Springer Monographs in Mathematics

M. Levine • F. Morel

Algebraic Cobordism

Springer

Marc Levine
Department of Mathematics
Northeastern University
Boston, MA 02115, USA
e-mail: marc@neu.edu

Fabien Morel
Mathematisches Institut der LMU
Theresienstr. 39
D - 80333 München, Germany
e-mail: morel@mathematik.uni-muenchen.de

Library of Congress Control Number: 2006936486

Mathematics Subject Classification (2000): 14F43, 55N22, 14C15, 14C17, 14C40

ISSN 1439-7382

ISBN-10 3-540-36822-1 Springer Berlin Heidelberg New York
ISBN-13 978-3-540-36822-9 Springer Berlin Heidelberg New York

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable for prosecution under the German Copyright Law.

Springer is a part of Springer Science+Business Media
springer.com
© Springer-Verlag Berlin Heidelberg 2007

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Typesetting by the authors and VTEX using a Springer LaTeX macro package
Cover design: Erich Kirchner, Heidelberg, Germany

Printed on acid-free paper SPIN: 11805953 VA 44/3100/VTEX - 5 4 3 2 1 0

To Rebecca, Anna and Ute–M.L.
To Juliette, Elise and Mymy–F.M.

Introduction

Motivation. This work grew out of our attempt to understand the analog in algebraic geometry of the fundamental paper of Quillen on the cobordism of differentiable manifolds [30]. In this paper, Quillen introduced the notion of a *(complex) oriented cohomology theory* on the category of differentiable manifolds, which basically means that the cohomology theory is endowed with suitable Gysin morphisms, and in particular gives the cohomology theory the additional structure of Chern classes for complex vector bundles. Quillen then observed that the complex cobordism theory $X \mapsto MU^*(X)$ is the universal such cohomology theory.

This new point of view allowed him to shed some new light on classical computations in cobordism theory. He made more precise the computation by Milnor and Novikov of the complex cobordism ring MU^* as a polynomial ring: it is in fact the Lazard ring $\mathbb{L}$, the coefficient ring of the universal formal group law defined and studied in [16]. The isomorphism

$$\mathbb{L} \cong MU^*$$

is obtained via the formal group law $F_{MU}(u,v)$ on MU^* defined as the expression of the Chern class $\mathrm{c}_1(L \otimes M)$ of a tensor product of line bundles as a power series in $\mathrm{c}_1(L)$ and $\mathrm{c}_1(M)$ by the formula

$$\mathrm{c}_1(L \otimes M) = F_{MU}(\mathrm{c}_1(L), \mathrm{c}_1(M)).$$

This result of Quillen is in fact a particular case of his main theorem obtained in [30]: for any differentiable manifold X, the $\mathbb{L}$-module $MU^*(X)$ is generated by the elements of non-negative degrees. We observe that this is highly non-trivial as the elements of $\mathbb{L}$, in the cohomological setting, are of negative degree!

Quillen's notion of oriented cohomology extends formally to the category $\mathbf{Sm}_k$ of smooth quasi-projective k-schemes, with k a fixed field, see section 1.1. Our main achievement here is to construct the universal oriented cohomology theory Ω^* on $\mathbf{Sm}_k$, which we call algebraic cobordism, and to prove the exact analogs of Quillen's theorems in this setting, at least over a field of characteristic zero. The computation

$$\mathbb{L} \cong \Omega^*(\operatorname{Spec} k)$$

is done in section 4.3, and the theorem asserting that $\Omega^*(X)$ is generated by elements of non-negative degrees is proved in section 4.4. Surprisingly, this result can be precisely reformulated, in algebraic geometry, as the *generalized degree formula* conjectured by Rost. We will give on the way other applications and examples, explaining for instance the relationship between our Ω^* and the K_0 functor of Grothendieck or the Chow ring functor CH^*.

It is fascinating to see that in the introduction of his paper Quillen acknowledges the influence of Grothendieck's philosophy of motives on his work. Here the pendulum swings back: our work is strongly influenced by Quillen's ideas, which we try to bring back to the "motivic" world. In some sense, this book is the result of putting Quillen's work [30] together with Grothendieck's work on the theory of Chern classes [11]. Indeed, if one relaxes the axiom from the paper of Grothendieck that the first Chern class $c_1 : \mathrm{Pic}(X) \to A^1(X)$ is a group homomorphism, then in the light of Quillen's work, one has to discover algebraic cobordism.

Overview. Most of the main results in this book were announced in [20, 21], and appeared in detailed form in the preprint [18] by Levine and Morel and the preprint [19] by Levine. This book is the result of putting these works together[1].

The reader should notice that we have made a change of convention on degrees from [20, 21]; there our cohomology theories were assumed to be take values in the category of graded commutative rings, and the push-forward maps were assumed to increase the degree by 2 times the codimension. This had the advantage of fitting well with the notation used in topology. But as is clear from our constructions, we only deal with the even part, and for notational simplicity we have divided the degrees by 2.

This book is organized as follows. In order to work in greater generality as in [9], instead of dealing only with cohomology theories on smooth varieties, we will construct Ω^* as an oriented Borel-Moore homology theory $X \mapsto \Omega_*(X)$ on the category of a finite type k-schemes.

In chapter 1, we introduce the notion of an oriented cohomology theory and state our main results. In chapter 2, we construct algebraic cobordism over any field as the universal "oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type" on the category of finite type k-schemes. Our construction is not merely an existence theorem, we define algebraic cobordism by giving explicit generators and relations.

[1] The second author wishes to thank the first author very much for incorporating his part, and for his work combining the two parts into a whole

An oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type has by definition projective push-forward, smooth pull-back, external products and 1st Chern class operators for line bundles, satisfying some natural axioms. However, this structure is not sufficient for our purposes, as one needs in addition a projective bundle formula and an extended homotopy property. In chapter 3 we establish our fundamental technical result: the localization theorem 1.2.8, when k admits resolution of singularities. The rest of the chapter 3 deduces from this theorem the projective bundle formula and extended homotopy invariance for algebraic cobordism.

Chapter 4 introduces the dual notions of oriented weak cohomology theories and oriented Borel-Moore weak homology theories. We develop the theory of Chern classes for these theories, give some applications, and then prove all the theorems announced in the introduction. One should note however that theorems 1.2.2 and 1.2.6 are only proven here in the weaker form where one replaces the notion of oriented cohomology theory by the notion of weak oriented cohomology theory. However, the proofs of the other theorems such as theorem 1.2.3, the various degree formulas and theorem 1.2.7 require only those weak forms.

Chapters 5 and 6 of this work deal with pull-backs. The essential difference between an oriented cohomology theory and an oriented weak cohomology theory is that the latter have only pull-backs for smooth morphisms while the former have pull-backs for any morphism between smooth k-schemes. It is convenient to work with the dual notion of an oriented Borel-Moore homology theory on the category of finite type k-schemes, which is introduced in chapter 5. Our main task in this setting is to construct pull-back maps for any local complete intersection morphism, which is done in chapter 6 (assuming k admits resolution of singularities). We conclude in chapter 7 by finishing the proofs of theorems 1.2.2 and 1.2.6, and extending many of our results on the oriented cohomology of smooth schemes to the setting of Borel-Moore homology of local complete intersection schemes.

Notations and conventions. We denote by $\mathbf{Sch}_S$ the category of separated schemes of finite type over S and by $\mathbf{Sm}_S$ its full subcategory consisting of schemes smooth and quasi-projective over S. For an S-scheme X, we shall denote by $\pi_X : X \to S$ the structural morphism. By a smooth morphism, we will always mean a smooth and quasi-projective morphism. In particular, a smooth S-scheme will always be assumed to be quasi-projective over S.

Throughout this paper, we let k be an arbitrary field, unless otherwise stated. We will usually, but not always, take $S = \operatorname{Spec} k$.

We denote by $\mathcal{O}_X$ the structure sheaf of a scheme X and by O_X, or simply O when no confusion can arise, the trivial line bundle over X. Given a Cartier divisor $D \subset X$ we let $\mathcal{O}_X(D)$ denote the invertible sheaf determined by D and $O_X(D)$ the line bundle whose $\mathcal{O}_X$-module of sections is $\mathcal{O}_X(D)$. For a vector bundle $E \to X$, we write $\mathcal{O}_X(E)$ for the sheaf of (germs of) sections of E. In general, we will pass freely between vector bundles over X and the corresponding locally free coherent sheaves of $\mathcal{O}_X$ modules.

For a locally free coherent sheaf $\mathcal{E}$ on a scheme X, we let $q : \mathbb{P}(\mathcal{E}) \to X$ denote the projective bundle $\mathrm{Proj}_{\mathcal{O}_X}(\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{E}))$, and $q^*\mathcal{E} \to \mathcal{O}(1)_{\mathcal{E}}$ the canonical quotient invertible sheaf. For a vector bundle $E \to X$, we write $\mathbb{P}(E)$ for $\mathbb{P}(\mathcal{O}(E))$, and $q^*E \to O(1)_E$ for the canonical quotient line bundle. For $n > 0$, O_X^n will denote the trivial vector bundle of rank n over X, and we write γ_n for the line bundle $O(1)_{O_X^{n+1}}$ on $\mathbb{P}^n_X$.

For a an element of a commutative ring R, we write a for the R-valued point $(1 : a)$ of $\mathbb{P}^1_R := \mathrm{Proj}_R R[X_0, X_1]$, and ∞ for the point $(0 : 1)$. Similarly, we use the coordinate $x := X_1/X_0$ to identify $\mathbb{P}^1_R \setminus \infty$ with $\mathbb{A}^1_R$. For a functor F defined on a sub-category of $\mathbf{Sch}_k$ we will usually write $F(k)$ instead of $F(\mathrm{Spec}\, k)$.

Acknowledgments. We would like to thank all those who made comments and suggestions on earlier versions of this text, especially Annette Huber-Klawitter, János Kollár, Alexander Merkurjev, Alexander Nenashev, Ivan Panin, Joël Riou and Jörg Wildeshaus. We would like to thank Jörg Schürmann for pointing out an error in our construction of Chern classes in oriented weak homology, and for discussions which helped lead us to the correct version. The sections on refined pull-back and the excess intersection formula were added to our original two papers following suggestions of Rahul Pandharipande. We also thank the participants in the Oberwolfach Arbeitsgemeinschaft on Algebraic Cobordism for presenting the material contained here in a seminar setting and giving it a critical run-through. Finally, we would like to thank the referee for several useful suggestions and the staff at Springer, especially Ute Motz, for their help in bringing this project to fruition.

Beside our obvious debt to Quillen, the reader will not fail to notice our repeated reliance on the ideas in Fulton's book [9]. In fact, one can view a large portion of this book as a revision of [9], replacing cycles with "cobordism cycles" and adding a liberal dash of Hironaka's resolution of singularities.

Marc Levine: Much of what went in to this book came out of discussions with Fabien Morel during my visit in the summer of 2000 to the Université de Paris 7, with subsequent work taking place during a number of visits to the Universität Duisburg–Essen. I would like thank both universities for their support and hospitality. Thanks are also due to Northeastern University for encouraging and supporting my research. Finally, I am grateful for support from the NSF via the grants DMS-987629, DMS-0140445 and DMS-0457195, and the Humboldt Foundation through the Wolfgang Paul program.

Boston, Essen, München, August, 2006

Marc Levine
Fabien Morel

Contents

1

Cobordism and oriented cohomology

In this chapter, we introduce the axiomatic framework of oriented cohomology theories, and state our main results.

1.1 Oriented cohomology theories

Fix a base scheme S. For $z \in Z \in \mathbf{Sm}_S$ we denote by $\dim_S(Z, z)$ the dimension over S of the connected component of Z containing z

Let $d \in \mathbb{Z}$ be an integer. A morphism $f : Y \to X$ in $\mathbf{Sm}_S$ has *relative dimension* d if, for each $y \in Y$, we have $\dim_S(Y, y) - \dim_S(X, f(y)) = d$. We shall also say in that case that f has relative codimension $-d$.

For a fixed base-scheme S, $\mathcal{V}$ will usually denote a full subcategory of $\mathbf{Sch}_S$ satisfying the following conditions

1. S and the empty scheme are in $\mathcal{V}$.
2. If $Y \to X$ is a smooth quasi-projective morphism in $\mathbf{Sch}_S$ with $X \in \mathcal{V}$, then $Y \in \mathcal{V}$.
3. If X and Y are in $\mathcal{V}$, then so is the product $X \times_S Y$.
4. If X and Y are in $\mathcal{V}$, so is $X \coprod Y$.

(1.1)

In particular, $\mathcal{V}$ contains $\mathbf{Sm}_S$. We call such a subcategory of $\mathbf{Sch}_S$ *admissible.*

Definition 1.1.1. *Let $f : X \to Z$, $g : Y \to Z$ be morphisms in an admissible subcategory $\mathcal{V}$ of* $\mathbf{Sch}_S$. *We say that f and g are* transverse *in $\mathcal{V}$ if*

1. $\mathrm{Tor}_q^{\mathcal{O}_Z}(\mathcal{O}_Y, \mathcal{O}_X) = 0$ *for all* $q > 0$.
2. *The fiber product $X \times_Z Y$ is in $\mathcal{V}$.*

If $\mathcal{V} = \mathbf{Sm}_S$ we just say f and g are transverse; if $\mathcal{V} = \mathbf{Sch}_S$, we sometimes say instead that f and g are Tor-independent.

We let $\mathbf{R}^*$ denote the category of *commutative, graded rings with unit*. Observe that a commutative graded ring is not necessarily *graded commutative*. We say that a functor $A^* : \mathbf{Sm}_S^{\mathrm{op}} \to \mathbf{R}^*$ is *additive* if $A^*(\emptyset) = 0$ and for any pair $(X, Y) \in \mathbf{Sm}_S^2$ the canonical ring map $A^*(X \amalg Y) \to A^*(X) \times A^*(Y)$ is an isomorphism.

The following notion is directly taken from Quillen's paper [30]:

Definition 1.1.2. *Let $\mathcal{V}$ be an admissible subcategory of* $\mathbf{Sch}_S$. *An* oriented cohomology theory *on $\mathcal{V}$ is given by*

(D1). An additive functor $A^* : \mathcal{V}^{\mathrm{op}} \to \mathbf{R}^*$.
(D2). For each projective morphism $f : Y \to X$ in $\mathcal{V}$ of relative codimension d, a homomorphism of graded $A^*(X)$-modules:

$$f_* : A^*(Y) \to A^{*+d}(X)$$

Observe that the ring homomorphism $f^* : A^*(X) \to A^*(Y)$ gives $A^*(Y)$ the structure of an $A^*(X)$-module.

These satisfy

(A1). One has $(\mathrm{Id}_X)_* = \mathrm{Id}_{A^*(X)}$ for any $X \in \mathcal{V}$. Moreover, given projective morphisms $f : Y \to X$ and $g : Z \to Y$ in $\mathcal{V}$, with f of relative codimension d and g of relative codimension e, one has

$$(f \circ g)_* = f_* \circ g_* : A^*(Z) \to A^{*+d+e}(X).$$

(A2). Let $f : X \to Z$, $g : Y \to Z$ be transverse morphisms in $\mathcal{V}$, giving the cartesian square

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

Suppose that f is projective of relative dimension d (thus so is f'). Then $g^* f_* = f'_* g'^*$.
(PB). Let $E \to X$ be a rank n vector bundle over some X in $\mathcal{V}$, $O(1) \to \mathbb{P}(E)$ the canonical quotient line bundle with zero section $s : \mathbb{P}(E) \to O(1)$. Let $1 \in A^0(\mathbb{P}(E))$ denote the multiplicative unit element. Define $\xi \in A^1(\mathbb{P}(E))$ by

$$\xi := s^*(s_*(1)).$$

Then $A^*(\mathbb{P}(E))$ is a free $A^*(X)$-module, with basis

$$(1, \xi, \ldots, \xi^{n-1}).$$

(EH). Let $E \to X$ be a vector bundle over some X in $\mathcal{V}$, and let $p : V \to X$ be an E-torsor. Then $p^* : A^*(X) \to A^*(V)$ is an isomorphism.

A morphism of oriented cohomology theories on $\mathcal{V}$ is a natural transformation of functors $\mathcal{V}^{\mathrm{op}} \to \mathbf{R}^*$ which commutes with the maps f_*.

The morphisms of the form f^* are called *pull-backs* and the morphisms of the form f_* are called *push-forwards.* Axiom (PB) will be referred to as the *projective bundle formula* and axiom (EH) as the *extended homotopy property.*

We now specialize to $S = \operatorname{Spec} k$, $\mathcal{V} = \mathbf{Sm}_k$, k a field. Given an oriented cohomology theory A^*, one may use Grothendieck's method [11] to define Chern classes $c_i(E) \in A^i(X)$ of a vector bundle $E \to X$ of rank n over X as follows: Using the notations of the previous definition, axiom (PB) implies that there exists unique elements $c_i(E) \in A^i(X)$, $i \in \{0, \ldots, n\}$, such that $c_0(E) = 1$ and

$$\sum_{i=0}^{n} (-1)^i \, c_i(E) \, \xi^{n-i} = 0.$$

One can check all the standard properties of Chern classes as in [11] using the axioms listed above (see §4.1.7 for details). Moreover, these Chern classes are characterized by the following properties:

1) For any line bundle L over $X \in \mathbf{Sm}_k$, $c_1(L)$ equals $s^* s_*(1) \in A^1(X)$, where $s : X \to L$ denotes the zero section.
2) For any morphism $Y \to X \in \mathbf{Sm}_k$, and any vector bundle E over X, one has for each $i \geq 0$
$$c_i(f^* E) = f^*(c_i(E)).$$
3) Whitney product formula: if
$$0 \to E' \to E \to E'' \to 0$$
is an exact sequence of vector bundles, then one has for each integer $n \geq 0$:
$$c_n(E) = \sum_{i=0}^{n} c_i(E') \, c_{n-i}(E'').$$

Sometime, to avoid confusion, we will write $c_i^A(E)$ for the Chern classes of E computed in the oriented cohomology theory A^*.

The fundamental insight of Quillen in [30], and the main difference with Grothendieck's axioms in [11], is that it is not true in general that one has the formula

$$c_1(L \otimes M) = c_1(L) + c_1(M)$$

for line bundles L and M over the same base. In other words the map

$$\begin{aligned} c_1 : \operatorname{Pic}(X) &\to A^1(X) \\ L &\mapsto c_1(L), \end{aligned}$$

is not assumed to be a group homomorphism, but is only a natural transformation of *pointed sets.* In fact, a classical remark due to Quillen [30, Proposition 2.7] describes the way c_1 is not additive as follows (see proposition 5.2.4 for a proof of this lemma):

Lemma 1.1.3. *Let A^* be an oriented cohomology theory on $\mathbf{Sm}_k$. Then for any line bundle L on $X \in \mathbf{Sm}_k$ the class $c_1(L)^n$ vanishes for n large enough*[1]*. Moreover, there is a unique power series*

$$F_A(u,v) = \sum_{i,j} a_{i,j}\, u^i\, v^j \in A^*(k)[[u,v]]$$

with $a_{i,j} \in A^{1-i-j}(k)$, such that, for any $X \in \mathbf{Sm}_k$ and any pair of line bundles L, M on X, we have

$$F_A(c_1(L), c_1(M)) = c_1(L \otimes M).$$

In addition, the pair $(A^*(k), F_A)$ is a commutative formal group law of rank one.

Recall from [16] that a commutative formal group law of rank one with coefficients in A is a pair (A, F) consisting of a commutative ring A and a formal power series

$$F(u,v) = \sum_{i,j} a_{i,j}\, u^i\, v^j \in A[[u,v]]$$

such that the following holds:

1. $F(u,0) = F(0,u) = u \in A[[u]]$.
2. $F(u,v) = F(v,u) \in A[[u,v]]$.
3. $F(u, F(v,w)) = F(F(u,v),w) \in A[[u,v,w]]$.

These properties of F_A reflect the fact that, for line bundles L, M, N on $X \in \mathbf{Sm}_k$, one has:

1′. $L \otimes O_X = O_X \otimes L = L \in \mathrm{Pic}(X)$.
2′. $L \otimes M = M \otimes L \in \mathrm{Pic}(X)$.
3′. $L \otimes (M \otimes N) = (L \otimes M) \otimes N \in \mathrm{Pic}(X)$.

Lazard pointed out in [16] that there exists a universal commutative formal group law of rank one $(\mathbb{L}, F_{\mathbb{L}})$ and proved that the ring $\mathbb{L}$ (now called the *Lazard ring*) is a polynomial ring with integer coefficients on a countable set of variables x_i, $i \geq 1$. The construction of $(\mathbb{L}, F_{\mathbb{L}})$ is rather easy. Set $\tilde{\mathbb{L}} := \mathbb{Z}[\{A_{i,j} | (i,j) \in \mathbb{N}^2\}]$, and $\tilde{F}(u,v) = \sum_{i,j} A_{i,j}\, u^i\, v^j \in \tilde{\mathbb{L}}[[u,v]]$. Then define $\mathbb{L}$ to be the quotient ring of $\tilde{\mathbb{L}}$ by the relations obtained by imposing the relations (1), (2) and (3) above to $\tilde{F}$, and let

$$F_{\mathbb{L}} = \sum_{i,j} a_{i,j}\, u^i\, v^j \in \mathbb{L}[[u,v]]$$

denote the image of F by the homomorphism $\tilde{\mathbb{L}} \to \mathbb{L}$. It is clear that the pair $(\mathbb{L}, F_{\mathbb{L}})$ is the universal commutative formal group law of rank one, which

[1] In fact we will prove later on that $n > \dim_k(X)$ suffices; this follows from theorem 2.3.13 and proposition 5.2.4.

means that to define a commutative formal group law of rank one (F, A) on A is equivalent to giving a ring homomorphism $\Phi_F : \mathbb{L} \to A$.

The Lazard ring can be graded by assigning the degree $i + j - 1$ to the coefficient $a_{i,j}$. We denote by $\mathbb{L}_*$ this commutative graded ring. We could as well have graded it by assigning the degree $1 - i - j$ to the coefficient $a_{i,j}$, in which case we denote by $\mathbb{L}^*$ the corresponding commutative graded ring. For instance $\mathbb{L}^0 = \mathbb{L}_0 = \mathbb{Z}$ and $\mathbb{L}^{-n} = \mathbb{L}_n = 0$ if $n < 0$.

One can then check that for any oriented cohomology theory A^* the homomorphism of rings induced by the formal group law given by lemma 1.1.3 is indeed a homomorphism of graded rings

$$\Phi_A : \mathbb{L}^* \to A^*(k)$$

Example 1.1.4. The Chow ring $X \mapsto \mathrm{CH}^*(X)$ is a basic example of an oriented cohomology theory on $\mathbf{Sm}_k$; this follows from [9]. In that case, the formal group law obtained on $\mathbb{Z} = \mathrm{CH}^*(k)$ by lemma 1.1.3 is the *additive* formal group law $F_a(u, v) = u + v$.

Example 1.1.5. Another fundamental example of oriented cohomology theory is given by the Grothendieck K^0 functor $X \mapsto K^0(X)$, where for X a smooth k-scheme, $K^0(X)$ denotes the Grothendieck group of locally free coherent sheaves on X. For $\mathcal{E}$ a locally free sheaf on X we denote by $[\mathcal{E}] \in K^0(X)$ its class. The tensor product of sheaves induces a unitary, commutative ring structure on $K^0(X)$. In fact we rather consider the graded ring $K^0(X)[\beta, \beta^{-1}] := K^0(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}]$, where $\mathbb{Z}[\beta, \beta^{-1}]$ is the ring of Laurent polynomial in a variable β of degree -1.

It is endowed with pull-backs for any morphism $f : Y \to X$ by the formula:

$$f^*([\mathcal{E}] \cdot \beta^n) := [f^*(\mathcal{E})] \cdot \beta^n$$

for $\mathcal{E}$ a locally free coherent sheaf on X and $n \in \mathbb{Z}$. We identify $K^0(X)$ with the Grothendieck group $G_0(X)$ of all coherent sheaves on X by taking a finite locally free resolution of a coherent sheaf (X is assumed to be regular). This allows one to define push-forwards for a projective morphism $f : Y \to X$ of pure codimension d by the formula

$$f_*([\mathcal{E}] \cdot \beta^n) := \Sigma_{i=0}^{\infty} (-1)^i [R^i f_*(\mathcal{E})] \cdot \beta^{n-d} \in K_0(X)[\beta, \beta^{-1}]$$

for $\mathcal{E}$ a locally free sheaf on Y and $n \in \mathbb{Z}$. One can easily check using standard results that this is an oriented cohomology theory.

