_t(2\pi) = 0$;

(2) Du_t is absolutely continuous and $D^2 u_t \in \mathbf{L}^2$ for each $t > 0$;

(3) $\dot{u}_t \equiv \mathbf{L}^2 - \lim_{\varepsilon \to 0} \varepsilon^{-1}(u_{t+\varepsilon} - u_t)$ and
$\ddot{u}_t \equiv \mathbf{L}^2 - \lim_{\varepsilon \to 0} \varepsilon^{-1}(\dot{u}_{t+\varepsilon} - \dot{u}_t)$ exist for each $t > 0$;

(4) $\ddot{u}_t = D^2 u_t$ as elements of $\mathbf{L}^2$ for each $t > 0$;

(5) $\mathbf{L}^2 - \lim_{t \to +0} u_t = f$ and $\mathbf{L}^2 - \lim_{t \to +0} \dot{u}_t = g$, f and g being given elements of $\mathbf{L}^2$.

Give a rigorous discussion of (a) conditions under which a solution exists, and (b) the uniqueness of this solution.

8.11. Writing $s_N(x) = \sum_{|n| \leqslant N} c_n e^{inx}$, show that if $c_n = O(|n|^{-1})$ as $|n| \to \infty$, then $(s_N(x))_{N=1}^{\infty}$ converges almost everywhere and that the limit function belongs to $\mathbf{L}^2$.

8.12. Let $(\rho_n)_{n \in Z}$ be a sequence of nonnegative numbers such that

$$\sum_{n \in Z} \rho_n = \infty \tag{1}$$

and

$$\sum_{n \in Z} \rho_n^2 < \infty . \tag{2}$$

Show that there exists a function $f \in \mathbf{L}^2$ such that

(i) $|\hat{f}(n)| = o(\rho_n)$ for $n \in Z$, $|n| \to \infty$;

(ii) f is essentially unbounded on every nondegenerate interval.

Hints: Write $N = \{n \in Z : \rho_n \neq 0\}$. Consider the linear space $\mathbf{E}$ of functions $f \in \mathbf{L}^2$ such that $\hat{f}(Z \backslash N) = \{0\}$ and $\hat{f}(n) = o(\rho_n)$ as $|n| \to \infty$.

Introduce into $\mathbf{E}$ the norm

$$\|f\|_{\mathbf{E}} = \sup_{n \in N} \rho_n^{-1}|\hat{f}(n)|$$

and verify that $\mathbf{E}$ is thereby made into a Banach space.

Assuming the assertion to be false, apply Appendix B.2.1 to show that

$$\|f\|_\infty \leqslant \text{const } \|f\|_{\mathbf{E}}. \tag{3}$$

Take any $g \in \mathbf{L}^1$ and consider the linear functional defined on $\mathbf{E}$ by

$$\Lambda(f) = \frac{1}{2\pi} \int f(-x) g(x)\, dx.$$

Use (3) and Appendix B.5.1 to show that there exists $\alpha \in \ell^1(Z)$ such that

$$\Lambda(f) = \sum_{n \in N} \alpha(n) \rho_n^{-1} \hat{f}(n) \qquad (f \in \mathbf{E}); \tag{4}$$

in doing this you will need to verify that each continuous linear functional on $\mathbf{c}_0(Z)$ (see 2.2.5) is expressible as

$$\phi \to \sum_{n \in Z} \alpha(n) \phi(n)$$

for some $\alpha \in \ell^1(Z)$. Deduce a contradiction of (1) from (4), using Exercise 3.14 on the way.

Remarks. The condition (2) is not essential: it is included merely to shorten the proof somewhat. One can in any case show that f may be chosen to satisfy (i) and (ii) and to belong to $\mathbf{L}^p$ for every $p < \infty$; for details, see Edwards [3].

8.13. Suppose that $f \in \mathbf{L}^2$. By applying the Parseval formula to the function

$$x \to f\left(x + \frac{k\pi}{r}\right) - f\left(x + \frac{(k-1)\pi}{r}\right),$$

show that

$$8r \sum_{n \in Z} |\hat{f}(n)|^2 \sin^2\left(\frac{n\pi}{2r}\right) = \frac{1}{2\pi} \int \sum_{k=1}^{2r} \left| f\left(x + \frac{k\pi}{r}\right) - f\left(x + \frac{(k-1)\pi}{r}\right)\right|^2 dx$$
$$\leqslant \Omega_\infty f\left(\frac{\pi}{r}\right) \cdot V(f), \tag{1}$$

where $\Omega_\infty f(\delta) = \sup\{\|T_a f - f\|_\infty : |a| \leqslant \delta\}$. Deduce that if f is of bounded variation, then it is continuous if and only if

$$\lim_{r \to \infty} r \sum_{n \in Z} |\hat{f}(n)|^2 \sin^2\left(\frac{n\pi}{2r}\right) = 0. \tag{2}$$

Hint: Observe that if $|f(\xi + 0) - f(\xi - 0)| = d > 0$, then, for all large r and almost all x, at least one term in the sum appearing on the right-hand side of (1) contributes an amount not less than $(d/3)^2$.

Remarks. It can be shown without too much difficulty that (2) is equivalent to

$$\lim_{N\to\infty} N^{-1} \sum_{|n|\leq N} |n\hat{f}(n)| = 0; \tag{3}$$

see, for example, [Ba$_1$], pp. 214–215. In this final form the criterion of continuity for functions of bounded variation is due to Wiener. It follows in particular that any function f of bounded variation, for which $\hat{f}(n) = o(1/|n|)$, is continuous. Compare with the remarks following 2.3.6. See also MR **38** # 487; **44** # 7220.

8.14. (1) Suppose that f, g, $h \in \mathbf{L}^2$. Prove the following extension of Appolonius' identity:

$$\|h - f\|^2 + \|h - g\|^2 = \tfrac{1}{2}\|f - g\|^2 + 2\|h - \tfrac{1}{2}(f + g)\|^2, \tag{1}$$

where we have written $\|\cdot\|$ in place of $\|\cdot\|_2$.

(2) Let M be a closed convex subset of $\mathbf{L}^2$. (Convexity of M means that $\alpha f + (1 - \alpha)g \in M$ whenever $f, g \in M$ and $0 < \alpha < 1$.) Show that there exists a unique $f_0 \in M$ such that

$$\|f_0\| = \inf\{\|f\| : f \in M\}.$$

Hints: The proof of (1) rests on direct calculation.

For (2), choose $f_n \in M$ $(n = 1, 2, \cdots)$ so that

$$\|f_n\| \equiv \delta_n \downarrow \delta \equiv \inf\{\|f\| : f \in M\}.$$

Apply (1) to conclude that the sequence (f_n) is Cauchy in $\mathbf{L}^2$. Consider the limit f_0 of this sequence. Prove uniqueness by another application of (1).

Remark. The result in (2) may be termed the "projection principle"; it has numerous interesting applications to problems in concrete analysis, for a discussion of some of which see Exercise 8.15 and [E], pp. 99 ff.

8.15. Suppose that $f \in \mathbf{H}^2$ (see Exercise 3.9) does not vanish almost everywhere (that is, that $\|f\| \equiv \|f\|_2 > 0$). Prove that $f(x) \neq 0$ for almost all x.

Hints: Assume (without loss of generality) that $\hat{f}(0) \neq 0$. Let M be the smallest closed convex set in $\mathbf{L}^2$ containing all functions $f \cdot t$, where t denotes a trigonometric polynomial belonging to $\mathbf{H}^2$ and $\hat{t}(0) = 1$. Take $g \in M$, minimizing the distance from 0 of elements of M; see Exercise 8.14. By comparing $\|g\|$ with $\|g + \lambda e^{inx}g\|$, where λ is any scalar and $n = 1, 2, \cdots$, deduce that $|g|$ is equal almost everywhere to some constant, c. Check that c cannot be 0. Observe finally that, if f vanished on a set S of positive measure, then the same would be true of g.

Remark. It can be shown that in fact

$$\frac{1}{2\pi}\int \log|f|dx > -\infty.$$

For this, as well as many other related results, see [Hel] and [Ho]. Compare also Exercise 15.17.

8.16. The so-called *isoperimetric problem* is that of determining, among all closed plane curves with given perimeter (here conveniently chosen to have the value 2π), that (or those) enclosing the greatest possible area.

Assuming sufficient smoothness on the part of the admissible curves, and expressing these curves in terms of arc length s as parameter, the problem can be formulated analytically as follows: among all pairs (f, g) of real-valued, absolutely continuous periodic functions f and g such that Df, $Dg \in \mathbf{L}^2$ and

$$(Df)^2 + (Dg)^2 = 1 \text{ a.e.},$$

determine those for which

$$A = \tfrac{1}{2} \int_0^{2\pi} (fDg - gDf)\, ds$$

is a maximum.

Use the Parseval formula to solve this version of the problem.

Notes: For a discussion of the isoperimetric problem for plane polygons, see [Ka], pp. 27–34. Chapter VII of [HLP] contains a brief account of the classical approach to this and some similar problems by the methods of the calculus of variations. Two other problems of this sort that have had a profound influence on mathematics are the Dirichlet problem and the Plateau problem; see [CH] and the references cited there, [E], Section 5.13, and [Am]. Variational methods are applicable to the study of many linear functional equations, in which connection they often suffice to yield useful information about the eigenvalue distributions in cases where the equation is not conveniently soluble explicitly; see, for example, [So], Chapter II. The existence theorems appropriate to many variational problems are crucial and difficult: they represent the concrete origins of, and the initial motivation for, the modern study of compactness in function spaces and of numerous other functional analytic techniques. Exercise 8.14 above includes a simple variational principle, and Exercise 8.15 illustrates a concrete application of this.

8.17. The reader is reminded that $\int F\, dx$ is defined (possibly ∞) for any nonnegative measurable function F on T to be $\lim_{k\to\infty} \int \inf(F, k)\, dx$; and that $\sum_{n\in Z} c_n$ is defined (possibly ∞) for any nonnegative function $(c_n)_{n\in Z}$ on Z to be $\lim_{k\to\infty} \sum_{|n|\leqslant k} c_n$.

Verify that, with these definitions, the Parseval formula

$$\frac{1}{2\pi} \int |f|^2\, dx = \sum_{n\in Z} |\hat{f}(n)|^2$$

holds for any $f \in \mathbf{L}^1$.

8.18. Suppose that $f \in \mathbf{L}^1$ and that S denotes the set of real numbers a such that $T_a f - f \in \mathbf{L}^2$. Show that S is a subgroup of R.

Define

$$N(a) = \{\frac{1}{2\pi}\int |T_a f - f|^2 \, dx\}^{1/2}$$

for $a \in R$, where (compare Exercise 8.17) $\infty^{1/2}$ is taken to mean ∞. Prove that N is lower semicontinuous (see Appendix A.4).

By using Exercise 3.16 and Appendix A.5 show that, if S *either* has positive interior measure *or* is nonmeager, then there exist positive numbers δ and Δ such that

$$\sum_{n \in Z} |\hat{f}(n)|^2 \sin^2 \tfrac{1}{2} na \leqslant \Delta$$

whenever $|a| \leqslant \delta$. Conclude that $f \in \mathbf{L}^2$.

Note: It is possible to reformulate the result so as to apply when f is assumed merely to be a distribution (see Chapter 12). Also, it is possible to replace $\mathbf{L}^2$ throughout by other spaces $\mathbf{L}^p$; the proof then becomes somewhat more complicated. The first consideration of this type of result appears to be due to de Bruijn [1], [2]. See also the work of Kemperman (Math. Rev. **20**, 1123), Carroll (Math. Rev. **28** #5137, **30** #1126, 2101), and a paper by the present writer entitled "Differences of functions and measures" (to appear J. Austr. Math. Soc.).

CHAPTER 9

Positive Definite Functions and Bochner's Theorem

9.1 Mise-en-Scène

Continuous (and not necessarily periodic) positive definite functions of a real variable were seemingly first studied by Bochner who, by using the existing theory of Fourier integrals, established for them a fundamental representation theorem now known by his name and which is the analogue for the group R of 9.2.8. These positive definite functions were not seen in their true perspective until some ten or fifteen years had elapsed. Then, as a result of the birth and growth of the theory of commutative Banach algebras and the applications of this theory to harmonic analysis on locally compact Abelian groups (see 11.4.18(3)), the central position of the Bochner theorem came to be appreciated. The developments in this direction were due largely to the Russian mathematicians Gelfand and Raikov, who enlarged still more the role played by positive definite functions by noticing their intimate relationship with the theory of representations of (not necessarily Abelian) locally compact groups. A similar path was hewn, independently, almost simultaneously, and from a slightly different point of view, by the French mathematicians H. Cartan and Godement; see [B], pp. 220 ff. It is now true to say that a considerable portion of our function-theoretical knowledge of locally compact groups rests upon a study of positive definite functions on such groups.

It turns out that continuity plus positive definiteness of an integrable function f is one of the very few known conditions which (a) ensures that the transform $\hat{f}$ is integrable, and (b) is expressible solely in terms of the topological group structure. For the group T, the result is contained in 9.2.8; for the group Z, see 12.13.3. Because of this, the positive definite concept is useful as a primitive tool in the development of harmonic analysis, rather than as an afterthought to the latter (which is how it appeared initially).

It is also true to say that positive definiteness of an integrable function f is equivalent to the demand that its transform $\hat{f}$ be nonnegative. (For the group T, see 9.2.4; the case of the group Z is covered by 12.13.2.) Granted the importance of the Fourier transformation, this fact adds to the significance of the positive definite concept.

In this brief chapter we shall deal rapidly with a part of the theory of positive definite functions on the group T, and observe how it might be used as an alternative approach to the $\mathbf{L}^2$ theory and the Parseval formula (see Chapter 8). The present situation, dealing as it does with a compact Abelian group, is technically much simpler than that involving a general locally compact group.

Another special case of the general setting is the dual one, in which the underlying group is Z rather than T. There is a perfect Bochner theorem applying to his case, but its discussion must be deferred until Section 12.13.

For remarks concerning the theory in a more general setting, see Section 9.4.

9.2 Toward the Bochner Theorem

We begin by framing our definition of positive definite functions. (Bochner's original definition was different; see 9.2.7.)

9.2.1. A function $f \in \mathbf{L}^1$ is said to be *positive definite* if and only if

$$f * u * u^*(0) = \frac{1}{4\pi^2} \int\int f(x - y)\overline{u(x)}u(y)\, dx\, dy \geqslant 0 \qquad (9.2.1)$$

for each continuous function u. (The reader is reminded that u^* denotes the function $x \to \overline{u(-x)}$; see the start of Section 2.3.)

9.2.2. The set of positive definite functions does not, of course, form a linear space. It is true, however, that any linear combination with real nonnegative coefficients of positive definite functions is again positive definite.

It is easily verified that any continuous character is positive definite; the same is therefore true of any trigonometric polynomial with nonnegative coefficients.

Further examples appear in 9.2.5.

9.2.3. A function $f \in \mathbf{L}^1$ is positive definite if and only if (9.2.1) holds for each trigonometric polynomial u.

Proof. The necessity is plain. Suppose, conversely, that (9.2.1) holds for each trigonometric polynomial u. If u is any continuous function, choose (as is possible by 6.1.1) a sequence $(u_n)_{n=1}^{\infty}$ of trigonometric polynomials converging uniformly to u. Then, by 3.1.4 and 3.1.6, $f * u_n * u_n^* \to f * u * u^*$ uniformly. Since, by hypothesis, $f * u_n * u_n^*(0) \geqslant 0$, it follows that $f * u * u^*(0) \geqslant 0$, showing that the hypothesis of 9.2.1 is fulfilled.

9.2.4. A function $f \in \mathbf{L}^1$ is positive definite if and only if $\hat{f} \geqslant 0$, so that in particular $f = f^*$ almost everywhere.

Proof. If f is positive definite, and if in (9.2.1) we take $u = e_n$, where $e_n(x) = e^{inx}$, we obtain exactly the inequality $\hat{f}(n) \geqslant 0$. If, on the other hand, $\hat{f} \geqslant 0$, and if

$$u = \sum c_n e_n$$

is a trigonometric polynomial (the range of summation being thus a finite subset of Z), then

$$\begin{aligned} f * u * u^*(0) &= \sum_{m,n} c_m \bar{c}_n \cdot f * e_m * e_n^*(0) \\ &= \sum_{m,n} c_m \bar{c}_n \cdot \delta_{mn} \hat{f}(n) \\ &= \sum_n |c_n|^2 \hat{f}(n) \geqslant 0, \end{aligned}$$

and 9.2.3 shows that f is positive definite.

9.2.5. By using the criterion 9.2.4 it is very simple to verify that for any $g \in \mathbf{L}^2$ the continuous function

$$f(x) = g * g^*(x) = \frac{1}{2\pi} \int g(x + y)\overline{g(y)}\, dy$$

is positive definite. The same conclusion also follows readily from 9.2.6 to follow. For the converse, see 9.2.10.

9.2.6. A continuous function f is positive definite if and only if

$$\sum_{m,n=1}^{k} f(x_m - x_n)\bar{z}_m z_n \geqslant 0 \tag{9.2.2}$$

holds for any finite sequence $(x_n)_{n=1}^k$ of real numbers and any finite sequence $(z_n)_{n=1}^k$ of complex numbers.

Proof. We leave it to the reader to show that (9.2.2) implies (9.2.1), remarking merely that it suffices to approximate the integral appearing in (9.2.1) by Riemann sums

$$\sum_{m,n=1}^{k} f(x_m - x_n)\overline{u(x_m)}u(x_n)(x_m - x_{m-1})(x_n - x_{n-1}).$$

It will thus be seen that (9.2.2) implies that f is positive definite.

Conversely suppose that f is continuous and positive definite. By 9.2.4, $\hat{f} \geqslant 0$. Hence $\sigma_N f$ is a trigonometric polynomial with nonnegative coefficients, and it is immediately verifiable that (9.2.2) holds when f is replaced by $\sigma_N f$. It now suffices to let $N \to \infty$ and use 6.1.1, in order to be led to (9.2.2) itself.

9.2.7. In 9.2.6 we have the exact analogue of Bochner's original definition of continuous positive definite functions on the additive group R of real numbers. It is not suitable for the study of discontinuous functions, which is why we have substituted 9.2.1.

We can now establish the form of Bochner's theorem appropriate to the group T.

9.2.8. (Bochner) Suppose that $f \in \mathbf{L}^1$ is positive definite, and that f is essentially bounded on some neighborhood of the origin. Then $\sum_{n\in Z}\hat{f}(n) < \infty$ and

$$f(x) = \sum_{n\in Z} \hat{f}(n)e^{inx} \qquad \text{a.e.}, \tag{9.2.3}$$

so that f is equal almost everywhere to a continuous positive definite function. If f is continuous, equality holds everywhere in (9.2.3).

Proof. In the first place we have

$$\sigma_N f(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)F_N(x)\,dx.$$

Now, if we suppose that $|f(x)| \leqslant m$ $(<\infty)$ for almost all x satisfying $|x| \leqslant a$ for some $a > 0$, it follows that

$$|\sigma_N f(0)| \leqslant \frac{m}{2\pi}\int_{-a}^{a} F_N(x)\,dx + \frac{1}{2\pi}\int_{a\leqslant|x|\leqslant\pi} |f(x)|\,dx\cdot(N+1)^{-1}\operatorname{cosec}^2 \tfrac{1}{2}a.$$

This shows that

$$\sup_N |\sigma_N f(0)| < \infty. \tag{9.2.4}$$

On the other hand,

$$\sigma_N f(0) = \sum_{|n|\leqslant N} \hat{f}(n)\left(1 - \frac{|n|}{N+1}\right). \tag{9.2.5}$$

This, together with (9.2.4) and the fact that $\hat{f} \geqslant 0$, shows that $\sum_{n\in Z}\hat{f}(n) < \infty$. The equality (9.2.3) is now a consequence of 2.4.2. Further, if f is known to be continuous, we may appeal to 2.4.3 to infer that (9.2.3) holds everywhere.

9.2.9. We notice two corollaries of 9.2.8 and its proof.

(1) If $f \in \mathbf{L}^1$ is positive definite, and if $|f(x)| \leqslant m$ almost everywhere on some neighborhood of 0, then the same inequality holds almost everywhere.

Proof. The inequality preceding (9.2.4) shows indeed that

$$\limsup_{N\to\infty} |\sigma_N f(0)| \leqslant m.$$

This, combined with 9.2.4 and (9.2.5), leads at once to

$$\sum_{n\in Z} \hat{f}(n) = \lim_{N\to\infty} \sum_{|n|\leqslant N} \hat{f}(n)\left(1 - \frac{|n|}{N+1}\right) \leqslant m;$$

then (9.2.3) and 9.2.4 show that $|f(x)| \leqslant m$ for almost all x.

(2) If f is a continuous positive definite function, then $|f(x)| \leqslant f(0)$ for all x.

Proof. This follows from (9.2.3), which now holds for all x, and 9.2.4. (It is also easily deducible from 9.2.6.)

9.2.10. The converse of 9.2.5 can now be established with ease: any continuous positive definite function f is expressible in the form $f = g * g^*$ for some $g \in \mathbf{L}^2$.

In fact, by 9.2.4 and 9.2.8, $\hat{f} \geqslant 0$ and

$$\sum_{n \in Z} \hat{f}(n) < \infty.$$

From 8.3.1 it therefore follows that a function $g \in \mathbf{L}^2$ exists such that $\hat{g} = \hat{f}^{1/2}$, in which case 2.3.1, (3.1.5), and 2.4.1 combine to show that $f = g * g^*$.

The reader should compare the conjunction of 9.2.5 and its converse (just established) with the criterion of M. Riesz mentioned in 10.6.2(4), and applying to functions with absolutely convergent Fourier series.

9.3 An Alternative Proof of the Parseval Formula

One can combine 9.2.5 and 9.2.8 so as to yield very rapidly a proof of the Parseval formula, on which all the $\mathbf{L}^2$ theory of Chapter 8 may then be founded.

Thus, if $g \in \mathbf{L}^2$, and if we apply to the continuous positive definite function $f = g * g^*$ the formula (9.2.3) and take $x = 0$, it appears that

$$f(0) = \sum_{n \in Z} \hat{f}(n). \tag{9.3.1}$$

Since

$$f(0) = \frac{1}{2\pi} \int |g(x)|^2 \, dx$$

and

$$\hat{f}(n) = |\hat{g}(n)|^2,$$

(9.3.1) reads

$$\frac{1}{2\pi} \int |g(x)|^2 \, dx = \sum_{n \in Z} |\hat{g}(n)|^2,$$

which is the Parseval formula.

In many developments of harmonic analysis on locally compact Abelian groups, the above procedure is precisely that by which the Parseval formula and the $\mathbf{L}^2$-theory are approached. See [B], p. 235 ff.