Moreover, for a line bundle L over X with with projection $\pi : L \to X$, zero section $s : X \to L$ and sheaf of sections $\mathcal{L}$, one has

$$s^*(s_*(1_X)) = s^*([\mathcal{O}_{s(X)}]\beta^{-1}) = s^*(1 - [\pi^*(\mathcal{L})^\vee])\beta^{-1} = (1 - [\mathcal{L}^\vee])\beta^{-1}$$

so that $c_1^K(L) := (1 - [\mathcal{L}^\vee])\beta^{-1}$. We thus find that the associated power series F_K is the *multiplicative formal group law*

$$F_m(u,v) := u + v - \beta uv$$

as this follows easily from the relation

$$(1 - [(\mathcal{L} \otimes \mathcal{M})^\vee]) = (1 - [\mathcal{L}^\vee]) + (1 - [\mathcal{M}^\vee]) - (1 - [\mathcal{L}^\vee])(1 - [\mathcal{M}^\vee])$$

in $K^0(X)$, where $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on X.

1.2 Algebraic cobordism

Definition 1.2.1. *Let A^* be an oriented cohomology theory on $\mathbf{Sm}_k$ with associated formal group law F_A.*

1) We shall say that A^* is ordinary *if $F_A(u,v)$ is the additive formal group law.*
2) We shall say that A^* is multiplicative *if $F_A(u,v) = u + v - buv$ for some (uniquely determined) $b \in A^{-1}(k)$; we shall say moreover that A^* is* periodic *if b is a unit in $A^*(k)$.*

Our main results on oriented cohomology theories are the following three theorems. In each of these statements, A^* denoted a fixed oriented cohomology theory on $\mathbf{Sm}_k$:

Theorem 1.2.2. *Let k be a field of characteristic zero. If A^* is ordinary then there exists one and only one morphism of oriented cohomology theories*

$$\vartheta_A^{\mathrm{CH}} : \mathrm{CH}^* \to A^*.$$

Theorem 1.2.3. *Let k be a field. If A^* is multiplicative and periodic then there exists one and only one morphism of oriented cohomology theories*

$$\vartheta_A^K : K^0[\beta, \beta^{-1}] \to A^*.$$

Theorem 1.2.2 says that, in characteristic zero, the Chow ring functor is the universal ordinary oriented cohomology theory on $\mathbf{Sm}_k$. It seems reasonable to conjecture that this statement still holds over any field. Theorem 1.2.3 says that $K^0[\beta, \beta^{-1}]$ is the universal multiplicative and periodic oriented cohomology theory on $\mathbf{Sm}_k$.

Remark 1.2.4. The classical Grothendieck-Riemann-Roch theorem can be easily deduced from theorem 1.2.3, see remark 4.2.11.

Remark 1.2.5. Using theorem 1.2.3 and the fact that for any smooth k-scheme the Chern character induces an isomorphism

$$ch : K^0(X) \otimes \mathbb{Q} \cong \mathrm{CH}(X) \otimes \mathbb{Q}$$

(where CH denotes the ungraded Chow ring), it is possible to prove $\mathbb{Q}$-versions of theorem 1.2.2 and theorem 1.2.6 below over *any* field.

Well-known examples of ordinary cohomology theories are given by the "classical" ones: the even part of étale ℓ-adic cohomology theory (with $\ell \neq \mathrm{char}(k)$ a prime number), the even de Rham cohomology theory over a field of characteristic zero, the even part of Betti cohomology associated to a complex embedding of the base field. In some sense theorem 1.2.2 and its rational analog over any field explains, a priori, the existence of the cycle map in all these classical cohomology theories.

The following result introduces our main object of study:

Theorem 1.2.6. *Assume k has characteristic zero. Then there exists a universal oriented cohomology theory on $\mathbf{Sm}_k$, denoted by*

$$X \mapsto \Omega^*(X),$$

which we call algebraic cobordism. *Thus, given an oriented cohomology theory A^* on $\mathbf{Sm}_k$, there is a unique morphism*

$$\vartheta : \Omega^* \to A^*$$

of oriented cohomology theories.

In addition, we have two main results describing properties of the universal theory Ω^* which do not obviously follow from universality. The first may be viewed as an algebraic version of Quillen's identification of $MU^*(pt)$ with $\mathbb{L}$:

Theorem 1.2.7. *For any field k of characteristic zero, the canonical homomorphism classifying F_Ω*

$$\Phi_\Omega : \mathbb{L}^* \to \Omega^*(k)$$

is an isomorphism.

The second reflects the strongly algebraic nature of Ω^*:

Theorem 1.2.8. *Let $i : Z \to X$ be a closed immersion between smooth varieties over k, d the codimension of Z in X and $j : U \to X$ the open immersion of the complement of Z. Then the sequence*

$$\Omega^{*-d}(Z) \xrightarrow{i_*} \Omega^*(X) \xrightarrow{j^*} \Omega^*(U) \to 0$$

is exact.

The construction of Ω^* is directly inspired by Quillen's description of complex cobordism [30]: For $f : Y \to X$ a projective morphism of codimension d from a smooth k-scheme Y to X denote by $[f : Y \to X]_A \in A^d(X)$ the element $f_*(1_Y)$. For each X, $\Omega^d(X)$ is generated as a group by the isomorphism classes of projective morphisms $Y \to X$ of codimension d with Y smooth. The morphism ϑ necessarily maps $f : Y \to X$ to $[f : Y \to X]_A$, which proves uniqueness of ϑ. Observe that $\Omega^n(X) = 0$ for $n > \dim(X)$. When $X = \operatorname{Spec} k$ we simply denote by $[Y] \in \Omega^{-d}(k)$ and $[Y]_A \in A^{-d}(k)$ the class of the projective smooth variety $Y \to \operatorname{Spec} k$ of dimension d.

Remark 1.2.9. One should note that the relations defining Ω^* are not just the obvious "algebraization" of the complex cobordism relations. Indeed, one can consider projective morphisms of the form $f : Y \to X \times \mathbb{P}^1$ with Y smooth and f transverse to the inclusion $X \times \{0,1\} \to X \times \mathbb{P}^1$. Letting $f_0 : Y_0 \to X$, $f_1 : Y_1 \to X$ be the pull-backs of f via $X \times 0 \to X \times \mathbb{P}^1$ and $X \times 1 \to X \times \mathbb{P}^1$, respectively, we do have the relation

$$[f_0 : Y_0 \to X] = [f_1 : Y_1 \to X]$$

in $\Omega^*(X)$. However, imposing only relations of this form on the free abelian group of isomorphism classes of projective morphisms $f : Y \to X$ (with Y irreducible and smooth over k) does not give $\Omega^*(X)$, even for $X = \operatorname{Spec} k$, and even for algebraically closed k. To see this, consider $\Omega^{-1}(k)$, i.e., the part of $\Omega^*(k)$ generated by the classes of smooth projective curves C over k. Clearly, the genus and the number of (geometrically) connected components is invariant under the "naive" cobordisms given by maps $Y \to \mathbb{P}^1$, but we know that $\mathbb{L}^{-1} \cong \mathbb{Z}$, generated by the class of $\mathbb{P}^1$. Thus, if one uses only the naive notion of algebraic cobordism, it would not be possible to make a curve of genus $g > 0$ equivalent to $(1-g)\mathbb{P}^1$, as it should be.

Example 1.2.10. In [30], Quillen defines a notion of *complex oriented cohomology theory* on the category of differentiable manifolds and pointed out that complex cobordism theory $X \mapsto MU^*(X)$ can be interpreted as the universal such theory. Our definition 1.1.2 is so inspired by Quillen's axioms that given a complex imbedding $\sigma : k \to \mathbb{C}$, it is clear that the functor $X \mapsto MU^{2*}(X_\sigma(\mathbb{C}))$ admits a canonical structure of oriented cohomology theory ($X_\sigma(\mathbb{C})$ denoting the differentiable manifold of complex points of $X \times_k \mathbb{C}$). From the universality of algebraic cobordism we get for any $X \in \mathbf{Sm}_k$ a canonical morphism of graded rings

$$\Omega^*(X) \to MU^{2*}(X_\sigma(\mathbb{C})).$$

Given a complex embedding $\sigma : k \to \mathbb{C}$ the previous considerations define a ring homomorphism

$$\Phi^{top} : \Omega^* \to MU^{2*}.$$

In very much the same way, given an extension of fields $k \subset K$ and a k-scheme X denote by X_K the scheme $X \times_{\operatorname{Spec} k} \operatorname{Spec} K$. For any oriented cohomology theory A^* on $\mathbf{Sm}_K$, the functor

$$(\mathbf{Sm}_k)^{op} \to R^*, X \mapsto A^*(X_K)$$

is an oriented cohomology theory on $\mathbf{Sm}_k$. In particular, we get natural morphisms $\Omega^*(X) \to \Omega^*(X_K)$, giving in the case $X = \operatorname{Spec} k$ a canonical ring homomorphism

$$\Omega^*(k) \to \Omega^*(K).$$

Theorem 1.2.7 easily implies:

Corollary 1.2.11. *Let k be a field of characteristic zero.*

(1) Given a complex embedding $\sigma : k \to \mathbb{C}$ the canonical homomorphism

$$\Phi^{top} : \Omega^*(k) \to MU^{2*}(pt)$$

is an isomorphism.

(2) Given a field extension $k \subset F$, the canonical homomorphism

$$\Omega^*(k) \to \Omega^*(F)$$

is an isomorphism

Remark 1.2.12. Suppose $\mathrm{char}(k) = 0$. Let X be a smooth irreducible quasi-projective k-scheme, with field of functions K. One then has a canonical homomorphism of rings $\Omega^*(X) \to \Omega^*(K)$ defined as the composition of the canonical morphism $\Omega^*(X) \to \Omega^*(X_K)$ (extension of scalars) with the restriction $\Omega^*(X_K) \to \Omega^*(K)$ to the tautological K-point of X_K. It corresponds to "taking the generic fiber" in the sense that given a projective morphism $f : Y \to X$ of relative codimension d and generic fiber $Y_K \to \mathrm{Spec}\, K$, a smooth projective K-scheme, its image by the previous homomorphism is the class $[Y_K] \in \Omega^d(K)$.

The composition $\Omega^*(k) \to \Omega^*(X) \to \Omega^*(K)$ is an isomorphism by corollary 1.2.11(2). We denote by

$$\deg : \Omega^*(X) \to \Omega^*(k)$$

the composition of $\Omega^*(X) \to \Omega^*(K)$ and the inverse isomorphism $\Omega^*(K) \to \Omega^*(k)$. Now, for a morphism $f : Y \to X$ of relative codimension 0, we have the *degree* of f, denoted $\deg(f)$, which is zero if f is not dominant and equal to the degree of the field extension $k(X) \to k(Y)$ if f is dominant. We observe that $\Omega^0(k)$ is canonically isomorphic to $\mathbb{Z}$ and that through this identification, $\deg([f : Y \to X]) = \deg(f)$ in case f has relative codimension zero.

From theorem 1.2.8 and corollary 1.2.11 we get the following result, which is a very close analogue of the fundamental results in [30] concerning complex cobordism.

Corollary 1.2.13. *Let k be a field of characteristic zero and let X be in* $\mathbf{Sm}_k$. *Then $\Omega^*(X)$ is generated as an $\mathbb{L}^*$-module by the classes of non-negative degrees (Recall that $\mathbb{L}^*$ is concentrated in degrees ≤ 0).*

Indeed corollary 1.2.11, with $F = k(X)$, implies that a given element $\eta \in \Omega^*(X)$ is "constant" over some open subscheme $j : U \to X$ of X:

$$j^*\eta = \deg(\eta) \cdot 1_U.$$

By theorem 1.2.8, the difference $\eta - \deg(\eta) \cdot 1_X$ comes from Ω^* of some proper closed subscheme Z (after removing the singular locus of Z), and noetherian induction completes the proof. In fact, since each reduced closed subscheme Z of X has a smooth birational model $\tilde{Z} \to Z$, we get the following more precise version, which we call the *generalized degree formula*:

Theorem 1.2.14. *Let k be a field of characteristic zero. Let X be in $\mathbf{Sm}_k$. For each closed integral subscheme $Z \subset X$ let $\tilde{Z} \to Z$ be a projective birational morphism with $\tilde{Z}$ smooth quasi-projective over k and let $[\tilde{Z} \to X] \in \Omega^*(X)$ denote the class of the projective morphism $\tilde{Z} \to X$. Then $\Omega^*(X)$ is generated as an $\mathbb{L}^*$-module by the classes $[\tilde{Z} \to X]$.*

In particular, for any irreducible $X \in \mathbf{Sm}_k$, $\Omega^*(X)$ is generated as an $\mathbb{L}^*$-module by the unit $1 \in \Omega^0(X)$ and by the elements $[\tilde{Z} \to X]$ with $\dim(\tilde{Z}) <$ $\dim(X)$, that is to say of degrees > 0 in $\Omega^*(X)$. More precisely, for $\eta \in$ $\Omega^*(X)$, there are integral proper closed subschemes Z_i of X, and elements $\alpha_i \in \Omega^*(k)$, $i = 1, \ldots, r$, such that

$$\eta = \deg(\eta) \cdot [\mathrm{Id}_X] + \sum_{i=1}^{r} \alpha_i \cdot [\tilde{Z}_i \to X]. \tag{1.2}$$

Given a smooth projective irreducible k-scheme X of dimension $d > 0$, Rost introduces (see [23]) the ideal $M(X) \subset \mathbb{L}^* = \Omega^*(\operatorname{Spec} k)$ generated by classes $[Y] \in \mathbb{L}^*$ of smooth projective k-schemes Y of dimension $< d$ for which there exists a morphism $Y \to X$ over k. The following result establishes Rost's degree formula as conjectured in [23]. It is an obvious corollary to theorem 1.2.14 and remark 1.2.12.

Theorem 1.2.15. *Let k be a field of characteristic zero. For any morphism $f : Y \to X$ between smooth projective irreducible k-schemes the class $[Y] -$ $\deg(f)[X]$ of $\mathbb{L}^*$ lies in the ideal $M(X)$. In other words, one has the following equality in the quotient ring $\mathbb{L}^*/M(X)$:*

$$[Y] = \deg(f) \cdot [X] \in \mathbb{L}^*/M(X).$$

We shall also deduce the following

Theorem 1.2.16. *Let k be a field of characteristic zero. Let X be a smooth projective k-variety.*

1. *The ideal $M(X)$ is a birational invariant of X.*
2. *The class of X modulo $M(X)$:*

$$[X] \in \mathbb{L}^*/M(X)$$

is a birational invariant of X as well.

For instance, let $d \geq 1$ be an integer and let N_d be the d-th Newton polynomial,

$$N_d(x_1, \ldots, x_d) \in \mathbb{Z}[x_1, \ldots, x_d].$$

Recall that if σ_i is the ith elementary symmetric function in variables $t_1, t_2, \ldots,$ then

$$N_d(\sigma_1, \ldots, \sigma_d) = \Sigma_i t_i^d.$$

If X is smooth projective of dimension d, we set

$$S_d(X) := -\deg N_d(c_1, \ldots, c_d)(T_X) \in \mathbb{Z},$$

T_X denoting the tangent bundle of X and $\deg : \mathrm{CH}^d(X) \to \mathbb{Z}$ the usual degree homomorphism. One checks that if X and Y are smooth projective k-schemes of dimension d and d', one has $S_{d+d'}(X \times Y) = 0$ if both $d > 0$ and $d' > 0$. We also know (see [3]) that if d is of the form $p^n - 1$ where p is a prime number and $n > 0$, then $s_d(X) := \frac{S_d(X)}{p}$ is always an integer. In that case, using theorem 1.2.14 and observing that if $\dim(Z) < d$ then $s_d([W] \cdot [Z]) \neq 0$ implies $\dim(Z) = 0$ one obtains the following result:

Corollary 1.2.17. *Let $f : Y \to X$ be a morphism between smooth projective varieties of dimensions $d > 0$. Assume that $d = p^n - 1$ where p is a prime number and $n > 0$. Then there exists a 0-cycle on X with integral coefficients whose degree is the integer*

$$s_d(Y) - \deg(f) \cdot s_d(X).$$

This formula was first proven by Rost[2], and then generalized further by Borghesi [4].

Consider now the graded ring homomorphisms

$$\Phi_a : \mathbb{L}^* \to \mathbb{Z}$$

and

$$\Phi_m : \mathbb{L}^* \to \mathbb{Z}[\beta, \beta^{-1}]$$

classifying respectively the additive and multiplicative formal group laws.

Theorem 1.2.6 obviously implies that, over a field of characteristic zero, the ordinary oriented cohomology theory

$$X \mapsto \Omega^*(X) \otimes_{\mathbb{L}^*} \mathbb{Z}$$

obtained by extension of scalars from Ω^* via Φ_a is the universal ordinary oriented cohomology theory. In the same way theorem 1.2.6 implies that, over a field of characteristic zero, the multiplicative oriented cohomology theory

$$X \mapsto \Omega^*(X) \otimes_{\mathbb{L}^*} \mathbb{Z}[\beta, \beta^{-1}]$$

obtained by extending the scalars from Ω^* via Φ_m is the universal multiplicative periodic oriented cohomology theory. Over a field of characteristic zero, we get from theorem 1.2.6 canonical morphisms of oriented cohomology theories

$$\Omega^* \to \mathrm{CH}^*$$

and

$$\Omega^* \to K^0[\beta, \beta^{-1}].$$

We immediately deduce from theorems 1.2.3 and 1.2.6 the following result:

[2] V. Voevodsky had considered weaker forms before.

Theorem 1.2.18. *Over a field of characteristic zero, the canonical morphism*

$$\Omega^* \to K^0[\beta, \beta^{-1}]$$

induces an isomorphism

$$\Omega^* \otimes_{\mathbb{L}^*} \mathbb{Z}[\beta, \beta^{-1}] \cong K^0[\beta, \beta^{-1}].$$

Theorem 1.2.18 is the analogue of a well-known theorem of Conner and Floyd [5]. Theorems 1.2.2 and 1.2.6 similarly imply the analogous relation between Ω^* and CH^*:

Theorem 1.2.19. *Let k be a field of characteristic zero. Then the canonical morphism*

$$\Omega^* \to \mathrm{CH}^*$$

induces an isomorphism

$$\Omega^* \otimes_{\mathbb{L}^*} \mathbb{Z} \to \mathrm{CH}^*.$$

In fact, we prove theorem 1.2.19 before theorem 1.2.2, using theorem 1.2.3, theorem 1.2.6, theorem 1.2.7 and some explicit computations of the class of a blow-up of a smooth variety along a smooth subvariety. We then deduce theorem 1.2.2 from theorems 1.2.6 and 1.2.19.

Remark 1.2.20. The hypothesis of characteristic zero in theorems 1.2.6, and the related theorem 1.2.18 is needed only to allow the use of resolution of singularities, and so these results are valid over any field admitting resolution of singularities in the sense of Appendix A. Theorem 1.2.7 uses resolution of singularities as well as the weak factorization theorem of [2] and [37]. Thus theorems 1.2.2 and 1.2.19 rely on both resolution of singularities and the weak factorization theorem.

Our definition of the homomorphism deg, on the other hand, relies at present on the generic smoothness of a morphism $Y \to X$ of smooth k-schemes, hence is restricted to characteristic zero, regardless of any assumptions on resolution of singularities. Thus, the explicit formula (1.2) in theorem 1.2.14 relies on characteristic zero for its very definition. Since the proof of theorem 1.2.19 relies on theorem 1.2.14, this result also requires characteristic zero in the same way.

Remark 1.2.21. Theorem 1.2.19, together with the natural transformation described in example 1.2.10, immediately implies a result of B. Totaro [34] constructing for any smooth $\mathbb{C}$-variety X, a map

$$\mathrm{CH}^*(X) \to MU^{2*}(X) \otimes_{\mathbb{L}^*} \mathbb{Z}$$

factoring the topological cycle class map $\mathrm{CH}^*(X) \to H^{2*}(X, \mathbb{Z})$ through the natural map $MU^{2*}(X) \otimes_{\mathbb{L}^*} \mathbb{Z} \to H^{2*}(X, \mathbb{Z})$.

Remark 1.2.22. Unoriented cobordism. Let $X \mapsto MO^*(X)$ denote unoriented cobordism theory and $MO^* := MO^*(pt)$ the unoriented cobordism of a point, as studied by Thom [33]. Given a real embedding $\sigma : k \to \mathbb{R}$, then for any smooth k-scheme X of dimension d denote by $X_\sigma(\mathbb{R})$ the differentiable manifold (of dimension d) of real points of X. Then clearly, the assignment

$$X \mapsto MO^*(X_\sigma(\mathbb{R}))$$

has a structure of oriented cohomology theory on $\mathbf{Sm}_k$ (one can use [30]; observe that the associated theory of Chern classes is this case is nothing but the theory of Stiefel-Whitney classes in $MO^*(X_\sigma(\mathbb{R}))$). Thus we get from the universality of Ω^* a natural transformation

$$\Omega^*(X) \to MO^*(X_\sigma(\mathbb{R})).$$

From theorem 1.2.7 we thus get for any real embedding $k \to \mathbb{R}$ a natural homomorphism:

$$\Psi_{k\to\mathbb{R}} : \mathbb{L}^* \cong \Omega^*(k) \to MO^*$$

which (using corollary 1.2.11) does not depend on k; to compute $\Psi_{k\to\mathbb{R}}$, we may thus assume $k = \mathbb{R}$. Concretely, $\Psi_\mathbb{R} : \mathbb{L}^* = \Omega^*(\operatorname{Spec}\mathbb{R}) \to MO^*$ is the map which sends the class $[X]$ of a smooth projective variety X over $\mathbb{R}$ to the unoriented class of the differentiable manifold $X(\mathbb{R})$ of real points.

From [30], the theory of Stiefel-Whitney classes in MO^* defines an isomorphism of rings

$$\mathbb{L}^*/[2] \to MO^*$$

where [2] denotes the ideal generated by the coefficients of the power series $[2](u) := F_\mathbb{L}(u, u)$. One easily checks that the induced epimorphism $\mathbb{L}^* \to MO^*$ is the homomorphism $\Psi_\mathbb{R}$ above.

From all this follows a geometric interpretation of the map $\Psi : \mathbb{L}^* \to \mathbb{L}^*/[2]$ using the identifications $\mathbb{L}^* = \Omega^*(\mathbb{R}) = MU^{2*}$ and $\mathbb{L}^*/[2] = MO^*$: let $x \in MU^{2n}$ be an element represented by a smooth projective variety X over $\mathbb{R}$. Then $\Psi(x)$ is equal to the unoriented cobordism class $[X(\mathbb{R})]$ (which thus only depends on x).

1.3 Relations with complex cobordism

At this point, let's give some heuristic explanation of the whole picture.

For X a finite CW-complex, one can define its singular cohomology groups with integral coefficients $H^*(X;\mathbb{Z})$, its complex K-theory $K^*(X)$, and its complex cobordism $MU^*(X)$ (see [3], for instance). These are complex oriented cohomology theories, they admit a theory of Chern classes and the analogue of lemma 1.1.3 implies the existence of a canonical ring homomorphism from $\mathbb{L}^*$ to the coefficient ring of the theory (which double the degrees with our conventions).

Quillen in [29] refined Milnor's [22] and Novikov's [26] computations that the complex cobordism MU^* of a point is a polynomial algebra with integral coefficient by showing that the map

$$\Phi^{top} : \mathbb{L}^* \to MU^{2*}$$

is an isomorphism (here we mean that Φ^{top} double the degrees and that the odd part of MU^* vanishes). Then in [30], Quillen produced a geometric proof of that fact emphasizing that MU^* is the universal complex oriented cohomology theory on the category of differentiable manifolds.

The theorem of Conner-Floyd [5] now asserts that for each CW-complex X the map

$$MU^*(X) \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \to K^*(X)$$

is an isomorphism (beware that in topology β has degree -2).

However, in general for a CW-complex X the homomorphism

$$MU^*(X) \otimes_{\mathbb{L}} \mathbb{Z} \to H^*(X; \mathbb{Z})$$

is not an isomorphism (not even surjective), even when restricted to the even part. Thus contrary to theorem 1.2.18, theorem 1.2.19 has no obvious counterpart in topology.

To give a heuristic explanation of our results we should mention that for smooth varieties over a field singular cohomology is replaced by motivic cohomology $H^{*,*}(X;\mathbb{Z})$, complex K-theory by Quillen's algebraic K-theory $K^{*,*}(X)$ and complex cobordism by the theory $MGL^{*,*}$ represented by the algebraic Thom complex MGL (in the setting of $\mathbb{A}^1$-homotopy theory, see [36]). One should note that these theories take values in the category of bigraded rings, the first degree corresponding to the cohomological degree and the second to the weight. In this setting, one should still have the Conner-Floyd isomorphism[3]

$$MGL^{*,*}(X) \otimes_{\mathbb{L}^*} \mathbb{Z}[\beta, \beta^{-1}] \cong K^{*,*}(X)[\beta, \beta^{-1}]$$

for any simplicial smooth k-variety X (beware here that β has bidegree $(-2,-1)$). However the map $MGL^{*,*}(X) \otimes_{\mathbb{L}^*} \mathbb{Z} \to H^{*,*}(X)$ would almost never be an isomorphism. Instead one expects a spectral sequence[4] from motivic cohomology to $MGL^{*,*}(X)$; the filtration considered in §4.5.2 should by the way be the one induced by that spectral sequence. Then theorem 1.2.19 is explained by the degeneration of this spectral sequence in the area computing the bidegrees of the form $(2n, n)$.

In fact, the geometric approach taken in the present work only deal with bidegrees of the form $(2n, n)$. Indeed, one can check that for any oriented

[3] This has been proven over any field by the second author jointly with M. Hopkins, unpublished.

[4] This spectral sequence has been announced in characteristic zero by the second author jointly with M. Hopkins, in preparation.

bigraded cohomology theory $A^{*,*}$ in the setting of $\mathbb{A}^1$ homotopy theory, the associated functor $X \mapsto \oplus_n A^{2n,n}(X)$ has a structure of oriented cohomology theory on $\mathbf{Sm}_k$ in our sense. In particular the universal property of Ω^* yields a natural transformation

$$\Omega^*(X) \to MGL^{2*,*}(X)$$

which we conjecture to be an isomorphism.

We are hopeful that our geometric approach can be extended to describe the whole bigraded algebraic cobordism, and that our results are only the first part of a general description of the functor $MGL^{*,*}$.

2

The definition of algebraic cobordism

The basic structures that we emphasize in chapter 2 consists of three types of operations: push-forwards f_* for projective morphisms, pull-backs f^* for smooth morphisms, and a first Chern class endomorphism $\tilde{c}_1(L)$ for each line bundle L.

In §2.1-2.3 we develop the formalism of oriented Borel-Moore functors, which encodes these three structures, and discuss their elementary properties. In §2.4 we give the definition of algebraic cobordism as the universal oriented Borel-Moore functor satisfying some explicit geometric axioms and make some elementary but important calculations in §2.5.

In this chapter k is an arbitrary field and $\mathcal{V}$ is an admissible subcategory of $\mathbf{Sch}_k$. We let $\mathcal{V}'$ denote the subcategory of $\mathcal{V}$ whose morphisms are the projective morphisms. $\mathbf{Ab}_*$ will denote the category of graded abelian groups.

2.1 Oriented Borel-Moore functors

The notion of an oriented Borel-Moore functor axiomatizes the basic relationships between projective push-forward, smooth pull-back and the first Chern class operators. This leads us directly to the universal such object, the functor of *cobordism cycles*.

2.1.1 Push-forwards, pull-backs and first Chern classes

Definition 2.1.1. *A functor $H_* : \mathcal{V}' \to \mathbf{Ab}_*$ is called* additive *if for any finite family $(X_1, \ldots, X_r)$ of finite type k-schemes, the homomorphism*

$$\oplus_{i=1}^r H_*(X_i) \to H_*(\amalg_{i=1}^r X_i)$$

induced by the (projective) morphisms $X_i \subset \amalg_{i=1}^r X_i$ is an isomorphism. Observe that in particular we must have

$$H_*(\emptyset) = 0.$$

Definition 2.1.2. *An* oriented Borel-Moore functor *on $\mathcal{V}$ is given by:*

(D1). An additive functor $H_* : \mathcal{V}' \to \mathbf{Ab}_*$.
(D2). For each smooth equi-dimensional morphism $f : Y \to X$ in $\mathcal{V}$ of relative dimension d a homomorphism of graded groups

$$f^* : H_*(X) \to H_{*+d}(Y).$$

(D3). For each line bundle L on X a homomorphism of graded abelian groups:

$$\tilde{c}_1(L) : H_*(X) \to H_{*-1}(X).$$

These data satisfy the following axioms:

(A1). For any pair of composable smooth equi-dimensional morphisms $(f : Y \to X, g : Z \to Y)$ respectively of dimension d and e, one has

$$(f \circ g)^* = g^* \circ f^* : H_*(X) \to H_{*+d+e}(Z).$$

In addition, $\mathrm{Id}_X^* = \mathrm{Id}_{H_*(X)}$ for any $X \in \mathcal{V}$.
(A2). Let $f : X \to Z$, $g : Y \to Z$ be morphisms in $\mathcal{V}$, giving the cartesian square

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z. \end{tikzcd}$$

Suppose that f is projective and g is smooth and equi-dimensional. Then

$$g^* f_* = f'_* g'^*.$$

(A3). Given a projective morphism $f : Y \to X$ and a line bundle L over X, one has

$$f_* \circ \tilde{c}_1(f^* L) = \tilde{c}_1(L) \circ f_*$$

(A4). Given a smooth equi-dimensional morphism $f : Y \to X$ and a line bundle L over X, one has

$$\tilde{c}_1(f^* L) \circ f^* = f^* \circ \tilde{c}_1(L).$$

(A5). Given line bundles L and M on $X \in \mathcal{V}$ one has:

$$\tilde{c}_1(L) \circ \tilde{c}_1(M) = \tilde{c}_1(M) \circ \tilde{c}_1(L).$$

Moreover, if L and M are isomorphic, then $\tilde{c}_1(L) = \tilde{c}_1(M)$.

Remark 2.1.3. Let $L \to X$ be a line bundle on some $X \in \mathcal{V}$, and let $\mathcal{L}$ be the invertible sheaf of sections of L. As a matter of notation, we define $\tilde{c}_1(\mathcal{L}) : H_*(X) \to H_{*-1}(X)$ to be $\tilde{c}_1(L)$.

Given an oriented Borel-Moore functor H_* and a projective morphism $f : Y \to X$ the homomorphism $f_* : H_*(Y) \to H_*(X)$ is called the *push-forward* along f. For a smooth equi-dimensional morphism $f : Y \to X$ of relative dimension d , the homomorphism $f^* : H_*(X) \to H_{*+d}(Y)$ is called the *pull-back* along g. And for a line bundle L on X, the homomorphism $\tilde{c}_1(L)$ is called the *first Chern class operator* of L.

A morphism $\vartheta : G_* \to H_*$ of oriented Borel-Moore functors is a natural transformation $\vartheta : G_* \to H_*$ of functors $\mathcal{V}' \to \mathbf{Ab}_*$ which moreover commutes with the smooth pull-backs and the operators $\tilde{c}_1$.

Remark 2.1.4. Any oriented Borel-Moore functor $A_*(-)$ on some admissible subcategory $\mathcal{W}$ of $\mathbf{Sch}_k$ defines by restriction an oriented Borel-Moore functor on all admissible $\mathcal{V} \subset \mathcal{W}$.

Conversely, an oriented Borel-Moore functor A_* on $\mathbf{Sm}_k$ determines an oriented Borel-Moore functor A_*^{BM} on $\mathbf{Sch}_k$ as follows.

One first takes $A_*^{BM}(Y)$ for Y smooth to be $A_*(Y)$. Then for any finite type k-scheme X, we consider the category $\mathcal{C}/X$ whose objects are projective morphisms $Y \to X$ with Y smooth and morphisms from $Z \to X$ to $Y \to X$ are (projective) morphisms $Z \to Y$ over X. Then one sets

$$A_*^{BM}(X) := \mathrm{colim}_{Y \to X \in \mathcal{C}/X} A_*(Y).$$

Push-forward morphisms $f_* : A_*^{BM}(Y) \to A_*^{BM}(X)$ for a projective morphism $f : Y \to X$ in $\mathbf{Sch}_k$, pull-backs along smooth equi-dimensional morphisms, action of $\tilde{c}_1(L)$ and external products are induced in the obvious way by the same operation on the subcategory of smooth k-schemes. All the axioms are easy consequences of those on the category $\mathbf{Sm}_k$.

Of course, the restriction of the oriented Borel-Moore functor A_*^{BM} to $\mathbf{Sm}_k$ equals A_*. But the converse of course is not true in general. Given an oriented Borel-Moore functor A_* on $\mathcal{V}$ and given $X \in \mathcal{V}$ there is a canonical morphism

$$A_*^{BM}(X) = \mathrm{colim}_{Y \to X \in \mathcal{C}/X} A_*(Y) \xrightarrow{\phi_A} A_*(X),$$

which defines a morphism of oriented Borel-Moore functors on $\mathcal{V}$. In general this is not an isomorphism. In case ϕ_A is surjective, we shall say that A_* is *generated by smooth schemes* and if ϕ_A is an isomorphism shall say that A_* is *detected by smooth schemes*

2.1.2 Cobordism cycles

For X a scheme of finite type over k, denote by $\mathcal{M}(X)$ the set of isomorphism classes (over X) of projective morphisms $Y \to X$ with Y in $\mathbf{Sm}_k$; $\mathcal{M}(X)$ becomes a monoid for the disjoint union. We let $\mathcal{M}_*^+(X)$ denote its group completion graded by the dimension over k of the Y's. Given a projective morphism $f : Y \to X$ with Y smooth, we let either $[f : Y \to X]$, or $[Y \to X]$ or $[f]$, depending on the context, denote the image of $f : Y \to X$ in $\mathcal{M}_*^+(X)$.

We observe that the class of the empty scheme $\emptyset \to X$ is $[\emptyset \to X] = 0$ and that $\mathcal{M}_*^+(X)$ is the free abelian group on classes $[Y \to X]$ with $Y \to X$ projective and Y smooth and irreducible.

Given a projective morphism $f : X \to X'$ in $\mathbf{Sch}_k$, composition with f defines a graded group homomorphism $f_* : \mathcal{M}_*^+(X) \to \mathcal{M}_*^+(X')$. Given a smooth equi-dimensional morphism $f : X' \to X$, of relative dimension d, one has the homomorphism $f^* : \mathcal{M}_*^+(X) \to \mathcal{M}_{*+d}^+(Y), [Z \to X] \mapsto [Z \times_X Y \to Y]$. It is easy to check that the operations f_* and f^* satisfies axioms (A1) and (A2) of definition 2.1.2.

Let H_* be an oriented Borel-Moore functor on some admissible $\mathcal{V} \subset \mathbf{Sch}_k$ and choose an element $a \in H_0(k)$. Following Quillen, we construct a canonical natural transformation

$$\vartheta_{H,a} : \mathcal{M}_*^+ \to H_*$$

of functors $\mathcal{V}' \to \mathbf{Ab}_*$ as follows: for each projective morphism $f : Y \to X$ in $\mathcal{V}$ with Y smooth and irreducible, set

$$\vartheta_{H,a}([f : Y \to X]) := f_* \circ \pi_Y^*(a),$$

where $\pi_Y : Y \to \operatorname{Spec} k$ denotes the structural morphism. Clearly $\vartheta_{H,a}$ is also natural with respect to smooth pull-backs in $\mathcal{V}$.

Remark 2.1.5. It would have been possible to define algebraic cobordism working only with $\mathcal{M}_*^+(X)$ (see lemma 2.5.11). Instead we will consider a slightly more sophisticated theory, mainly the universal oriented Borel-Moore functor $X \mapsto \mathcal{Z}_*(X)$ which we are going to construct; $\mathcal{Z}_*$ is obtained from $\mathcal{M}_*^+$ by formally adding the first Chern class operators. This approach simplifies the definition of algebraic cobordism.

Definition 2.1.6. *Let X be a k-scheme of finite type.*

(1) A cobordism cycle *over X is a family $(f : Y \to X, L_1, \ldots, L_r)$ consisting of:*

(a) a projective morphism $f : Y \to X$ with Y in $\mathbf{Sm}_k$ and integral.
(b) a finite sequence $(L_1, \ldots, L_r)$ of r line bundles over Y (this sequence is to be interpreted as empty if $r = 0$).

The dimension of $(f : Y \to X, L_1, \ldots, L_r)$ is $\dim_k(Y) - r \in \mathbb{Z}$.

(2) An isomorphism *Φ of cobordism cycles*

$$(Y \to X, L_1, \ldots, L_r) \cong (Y' \to X, L'_1, \ldots, L'_{r'})$$

is a triple $\Phi = (\phi : Y \to Y', \sigma, (\psi_1, \ldots, \psi_r))$ consisting of:

(a) an isomorphism $\phi : Y \to Y'$ of X-schemes.
(b) a bijection $\sigma : \{1, \ldots, r\} \cong \{1, \ldots, r'\}$ (so that r must equal r').

(c) for each $i \in \{1, \ldots, r\}$ an isomorphism of line bundles over Y: $\psi_i : L_i \cong \phi^*(L'_{\sigma(i)})$.

(3) We let $\mathcal{C}(X)$ be the set of isomorphism classes of cobordism cycles over X and $\mathcal{Z}(X)$ be the free abelian group on $\mathcal{C}(X)$. We observe that the dimension of cobordism cycles makes $\mathcal{Z}_*(X)$ into a graded abelian group called the group of cobordism cycles on X. The image of a cobordism cycle $(f : Y \to X, L_1, \ldots, L_r)$ in this group is denoted $[f : Y \to X, L_1, \ldots, L_r]$, or simply $[f, L_1, \ldots, L_r]$, or $[Y \to X, L_1, \ldots, L_r]$, depending on the context.

If $Y \to X$ is a projective morphism with Y smooth over k, we denote by $[Y \to X] \in \mathcal{Z}_*(X)$ the sum of the classes $[Y_\alpha \to X]$ corresponding to the irreducible components Y_α of Y. We thus have a natural graded homomorphism

$$\mathcal{M}_*^+(X) \to \mathcal{Z}_*(X),$$

which is easily seen to be a monomorphism. When X is smooth and equi-dimensional of dimension d, the class $[\mathrm{Id}_X : X \to X] \in \mathcal{Z}_d(X)$ is simply denoted 1_X.

More generally, given $Y = \amalg_j Y_j$ in $\mathbf{Sm}_k$ with line bundles $L_1, \ldots, L_r$ on Y and a projective morphism $f : Y \to X$ in $\mathbf{Sch}_k$, we write $[f : Y \to X, L_1, \ldots, L_r]$ for the sum $\sum_j [f_j : Y_j \to X, L_{j1}, \ldots, L_{jr}]$ in $\mathcal{Z}_*(X)$, where f_j and L_{ji} are the restrictions of f and L_i to Y_j.

Remark 2.1.7. Clearly given finite type k-schemes X and X', the natural homomorphism

$$\mathcal{Z}_*(X) \oplus \mathcal{Z}_*(X') \to \mathcal{Z}_*(X \amalg X')$$

is an isomorphism of graded abelian groups, so that $\mathcal{Z}_*$ is additive.

Moreover if X is a finite type k-scheme and X_α are the irreducible components of X then clearly the homomorphism

$$\oplus_\alpha \mathcal{Z}_*(X_\alpha) \to \mathcal{Z}_*(X)$$

is an epimorphism.

Let $g : X \to X'$ be a projective morphism in $\mathcal{V}$. Composition with g defines the map of graded groups

$$g_* : \mathcal{Z}_*(X) \to \mathcal{Z}_*(X')$$
$$[f : Y \to X, L_1, \ldots, L_r] \mapsto [g \circ f : Y \to X', L_1, \ldots, L_r],$$

called the *push-forward along g*.

If $g : X \to X'$ is a smooth equi-dimensional morphism of relative dimension d, sending $[f : Y \to X, L_1, \ldots, L_r]$ to $[p_2 : (Y \times_X X') \to X', p_1^*(L_1), \ldots, p_1^*(L_r)]$ defines the homomorphism

$$g^* : \mathcal{Z}_*(X) \to \mathcal{Z}_{*+d}(X')$$

called *pull-back along g.*.

Let X be k-scheme of finite type and let L be a line bundle on X. We call the homomorphism

$$\begin{aligned} \tilde{c}_1(L) : \mathcal{Z}_*(X) &\to \mathcal{Z}_{*-1}(X) \\ [f : Y \to X, L_1, \dots, L_r] &\mapsto [Y \to X, L_1, \dots, L_r, f^*(L)] \end{aligned}$$

the *first Chern class homomorphism* of L.

Remark 2.1.8. Let $(f : Y \to X, L_1, \dots, L_r)$ be a standard cobordism cycle on X. Then one obviously has the formulas (in $\mathcal{Z}_*(X)$):

$$\begin{aligned} [f : Y \to X, L_1, \dots, L_r] &= f_* \circ [\mathrm{Id}_Y, L_1, \dots, L_r] \\ &= f_* \circ \tilde{c}_1(L_r)([\mathrm{Id}_Y, L_1, \dots, L_{r-1}]) \\ &\vdots \\ &= f_* \circ \tilde{c}_1(L_r) \circ \dots \circ \tilde{c}_1(L_1)(1_Y) \\ &= f_* \circ \tilde{c}_1(L_r) \circ \dots \circ \tilde{c}_1(L_1) \circ \pi_Y^*(1), \end{aligned}$$

where $\pi_Y : Y \to \operatorname{Spec} k$ is the structure morphism and $1 \in \mathcal{Z}_0(k)$ is the class of the identity map on $\operatorname{Spec} k$.

Lemma 2.1.9. *Let $\mathcal{V}$ be an admissible subcategory of $\mathbf{Sch}_k$. The functor*

$$\begin{aligned} \mathcal{Z}_* : \mathcal{V}' &\to \mathbf{Ab}_* \\ X &\mapsto \mathcal{Z}_*(X), \end{aligned}$$

endowed with the above operations of smooth pull-backs and first Chern classes is an oriented Borel-Moore functor on $\mathcal{V}$. Moreover, $\mathcal{Z}_*$ with the distinguished element $1 \in \mathcal{Z}_0(k)$ is universal in the following sense: given an oriented Borel-Moore functor H_* on $\mathcal{V}$ and an element $a \in H_0(k)$ there is one and only one morphism of oriented Borel-Moore functors

$$\vartheta_{H,a} : \mathcal{Z}_* \to H_*$$

such that $\vartheta_{H,a}(1) = a \in H_*(k)$.

The proof is rather easy. To prove the universality one uses remark 2.1.8 to show that one must have

$$\vartheta_{H,a}([f : Y \to X, L_1, \dots, L_r]) = f_* \circ \tilde{c}_1(L_r) \circ \dots \circ \tilde{c}_1(L_1)(\pi_Y^*(a))$$

2.1.3 External products

We define an external product

$$\times : \mathcal{Z}_*(X) \times \mathcal{Z}_*(Y) \to \mathcal{Z}_*(X \times_k Y)$$

on the functor $\mathcal{Z}_*$ by:

$$\begin{aligned}&[f : X' \to X, L_1, \ldots, L_r] \times [g : Y' \to Y, M_1, \ldots, M_s])\\ &\quad := [f \times g : X' \times Y' \to X \times Y, p_1^*(L_1), \ldots, p_1^*(L_r), p_2^*(M_1), \ldots, p_2^*(M_s)]\end{aligned}$$

$\times$ is associative and commutative. Moreover $1 := [\mathrm{Id}_k] \in \mathcal{Z}_0(k)$ is a unit element for this product. In particular, $\mathcal{Z}_*(k)$ becomes a unitary, associative, commutative graded ring and for $X \in \mathcal{V}$ the external product gives the group $\mathcal{Z}_*(X)$ the structure of a graded $\mathcal{Z}_*(k)$-module. We are led to the following definition:

Definition 2.1.10. *An* oriented Borel-Moore functor on $\mathcal{V}$ with product *consists of an oriented Borel-Moore functor on $\mathcal{V}$, H_*, together with:*

(D4). An element $1 \in H_0(k)$ and, for each pair (X, Y) of k-schemes in $\mathcal{V}$, a bilinear graded pairing (called the external product)

$$\begin{aligned}\times : H_*(X) \times H_*(Y) &\to H_*(X \times Y)\\ (\alpha, \beta) &\mapsto \alpha \times \beta\end{aligned}$$

which is (strictly) commutative, associative, and admits 1 as unit.

These satisfy

(A6). Given projective morphisms f and g one has

$$\times \circ (f_* \times g_*) = (f \times g)_* \circ \times.$$

(A7). Given smooth equi-dimensional morphisms f and g, one has

$$\times \circ (f^* \times g^*) = (f \times g)^* \circ \times.$$

(A8). Given k-schemes X and Y in $\mathcal{V}$ and a line bundle L on X one has for any classes $\alpha \in H_*(X)$ and $\beta \in H_*(Y)$

$$\tilde{c}_1(L)(\alpha) \times \beta = \tilde{c}_1(p_1^*(L))\big(\alpha \times \beta\big).$$

Given an oriented Borel-Moore functor with product A_* we observe that the axioms give $A_*(k)$ a commutative, graded ring structure, give to each $A_*(X)$ a structure of $A_*(k)$-module, and imply that all the operations f_*, f^* and $\tilde{c}_1(L)$ preserve the $A_*(k)$-module structure. For $p : Y \to \mathrm{Spec}\, k$ in $\mathbf{Sm}_k$, we denote the element $p^*1 \in A_*(Y)$ by 1_Y^A, or just 1_Y if the meaning is clear.

Remark 2.1.11. Endowed with its external product, $\mathcal{Z}_*$ is an oriented Borel-Moore functor on $\mathcal{V}$ with product, for all admissible $\mathcal{V}$. Moreover, one easily checks that it is in fact the universal one: given an oriented Borel-Moore functor on $\mathcal{V}$ with product A_*, there exists one and only one morphism of oriented Borel-Moore functors with product

$$\vartheta_A : \mathcal{Z}_* \to A_*.$$

Indeed, one easily checks that the transformation

$$\vartheta_{A,1} : \mathcal{Z}_* \to A_*$$

given by lemma 2.1.9 is compatible with the external products.

Definition 2.1.12. *Let R_* be a commutative graded ring with unit. An oriented Borel-Moore R_*-functor on $\mathcal{V}$, A_*, is an oriented Borel-Moore functor on $\mathcal{V}$ with product, together with a graded ring homomorphism*

$$\Phi : R_* \to A_*(k).$$

For such a functor, one gets the structure of an R_*-module on $A_*(X)$ for each $X \in \mathcal{V}$, by using Φ and the external product. All the operations of projective push-forward, smooth pull-back, and $\tilde{c}_1$ of line bundles are R_*-linear.

For instance, given an oriented Borel-Moore R_*-functor A_* and a homomorphism of commutative graded rings $R_* \to S_*$, one can construct an oriented Borel-Moore S_*-functor, denoted by $A_* \otimes_{R_*} S_*$, by the assignment $X \mapsto A_*(X) \otimes_{R_*} S_*$. The push-forward, smooth pull-back, and $\tilde{c}_1$ of line bundles are obtained by extension of scalars $(-) \otimes_{R_*} S_*$.

2.1.4 Standard cycles

Let A_* be an oriented Borel-Moore R_*-functor with product on $\mathcal{V}$. By lemma 2.1.9, we have the morphism of oriented Borel-Moore functors $\vartheta_{A,1} : \mathcal{Z}_* \to A_*$. For X in $\mathcal{V}$, let $\bar{A}_*(X)$ be the sub-$A_*(k)$-module of $A_*(X)$ generated by $\vartheta_{A,1}(\mathcal{Z}_*(X))$. It is easy to see that this defines an oriented Borel-Moore R_*-functor on $\mathcal{V}$ with product, $\bar{A}_*$. We call the element

$$\vartheta_{A,1}([f : Y \to X, L_1, \dots, L_r]) = f_*(\tilde{c}_1(L_1) \circ \dots \tilde{c}_1(L_r)(1_Y))$$

a *standard cycle* on X, and write this element as $[f : Y \to X, L_1, \dots, L_r]_A$.

Remark 2.1.13. For $Y \in \mathcal{V}$ with structure morphism $f : Y \to \operatorname{Spec} k$, we write $[Y]_A$ for $[f : Y \to \operatorname{Spec} k]_A$. Note that $1_Y = [\mathrm{Id}_Y]_A$ and that $[Y]_A = f_*(1_Y)$. It follows from the additivity of A_* that $1_{Y \amalg Y'} = 1_Y + 1_{Y'}$ in $A_*(Y \amalg Y') = A_*(Y) \oplus A_*(Y')$, and $[Y \amalg Y']_A = [Y]_A + [Y']_A$.

2.1.5 Imposing relations

Let H_* be an oriented Borel-Moore functor and, for each $X \in \mathcal{V}$, let $\mathcal{R}_*(X) \subset H_*(X)$ be a set of homogeneous elements. We will construct a new oriented Borel-Moore functor denoted by $H_*/\mathcal{R}_*$ together with a morphism of oriented Borel-Moore functors $\pi : H_* \to H_*/\mathcal{R}_*$ with the following universal property: given any oriented Borel-Moore functor G_* and any morphism of oriented functors $\vartheta : H_* \to G_*$ such that, for each X, the homomorphism $\vartheta(X) : H_*(X) \to G_*(X)$ vanishes on $\mathcal{R}_*(X)$, then there is one and only one morphism of oriented Borel-Moore functors $\varphi : H_*/\mathcal{R}_* \to G_*$ such that $\varphi \circ \pi = \vartheta$. This oriented Borel-Moore functor will then be said to be obtained from H_* by *killing the elements in the* $\mathcal{R}_*(X)$, or that $H_*/\mathcal{R}_*$ is the *quotient of* H_* *by the relations* $\mathcal{R}_*$.