9.4 Other Versions of the Bochner Theorem

We have mentioned in Section 9.1 that the concept of positive definite function extends altogether naturally to quite general groups and that a great deal of work has been done in this direction. Here we insert a few bibliographical indications for the benefit of the interested reader with an ample supply of ambition and energy.

What would appear to be perhaps the "natural" extension of the Bochner theorem is the perfect version applying to any locally compact Abelian group. For the details, see [B], pp. 220 ff: [R], p. 19; [N], pp. 404 ff.; [We], Chapitre VI; [E], Sections 10.3 and 10.4; Bucy and Maltese [1]; [Ph]. The third reference displays the use of Banach-algebra techniques; see the remarks in 11.4.18(1). Infinitesimal fragments of such extensions of the Bochner theorem appear in Section 12.13 and in Exercises 12.34 and 12.35.

Slightly less complete versions of the theorem have been worked out for non-Abelian locally compact groups; see [N], pp. 394 ff. and Godement [1], especially pp. 50–53.

In pursuit of still more generality, Itô and M. G. Krein have devised formulations of the theorem applying to functions on sets and spaces that are not groups. For an example, see [N], pp. 427–428; for a brief survey, see [Hew], pp. 145–149 and the references cited there.

Owing in part to impulses transmitted by mathematical physicists with interests in the quantum theory of fields, special attention has been paid to forms of the Bochner theorem and of harmonic analysis in general applying to Abelian nonlocally compact topological groups. A special, but particularly relevant, such group is the underlying additive group of an infinite-dimensional Hilbert space, for which case the reader should consult [G] and the references there, and [GV], Chapter IV. Possibly the Hilbert space example, although especially significant, is too special to be genuinely typical—a state of affairs due perhaps to the existing variety of possible methods of approach tending to obscure the underlying essentials. Be that as it may, more general cases have been examined with some success. For a nonlocally compact Abelian group there may exist no genuine invariant (= Haar) integral having all the customary properties (as described in, for example, Section 15 of [HR]). However Shah [1] has shown that considerable progress can be made if there exists a suitable stand-in for the missing invariant integral. Such a stand-in does exist for Hilbert space and this is probably the underlying reason for success in this case.

For further developments, see MR **37** ## 1893, 5610, 5611; **40** # 6224; **51** # 13582; **52** # 14870; **55** ## 3678, 3679.

EXERCISES

9.1. Suppose that $f \in \mathbf{L}^1$ and that $(f_n)_{n=1}^{\infty}$ is a sequence of integrable functions such that $\lim_{n\to\infty} \hat{f}_n = \hat{f}$ on Z. Given that each f_n is positive definite, show that f is positive definite.

9.2. Suppose that f is continuous and positive definite and that

$$\lambda \equiv \liminf_{a\to 0} a^{-2}[2f(0) - f(a) - f(-a)] < \infty.$$

Prove that

$$\sum_{n \in Z} (1 + n^2)\hat{f}(n) \leqslant f(0) + \lambda.$$

9.3. Given that f is positive definite and that $f \in \mathbf{C}^\infty(U)$ (or is analytic on U), for some open neighborhood U of 0, show that f is $\mathbf{C}^\infty$ (or analytic) everywhere.

Hint: Use Exercise 9.2 to majorize the sums $\sum_{n \in Z} |n|^{2k}\hat{f}(n)$ for $k = 0, 1, 2, \cdots$.

9.4. Prove that the (pointwise) product of two positive definite functions in $\mathbf{L}^\infty$ is again positive definite.

9.5. A positive definite function $f \in \mathbf{L}^\infty$ is said to be *minimal* if, whenever $g \in \mathbf{L}^\infty$ is positive definite and such that $f - g$ is positive definite, g is equal almost everywhere to a scalar multiple of f. Verify that the only minimal functions are the nonnegative scalar multiples of the continuous characters e^{inx}.

Note: This characterization of the continuous characters is important in the Cartan-Godement account mentioned in Section 9.1.

9.6. Suppose that $f \in \mathbf{L}^\infty$ is positive definite. Show that $1/f$ is equal almost everywhere to a positive definite function in $\mathbf{L}^\infty$, if and only if f is equal almost everywhere to a positive multiple of a continuous character e^{inx}.

9.7. Show that the linear subspace of $\mathbf{C}$ generated by the set of continuous positive definite functions is precisely the set $\mathbf{A}$ of continuous functions f such that

$$\sum_{n \in Z} |\hat{f}(n)| < \infty.$$

(The space $\mathbf{A}$ will be encountered again in Section 10.6, 11.4.17, and Section 12.11.)

9.8. It is known (see 10.6.3) that there exist functions $g \in \mathbf{L}^1$ such that $|\hat{g}|$ is not the Fourier transform of any $\mathbf{L}^1$-function. Give examples of this phenomenon for the case in which $\mathbf{L}^1$ is replaced throughout by $\mathbf{C}$ and by $\mathbf{L}^\infty$.

CHAPTER 10

Pointwise Convergence of Fourier Series

In this chapter we shall deal rather summarily with some positive and negative results about the pointwise convergence of Fourier series. The reasons for not according this topic a fuller treatment are discussed in Chapter 1. The reader who is particularly attracted by these aspects may consult [Z], especially Chapters II, VIII, and XIII; [Ba], especially Chapters I, III–V, VII, IX; [HaR], especially Chapter IV; [I], pp. 23 ff., pp. 103 ff.; [A]; and the work of Carleson mentioned in 10.4.5.

Our account is particularly terse in relation to the many known sufficient conditions for convergence at a particular point. Out of a veritable multitude of such results, increasing almost daily, we shall in fact prove only the very familiar criteria associated with the names of Jordan and Dini, respectively. These are perhaps the most useful aids to handling the functions that occur naturally and with appreciable frequency.

On the other hand, partly in order to reinforce the remarks in Chapter 1 that bear upon the difficulties of characterizing Fourier series directly in terms of pointwise convergence, and partly to exhibit some of the charateristic functional analytic techniques of modern analysis, we shall devote quite a large fraction of the chapter to telling part of the story of the misbehavior of Fourier series in relation to pointwise convergence.

Although our main concern throughout this book is with Fourier series of functions on T, it is worth noting at this point that, rather late in the historical development of harmonic analysis, it came to be recognized that the classically natural groups T and its finite powers are "difficult" in respect of questions about pointwise convergence of Fourier series. More precisely, there are compact Abelian groups which are, from a classical viewpoint, rather bizarre, and yet relative to which Fourier series behave in a fashion simpler and more civilized than they do when the underlying group is T. The simplest (infinite) instance of such a group is the so-called Cantor group $\mathscr{C}$, which will be discussed in Chapter 14 as an aid in the study of Fourier series of functions on T. It turns out that, if f is any continuous complex-valued function on $\mathscr{C}$, then the Fourier series of f converges uniformly to f.

Concerning the dual aspect of the problems handled in this chapter, see the remarks in Section 6.7.

The reader is again reminded that, in the absence of any statement to the contrary, the convergence of a numerical series $\sum_{n\in Z} c_n$ is defined to signify the existence of a finite limit

$$\lim_{N\to\infty} \sum_{|n|\leqslant N} c_n .$$

A similar convention applies if the c_n are elements of any one of the topological linear spaces of functions, measures, or distributions to be encountered in due course. If the c_n denote arbitrary nonnegative real numbers, the above definition of the sum $\sum_{n\in Z} c_n$ makes it equal to

$$\sup_F \sum_{n\in F} c_n ,$$

where the supremum is taken with respect to all finite subsets F of Z. In this case, if the series is not convergent, the set of such finite partial sums is unbounded above and, by universal convention, one then writes

$$\sum_{n\in Z} c_n = \infty .$$

The sum of a series of nonnegative real numbers is thus assigned a finite value if and only if it is convergent.

10.1 Functions of Bounded Variation and Jordan's Test

10.1.1. (Jordan's test) If $f \in \mathbf{L}^1$ is of bounded variation on some neighborhood of a point x, then

$$\lim_{N\to\infty} s_N f(x) = \tfrac{1}{2}[f(x+0) + f(x-0)].$$

Proof. Since ([W], Lemma 6.4b) the real and imaginary parts of f are each expressible, throughout some neighborhood of x, as the difference of two monotone functions, both limits $f(x+0) \equiv \lim_{a\downarrow 0} f(x+a)$ and $f(x-0) \equiv \lim_{a\downarrow 0} f(x-a)$ exist finitely.

It is in any case enough to handle the case in which f is real-valued, which we assume henceforth. Now

$$\begin{aligned} s_N f(x) &= \frac{1}{2\pi}\int f(x-y)D_N(y)\,dy \\ &= \frac{1}{2\pi}\int_0^\pi [f(x+y) + f(x-y)]D_N(y)\,dy, \end{aligned}$$

and it will therefore suffice to show that

$$\lim_{N\to\infty} \frac{1}{2\pi}\int_0^\pi g(y)D_N(y)\,dy = \tfrac{1}{2}g(+0) \tag{10.1.1}$$

for any real-valued function g which is integrable over $(0, \pi)$ and of bounded variation on some right-hand neighborhood $[0, \delta]$ of 0, where $0 < \delta < \pi$. In doing this we may, without loss of generality, assume that g is increasing on

$[0, \delta]$ and that $g(+0) = 0$. [The latter reduction is possible in view of the obvious fact that (10.1.1) holds when g is a constant; see (5.1.3).] This being so, the second mean value theorem of the integral calculus gives

$$\begin{aligned}\frac{1}{2\pi}\int_0^\pi g(y)D_N(y)\,dy &= \frac{1}{2\pi}\int_0^\delta + \frac{1}{2\pi}\int_\delta^\pi \\ &= \frac{1}{2\pi}g(\delta - 0)\int_\xi^\delta D_N(y)\,dy + \frac{1}{2\pi}\int_\delta^\pi g(y)D_N(y)\,dy\end{aligned}$$

for some ξ satisfying $0 \leqslant \xi \leqslant \delta$. This ξ may depend upon N. Thus

$$\left|\frac{1}{2\pi}\int_0^\pi g(y)D_N(y)\,dy\right| \leqslant \frac{1}{2\pi}g(\delta - 0)\left|\int_\xi^\delta D_N(y)\,dy\right| + \left|\frac{1}{2\pi}\int_\delta^\pi g(y)D_N(y)\,dy\right|. \tag{10.1.2}$$

Now

$$\begin{aligned}\left|\frac{1}{2\pi}\int_\xi^\delta D_N(y)\,dy\right| &= \left|\frac{1}{\pi}\int_\xi^\delta \frac{\sin(N + \tfrac{1}{2})y\,dy}{2\sin\tfrac{1}{2}y}\right| \leqslant \left|\frac{1}{\pi}\int_\xi^\delta \frac{\sin(N + \tfrac{1}{2})y\,dy}{y}\right| \\ &\quad + \frac{1}{\pi}\int_\xi^\delta |y^{-1} - \tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}y|\,dy \leqslant \left|\frac{1}{\pi}\int_{(N+\frac{1}{2})\xi}^{(N+\frac{1}{2})\delta} \frac{\sin t\,dt}{t}\right| \\ &\quad + \frac{1}{\pi}\int_0^\pi |y^{-1} - \tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}y|\,dy.\end{aligned}$$

Since $y^{-1} - \tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}y$ is integrable over $(0, \pi)$, 10.1.2 to follow shows that

$$\left|\frac{1}{2\pi}\int_\xi^\delta D_N(y)\,dy\right| \leqslant A, \tag{10.1.3}$$

where A is independent of ξ, δ, and N.

Reverting to (10.1.2), and assuming that $\varepsilon > 0$ is assigned, we first fix $\delta > 0$ so small that $Ag(\delta) \leqslant \varepsilon$ [which is possible since $g(+0) = 0$], so obtaining

$$\left|\frac{1}{2\pi}\int g(y)D_N(y)\,dy\right| \leqslant \varepsilon + \left|\frac{1}{2\pi}\int_\delta^\pi g(y)D_N(y)\,dy\right|. \tag{10.1.4}$$

Since $g(y)\operatorname{cosec}\tfrac{1}{2}y$ is integrable over (δ, π), we may allow N to tend to infinity and apply 2.3.8 to the second term on the right in (10.1.4) and so conclude that

$$\limsup_{N\to\infty}\left|\frac{1}{2\pi}\int_0^\pi g(y)D_N(y)\,dy\right| \leqslant \varepsilon.$$

Since ε is freely chosen, (10.1.1) is thus established. See also Izumi [1].

10.1.2. The integral $\int_a^b t^{-1}\sin t\,dt$ is bounded for all real values of a and b, that is,

$$\sup_{a,b\in R}\left|\int_a^b t^{-1}\sin t\,dt\right| < \infty.$$

Proof. This is left as Exercise 10.1 for the reader.

10.1.3. As has been indicated in 5.3.5, by using 2.3.6, 5.2.1 and a Tauberian theorem of Hardy (see Exercise 5.8), 10.1.1 could be inferred directly from 6.3.1.

An examination of the proof of 10.1.1 leads to the following global version thereof.

10.1.4. If f is of bounded variation, then

$$\lim_{N\to\infty} s_N f(x) = \tfrac{1}{2}[f(x+0) + f(x-0)]$$

for all x, and the convergence is bounded:

$$|s_N f(x)| \leqslant \text{const}\,(\|f\|_\infty + V(f)), \tag{10.1.5}$$

where $V(f)$ denotes the total variation of f over any interval of length 2π.

Proof. Only the inequality has to be proved. A perusal of the proof of 10.1.1, and a glance at the way in which a function of bounded variation can be expressed as the difference of two increasing functions ([W], p. 105), show that it will suffice to prove that

$$\left|\frac{1}{2\pi}\int_0^\pi g(y)D_N(y)\,dy\right| \leqslant \text{const}\sup\{|g(y)| : 0 \leqslant y \leqslant \pi\} \tag{10.1.6}$$

for any increasing function g on $[0, \pi]$ such that $g(+0) = 0$. Now the second mean value theorem of the integral calculus gives for such g the relation

$$\frac{1}{2\pi}\int_0^\pi g(y)D_N(y)\,dy = \frac{1}{2\pi}g(\pi - 0)\int_\alpha^\pi D_N(y)\,dy,$$

for some α in $[0, \pi]$, so that (10.1.6) is an immediate consequence of (10.1.3).

Remarks. (1) A different proof is easily derived from Exercise 10.12.

(2) For functions f which are both continuous and of bounded variation, it is true that $\lim_{N\to\infty} s_N f = f$ uniformly. See Exercises 10.13 and 10.14.

10.1.5. If $f \in \mathbf{L}^1$, then $\sum_{n\neq 0} \hat{f}(n)e^{inx}/n$ is uniformly convergent for all x.

Proof. The function g defined by

$$g(x) = \int_0^x f(y)\,dy - \hat{f}(0)x$$

is periodic and absolutely continuous, and its Fourier series is

$$\sum_{n\neq 0} (in)^{-1}\hat{f}(n)e^{inx};$$

see 2.3.4. The result follows on applying Remark (2) following 10.1.4.

10.1.6. **Remarks.** (1) The proof of 10.1.5 can be extended to show that $\sum_{n\neq 0}\hat{\mu}(n)/n$ is convergent for any measure μ (see Sections 12.2 and 12.5).

(2) From 10.1.5 it follows that $\sum_{n=2}^{\infty} \sin nx/\log n$ is not a Fourier-Lebesgue series. (Nor, by Remark (1) immediately above, is it even a so-called Fourier-Stieltjes series; see 12.5.2.) See also 7.3.4 and Exercise 7.7.

10.2 Remarks on Other Criteria for Convergence; Dini's Test

We adopt the notation introduced in Section 6.3, writing in particular

$$f_s^*(y) \equiv f_s^*(y, x) = \tfrac{1}{2}[f(x + y) + f(x - y) - 2s]. \tag{10.2.1}$$

Then, parallel to (6.3.2), we have

$$s_N f(x) - s = \frac{1}{\pi}\int_0^\pi f_s^*(y)D_N(y)\,dy. \tag{10.2.2}$$

Since $f_s^*(y)$ cosec $\frac{1}{2}y$ is integrable over (δ, π) for any δ satisfying $0 < \delta \leqslant \pi$, 2.3.8 shows that

$$\lim_{N\to\infty} \frac{1}{\pi}\int_\delta^\pi f_s^*(y)D_N(y)\,dy = 0 \qquad (0 < \delta \leqslant \pi). \tag{10.2.3}$$

An immediate corollary of this is the following statement.

10.2.1. If $f \in \mathbf{L}^1$, in order that

$$\lim_{N\to\infty} s_N f(x) = s, \tag{10.2.4}$$

it is necessary and sufficient that for some δ satisfying $0 < \delta \leqslant \pi$ it is true that

$$\lim_{N\to\infty} \frac{1}{\pi}\int_0^\delta f_s^*(y)D_N(y)\,dy = 0. \tag{10.2.5}$$

Beside this we may infer

10.2.2. In order that (10.2.4) be true, it is sufficient that to any $\varepsilon > 0$ shall correspond a number $\delta(\varepsilon)$ satisfying $0 < \delta(\varepsilon) \leqslant \pi$ and a positive integer $N(\varepsilon)$ such that

$$\left|\frac{1}{\pi}\int_0^{\delta(\varepsilon)} f_s^*(y)D_N(y)\,dy\right| \leqslant \varepsilon \qquad \text{for } N \geqslant N(\varepsilon). \tag{10.2.6}$$

Proof. In this case

$$\left|\frac{1}{\pi}\int_0^\pi f_s^*(y)D_N(y)\,dy\right| \leqslant \varepsilon + \left|\frac{1}{\pi}\int_{\delta(\varepsilon)}^\pi f_s^*(y)D_N(y)\,dy\right|$$

for all $N \geqslant N(\varepsilon)$, whence use of 2.3.8 yields from (10.2.2)

$$\limsup_{N \to \infty} |s_N f(x) - s| \leqslant \varepsilon.$$

On letting $\varepsilon \to 0$, we obtain (10.2.4).

10.2.3. (Dini's test) In order that (10.2.4) be true, it is sufficient that for some δ satisfying $0 < \delta \leqslant \pi$ one has

$$\int_0^\delta \frac{|f_s^*(y)|\,dy}{y} < \infty. \tag{10.2.7}$$

(Compare this with 5.2.3.)

Proof. Since $y^{-1} - \frac{1}{2}\operatorname{cosec}\frac{1}{2}y$ is bounded over $(0, \delta)$, (10.2.7) and 2.3.8 combine to yield (10.2.5). The result therefore follows from 10.2.1.

10.2.4. **Remarks.** (1) If (10.2.7) holds for any value of s at all, and if f has at x at worst a jump discontinuity, the value of s must be $\frac{1}{2}[f(x + 0) + f(x - 0)]$.

(2) It is evident that (10.2.7) is fulfilled, with $s = f(x)$, if, for example, $f(x + y) - f(x) = O(|y|^\alpha)$ for some $\alpha > 0$.

10.3 The Divergence of Fourier Series

In this section we shall assemble a few results concerning the pointwise divergence of the Fourier series of functions of various types. For many more details the reader is referred to [Z_1], Chapter VIII and [Ba_1], Chapters I and V. Concerning mean convergence in $\mathbf{L}^1$, see Exercise 10.2 and 12.10.2.

One may seek to support a statement of the type: "The Fourier series of a continuous function may diverge" in one of at least two ways. *Either* one may try to construct in as explicit a manner as possible a specific continuous function with the desired property; *or* one may use a *reductio ad absurdum* argument by showing that the hypothesis, that no such functions exist, leads to a contradiction of what is already known. In the former case one has (if the alleged construction is successful) a constructive proof of the statement; in the latter case one has (assuming that no errors are made on the way) an existential proof of the statement, so-called because one has shown that functions of the specified type must exist without prescribing any way of finding one. In 6.1.4, we have already encountered an example of the second type of argument. Both types of proof have their appeal. A constructive proof is usually more satisfying, but it is often ruled out by lack of enough detailed information about certain elements of the proposed construction. This is frequently the case in abstract analysis, and this is where the existential type of proof often comes to the rescue.

A further illustration is provided by the assertion: "There exist integrable functions whose Fourier transforms tend to zero arbitrarily slowly at infinity." For the group T, a constructive proof of a strong form of this assertion is included in Section 7.4 and depends on many considerations peculiar to this group. There is a meaningful analogous assertion for any nondiscrete locally compact Abelian group. This analogue can be painlessly established (see Exercise 10.22) by an existential proof using abstract arguments; by expending more effort and using the properties of Sidon sets (see the introductory remarks to Chapter 15), more or less constructive proofs may also be furnished.

We now proceed to illustrate both types of proof in connection with statements about the divergence of Fourier series.

10.3.1. Fejér's Example. We begin with a construction due to Fejér leading to continuous functions whose Fourier series diverge at a given point, which may without loss of generality be taken to be the origin.

If p and q are integers satisfying $p \geqslant q \geqslant 1$, let $t_{p,q}$ denote the trigonometric polynomial defined by

$$t_{p,q}(x) = \frac{\cos(p-q)x}{q} + \cdots + \frac{\cos(p-1)x}{1} - \frac{\cos(p+1)x}{1} - \cdots - \frac{\cos(p+q)x}{q}$$

$$= 2\sin px \sum_{k=1}^{q} k^{-1} \sin kx. \tag{10.3.1}$$

In view of Exercise 1.4, the $t_{p,q}$ are uniformly bounded.

Suppose now that $(p_k)_{k=1}^{\infty}$ and $(q_k)_{k=1}^{\infty}$ are sequences of integers satisfying

$$1 \leqslant q_k \leqslant p_k, \qquad p_k + q_k < p_{k+1} - q_{k+1}. \tag{10.3.2}$$

Suppose further that $(\alpha_k)_{k=1}^{\infty}$ is any sequence of complex numbers such that

$$\sum_{k=1}^{\infty} |\alpha_k| < \infty, \tag{10.3.3}$$

$$\liminf_{k\to\infty} |\alpha_k| \log q_k > 0. \tag{10.3.4}$$

One might take, for example,

$$p_k = 2^{k^3+1}, \qquad q_k = 2^{k^3}, \qquad \alpha_k = k^{-2}.$$

Consider the function

$$f(x) = \sum_{k=1}^{\infty} \alpha_k \cdot t_{p_k,q_k}(x). \tag{10.3.5}$$

Because of (10.3.3) the series converges uniformly, so that f is continuous.