To construct $H_*/\mathcal{R}_*$, we proceed as follows: For $X \in \mathcal{V}$ denote by $\langle \mathcal{R}_* \rangle(X) \subset H_*(X)$ the subgroup generated by elements of the following form:

$$f_* \circ \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r) \circ g^*(\rho) \tag{2.1}$$

with $f : Y \to X$ a projective morphism in $\mathcal{V}$, $(L_1, \ldots, L_r)$, $r \geq 0$, a family of line bundles over Y, $g : Y \to Z$ a smooth equi-dimensional morphism and $\rho \in \mathcal{R}_*(Z)$. Then $\langle \mathcal{R}_* \rangle$ is an oriented Borel-Moore sub-functor of H_*, and is the smallest one which contains each of the $\mathcal{R}_*(X)$.

The assignment $X \mapsto H_*(X)/\langle \mathcal{R}_* \rangle(X)$ thus has a unique structure of oriented Borel-Moore functor which makes the canonical projection $\pi : H_*(X) \to H_*(X)/\langle \mathcal{R}_* \rangle(X)$ a morphism of oriented Borel-Moore functors. We denote by $H_*/\mathcal{R}_*$ this oriented Borel-Moore functor. It is clear that the morphism $\pi : H_* \to H_*/\mathcal{R}_*$ is a solution to our problem.

Remark 2.1.14. Let A_* be an oriented Borel-Moore functor with product. Assume we are given for each X a set $\mathcal{R}_*(X)$ of homogeneous elements in $A_*(X)$ such that for $\rho \in A_*(X)$ and $\sigma \in A_*(Y)$ one has

$$\rho \times \sigma \in \mathcal{R}_*(X \times Y)$$

if either $\rho \in \mathcal{R}_*(X)$ or $\sigma \in \mathcal{R}_*(Y)$. Then there is one and only one external product on the oriented Borel-Moore functor $H_*/\mathcal{R}_*$ compatible with the projection $H_* \to H_*/\mathcal{R}_*$. This statement easily follows from (2.1).

2.1.6 Cohomological notations

Let A_* be a oriented Borel-Moore functor on $\mathbf{Sm}_k$. For X in $\mathbf{Sm}_k$, let the X_α denote the irreducible components of X and set $d_\alpha := \dim_k(X_\alpha)$. We introduce the following notation:

$$A^n(X) = \oplus_\alpha A_{d_\alpha - n}(X_\alpha).$$

Given a smooth morphism $f : Y \to X$, the pull-back morphism in A_* associated to f defines a homomorphism of degree zero, $A^*(X) \to A^*(Y)$. Given a projective morphism $f : Y \to X$ of relative codimension d, the push-forward morphism in A_* defines the push-forward $f_* : A^*(Y) \to A^{*+d}(X)$. The endomorphism $\tilde{c}_1(L)$ of $A_*(X)$ associated to a line bundle L on X induces the endomorphism $\tilde{c}_1(L)$: $A^*(X) \to A^{*+1}(X)$. Finally, an external product on A_* induces an external product $A^*(X) \otimes A^*(Y) \to A^*(X \times Y)$.

The assignment $X \mapsto A^*(X)$ will be called the *oriented cohomological functor* on $\mathbf{Sm}_k$ associated to A_*. One can rewrite all the axioms for an oriented Borel-Moore functor A_* on $\mathbf{Sm}_k$ in terms of A^*. Clearly A^* and A_* are thus determined by each other and the category of oriented Borel-Moore functors on $\mathbf{Sm}_k$ is equivalent to that of oriented cohomological functors on $\mathbf{Sm}_k$. We have as well the notion of an oriented cohomological functor with product on $\mathbf{Sm}_k$, with the analogous equivalence to the category of oriented Borel-Moore functors with product on $\mathbf{Sm}_k$.

2.2 Oriented functors of geometric type

We introduce three additional axioms for an oriented Borel-Moore functor.

Recall from §1.1 that $\mathbb{L}_*$ denotes the Lazard ring homologically graded (which means that $\mathbb{L}_n = 0$ if $n < 0$) and that $F_{\mathbb{L}}(u, v) \in \mathbb{L}_*[[u, v]]$ denotes the universal formal group law.

Given a line bundle $L \to X$ on some $X \in \mathbf{Sm}_k$ with zero-section s_0, a section s of L is *transverse to the zero-section* if the cartesian diagram

$$\begin{array}{ccc} X \times_L X & \longrightarrow & X \\ \downarrow & & \downarrow s \\ X & \xrightarrow[s_0]{} & L \end{array}$$

is transverse in $\mathbf{Sm}_k$, i.e., if the subscheme of X defined by s is a smooth codimension one closed subscheme of X.

Definition 2.2.1. *An oriented Borel-Moore $\mathbb{L}_*$-functor A_* on $\mathcal{V}$ is said to be of* geometric type *if the following three axioms holds:*

(Dim). For any smooth k-scheme Y and any family $(L_1, \ldots, L_n)$ of line bundles on Y with $n > \dim_k(Y)$, one has

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n)(1_Y) = 0 \in A_*(Y).$$

(Sect). For any smooth k-scheme Y, any line bundle L on Y and any section s of L which is transverse to the zero section of L, one has

$$\tilde{c}_1(L)(1_Y) = i_*(1_Z),$$

where $i : Z \to Y$ is the closed immersion of the zero-subscheme of s.

(FGL). Let $\phi_A : \mathbb{L}_* \to A_*(k)$ be the ring homomorphism giving the $\mathbb{L}_*$-structure and let $F_A \in A_*(k)[[u,v]]$ be the image of the universal formal group law $F_{\mathbb{L}} \in \mathbb{L}_*[[u,v]]$ by ϕ_A. Then for any smooth k-scheme Y and any pair (L,M) of line bundles on Y, one has

$$F_A(\tilde{c}_1(L), \tilde{c}_1(M))(1_Y) = \tilde{c}_1(L \otimes M)(1_Y) \in A_*(Y).$$

Remark 2.2.2. The axioms (Dim) and (Sect) make sense for any oriented Borel-Moore functor with product, and we will sometimes use them in this less restrictive context.

Remark 2.2.3. To make sense of the left-hand side of axiom (FGL), we use the vanishing stated in (Dim) and the fact that $\tilde{c}_1(L)$ and $\tilde{c}_1(M)$ commute. In fact, only the following weak form of axiom (Dim) is needed for the left-hand side of (FGL) to make sense:

(Nilp). For each smooth k-scheme Y there exists an integer N_Y such that, for each family $(L_1, \ldots, L_n)$ of line bundles on Y with $n > N_Y$, one has

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n)(1_Y) = 0 \in A_*(Y).$$

For instance, this property holds in A_* if, for each given X, $A_*(X)$ vanishes below some degree.

We will prove below (in theorem 2.3.13) that an oriented Borel-Moore $\mathbb{L}_*$-functor H_* which satisfies (Nilp), (Sect) and (FGL) also satisfies (Dim).

Example 2.2.4. The Chow group functor

$$X \mapsto \mathrm{CH}_*(X)$$

endowed with projective push-forward, smooth pull-back, the action of the $\tilde{c}_1$ of line bundles and the external product of cycles (see [9, Chapters 1 & 2]) is an oriented Borel-Moore functor on $\mathbf{Sch}_k$. Moreover, given line bundles L and M (over the same base) one has the formula

$$\tilde{c}_1(L \otimes M) = \tilde{c}_1(L) + \tilde{c}_1(M).$$

This gives CH_* an $\mathbb{L}_*$-structure: the formal group law is given by

$$F_{\mathrm{CH}}(u,v) = F_a(u,v) = u + v.$$

One easily checks (using the results of [9]) that CH_* is of geometric type. Moreover, using resolution of singularities, one can see that the Chow group functor $X \mapsto \mathrm{CH}_*(X)$ is detected by smooth k-schemes (in the sense of section 2.1.4) when k has characteristic zero.

Example 2.2.5. Another fundamental example of an oriented Borel-Moore functor on $\mathbf{Sch}_k$ is given by G_0-theory, $X \mapsto G_0(X)$, where $G_0(X)$ denotes the

Grothendieck K-group of the category of coherent $\mathcal{O}(X)$-modules (see Fulton [9, Chapter 15] for instance).

We have already introduced $K^0(X)$, the Grothendieck K-group of locally free sheaves on X. For X an arbitrary finite type k-scheme, the tensor product of coherent sheaves induces a unitary, commutative ring structure on $K^0(X)$ and gives $G_0(X)$ a natural structure of $K^0(X)$-module.

Let $G_0(X)[\beta, \beta^{-1}]$ be $G_0(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}]$, where $\mathbb{Z}[\beta, \beta^{-1}]$ is the ring of Laurent polynomials in a variable β of degree $+1$. Define push-forward for a projective morphism $f : Y \to X$ by

$$f_*([\mathcal{M}] \cdot \beta^n) := \Sigma_{i=0}^{\infty} (-1)^i [R^i f_*(\mathcal{M})] \cdot \beta^n \in G_0(X)[\beta, \beta^{-1}]$$

for $\mathcal{M}$ a coherent $\mathcal{O}(Y)$-module and $n \in \mathbb{Z}$, thus defining a functor

$$\begin{aligned} \mathbf{Sch}'_k &\to \mathbf{Ab}_* \\ X &\mapsto G_0(X)[\beta, \beta^{-1}]. \end{aligned}$$

It is endowed with pull-back along a smooth equi-dimensional morphism $f : Y \to X$ of relative dimension d by the formula:

$$f^*([\mathcal{M}] \cdot \beta^n) := [f^*(\mathcal{M})] \cdot \beta^{n+d}.$$

The first Chern class endomorphism associated to the line bundle L on $X \in \mathbf{Sch}_k$ is defined by the multiplication by $(1 - [\mathcal{L}^\vee]) \cdot \beta^{-1}$:

$$\tilde{c}_1(L) := - \cup (1 - [\mathcal{L}^\vee]) \cdot \beta^{-1} : G_0(X)[\beta, \beta^{-1}] \to G_0(X)[\beta, \beta^{-1}],$$

where $\mathcal{L}$ is the invertible sheaf of sections of L, using the $K^0(X)$-module structure on $G_0(X)$. One can easily check that together with the external product

$$\begin{aligned} G_0(X)[\beta, \beta^{-1}] \times G_0(Y)[\beta, \beta^{-1}] &\to G_0(X \times Y)[\beta, \beta^{-1}] \\ ([\mathcal{M}]\beta^a, [\mathcal{N}]\beta^b) &\mapsto [\pi_X^*(\mathcal{M}) \otimes_{X \times Y} \pi_Y^*(\mathcal{N})]\beta^{a+b} \end{aligned}$$

our functor $X \mapsto G_0(X)[\beta, \beta^{-1}]$ is an oriented Borel-Moore functor with product on $\mathbf{Sch}_k$. Moreover, if L and M are line bundles over X, the formula

$$\tilde{c}_1(L \otimes M) = \tilde{c}_1(L) + \tilde{c}_1(M) - \beta \circ \tilde{c}_1(L) \circ \tilde{c}_1(M)$$

mentioned in example 1.1.5 gives $G_0[\beta, \beta^{-1}]$ an $\mathbb{L}_*$-structure with associated formal group law

$$F_K(u, v) = F_m(u, v) := u + v - \beta uv.$$

One can then check that $G_0[\beta, \beta^{-1}]$ is of geometric type. However, we do not know whether or not the functor $X \mapsto G_0(X)[\beta, \beta^{-1}]$ is detected by smooth k-schemes, even in characteristic zero.

$G_0[\beta, \beta^{-1}]$ is however generated by smooth k-schemes if k admits resolution of singularities. This follows easily from the well-known facts that $G_0(X)$ is generated by the classes $[\mathcal{O}_Z]$, with $Z \subset X$ an integral closed subscheme, and for $p : \tilde{Z} \to Z$ a resolution of singularities, $p_*([\mathcal{O}_{\tilde{Z}}]) = [\mathcal{O}_Z] + \eta$ in $G_0(Z)$, where $\eta \in G_0(Z)$ is some class in the image of $G_0(W)$ for some proper closed subscheme W of the integral scheme Z. Thus an easy noetherian induction, using the right-exact localization sequence for G_0, gives the desired generation of $G_0(X)$.

2.3 Some elementary properties

The imposition of the axiom (Sect) gives an oriented Borel-Moore functor with product some of the flavor of classical cobordism. In this section, we derive some of these properties, and also show how the axiom (Dim) can be replaced by the seemingly weaker axiom (Nilp).

2.3.1 The axiom (Sect)

We fix an admissible subcategory $\mathcal{V}$ of $\mathbf{Sch}_k$ and let A_* be an oriented Borel-Moore functor with product on $\mathcal{V}$, satisfying the axiom (Sect).

Remark 2.3.1. Let $[f : Y \to X, L_1, \ldots, L_r, L]_A$ be a standard cycle over some $X \in \mathcal{V}$. Suppose that L has a section with smooth divisor $i : Z \to Y$. Then

$$[f : Y \to X, L_1, \ldots, L_r, L]_A = [f \circ i : Z \to X, i^*L_1, \ldots, i^*L_r]_A.$$

Indeed, applying $f_* \circ \tilde{c}_1(L_1) \circ \ldots \circ \tilde{c}_1(L_r)$ to the identity $\tilde{c}_1(L)(1_Y) = i_*(1_Z)$ given by (Sect) yields this relation.

Definition 2.3.2. *Let Y be in $\mathbf{Sm}_k$. A* geometric cobordism over Y *is a projective morphism $f : W \to Y \times \mathbb{P}^1$, with $W \in \mathbf{Sm}_k$ and with $p_2 \circ f$ transverse (in $\mathbf{Sm}_k$) to the inclusion $\{0, \infty\} \to \mathbb{P}^1$.*

Lemma 2.3.3. *Take $Y \in \mathbf{Sm}_k$ and let $f : W \to Y \times \mathbb{P}^1$ be a geometric cobordism over Y. Let $f_0 : W_0 \to Y$, $f_\infty : W_\infty \to Y$ be the fibers of $p_2 \circ f : W \to \mathbb{P}^1$ over $0, \infty$, resp., with morphisms f_0, f_∞ induced by $p_1 \circ f$. Then*

$$[f_0 : W_0 \to Y]_A = [f_\infty : W_\infty \to Y]_A.$$

Proof. Both $i_0 : W_0 \to W$ and $i_\infty : W_\infty \to W$ are subschemes of W defined by sections of $(p_2 \circ f)^*\mathcal{O}_{\mathbb{P}^1}(1)$, hence by axiom (Sect) we have

$$i_{0*}(1_{W_0}) = \tilde{c}_1((p_2 \circ f)^*\mathcal{O}_{\mathbb{P}^1}(1))(1_W) = i_{\infty *}(1_{W_\infty}).$$

The result follows by pushing forward to Y by $p_1 \circ f$. $\square$

Lemma 2.3.4. *Let L be a finite separable k-algebra. Then*

$$[\operatorname{Spec} L]_A = [L:k]\cdot 1$$

in $A_0(k)$.

Proof. Since $[Y \amalg Y']_A = [Y]_A + [Y']_A$ (see remark 2.1.13), we may reduce to the case of connected $\operatorname{Spec} L$, which means that L is a finite separable field extension of k. Thus there exists $x \in L$ such that $L = k[x]$. Let $f \in k[X]$ be the monic irreducible polynomial of x, of degree $d = [L:k]$. If $f(X) = X^d + \sum_{i=0}^{d-1} b_i X^i$, let $F(X_0, X_1)$ be the homogenized form of f, $F(X_0, X_1) = X_1^d + \sum_{i=0}^{d-1} b_i X_1^i X_0^{d-i}$.

Assume first that k is infinite. Choose distinct elements $a_1, \ldots, a_d \in k$, and let $G = \prod_{i=1}^d (X_1 - a_i X_0)$. Since f is irreducible, no a_i is a root of f. Let $H = Y_0 \cdot F + Y_1 \cdot G$, and let

$$W \subset \mathbb{P}^1 \times \mathbb{P}^1 = \operatorname{Proj}_k(k[X_0, X_1]) \times \operatorname{Proj}_k(k[Y_0, Y_1])$$

be the closed subscheme defined by H. Using the Jacobian criterion, one checks that W is smooth over k. Via the projection on the second factor, W is finite over $\mathbb{P}^1_k$, and defines a geometric cobordism over $\operatorname{Spec} k$, with fibers $\operatorname{Spec} L$ over $(Y_0 : Y_1) = (1:0)$ and d disjoint copies of $\operatorname{Spec} k$ over $(Y_0 : Y_1) = (0:1)$. By lemma 2.3.3, we have the identity

$$[\operatorname{Spec} L]_A = d \cdot [\operatorname{Spec} k]_A = d \cdot 1$$

in $A_0(k)$.

If k is finite, we proceed by induction on $d = [L:k]$. If $d = 2$ the same argument as above applies. So we may assume $d > 2$. We choose an irreducible polynomial $h \in k[U]$ of degree $d-1$ (such an h always exists) and an $a \in k$. We set $g = (X-a) \times h$. Note that h is automatically separable since k is perfect. Then the above reasoning applies to show that $[\operatorname{Spec} L]_A = 1 + [\operatorname{Spec} k[X]/h]_A$, and the inductive hypothesis gives the result. □

Lemma 2.3.5. *(1) Let $k \subset L$ be a finite separable extension of fields. For a scheme $X \in \mathcal{V}$, denote by $\pi(L/k) : X_L \to X$ the natural morphism. Then the composition:*

$$A_*(X) \xrightarrow{\pi(L/k)^*} A_*(X_L) \xrightarrow{\pi(L/k)_*} A_*(X)$$

is multiplication by $[L:k]$.

(2) Let $k \subset F_1$ and $k \subset F_2$ be finite separable fields extensions of k, of relatively prime degree. Then for any scheme X in $\mathcal{V}$ the homomorphism:

$$A_*(X) \xrightarrow{\pi(F_1/k)^* + \pi(F_2/k)^*} A_*(X_{F_1}) \oplus A_*(X_{F_2})$$

is a split monomorphism.

Proof. (1) Let $p : \mathrm{Spec}\, L \to \mathrm{Spec}\, k$ be the morphism induced by $k \to L$. Then $\pi_{L/k} = p \times \mathrm{Id}_X$. Thus, for $x \in A_*(X) = A_*(\mathrm{Spec}\, k \times X)$, we have

$$\begin{aligned}\pi_{L/k*}\pi^*_{L/k} &= (p \times \mathrm{Id}_X)_*(p \times \mathrm{Id}_X)^*(1 \times x)\\ &= p_*(p^*(1)) \times x\\ &= [\mathrm{Spec}\, L]_A \times x,\end{aligned}$$

using axioms (A6) and (A7) of definition 2.1.10. The result thus follows from lemma 2.3.4.

(2) Let u and v be integers such that $u[F_1 : k] + v[F_2 : k] = 1$ Then the homomorphism

$$A_*(X_{F_1}) \oplus A_*(X_{F_2}) \xrightarrow{u\pi(F_1/k)_* + v\pi(F_2/k)_*} A_*(X)$$

is a left inverse to the given homomorphism. □

Recall from §2.1.4 the sub-functor $\bar{A}_*$ of A_*, with $A_*(X)$ generated over $A_*(k)$ by the standard cycles $[f : Y \to X, L_1, \ldots, L_r]_A$.

Lemma 2.3.6. *Let O_X be the trivial line bundle on $X \in \mathbf{Sch}_k$. Then the homomorphism*

$$\tilde{c}_1(O_X) : \bar{A}_*(X) \to \bar{A}_{*-1}(X)$$

is the zero homomorphism. In particular, if X is in $\mathbf{Sm}_k$, then $\tilde{c}_1(O_X)(1_X) = 0$.

Proof. For any standard cycle $[Y \to X, L_1, \ldots, L_r]_A$ on X, one has

$$\begin{aligned}\tilde{c}_1(O_X)([Y \to X, L_1, \ldots, L_r]_A) &= [Y \to X, L_1, \ldots, L_r, O_Y]_A\\ &= [\emptyset \to X, L_1, \ldots, L_r]_A = 0,\end{aligned}$$

by (Sect), because the constant unit section of O_X never vanishes. □

Another useful result is the following:

Lemma 2.3.7. *Take $X \in \mathcal{V}$, let α be a k-point of $\mathbb{P}^1$ and take $x \in A_*(X)$. Let $i^X_\alpha : X \to X \times \mathbb{P}^1$ be the section with constant value α. Then*

$$i^X_{\alpha*}(x) = \tilde{c}_1(p_2^* O(1))(p_1^*(x)).$$

In particular, if α and β are any two k-points of $\mathbb{P}^1$, then

$$i^X_{\alpha*}(x) = i^X_{\beta*}(x).$$

Proof. By definition 2.1.10(A7), with $f = \mathrm{Id}_X$, $g : \mathbb{P}^1 \to \mathrm{Spec}\, k$ the structure morphism, we have

$$p_1^*(x) = x \times 1_{\mathbb{P}^1}$$

By (A8), we have

$$\tilde{c}_1(p_2^* O(1))(p_1^*(x)) = x \times \tilde{c}_1(O(1))(1_{\mathbb{P}^1}).$$

The axiom (Sect) implies $\tilde{c}_1(O(1))(1_{\mathbb{P}^1}) = i_{\alpha *}^{\operatorname{Spec} k}(1)$; by (A6), we have

$$x \times i_{\alpha *}^{\operatorname{Spec} k}(1) = i_{\alpha *}^{X}(x)$$

completing the proof. □

Remark 2.3.8. Suppose that k has characteristic zero. Let $f : Y \to X$ be a projective morphism with X finite type over k and Y smooth quasi-projective over k. Choose a line bundle L on X generated by sections $s_1, \ldots, s_n$. Then the set of points $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{A}^n$ such that $\sum_{i=1}^n x_i f^*(s_i)$ is a section transverse to the zero section of $f^*(L)_{|Y_{\kappa(\underline{x})}}$ on $Y_{\kappa(\underline{x})}$ is an open dense subset; see [25, Bertini's theorem 2.3] for instance.

Suppose that k has positive characteristic, $Y = X$ and L is very ample. If $s_1, \ldots, s_n$ are sections of L which give a closed immersion of Y into $\mathbb{P}^{n-1}$, then as above, the subscheme of sections $\sum_{i=1}^n x_i f^*(s_i)$ of L which are transverse to the zero section is a dense open subset of $\mathbb{A}^n$.

Thus:

(1) If k has characteristic zero and L is globally generated, or if k is an infinite field of positive characteristic, $Y = X$ and L is very ample, such an open set has always a rational point over k, and thus there is always a section of L transverse to the zero section.

(2) When k is finite, the open set may not have a rational point but will always have two closed point x and y with $[\kappa(x) : k]$ and $[\kappa(y) : k]$ relatively prime. Thus, assuming $Y = X$ and L is very ample, there are finite extensions $k \subset F_1$ and $k \subset F_2$ such that $[F_1 : k]$ and $[F_2 : k]$ are relatively prime and such that the pull-back of L to X_{F_1} and X_{F_2} have sections transverse to the zero section.

Lemma 2.3.9. *(1) Let Y be in $\mathbf{Sm}_k$, and let $(L_1, \ldots, L_n)$ be a family of line bundles over Y. Suppose that each of the L_i is very ample and that $n > \dim_k Y$. Then for all line bundles $M_1, \ldots, M_r$ on Y,*

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n)([\mathrm{Id}_Y, M_1, \ldots, M_r]_A) = 0.$$

(2) Suppose k has characteristic zero. Let X be in $\mathcal{V} \subset \mathbf{Sch}_k$. n be an integer such that $n > \dim_k(X)$ and $(L_1, \ldots, L_n)$ be a family of line bundles over X. Assume that each of the L_i is generated by its global sections. Then

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n) = 0 \in \mathrm{End}(\bar{A}_*(X)).$$

In particular, for a globally generated line bundle L, one has

$$\tilde{c}_1(L)^n = 0 \in \mathrm{End}(\bar{A}_*(X)).$$

Proof. For (2), choose a standard cycle $x := [f : Y \to X, M_1, \ldots, M_r]_A$ and let $\bar{X} = f(Y)$. We proceed by induction on $N := \dim_k \bar{X}$ to show that $(\prod_{i=1}^n \tilde{c}_1(L_i))(x) = 0$ for $n > N$. If $\dim_k \bar{X} = 0$, each f^*L_i is trivial and the result follows from lemma 2.3.6. We assume $\dim_k \bar{X} > 0$.

We know by remark 2.3.8 that for a general section s of L_n, the pull-back of s to Y is transverse to the zero section of $f^*(L_n)$. Let $j : Z \to Y$ denote the closed immersion of the zeros of $f^*(s)$ and let $S := f(j(Z))$ be the image of Z. Observe that either $Z = \emptyset$, which means f^*L_n is trivial, or $\dim_k(S) < \dim_k(\bar{X})$, because we may assume that s is non-zero at the generic point of $\bar{X}$. In the first case $\tilde{c}_1(f^*L_n) = 0$ and there is nothing to prove, so we assume we are in the second case. One then has (using remark 2.3.1):

$$
\begin{aligned}
&\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n)([f : Y \to X, M_1, \ldots, M_r]_A) \\
&\qquad = \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_{n-1})([f : Y \to X, M_1, \ldots, M_r, f^*L_n]_A) \\
&\qquad = \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_{n-1})([f \circ j : Z \to X, j^*M_1, \ldots, j^*M_r]_A)
\end{aligned}
$$

Since $\dim_k(S) < \dim_k(\bar{X})$, the last term vanishes by the inductive hypothesis.

For (1), if k has characteristic zero, we may use (2). If k has positive characteristic and is infinite, the same argument as above, using remark 2.3.8(1) proves (1). If k is finite, then one chooses (using 2.3.8(2)) two finite extensions $k \subset F_1$ and $k \subset F_2$ such that $[F_1 : k]$ and $[F_2 : k]$ are prime together and such that the pull-back of L to X_{F_1} and X_{F_2} have sections transverse to the zero section. The same reasoning shows that the element

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n)([\mathrm{Id}_Y, M_1, \ldots, M_r]_A)$$

maps to zero into both $\bar{A}_*(X_{F_1})$ and $\bar{A}_*(X_{F_1})$. The result then follows easily from lemma 2.3.5.

□

2.3.2 The axiom (Dim)

Throughout this section A_* will be an oriented Borel-Moore $\mathbb{L}_*$-functor with product on an admissible subcategory $\mathcal{V}$ of $\mathbf{Sch}_k$, satisfying the axioms (Nilp), (Sect) and (FGL). The main result of this section shows that such an A_* also satisfies (Dim).

Let $F(u_1, \ldots, u_n) \in A_*(k)[[u_1, \ldots, u_n]]$ be a power series, which we expand as

$$F(u_1, \ldots, u_n) = \sum_I a_I u^I,$$

where I runs over the set of r-tuples $I = (n_1, \ldots, n_r)$ of integers and $a_I \in A_*(k)$,. We will say that F is *absolutely homogeneous of degree n* if for each I, a_I is in $A_{|I|-n}(k)$, where $|I| = n_1 + \cdots + n_r$.

We have the universal formal group law $(\mathbb{L}_*, F_{\mathbb{L}})$. Recall the existence of a (unique) power series $\chi(u) \in \mathbb{L}_*[[u]]$, with leading term $-u$ and which satisfies the equality

$$F_{\mathbb{L}}(u, \chi(u)) = 0.$$

Note that $\chi(u)$ is absolutely homogeneous of degree 1. In the sequel, we will freely use the following notations:

$$\begin{aligned} u +_F v &:= F_{\mathbb{L}}(u, v) \in \mathbb{L}_*[[u, v]], \\ [-1]_F u &:= \chi(u) \in \mathbb{L}_*[[u]], \\ u -_F v &:= F_{\mathbb{L}}(u, \chi(v)) \in \mathbb{L}_*[[u, v]]. \end{aligned}$$

We will also use the same notation for the image of these power series by the homomorphism $\mathbb{L}_* \to A_*(k)$. If f, g are locally nilpotent commuting endomorphisms of an $A_*(k)$-module, we also denote by $f +_F g$ the endomorphism obtained by substituting f for u and g for v in $u +_F v$, $f -_F g$ and $[-1]_F f$ for those obtained in the same way from $u -_F v$ and $[-1]_F u$.

The following lemma is an immediate consequence of the definitions, noting that the endomorphisms $\tilde{c}_1(L)$, $\tilde{c}_1(M)$ are locally nilpotent and commute with each other:

Lemma 2.3.10. *Suppose A_* satisfies (Nilp), (Sect) and (FGL). For any finite type k-scheme X and any pair (L, M) of line bundles on X one has the following relations in $\mathrm{End}(\bar{A}_*(X))$:*

1. $\tilde{c}_1(L) +_F \tilde{c}_1(M) = \tilde{c}_1(L \otimes M)$
2. $\tilde{c}_1(L) -_F \tilde{c}_1(M) = \tilde{c}_1(L \otimes M^\vee)$
3. $[-1]_F \tilde{c}_1(L) = \tilde{c}_1(L^\vee)$

where $^\vee$ denotes the operation of dualization.

Remark 2.3.11. Let X be a finite type k-scheme. Let $\mathrm{End}^{\tilde{c}_1}(\bar{A}_*(X))$ denote the sub-$A_*(k)$-algebra of the $A_*(k)$-algebra $\mathrm{End}_{\bar{A}_*(k)}(\bar{A}_*(X))$ generated by the $\tilde{c}_1(L)$. Observe that this algebra is commutative graded. Then each element in $\mathrm{End}^{\tilde{c}_1}_{-1}(\bar{A}_*(X))$ (the subgroup of elements of degree -1 of that algebra) is locally nilpotent, and the map $(f, g) \mapsto f +_F g$ defines an abelian group structure on this set, with $f \mapsto [-1]_F f$ as inverse. Moreover, the map

$$\tilde{c}_1 : \mathrm{Pic}(X) \to \mathrm{End}^{\tilde{c}_1}_{-1}(\bar{A}_*(X))$$

is a group homomorphism. This is a reformulation of the previous lemma.

Remark 2.3.12. Let X be a quasi-projective k-scheme. For each line bundle L on X, there is a very ample line bundle M such that $M \otimes L$ is very ample. From lemma 2.3.10, the endomorphism $\tilde{c}_1(L)$ of $\bar{A}_*(X)$ can be computed as

$$\tilde{c}_1(L) = \tilde{c}_1(M \otimes L) -_F \tilde{c}_1(M)$$

Theorem 2.3.13. *Let A_* be an oriented Borel-Moore $\mathbb{L}_*$-functor with product on an admissible subcategory $\mathcal{V}$ of $\mathbf{Sch}_k$, satisfying the axioms (Nilp), (Sect) and (FGL).*
(1) Let k be a field of characteristic zero. Let X be in $\mathcal{V}$ and let $(L_1, \ldots, L_n)$ be a family of line bundles on X with $n > \dim_k(X)$, such that one of the following two conditions is satisfied:

(a) The line bundles are all globally generated.
(b) X is a quasi-projective k-scheme.

Then one has

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n) = 0$$

in $\mathrm{End}(\bar{A}_*(X))$.
(2) Let k be an arbitrary field. Let Y be in $\mathbf{Sm}_k$. Then for any family $(L_1, \ldots, L_n)$ of line bundles on Y with $n > \dim_k(Y)$, one has

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n)(1_Y) = 0$$

In particular, A_* satisfies the axiom (Dim).

Proof. Case (1a) follows from lemma 2.3.9(2). Assume now that X is a quasi-projective scheme. Using remark 2.3.12, we see that for each $i \in \{1, \ldots, n\}$ there exists two very ample line bundles M_i and N_i such that $L_i \cong M_i \otimes N_i^\vee$. But then from lemma 2.3.10 we have

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n) = \Pi_{i=1}^n \big(\tilde{c}_1(M_i) -_F \tilde{c}_1(N_i)\big).$$

Thus the endomorphism $\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n)$ is a sum of terms of the form $a \cdot \tilde{c}_1(L'_1) \circ \cdots \circ \tilde{c}_1(L'_m)$, with $a \in A_*(k)$, each L'_j very ample, and $m \geq n$. Using lemma 2.3.9 again completes the proof of (1b)

The proof of (2) is essentially the same as the proof of (1b), where we use lemma 2.3.9(1) instead of lemma 2.3.9(2). □

Remark 2.3.14. We do not know in general whether or not the previous theorem holds for any finite type k-scheme and any family of line bundles.

Remark 2.3.15. Beware that the theorem 2.3.13(1) doesn't hold in $\mathrm{End}(A_*)$ in general.

2.4 The construction of algebraic cobordism

Starting with the cobordism cycles functor $\mathcal{Z}_*$, we complete the construction of algebraic cobordism by imposing the axioms (Dim), (Sect) and (FGL).

2.4.1 Power series.

Suppose that A_* satisfies the axiom (Dim). Let

$$F(u_1,\ldots,u_r) \in A_*(k)[[u_1,\ldots,u_r]]$$

be a formal power series in $(u_1,\ldots,u_r)$ with coefficients in the graded ring $A_*(k)$. Suppose that F is absolutely homogeneous of degree n.

Given line bundles $(L_1,\ldots,L_r)$ on $X \in \mathcal{V}$, the operations $\tilde{c}_1(L_1)$, ..., $\tilde{c}_1(L_r)$ are locally nilpotent on $\bar{A}_*(X)$ (by axiom (Dim)) and commute with each other. In the endomorphism ring $\mathrm{End}(\bar{A}_*(X))$ we may thus substitute $\tilde{c}_1(L_i)$ for u_i in F and get a well-defined homogeneous element of degree $-n$:

$$F(\tilde{c}_1(L_1),\ldots,\tilde{c}_1(L_r)) : \bar{A}_*(X) \to \bar{A}_{*-n}(X) \subset A_{*-n}(X). \tag{2.2}$$

If X is a smooth equi-dimensional k-scheme of dimension d, we have the class $1_X \in \bar{A}_d(X)$ and we set

$$[F(L_1,\ldots,L_r)]_A := F(\tilde{c}_1(L_1),\ldots,\tilde{c}_1(L_r))(1_X) \in A_{d-n}(X)$$

Similarly, if $f : Y \to X$ is in $\mathcal{M}(X)$, with X in $\mathcal{V}$, we write $[f : Y \to X]_A$ for $f_*(1_Y)$. Using these notations, the equation in axiom (Sect) can be written as

$$[L]_A = [Z \to Y]_A$$

and that in axiom (FGL) as

$$[F_A(L,M)]_A = [L \otimes M]_A.$$

We sometimes omit the subscript A if the meaning is clear from the context.

2.4.2 Imposing the axiom (Dim)

Definition 2.4.1. *Let X be a smooth and irreducible k-scheme. We let*

$$\mathcal{R}^{Dim}_*(X) \subset \mathcal{Z}_*(X)$$

denote the subset consiting of all elements of the form

$$[Y \to X, L_1,\ldots,L_r], \text{ where } \dim_k(Y) < r.$$

We denote by $\underline{\mathcal{Z}}_*$ the oriented Borel-Moore functor $\mathcal{Z}_*/\mathcal{R}^{Dim}_*$.

Of course,$\underline{\mathcal{Z}}_*$ is by construction the universal oriented Borel-Moore functor on $\mathcal{V}$ satisfying (Dim).

The following lemma is easy to prove using formula (2.1) of §2.1.5:

Lemma 2.4.2. *Let X be a finite type k-scheme. Then $\langle\mathcal{R}_*^{Dim}\rangle(X)$ is the subgroup of $\mathcal{Z}_*(X)$ generated by standard cobordism cycles of the form:*

$$[Y \to X, \pi^*(L_1), \ldots, \pi^*(L_r), M_1, \ldots, M_s],$$

where $\pi : Y \to Z$ is a smooth quasi-projective equi-dimensional morphism, Z is a smooth quasi-projective irreducible k-scheme, $(L_1, \ldots, L_r)$ are line bundles on Z and $r > \dim_k(Z)$.

One easily checks using this lemma and remark 2.1.14 that the external product on $\mathcal{Z}_*$ descends to give $\underline{\mathcal{Z}}_*$ an external product which makes $\underline{\mathcal{Z}}_*$ an oriented Borel-Moore functor with product on $\mathbf{Sch}_k$.

Remark 2.4.3. Of course, for any $X \in \mathbf{Sch}_k$ one has $\underline{\mathcal{Z}}_n(X) = 0$ if $n < 0$ by construction.

The following lemma is immediate:

Lemma 2.4.4. *Let X be a k-scheme of finite type and L a line bundle on X. Then the endomorphism $\tilde{c}_1(L)$ of $\underline{\mathcal{Z}}_n(X)$ is locally nilpotent, i.e., for each $a \in \underline{\mathcal{Z}}_n(X)$ there is an $n \in \mathbb{N}$ such that $(\tilde{c}_1(L))^n(a) = 0$.*

2.4.3 Imposing the axiom (Sect): algebraic pre-cobordism

Definition 2.4.5. *Let Y be a smooth and irreducible k-scheme. We let*

$$\mathcal{R}_*^{Sect}(Y) \subset \underline{\mathcal{Z}}_*(Y)$$

denote the subset consisting of all elements of the form

$$[L] - [Z \to Y],$$

where L is a line bundle over Y, $s : Y \to L$ is a section transverse to the zero section and $Z \to Y$ is the closed subscheme of zeroes of s (which is smooth over k by assumption on s). We denote by $\underline{\Omega}_*$ the oriented Borel-Moore functor $\underline{\mathcal{Z}}_*/\mathcal{R}_*^{Sect}$. It is called algebraic pre-cobordism.

Remark 2.4.6. In other words, the elements in $\mathcal{R}_*^{Sect}(Y)$ are exactly those of the form $[O_Y(Z)] - [Z \to Y]$ for $Z \to Y$ a smooth divisor.

The following lemma is easy to prove using formula (2.1) of §2.1.5:

Lemma 2.4.7. *Let X be a finite type k-scheme. Then $\langle R_*^{Sect}\rangle(X)$ is the subgroup of $\underline{\mathcal{Z}}_*(X)$ generated by elements of the form:*

$$[Y \to X, L_1, \ldots, L_r] - [Z \to X, i^*(L_1), \ldots, i^*(L_{r-1})]$$

with $r > 0$, $[Y \to X, L_1, \ldots, L_r]$ a standard cobordism cycle on X, and $i : Z \to Y$ the closed immersion of the subscheme defined by the vanishing of a transverse section $s : Y \to L_r$. Thus one has

$$[Y \to X, L_1, \ldots, L_r] = [Z \to X, i^*(L_1), \ldots, i^*(L_{r-1})]$$

in $\underline{\Omega}_*(X)$.

The elements of the above form are called *elementary cobordisms*. One easily checks using the previous lemma and remark 2.1.14 that the external product on $\underline{\mathcal{Z}}_*$ descends to give $\underline{\Omega}_*$ an external product which makes $\underline{\Omega}_*$ an oriented Borel-Moore functor with product on $\mathbf{Sch}_k$.

Remark 2.4.8. Let $f : W \to X \times \mathbb{P}^1$ be a geometric cobordism over X. In particular, the closed subschemes $W_0 := f^{-1}(0)$ and $W_1 := f^{-1}(1)$ are smooth over k. We call the difference $[W_0 \to X] - [W_1 \to X] \in \mathcal{M}(X)^+$ a *naive cobordism*, we let $\mathcal{N}_*(X) \subset \mathcal{M}_*^+(X)$ denote the subgroup generated by the naive cobordisms and we denote by $\Omega_*^{naive}(X)$ the quotient $\mathcal{M}_*^+(X)/\mathcal{N}_*(X)$. It is clear that the image of $\mathcal{N}_*(X)$ vanishes in $\underline{\Omega}_*(X)$. In fact, by lemma 2.3.3, the difference $[W_0 \to W] - [W_1 \to W]$ even vanishes in $\underline{\Omega}_*(W)$. We thus get a homomorphism $\Omega_*^{naive}(X) \to \underline{\Omega}_*(X)$. This homomorphism is not in general an isomorphism. In fact it is not in general a surjection because there are line bundles which have no sections transverse to the zero section.

2.4.4 Imposing (FGL): algebraic cobordism

Recall from §1.1 the Lazard ring $\mathbb{L}_*$, and the universal formal group law $F_{\mathbb{L}}(u, v) = \sum_{i,j\geq 0} a_{i,j}u^iv^j$. $\mathbb{L}_*$ is graded and the degree of $a_{i,j}$ is $i + j - 1$. Thus $F_{\mathbb{L}}(u, v)$ is absolutely homogeneous of degree 1. We also observe that $a_{i,j} = 0$ when $ij = 0$, except for $a_{1,0} = a_{0,1} = 1$. In the sequel we will consider the oriented Borel-Moore $\mathbb{L}_*$-functor $X \mapsto \mathbb{L}_* \otimes \underline{\Omega}_*(X)$ obtained from $\underline{\Omega}_*$ by extension of scalars. This functor satisfies the axioms (Dim) and (Sect) by construction.

Definition 2.4.9. *Let Y be a smooth irreducible k-scheme. We let*

$$\mathcal{R}_*^{FGL}(Y) \subset \mathbb{L}_* \otimes \underline{\Omega}_*(Y)$$

be the subset of elements of the form

$$[F_{\mathbb{L}}(L, M)] - [L \otimes M]$$

where L and M are line bundles over Y.

If $S_* \subset \mathbb{L}_* \otimes \underline{\Omega}_*(X)$ is a graded subset, we denote by $\mathbb{L}_*S \subset \mathbb{L}_* \otimes \underline{\Omega}_*(X)$ the subset of elements of the form $a\rho$ with $a \in \mathbb{L}_*$ and $\rho \in S_*$.

Definition 2.4.10. *We define* algebraic cobordism Ω_*,

$$X \mapsto \Omega_*(X),$$

to be the oriented Borel-Moore $\mathbb{L}_*$-functor on $\mathbf{Sch}_k$ which is the quotient of $\mathbb{L}_* \otimes \underline{\Omega}_*$ by the relations $\mathbb{L}_*\mathcal{R}_*^{FGL}$,

$$\Omega_* := \mathbb{L}_* \otimes \underline{\Omega}_* / \langle \mathbb{L}_*\mathcal{R}_*^{FGL} \rangle.$$

Remark 2.4.11. It is easy to see that $\langle \mathbb{L}_* \mathcal{R}^{FGL}_* \rangle(X)$ has the explicit description as the $\mathbb{L}_*$-submodule of $\mathbb{L}_* \otimes \underline{\Omega}_*(X)$ generated by elements of the form

$$f_*\big(\tilde{c}_1(L_1) \circ \ldots \circ \tilde{c}_1(L_n)([F_{\mathbb{L}}(L, M)] - [L \otimes M])\big),$$

where $f : Y \to X$ is in $\mathcal{M}(X)$, and $L_1, \ldots, L_n$, L and M are line bundles on Y.

Remark 2.4.12. One could imagine that we could have defined algebraic cobordism directly as the quotient of $\mathbb{L}_* \otimes \underline{\mathcal{Z}}_*$ by $\mathbb{L}_*\big(\mathcal{R}^{Sect}_* \bigcup \mathcal{R}^{FGL}_*\big)$. However, for the elements in $\mathcal{R}^{FGL}_*$ to be defined, we need to have some vanishing of products of the $\tilde{c}_1$, which is guaranteed by axiom (Dim). This forces us to start by killing $\mathcal{R}^{Dim}_*$ first.

One easily observes that the sets $< \mathbb{L}_* \mathcal{R}^{FGL}_* > (X)$ satisfy the conditions of remark 2.1.14. The external product on $\mathbb{L}_* \otimes \underline{\Omega}_*$ thus descends to Ω_*, making algebraic cobordism an oriented Borel-Moore $\mathbb{L}_*$-functor with product on $\mathbf{Sch}_k$.

Let's denote the composite homomorphism $\mathbb{L}_* \to \mathbb{L}_* \otimes \underline{\Omega}_*(k) \to \Omega_*(k)$ by

$$\begin{aligned} \Phi(k) : \mathbb{L}_* &\to \Omega_*(k) \\ a &\mapsto [a]. \end{aligned}$$

$\Phi(k)$ turns Ω_* into an oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type. The following theorem is clear by construction.

Theorem 2.4.13. *Let $\mathcal{V}$ be an admissible subcategory of $\mathbf{Sch}_k$. Algebraic cobordism is the universal oriented Borel-Moore $\mathbb{L}_*$-functor on $\mathcal{V}$ of geometric type. More precisely, given an oriented Borel-Moore $\mathbb{L}_*$-functor on $\mathcal{V}$ of geometric type, A_*, there is a unique morphism of oriented Borel-Moore $\mathbb{L}_*$-functors*

$$\vartheta_A : \Omega_* \to A_*.$$

Remark 2.4.14. (1) The functor ϑ_A is given by

$$\vartheta_A(a \otimes [f : Y \to X, L_1 \ldots, L_r]) = a \times [f : Y \to X, L_1 \ldots, L_r]_A.$$

(2) Given any oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type A_*, the morphism ϑ_A clearly induces a morphism

$$\tilde{\vartheta}_A : \Omega_* \otimes_{\mathbb{L}_*} A_*(k) \to A_*$$

from the oriented Borel-Moore $\mathbb{L}_*$-functor obtained from Ω_* by the extension of scalars $\mathbb{L}_* \to A_*$. Observe that the left hand side $X \mapsto \Omega_*(X) \otimes_{\mathbb{L}_*} A_*(k)$ is still an oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type.

We shall say that A_* is *free* if the morphism $\tilde{\vartheta}_A : \Omega_* \otimes_{\mathbb{L}_*} A_*(k) \to A_*$ is an isomorphism. This means that to give a morphism of oriented Borel-Moore

$\mathbb{L}_*$-functors from A_* to an oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type B_* is the same as to give a factorization $\mathbb{L}_* \to A_*(k) \to B_*(k)$.

Many of our main results can be rephrased by saying that a given theory A_* is free. For instance, we conjecture that the Chow groups functors as well as the K-theory functor are free, over any field. We will prove this conjecture in characteristic zero in chapter 4. For the K-theory functor we "only" need the resolution of singularities, for the Chow groups we need the resolution of singularities and the weak factorization theorem of [2] and [37].

Algebraic cobordism is detected by smooth k-schemes:

Lemma 2.4.15. *For any $X \in \mathbf{Sch}_k$ the homomorphism*

$$\Omega_*^{BM}(X) = \mathrm{colim}_{Y \to X \in \mathcal{C}/X} \Omega_*(Y) \to \Omega_*(X)$$

is an isomorphism.

Proof. Indeed generators of $\Omega_*(X)$ clearly come from the left hand side, and this is still true for the relations: they all come from explicit relations on smooth k-schemes. □

Remark 2.4.16. Clearly $\Omega_*(X)$ is generated as an $\mathbb{L}_*$-module by the standard cycles $[f : Y \to X, L_1, \ldots, L_r]$ for every $X \in \mathbf{Sch}_k$, i.e., $\Omega_* = \bar{\Omega}_*$. Thus, all the identities in $\mathrm{End}(\bar{A}_*)$ found in §2.3 are valid in $\mathrm{End}(\Omega_*)$.

2.5 Some computations in algebraic cobordism

In this section, A_* will be an oriented Borel-Moore functor of geometric type on an admissible subcategory $\mathcal{V}$ of $\mathbf{Sch}_k$, and $F_A(u,v) = \sum_{i,j} a_{i,j} u^i v^j \in A_*(k)[[u,v]]$ is the associated formal group law. We will drop the subscript A in the notation $[f : Y \to X]_A$ and $[F(L_1, \ldots, L_r)]_A$. Sometimes an alternate notation is more useful: for line bundles $L_i \to X$ on $X \in \mathbf{Sm}_k$, $i = 1, \ldots, r$, we write $c_1(L_1)^{n_1} \ldots c_1(L_r)^{n_r}$ for $\tilde{c}_1(L_1)^{n_1} \circ \ldots \circ \tilde{c}_1(L_r)^{n_r}(1_X)$.

We will compute the class of a blow-up of X in $A_*(X)$ and give a geometric description of the coefficients $a_{i,j}$ of F_A. Of course, since Ω_* is universal, such computations are equivalent to making the computation for $A_* = \Omega_*$, so working in full generality is somewhat illusory. We will also derive some useful facts about generators and relations in Ω_*.