Due to uniform convergence, the Fourier coefficients of f may be calculated by termwise integration of the series appearing in (10.3.5). One finds in this way that

$$\begin{aligned}|s_{p_k+q_k}f(0) - s_{p_k}f(0)| &= |\alpha_k||1^{-1}\cos(p_k+1)0 + \cdots + q_k^{-1}\cos(p_k+q_k)0| \\ &= |\alpha_k|\sum_{r=1}^{q_k} r^{-1} \\ &\sim |\alpha_k|\log q_k. \end{aligned} \tag{10.3.6}$$

In the course of this calculation, the relations (10.3.2) are used in ensuring that the various trigonometric polynomials t_{p_k,q_k} have no "overlapping" harmonics. The relations (10.3.6) and (10.3.4) shows that the sequence $(s_N f(0))_{N=1}^{\infty}$ is not convergent, that is, that the Fourier series of f is not convergent at the origin. Indeed, if we choose the α_k so that

$$\limsup_{k\to\infty} |\alpha_k|\log q_k = \infty,$$

it follows from (10.3.6) that the partial sums of the Fourier series of f are unbounded at the origin.

Numerous variations may be played on the preceding construction; see [Z$_1$], pp. 299–300; [Ba$_1$], §45; [Kz], p. 51, Proof B; Edwards and Price [1].

10.3.2. **Existential Proofs.** In this and the following subsection we shall use the uniform boundedness principle as the basis of existential proofs of the statement asserting the possible divergence of the Fourier series of continuous functions.

The aim of this subsection is as follows. Let $(x_k)_{k=1}^{\infty}$ be any sequence of real numbers and $(\beta_N)_{N=1}^{\infty}$ any sequence of positive real numbers such that

$$\lim_{N\to\infty} \beta_N = 0. \tag{10.3.7}$$

$\mathbf{C}$ denotes the Banach space of continuous (periodic) functions; see 2.2.4. Our claim is that for each $f \in \mathbf{C}$, save perhaps those of a meager (= first category; see Appendix A.1) subset of $\mathbf{C}$, it is the case that

$$\limsup_{N\to\infty} \frac{|s_N f(x_k)|}{\beta_N \log(N+1)} = \infty \qquad (k = 1, 2, \cdots). \tag{10.3.8}$$

Since $\mathbf{C}$ is not a meager subset of itself (Appendix A.3), this shows that continuous functions f certainly exist for which (10.3.8) is true.

Proof. We shall apply the result stated in Appendix B.2.1, taking for the Fréchet space featuring therein the Banach space $\mathbf{C}$, and defining

$$p_k(f) = \sup_{N\geqslant 1} \frac{|s_N f(x_k)|}{\beta_N \log(N+1)}.$$

This p_k has all the properties demanded, lower semicontinuity being a consequence of the fact that

$$f \to s_N f(x_k)$$

is evidently a continuous linear functional on $\mathbf{C}$. The set of $f \in \mathbf{C}$ satisfying (10.3.8) is exactly the complement, relative to $\mathbf{C}$, of the set

$$\mathbf{S} = \{f \in \mathbf{C} : \inf_k p_k(f) < \infty\}.$$

It therefore suffices to show that $\mathbf{S}$ is meager.

Now, if $\mathbf{S}$ were nonmeager, Appendix B.2.1 would entail that for some k there is a constant c such that

$$p_k(f) \leqslant c\|f\|$$

for all $f \in \mathbf{C}$. This would signify that

$$\left|\frac{1}{2\pi}\int f(x_k - y)D_N(y)\,dy\right| \leqslant c\|f\|\beta_N \log(N+1). \qquad (10.3.9)$$

Appealing to translation invariance of the integral, combined with the converse of Hölder's inequality (see Exercise 3.6), it would appear from (10.3.9) that

$$\frac{1}{2\pi}\int |D_N(x_k - y)|\,dy \leqslant c\beta_N \log(N+1);$$

or, by translation invariance of the integral once again, that

$$\|D_N\|_1 \leqslant c\beta_N \log(N+1). \qquad (10.3.10)$$

But (10.3.7) and (10.3.10) flatly contradict (5.1.10), which says that

$$\|D_N\|_1 \sim \frac{4}{\pi^2}\log N \qquad (N \to \infty).$$

This contradiction establishes the desired result. See also [Kz], p. 51, Proof A.

10.3.3. Let $(x_k)_{k=1}^{\infty}$ and $(\beta_N)_{N=1}^{\infty}$ be as in 10.3.2. There exists a meagre subset $\mathbf{S}$ of $\mathbf{C}$ with the following property: if $f \in \mathbf{C}\backslash\mathbf{S}$, one has

$$\limsup_{N\to\infty} \frac{|s_N f(x)|}{\beta_N \log(N+1)} = \infty \qquad (x \in E), \qquad (10.3.11)$$

where E is a (possibly f-dependent) set which is everywhere dense and whose complement is meager (so that E is nonmeager and therefore uncountable), and such that $x_k \in E$ for all k.

Proof. We may suppose from the outset that the points x_k are everywhere dense in $[-\pi, \pi]$ and that all values of x considered lie in this same interval. For S we choose the set specified in the proof if 10.3.2, where it is shown that S is meager. Take $f \in \mathbf{C} \backslash \mathbf{S}$ and define

$$w(x) = \sup_{N \geq 1} \frac{|s_N f(x)|}{\beta_N \log (N + 1)},$$

so that $w(x_k) = \infty$ for $k = 1, 2, \cdots$. If E is defined to be the set of points $x \in [-\pi, \pi]$ for which $w(x) = \infty$, then E contains every x_k and is therefore everywhere dense in $[-\pi, \pi]$. Moreover, if

$$E_r = \{x \in [-\pi, \pi] : w(x) > r\} \qquad (r = 1, 2, \cdots),$$

then E_r is open relative to $[-\pi, \pi]$ (since w is evidently lower semicontinuous), E_r contains all the x_k, and is thus everywhere dense in $[-\pi, \pi]$, and $E = \bigcap_{r=1}^{\infty} E_r$. Taking complements relative to $[-\pi, \pi]$, the complement of E is the union of the complements of the E_r. Each of the latter is closed and nowhere dense, so that the complement of E is meager. By the category theorem (Appendix A.3), E must therefore be nonmeager; and since $[-\pi, \pi]$ is nondiscrete, this entails that E is uncountable.

10.3.4. **Further Results.** Evidently, 10.3.3 entails the existence of continuous functions f whose Fourier series diverge on sets E that are uncountable, nonmeager, and everywhere dense. (The existence of continuous functions whose Fourier series diverge at a specified point was known to du Bois-Reymond in 1872.)

The examples in 10.3.2 and 10.3.3 are all such that $|s_N f(x)|$ is unbounded with respect to N for certain values of x. It can, however, happen that $(s_N f(x))_{N=1}^{\infty}$ is boundedly divergent for each point x in a set with the power of the continuum, and this for suitable continuous functions f; for an example, see [Ba$_1$], p. 348.

It results from the work of Carleson, described in 10.4.5, that the Fourier series of any continuous function converges pointwise almost everywhere to that function.

Men'shov showed in 1947 that there exists a continuous function f such that *any* subsequence of $(s_N f)_{N=1}^{\infty}$ diverges at some point. He also established the curious fact that any $f \in \mathbf{C}$ can be decomposed into a sum $f_1 + f_2$, where $f_i \in \mathbf{C}$ and some subsequence of $(s_N f_i)_{N=1}^{\infty}$ is uniformly convergent ($i = 1, 2$).

There are very few simple operations on (say) continuous functions that preserve convergence of the Fourier series. Thus, for example, there exists at least one $f \in \mathbf{C}$ having a uniformly convergent Fourier series, while the Fourier series of f^2 diverges on a set having the power of the continuum; and also a similar f such that the Fourier series of $|f|$ diverges at some points ([Ba$_1$], p. 350; p. 360, Problem 14). For further results of this type, see Kahane and Katznelson [1].

In 1926 Kolmogorov showed that there exist integrable functions f whose Fourier series diverge everywhere. The proof, which is more or less constructive but complicated and difficult, is given in [Z$_1$], pp. 310–314 and [Ba$_1$], pp. 455–464; [HaR], Theorem 79, is an earlier construction of Kolmogorov designed to produce integrable functions whose Fourier series diverge almost everywhere. That such functions exist has been established by Stein ([1], Theorem 6) on the basis of a general theorem which is cited in 16.2.8 and which provides a powerful general approach to many existence theorems of this type. Granted the existence of one such function, it is relatively simple to deduce that they exist in abundance; see Exercise 10.21. See also Chen [1] and M. and S.-I. Izumi [2].

Regarding similar results when the operators s_N are replaced by something similar but more general, see Exercises 10.23 and 10.24.

For Kolmogorov's function f, as for those in 10.3.2 and 10.3.3, it is the case that $s_N f(x)$ is unbounded. In 1936 Marcinkiewicz showed that integrable functions f exist for which $s_N f(x)$ is boundedly divergent for almost all x (see [Z_1], p. 308, and [Ba_1], pp. 430–443), but it is apparently still unknown whether "almost all" can here be replaced by "all."

There is an elegant discussion of divergence for homogeneous Banach spaces (including the Kolmogorov example cited above) in [Kz], pp. 55–61; see also [Moz], Appendix C.

See also MR **33** # 6266; **35** # 3349; **39** # 7342; **41** # 8906; **51** # 13578.

10.3.5. Majorants for the $s_N f$.

The example given by Kolmogorov and mentioned in 10.3.4 shows that, if we write

$$s^*f(x) \equiv \sup_N |s_N f(x)| \ (\leqslant \infty),$$

then there exist integrable functions f such that

$$s^*f(x) = \infty$$

for all x. The reader should compare this with the results quoted in 6.4.7 and relating to the majorant $\sigma^* f$ of the Cesàro means $\sigma_N f$. See also 10.4.5 and Exercise 10.2, and 13.10 below.

In spite of this disconcerting situation, there are some remarkable assertions of a type which say that, if f is real-valued and if the $s_N f$ are bounded below in a suitable sense, then they are also bounded above in a corresponding sense. A typical statement of this type affirms that if $f \in \mathbf{L}^1$ is real-valued and such that $\inf_N s_N f \in \mathbf{L}^1$, then $\sup_N s_N f \in \mathbf{L}^p$ for any p satisfying $0 < p < 1$. For other similar results and the relevant details, see [Z_2], pp. 173–175.

10.3.6. Topological Bases of Trigonometric Polynomials.

Consider a Banach space $\mathbf{E}$ and a sequence $(a_n)_{n \in Z}$ of elements of $\mathbf{E}$. This sequence is said to form a *topological base* for $\mathbf{E}$ if to each $f \in \mathbf{E}$ corresponds precisely one sequence $(\alpha_n)_{n \in Z}$ of scalars such that

$$f = \sum_{n \in Z} \alpha_n a_n \equiv \lim_{N \to \infty} \sum_{|n| \leqslant N} \alpha_n a_n, \tag{10.3.12}$$

the limit being taken with respect to the norm on $\mathbf{E}$. A similar definition applies to sequences $(a_n)_{n=1}^{\infty}$. (The reader will recall that a not necessarily countable family (a_n) of elements of $\mathbf{E}$ forms an *algebraic*, or *Hamel, base* for $\mathbf{E}$ if each

$f \in \mathbf{E}$ is expressible uniquely as a *finite* sum of terms $\alpha_n a_n$, the coefficients α_n being f-dependent scalars.) For more details about topological bases in general, see, for example, [E], Section 6.8. The existence of topological bases has been settled only for a number of particular (albeit especially important) Banach spaces; but see **MR 35** # 700.

Let us now specialize by assuming that $\mathbf{E}$ is one of the spaces $\mathbf{L}^p$, where $1 \leqslant p \leqslant \infty$, or $\mathbf{C}$, and $a_n = e_n$, the function $x \to e^{inx}$. It will appear in 12.10.1 that $(e_n)_{n \in Z}$ is a topological base for $\mathbf{L}^p$ whenever $1 < p < \infty$. On the contrary, however, this sequence is *not* a topological base for any one of the spaces $\mathbf{L}^1$, $\mathbf{L}^\infty$, or $\mathbf{C}$; see Exercise 10.15.

Taking the case of $\mathbf{C}$, the question arises as to whether there exists any topological base $(t_n)_{n=1}^\infty$ for $\mathbf{C}$ in which each t_n is a trigonometric polynomial; and, if so, what can be said about the degree d_n of t_n. It has emerged (see [Ba$_1$], p. 360, Problem 16) that one cannot have $d_n \leqslant n$ for all n; and yet (*loc. cit.*, Problem 17) that there exists a topological base $(t_n)_{n=1}^\infty$ for $\mathbf{C}$ comprised of trigonometric polynomials for which $d_n = o(n^{2+\varepsilon})$ for every $\varepsilon > 0$. As yet no conditions are known concerning the degrees d_n which are necessary and sufficient to ensure the existence of a topological base for $\mathbf{C}$ formed by trigonometric polynomials t_n of degree d_n.

In 2.2.1 we mentioned the question of the possibility of decomposing $\mathbf{C}$ into a direct sum of minimal translation-invariant subspaces. In view of 2.2.1(2), such a decomposition is possible if and only if $(e_n)_{n \in Z}$ is a topological base for $\mathbf{C}$. Accordingly, by Exercise 10.15, no such decomposition is possible for $\mathbf{C}$. By the same token, the analogous procedure is impossible for $\mathbf{L}^1$ and $\mathbf{L}^\infty$, too. On the other hand, the fact that the e_n do form a topological base in $\mathbf{L}^p$ $(1 < p < \infty)$, ensures that the analogous decomposition is possible in each of these spaces.

10.4 The Order of Magnitude of $s_N f$. Pointwise Convergence Almost Everywhere

This section begins with a result to the effect that the misbehavior of Fourier series shown in 10.3.2 and 10.3.3 to occur for certain continuous functions is, in a sense, the worst that can happen even for arbitrary integrable functions. Knowing this, it is possible to infer the convergence everywhere or almost everywhere of certain series simply related to Fourier-Lebesgue series (see 10.4.3 and 10.4.4). This in turn leads to the consideration of conditions on a sequence $(c_n)_{n \in Z}$ sufficient to ensure the convergence almost everywhere of the trigonometric series

$$\sum_{n \in Z} c_n e^{inx}.$$

Within the circle of ideas thus suggested appear certain problems concerning the pointwise convergence of Fourier series which, while remaining intractable for a very long while, provided the incentive for a great deal of fruitful research; see 10.4.2 and 10.4.4 to 10.4.6.

10.4.1. If $f \in \mathbf{L}^1$, then the relation

$$s_N f(x) = o(\log N) \tag{10.4.1}$$

holds

(1) whenever $f(x+0) + f(x-0) \equiv \lim_{y \downarrow 0}[f(x+y) + f(x-y)]$ exists finitely, and

(2) for almost all x in any case.

Proof. (1) Suppose that the said limit exists, and let s denote its value. By (10.2.2),

$$\begin{aligned} |s_N f(x) - s| &= \left| \frac{1}{\pi} \int_0^\pi f_s^*(y) D_N(y)\, dy \right| \\ &\leqslant \left| \frac{1}{\pi} \int_0^\delta \right| + \left| \frac{1}{\pi} \int_\delta^\pi \right|, \end{aligned}$$

where $\delta > 0$ will be chosen in a moment.

By hypothesis, $f_s^*(y) \to 0$ as $y \downarrow 0$. So, given $\varepsilon > 0$, we may choose and fix $\delta > 0$ so small that $|f_s^*(y)| \leqslant \varepsilon$ for $0 \leqslant y \leqslant \delta$. Then

$$\left| \frac{1}{\pi} \int_0^\delta \right| \leqslant \varepsilon \cdot \frac{1}{\pi} \int_0^\pi |D_N(y)|\, dy \leqslant A\varepsilon \log N,$$

by 5.1.1, A denoting an absolute constant. Having fixed δ, 2.3.8 shows that $(1/\pi)\int_\delta^\pi = o(1)$ as $N \to \infty$. One thus obtains

$$|s_N f(x) - s| \leqslant A\varepsilon \log N + \varepsilon$$

for all sufficiently large N, which implies (10.4.1).

(2) In the general case, we proceed as before save that the range of integration $(0, \pi)$ is divided into $(0, \pi/N)$ and $(\pi/N, \pi)$. Using the estimates $|D_N(y)| \leqslant AN$, $|D_N(y)| \leqslant A/y$, where A is again a positive absolute constant, we find that

$$\begin{aligned} A^{-1}|s_N f(x) - s| &\leqslant \frac{1}{\pi} \int_0^{\pi/N} |f_s^*(y)| N\, dy + \frac{1}{\pi} \int_{\pi/N}^\pi \frac{|f_s^*(y)|\, dy}{y} \\ &= I_1 + I_2. \end{aligned} \tag{10.4.2}$$

The choice of the number s is so far immaterial. Defining, as in (6.4.6),

$$I(h) = \int_0^h |f_s^*(y)|\, dy,$$

we have

$$I_1 \leqslant \frac{1}{\pi} N I\left(\frac{\pi}{N}\right).$$

Moreover, by partial integration,

$$I_2 = \frac{1}{\pi} \left[\frac{I(y)}{y} \right]_{\pi/N}^\pi + \frac{1}{\pi} \int_{\pi/N}^\pi \frac{I(y)\, dy}{y^2}.$$

Thus, by (10.4.2),

$$A^{-1}|s_N f(x) - s| \leqslant \frac{1}{\pi} NI\left(\frac{\pi}{N}\right) + \frac{I(\pi)}{\pi^2} + \frac{1}{\pi}\int_{\pi/N}^{\pi} \frac{I(y)\,dy}{y^2}. \qquad (10.4.3)$$

According to 6.4.2, if we take $s = f(x)$, then $I(y) = o(y)$ as $y \downarrow 0$ for almost all x. For any such x, (10.4.3) shows that

$$|s_N f(x) - f(x)| = O(1) + o\left(\int_{\pi/N}^{\pi} \frac{dy}{y}\right) = o(\log N),$$

as required.

10.4.2. **Remarks.** It is apparently unknown whether 10.4.1 is the best-possible result of its kind. However, it is known (Stein [1], Theorem 6; Carleson [1]) that, if $\varepsilon_N \downarrow 0$ as $N \uparrow \infty$, then there exists at least one $f \in \mathbf{L}^1$ for which the relation

$$s_N f(x) - s_{N'} f(x) = O\{\varepsilon_{N-N'} \log (N - N')\} \qquad (N > N' + 1, N' \to \infty)$$

is false for almost all x; and that there exists at least one $f \in \mathbf{L}^1$ for which the relation

$$s_N f(x) = O(\varepsilon_N \log\log N) \qquad (N \to \infty)$$

is false for almost all x.

10.4.3. If $f \in \mathbf{L}^1$ and $\alpha > 0$, each of the series

$$\sum_{n \in Z} \frac{\hat{f}(n) e^{inx}}{\log(2 + |n|)}, \qquad \sum_{n \in Z} \frac{\hat{f}(n) e^{inx}}{(1 + |n|)^{\alpha}}$$

is convergent

(1) wherever $f(x + 0) + f(x - 0)$ exists finitely, and

(2) for almost all x, in any case.

Proof. This follows from 6.3.1, 6.4.4, 10.4.1, and Exercise 7.4.

10.4.4. **The Case $p > 1$.** With somewhat more effort it is possible to improve (10.4.1) for the case in which $f \in \mathbf{L}^p$ for some $p > 1$. For example, it was proved long ago by Littlewood and Paley that

$$s_N f(x) = o\{(\log N)^{1/p}\} \qquad \text{if } f \in \mathbf{L}^p,\ 1 \leqslant p \leqslant 2 \qquad (10.4.4)$$

for almost all x. Yet, for an equally long time, no success attended attempts to establish (10.4.4) for the case where $p > 2$, or even to show that (10.4.4) holds for $f \in \mathbf{C}$ and $p > 2$; see [Z_2], pp. 161–162, 166–167. The present position will be outlined in Subsection 10.4.5.

It is relatively simple to show on the basis of 10.4.3 that the trigonometric series

$$\sum_{n \in Z} c_n e^{inx} \qquad (10.4.5)$$

converges almost everywhere whenever

$$\sum_{n\in Z} |c_n|^2 \log^2 (1 + |n|) < \infty; \tag{10.4.6}$$

this result, due to Hardy, is the forerunner of more elaborate investigations mentioned immediately below and in the next two subsections.

On the basis of (10.4.4) it can be shown ([Z₂], p. 170) that

$$\sum_{n\in Z} \frac{\hat{f}(n)e^{inx}}{[\log (2 + |n|)]^{1/p}}$$

converges almost everywhere whenever $f \in \mathbf{L}^p$ and $1 \leqslant p \leqslant 2$; and that (10.4.5) converges almost everywhere provided

$$\sum_{n\in Z} |c_n|^2 \log (1 + |n|) < \infty. \tag{10.4.7}$$

This last result is due to Kolmogorov, and Seliverstov, and Plessner; see $[Z_2]$, p. 163 or $[Ba_1]$, p. 363.

On the other hand, it is known (see $[Ba_1]$, p. 483, Problem 1) that there exist functions $f \in \mathbf{C}$ such that

$$\sum_{n\in Z} |\hat{f}(n)|^2 \log (1 + |n|) < \infty$$

and yet the Fourier series of f diverges at infinitely many points.

10.4.5. **Lusin's Problem and Carleson's Theorems.** The results mentioned in Subsections 10.4.1 to 10.4.4 were by-products of a prolonged and arduous study of a problem posed by Lusin in 1915, namely: does the condition

$$\sum_{n\in Z} |c_n|^2 < \infty \tag{10.4.8}$$

suffice to ensure the convergence almost everywhere of the series (10.4.5)? In other words, does the Fourier series of any function in $\mathbf{L}^2$ converge almost everywhere? It is natural to ask more generally: does the Fourier series of any function belonging to $\mathbf{L}^p$ for some $p > 1$ converge almost everywhere?

Over a period of fifty years, an enormous amount of effort was expended on efforts to solve these problems. Some description of the situation prevailing until 1966 appears in $[Z_2]$, pp. 165–166 and in the relevant portions of [Li].

In 1966 Carleson [1] announced an affirmative answer to Lusin's question, together with major improvements on (10.4.4). His results, the proofs of which are far too long and complicated to be given here, are as follows:

(1) if $f(\log^+ |f|)^{1+\delta} \in \mathbf{L}^1$ for some $\delta > 0$, then

$$s_N f(x) = o(\log \log N)$$

for almost all x;

(2) if $f \in \mathbf{L}^p$ for some $p > 1$, then

$$s_N f(x) = o(\log \log \log N)$$

for almost all x;

(3) if $f \in \mathbf{L}^2$, then

$$\lim_{N \to \infty} s_N f(x) = f(x)$$

for almost all x.

Subsequently, R. A. Hunt proved that in (3) the exponent 2 may be replaced by any exponent $p > 1$. For proofs, see [Ga] and/or [Moz], MR **49** # 5676. See also MR **52** # 6300 for an illuminating discussion.

We here remark merely that, granted the convergence almost everywhere of the Fourier series of each $f \in \mathbf{L}^p$ for any fixed p satisfying $1 < p \leqslant 2$, a theorem of Stein stated in Subsection 16.2.8 entails that the maximal operator

$$s^* : f \to s^* f$$

defined in Subsection 10.3.5 is of weak type (p, p) on $\mathbf{L}^p$, that is, there exists a number A_p such that

$$m(\{x \in [0, 2\pi) : s^*f(x) > \lambda\}) \leqslant A_p \lambda^{-p} \|f\|_p{}^p \tag{10.4.9}$$

for each $\lambda > 0$ and each $f \in \mathbf{L}^p$; and that to each number q satisfying $0 < q < p$ corresponds a number $A_{p,q}$ such that

$$\|s^* f\|_q \leqslant A_{p,q} \|f\|_p \tag{10.4.10}$$

for each $f \in \mathbf{L}^p$. The numbers A_p and $A_{p,q}$ are independent of f. In (10.4.9), m denotes Lebesgue measure. Concerning these matters, see Subsection 13.10.2.

Beside this, if $1 < p \leqslant \infty$ and $0 < q \leqslant \infty$, and if s^* is known to be of weak type (p, q) on $\mathbf{L}^p$, that is,

$$m(\{x \in [0, 2\pi) : s^*f(x) > \lambda\}) \leqslant (\text{const } \|f\|_p/\lambda)^q$$

for each number $\lambda > 0$ and each $f \in \mathbf{L}^p$, it is a relatively easy task to deduce that $\lim_{N\to\infty} s_N f(x) = f(x)$ for almost all x whenever $f \in \mathbf{L}^p$; see Exercise 13.26.

10.4.6. Sets of Divergence. What has been said about consequences of conditions like (10.4.6) and (10.4.7) will explain the intense interest that has been shown in the following type of problem. Take a *weight sequence* W defined on the nonnegative integers and such that $W(N)$ increases to infinity with N. If we suppose that

$$\sum_{n \in Z} |c_n|^2 W(|n|) < \infty, \tag{10.4.11}$$

what can be said regarding the size of the set E of points of divergence of the series

$$\sum_{n \in Z} c_n e^{inx} ?$$

It has turned out that, at least when $W(N)$ increases more rapidly than $\log N$ (compare (10.4.7)), new concepts of the thinness of sets, more refined than that provided by vanishing of the Lebesgue measure, come into play. Owing to a strong formal similarity with definitions in potential theory, the appropriate refined set functions are called *capacities*; see Section 12.12. The general type of conclusion to be drawn from (10.4.11) is that E is a set of zero capacity, where the particular type of capacity referred to depends on the weight function W. Several results of this sort are known and due to Beurling, Salem, and Zygmund and Temko (1940 onward). For details, see [Ba$_1$], pp. 398–414; [KS], Chapitre IV.

As has been seen in Section 8.6, if (10.4.5) is the Fourier series of some function $f \in \mathbf{L}^2$, then $s_{N_k}f \to f$ almost everywhere for any Hadamard sequence $(N_k)_{k=1}^{\infty}$ of positive integers (that is, any sequence of positive integers for which $\inf N_{k+1}/N_k > 1$). Also, by the Kolmogorov-Seliverstov-Plessner theorem, if (10.4.7) holds, then $s_N f \to f$ almost everywhere. To Salem we owe a general investigation of such assertions, based upon hypotheses of the type (10.4.11). This study produced further conditions on the sequence $(N_k)_{k=1}^{\infty}$ sufficient to ensure that (10.4.11) entails that $s_{N_k}f \to f$ almost everywhere. One such condition is that

$$\sum_{k=1}^{\infty} \frac{A^{-W(N_k)}}{W(N_k)} < \infty$$

for some $A > 0$; the case $W(N) = \log N$ includes the Kolmogorov-Seliverstov-Plessner result. Perhaps even more remarkable is the fact that Salem obtained conditions on the N_k sufficient to ensure that $s_{N_k}f \to f$ almost everywhere for *any* preassigned $f \in \mathbf{L}^1$: the condition reads

$$\sum_k \omega_1 f\left(\frac{1}{N_k}\right) \left|\log \omega_1 f\left(\frac{1}{N_k}\right)\right| < \infty,$$

where $\omega_1 f$ is defined as in 2.3.7. The details are presented in [Ba$_1$], pp. 389–397.

10.5 More about the Parseval Formula

Certain cases of the Parseval formula have already been discussed in 6.2.5 and Section 8.2 and we now extend the discussion a little. In all the cases we have to consider it is immaterial whether we take the (polarized) formula to be (8.2.4) or (8.2.5). For definiteness, we choose the latter. Thus we shall be concerned with the formula

$$\frac{1}{2\pi} \int f(x)g(x)\,dx = \sum_{n \in Z} \hat{f}(n)\hat{g}(-n). \tag{10.5.1}$$

10.5.1. If $f \in \mathbf{L}^1$ and $g \in \mathbf{BV}$, then (10.5.1) holds, the series being convergent.

Proof. We have seen in 10.1.4 that

$$s_N g(x) \to \tfrac{1}{2}\{g(x+0) + g(x-0)\}$$

boundedly. Moreover, the right-hand side here is equal to $g(x)$ save perhaps for a countable (and therefore null) set of x. So ([W], Theorem 4.1b) we have

$$\begin{aligned}\frac{1}{2\pi}\int fg\,dx &= \lim_{N\to\infty}\frac{1}{2\pi}\int f\cdot s_N g\cdot dx\\ &= \lim_{N}\sum_{|n|\leqslant N}\hat{f}(-n)\hat{g}(n),\end{aligned}$$

which is equivalent to (10.5.1).

Remark. As was remarked to me by Professor Goes (private correspondence) (10.5.1) holds for all $f \in \mathbf{L}^1$, if and only if

$$\sup_N \|s_N g\|_\infty < \infty;$$

cf. Remark 6.2.6(2) above.

10.5.2. If $f \in \mathbf{L}^1$ and $g \in \mathbf{L}^\infty$, the series $\sum_{n\in Z}\hat{f}(n)\hat{g}(-n)$ is Cesàro-summable to $(1/2\pi)\int fg\,dx$.

Proof. This proof is similar to that of 10.5.1, the sole difference being that $s_N g$ is replaced by $\sigma_N g$ and that appeal is made to 6.4.4 and 6.4.7 in place of 10.1.4. More simply, an appeal to 6.1.1 suffices; see 10.5.3.

10.5.3. If $1 < p < \infty$ and $f \in \mathbf{L}^p$, $g \in \mathbf{L}^{p'}$, then the series $\sum_{n\in Z}\hat{f}(n)\hat{g}(-n)$ is Cesàro-summable to $(1/2\pi)\int fg\,dx$.

Proof. Once again the same method is used, but now we appeal to the fact that $\sigma_N g \to g$ in mean in $\mathbf{L}^{p'}$ (see 6.1.1) coupled with Hölder's inequality, which shows that

$$\left|\frac{1}{2\pi}\int fg\,dx - \frac{1}{2\pi}\int f\cdot\sigma_N g\,dx\right| \leqslant \|f\|_p\cdot\|g-\sigma_N g\|_{p'}.$$

10.5.4. **Remark.** It is actually the case that, under the hypotheses of 10.5.3, the series $\sum_{n\in Z}\hat{f}(n)\hat{g}(-n)$ is convergent. This is so because, if $f \in \mathbf{L}^p$ and $1 < p < \infty$, then $s_N f$ converges in mean in $\mathbf{L}^p$ to f (which assertion is false if $p = 1$ or ∞). The proof of this result will be given later, in Section 12.10.

10.5.5. **Negative Results.** (1) It is not true that the series $\sum_{n\in Z}\hat{f}(n)\hat{g}(-n)$ is convergent whenever $f \in \mathbf{L}^1$ and $g \in \mathbf{C}$; see Exercise 10.7.

(2) By 10.5.4 we know that $\sum_{n\in Z}\hat{f}(n)\hat{g}(-n)$ is convergent whenever $1 < p < \infty$ and $f \in \mathbf{L}^p$, $g \in \mathbf{L}^{p'}$; and if $p = p' = 2$, the series is even absolutely convergent (see 8.2.2). It can be shown that for no value of $p \neq 2$ does $\sum_{n\in Z}\hat{f}(n)\hat{g}(-n)$ converge absolutely for all $f \in \mathbf{L}^p$ and all $g \in \mathbf{L}^{p'}$. [Notice that (1) immediately above rules out the possibility that absolute convergence might obtain for all $f \in \mathbf{L}^1$ and all $g \in \mathbf{L}^\infty$.]

10.6 Functions with Absolutely Convergent Fourier Series

10.6.1. The Space A. As has been noted in 2.5.3, the entity associated with the group T which is analogous and dual to the space $\mathbf{A}(Z)$ introduced in 2.3.9 is the space $\mathbf{A} = \mathbf{A}(R/2\pi Z)$ of functions on T of the form $\hat{\phi}$, where ϕ ranges over $\ell^1(Z)$. It is trivial to assert that $\mathbf{A}$ consists exactly of those continuous functions f on T such that

$$\|f\|_{\mathbf{A}} \equiv \sum_{n \in Z} |\hat{f}(n)| < \infty; \tag{10.6.1}$$

see also the characterization afforded by Exercise 9.7.

(1) It is left as a simple exercise for the reader to verify that $\mathbf{A}$ is a Banach space under pointwise linear operations and the norm defined in (10.6.1); that $\mathbf{A}$ is also an algebra under pointwise multiplication; and that

$$\|fg\|_{\mathbf{A}} \leqslant \|f\|_{\mathbf{A}} \cdot \|g\|_{\mathbf{A}}; \tag{10.6.2}$$

see Exercise 10.16. This means that $\mathbf{A}$, taken with pointwise operations and the norm (10.6.1), forms a commutative complex Banach algebra with the constant function 1 as its identity element. To this aspect we shall return in 11.4.1 and 11.4.17.

(2) That $\mathbf{A}$ is a proper subset of $\mathbf{C}$ is seen from 10.3.1, or by applying 7.2.2(1) to the sum function of the series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{(n \cdot \log n)};$$

the sum function of this series is even absolutely continuous (see 12.8.3(2)) and does not belong to $\mathbf{A}$. See also Exercise 10.18.

Incidentally, since the sum function f is absolutely continuous (hence of bounded variation), it ensues from the Remark following 10.6.2(1) below that f satisfies no Lipschitz condition of order $\alpha > 0$.

(3) It will appear in 12.11.3 that the problem of determining all the continuous linear functionals on $\mathbf{A}$ leads to a significant class of distributions.

10.6.2. The Classical Approach. The classical approach to $\mathbf{A}$ (for which the reader is referred to [Z_1], Chapter VI; [Ba_2], Chapter IX; [KS], Chapitre X; [Kah_2], especially Chapters I, II; and [I], pp. 66 ff.) has in the main concentrated attention on seeking conditions on an individual function f which are sufficient or necessary to ensure that $f \in \mathbf{A}$. Problems concerning the algebraic-topological structure of $\mathbf{A}$ belong to the modern approach mentioned in 10.6.3. For both aspects [Kah_2] is the most recent and perhaps the most incisive account.

Of the classical results, we shall handle in detail only two, contenting ourselves with brief references to the many others.

We begin with some calculations. By (8.5.7), if $f \in \mathbf{L}^2$,

$$[\omega_2 f(a)]^2 = \sum_{n \in Z} 4 \sin^2 \tfrac{1}{2}na \cdot |\hat{f}(n)|^2$$

and so

$$[\omega_2 f(2a)]^2 = 4 \sum_{n \in Z} \sin^2 na \cdot |\hat{f}(n)|^2.$$

Integrating with respect to a over $[0, \pi/N]$, where $N \in \{1, 2, \cdots\}$,

$$4 \sum_{n \in Z} \left(\int_0^{\pi/N} \sin^2 na \, da \right) |\hat{f}(n)|^2 \leqslant \pi N^{-1} [\Omega_2 f(2\pi N^{-1})]^2.$$

Now, if $1 \leqslant s \leqslant N^{-1}|n| < s + 1$, then

$$\begin{aligned}
\int_0^{\pi/N} \sin^2 na \, da &= |n|^{-1} \int_0^{|n|N^{-1}\pi} \sin^2 t \, dt \\
&\geqslant (s + 1)^{-1} N^{-1} \int_0^{s\pi} \sin^2 t \, dt \\
&= (s + 1)^{-1} N^{-1} s \int_0^{\pi} \sin^2 t \, dt \\
&= (s + 1)^{-1} N^{-1} s \cdot \pi/2 \\
&\geqslant (4N)^{-1} \pi.
\end{aligned}$$

Hence

$$\gamma_N^2 \equiv \sum_{|n| \geqslant N} |\hat{f}(n)|^2 \leqslant [\Omega_2 f(2\pi N^{-1})]^2. \tag{10.6.3}$$

Moreover

$$\begin{aligned}
\|\hat{f}\|_1 \equiv \sum_{n \in Z} |\hat{f}(n)| &= |\hat{f}(0)| + \sum_{n=1}^{\infty} (|\hat{f}(n)| + |\hat{f}(-n)|) \\
&= |\hat{f}(0)| + \sum_{n=1}^{\infty} \sum_{N=1}^{n} n^{-1} (|\hat{f}(n)| + |\hat{f}(-n)|) \\
&= |\hat{f}(0)| + \sum_{N=1}^{\infty} \sum_{n \geqslant N} n^{-1} (|\hat{f}(n)| + |\hat{f}(-n)|) \\
&\leqslant |\hat{f}(0)| + \sum_{n=1}^{\infty} \sum_{n \geqslant N} n^{-1} 2^{1/2} (|\hat{f}(n)|^2 + |\hat{f}(-n)|^2)^{1/2} \\
&\leqslant |\hat{f}(0)| + \sum_{N=1}^{\infty} \left(\sum_{n \geqslant N} 2n^{-2} \right)^{1/2} \left(\sum_{n \geqslant N} (|\hat{f}(n)|^2 + |\hat{f}(-n)|^2) \right)^{1/2} \\
&= |\hat{f}(0)| + \sum_{N=1}^{\infty} 2N^{-1/2} \left(\sum_{|n| \geqslant N} |\hat{f}(x)|^2 \right) \\
&= |\hat{f}(0)| + 2 \sum_{N=1}^{\infty} N^{-1/2} r_N.
\end{aligned}$$

By (10.6.3), therefore

$$\|\hat{f}\|_1 \leqslant |\hat{f}(0)| + 2 \sum_{N=1}^{\infty} N^{-1/2}\Omega_2 f(2\pi N^{-1}). \tag{10.6.4}$$

Next, supposing f to be of bounded variation and a to satisfy $0 < a \leqslant 2\pi/N$, one has

$$\sum_{k=1}^{4N} |f(x + ka) - f(x + (k - 1)a)|^2 \leqslant \Omega_\infty f(2\pi N^{-1})\cdot V(f).$$

Integrating with respect to x and using translation invariance of the integral,

$$4N\cdot[\omega_2 f(a)]^2 \leqslant (2\pi)^{-1}\Omega_\infty f(2\pi N^{-1})\cdot V(f)$$

and so

$$\Omega_2 f(2\pi N^{-1})^2 \leqslant (8\pi N)^{-1}\Omega_\infty f(2\pi N^{-1})\cdot V(f).$$

Hence, (10.6.4) yields

$$\|\hat{f}\|_1 \leqslant |\hat{f}(0)| + 2(8\pi)^{-1/2} V(f) \sum_{N=1}^{\infty} N^{-1}\Omega_\infty f(2\pi N^{-1})^{1/2}. \tag{10.6.5}$$

From (10.6.4) and (10.6.5) one can read off as corollaries a number of results.

(1) If f is of bounded variation and

$$\sum_{k=1}^{\infty} (\Omega_\infty f(2^{-k}\pi))^{1/2} < \infty, \tag{10.6.6}$$

then $\hat{f} \in \ell^1$; in particular, if f is also continuous, then $f \in \mathbf{A}$. Notice that (10.6.6) holds whenever $\Omega_\infty f(a) = O(|a|^\alpha)$ as $a \to 0$ for some $\alpha > 0$.

This result is due to Zygmund; see [Kah$_2$], p. 13. See also Exercise 10.17.

Proof. Apply (10.6.5), noticing that

$$\begin{aligned}\sum_{N=2}^{\infty} N^{-1}\Omega_\infty f(2\pi N^{-1})^{1/2} &= \sum_{k=2}^{\infty} \sum_{2^{k-1}\leqslant N < 2^k} N^{-1}\Omega_\infty f(2\pi N^{-1})^{1/2} \\ &\leqslant \sum_{k=2}^{\infty} 2^{k-1}\cdot(2^{k-1})^{-1}\Omega_\infty f(\pi 2^{-k+2})^{1/2} \\ &= \sum_{k=2}^{\infty} \Omega_\infty f(2^{-k+2}\pi)^{1/2}.\end{aligned}$$

(2) If $f \in \mathbf{L}^2$ and

$$\sum_{N=1}^{\infty} N^{-1/2}\Omega_2 f(2\pi N^{-1}) < \infty, \tag{10.6.7}$$

then $\hat{f} \in \ell^1$. Notice that (10.6.7) holds whenever

$$\sum_{k=1}^{\infty} 2^{k/2}\Omega_2 f(2^{-k}\pi) < \infty,$$

and that both criteria are fulfilled whenever

$$\Omega_2 f(a) = O(|a|^\alpha) \quad \text{as } a \to 0 \text{ for some } \alpha > \tfrac{1}{2}.$$

These results are due to Bernstein and Szász; see [Ba_2], pp. 154–155 and [Kah_2], pp. 13–14. See also Exercises 10.25 and 13.2.

Proof. This follows from (10.6.4), just as (1) followed from (10.6.5).

Remarks and further results.

The example cited in 10.6.1(2) shows that conditions like (10.6.6), bearing upon moduli of continuity of f, cannot be entirely suppressed in the hypotheses of (1). For somewhat similar results, see Hirschman [1], Lemmas 2d and 3c (the latter applying to the dual situation) and also Boas [1].

There is a converse to (2): it is known that there exist continuous functions f such that $\Omega_2 f(a) = O(|a|^{1/2})$ as $a \to 0$ and which nonetheless do *not* belong to **A**. See [Z_1], pp. 240–243; [KS], p. 129; [Kah_2], pp. 14–45. See also Mitjagin [1], Yadov and Goyal [1].

There is also a similar converse to (1); see [Kah_2], p. 16.

It is known that no condition involving only $\Omega_\infty f$ can be at once necessary and sufficient in order that f be equal a.e. to a member of **A**.

Further reading: MR **34** ## 2790, 3197, 4797; **35** # 7081; 39 ## 1912, 6015; **40** # 7729.

(3) Other sufficient conditions for membership of f to **A** involve the numbers

$$E_N^{(p)} f = \inf \{\|f - t\|_p : t \in \mathbf{T}_N\};$$

cf. Section 6.5, where $E_N^{(\infty)} f$ is written as $E_N f$, and Exercise 6.10. Among such results we note that of Stečkin, which asserts that, if $f \in \mathbf{L}^2$, then $\hat{f} \in \ell^1$ whenever

$$\sum_{N=1}^{\infty} N^{-1/2} E_N^{(2)} f < \infty;$$

see [Ba_2], p. 155. Some of Stečkin's results apply to general orthogonal expansions and thus yield criteria in order that

$$\sum_{n \in S}^{\infty} |\hat{f}(n)| < \infty$$

for preassigned subsets S of Z. See also Yadov [1] and Žuk [1].

(4) It follows easily from 8.2.1 and 8.3.1 that $f \in \mathbf{A}$ if and only if it can be expressed in the form $f = u * v$ with $u, v \in \mathbf{L}^2$. This criterion was first noted by M. Riesz; unfortunately, it is difficult to apply in specific cases which are not already decidable in more evident ways.

Somewhat similar results have been given by M. and S.-I. Izumi [4] and others.

(5) Another necessary and sufficient condition, discovered by Stečkin in 1951, asserts that, if $f \in \mathbf{L}^2$, then $f \in \ell^1$ if and only if

$$\sum_{N=1}^{\infty} N^{-1/2} \cdot {}_*E_N f < \infty,$$

where

$${}_*E_N f = \inf \|f - S\|_2,$$

the infimum being taken as S ranges over all (not necessarily harmonic) trigonometric polynomials of the form

$$S(x) = \sum_{n=1}^{N} c_n \cdot \exp(i\lambda_n x),$$

where the c_n are complex numbers and the λ_n are distinct real numbers; see [Ba_2], p. 186; [Kah_2], p. 10. Compare this with the sufficient condition given in (3), noting that evidently

$${}_*E_{2N+2} f \leqslant {}_*E_{2N+1} f \leqslant E_N^{(2)} f.$$

(6) Wiener showed that a function on T, which agrees on a neighborhood of each point x with some (possibly x dependent) element of $\mathbf{A}$, itself belongs to $\mathbf{A}$: in brief, a function which belongs locally to $\mathbf{A}$ belongs globally to $\mathbf{A}$. We shall not give the original proof of this (for which see [Z_1], p. 245; [Ba_2], p. 188; [Kah_2], p. 11) but rather a proof based upon Banach algebra theory; see Exercise 11.19.

A simple sufficient condition for local membership of $\mathbf{A}$ may be derived from Exercise 13.3.

(7) If one introduces prematurely the conjugate function $\tilde{f}$, defined in Section 12.8, one can state the remarkable result of Hardy and Littlewood asserting that, if both f and $\tilde{f}$ are of bounded variation, then

$$\sum_{n \in Z} |\hat{f}(n)| < \infty.$$

For a proof, see [Z_1], pp. 242 and 287. Compare this with F. and M. Riesz' theorem mentioned in 12.8.5(4). See also Exercise 12.19.

(8) Let E denote a closed subset of T. Almost every question so far posed in relation to $\mathbf{A}$ can also be posed in relation to $\mathbf{A}(E)$, the set of restrictions to E of elements of $\mathbf{A}$. For such variants we must refer the reader to [KS], Chapitre X, especially pp. 130 ff and [Kah_2], p. 19.

It is of course evident that in all cases $\mathbf{A}(E)$ is a subset of $\mathbf{C}(E)$, the space of all continuous complex-valued functions on E. Much less evident is the fact that there are infinite closed sets E such that $\mathbf{A}(E)$ exhausts $\mathbf{C}(E)$: such sets E are termed "Helson sets," concerning which a little more will be said in Section 15.7; see especially Subsection 15.7.3. See also [Kah_2], Chapitres III, IV, IX; MR **40** ## 630, 7731; **43** # 6660; **55** ## 966, 970.

10.6.3. The Modern Approach. (1) In more recent times emphasis has been placed on structural properties of **A** as a whole. Thus, one of the problems on which attention has been centered is the following: under what conditions upon the function F, defined on some subset D of the complex plane, is it true that $F \circ f \in \mathbf{A}$ whenever $f \in \mathbf{A}$ and $f(R) = f(T) \subset D$?