2.5.1 The inverse for F_A

We have the inverse for the group law F_A, $\chi_A(u) = \sum_{i>0} \alpha_i u^i \in A_*(k)[[u]]$, which satisfies the equality

$$F_A(u, \chi_A(u)) = 0$$

Thus, for each line bundle L on $X \in \mathbf{Sm}_k$, we have the relation in $A_*(X)$:

$$[\chi_{\mathbb{A}}(L)] = [L^\vee]$$

An easy computation gives:

$$\chi_{\mathbb{A}}(u) = -u + a_{1,1}u^2 - (a_{1,1})^2u^3 + \big((a_{1,1})^3 + a_{1,1} \cdot a_{2,1} + 2a_{3,1} - a_{2,2}\big)\, u^4 + \text{ terms of degree } \geq 5 \quad (2.3)$$

2.5.2 A universal formula for the blow-up

Let $i : Z \to X$ be a closed embedding between smooth k-varieties. We recall the Fulton–MacPherson *deformation to the normal bundle*[1]; we present the entire diagram first and then explain the various terms:

$$\begin{tikzcd}
Z \times 1 \arrow[rr, hook, "i"] \arrow[d, hook] & & Y_1 \arrow[r, equal] \arrow[d, hook, "i_1"'] & X \times 1 \arrow[d, hook] & \\
Z \times \mathbb{P}^1 \arrow[rr, hook, "\tilde{i}"] & & Y \arrow[r, "\pi"'] \arrow[dddd, "p_1"] & X \times \mathbb{P}^1 \arrow[r, "p_2"'] & \mathbb{P}^1 \\
& X_Z \arrow[rr, "p"] \arrow[dr, hook] & & X \times 0 \arrow[u, hook] & \\
Z \times 0 \arrow[uu, hook] \arrow[dr, hook, "s"] \arrow[d, "s_0"] & & Y_0 \arrow[uu, hook] \arrow[ur, "\pi_0"'] & & \\
N_i \arrow[r, "j"] & \mathbb{P} \arrow[ur, hook] \arrow[rr, "q"'] & & Z \times 0 \arrow[uu, hook, "i"'] & \\
& & X & &
\end{tikzcd} \quad (2.4)$$

Here $p : X_Z \to X \times 0 = X$ is the blow-up of X along Z, $\pi : Y \to X \times \mathbb{P}^1$ is the blow-up of $X \times \mathbb{P}^1$ along $Z \times 0$, $\mathbb{P} \to Y$ is the inclusion of the exceptional divisor $\pi^{-1}(Z \times 0)$ with $q : \mathbb{P} \to Z \times 0$ the map induced by π. $Y_0 := \pi^{-1}(X \times 0)$ and $Y_1 := \pi^{-1}(X \times 1)$. The maps p_1 and p_2 are the ones induced by the projections $p_1 : X \times \mathbb{P}^1 \to X$ and $p_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1$.

The restriction of π to the proper transform $\pi^{-1}[X \times 0]$ identifies $\pi^{-1}[X \times 0]$ with X_Z as X-schemes; under this identification, $Y_0 = X_Z \cup \mathbb{P}$.

Since $Z \times 0$ is a Cartier divisor on $Z \times \mathbb{P}^1 \subset X \times \mathbb{P}^1$, the closed immersion $i \times \mathrm{id} : Z \times \mathbb{P}^1 \to X \times \mathbb{P}^1$ lifts to the closed immersion $\tilde{i} : Z \times \mathbb{P}^1 \to Y$. Since π is an isomorphism away from $Z \times 0$, the inclusion $X \times 1 \to X \times \mathbb{P}^1$ defines the closed immersion $i_1 : X = Y_1 \to Y$.

Let η_i be the conormal sheaf $\mathcal{I}_Z/\mathcal{I}_Z^2$ of i; $q : \mathbb{P} \to Z$ is identified with the projective bundle of the conormal sheaf of $Z \times \{0\} \subset X \times \mathbb{P}^1$, i.e.,

$$\mathbb{P} = \mathbb{P}(\eta_i \oplus \mathcal{O}_Z).$$

[1] see [9].

The intersection $\mathbb{P} \cap X_Z$ is the subbundle $\mathbb{P}(\eta_i) \subset \mathbb{P}$ corresponding to the projection $\eta_i \oplus \mathcal{O}_Z \to \eta_i$; as a subscheme of X_Z, $\mathbb{P} \cap X_Z$ is the exceptional divisor E of the blow-up $X_Z \to X$. The complement of E in $\mathbb{P}$ is the normal bundle $N_i \to Z$ and the inclusion $s : Z \times 0 \to \mathbb{P}$ is the zero-section $s_0 : Z \to N_i$ followed by the open immersion $j : N_i \to \mathbb{P}$.

Proposition 2.5.1. *Let $i : Z \to X$ be a closed embedding of smooth k-schemes. Let $p : X_Z \to X$ the blow-up of X along Z, η_i the conormal sheaf of i, $\mathbb{P} := \mathbb{P}(\eta_i \oplus \mathcal{O}_Z)$ and $q : \mathbb{P} \to Z$ the projection. Then we have the equality in $A_d(Y)$:*

$$[X_Z \xrightarrow{p} Y] = [X \xrightarrow{i_1} Y] + [\chi_{\mathbb{A}}(O_Y(\mathbb{P}))]$$

Proof. We have the identity of divisors $\pi^*(X \times \{0\}) = \mathbb{P} + X_Z$. Thus we have the isomorphism

$$O_Y(\mathbb{P}) \otimes O_Y(X_Z) \cong p_2^*(O(1)),$$

where $O_Y(\mathbb{P})$ and $O_Y(X_Z)$ are the line bundles on Y of the divisors $\mathbb{P}$ and X_Z, respectively, and $O(1)$ is the tautological quotient line bundle on $\mathbb{P}^1$.

Similarly, we have $O_Y(X) \cong p_2^*(O(1))$, and hence

$$O_Y(X_Z) \cong O_Y(X) \otimes O_Y(\mathbb{P})^\vee. \tag{2.5}$$

From lemma 2.3.10 and the isomorphism (2.5)

$$\begin{aligned}
[X_Z \to Y] &= \tilde{c}_1(O_Y(X_Z))(1_Y) \\
&= \sum_{i,j} a_{i,j} \tilde{c}_1(O_Y(X))^i \circ (\chi_{\mathbb{A}}(\tilde{c}_1(O_Y(\mathbb{P})))^j \, (1_Y) \\
&= [X \to Y] + [\chi_{\mathbb{A}}(O_Y(\mathbb{P}))] \\
&\qquad + \sum_{i \geq 1, j \geq 1} a_{i,j} \tilde{c}_1(O_Y(X))^i \circ (\chi_{\mathbb{A}}(\tilde{c}_1(O_Y(\mathbb{P}))))^j \, (1_Y).
\end{aligned} \tag{2.6}$$

Now the restriction of $O_Y(\mathbb{P})$ to X is clearly trivial since $\mathbb{P} \cap X = \emptyset$. In particular, in $A_*(Y)$ one has

$$\begin{aligned}
\tilde{c}_1(O_Y(\mathbb{P})) \circ \tilde{c}_1(O_Y(X))(1_Y) &= \tilde{c}_1(O_Y(\mathbb{P}))[X \to Y] \\
&= \tilde{c}_1(O_Y(\mathbb{P}))(i_{1*}(1_X)) \\
&= i_{1*}\big(\, \tilde{c}_1(i_1^* O_Y(\mathbb{P}))(1_X)\big) \\
&= i_{1*}\big(\, \tilde{c}_1(O_X)(1_X)\big) = 0
\end{aligned}$$

by lemma 2.3.6. Thus, in our formula (2.6), the terms with $i \geq 1$ and $j \geq 1$ all vanish, completing the proof. □

We can simplify this formula further. Let $g(u) \in A_*(k)[[u]]$ be the power series uniquely determined by the equation

$$\chi_{\mathbb{A}}(u) = u \cdot g(u).$$

We thus have by (2.3)

$$g(u) = -1 + a_{1,1}u - (a_{1,1})^2u^2 + \big((a_{1,1})^3 + a_{1,1}\cdot a_{2,1} + 2a_{3,1} - a_{2,2}\big)\,u^3 \\ + \text{ terms of degree } \geq 4 \quad (2.7)$$

Proposition 2.5.2. *Let $i : Z \to X$ be a closed immersion of smooth k-schemes. Let $X_Z \to X$ the blow-up of X at Z, η_i the conormal sheaf of i, $\mathbb{P} := \mathbb{P}(\eta_i \oplus \mathcal{O}_Z) \subset Y$ the exceptional divisor of π, and $q : \mathbb{P} \to Z$ the projection. Let $O_{\mathbb{P}}(-1)$ denote the dual of the canonical quotient line bundle $O_{\mathbb{P}}(1)$. Then one has the equality in $A_d(X)$:*

$$[X_Z \to X] = [\mathrm{Id}_X] + i_* \circ q_*\big([g(O_{\mathbb{P}}(-1))]\big) \qquad (2.8)$$

Proof. Let $\phi : \mathbb{P} \to Y$ denote the inclusion, and let $E = \mathbb{P} \cap X_Z$; E is the exceptional divisor of the blow-up $X_Z \to X$. It is easy to see that E is defined by the vanishing of the composition

$$\mathcal{O}_{\mathbb{P}} \cong q^*\mathcal{O}_Z \to q^*(\eta_i \oplus \mathcal{O}_Z) \to \mathcal{O}(1),$$

so $\mathcal{O}_{\mathbb{P}}(E) \cong \mathcal{O}(1)$.

Note that $\phi_*(1_{\mathbb{P}}) = \tilde{c}_1(O_Y(\mathbb{P}))(1_Y)$. Using the axiom (A3) of definition 2.1.2, it follows that

$$\phi_*([F(\phi^*O_Y(\mathbb{P}))]) = \tilde{c}_1(O_Y(\mathbb{P}))([F(O_Y(\mathbb{P}))]) = [(uF)(O_Y(\mathbb{P})]$$

for any power series $F(u) \in A_*(k)[[u]]$. In particular, one has the equality in $A_*(Y)$:

$$[\chi_A(O_Y(\mathbb{P}))] = \phi_*\big([g(\phi^*O_Y(\mathbb{P}))]\big).$$

Since $X \cap \mathbb{P} = \emptyset$, we have $\phi^*O_Y(X) \cong O_{\mathbb{P}}$. Thus, from (2.5) we see that

$$\phi^*O_Y(\mathbb{P}) \cong \phi^*O_Y(-X_Z) \cong O_{\mathbb{P}}(-E).$$

Thus $\phi^*O_Y(\mathbb{P}) \cong O_{\mathbb{P}}(-1)$, giving us the identity in in $A_*(Y)$:

$$[\chi_A(O_Y(\mathbb{P}))] = \phi_*\big([g(O_{\mathbb{P}}(-1))]\big).$$

Substituting this identity in the formula of proposition 2.5.1, and pushing forward to $A_*(X)$ by the projective morphism $Y \to X \times \mathbb{P}^1 \to X$ yields the desired formula. □

Remark 2.5.3. It is useful to have a formula using $O_{\mathbb{P}}(1)$ instead of $O_{\mathbb{P}}(-1)$, as the former is sometimes very ample, and one can then use the relation (Sect) to give an explicit formula in terms of subvarieties of $\mathbb{P}$. To rewrite the formula (2.8) in this way, we note that $\chi(\chi(u)) = u$ implies that

$$g(\chi(u)) = \frac{1}{g(u)}$$

so letting $h(u) = 1/g(u)$, we have

$$[X_Z \to X] = [\mathrm{Id}_X] + i_* \circ q_*\big([h(O_{\mathbb{P}}(1))]\big) \qquad (2.9)$$

2.5.3 Projective spaces and Milnor's hypersurfaces

Let $n > 0$ and $m > 0$ be integers. Recall that γ_n denotes the line bundle on $\mathbb{P}^n$ whose sheaf of sections is $\mathcal{O}(1)$. Write $\gamma_{n,m}$ for the line bundle $p_1^*(\gamma_n)\otimes p_2^*(\gamma_m)$ on $\mathbb{P}^n \times \mathbb{P}^m$.

We let $i : H_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m$ denote the smooth closed subscheme defined by the vanishing of a section of $\gamma_{n,m}$ transverse to the zero-section.

Remark 2.5.4. The smooth projective k-schemes $H_{n,m}$ are known as the Milnor hypersurfaces [3]. Taking $m \le n$, it is easy to see that, choosing suitable homogeneous coordinates $X_0, \dots, X_n$ for $\mathbb{P}^n$ and $Y_0, \dots, Y_m$ for $\mathbb{P}^m$, $H_{n,m}$ is defined by the vanishing of $\sum_{i=0}^m X_iY_i$; this also shows that the isomorphism class of $H_{n,m}$ as a k-scheme is independent of the choice of section. This also shows that, for $m = 1 < n$, $H_{n,m} \subset \mathbb{P}^n \times \mathbb{P}^1$ is the standard embedding of the blow-up of a linear $\mathbb{P}^{n-2}$ in $\mathbb{P}^n$. In general, the projection $H_{n,m} \to \mathbb{P}^m$ makes $H_{n,m}$ a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^m$. In the special case $n = m = 1$, we see that $H_{1,1}$ is (up to a change of coordinates) the diagonal $\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 2.5.5. *Write the formal group law of A_* as $F_A(u,v) = \sum_{i,j} a_{ij}\cdot u^i \cdot v^j$. Then we have the equation*

$$[H_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m] = \sum_{i\ge 0}^{n}\sum_{j\ge 0}^{m} a_{ij}[\mathbb{P}^{n-i} \times \mathbb{P}^{m-j} \to \mathbb{P}^n \times \mathbb{P}^m] \tag{2.10}$$

in $A_*(\mathbb{P}^n \times \mathbb{P}^m)$, where $\mathbb{P}^{n-i} \times \mathbb{P}^{m-j} \to \mathbb{P}^n \times \mathbb{P}^m$ is the product of linear embeddings $\mathbb{P}^{n-i} \to \mathbb{P}^n$, $\mathbb{P}^{m-j} \to \mathbb{P}^m$. We also have the equation

$$[H_{n,m}] = [\mathbb{P}^n]\cdot[\mathbb{P}^{m-1}] + [\mathbb{P}^{n-1}]\cdot[\mathbb{P}^m] + \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}[\mathbb{P}^{n-i}]\cdot[\mathbb{P}^{m-j}] \tag{2.11}$$

in $A_*(k)$.

Proof. The formula (2.11) follows from (2.10) by pushing forward from $\mathbb{P}^n \times \mathbb{P}^m$ to $\operatorname{Spec} k$, after noting that $[\mathbb{P}^{n-i}]\cdot[\mathbb{P}^{m-j}] = [\mathbb{P}^{n-i} \times \mathbb{P}^{m-j}]$, and that $a_{10} = a_{01} = 1$, $a_{n0} = a_{0m} = 0$ for $n > 1$ or $m > 1$.

To prove (2.10), one has the following computation in $A_*(\mathbb{P}^n \times \mathbb{P}^m)$:

$$\begin{aligned}
[H_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m] &= \tilde{c}_1(\gamma_{n,m})(1_{\mathbb{P}^n\times\mathbb{P}^m}) \\
&= \tilde{c}_1(p_1^*(\gamma_n)\otimes p_2^*(\gamma_m)))(1_{\mathbb{P}^n\times\mathbb{P}^m}) \\
&= F_A(\tilde{c}_1(p_1^*(\gamma_n)), \tilde{c}_1(p_2^*(\gamma_m))))(1_{\mathbb{P}^n\times\mathbb{P}^m}) \\
&= \tilde{c}_1(p_1^*(\gamma_n))(1_{\mathbb{P}^n\times\mathbb{P}^m}) + \tilde{c}_1(p_2^*(\gamma_m))(1_{\mathbb{P}^n\times\mathbb{P}^m}) \\
&\quad + \Big(\sum_{i\ge 1, j\ge 1} a_{ij}\cdot \tilde{c}_1(p_1^*(\gamma_n))^i \circ \tilde{c}_1(p_2^*(\gamma_m))^j\Big)(1_{\mathbb{P}^n\times\mathbb{P}^m}).
\end{aligned}$$

The last expression can be computed easily: since the sections of γ_n define hyperplanes in $\mathbb{P}^n$, applying the axioms (Sect) and definition 2.1.2(A3) repeatedly yields

$$\tilde{c}_1(p_1^*(\gamma_n))^i \circ \tilde{c}_1(p_2^*(\gamma_m))^j(1_{\mathbb{P}^n\times\mathbb{P}^m}) = [\mathbb{P}^{n-i} \times \mathbb{P}^{m-j} \to \mathbb{P}^n \times \mathbb{P}^m].$$

One thus gets the right hand side of formula (2.10), proving the lemma. □

Remark 2.5.6. We observe that equation (2.11) for $n = m = 1$ gives:

$$[H_{1,1}] = [\mathbb{P}^1] + [\mathbb{P}^1] + [a_{1,1}] \cdot 1.$$

By remark 2.5.4, $H_{1,1}$ is isomorphic to the diagonal $\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$, and the formula (2.11) yields

$$[a_{1,1}] = -[\mathbb{P}^1] \in A_1(k).$$

Remark 2.5.7. Using remark 2.5.4 again, we find that $H_{2,1} \subset \mathbb{P}^2\times\mathbb{P}^1$ is isomorphic to the blow-up of a k-rational point $p = \operatorname{Spec} k$ in $\mathbb{P}^2$. Using the notation of proposition 2.5.2, we have $\eta_i \cong \mathcal{O}_p^2$, and hence $\mathbb{P} = \mathbb{P}(k \oplus k \oplus k) = \mathbb{P}^2$ and $O_{\mathbb{P}}(-1) = \gamma_2^\vee$. Using this with our formula for g and proposition 2.5.2, we find

$$[H_{2,1} \to \mathbb{P}^2] = [\mathrm{Id}_{\mathbb{P}^2}] + i_*q_*\big(- [\mathrm{Id}_{\mathbb{P}^2}] - [\mathbb{P}^1] \cdot c_1(\gamma_2^\vee) - [\mathbb{P}^1]^2 \cdot c_1(\gamma_2^\vee)^2\big).$$

Let $\mathbb{P}^1 \to \mathbb{P}^2$ be the linear embedding, $pt \to \mathbb{P}^2$ the inclusion of a k-point. By (Sect), $c_1(\gamma_2) = [\mathbb{P}^1 \to \mathbb{P}^2]$, $c_1(\gamma_2)^2 = [pt \to \mathbb{P}^2]$ and $c_1(\gamma_2)^3 = 0$. Since $c_1(\gamma_2^\vee) = \chi(c_1(\gamma_2))$, we have from (2.3)

$$\begin{aligned} c_1(\gamma_2^\vee) &= -c_1(\gamma_2) + a_{1,1}c_1(\gamma_2)^2 \\ &= -[\mathbb{P}^1 \to \mathbb{P}^2] - [\mathbb{P}^1] \cdot [pt \to \mathbb{P}^2] \end{aligned}$$

$$\begin{aligned} c_1(\gamma_2^\vee)^2 &= (-c_1(\gamma_2) + a_{1,1}c_1(\gamma_2)^2)^2 \\ &= [pt \to \mathbb{P}^2] \end{aligned}$$

and thus

$$\begin{aligned} [H_{2,1} \to \mathbb{P}^2] &= [\mathrm{Id}_{\mathbb{P}^2}] - i_*([\mathbb{P}^2 \to p]) - [\mathbb{P}^1] \cdot i_*(-[\mathbb{P}^1 \to p] - [\mathbb{P}^1] \cdot [\mathrm{Id}_p]) \\ &\quad - [\mathbb{P}^1]^2 \cdot i_*([\mathrm{Id}_p]) \\ &= [\mathrm{Id}_{\mathbb{P}^2}] + (-[\mathbb{P}^2] + [\mathbb{P}^1]^2) \cdot [p \to \mathbb{P}^2] \end{aligned}$$

Pushing forward to $\operatorname{Spec} k$ yields $[H_{2,1}] = [\mathbb{P}^1]^2$.

The formula of lemma 2.5.5 gives $[H_{2,1}] = [\mathbb{P}^2] + [\mathbb{P}^1] \cdot [\mathbb{P}^1] + a_{1,1} \cdot [\mathbb{P}^1] + a_{2,1}$ which finally yields the formula in $A_*(k)$:

$$a_{2,1} = [\mathbb{P}^1]^2 - [\mathbb{P}^2].$$

Remark 2.5.8. We may solve for $a_{m,n}$ in the formula (2.11), giving

$$\begin{aligned} a_{n,m} = [H_{n,m}] - [\mathbb{P}^n] \cdot [\mathbb{P}^{m-1}] - [\mathbb{P}^{n-1}] \cdot [\mathbb{P}^m] \\ - \sum_{\substack{1\le i\le n,\ 1\le j\le m \\ (i,j)\neq(n,m)}} a_{ij}[\mathbb{P}^{n-i}] \cdot [\mathbb{P}^{m-j}]. \end{aligned}$$

This gives an inductive formula for $a_{n,m}$ in terms of the classes $[\mathbb{P}^i]$ and $[H_{i,j}]$. Thus, the image of $\mathbb{L}_*$ in $A_*(k)$ is contained in the subring generated by these classes. In addition, for $n > 1$ and $m > 1$, this formula shows that in $A_*(k)$, $[H_{n,m}]$ equals $a_{n,m}$ modulo decomposable elements (sums of products of element of degree > 0).

2.5.4 Generators for algebraic cobordism

We have the graded subgroup $\mathcal{M}_*^+(k)$ of $\mathcal{Z}_*(k)$ generated by the cobordism cycles $[Y \to \operatorname{Spec} k] = [Y]$; the product on $\mathcal{Z}_*(k)$ makes $\mathcal{M}_*^+(k)$ a graded subring. We have the canonical ring homomorphism $\mathbb{L}_* \to \Omega_*(k)$ induced by the quotient map $\mathbb{L}_* \otimes \underline{\Omega}_*(k) \to \Omega_*(k)$.

Lemma 2.5.9. *Let X be a finite type k-scheme. Then $\Omega_*(X)$ is generated as a group by standard cobordism cycles*

$$[Y \to X, L_1, \ldots, L_r]$$

In other words, the evident homomorphism $\underline{\Omega}_*(X) \to \Omega_*(X)$ is surjective.

Proof. The $\mathbb{L}_*$-module $\Omega_*(X)$ is clearly generated by the standard cobordism cycles. Since the $\mathbb{L}_*$ action on Ω_* factors through the canonical homomorphism $\mathbb{L}_* \to \Omega_*(k)$, via the external product action $\Omega_*(k) \otimes \Omega_*(X) \to \Omega_*(X)$, it suffices to show that the ring homomorphism

$$\mathcal{Z}_*(k) \to \Omega_*(k)$$

is surjective. As the ring homomorphism $\mathbb{L}_* \otimes \mathcal{Z}_*(k) \to \Omega_*(k)$ is surjective by definition, it suffices to prove that the image of $\mathbb{L}_* \to \Omega_*(k)$ is in the image of $\mathcal{Z}_*(k) \to \Omega_*(k)$. In fact, the image of $\mathbb{L}_* \to \Omega_*(k)$ is generated by the image of the subring $\mathcal{M}_*^+(k)$ of $\mathcal{Z}_*(k)$, as one sees by applying remark 2.5.8 to the case $A_* = \Omega_*$. $\square$

Lemma 2.5.10. *For $X \in \mathbf{Sch}_k$ $\Omega_*(X)$ is generated as an $\mathcal{M}_*^+(k)$-module by classes of the form $[Y \to X, L_1, \ldots, L_r]$, where each of the line bundles L_i on Y is very ample.*

Proof. Given any cobordism cycle $(f : Y \to X, L_1, \ldots, L_r)$ on $X \in \mathbf{Sch}_k$, we have the formula from remark 2.1.8

$$[f : Y \to X, L_1, \ldots, L_r] = f_* \circ \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r)(1_Y).$$

This identity and remark 2.3.12 imply that $\Omega_*(X)$ is generated as an $\mathbb{L}_*$-module by classes of the desired form; remark 2.5.8 implies that $\Omega_*(X)$ is generated as an $\mathcal{M}_*^+(k)$-module by classes of the desired form. $\square$

Lemma 2.5.11. *Let X be a finite type k-scheme. Then the canonical homomorphism*

$$\mathcal{M}_*^+(X) \to \Omega_*(X)$$

is an epimorphism. In other words, the graded abelian group $\Omega_*(X)$ is generated as a group by the classes $[Y \to X]$ of projective morphisms $Y \to X$ with Y smooth quasi-projective and irreducible.

Proof. From lemma 2.5.10 we know that $\Omega_*(X)$ is generated as a group by classes of the form

$$[f \circ p_1 : Y \times Z \to X, p_1^* L_1, \ldots, p_1^* L_r],$$

where Z is smooth and projective over k, $f : Y \to X$ is in $\mathcal{M}(X)$ and each L_i is a very ample line bundle on Y. If k is infinite, using remark 2.3.8 we may find sections s_i of L_i such that the subschemes $s_i = 0$ are smooth and intersect transversely on Y, for $i = 1, \ldots, r$. Using lemma 2.4.7, we see that

$$\begin{aligned}
&[Y \times Z \to X, p_1^* L_1, \ldots, p_1^* L_r] \\
&\qquad\qquad = [Y_1 \times Z \to X, p_1^* L_2, \ldots, p_1^* L_r] \\
&\qquad\qquad\qquad\qquad = \cdots = [Y_r \times Z \to X]
\end{aligned}$$

where $Y_i = \bigcap_{j=1}^{i} \{s_j = 0\}$, proving the statement.