As a matter of fact, some aspects of this problem were first considered by Lévy as long ago as 1934; and even prior to this Wiener had obtained a special case of Lévy's result. Wiener showed that $f^{-1} \in \mathbf{A}$ whenever $f \in \mathbf{A}$ and f is nonvanishing, while Lévy showed that a function F has the property mentioned in the preceding paragraph whenever D is open and F is analytic at each point of D. The modern approach to these problems differs from the original (for which see [Z_1], pp. 245–247; [Ba_2], pp. 186–194; [Kah_2], pp. 57, 58) in the methods utilized, namely, the theory of Banach algebras. We shall deal with these theorems by use of this technique in 11.4.17; the dual results appear in 11.4.13 and 11.4.16.

In 1958 Katznelson discussed the necessity of Lévy's sufficient condition, and variants and analogues of the Lévy-Katznelson results have been examined for cases in which the underlying group T is replaced by a more general group. Katznelson, Helson, Rudin, and Kahane have all contributed to these problems; see Herz [1], Rudin [2], [5], [R], Chapter 6; [Kah], Capítulos IV to VI; [Kah_2], Chapitre VI; [Kz], Chapter VIII. Here we mention merely the versions of the Lévy-Katznelson results appropriate to the groups T and Z, namely:

(a) If F is defined on $[-1, 1]$ and $F \circ f \in \mathbf{A}$ whenever $f \in \mathbf{A}$ and $f(T) \subset [-1, 1]$, then F is analytic on $[-1, 1]$.

(b) If F is defined on $[-1, 1]$ and $F \circ \phi \in \mathbf{A}(Z)$ whenever $\phi \in \mathbf{A}(Z)$ and $\phi(Z) \subset [-1, 1]$, then F is analytic at 0 and $F(0) = 0$.

Of these statements, (a) is due to Katznelson and (b) to Helson and Kahane jointly. The assertion that F is analytic on $[-1, 1]$ (respectively at 0) signifies that F is extendible into a function analytic on some open subset of the complex plane containing $[-1, 1]$ (respectively 0).

From (a) it follows as a corollary that there exist functions $f \in \mathbf{A}$ such that $|f|$ does not belong to **A**; the analogous assertion with $\mathbf{A}(Z)$ in place of **A** follows likewise from (b). (Both of these corollaries were established a little earlier by Kahane [2], [3].) On the other hand, Beurling has shown that if $f \in \mathbf{A}$ is such that $|\hat{f}(\pm n)| \leqslant c_n$ $(n = 0, 1, 2, \cdots)$, where $c_n \downarrow 0$ and $\sum_{n=0}^{\infty} c_n < \infty$, then $|f| \in \mathbf{A}$; see [I], p. 78.

(2) There are analogous problems in which $\mathbf{A}(Z)$ is replaced by $\mathbf{A}^p(Z) \equiv \{\hat{f} : f \in \mathbf{L}^p\}$. The case in which $2 \leqslant p \leqslant \infty$ is solved by Rider [2], who shows (among other things) that a complex-valued function F on the complex plane has the property that $F \circ \phi \in \mathbf{A}^p(Z)$ whenever $\phi \in \mathbf{A}^p(Z)$ if and only if it has the form

$$F(z) = az + b\bar{z} + |z|^{2/p'}c(z),$$

where a and b are complex numbers and the function c is bounded on a neighborhood of the origin. (For the "if" assertion, see Exercise 13.24.) See also Rudin [4].

The same problem has been studied for yet other important algebras; see the end of 11.4.17 below.

(3) This is a convenient place to comment further on the dual form of the problem stated in 4.2.5, namely: which maps Φ of T into itself have the property that $f \circ \Phi \in \mathbf{A}$ whenever $f \in \mathbf{A}$? By dualizing the substance of Chapter 4, it may be seen that these maps Φ correspond to the homomorphisms of the convolution algebra $\ell^1(Z)$. Insofar as this can be and has been subsumed under the study of the $\mathbf{L}^1$ homomorphism problem for general groups, the appropriate reference is again [R], Chapter 4. For the particular group we have in mind, an independent and more direct solution is due in part to Leibenson and in part to Kahane (1954–56); an account of Kahane's approach will be found in Capítulo III of [Kah]; see also [Kah$_2$], p. 86 and Chapitre IX. A mapping Φ having the stated property may be said to define a *permissible change of variable* (relative to A), and the Leibenson-Kahane result asserts that the permissible changes of variable are precisely those defined by maps Φ having the form $\Phi: \dot{x} \to (nx + a)^{\cdot}$, where $n \in Z$ and $a \in R$.

In Kahane's approach to this problem one first thinks of Φ, which must obviously be continuous, as a (periodic) map from R into T. For each $x \in R$, $\Phi(x) \in T$ and $e^{i\Phi(x)}$ is uniquely defined. In $x \to e^{i\Phi(x)}$ one has a continuous map of R into the multiplicative group of complex numbers having unit absolute value. A simple argument (using local branches of the logarithm) shows that there exists a continuous real-valued function ϕ on R such that $e^{i\Phi(x)} = e^{i\phi(x)}$, so that $\Phi(x) = (\phi(x))^{\cdot}$, the coset modulo T containing the real number $\phi(x)$. Since Φ is periodic, ϕ must have the property that

$$\phi(x + 2\pi) - \phi(x) = 2n\pi$$

identically in x, n being some integer. The crucial point of this transformation is that ϕ is a complex- (actually real-) valued function on R, which Φ is not. However, ϕ is not necessarily periodic and we make one further change of focus to take care of this, namely, we look at the periodic real-valued function ϕ_0 defined by

$$\phi_0(x) = \phi(x) - nx.$$

By using the fact that $f \circ \phi = f \circ \Phi \in \mathbf{A}$ whenever $f \in \mathbf{A}$, it is relatively easy to deduce that

(c) $\sup_{k\in Z}\|e^{ik\phi_0}\|_{\mathbf{A}} = \sup_{k\in Z}\|e^{ik\phi}\|_{\mathbf{A}} < \infty$

and that

(d) $\phi_0 \in \mathbf{A}$;

for (c), see Exercise 10.19; the proof of (d) depends upon and is an easy corollary of the result stated in 10.6.2(6).

The crux of Kahane's argument is the difficult deduction from (c) and (d) that ϕ_0 is a constant.

(4) For a study of isomorphisms and homomorphisms of the algebras $\mathbf{A}(E)$

defined in Subsection 10.6.2(8), see de Leeuw and Katznelson [1], McGehee [1], and [Kah$_2$], Chapitre IX. (In the terminology introduced in Chapter 11, $A(E)$ is isomorphic to the quotient algebra of $\ell^1(Z)$ modulo the ideal $\mathbf{I}_E$ composed of those $\phi \in \ell^1(Z)$ such that $\hat{\phi}$ vanishes on E.)

For further reading, consult MR **34** ## 3226, 3365, 3366, 4793, 4796; **35** ## 670, 3373; **36** ## 1924, 6880; **37** ## 6672, 6690; **41** # 744; **49** ## 5723, 7700; **50** ## 895, 5338; **51** ## 1289, 6627; **52** ## 8777, 8798, 8805, 14816; **53** ## 8791, 8792; **54** ## 856, 858, 8160, 8163.

EXERCISES

10.1. Prove the statement in 10.1.2, namely

$$\sup\left\{\left|\int_a^b t^{-1}\sin t\,dt\right| : a \in R, \quad b \in R\right\} < \infty.$$

10.2. Let $(\beta_N)_{N=1}^{\infty}$ be a sequence of positive numbers converging to zero. Show that there exists an $f \in \mathbf{L}^1$ such that

$$\limsup_{N\to\infty} \|s_N f\|_1/(\beta_N \log N) = \infty.$$

Verify that nonetheless

$$\|s_N f\|_1 = o(\log N) \qquad \text{as } N \to \infty$$

for any $f \in \mathbf{L}^1$.

Hints: For the first part, adapt the type of proof used in 10.3.2. For the second part, use the Fubini-Tonelli theorem to show that

$$\|s_N f - f\|_1 \leqslant \frac{1}{2\pi}\int \|T_{-y}f - f\|_1 |D_N(y)|\,dy,$$

and use the fact that $\|T_{-y}f - f\|_1 \to 0$ with y.

10.3. By examining the proof of 10.4.1, show that the relation (10.4.1) holds uniformly with respect to x when f is continuous.

10.4. Show that if f is continuous, then the series

$$\sum_{n\in Z} \frac{\hat{f}(n)e^{inx}}{\log(2 + |n|)}$$

is uniformly convergent.

Hints: Use the preceding exercise and examine the proof of 10.4.3. Alternatively, put $S(f) = \sum_{n\in Z}\hat{f}(n)/\log(2 + |n|)$ and $S_N(f) = \sum_{|n|\leqslant N}\hat{f}(n)/\log(2 + |n|)$. Verify that $S_N(f) \to S(f)$ for each $f \in \mathbf{C}$ and use the uniform boundedness principle (Appendix B.2.1(2)) to deduce that the S_N are equicontinuous on $\mathbf{C}$. Conclude that $S_N(f) \to S(f)$ uniformly when f ranges over a relatively compact subset of $\mathbf{C}$, and hence that $S_N(T_x f) \to S(T_x f)$ uniformly in x for a given $f \in \mathbf{C}$.

10.5. Prove that $\sum_{n \in Z} \hat{f}(n)\hat{g}(n)/\log(2 + |n|)$ is convergent whenever $f \in \mathbf{C}$ and $g \in \mathbf{L}^1$. Deduce that there exists a number c such that

$$\Big\| \sum_{|n| \leqslant N} \frac{\hat{g}(e)e_n}{\log(2 + |n|)} \Big\|_1 \leqslant c \cdot \|g\|_1$$

for all N and all $g \in \mathbf{L}^1$ (c being independent of N and g), and that the series $\sum_{n \in Z} \hat{g}(n)e_n/\log(2 + |n|)$ is convergent in mean in $\mathbf{L}^1$ for each $g \in \mathbf{L}^1$.

Conclude finally that the series $\sum_{n \in Z} \hat{f}(n)\hat{g}(n)/\log(2 + |n|)$ is convergent whenever $f \in \mathbf{L}^\infty$ and $g \in \mathbf{L}^1$.

Hints: For the first statement use the uniform boundedness principle (Appendix B.2.1(2)) and Exercise 3.6. Deduce from this that the set of $g \in \mathbf{L}^1$, for which $\sum_{n \in Z} \hat{g}(n)e_n/\log(2 + |n|)$ is convergent in mean in $\mathbf{L}^1$, is closed in $\mathbf{L}^1$.

10.6. Prove that if $(\lambda_n)_{n=0}^{\infty}$ is a sequence such that $\sum_{n=0}^{\infty} |\Delta\lambda_n| < \infty$, then the series $\sum_{n \in Z} \lambda_{|n|}\hat{g}(n)e_n/\log(2 + |n|)$ is convergent in mean in $\mathbf{L}^1$ whenever $g \in \mathbf{L}^1$.

Hints: Use the preceding exercise, together with an adaptation of Exercise 7.1.

10.7. Show that the series $\sum_{n \in Z} \hat{f}(n)\hat{g}(-n)$ diverges for suitably chosen $f \in \mathbf{L}^1$ and all $g \in \mathbf{C}$. (Compare with 10.5.5(1).)

Hint: Argue by contradiction, using the uniform boundedness principle and Exercise 10.2.

10.8. Let j be the periodic function such that $j(x) = \frac{1}{2}(\pi - x)$ for $0 \leqslant x < 2\pi$. Show that the Fourier series of j is $\sum_{n=1}^{\infty} \sin nx/n$. Verify that for $0 \leqslant x \leqslant \pi$,

$$s_N j(x) + \tfrac{1}{2}x = \tfrac{1}{2}\int_0^x D_N(y)\,dy$$

and deduce that

(1) $\displaystyle\lim_{N \to \infty} \Big\{s_N j(x) + \tfrac{1}{2}x - \int_0^{(N+\frac{1}{2})x} \frac{\sin t\,dt}{t}\Big\} = 0$ uniformly for $0 \leqslant x \leqslant \pi$;

(2) $\displaystyle\lim_{N \to \infty} s_N j(x) = j(x)$ for $0 < x < \pi$;

(3) $\displaystyle\lim_{N \to \infty} s_N j\left(\frac{a}{N}\right) = \int_0^a \frac{\sin t\,dt}{t}$ for $a > 0$;

(4) $\displaystyle\limsup_{N \to \infty, x \to +0} s_N j(x) \geqslant \int_0^{\pi} \frac{\sin t\,dt}{t} > \tfrac{1}{2}\pi = j(+0).$

Notes: Conclusion (4) shows that the sequence $(s_N j)$ of functions exhibits the so-called *Gibbs phenomenon* on right-hand neighborhoods of zero. A similar situation prevails on left-hand neighborhoods of zero. This exhibition is typical of the sequence $(s_N f)$ whenever f is of bounded variation and has jump discontinuities; see [Z_1], pp. 61–62. The Gibbs phenomenon at a point x_0 of the sequence $(s_N f)$ is a feature of the nonuniformity of the convergence on neighborhoods of a point of discontinuity.

For many more details, plus a most interesting survey of the history of the Gibbs phenomenon, see Hewitt and Hewitt [1].

10.9. Discuss the following suggested procedure, and in particular frame hypotheses sufficient to justify the steps: take a function F on R and form the periodic function

$$f(x) = \sum_{k \in Z} F(x + 2\pi k).$$

Then

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(y) e^{-iny}\, dy \qquad (n \in Z),$$

and so, since $f(0) = \sum_{n \in Z} \hat{f}(n)$, we obtain *Poisson's summation formula*:

$$\sum_{k \in Z} F(2k\pi) = \frac{1}{2\pi} \sum_{n \in Z} \int_{-\infty}^{\infty} F(y) e^{-iny}\, dy.$$

See also [Kz], p. 129; MR **36** # 4265; **54** # 5734.

10.10. Justify the use of Poisson's summation formula (Exercise 10.9) in case $F(y) = (a^2 + y^2)^{-1}$ (a real and nonzero) and so deduce that

$$\sum_{k \in Z} (a^2 + 4\pi^2 k^2)^{-1} = \frac{(2|a|)^{-1}(1 + e^{-|a|})}{1 - e^{-|a|}}$$

for such values of a. Conclude that

$$\sum_{k=1}^{\infty} k^{-2} = \frac{\pi^2}{6}.$$

10.11. Justify the use of Poisson's summation formula (Exercise 10.9) in case $F(y) = \exp(-a^2 y^2)$ (a real and nonzero), and so deduce that

$$\sum_{k \in Z} e^{-k^2 s} = \left(\frac{\pi}{s}\right)^{1/2} \sum_{n \in Z} e^{-\pi^2 n^2 / s}$$

for real $s > 0$.

Note: This is a famous transformation formula for one of the so-called theta functions; see [Be], p. 11.

10.12. Denote by **CBV** the linear space of (periodic) functions that are continuous and of bounded variation. Show that **CBV** is a Banach space when endowed with the norm

$$\|f\| = \|f\|_\infty + V(f).$$

Using the uniform boundedness principle (Appendix B.2.1) and the result (included in 10.1.4) that $\sup_N |s_N f(0)| < \infty$ for $f \in \mathbf{CBV}$, deduce inequality (10.1.5), that is, the existence of a number $m \geqslant 0$, independent of f, such that

$$\sup_N \|s_N f\|_\infty \leqslant m \cdot [\|f\|_\infty + V(f)]$$

for $f \in \mathbf{CBV}$.

Remark. I am grateful to Professor G. Goes for the remark that, by 7.2.2(2), the series $h(x) = 2\sum_{n=1}^\infty n^{-1} \sin nx$ is boundedly convergent. If f is of bounded variation, integration by parts yields

$$s_N f(x) = \hat{f}(0) + \frac{1}{2\pi}\int_0^{2\pi} s_N h(x - y)\, df(y)$$

and so

$$\|s_N f\|_\infty \leqslant |\hat{f}(0)| + \|s_N h\|_\infty \cdot V(f) \leqslant |\hat{f}(0)| + \text{const. } V(f).$$

10.13. Prove that $\lim_{N\to\infty} s_N f = f$ uniformly for each $f \in \mathbf{CBV}$.

Hint: Re-examine the proof of 10.1.1.

Alternatively, use Exercises 5.5 and 8.13 and Theorem 6.1.1. See also MR 50 #10657.

10.14. Is it true to assert that $\lim_{N\to\infty} s_N f = f$, in the sense of the norm on **CBV** defined in Exercise 10.12, whenever $f \in \mathbf{CBV}$?

Is it true to assert that the trigonometric polynomials are everywhere dense in the space **CBV** (relative to the norm defined in Exercise 10.12)?

Full justification is required for your answers. You may assume and use the fact that there exist continuous functions of bounded variation which are not absolutely continuous; see Remark (2) following 2.3.6.

10.15. Prove that the sequence $(e_n)_{n\in Z}$ is not a topological base for any one of the spaces $\mathbf{L}^1$, $\mathbf{L}^\infty$, and $\mathbf{C}$.

Hint: Assuming the contrary, determine the form of the coefficients α_n in the associated expansion (10.3.12).

10.16. Verify in detail the statements made in 10.6.1(1) concerning **A**.

10.17. Prove that, if f is of bounded variation and

$$\Omega_\infty f(a) = O((\log |a|^{-1})^{-\beta})$$

as $a \to 0$ for some $\beta < 2$, then $\hat{f} \in \ell^1$.

Hint: Use (10.6.5).

10.18. (1) Suppose that $\sum_{n\in Z}|c_n| < \infty$ and that $nc_n \neq o(1)$ as $|n| \to \infty$. Show that $f(x) = \sum_{n\in Z} c_n e^{inx}$ belongs to **A** and is not absolutely continuous.

(2) Use Exercise 3.14 to prove that there exist absolutely continuous functions which do not belong to **A** (compare 10.6.1(2) and Exercise 10.25).

10.19. Suppose ϕ is as in 10.6.3(3), namely, a real-valued function on R such that

$$\phi(x + 2\pi) - \phi(x) = 2n\pi$$

for some integer n and all $x \in R$, and having the property that $f \circ \phi \in \mathbf{A}$ whenever $f \in \mathbf{A}$. Prove that

$$\sup_{k \in Z} \|e^{ik\phi}\|_{\mathbf{A}} < \infty.$$

Hint: Consider the linear mapping T of $\mathbf{A}$ into itself defined by $Tf = f \circ \phi$. Show that T has a closed graph and apply Appendix B.3.3.

10.20. (S. Saks) Let $\mathbf{E}$ be a Banach space and $(S_N)_{N=1}^{\infty}$ a sequence of continuous linear operators from $\mathbf{E}$ into $\mathbf{L}^p$, where $0 < p \leqslant \infty$. Suppose further that $\mathbf{F}$ is a nonmeager subset of $\mathbf{E}$, that α is a positive number, and that to each $f \in \mathbf{F}$ corresponds a set A_f with Lebesgue measure $m(A_f) > \alpha$ such that $\limsup_{N \to \infty} |S_N f(x)| < \infty$ for almost all $x \in A_f$.

Prove the following two statements:

(1) if $0 < \varepsilon < \alpha$, there exists a set A such that $m(A) > \alpha - \varepsilon$ and $\limsup_{N \to \infty} |S_N f(x)| < \infty$ for almost all $x \in A$ for each $f \in \mathbf{E}$;

(2) there exists a set A and a meager subset $\mathbf{E}_0$ of $\mathbf{E}$ such that

$$\limsup_{N \to \infty} |S_N f(x)| < \infty \quad \text{for almost all } x \in A$$

whenever $f \in \mathbf{E}$, and

$$\limsup_{N \to \infty} |S_N f(x)| = \infty \quad \text{for almost all } x \notin A$$

whenever $f \in \mathbf{E} \backslash \mathbf{E}_0$.

Hints: See [KSt], p. 25 or [GP], p. 153.

Remark. The results are true if $\mathbf{E}$ is a Fréchet space; and $\mathbf{L}^p$ can be replaced by more general spaces of functions.

10.21. Assuming Kolmogorov's theorem (asserting the existence of at least one integrable function whose Fourier series diverges almost everywhere), deduce that the set of functions $f \in \mathbf{L}^1$, whose Fourier series diverge almost everywhere, is a comeager subset of $\mathbf{L}^1$ (that is, is a subset of $\mathbf{L}^1$ whose complement is meager and which is therefore itself nonmeager).

Hint: Use the preceding exercise.

10.22. Employ the uniform boundedness principle (see Appendix B.2.1) to prove the following statement: if ϕ is a nonnegative function on Z such that the set $N = \{n \in Z : \phi(n) > 0\}$ is infinite and

$$\liminf_{n \in N, |n| \to \infty} \phi(n) = 0,$$

then there exist functions $f \in \mathbf{L}^1$ such that

$$\limsup_{n \in N, |n| \to \infty} \frac{|\hat{f}(n)|}{\phi(n)} = \infty.$$

(See the opening remarks in Section 10.3.)

10.23. Let U be a linear operator from $\mathbf{T}$ onto $\mathbf{T}_N$ such that $Uf = f$ for $f \in \mathbf{T}_N$ (so that U is a linear projection of $\mathbf{T}$ onto $\mathbf{T}_N$). Prove that

$$\frac{1}{2\pi}\int UT_a f(x+a)\,da = s_N f(x)$$

for $f \in \mathbf{T}$.

Defining

$$\|U\|_{p,p} = \sup\{\|Uf\|_p : f \in \mathbf{T},\ \|f\|_p \leqslant 1\},$$

and similarly for $\|s_N\|_{p,p}$, deduce that

$$\|U\|_{1,1} \geqslant \|s_N\|_{1,1},\ \|U\|_{\infty,\infty} \geqslant \|s_N\|_{\infty,\infty}.$$

10.24. Let $\mathbf{E}$ denote $\mathbf{C}$ or $\mathbf{L}^1$, and let $(N_k)_{k=1}^{\infty}$ be a sequence of positive integers such that $\sup_k N_k = \infty$. Suppose that U_k is a linear projection from $\mathbf{E}$ onto $\mathbf{T}_{N_k}$ which is continuous from $\mathbf{E}$ into $\mathbf{E}$. Prove that there exists an $f \in \mathbf{E}$ such that

$$\limsup_{k\to\infty} \|U_k f\|_{\mathbf{E}} = \infty. \tag{1}$$

Remarks. In case $\mathbf{E} = \mathbf{C}$, it is not known whether in all cases $f \in \mathbf{C}$ and $x \in T$ exist such that $\limsup_{k\to\infty} |U_k f(x)| = \infty$, but this is easily deduced from (1) if each U_k commutes with translations. In case $\mathbf{E} = \mathbf{L}^1$, it is likewise in doubt whether $f \in \mathbf{L}^1$ exists such that $\limsup_{k\to\infty}|U_k f(x)| = \infty$ for almost all x (or even for those x in some set of positive measure). See 10.3.4 for the case in which $N_k = k$ and $U_k = s_k$.

Hints: Use the preceding exercise and (5.1.10) in combination with the uniform boundedness principle (Appendix B.2.2).

10.25. Suppose that f is absolutely continuous and that $Df \in \mathbf{L}^2$. Prove that $f \in \mathbf{A}$ and that

$$\|f\|_{\mathbf{A}} \leqslant |\hat{f}(0)| + B\|Df\|_2.$$

where B is an absolute constant.

Remark. Stronger results appear in Exercises 13.2 and 13.3; compare also Exercise 10.18.

Hint: Examine the proof of 8.5.4 and use a similar method.

10.26. Suppose that $0 < \varepsilon < \pi/2$ and let v_ε be the continuous nonnegative even periodic function that vanishes outside $[-\varepsilon, \varepsilon]$ (mod 2π), is linear on $[0, \varepsilon]$, and satisfies $\|v_\varepsilon\|_1 = 1/2\pi$. Verify that $v_\varepsilon \in \mathbf{A}$ and $\|v_\varepsilon\|_{\mathbf{A}} = \varepsilon^{-1}$.

Define $u_\varepsilon = 4\varepsilon v_{2\varepsilon} - \varepsilon v_\varepsilon$ and verify that $u_\varepsilon = 1$ on $[-\varepsilon, \varepsilon]$, that u_ε vanishes outside $[-2\varepsilon, 2\varepsilon]$ (mod 2π), and that $\|u_\varepsilon\|_{\mathbf{A}} \leqslant 3$.

Suppose that $f \in \mathbf{A}$ satisfies $f(0) = 0$. Show that $\lim_{\varepsilon\to 0}\|u_\varepsilon f\|_{\mathbf{A}} = 0$.

Remark. This construction will find a use in connection with spectral synthesis sets; see Exercise 12.52 and compare [R], Theorem 2.6.4.

Hints: For the second part, show first that it is enough to deal with the case in which $f \in \mathbf{T}$. In this case, estimate the $\mathbf{L}^1$-norm of $u_\varepsilon f$ and the $\mathbf{L}^\infty$-norm of $D(u_\varepsilon f)$, and then use the preceding exercise.

APPENDIX A

Metric Spaces and Baire's Theorem

It is assumed that the reader is familiar with the definition and simple properties of a metric space and its metric topology. The aim of this appendix is to introduce the concepts of meager and nonmeager sets, to prove Baire's "category theorem," and to give some corollaries thereof (some of which form the basis of results given in Appendix B). See also [K], pp. 200–203, or [HS], p. 68.

A.1 Some Definitions

Let $\mathbf{E}$ be a metric space (or any topological space). A subset A of $\mathbf{E}$ is said to be:

(1) *nowhere dense* (or *nondense*) if the closure $\bar{A}$ contains no interior points;

(2) *everywhere dense* if $\bar{A} = \mathbf{E}$;

(3) *meager* (or *of first category*) it is expressible as the union of countably many nowhere dense sets;

(4) *nonmeager* (or *of second category*) if it is not meager;

(5) *comeager* if its complement is meager.

The reader will notice that a set A is nowhere dense if and only if the open set $\mathbf{E}\backslash\bar{A}$ is everywhere dense.

A.2 Baire's Category Theorem

If $\mathbf{E}$ is a complete metric space, then

(1) the intersection $U = \bigcap_{n=1}^{\infty} U_n$ of a sequence of everywhere dense open subsets U_n of $\mathbf{E}$ is everywhere dense;

(2) a meager subset of $\mathbf{E}$ has no interior points (or, equivalently, a comeager subset is everywhere dense).

Proof. (1) Let us first show that U is nonvoid. For any $x \in \mathbf{E}$ and any number $\varepsilon > 0$ we write

$$B(x, \varepsilon) = \{y \in \mathbf{E} : d(y, x) < \varepsilon\}$$

and

$$\bar{B}(x, \varepsilon) = \{y \in \mathbf{E} : d(y, x) \leqslant \varepsilon\}$$

where d denotes the metric on $\mathbf{E}$. (To allay possible confusion resulting from the notation, we point out that $\bar{B}(x, \varepsilon)$, although closed in the topology defined by d, is in general not the closure of $B(x, \varepsilon)$.)

We may assume that $\mathbf{E}$ is nonvoid. The same is therefore true of U_1 (since U_1 is everywhere dense). Choose freely $x_1 \in U_1$ and then $\varepsilon_1 > 0$ so that

$$\varepsilon_1 < 1, \qquad \bar{B}(x_1, \varepsilon_1) \subset U_1.$$

Since U_2 is everywhere dense, U_2 meets $B(x_1, \varepsilon_1)$. Choose $x_2 \in U_2 \cap B(x_1, \varepsilon_1)$ and then $\varepsilon_2 > 0$ so that

$$\varepsilon_2 < \tfrac{1}{2}, \qquad \bar{B}(x_2, \varepsilon_2) \subset B(x_1, \varepsilon_1) \cap U_2,$$

which is possible since $B(x_1, \varepsilon_1) \cap U_2$ is open. Proceeding thus, we obtain numbers ε_n satisfying $0 < \varepsilon_n < 1/n$ and points $x_n \in \mathbf{E}$ such that

$$\bar{B}(x_n, \varepsilon_n) \subset B(x_{n-1}, \varepsilon_{n-1}) \cap U_n. \tag{A.2.1}$$

If $n > m$ we have $x_n \in B(x_n, \varepsilon_n) \subset B(x_m, \varepsilon_m)$, by (A.2.1), so that $d(x_n, x_m) < \varepsilon_m < 1/m$. The sequence (x_n) is thus Cauchy and so, $\mathbf{E}$ being complete by hypothesis, $x = \lim x_n$ exists in $\mathbf{E}$. Since $x_n \in B(x_m, \varepsilon_m)$ for $n > m \geqslant 1$, so $x \in \bar{B}(x_m, \varepsilon_m)$ for all m, and (A.2.1) shows then that $x \in U$. Thus U is nonvoid.

Take now any closed ball $\bar{B} = \bar{B}(x_0, \delta)$ in $\mathbf{E}$. It is easy to verify that $U_n \cap \bar{B}$ is everywhere dense in the complete metric space $\bar{B}$ (a subspace of $\mathbf{E}$). So, by what we have proved, $\bigcap_{n=1}^{\infty} (\bar{B} \cap U_n)$ is nonvoid, that is, $\bigcap_{n=1}^{\infty} U_n = U$ meets $\bar{B}$. This being so for any $\bar{B}$, U is everywhere dense in $\mathbf{E}$. This proves (1).

(2) Let M be any meager subset of $\mathbf{E}$. Then we can write $M = \bigcup_{n=1}^{\infty} A_n$, where A_n is nowhere dense. So $U_n = \mathbf{E} \backslash \bar{A}_n$ is everywhere dense and open. By (1), $\bigcap_{n=1}^{\infty} U_n$ is everywhere dense. That is, $\mathbf{E} \backslash \bigcup_{n=1}^{\infty} \bar{A}_n$ is everywhere dense, so that $\bigcup_{n=1}^{\infty} \bar{A}_n$ has no interior points. The same is therefore true of $M \subset \bigcup_{n=1}^{\infty} \bar{A}_n$.

A.3 Corollary

If $\mathbf{E}$ is a complete metric space, it is nonmeager in itself.

A.4 Lower Semicontinuous Functions

Let $\mathbf{E}$ be a metric space (or a topological space). A function f on $\mathbf{E}$ with values in $(-\infty, \infty]$ is said to be *lower semicontinuous* if and only if for each real number α the set $\{x \in \mathbf{E} : f(x) > \alpha\}$ is an open subset of $\mathbf{E}$.

The reader will verify that the upper envelope of an arbitrary family of lower semicontinuous functions is again lower semicontinuous.

A.5 A Lemma

Let $\mathbf{E}$ be a metric space (or a topological space) and $(f_\iota)_{\iota \in I}$ an arbitrary family of lower semicontinuous functions on $\mathbf{E}$. If

$$\sup_{\iota \in I} f_\iota(x) < \infty$$

holds for each x in some nonmeager subset S of $\mathbf{E}$, then there exists a number $m < \infty$ and a nonvoid open subset U of $\mathbf{E}$ such that

$$\sup_{\iota \in I} f_\iota(x) \leqslant m \qquad (x \in U).$$

Proof. For each natural number m let

$$S_m = \{x \in \mathbf{E} : f(x) \leqslant m\},$$

where f is the upper envelope of the f_ι. Since f is lower semicontinuous, S_m is closed in $\mathbf{E}$. Also, $S = \bigcup_{m=1}^{\infty} S_m$. Since S is nonmeager, S_m must fail to be nowhere dense for some m. For this m, S_m contains a nonvoid open set U.

APPENDIX B

Concerning Topological Linear Spaces

B.1 Preliminary Definitions

B.1.1. All linear spaces involved are over the real or complex field of scalars, the scalar field being denoted by Φ.

By a *topological linear space* we mean a linear space $\mathbf{E}$, together with a designated topology on $\mathbf{E}$ relative to which the functions $(x, y) \to x + y$ and $(\lambda, x) \to \lambda x$ are continuous from $\mathbf{E} \times \mathbf{E}$ into $\mathbf{E}$ and from $\Phi \times \mathbf{E}$ into $\mathbf{E}$, respectively. See [E], Chapter 1.

The specific results about topological linear spaces needed in the main text refer solely to a particular type of such space, namely, those classified as Fréchet spaces (see [E], Chapter 6). Our definition of these will be made in terms of seminorms.

B.1.2. By a *seminorm* (or *prenorm*) on a linear space $\mathbf{E}$ is meant a function p from $\mathbf{E}$ into $[0, \infty)$ having the following properties:

$$p(x + y) \leqslant p(x) + p(y), \qquad p(\lambda x) = |\lambda| p(x)$$

for $x, y \in \mathbf{E}$ and $\lambda \in \Phi$.

A *norm* is a seminorm p for which $p(x) > 0$ whenever $x \neq 0$. Norms will usually be denoted by $\|\cdot\|$.

B.1.3. **Fréchet Spaces.** A *Fréchet space* is a topological linear space $\mathbf{E}$ satisfying the following conditions:

(a) There is a finite or denumerably infinite family (p_k) of seminorms on $\mathbf{E}$ which define the topology of $\mathbf{E}$ in the sense that the sets

$$\{x \in \mathbf{E} : p_k(x) < \varepsilon \text{ for all } k \in J\}, \tag{B.1.1}$$

obtained when ε ranges over all positive numbers and J over all finite sets of indices k, constitute a base (or fundamental system) of neighborhoods of 0 for the topology of $\mathbf{E}$.

(b) The topology of $\mathbf{E}$ is Hausdorff, that is (what is easily seen to be equivalent), $x = 0$ is the only element of $\mathbf{E}$ for which $p_k(x) = 0$ for all indices k.

(c) **E** is complete in the sense that to any sequence $(x_n)_{n=1}^{\infty}$ of points of **E** for which

$$\lim_{m,n\to\infty} p_k(x_m - x_n) = 0$$

for each index k, corresponds an $x \in \mathbf{E}$ to which the sequence (x_n) converges, that is, for which

$$\lim_{n\to\infty} p_k(x - x_n) = 0$$

for each index k.

We then speak of (p_k) as a *defining family of seminorms* for the Fréchet space **E**.

B.1.4. **Remarks.** (1) For a given Fréchet space there are many different defining families of seminorms.

(2) Given a linear space **E** and a countable family (p_k) of seminorms on **E** such that (b) and (c) are fulfilled, we can topologize **E** in just one way as a Fréchet space for which (p_k) is a defining family of seminorms. Namely, we agree (as a matter of definition) that the sets (B.1.1) shall constitute a base of neighborhoods of 0; and that, for any $x_0 \in \mathbf{E}$, the images of these sets under the translation $x \to x + x_0$ shall constitute a base of neighborhoods of x_0. The properties of seminorms ensure that in this way one does indeed obtain a topological linear space which satisfies conditions (a) to (c).

(3) If **E** is a Fréchet space, it can be made into a complete metric space whose topology is identical with the initial topology on **E**, and this in several ways. One way is to define the metric

$$d(x, y) = \sum_k k^{-2} \frac{p_k(x - y)}{1 + p_k(x - y)}.$$

(4) In a Fréchet space one can always choose the defining family (p_k) so that the index set is the set of positive integers and so that $p_1 \leqslant p_2 \leqslant \cdots$. (By repeating seminorms we may suppose that the original set of indices is the set of natural numbers, define new seminorms $q_h = \sup\{p_k : 1 \leqslant k \leqslant h\}$, and take the (q_h) as the desired defining family.] If this be done, a neighborhood base at 0 is comprised of the sets $\{x \in \mathbf{E} : p_k(x) < \varepsilon\}$ when k and $\varepsilon > 0$ vary; a neighborhood base at 0 is also obtained if ε is restricted to range over any sequence of positive numbers tending to 0; or again, if the strict inequalities $p_k(x) < \varepsilon$ are replaced throughout by $p_k(x) \leqslant \varepsilon$.

(5) The product of two Fréchet spaces, or a closed linear subspace of a Fréchet space, is a Fréchet space. If (p_k) and (q_h) are defining families for Fréchet spaces **E** and **F**, respectively, the seminorms $r_{kh}(x, y) = p_k(x) + q_h(y)$ constitute a defining family for $\mathbf{E} \times \mathbf{F}$.

B.1.5. **Banach Spaces.** A *Banach space* is a Fréchet space possessing a defining family comprising just one element, which must of necessity be a norm. Expressed in another way, a Banach space is a normed linear space that is complete for its norm.

The reader will note that the scalar field Φ is itself a Banach space, the norm being equal to the absolute value.

B.1.6. **Bounded Sets.** Let $\mathbf{E}$ be a topological linear space and A a subset of $\mathbf{E}$. A is said to be *bounded* if and only if to each neighborhood U of 0 in $\mathbf{E}$ corresponds a positive scalar λ such that $A \subset \lambda U$ (the set of multiples by λ of all elements of U). Except when $\mathbf{E}$ is a Banach space, this concept of boundedness is different from metric boundedness (which is equivalent to finiteness of the diameter).

If $\mathbf{E}$ is a Fréchet space and (p_k) a defining family of seminorms for $\mathbf{E}$, the set $A \subset \mathbf{E}$ is bounded if and only if $\sup \{p_k(x) : x \in A\} < \infty$ for each index k; if $\mathbf{E}$ is a Banach space, the condition is merely that $\sup \{\|x\| : x \in A\} < \infty$.

B.1.7. **The Dual Space.** If $\mathbf{E}$ is a topological linear space we denote by $\mathbf{E}'$ the linear space of all continuous linear functionals on $\mathbf{E}$ (the algebraic operations in $\mathbf{E}'$ being "pointwise"). $\mathbf{E}'$ is termed the (topological) *dual* of $\mathbf{E}$; some authors use the terms "adjoint" or "conjugate" where we use the term "dual."

A sequence $(f_n)_{n=1}^{\infty}$ of elements of $\mathbf{E}'$ is said to *converge weakly in* $\mathbf{E}'$ to $f \in \mathbf{E}'$ if and only if $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in \mathbf{E}$.

If $\mathbf{E}$ is a Banach space, $\mathbf{E}'$ is also a Banach space for the so-called *dual norm*

$$\|f\| = \sup \{|f(x)| : x \in \mathbf{E},\ \|x\| \leqslant 1\};$$

the proof of completeness of $\mathbf{E}'$ is exactly like that in the special case dealt with in 12.7.1.

B.1.8. **Quotient Spaces and Quotient Norms.** Let $\mathbf{E}$ be a linear space and L a linear subspace of $\mathbf{E}$. The *quotient space* $\mathbf{E}/L$, whose elements are cosets $x + L$, is defined in purely algebraic terms and is turned into a linear space by defining

$$\begin{aligned}(x + L) + (y + L) &= (x + y) + L,\\ \lambda(x + L) &= (\lambda x) + L\end{aligned}$$

for $x, y \in \mathbf{E}$ and $\lambda \in \Phi$. Denote by ϕ the *quotient map* $x \to x + L$ of $\mathbf{E}$ onto $\mathbf{E}/L$; ϕ is linear.

If $\mathbf{E}$ is a topological linear space, one can make $\mathbf{E}/L$ into a topological linear space (again spoken of as the quotient space) by taking a base of neighborhoods of zero in $\mathbf{E}/L$ to be formed on the sets $\phi(U)$, where U ranges over a base of neighborhoods of zero in $\mathbf{E}$. The quotient space $\mathbf{E}/L$ is Hausdorff if and only if L is closed in $\mathbf{E}$.

A special case of importance is that in which $\mathbf{E}$ is normed, in which case the aforesaid quotient topology of $\mathbf{E}/L$ is derivable from the *quotient seminorm*

$$\|x + L\| = \inf_{y \in L} \|x + y\|.$$

If L is closed in $\mathbf{E}$, this quotient seminorm is actually a norm on $\mathbf{E}/L$ and is termed the *quotient norm on* $\mathbf{E}/L$.

If $\mathbf{E}$ is a Banach space and L is closed in $\mathbf{E}$, $\mathbf{E}/L$ is a Banach space (for its quotient norm); compare Subsection 11.4.7.

For more details see, for example, [E], Sections 1.8.5 and 1.10.5.

B.2 Uniform Boundedness Principles

We require two such principles stemming from Baire's theorem (see Section A.2) as a common source. The first concerns seminorms and linear functionals, and the second refers to linear operators.

B.2.1. (1) Let $\mathbf{E}$ be a Fréchet space and p_k $(k = 1, 2, \cdots)$ a lower semicontinuous function on $\mathbf{E}$ with values in $[0, \infty]$ such that

$$p_k(x + y) \leqslant p_k(x) + p_k(y), \qquad p_k(\lambda y) = |\lambda| p_k(x)$$

for $x, y \in \mathbf{E}$ and $\lambda \in \Phi$ (these conditions on p_k being fulfilled whenever p_k is a lower semicontinuous seminorm on $\mathbf{E}$). If $\inf_k p_k(x) < \infty$ for each x in a nonmeager subset S of $\mathbf{E}$ (in particular, if $\inf_k p_k(x) < \infty$ for each $x \in \mathbf{E}$), then there exists an index k such that p_k is finite valued and continuous on $\mathbf{E}$.

(2) Let $\mathbf{E}$ be a Fréchet space and $(f)_{i \in I}$ an arbitrary family of continuous linear functionals on $\mathbf{E}$. If

$$\sup \{|f_i(x)| : i \in I\} < \infty \tag{B.2.1}$$

for each x in a nonmeager subset S of $\mathbf{E}$ (in particular, if (B.2.1) holds for each $x \in \mathbf{E}$), then the f_i are equicontinuous on $\mathbf{E}$, that is, to each $\varepsilon > 0$ corresponds a neighborhood U of 0 in $\mathbf{E}$ such that $x \in U$ entails $\sup_i |f_i(x)| < \varepsilon$. If $\mathbf{E}$ is a Banach space, the conclusion reads simply $\sup_{i \in I} \|f_i\| < \infty$.

Proof. (1) In interpreting the hypotheses on p_k we agree that $\alpha + \infty = \infty + \infty = \infty$ for any real $\alpha \geqslant 0$, that $0 \cdot \infty = 0$, and that $\alpha \cdot \infty = \infty$ for any real $\alpha > 0$. For $k, r = 1, 2, \cdots$, let $S_{k,r}$ denote the set of $x \in \mathbf{E}$ such that $p_k(x) \leqslant r$. Since p_k is lower semicontinuous, each $S_{k,r}$ is closed in $\mathbf{E}$. Plainly, $S \subset \bigcup_{k,r=1}^{\infty} S_{k,r}$ so that, since S is nonmeager by hypothesis (see A.3 for the particular case in which $S = \mathbf{E}$), some $S_{k,r}$ has interior points. Thus there exists $x_0 \in \mathbf{E}$ and a neighborhood U of 0 in $\mathbf{E}$ such that $p_k(x) \leqslant r$ for $x \in x_0 + U$. Then the identity $x = \frac{1}{2}(x_0 + x) - \frac{1}{2}(x_0 - x)$ combines with the properties of p_k to show that $p_k(x) \leqslant r$ for $x \in U$. Consequently, given $\varepsilon > 0$, we have $p_k(x) \leqslant \varepsilon$ provided that $x \in U_1 \equiv (r^{-1}\varepsilon)U$. Now U_1 is again a neighborhood of 0 in $\mathbf{E}$, because the function $x \to (r\varepsilon^{-1})x$ is continuous from

E into **E** (see the axioms of a topological linear space in B.1.1). This shows that p_k is finite and continuous on **E** and so completes the proof of (1).

(2) This follows immediately from (1) on defining $p_k = p$ for all k, where

$$p(x) = \sup\{|f_i(x)| : i \in I\}.$$

An important corollary of B.2.1 is the following statement about families of continuous linear operators.

B.2.2. **Banach-Steinhaus Theorem.** Let **E** and **F** be Fréchet spaces and $(T_i)_{i\in I}$ an arbitrary family of continuous linear operators from **E** into **F**. Suppose that, for each x in a nonmeager subset S of **E** (for example, for each $x \in \mathbf{E}$), the set $\{T_i x : i \in I\}$ is a bounded subset of **F**. Then the T_i are equicontinuous on **E**, that is, to each neighborhood V of 0 in **F** corresponds a neighborhood U of 0 in **E** such that $T_i U \subset V$ for all $i \in I$.

Proof. We may assume that the given neighborhood V is of the form $V = \{y \in \mathbf{F} : q(y) \leqslant \varepsilon\}$, where q is some continuous seminorm on **F** (being, for example, a member of some defining family for **F**; see B.1.4(4)). Now apply B.2.1(1) to the situation in which

$$p_k = \sup_{i \in I} |q| \circ T_i \qquad (k = 1, 2, \cdots).$$

Remark. In the proof of B.2.2 no essential use is made of the fact that **F** is a Fréchet space: all that is necessary is that **F** be a topological linear space whose topology can be defined in terms of continuous seminorms. In other words, B.2.2 extends to the case in which **F** belongs to the category of so-called "locally convex" topological linear spaces. For further developments, see [E], Chapter 7.

B.3 Open Mapping and Closed Graph Theorems

B.3.1. **Some Definitions.** Let **E** and **F** be Fréchet spaces and T a linear operator from **E** into **F**. The *graph* of T is the subset of $\mathbf{E} \times \mathbf{F}$ comprising those pairs (x, y) in which $y = Tx$. The operator T is said to be *closed* (or to have a *closed graph*) if and only if its graph is a closed subset of the product space $\mathbf{E} \times \mathbf{F}$. This signifies that, if $\lim_{n\to\infty} x_n = 0$ in **E** and $\lim_{n\to\infty} Tx_n = y$ in **F**, then necessarily $y = 0$.