If k is finite, we apply the same reasoning, using remark 2.3.8(2) and lemma 2.3.5. □

Theorem 2.5.12. *Let k be a field. Then the homomorphism $\Phi_0(k) : \mathbb{L}_0 \to \Omega_0(k)$ is an isomorphism and $\Omega_0(k)$ is the free abelian group on $1 = [\operatorname{Spec} k]$. Moreover, given any smooth variety $X = \operatorname{Spec} A$ of dimension zero over k, then $[X] = \dim_k(A) \cdot 1$ in $\Omega_0(k)$.*

Proof. The last formula has been established in lemma 2.3.4. The surjectivity of $\Phi_0(k)$ follows from that formula and from lemma 2.5.11 which implies that $\Omega_0(k)$ is generated by classes $[\operatorname{Spec} L]$ where L is a (separable) finite field extension of k. Thus $\mathbb{Z} = \mathbb{L}_0 \to \Omega_0(k)$ is surjective.

For injectivity, we have the natural transformation $\Omega_* \to \mathrm{CH}_*$ given by the universality of Ω_* (theorem 2.4.13); the map $\Omega_0(k) \to \mathrm{CH}_0(k) = \mathbb{Z}$ is a left inverse to $\mathbb{L}_0 \to \Omega_0(k)$, which is thus injective. □

2.5.5 Relations defining algebraic cobordism

It will be useful to give explicit generators for the kernel of the natural surjection (see lemma 2.5.11):

$$\underline{\Omega}_* \to \Omega_*.$$

For this, first use remark 2.5.8 to choose for each (i, j) with $i \leq j$ an element $a'_{ij} \in \mathcal{M}_{i+j-1}^+(k)$ lifting $[a_{ij}] \in \Omega_{i+j-1}(k)$; for $j < i$ we set $a'_{ji} = a'_{ij}$. Let $F(u, v) \in \underline{\Omega}_*(k)[[u, v]]$ be the power series

$$F(u,v) = u + v + \sum_{i,j\geq 1} a'_{ij}u^i v^j.$$

Definition 2.5.13. *Let $X \in \mathbf{Sch}_k$. Let $\tilde{\mathcal{R}}_*(X)$ denote the subgroup of $\underline{\Omega}_*(X)$ generated by elements of the form*

$$f_* \circ \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r)\big([F(L,M)] - [L \otimes M]\big),$$

where $f : Y \to X$ is in $\mathcal{M}(X)$, and $(L_1, \ldots, L_r, L, M)$ are line bundles on Y. We denote by $\tilde{\Omega}_*(X)$ the quotient group $\underline{\Omega}_*(X)/\tilde{\mathcal{R}}_*(X)$.

Lemma 2.5.14. *Let $X \in \mathbf{Sch}_k$. Then $\tilde{\mathcal{R}}_*(X)$ is also the subgroup of $\underline{\Omega}_*(X)$ generated by elements of the form*

$$f_* \circ \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r)\big(F(\tilde{c}_1(L), \tilde{c}_1(M))(\eta) - \tilde{c}_1(L \otimes M)(\eta)\big),$$

where $(f : Y \to X, L_1, \ldots, L_r,)$ is a standard cobordism cycle on X, L, M are line bundles on Y, and η is in $\underline{\Omega}_*(Y)$.

Proof. Indeed, $\underline{\Omega}_*(Y)$ is generated by the standard cobordism cycles $(g : Z \to Y, M_1, \ldots, M_s)$ on Y. But then using the notations of the lemma with $\eta = [g : Z \to Y, M_1, \ldots, M_s]$ we have

$$\begin{aligned} &f_* \circ \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r) \circ F(\tilde{c}_1(L), \tilde{c}_1(M))(\eta) \\ &= (f \circ g)_* \circ \tilde{c}_1(g^*L_1) \circ \cdots \circ \tilde{c}_1(g^*L_r) \circ \tilde{c}_1(M_1) \circ \cdots \circ \tilde{c}_1(M_s)([F(g^*L, g^*M)]) \end{aligned}$$

and

$$\begin{aligned} &f_* \circ \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r) \circ \tilde{c}_1(L \otimes M)(\eta) \\ &= (f \circ g)_* \circ \tilde{c}_1(g^*L_1) \circ \cdots \circ \tilde{c}_1(g^*L_r) \circ \tilde{c}_1(M_1) \circ \cdots \circ \tilde{c}_1(M_s)([g^*L \otimes g^*M]) \end{aligned}$$

which verifies our assertion.

□

As a consequence of the lemma, together with remark 2.1.14, one sees that $X \mapsto \tilde{\Omega}_*(X)$ is an oriented Borel-Moore functor with product on $\mathbf{Sch}_k$.

It is easy to see that the elements in $\tilde{\mathcal{R}}_*(X)$ become zero in $\Omega_*(X)$ through the projection $\underline{\Omega}_*(X) \to \Omega_*(X)$, giving a natural epimorphism $\tau_X : \tilde{\Omega}_*(X) \to \Omega_*(X)$.

Proposition 2.5.15. *Let X be a finite type k-scheme. Then the homomorphism*

$$\tau_X : \tilde{\Omega}_*(X) \to \Omega_*(X)$$

is an isomorphism.

Proof. It suffices to show that $(F, \tilde{\Omega}_*(k))$ is a commutative formal group. Indeed, if this is so, we have the canonical homomorphism

$$\phi : \mathbb{L}_* \to \tilde{\Omega}_*(k),$$

with $\phi(a_{ij}) = a'_{ij}$. Using the $\tilde{\Omega}_*(k)$-module structure on $\tilde{\Omega}_*$, we get the surjective morphism of oriented Borel-Moore functors with product

$$\vartheta : \mathbb{L}_* \otimes \underline{\Omega}_* \to \tilde{\Omega}_*,$$
$$\vartheta(a \otimes b) = \phi(a)b,$$

extending the natural transformation $\underline{\Omega}_* \to \tilde{\Omega}_*$. If L and M are line bundles on X, we have

$$\vartheta(F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))) = F(\tilde{c}_1(L), \tilde{c}_1(M))$$

as endomorphisms of $\tilde{\Omega}_*(X)$, hence ϑ descends to a surjective natural transformation $\Omega_* \to \tilde{\Omega}_*$, which is easily seen to be inverse to τ.

Now, to show that $(F, \tilde{\Omega}_*(k))$ is a commutative formal group, we need only verify the associativity $F(F(u,v),w) = F(u, F(v,w))$ in $\tilde{\Omega}_*(k)[[u,v,w]]$.

Suppose the associativity relation $F(F(u,v),w) = F(u, F(v,w))$ is satisfied modulo (u^n, v^{m+1}, w^{p+1}). Write

$$F(F(u,v),w) = \sum_{ijl} a_{ijl} u^i v^j w^l; \quad F(u, F(v,w)) = \sum_{ijl} a'_{ijl} u^i v^j w^l.$$

Let (a, b, c) be integers, and let $O_X(a,b,c)$ denote the line bundle whose sheaf of section is $\mathcal{O}(a,b,c) := p_1^*\mathcal{O}(a) \otimes p_2^*\mathcal{O}(b) \otimes p_3^*\mathcal{O}(c)$ on $X := \mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^p$. Then, as endomorphisms of $\tilde{\Omega}_*(X)$, we have

$$\begin{aligned}
F(F(\tilde{c}_1(O_X(1,0,0)), &\tilde{c}_1(O_X(0,1,0))), \tilde{c}_1(O_X(0,0,1))) \\
&= F(\tilde{c}_1(O_X(1,1,0)), \tilde{c}_1(O_X(0,0,1))) \\
&= \tilde{c}_1(O_X(1,1,1)) \\
&= F(\tilde{c}_1(O_X(1,0,0)), \tilde{c}_1(O_X(0,1,1)) \\
&= F(\tilde{c}_1(O_X(1,0,0)), F(\tilde{c}_1(O_X(0,1,0), \tilde{c}_1(O_X(0,0,1))).
\end{aligned}$$

Evaluating both sides on $\mathrm{Id}_{\mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^p}$ gives

$$\sum_{ijl} a_{ijl}[H^{(i)} \times H^{(j)} \times H^{(l)} \to X] = \sum_{ijl} a'_{ijl}[H^{(i)} \times H^{(j)} \times H^{(l)} \to X]$$

in $\tilde{\Omega}_*(X)$, where $H^{(i)}$ stands for the intersection of i independent hyperplanes. Pushing forward to $\operatorname{Spec} k$, using our induction hypothesis, and the fact that $H^{(n)} \times H^{(m)} \times H^{(p)}$ pushes forward to the identity in $\tilde{\Omega}_*(k)$, we find that the associativity relation $F(F(u,v),w) = F(u, F(v,w))$ is satisfied modulo $(u^{n+1}, v^{m+1}, w^{p+1})$. The same argument allows us to increase the degree in v and in w, which completes the proof. □

3

Fundamental properties of algebraic cobordism

In this chapter we show, assuming the base-field admits resolution of singularities, that algebraic cobordism has the properties expected of a reasonable Borel-Moore homology theory, namely, the projective bundle formula (theorem 3.5.4) and the extended homotopy property (theorem 3.6.3). Crucial to the proofs of these results is the fundamental right-exact *localization sequence* (see theorem 3.2.7), which is the main technical result of this chapter. As preparation, we construct in § 3.1 the class of a strict normal crossing divisor E on some smooth W as an element of the algebraic cobordism of the support of E.

In addition, we prove a moving lemma (proposition 3.3.1) which shows that, for $W \in \mathbf{Sm}_k$ with smooth closed subscheme Z, $\Omega_*(W)$ is generated by $f : Y \to W$ in $\mathcal{M}(W)$ which are transverse to the inclusion $Z \to W$. This will be important in showing that Ω_* is the universal Borel-Moore homology theory on $\mathbf{Sch}_k$ in chapter 7.

3.1 Divisor classes

For a Cartier divisor E on some $W \in \mathbf{Sm}_k$, we have the class $\tilde{c}_1(O_W(E))(1_W)$ in $\Omega_*(W)$, denoted $[O_W(E)]$. Letting $i : |E| \to W$ be the support of E, the main object of this section is to define a class $[E \to |E|] \in \Omega_*(|E|)$, in case E is a strict normal crossing divisor, such that $i_*([E \to |E|]) = [O_W(E)]$.

3.1.1 Some power series

For a formal group law (F, R), recall that we simply write $u +_F v$ for $F(u, v)$, and extend this notation in the evident way for the other formal group operations such as *formal opposite*, denoted $[-1] \cdot_F v$ (which satisfies $u +_F [-1] \cdot_F u = 0$), *formal difference*, denoted $u -_F v$ (equal to $u +_F [-1] \cdot_F (v)$), *n-fold formal sum* $u_1 +_F + \cdots +_F u_n$, and *formal multiplication by* $n \in \mathbb{Z}$, denoted $[n] \cdot_F u$.

In this section we will be using the formal group law $(F, R) = (F_\Omega, \Omega_*(k))$ unless explicit mention to the contrary is made.

Given integers $n_1, \ldots, n_m$, we will use the notation

$$F^{n_1,\ldots,n_m}(u_1, \ldots, u_m) := [n_1] \cdot_F u_1 +_F \ldots. +_F [n_m] \cdot_F u_m$$

Example 3.1.1. By the relations encoded in the definition of Ω_*, we have for line bundles $L_1, \ldots, L_r$ on a finite type k-scheme X:

$$F^{n_1,\ldots,n_m}(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_r)) = \tilde{c}_1(L_1^{\otimes n_1} \otimes \ldots \otimes L_r^{\otimes n_r}).$$

For a sequence $J = (j_1, \ldots, j_m) \in \mathbb{N}^m$ of non-negative integers j_i, let $u^J = u_1^{j_1} \cdot \ldots \cdot u_m^{j_m}$. We set $||J|| := Sup_i(j_i)$.

Lemma 3.1.2. *Any power series $H \in R[[u_1, \ldots, u_m]]$ can be uniquely written as*

$$H(u_1, \ldots, u_m) = \sum_{J,\ ||J|| \leq 1} u^J \cdot H_J(u_1, \ldots, u_m),$$

where each monomial $h_{J,J'}u^{J'}$, $J' = (j'_1, \ldots, j'_m)$, occurring in H_J has $j'_s = 0$ if $j_s = 0$.

Proof. Write $H(u_1, \ldots, u_m) = \sum_K h_K \cdot u^K$, and let $J = (j_1, \ldots, j_m)$ with $||J|| \leq 1$. Let $H_J = \sum'_K h_K u^{K-J}$, where the sum is over all $K = (k_1, \ldots, k_m)$ such that $k_i \geq j_i$ for all i, and $k_i = 0$ if $j_i = 0$. Uniqueness is easy and left to the reader. □

Applying the above lemma to $F^{n_1,\ldots,n_m}$, we have power series

$$F_J^{n_1,\ldots,n_m}(u_1, \ldots, u_m) \in \Omega_*(k)[[u_1, \ldots, u_m]];\ ||J|| \leq 1,$$

with

$$F^{n_1,\ldots,n_m}(u_1, \ldots, u_m) = \sum_{J,\ ||J|| \leq 1} u^J \cdot F_J^{n_1,\ldots,n_m}(u_1, \ldots, u_m).$$

Example 3.1.3. Assume $m = 2$ and $n_1 = n_2 = 1$. Then

$$\begin{aligned} F^{1,1}(u, v) = F(u, v) &= \sum_{i,j} a_{i,j} u^i v^j \\ &= u + v + \sum_{i \geq 1, j \geq 1} a_{i,j} u^i v^j \\ &= u + v + uv\Big(\sum_{i \geq 1, j \geq 1} a_{i,j} u^{i-1} v^{j-1}\Big), \end{aligned}$$

so $F^{1,1}_{(0,0)} = 0$, $F^{1,1}_{(1,0)}(u, v) = 1$, $F^{1,1}_{(0,1)}(u, v) = 1$ and

$$F^{1,1}_{(1,1)}(u, v) = \sum_{i \geq 1, j \geq 1} a_{i,j} u^{i-1} v^{j-1}.$$

Similarly, $F^{n_1,\ldots,n_r}_{(0,\ldots,0)} = 0$ for all $(n_1, \ldots, n_r)$.

3.1.2 Strict normal crossing divisors

Definition 3.1.4. *Let W be in $\mathbf{Sm}_k$. Recall that a* strict normal crossing divisor *E on W is a Weil divisor $E = \sum_{i=1}^m n_i \cdot E_i$ where each n_i is ≥ 1, each E_i is an integral closed subscheme of W and for each $I \subset \{1, \ldots, m\}$ the subscheme*

$$E_I := \bigcap_{i \in I} E_i$$

is smooth over k of codimension $|I|$ on W. E_I is called a face *of E.*

We denote by $i : |E| \to W$ the support of E, i.e., the reduced closed subscheme whose underlying space is the union of the E_i. We remind the reader that $O_W(E)$ denotes the line bundle on W corresponding to E, which means that its $\mathcal{O}_W$-module of sections is $\mathcal{O}_W(E)$.

Let W be in $\mathbf{Sm}_k$ and let E be a strict normal crossing divisor on W. Write $E = \sum_{j=1}^m n_j E_j$, with the E_j integral. For each index $J = (j_1, \ldots, j_m)$ with $||J|| \leq 1$, we have the face

$$E^J := \cap_{i,\ j_i=1} E_i$$

of E. Of course, $E_I = E^{J(I)}$ where $J(I) = (j_1, \ldots, j_m)$ with $j_i = 0$ if $i \notin I$ and $j_i = 1$ if $i \in I$. Let $i_J : E^J \to W$ be the inclusion.

Let $\iota^J : E^J \to |E|$ be the inclusion. Let $L_i := O_W(E_i)$, and let $L_i^J = (i^J)^* L_i$.

Definition 3.1.5. *Define the class $[E \to |E|] \in \Omega_*(|E|)$ by the formula*

$$[E \to |E|] := \sum_{J,\ ||J|| \leq 1} \iota_*^J\big([F_J^{n_1,\ldots,n_m}(L_1^J, \ldots, L_m^J)]\big). \tag{3.1}$$

If $f : |E| \to X$ is a projective morphism, we write $[E \to X] \in \Omega_*(X)$ for $f_*([E \to |E|])$.

Example 3.1.6. Assume $m = 2$ and $n_1 = n_2 = 1$, so that $E = E_1 + E_2$ and $|E| = E_1 \bigcup E_2$. From example 3.1.3, we see that

$$\begin{aligned} [E \to |E|] &= \iota_*^{(1,0)}(1_{E_1}) + \iota_*^{(0,1)}(1_{E_2}) + \iota_*^{(1,1)}[F_{(1,1)}^{1,1}(L_1^{(1,1)}, L_2^{(1,1)})] \\ &= [E_1 \to |E|] + [E_2 \to |E|] + \iota_*^{(1,1)}[F_{(1,1)}^{1,1}(L_1^{(1,1)}, L_2^{(1,1)})] \end{aligned}$$

In particular, assume that L_1 and L_2 are trivial. Then equation (3.1) becomes

$$[E \to |E|] = [E_1 \to |E|] + [E_2 \to |E|] + [a_{1,1}] \cdot [E_{\{1,2\}} \to |E|]$$

which equals (using remark 2.5.6):

$$[E \to |E|] = [E_1 \to |E|] + [E_2 \to |E|] - [\mathbb{P}^1] \cdot [E_{\{1,2\}} \to |E|]$$

Remark 3.1.7. Let E be a strict normal crossing divisor on some W in $\mathbf{Sm}_k$. Write $E = \sum_{i=1}^r n_i E_i$, with the E_i the irreducible components of $|E|$. Let $\mathbb{L}_{\geq 1} \subset \mathbb{L}_*$ be the ideal of elements of positive degree, and let $\mathbb{L}_* \to \mathbb{Z}$ be the ring homomorphism with kernel $\mathbb{L}_{\geq 1}$. Then, in $\Omega_*(|E|) \otimes_{\mathbb{L}_*} \mathbb{Z}$, we have

$$[E \to |E|] = \sum_{i=1}^r n_i [E_i \to |E|].$$

Indeed, it follows directly from the definition of $F^{n_1,\ldots,n_r}$ that

$$F_J^{n_1,\ldots,n_r}(u_1,\ldots,u_r) \equiv 0 \bmod \mathbb{L}_{\geq 1}$$

if $|J| > 1$. If $J = (j_1,\ldots,j_r)$ with $j_i = 1$ and $j_l = 0$ for $l \neq i$, then

$$F_J^{n_1,\ldots,n_r}(u_1,\ldots,u_r) \equiv n_i \bmod \mathbb{L}_{\geq 1}.$$

This yields the desired formula.

Lemma 3.1.8. *Let W be in $\mathbf{Sm}_k$ and let $E = \sum_{i=1}^m E_i$ be a strict normal crossing divisor on W. Let $i : E_{\{1,\ldots,m\}} \to W$ be the inclusion. Let $H(u_1,\ldots,u_n)$ be a power series with $\Omega_*(k)$ coefficients, and let $L_1,\ldots,L_n$ be line bundles on W. Then*

$$i_*[H(i^*L_1,\ldots,i^*L_n)] = \tilde{c}_1(O_W(E_1)) \circ \ldots \circ \tilde{c}_1(O_W(E_m))[H(L_1,\ldots,L_n)].$$

in $\Omega_*(W)$.

Proof. By induction on m it suffices to prove the case $m = 1$; write E for E_1. We have

$$\begin{aligned}
i_*[H(i^*L_1,\ldots,i^*L_n)] &= i_*\big(H(\tilde{c}_1(i^*L_1),\ldots,\tilde{c}_1(i^*L_n))(\mathrm{Id}_E)\big) \\
&= H(\tilde{c}_1(L_1),\ldots,\tilde{c}_1(L_n))([E \to W]) \\
&= H(\tilde{c}_1(L_1),\ldots,\tilde{c}_1(L_n)) \circ \tilde{c}_1(O_W(E))(\mathrm{Id}_W) \\
&= \tilde{c}_1(O_W(E))\big(H(\tilde{c}_1(L_1),\ldots,\tilde{c}_1(L_n))(\mathrm{Id}_W)\big) \\
&= \tilde{c}_1(O_W(E))[H(L_1,\ldots,L_n)]
\end{aligned}$$

□

Proposition 3.1.9. *Let W be in $\mathbf{Sm}_k$, let E be a strict normal crossing divisor on W. Then*

$$[E \to W] = [O_W(E)].$$

In particular, let X be a finite type k-scheme, let $f : W \to X$ be a projective morphism, and let E, E' be strict normal crossing divisors on W with $O_W(E) \cong O_W(E')$. Then

$$[E \to X] = [E' \to X]$$

in $\Omega_*(X)$.

Proof. Write $E = \sum_{i=1}^m n_i E_i$ with the E_i smooth and irreducible and let $L_i = O_W(E_i)$. Write F for $F^{n_1,\ldots,n_m}$, F_J for $F_J^{n_1,\ldots,n_m}$. Let $F'_J = u^J F_J$, so that

$$F = \sum_J F'_J.$$

Let $i^J : E^J \to W$ be the inclusion. By lemma 3.1.8, we have

$$i^J_*([F_J(L_1^J, \ldots, L_m^J)]) = [F'_J(L_1, \ldots, L_m)].$$

Thus

$$\begin{aligned}
[E \to W] &= \sum_J i^J_*\big([F_J(\tilde{c}_1(L_1^J), \ldots, \tilde{c}_1(L_m^J)]\big) \\
&= \sum_J [F'_J(L_1, \ldots, L_m)] \\
&= [F(L_1, \ldots, L_m)] \\
&= F\big(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_m)\big)(\mathrm{Id}_W) \\
&= \tilde{c}_1\big(L_1^{\otimes n_1} \otimes \ldots \otimes L_m^{\otimes n_m}\big)(\mathrm{Id}_W) \quad \text{(cf. example 3.1.1)} \\
&= [O_W(E)].
\end{aligned}$$

□

3.2 Localization

Let X be a finite type k-scheme $i : Z \to X$ a closed subscheme, and $j : U \to X$ the open complement. It is obvious that the composite $\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U)$ is zero, so the sequence

$$\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U) \to 0 \tag{3.2}$$

is a complex. Our main task in this section will be to show that (3.2) is exact (theorem 3.2.7), at least under the assumption that the base field k admits resolution of singularities. The most difficult part of the argument concerns the exactness at $\Omega_*(X)$; one reduces fairly easily to two types of classes:

1. the difference of two cobordism cycles $[Y \to X] - [Y' \to X]$ with $Y \cong Y'$ over U.
2. the difference of two cobordism cycles $[f : Y \to X, L] - [f : Y \to X, L']$ with $L \cong L'$ on $f^{-1}(U)$.

Thus, we begin by analyzing the effect of a birational transformation on the class $[Y \to X]$ and the effect of twisting a line bundle L by $O_Y(D)$ on the class $[Y \to X, L]$.

3.2.1 Blow-ups

Let $f : W' \to W$ be a projective birational map, with W' and W smooth over k. In this section, we consider the problem of writing the class $[W' \to W]$ in $\Omega_*(W)$.

By [12], there is a closed subscheme T of W such that f is the blow-up of W along T. Since W is smooth, we may assume[1] that the support of T is the fundamental locus of f, i.e., $W \setminus |T|$ is exactly the set of $x \in W$ over which f is an isomorphism. Let $E \subset W'$ be the exceptional divisor $f^{-1}(T)$.

We form a version of the "deformation to the normal cone" as follows: let $\mu : Y \to W \times \mathbb{P}^1$ be the blow-up of $W \times \mathbb{P}^1$ along $T \times 0$. Let $\langle W \times 0 \rangle$ denote the proper transform of $W \times 0$, let $\langle T \times \mathbb{P}^1 \rangle$ denote the proper transform of $T \times \mathbb{P}^1$, let $\tilde{E}$ be the exceptional divisor of μ and set

$$
\begin{aligned}
Y_1 &:= \mu^{-1}(W \times 1) \\
Y_0 &:= \mu^{-1}(W \times 0) = \langle W \times 0 \rangle \cup \tilde{E}.
\end{aligned}
$$

This gives us the deformation diagram (compare with the deformation diagram (2.4))

$$
\begin{tikzcd}
T \times 1 \arrow[rr, hook, "i"] \arrow[d, hook] & & Y_1 \arrow[r, equal] \arrow[d, hook, "i_1"'] & W \times 1 \arrow[d, hook] & \\
\langle T \times \mathbb{P}^1 \rangle \arrow[rr, hook, "\tilde{i}"] & & Y \arrow[r, "\mu"'] & W \times \mathbb{P}^1 \arrow[r, "p_2"'] \arrow[dddd, bend left=60, "p_1"] & \mathbb{P}^1 \\
& \langle W \times 0 \rangle \arrow[rr, "\bar{\mu}"] \arrow[dr, hook] & & W \times 0 \arrow[u, hook] & \\
T \times 0 \arrow[uu, hook] \arrow[dr, hook, "s"'] & & Y_0 \arrow[uu] \arrow[ur, "\mu_0"'] & & \\
& \tilde{E} \arrow[ur, hook] \arrow[rr, "q"'] & & T \times 0 \arrow[uu, hook, "i"'] & \\
& & & W &
\end{tikzcd}
\tag{3.3}
$$

Lemma 3.2.1. *The restriction of $\mu : Y \to W \times \mathbb{P}^1$ to*

$$\bar{\mu} : \langle W \times 0 \rangle \to W,$$

is isomorphic over W to $f : W' \to W$. In addition, $Y \setminus \tilde{E} \cap \langle T \times \mathbb{P}^1 \rangle$ is smooth, and contains $\langle W \times 0 \rangle$. Finally, if E is a strict normal crossing divisor, then $\tilde{E} + \langle W \times 0 \rangle \setminus \langle T \times \mathbb{P}^1 \rangle$ is a strict normal crossing divisor on $Y \setminus \tilde{E} \cap \langle T \times \mathbb{P}^1 \rangle$

Proof. Via the projection $W \times \mathbb{P}^1 \to \mathbb{P}^1$, all schemes and morphisms involved are over $\mathbb{P}^1$. Since μ is an isomorphism over $\mathbb{P}^1 \setminus \{0\}$, we may restrict everything

[1] See [12, Exercise 7.11(c)].

over the open subscheme $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$ of $\mathbb{P}^1$. We indicate this by replacing $\mathbb{P}^1$ with $\mathbb{A}^1$ in the notation, but leaving all other notations the same.