Evidently, if T is continuous, then it is closed. One of the two theorems we aim to prove asserts the converse of this. This will be deduced from another rather surprising result involving the concept of an open mapping.

If, as before, T is a linear operator from **E** into **F**, we shall say that T is *open* if TU is an open subset of **F** whenever U is an open subset of **E**. Because of linearity, this signifies exactly that TU is a neighborhood of 0 in **F** whenever U is a neighborhood of 0 in **E**.

B.3.2. **Open Mapping Theorem.** Let $\mathbf{E}$ and $\mathbf{F}$ be Fréchet spaces and T a closed linear operator from $\mathbf{E}$ into $\mathbf{F}$ such that $T\mathbf{E}$ is nonmeager in $\mathbf{F}$. Then T is open; in particular, $T\mathbf{E} = \mathbf{F}$. (Notice that the hypotheses are fulfilled whenever T is continuous and $T\mathbf{E} = \mathbf{F}$; see Section A.3.)

Proof. It has to be shown that, U being any neighborhood of 0 in $\mathbf{E}$, TU is a neighborhood of 0 in F. Now [see B.1.4(4)] we can choose an increasing defining family $(p_k)_{k=1}^{\infty}$ for $\mathbf{E}$ so that U contains the set $\{x \in \mathbf{E} : p_1(x) \leqslant 1\}$. Let $(q_h)_{h=1}^{\infty}$ be a defining family for $\mathbf{F}$, also increasing. We will show that there exists an index h_1 and numbers $r > 0$ and $\varepsilon_1 > 0$ such that each $y \in \mathbf{F}$ satisfying $q_{h_1}(y) \leqslant \varepsilon_1$ is expressible as $y = Tx$ for some $x \in \mathbf{E}$ satisfying $p_1(x) \leqslant r$. This is clearly enough to establish the required result.

To do this, write $U_k = \{x \in \mathbf{E} : p_k(x) \leqslant k^{-2}\}$ for $k = 1, 2, \cdots$. Since $\mathbf{E} = \bigcup_{n=1}^{\infty}(nU_k)$, $T\mathbf{E} \subset \bigcup_{n=1}^{\infty}T(nU_k)$. Then, $T\mathbf{E}$ being nonmeager by hypothesis, an n exists such that the closure of $T(nU_k) = n \cdot TU_k$ contains interior points. This entails, as is easily verified, that $\overline{TU_k}$ is a neighborhood of 0 in $\mathbf{F}$. Thus there exist an index h_k and a number $\varepsilon_k > 0$ such that $\overline{TU_k}$ contains the set $V_k = \{y \in \mathbf{F} : q_{h_k}(y) \leqslant \varepsilon_k\}$. One may assume without loss of generality that $h_k \geqslant k$ and that $\lim_{k\to\infty}\varepsilon_k = 0$.

Suppose now that $y \in V_1$. Since $\overline{TU_1} \supset V_1$, $x_1 \in U_1$ may be chosen so that $q_{h_2}(y - Tx_1) \leqslant \varepsilon_2$. Then $y - Tx_1 \in V_2$ and, accordingly, since $\overline{TU_2} \supset V_2$, $x_2 \in U_2$ may be chosen so that $q_{h_3}(y - Tx_1 - Tx_2) \leqslant \varepsilon_3$. Proceeding in this way, we obtain points x_n of $\mathbf{E}$ so that

$$p_n(x_n) \leqslant n^{-2}, \qquad q_{h_{n+1}}(y - Tx_1 - Tx_2 - \cdots - Tx_n) \leqslant \varepsilon_{n+1}. \tag{B.3.1}$$

The completeness of $\mathbf{E}$ entails that

$$x = \lim_{n\to\infty} (x_1 + x_2 + \cdots + x_n)$$

exists in $\mathbf{E}$, since the first relation in (B.3.1) ensures that $\sum_n p_k(x_n) < \infty$ and therefore that the sequence $(x_1 + \cdots + x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbf{E}$. The second inequality in (B.3.1) shows that for $n \geqslant h$

$$q_h(y - Tx_1 - \cdots - Tx_n) \leqslant q_{h_n}(y - Tx_1 - \cdots - Tx_n) \leqslant \varepsilon_{n+1},$$

so that $Tx_1 + \cdots + Tx_n = T(x_1 + \ldots + x_n) \to y$ in $\mathbf{F}$. Since T is closed, it follows that $y = Tx$. Finally,

$$p_1(x) = \lim_{n\to\infty} p_1(x_1 + \cdots + x_n) \leqslant \liminf_{n\to\infty} \sum_{k=1}^{n} p_1(x_k)$$
$$\leqslant \liminf_{n\to\infty} \sum_{k=1}^{n} p_k(x_k) \leqslant \sum_{k=1}^{\infty} k^{-2} = r,$$

say, by (B.3.1) again. The proof is thus complete.

B.3.3. **Closed Graph Theorem.** Let $\mathbf{E}$ and $\mathbf{F}$ be Fréchet spaces. Any closed linear operator from $\mathbf{E}$ into $\mathbf{F}$ is continuous.

Proof. Let $\mathbf{G}$ be the graph of T. Then $\mathbf{G}$ is a closed linear subspace of $\mathbf{E} \times \mathbf{F}$ and is therefore (see B.1.4(5)) itself a Fréchet space. Define the linear operator S from $\mathbf{G}$ into $\mathbf{E}$ by $S(x, Tx) = x$ for $x \in \mathbf{E}$. It is clear that S is continuous and $S\mathbf{G} = \mathbf{E}$. So, by B.3.2, S is open. This entails that T is continuous from $\mathbf{E}$ into $\mathbf{F}$: indeed, if V is any neighborhood of 0 in $\mathbf{F}$, the set W of pairs (x, Tx) for which $x \in \mathbf{E}$ and $Tx \in V$, is a neighborhood of 0 in $\mathbf{G}$, so that SW must be a neighborhood U of 0 in $\mathbf{E}$; but $x \in U$ entails that $Tx \in V$, so that $TU \subset V$ and T is thereby seen to be continuous.

B.4 The Weak Compacity Principle

B.4.1. Let $\mathbf{E}$ be a separable Fréchet space (that is, a Fréchet space in which there exists a countable, everywhere dense subset) and $(f_n)_{n=1}^{\infty}$ a sequence of continuous linear functionals on $\mathbf{E}$ such that

$$\limsup_{n \to \infty} |f_n(x)| < \infty$$

for each x in a nonmeager subset S of $\mathbf{E}$. Then there exists a subsequence $(f_n)_{k=1}^{\infty}$ which converges weakly in $\mathbf{E}'$ to some $f \in \mathbf{E}'$. (See B.1.7.)

Proof. From B.2.1 it follows that the f_n are equicontinuous, and in particular that $\sup_n |f_n(x)| < \infty$ for each $x \in \mathbf{E}$.

Now choose and enumerate as $(x_k)_{k=1}^{\infty}$ a countable everywhere dense subset of $\mathbf{E}$. The numerical sequence $(f_n(x_1))_{n=1}^{\infty}$ being bounded, a subsequence $(f_{\alpha_1(n)})_{n=1}^{\infty}$ may be extracted so that $\lim_{n\to\infty} f_{\alpha_1(n)}(x_1)$ exists finitely. (Here, α_1 is a strictly increasing map of the set N of natural numbers into itself.) Again, the numerical sequence $(f_{\alpha_1(n)}(x_2))_{n=1}^{\infty}$ being bounded, a subsequence $(f_{\alpha_1\alpha_2(n)})_{n=1}^{\infty}$ may be extracted so that $(f_{\alpha_1\alpha_2(n)}(x_2))_{n=1}^{\infty}$ is convergent to a finite limit. (We are writing $\alpha_1\alpha_2$ for the composite map $\alpha_1 \circ \alpha_2$.) Proceeding in this way, we obtain iterated subsequences $\alpha_1, \alpha_1\alpha_2, \cdots$ so that

$$\lim_{n \to \infty} f_{\alpha_1 \cdots \alpha_i(n)}(x_j)$$

exists finitely for $1 \leqslant j \leqslant i$ and $i = 1, 2, \cdots$. We now take the "diagonal subsequence" β defined by

$$\beta(n) = \alpha_1 \cdots \alpha_n(n).$$

This has the crucial property that $(\beta(n))_{n=1}^{\infty}$ is a subsequence of $(\alpha_1 \cdots \alpha_i(n))_{n=1}^{\infty}$ so that

$$\lim_{n \to \infty} f_{\beta(n)}(x_i) \tag{B.4.1}$$

exists finitely for each i.

Take now any $x \in \mathbf{E}$ and any $\varepsilon > 0$. By equicontinuity, there is a neighborhood U of 0 such that $|f_n(y)| \leqslant \varepsilon/4$ for all n and all $y \in U$. Next choose i so that $x - x_i \in U$ (possible since the x_i are everywhere dense in $\mathbf{E}$). We then have

$$\begin{aligned} |f_{\beta(n)}(x) - f_{\beta(n')}(x)| &\leqslant |f_{\beta(n)}(x) - f_{\beta(n)}(x_i)| + |f_{\beta(n')}(x) - f_{\beta(n')}(x_i)| \\ &\qquad + |f_{\beta(n)}(x_i) - f_{\beta(n')}(x_i)| \\ &= |f_{\beta(n)}(x - x_i)| + |f_{\beta(n')}(x - x_i)| + |f_{\beta(n)}(x_i) - f_{\beta(n')}(x_i)| \\ &\leqslant 2 \cdot \frac{\varepsilon}{4} + |f_{\beta(n)}(x_i) - f_{\beta(n')}(x_i)| \end{aligned}$$

uniformly in n and n'. The existence of the limit (B.4.1) then shows that

$$|f_{\beta(n)}(x) - f_{\beta(n')}(x)| \leqslant \varepsilon$$

provided $n, n' > n_0(\varepsilon)$. We thus infer that

$$f(x) = \lim_{n \to \infty} f_{\beta(n)}(x) \tag{B.4.2}$$

exists finitely for each $x \in \mathbf{E}$. It is plain that f is a linear functional on $\mathbf{E}$. That $f \in \mathbf{E}'$ (that is, is continuous) follows from the fact that (with the above notation) $|f(y)| \leqslant \varepsilon/4$ for $y \in U$. Thus $f \in \mathbf{E}'$ and the definition (B.4.2) ensures that $\lim_{n \to \infty} f_{\beta(n)} = f$ weakly in $\mathbf{E}'$. So $(f_{\beta(n)})_{n=1}^{\infty}$ is a subsequence of the type whose existence is asserted.

B.4.2. We remark that there is an analogue of B.4.1 which is valid for any (not necessarily separable) Fréchet space $\mathbf{E}$, and indeed in a still wider context.

Suppose that $\mathbf{E}$ is any topological linear space and that $(f_n)_{n=1}^{\infty}$ is any equicontinuous sequence of linear functionals on $\mathbf{E}$. Although it may not be possible to extract a subsequence of (f_n) which converges weakly in $\mathbf{E}'$, the following statement is true.

There exists a continuous linear functional f on $\mathbf{E}$ with the property that, corresponding to any given $\varepsilon > 0$, any finite subset $\{x_1, \cdots, x_r\}$ of $\mathbf{E}$, and any integer n_0, there is an integer $n > n_0$ for which

$$|f(x_i) - f_n(x_i)| < \varepsilon \qquad (i = 1, 2, \cdots, r).$$

(Such an f is nothing other than a limiting point of the given sequence in relation to the so-called weak topology on $\mathbf{E}'$ generated by $\mathbf{E}$; see [E], pp. 88–89.)

B.5 The Hahn-Banach Theorem

The only aspects of this many-headed theorem that are used in this book are stated in B.5.1 to B.5.3. For further discussion of the theorem and its applications, see [E], Chapter 2.

B.5.1. Let $\mathbf{E}$ be a linear space, p a seminorm on $\mathbf{E}$, L a linear subspace of $\mathbf{E}$, and f_0 a linear functional defined on L and satisfying

$$|f_0(x)| \leqslant p(x) \qquad (x \in L). \tag{B.5.1}$$

Then there exists a linear functional f on $\mathbf{E}$ such that

$$f(x) = f_0(x) \qquad (x \in L), \tag{B.5.2}$$

$$|f(x)| \leqslant p(x) \qquad (x \in \mathbf{E}). \tag{B.5.3}$$

Proof. This may be taken *verbatim* from pp. 53–55 of [E], but note the misprint on the last line of p. 54, where "$\leqslant$" should read "$\geqslant$."

B.5.2. Let $\mathbf{E}$ be a Fréchet space (or, indeed, any locally convex topological linear space; see the Remark following Subsection B.2.2), A any nonvoid subset of $\mathbf{E}$, and x_0 an element of $\mathbf{E}$. Then x_0 is the limit in $\mathbf{E}$ of finite linear combinations of elements of A if and only if the following condition is fulfilled: if f is any continuous linear functional $\mathbf{E}$ (that is, if $f \in \mathbf{E}'$) such that $f(A) = \{0\}$, then $f(x_0) = 0$.

Proof. The "only if" assertion is trivial in view of the linearity and continuity of f.

Suppose, conversely, that the condition is fulfilled. Let L_0 denote the closed linear subspace of $\mathbf{E}$ generated by A and suppose, if possible, that $x_0 \notin L_0$. Since L_0 is closed, and since the topology of $\mathbf{E}$ is defined by a family of continuous seminorms, there is a continuous seminorm p on $\mathbf{E}$ such that $p(y - x_0) > 1$ for all $y \in L_0$, and hence also

$$p(y + x_0) = p(-y - x_0) > 1 \qquad (y \in L_0). \tag{B.5.4}$$

Let $L = L_0 + \Phi x_0$ and define the linear functional f_0 on L by the formula $f_0(y + \lambda x_0) = \lambda$ $(y \in L_0, \lambda \in \Phi)$. Then, for $x = y + \lambda x_0 \in L$, (B.5.4) gives for $\lambda \neq 0$

$$|f_0(x)| = |\lambda| \leqslant |\lambda| \cdot p\left(\frac{y}{\lambda} + x_0\right) = p(y + \lambda x_0) = p(x);$$

and the same inequality is trivially valid for $\lambda = 0$. Appeal to B.5.1 shows that f_0 can be extended into a linear functional f on $\mathbf{E}$ such that (B.5.3) is true, which, since p is continuous, entails that f is continuous. On the other hand, it is evident that f_0, and therefore f too, vanishes on L_0, a fortiori vanishes on A, while $f(x_0) = f_0(x_0) = 1 \neq 0$. This contradicts the main hypothesis.

B.5.3. Let $\mathbf{E}$ be a normed linear space and $\mathbf{E}'$ its dual (see B.1.7). If $x_0 \in \mathbf{E}$, there exists $f \in \mathbf{E}'$ such that

$$\|f\| \leqslant 1$$

and

$$f(x_0) = \|x_0\|.$$

Proof. Let L be the linear subspace of $\mathbf{E}$ generated by x_0. Define f_0 on L by $f_0(\lambda x_0) = \lambda \|x_0\|$ $(\lambda \in \Phi)$. It now suffices to apply B.5.1, taking $p(x) = \|x\|$: any extension f of f_0 of the type specified in B.5.1 satisfies all requirements.

APPENDIX C

The Dual of $\mathbf{L}^p$ $(1 \leqslant p < \infty)$; Weak Sequential Completeness of $\mathbf{L}^1$

The aim of this appendix is to give a proof of two results used in the main text.

C.1 The Dual of $\mathbf{L}^p$ $(1 \leqslant p < \infty)$

Theorem. Let $1 \leqslant p < \infty$ and let F be any continuous linear functional on $\mathbf{L}^p$. Then there exists an essentially unique function $g \in \mathbf{L}^{p'}$ $(1/p + 1/p' = 1)$ such that

$$F(f) = \frac{1}{2\pi} \int fg \, dx \tag{C.1}$$

for all $f \in \mathbf{L}^p$. For any such function g one has

$$\|g\|_{p'} = \|F\| \equiv \sup \{|F(f)| : f \in \mathbf{L}^p, \ \|f\|_p \leqslant 1\}. \tag{C.2}$$

Proof. That any $g \in \mathbf{L}^{p'}$ satisfying (C.1) also satisfies (C.2), follows at once from Hölder's inequality and its converse (see Exercise 3.6). So we confine our attention to the proof of the existence of g [its essential uniqueness being a corollary of (C.2)]. This proof will be based upon use of the Radon-Nikodým theorem (see [W], Chapter 6; [HS], Section 19; [AB], p. 406).

For this purpose we consider Borel subsets E of the interval $X = (0, 2\pi)$ (compare [W], p. 93). The characteristic function of E may be extended by periodicity, the result being denoted by χ_E and being a member of $\mathbf{L}^p$. The number

$$\nu(E) = F(\chi_E) \tag{C.3}$$

is thus well-defined. Since F is linear, ν is (finitely) additive. If we can show that ν is countably additive, it will follow that ν is a complex Borel measure on X ([W], p. 95; [HS], p. 329). Now, if E is the union of disjoint Borel sets E_n $(n = 1, 2, \cdots)$, then

$$\chi_E = \sum_{n=1}^{\infty} \chi_{E_n},$$

the series being convergent in $\mathbf{L}^p$ since $p < \infty$. (Here is the major reason for breakdown of the theorem for $p = \infty$.) The continuity of F, together with its linearity, shows that therefore

$$\nu(E) = \sum_{n=1}^{\infty} \nu(E_n),$$

that is, that ν is indeed countably additive.

Again since $p < \infty$, the continuity of F shows that ν is absolutely continuous relative to the restriction μ of Lebesgue measure to X (see [W], p. 98; [HS], p. 312).

At this stage the Radon-Nikodým theorem ([W], Theorem 6.3d; [HS], p. 315) may be invoked to ensure the existence of an integrable function g on X (which may be extended by periodicity) such that

$$\nu(E) = F(\chi_E) = \frac{1}{2\pi} \int_E g \, dx = \frac{1}{2\pi} \int \chi_E g \, dx$$

for all Borel sets $E \subset X$. The linearity of F then shows that

$$F(f) = \frac{1}{2\pi} \int fg \, dx$$

holds for all f which are finite linear combinations of functions χ_E. Knowing this, it is easy to conclude that $g \in \mathbf{L}^{p'}$ and that

$$\|g\|_{p'} \leqslant \|F\|;$$

compare Exercise 3.6. Hölder's inequality shows then that the difference

$$F_0(f) = F(f) - \frac{1}{2\pi} \int fg \, dx$$

is a continuous linear functional on $\mathbf{L}^p$ which vanishes for all f in the everywhere dense subset of $\mathbf{L}^p$ formed of the finite linear combinations of characteristic functions of Borel sets. F_0 must therefore vanish identically, that is, (C.1) holds for all $f \in \mathbf{L}^p$.

Note: For $1 < p < \infty$ a different proof is possible; see [HS], pp. 222 ff., [AB], p. 246. The proof given above is that used ([HS], p. 351), to deal with the case $p = 1$.

C.2 Weak Sequential Completeness of $\mathbf{L}^1$

Although 12.3.10(2) is false when $p = 1$, there is a sort of partial substitute that sometimes will save the day.

Theorem. Suppose that $(f_i)_{i=1}^{\infty}$ is a sequence of integrable functions forming a weak Cauchy sequence in $\mathbf{L}^1$, that is, for which

$$\lim_i \frac{1}{2\pi} \int f_i g \, dx$$

exists finitely for each $g \in \mathbf{L}^\infty$. Then there exists a function $f \in \mathbf{L}^1$ to which (f_i) is weakly convergent, that is, which is such that

$$\lim_i \frac{1}{2\pi} \int f_i g \, dx = \frac{1}{2\pi} \int fg \, dx$$

for each $g \in \mathbf{L}^\infty$.

The proof of this result is rather lengthy and the reader is referred to [E], p. 275.

APPENDIX D

A Weak Form of Runge's Theorem

We aim to employ the Hahn-Banach theorem (Appendix B.5.2) to prove a simplified version of Runge's theorem adequate for the purposes of 12.9.8(3).

In what follows we denote by Δ the complex plane, by $\overline{\Delta}$ the compactified complex plane (that is, the Riemann sphere), by Ω a nonvoid open subset of Δ, and by K a nonvoid compact subset of Ω. It will be assumed that

(1) $\Delta \backslash K$ is connected (or, equivalently, that K is simply connected), and that

(2) there lies in $\Omega \backslash K$ a smooth closed path Γ such that the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)\, dt}{t - z} \tag{D.1}$$

holds for each $z \in K$ and each function f which is holomorphic on Ω.

We do not intend discussing in detail conditions that guarantee (2) (see the Remarks below); suffice it to say here that the condition is evidently fulfilled in the conditions prevailing in 12.9.8.

Theorem. With the above notations and hypotheses, each function f which is holomorphic on Ω is the limit, uniformly on K, of polynomials in z. (As is customary, the symbol "z" is used to denote a complex number and also the natural complex coordinate function on Δ.)

Proof. Denote by $\mathbf{C}(K)$ the complex linear space of continuous, complex-valued functions on K, into which we introduce the norm

$$\|f\| = \sup \{|f(z)| : z \in K\}.$$

Also, let $\mathbf{H}$ denote the subspace of $\mathbf{C}(K)$ formed of those $f \in \mathbf{C}(K)$ which are restrictions to K of functions holomorphic on Ω. According to B.5.1 and B.5.2, our task will be finished as soon as it is shown that any continuous linear functional F on $\mathbf{C}(K)$ with the property that

$$F(u_n) = 0 \qquad (n = 0, 1, 2, \cdots), \tag{D.2}$$

where u_n is the restriction to K of the function $z \to z^n$, satisfies also

$$F(f) = 0 \tag{D.3}$$

for any $f \in \mathbf{H}$.

For any $t \in \Delta \backslash K$, let f_t denote that element of $\mathbf{C}(K)$ obtained by restricting to K the function $z \to (t - z)^{-1}$. If $f \in \mathbf{H}$, the formula (D.1) may be applied to show that f is the limit in $\mathbf{C}(K)$ (that is, uniformly on K) of linear combinations of functions f_t with $t \in \Gamma$; for this it suffices to approximate the integral appearing in (D.1) by Riemann sums. Thus (D.3) is implied by (and is actually equivalent to) the assertion

$$F(f_t) = 0 \qquad (t \in \Gamma). \tag{D.4}$$

In order to show, finally, that (D.2) implies (D.4), we examine the function

$$\phi(t) = F(f_t), \tag{D.5}$$

defined for $t \in \Delta \backslash K$. Using the continuity of F on $\mathbf{C}(K)$, it is easy to verify that ϕ is holomorphic at all points of $\Delta \backslash K$: we leave this as a simple exercise for the reader. Moreover, if $|t|$ is sufficiently large,

$$(t - z)^{-1} = \sum_{n=0}^{\infty} \frac{z^n}{t^{n+1}},$$

the series converging uniformly for $z \in K$, so that continuity of F shows that

$$\phi(t) = \sum_{n=0}^{\infty} F(u_n) t^{-n-1}, \tag{D.6}$$

again for sufficiently large $|t|$. Now (D.6) shows first that ϕ can be extended to $\bar{\Delta} \backslash K$ in such a way as to be holomorphic on a neighborhood of ∞ (simply by setting $\phi(\infty) = 0$); and second, in conjunction with (D.2), that ϕ vanishes on a neighborhood of ∞. Since, by (1), $\Delta \backslash K$ is connected, the same is true of $\bar{\Delta} \backslash K = (\Delta \backslash K) \cup \{\infty\}$, and it follows that ϕ must vanish throughout the whole of $\Delta \backslash K$. In particular, (D.4) is true. This completes the proof.

Remarks. For a different approach to Runge's theorem and related questions, see [He], pp. 149–153, and the references cited there. The theorem has ramifications and analogues extending to Riemann surfaces (see, for example, the relevant portions of Behnke and Stein's "Theorie der analytischen Funktionen einer komplexen Veränderlichen") and to functions of several complex variables or on complex analytic manifolds (see Gunning and Rossi's "Analytic Functions of Several Complex Variables," Chapter I, Section F, and Chapter VII).

There are also real-variable analogues of Runge's theorem, applicable to solutions of linear partial differential equations; see [E], p. 396, and the references cited there.

Bibliography

Background reading in general topology, functional analysis, and integration theory.

[AB] ASPLUND, E. AND BUNGART, L. *A First Course in Integration.* Holt, Rinehart and Winston, Inc., New York (1966).

[$DS_{1,2}$] DUNFORD, N. AND SCHWARTZ, J. T. *Linear Operators, Parts I, II.* Interscience Publishers, Inc., New York (1958, 1963).

[E] EDWARDS, R. E. *Functional Analysis: Theory and Applications.* Holt, Rinehart and Winston, Inc., New York (1965).

[GP] GOFFMAN, C. AND PEDRICK, G. *First Course in Functional Analysis.* Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1965).

[HS] HEWITT, E. AND STROMBERG, K. *Real and Abstract Analysis.* Springer-Verlag, Berlin (1965).

[K] KELLEY, J. L. *General Topology.* D. Van Nostrand Company, Inc., Princeton, New Jersey (1955).

[Na] NATANSON, I. P. *Theory of Functions of a Real Variable.* Frederick Ungar Publishing Co., New York (1965).

[W] WILLIAMSON, J. H. *Lebesgue Integration.* Holt, Rinehart and Winston, Inc., New York (1962).

References to Fourier series and other specialized topics.

[A] ALEXITS, G. *Convergence Problems of Orthogonal Series.* Pergamon Press, Inc., New York (1961).

[Am] ALMGREN, F. J. *Plateau's Problem: an Invitation to Varifold Geometry.* W. A. Benjamin, Inc., New York (1966).

[B] BACHMAN, G. *Elements of Abstract Harmonic Analysis.* Academic Press, Inc., New York (1964).

[$Ba_{1,2}$] BARY, N. *A Treatise on Trigonometric Series, Vols. 1 and 2.* Pergamon Press, Inc., New York (1964).

[Be] Bellman, R. *A Brief Introduction to Theta Functions*. Holt, Rinehart and Winston, Inc., New York (1961).

[Bes] Besicovitch, A. S. *Almost Periodic Functions*. Cambridge University Press, New York (1932).

[BK] Butzer, P. L. and Korevaar, J. (editors). *On Approximation Theory*. Birkhauser Verlag, Basel und Stuttgart (1964).

[Bo] Boerner, H. *Representations of Groups with Special Consideration for the Needs of Modern Physics*. North Holland Publishing Co. and Interscience Publishers, Inc., New York (1963).

[Boa] Boas, R. P. Jr. *Integrability Theorems for Trigonometric Transforms*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Bd. 38. Springer-Verlag, New York (1967).

[CH] Courant, R. and Hilbert, D. *Methods of Mathematical Physics, Vol. II*. Interscience Publishers, Inc., New York (1962).

[dBR] de Branges, L. and Rovnyak, J. *Square Summable Power Series*. Holt, Rinehart and Winston, Inc., New York (1966).

[DW] Doran, R. S. and Wichmann, J. *Approximate Identities and Factorization in Banach Modules*. To appear Springer-Verlag Lecture Notes in Mathematics Series.

[E_1] Edwards, R. E. *Integration and Harmonic Analysis on Compact Groups*. London Math. Soc. Lecture Note Series 8, Cambridge University Press, New York (1972).

[G] Gross, L. *Harmonic Analysis on Hilbert Space*. Mem. Amer. Math. Soc., No. 46, Amer. Math. Soc., Providence (1963).

[Ga] Garsia, A. M. *Topics in Almost Everywhere Convergence*. Lectures in Advanced Mathematics, 4. Markham Publishing Co., Chicago (1970).

[GP] Goffman, C. and Pedrick, G. *First Course in Functional Analysis*. Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1965).

[GV] Gelfand, I. M. and Vilenkin, N. Ya. *Generalized Functions, Vol. 4 Applications of Harmonic Analysis*. Academic Press, Inc., New York (1964).

[H] Hirschman, I. I., Jr. *Infinite Series*. Holt, Rinehart and Winston, Inc., New York (1962).

[Ha] Hardy, G. H. *Divergent Series*. Oxford University Press, New York (1949).

[HaR] Hardy, G. H. and Rogosinski, W. *Fourier Series*. Cambridge University Press, New York (1944).

[He] Heins, M. *Topics in Complex Function Theory*. Holt, Rinehart and Winston, Inc., New York (1962).

[Hel] Helson, H. *Lectures on Invariant Subspaces*. Academic Press, Inc., New York (1964).

[Hew] Hewitt, E. *A Survey of Abstract Harmonic Analysis. Surveys in Applied Mathematics, IV*. John Wiley & Sons, Inc., New York (1958), 107–168.

[Hi] Hirschman, I. I., Jr. (editor). *Studies in Real and Complex Analysis. Studies in Mathematics, Vol. 3*. The Math. Association of America; Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1965).

[HLP] Hardy, G. H., Littlewood, J. E. and Pólya, G. *Inequalities*. Cambridge University Press, New York (1934).

[Ho] Hoffman, K. *Banach Spaces of Analytic Functions*. Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1962).

[HR] Hewitt, E. and Ross, K. A. *Abstract Harmonic Analysis, I, II*. Springer-Verlag, Berlin (1963, 1970).

[HW] Hardy, G. H. and Wright, E. M. *An Introduction to the Theory of Numbers*. Oxford University Press, New York (1938).

[I] Izumi, S.-I. *Introduction to the Theory of Fourier Series*. Institute of Mathematics, Academia Sinica, Taipei (1965).

[Ka] Kazarinoff, N. D. *Analytic Inequalities*. Holt, Rinehart and Winston, Inc., New York (1961).

[Kah] Kahane, J.-P. *Algebras de Convolucion de Sucesiones, Funciones y Medidas Sumables*. Cursos y Seminarios de Matematica, Fasc. 6. Univ. de Buenos Aires (1961).

[Kah_1] Kahane, J.-P. *Lectures on Mean Periodic Functions*. Tata Institute of Fundamental Research Lectures on Mathematics and Physics, No. 15. Bombay (1959).

[Kah_2] Kahane, J.-P. *Séries de Fourier absolument convergentes*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Bd. 50. Springer-Verlag, Berlin–Heidelberg–New York (1970).

[KS] Kahane, J.-P. and Salem, R. *Ensembles Parfaits et Séries Trigonométriques*. Hermann & Cie, Paris (1963).

[KSt] Kaczmarz, S. and Steinhaus, H. *Theorie der Orthogonalreihen*. Chelsea Publishing Company, New York (1951).

[Kz] Katznelson, Y. *Introduction to Harmonic Analysis*. John Wiley & Sons, Inc., New York (1968).

[L_1] LORENTZ, G. G. *Bernstein Polynomials.* University of Toronto Press (1955).

[L_2] LORENTZ, G. G. *Approximation of Functions.* Holt, Rinehart and Winston, Inc., New York (1966).

[Li] LITTLEWOOD, J. E. *Some Problems in Real and Complex Analysis.* Lecture notes, Univ. of Wisconsin, Madison (1962).

[Lo] LOOMIS, L. H. *An Introduction to Abstract Harmonic Analysis.* D. Van Nostrand Company, Inc., Princeton, New Jersey (1953).

[M] MANDELBROJT, M. *Séries de Fourier et Classes Quasianalytiques de Fonctions.* Gauthier-Villars, Paris (1935).

[Mk] MAAK, W. *Fastperiodische Funktionen.* Springer-Verlag, Berlin (1950).

[Moz] MOZZOCHI, C. P. *On the pointwise convergence of Fourier Series.* Lecture Notes in Mathematics, 199. Springer-Verlag, Berlin–Heidelberg–New York (1971).

[N] NAIMARK, M. A. *Normed Rings.* P. Noordhoff, N. V., Groningen, Netherlands (1959).

[P] PITT, H. R. *Tauberian Theorems.* Oxford University Press, New York (1958).

[Ph] PHELPS, R. R. *Lectures on Choquet's Theorem.* D. Van Nostrand Company, Inc., Princeton, New Jersey (1966).

[PS] PÓLYA, G. AND SZEGÖ, G. *Aufgaben und Lehresätze aus der Analysis, Bd. I, II* Springer-Verlag, Berlin (1925).

[R] RUDIN, W. *Fourier Analysis on Groups.* Interscience Publishers, Inc., New York (1962).

[Re] REITER, H. *Classical Harmonic Analysis and Locally Compact Groups.* Oxford Mathematical Monographs. Oxford University Press, New York (1968).

[Ri] RICKART, C. E. *Banach Algebras.* D. Van Nostrand Company, Inc., Princeton, New Jersey (1960).

[SMA] *Studies in Modern Analysis, Vol. I.* (edited by R. C. Buck) The Math. Assoc. of America, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1962).

[So] SOBOLEV, S. L. *Applications of Functional Analysis in Mathematical Physics.* Amer. Math. Soc. Translations of Math. Monographs, Vol. VII, Providence (1963).

[T] TONELLI, L. *Serie Trigonometriche.* Bologna (1928).

[Ti] TIMAN, A. F. *Theory of Approximation of Functions of a Real Variable.* The Macmillan Company, New York (1963).

[TV] TOLSTOV, G. P. *Fourier Series.* Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1962).

[vW] VAN DER WAERDEN, B. L. *Moderne Algebra.* Springer-Verlag, Berlin (1937).

[We] WEIL, A. *L'Intégration dans les Groupes Topologiques et Ses Applications.* Hermann & Cie, Paris (1951).

[Wi] WIENER, N. *The Fourier Integral and Certain of Its Applications.* Cambridge University Press, New York (1933).

[$Z_{1,2}$] ZYGMUND, A. *Trigonometrical Series, Vols. I and II.* Cambridge University Press, New York (1959).

Research Publications

The following list makes no attempt at completeness; it is rather a supplement to the bibliographies of such standard references as [Z], [Ba], *and* [R].

ALJANČIČ, S.

[1] On the integral moduli of continuity in L_p $(1 < p < \infty)$ of Fourier series with monotone coefficients. *Proc. Amer. Math. Soc.* **17** (1966), 287–294.

BLOOM, W. R.

[1] Bernstein's inequality for locally compact Abelian groups. *J. Austr. Math. Soc.* **XVII**(1) (1974), 88–102.

[2] A converse of Bernstein's inequality for locally compact Abelian groups. *Bull. Austr. Math. Soc.* **9**(2) (1973), 291–298.

[3] Jackson's theorem for locally compact Abelian groups. ibid. **10**(1) (1974), 59–66.

[4] Jackson's theorem for finite products and homomorphic images of locally compact Abelian groups. ibid. **12**(2) (1975), 301–309.

BOAS, R. P. JR.

[1] Beurling's test for absolute convergence of Fourier series. *Bull. Amer. Math. Soc.* **66** (1960), 24–27.

[2] Fourier Series with positive coefficients. *J. Math. Anal. Appl.* **17** (1967), 463–483.

BREDON, G. E.

[1] A new treatment of the Haar integral. *Mich. Math. J.* **10** (1963), 365–373.

BUCY, R. S. AND MALTESE, G.

[1] Extreme positive definite functions and Choquet's representation theorem. *J. Math. Anal. Appl.* **12** (1965), 371–377.

BUTZER, P. L. AND NESSEL, R. J.

[1] Über eine verallgemeinerung eines satzes von de la Vallée Poussin. *International Series in Numerical Mathematics*, **5**, pp. 45–48. Birkhauser Verlag (1964).

CARLESON, L.

[1] On convergence and growth of partial sums of Fourier series. *Acta Math.* **116** (1966), 135–157.

CHEN, Y. M.

[1] An almost everywhere divergent Fourier series of the class $L(\log^+ \log^+ L)^{1-\varepsilon}$. *J. London Math. Soc.* **44** (1969), 643–654.

COHEN, P. J.

[1] Factorization in group algebras. *Duke Math. J.* **26** (1959), 199–205.

[2] On homomorphisms of group algebras. *Amer. J. Math.* **82** (1960), 213–226.

[3] On a conjecture of Littlewood and idempotent measures. *Amer. J. Math.* **82** (1960), 191–212.

[4] A note on constructive methods in Banach algebras. *Proc. Amer. Math. Soc.* **12** (1961), 159–163.

CURTIS, P. C. JR. AND FIGÀ-TALAMANCA, A.

[1] Factorization theorems for Banach algebras. *Function Algebras*, pp. 169–185. Scott, Foresman and Company, Chicago (1966).

DE BRUIJN, N. G.

[1] Functions whose differences belong to a given class. *Nieuw Arch. Wiskunde* **23** (1951), 194–218.

[2] A difference property for Riemann integrable functions and for some similar classes of functions. *Nederl. Akad. Wetensch. Proc. Ser.* **A55** (1952), 145–151.

DE LEEUW, K. AND KATZNELSON, Y.

[1] On certain homomorphisms of quotients of group algebras. *Israel J. Math.* **2** (1964), 120–126.

EDWARDS, D. A.

[1] On translates of L_∞ functions. *J. London Math. Soc.* **36** (1961), 431–432.

EDWARDS, R. E.

[1] Criteria for Fourier transforms. *J. Austr. Math. Soc.* **7** (1967), 239–246.

[2] Translates of L^∞ functions and of bounded measures. *J. Austr. Math. Soc.* *IV* (1964), 403–409.

[3] Boundedness principles and Fourier theory. To appear *Pacific J. Math.*

EDWARDS, R. E. AND HEWITT, E.

[1] Pointwise limits for sequences of convolution operators. *Acta Math.* **113** (1965), 181–218.

EDWARDS, R. E. AND PRICE, J. F.

[1] A naively constructive approach to boundedness principles, with applications to harmonic analysis. *L'Ens. math.* **XVI**, fasc. 3–4 (1970), 255–296.

ESSÉN, M.

[1] Studies on a convolution inequality. *Ark. Mat.* **5** (9) (1963), 113–152.

GAUDET, R. J. AND GAMLEN J. L. B.

[1] An elementary proof of part of a classical conjecture. *Bull. Austr. Math. Soc.* **3** (1970), 289–291.

GODEMENT, R.

[1] Les fonctions de type positif et la théorie des groupes. *Trans. Amer. Math. Soc.* **63** (1948), 1–84.

GOES, G. AND GOES, S.

[1] Sequences of bounded variation and sequences of Fourier coefficients. I. *Math. Z.* **118** (1970), 93–102.

HARDY, G. H.

[1] Notes on some points in the integral calculus, LXVI, The arithmetic mean of a Fourier constant. *Messenger of Math.* **58** (1928), 50–52.

HERZ, C. S.

[1] Fonctions opérant sur certains semi-groupes. *C. R. Acad. Sci. Paris* **255** (1962), 2046–2048.

HEWITT, E.

[1] The ranges of certain convolution operators. *Math. Scand.* **15** (1964), 147–155.

HEWITT, E. AND HEWITT, R. E.

[1] The Gibbs-Wilbraham phenomenon: an episode in Fourier analysis. To appear *The Mathematical Intelligencer*.

HIRSCHMAN, I. I., JR.

[1] On multiplier transformations. *Duke Math. J.* **26** (1959), 221–242.

IZUMI, M. AND IZUMI, S.-I.

[1] Fourier series of functions of bounded variation. *Proc. Japan Acad.* **44** (6) (1968), 415–417.

[2] Divergence of Fourier series. *Bull. Austr. Math. Soc.* **8** (1973), 289–304.

[3] On absolute convergence of Fourier series. *Ark. för Mat.* **7**(12) (1967), 177–184.

[4] Absolute Convergence of Fourier series of convolution functions. *J. Approx. Theory* **1** (1968), 103–109.

KAHANE, J.-P.

[1] Idempotents and closed subalgebras of $A(Z)$. *Function Algebras*, pp. 198–207. Scott, Foresman and Company, Chicago (1966).

[2] Sur certaines classes de séries de Fourier absolument convergentes. *J. Math. Pures et Appl.* **35** (1956), 249–258.

[3] Sur un problème de Littlewood. *Nederl. Akad. Wetensch. Proc. Ser.* **A60** = *Indag. Math.* **19** (1957), 268–271.

[4] Sur les réarrangements des suites de coefficients de Fourier-Lebesgue. *C. R. Acad. Sci. Paris.* Sér. A-B **265** (1967), A310–A312.

KAHANE, J.-P. AND KATZNELSON, Y.

[1] Sur les séries de Fourier uniformément convergentes. *C. R. Acad. Sci. Paris* **261** (1965), 3025–3028.

KANO, T. AND UCHIYAMA, S.

[1] On the convergence in L of some special Fourier series. *Proc. Japan Acad.* **53** A(2) (1977), 72–77.

McGehee, O. C.

[1] Certain isomorphisms between quotients of group algebras. *Notices Amer. Math. Soc.* **13** (1966), 475.

Men'shov, D. E.

[1] On convergence in measure of trigonometric series. (In Russian) *Trudy Mat. Inst. im V. A. Steklova* **32** (1950), 3–97.

Mitjagin, B. S.

[1] On the absolute convergence of the series of Fourier coefficients. (In Russian) *Dokl. Akad. Nauk SSSR* **157** (1964), 1047–1050.

Mueller, M. J.

[1] Three results for locally compact groups connected with the Haar measure density theorem. *Proc. Amer. Math. Soc.* **16** (1965), 1414–1416.

Newman, D. J.

[1] The non-existence of projections from $\mathbf{L}^1$ to $\mathbf{H}^1$. *Proc. Amer. Math. Soc.* **12** (1961), 98–99.

[2] An $\mathbf{L}^1$ extremal problem for polynomials. *Proc. Amer. Math. Soc.* **16** (1965), 1287–1290.

Ray, K. C. and Lahiri, B. K.

[1] An extension of a theorem of Steinhaus. *Bull. Calcutta Math. Soc.* **56** (1964), 29–31.

Rider, D.

[1] Central idempotents in group algebras. To appear *Bull. Amer. Math. Soc.*

[2] Transformations of Fourier coefficients. To appear *Pacific J. Math.*

Rooney, P. G.

[1] On the representation of sequences as Fourier coefficients. *Proc. Amer. Math. Soc.* **11** (1960), 762–768.

Rubel, L. A.

[1] Uniform distributions and densities on locally compact Abelian groups. *Symposia on Theor. Physics and Math.*, Vol. **9**, 183–193. Plenum Press, New York (1969).

Rudin, W.

[1] Representation of functions by convolutions. *J. Math. Mech.* **7** (1958), 103–116.

[2] A strong converse of the Wiener-Lévy theorem. *Canad. J. Math.* **14** (1962), 694–701.

[3] Idempotents in group algebras. *Bull. Amer. Math. Soc.* **69** (1963), 224–227.

[4] Some theorems on Fourier coefficients. *Proc. Amer. Math. Soc.* **10** (1959), 855–859.

[5] Positive definite sequences and absolutely monotonic functions. *Duke Math. J.* **26** (1959), 617–622.

Rudin, W. and Schneider, H.

[1] Idempotents in group rings. *Duke Math. J.* **31** (1964), 585–602.

RYAN, R.

[1] Fourier transforms of certain classes of integrable functions. *Trans. Amer. Math. Soc.* **105** (1962), 102–111.

SHAH, T.-S.

[1] Positive definite functions on commutative topological groups. *Scientia Sinica* (5)**XIV** (1965), 653–665.

SRINIVASAN, T. P. AND WANG, J.-K.

[1] Weak* Dirichlet Algebras. *Function Algebras*, pp. 216–249. Scott, Foresman and Company, Chicago (1966).

STEIN, E. M.

[1] On limits of sequences of operators. *Ann. Math.* **74** (1961), 140–170.

TAIBLESON, M.

[1] Fourier coefficients of functions of bounded variation. *Proc. Amer. Math. Soc.* **18** (1967), 766.

TELJAKOWSKĬI, S. A.

[1] Some estimates for trigonometric series with quasiconvex coefficients (Russian). *Mat. Sbornik* **63** (105) (1964), 426–444 = A.M.S. Translations (2) **86** (1970), 109–131.

[2] On the question of convergence of Fourier series in the L-metric (Russian). *Mat. Zametki* **1** (1967), 91–98. MR **34** # 8067.

TIMAN, M. F.

[1] Best approximation of a function and linear methods of summing Fourier series. (In Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **29** (1965), 587–604.

WEISS, G.

[1] Harmonic Analysis. *Studies in Mathematics, Vol. 3*, pp. 124–178. The Math. Assoc. of America; Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1965).

YADAV, B. S.

[1] On a theorem of Titchmarsh. *Publ. Inst. Math.* (*Beograd*) (*N.S.*) **2(16)** (1962), 87–92 (1963).

YADAV, B. S. AND GOYAL, O. P.

[1] On the absolute convergence of Fourier series. *J. Maharaja Sayajirao Univ. Baroda* **10** (1961), 29–34.

ZAMANSKY, M.

[1] Approximation et Analyse Harmonique, I. *Bull. Sci. math.*, 2e série, **101** (1977), 3–70.

[2] Approximation et Analyse Harmonique, II. *Bull. Sci. math.*, 2e série, **101** (1977), 149–180.

ZUK, V. V.

[1] On the absolute convergence of Fourier series. (In Russian) *Dokl. Akad. Nauk SSSR* **160** (1965), 519–522.

Symbols

Numerals in boldface type refer to the exercises.

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