The assertions being local over W, we may assume that $W = \operatorname{Spec} A$ for some smooth integral k-algebra A. Let $I \subset A$ be the ideal defining T, $J \subset A[t]$ the ideal defining $T \times 0$, so $J = (I, t)$.

Suppose $I = (f_0, \dots, f_m)$. Then $W' = \operatorname{Proj}_A(\oplus_n I^n)$ is the subscheme of $\mathbb{P}^m_A$ defined by the kernel $N_{W'}$ of the surjection of graded rings

$$\begin{aligned} A[X_0, \dots, X_m] &\to \oplus_n I^n \\ g(X_0, \dots, X_m) &\mapsto g(f_0, \dots, f_m). \end{aligned}$$

Similarly, Y is the subscheme of $\mathbb{P}^{m+1}_{A[t]}$ defined by the kernel N_Y of the surjection of graded rings

$$\begin{aligned} A[t][X_0, \dots, X_m, X_{m+1}] &\to \oplus_n J^n \\ g(X_0, \dots, X_m, X_{m+1}) &\mapsto g(f_0, \dots, f_m, t). \end{aligned}$$

We claim that N_Y is the ideal generated by $N_{W'}$ and the elements $f_j X_{m+1} - tX_j$, $j = 0, \dots, m$. To see this, take $g \in N_Y$ of degree d, and expand g as a sum of monomials $X^I X^j_{m+1}$, $(I = (i_0, \dots, i_m)$, $|I| = \sum_j i_j)$:

$$g = \sum_{|I|+i=d} g_I X^I X^i_{m+1},$$

with

$$g_I = \sum_{j=0}^{N_I} g^j_I t^j; \quad g^j_I \in A,\ g^{N_I}_I \neq 0.$$

Modulo the elements $f_j X_{m+1} - tX_j$, we may rewrite g as

$$g' := \sum_{i=0}^{d} g'_i X^i_{m+1} + (\sum_{j=1}^{M} a_j t^j) X^d_{m+1},$$

with $g'_i \in A[X_0, \dots, X_m]$ homogeneous of degree $d-i$, and $a_j \in A$. Evaluating g', we have

$$0 = g'(f_0, \dots, f_m, t) = \sum_{i=0}^{d} g'_i(f_0, \dots, f_m) t^i + \sum_{i=1}^{M} a_i t^{d+i},$$

hence each $a_i = 0$, and each g'_i is in $N_{W'}$, proving our claim.

Next, we note that multiplication by X_{m+1}, $\times X_{m+1} : \mathcal{O}_Y \to \mathcal{O}_Y(1)$, is injective. Indeed, multiplication by X_{m+1} on the homogeneous coordinate ring of Y is just multiplication by t on $\oplus_n J^n$, which is evidently injective. This implies the injectivity of $\times X_{m+1}$ on the sheaf level. Additionally, it is clear from our description of N_Y that, for $0 \le i \le n$,

$$A[X_0, \ldots, X_m, X_i^{-1}]/N_{W'} \cong A[t][X_0, \ldots, X_{m+1}, X_i^{-1}]/(N_Y, X_{m+1}),$$

so W' is isomorphic to the subscheme of Y defined by $X_{m+1} = 0$. Since W' is smooth, and X_{m+1} is a non-zero divisor on $\mathcal{O}_Y$, this implies that Y is smooth in a neighborhood of $X_{m+1} = 0$. Finally, since the proper transform $\langle W \times 0\rangle$ is irreducible and dense in $(X_{m+1} = 0)$, we see that the equation $X_{m+1} = 0$ defines the subscheme $\langle W \times 0\rangle$.

We now look at the proper transform $\langle T \times \mathbb{A}^1\rangle$. Let $\mathcal{N}$ denote the sheaf of ideals on Y defining $\langle T \times \mathbb{A}^1\rangle$, and let $\mathcal{I}$ be the sheaf of ideals defined by the homogeneous ideal $(X_0, \ldots, X_m)$. We claim that $\mathcal{N} = \mathcal{I}$. Indeed, on the subscheme of Y defined by $(X_0, \ldots, X_m)$, X_{m+1} is invertible, hence the relations $f_j X_{m+1} - tX_j$ in N_Y imply that $\mathcal{I} \supset (f_0, \ldots, f_m)\mathcal{O}_Y$, and we have equality of ideal sheaves after inverting t. Thus $\mathcal{N} = \mathcal{I}$ after inverting t, hence $\mathcal{N} \supset \mathcal{I}$. Since the subscheme of Y defined by $\mathcal{I}$ is evidently isomorphic to $T \times \mathbb{A}^1$ via the projection to $X \times \mathbb{A}^1$, $\mathcal{I}$ is thus maximal among ideal sheaves $\mathcal{J}$ with $\mathcal{J}[t^{-1}] = \mathcal{I}[t^{-1}]$. Thus $\mathcal{N} = \mathcal{I}$, as claimed.

On the other hand, consider the affine open subscheme U_i of Y defined by $X_i \neq 0$, and the similarly defined subscheme V_i of W'. Our description of N_Y in terms of $N_{W'}$ implies that $U_i \cong V_i \times \mathbb{A}^1$, using X_{m+1}/X_i as the map to $\mathbb{A}^1$. Thus U_i is smooth. Since $Y \to X \times \mathbb{A}^1$ is an isomorphism over $X \times \mathbb{A}^1 \setminus T \times 0$, this proves that $Y \setminus \tilde{E} \cap \langle T \times \mathbb{A}^1\rangle$ is smooth.

Finally, both $\tilde{E} \cap U_i \subset U_i$ and $E \cap V_i \subset V_i$ are the subschemes of U_i and V_i, respectively, defined by f_i. Again referring to the explicit equations defining Y, we see that $\tilde{E} \cap U_i \subset U_i$ is isomorphic to $E \times \mathbb{A}^1 \subset V_i \times \mathbb{A}^1$, using as above the coordinate X_{n+1}/X_i. Thus $\tilde{E} \setminus \langle T \times \mathbb{A}^1\rangle$ is a strict normal crossing divisor on $Y \setminus \tilde{E} \cap \langle T \times \mathbb{A}^1\rangle$. We note that $\langle W \times 0\rangle$ is smooth, and $\langle W \times 0\rangle \cap \tilde{E}$ is the strict normal crossing divisor E on W'. Write $E = \sum_i n_i D_i$. Since $\tilde{E} = E \times \mathbb{A}^1$ in a neighborhood of $\langle W \times 0\rangle$, this implies that $\tilde{E} = \sum_i n_i D_i \times \mathbb{A}^1$ in a neighborhood of $\langle W \times 0\rangle$. Thus $\langle W \times 0\rangle$ intersects each irreducible component of $\tilde{E}$ transversely, hence $\tilde{E} + \langle W \times 0\rangle \setminus \langle T \times \mathbb{A}^1\rangle$ is a strict normal crossing divisor on $Y \setminus \tilde{E} \cap \langle T \times \mathbb{A}^1\rangle$. □

3.2.2 Preliminaries on classes of divisors

Let E be a strict normal crossing divisor on some $W \in \mathbf{Sm}_k$. We have defined in §3.1 the class $[E \to |E|]$ in $\Omega_*(|E|)$; for later use we will need a somewhat more detailed description of this class. Write $E = \sum_{i=1}^m n_i D_i$ with the D_i distinct and integral, so $D := \sum_{i=1}^m D_i$ is a reduced strict normal crossing divisor. Write $E_{\text{mult}} := E - D$, and $E_{\text{red}} = \sum_{n_i=1} D_i$. Define

$$|E|_{\text{sing}} := \big(\cup_{i<j} |D_i| \cap |D_j|\big) \cup |E_{\text{mult}}|,$$

so $|E|_{\text{sing}}$ is the singular locus of $|D|$, together with the union of the $|D_j|$ with $n_j > 1$.

Lemma 3.2.2. *Let W be in $\mathbf{Sm}_k$, and let $E = \sum_{i=1}^m n_i D_i$ be a strict normal crossing divisor on W; suppose $n_i = 1$ for $i = 1, \ldots s$, and $n_i > 1$ for $i > s$. Let $\iota_{\mathrm{sing}} : |E|_{\mathrm{sing}} \to |E|$ be the inclusion. Then there is an element η of $\Omega_*(|E|_{\mathrm{sing}})$ such that*

$$[E \to |E|] = \sum_{i=1}^{s} [E_i \to |E|] + \iota_{\mathrm{sing}*}\eta.$$

Proof. Let $n = \sum_i n_i$, and let F_n denote the n-fold sum

$$F_n(u_1, \ldots, u_n) = u_1 +_F \ldots +_F u_n.$$

We have $F(u, v) = u + v + \sum_{i \geq 1,\ j \geq 1} a_{ij} u^i v^j$, from which it follows that

$$F_n(u_1, \ldots, u_n) = \sum_{i=1}^{n} u_i + \sum_{i_1 \geq 1, \ldots, i_n \geq 1} a_{i_1 \ldots i_n} u_1^{i_1} \cdots u_n^{i_n}.$$

Thus, if we expand

$$F^{1,\ldots,1,n_{s+1},\ldots,n_m} := F(u_1, \ldots, u_s, u_{s+1}, \ldots, u_{s+1}, \ldots, u_m, \ldots, u_m)$$

as required for the definition of $[E \to |E|]$,

$$F^{1,\ldots,1,n_{s+1},\ldots,n_m} = \sum_{J,\ ||J|| \leq 1} u^J F_J(u_1, \ldots, u_m),$$

we find

$$F^{1,\ldots,1,n_{s+1},\ldots,n_m} = \sum_{i=1}^{s} u_i + \sum_{J'} u^{J'} F_{J'}(u_1, \ldots, u_m),$$

where each J' has either two nonzero entries, or an entry $j'_l = 1$ with $l > s$. We thus have

$$\begin{aligned}
[E \to |E|] &= \sum_{J,\ ||J|| \leq 1} \iota_*^J [F_J(i^{J*} O_W(D_1), \ldots, i^{J*} O_W(D_m))] \\
&= \sum_{j=1}^{s} \iota_{j*}(\mathrm{Id}_{D_j}) \\
&\qquad + \sum_{J'} \iota_*^{J'} [F_{J'}(i^{J'*} O_W(D_1), \ldots, i^{J'*} O_W(D_m))],
\end{aligned}$$

where $\iota_j : D_j \to |E|$, $\iota^J : D^J \to |E|$, $i^J : D^J \to W$ are the inclusions. This gives us the desired decomposition, because each $\iota_*^{J'}$ factors through $\iota_{\mathrm{sing}*}$. □

Lemma 3.2.3. *Let $j : V \to Y$ be an open subscheme of some $Y \in \mathbf{Sm}_k$, and let $\tilde{D}$ be a strict normal crossing divisor on Y such that $D := j^*\tilde{D}$ is smooth (and reduced). Let $j_D : D \to |\tilde{D}|$ be the inclusion. Then there is a class $[\tilde{D}]* \in \mathcal{M}^+(|\tilde{D}|)$ such that*

1. $j_D^*[\tilde{D}]* = \mathrm{Id}_D \in \mathcal{M}^+(D)$.
2. The image of $[\tilde{D}]*$ *in* $\Omega_*(|\tilde{D}|)$ *is* $[\tilde{D} \to |\tilde{D}|]$.

Proof. We may write $\tilde{D}$ as $\tilde{D} = \sum_{i=1}^s \tilde{D}_i + \sum_{j=s+1}^m n_j \tilde{D}_j$, with the $\tilde{D}_j$ smooth and irreducible, and with $\sum_{i=1}^s \tilde{D}_i$ the closure of D in Y. We may suppose that $n_j > 1$ for all $j > s$; otherwise, just enlarge V. Let $i : |\tilde{D}|_{\mathrm{sing}} \to |\tilde{D}|$ be the inclusion. By lemma 3.2.2, there is a class $\eta \in \Omega_*(|\tilde{D}|_{\mathrm{sing}})$ such that

$$[\tilde{D} \to |\tilde{D}|] = \sum_{j=1}^s [\tilde{D}_j \to |\tilde{D}|] + i_*\eta.$$

Let $\eta*$ be a lifting of η to an element of $\mathcal{M}^+(|\tilde{D}|_{\mathrm{sing}})$ (use lemma 2.5.11). Since $\sum_{i=1}^s j_D^*[\tilde{D}_i \to |\tilde{D}|] = \mathrm{Id}_D$, and $|\tilde{D}|_{\mathrm{sing}}$ and $\tilde{D}_j$, $j > s$ are contained in $Y \setminus V$, taking

$$[\tilde{D}]* = \sum_{j=1}^s [\tilde{D}_j \to |\tilde{D}|] + i_*\eta*$$

gives the desired element of $\mathcal{M}^+(|\tilde{D}|)$. □

3.2.3 Main result

In this section we assume that k admits resolution of singularities.

Proposition 3.2.4. *Let $\mu : W' \to W$ be a birational projective morphism, with W and W' in $\mathbf{Sm}_k$. Let $i_F : F \to W$ be a reduced closed subscheme containing the fundamental locus of μ, and let E be the exceptional divisor of μ. Suppose that E is a strict normal crossing divisor. Then there is an element $\eta \in \Omega_*(F)$ such that*

$$[\mu : W' \to W] = \mathrm{Id}_W + i_{F*}(\eta)$$

Proof. We may suppose that F is the fundamental locus of μ. Let T be a closed subscheme of W supported in F such that μ is the blow-up of W along T. Let $q : Y \to W \times \mathbb{P}^1$ be the blow-up of $W \times \mathbb{P}^1$ along $T \times 0$, and let $\tilde{E}$ be the exceptional divisor. By lemma 3.2.1, we have the identification of $W' \to W$ with the restriction of q to $q_0 : \langle W \times 0\rangle \to W \times 0$. Furthermore, the singular locus of Y is contained in $|\tilde{E}| \cap \langle F \times \mathbb{P}^1\rangle$, which is disjoint from $\langle W \times 0\rangle$. Finally, $\langle W \times 0\rangle + \tilde{E}$ is a reduced strict normal crossing divisor away from $Y_{\mathrm{sing}} := |\tilde{E}| \cap \langle F \times \mathbb{P}^1\rangle$.

Thus, by the resolution of singularities, we may find a projective birational map $p : \tilde{Y} \to Y$ in $\mathbf{Sm}_k$, which is an isomorphism over $Y \setminus Y_{\mathrm{sing}}$, such that $(qp)^*(W \times 0)$ is a strict normal crossing divisor. Thus, $(qp)^*(W \times 0) = 1 \cdot \langle W \times 0\rangle + \sum_i n_i \tilde{D}_i$ with $\langle W \times 0\rangle + \sum_{i=1}^m \tilde{D}_i$ a reduced normal crossing divisor, and with $(qp)(\tilde{D}_i) \subset F$ for all i.

Let $\tilde{D} = \sum_{i=1}^m \tilde{D}_i$, and let

$$
\begin{aligned}
&i : |\langle W \times 0\rangle + \tilde{D}| \to \tilde{Y},\\
&i_{\langle W\times 0\rangle} : \langle W \times 0\rangle \to |\langle W \times 0\rangle + \tilde{D}|,\\
&i_{|\tilde{D}|} : |\tilde{D}| \to |\langle W \times 0\rangle + \tilde{D}|
\end{aligned}
$$

be the inclusions. Let $f : \tilde{Y} \to W$ be the morphism p_1qp and let $f^F : |\tilde{D}| \to F$ be the restriction of f.

Since the divisors $(qp)^*(W \times \infty)$ and $(qp)^*(W \times 0)$ are linearly equivalent strict normal crossing divisors on $\tilde{Y}$, it follows from proposition 3.1.9 that

$$
f_*([\langle W \times 0\rangle + \sum_i n_i\tilde{D}_i \to \tilde{Y}]) = f_*[(qp)^*(W \times \infty) \to \tilde{Y}]
$$

in $\Omega_*(W)$. Since qp is an isomorphism over $W \times (\mathbb{P}^1 \setminus \{0\})$, we have

$$
f_*([(qp)^*(W \times \infty) \to \tilde{Y}]) = \mathrm{Id}_W.
$$

By lemma 3.2.3, there is an element $\tau \in \Omega_*(|\tilde{D}|)$ such that

$$
\begin{aligned}
&[\langle W \times 0\rangle + \sum_i n_i\tilde{D}_i \to |\langle W \times 0\rangle + \tilde{D}|]\\
&\qquad\qquad = [\langle W \times 0\rangle \to |\langle W \times 0\rangle + \tilde{D}|] + i_{|\tilde{D}|*}(\tau)
\end{aligned}
$$

in $\Omega_*(|\langle W \times 0\rangle + \tilde{D}|)$. Let $\eta = f^F_*(\tau) \in \Omega_*(F)$. We thus have

$$
f_*([\langle W \times 0\rangle \to \tilde{Y}]) + i_{F*}(\eta) = \mathrm{Id}_W.
$$

Since $f : \langle W \times 0\rangle \to W$ is isomorphic to $\mu : W' \to W$, this proves the proposition. □

Lemma 3.2.5. *Take Y in $\mathbf{Sm}_k$, let $j : U \to Y$ be an open subscheme, and let L_1', $L_1, \ldots, L_m$ be line bundles on Y. Suppose that*

1. *The complement $i : Z \to Y$ of U is a strict normal crossing divisor.*
2. *$j^*L_1 \cong j^*L_1'$.*

Then $(\mathrm{Id}_Y, L_1', \ldots, L_m) - (\mathrm{Id}_Y, L_1, \ldots, L_m)$ is in $i_*(\Omega_*(Z))$.

Proof. It suffices to show that $[L_1'] - [L_1] = i_*x$ for some $x \in \Omega_*(Z)$. Indeed, if this is the case, then

$$
\begin{aligned}
(\mathrm{Id}_Y, L_1', \ldots, L_m) - (\mathrm{Id}_Y, L_1, \ldots, L_m) &= \tilde{c}_1(L_m) \circ \ldots \circ \tilde{c}_1(L_2)([L_1'] - [L_1])\\
&= \tilde{c}_1(L_m) \circ \ldots \circ \tilde{c}_1(L_2)(i_*x)\\
&= i_*(\tilde{c}_1(i^*L_m) \circ \ldots \circ \tilde{c}_1(i^*L_2)(x)).
\end{aligned}
$$

The kernel of $j^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(U)$ is the set of line bundles of the form $O_Y(D)$, where D is a divisor supported on the normal crossing divisor Z. Thus, there are effective divisors A and B, supported on Z, such that $L_1 \otimes O_Y(A) \cong L_1' \otimes O_Y(B)$. It clearly suffices to handle the case $L_1' = L_1 \otimes O_Y(A)$.

In this case, $[L_1'] = [L_1] +_F [O_Y(A)]$. Since

$$F(u,v) \equiv u \mod (v)\Omega_*(k)[[u,v]],$$

there is a polynomial $g(u,v)$ in $\Omega_*(k)[u,v]$ with

$$[L_1'] = [L_1] + g(\tilde{c}_1(L_1), \tilde{c}_1(O_Y(A)))([O_Y(A)]).$$

Arguing as above, it suffices to show that $[O_Y(A)]$ is in $i_*(\Omega_*(Z))$. But from proposition 3.1.9 we have

$$[O_Y(A)] = i_*[A \to Z],$$

which completes the proof of the lemma. □

Lemma 3.2.6. *Let X be a finite type k-scheme, $(Y \xrightarrow{f} X, L_1, \ldots, L_m)$ a standard cobordism cycle on X. Let $i : Z \to X$ be a closed subscheme with complement $j : U \to X$ and let $j_Y : Y_U \to Y$ denote the inclusion of the open subscheme $Y_U := Y \times_X U$. Suppose there is a smooth, quasi-projective k-scheme T, a smooth morphism $\pi : Y_U \to T$ and line bundles $M_1, \ldots, M_r$ on T with $j_Y^* L_i \cong \pi^* M_i$, $i = 1, \ldots, r$ and with $r > \dim_k T$. Then the class of $[Y \xrightarrow{f} X, L_1, \ldots, L_m]$ in $\Omega_*(X)$ is in $i_*(\Omega_*(Z))$.*

Proof. First assume that k is infinite. We proceed by induction on $\dim_k T$. We may assume that $Y = X$ and $f = \mathrm{Id}_Y$, so that π is a morphism $\pi : U \to T$. It suffices to prove the case $r = m = \dim_k T + 1$.

We note that

$$[\mathrm{Id}_Y, L_1, \ldots, L_r] = \tilde{c}_1(L_1) \circ \ldots \circ \tilde{c}_1(L_r)([\mathrm{Id}_Y]).$$

If L is a line bundle on Y, then $\tilde{c}_1(L)(i_*\eta) = i_*\tilde{c}_1(i^*L)(\eta)$, for $\eta \in \Omega_*(Z)$. Thus $\tilde{c}_1(L)$ sends $i_*(\Omega_*(Z))$ into itself. Since $\tilde{c}_1(L \otimes M^{\pm 1}) = \tilde{c}_1(L) \pm_F \tilde{c}_1(M)$, the result for $L_1 = L$ and $L_1 = M$ implies the result for $L_1 = L \otimes M^{\pm 1}$.

Next, let $g : \tilde{Y} \to Y$ be a projective birational morphism in $\mathbf{Sm}_k$ which is an isomorphism over U, and with exceptional divisor a strict normal crossing divisor. We identify U with $g^{-1}(U)$ and let $\tilde{Z} = \tilde{Y} \setminus U$. Since $[\tilde{Y} \to Y] - [\mathrm{Id}_Y]$ is in $i_*(\Omega_*(Z))$ (proposition 3.2.4) and since

$$\begin{aligned} g_*\tilde{c}_1(g^*L_1) \circ \ldots \circ \tilde{c}_1(g^*L_r)([\mathrm{Id}_{\tilde{Y}}]) - \tilde{c}_1(L_1) \circ \ldots \circ \tilde{c}_1(L_r)([\mathrm{Id}_Y]) \\ = \tilde{c}_1(L_1) \circ \ldots \circ \tilde{c}_1(L_r)([\tilde{Y} \to Y] - [\mathrm{Id}_Y]), \end{aligned}$$

it suffices to prove the result with $\tilde{Y}$ replacing Y and g^*L_j replacing L_j. Thus, using resolution of singularities, we may assume that Z is a strict normal crossing divisor on Y.

By resolution of singularities, there is a smooth projective k-scheme $\tilde{T}$ containing T as a dense open subscheme. Also by resolution of singularities, there is a projective birational morphism $g : \tilde{Y} \to Y$, which is an isomorphism over U, and with the exceptional divisor of g a strict normal crossing divisor, such that π extends to a morphism $\tilde{\pi} : \tilde{Y} \to \tilde{T}$. Since $\mathrm{Pic}(\tilde{T}) \to \mathrm{Pic}(T)$ is surjective, we may replace T with $\tilde{T}$. As above, we may replace Y with $\tilde{Y}$. Thus, changing notation, we may assume that $\pi : U \to T$ extends to $\tilde{\pi} : Y \to T$. By lemma 3.2.5, we may assume that $L_j = \tilde{\pi}^*(M_j)$ for $j = 1, \ldots, r$.

Suppose that $\dim_k T = 0$, then $M_1 = O_Z$, so $L_1 = O_Y$. Since $\tilde{c}_1(O_Y)$ is the zero endomorphism, the case $\dim_k T = 0$ is settled.

Suppose that $\dim_k T = r > 0$. We may write $M_1 = N \otimes M^{-1}$, with both N and M very ample line bundles on T. Using the formal group law as above, we may assume that M_1 is very ample on T.

Since k is infinite, the Bertini theorem tells us that there is a section s of M_1 with smooth divisor T_1. Let $\bar{Y}$ be the subscheme of Y defined by $\tilde{\pi}^* s = 0$. Since $\bar{Y}$ is a divisor on Y, we may write $\bar{Y} = Y_1 + A$, with A and Y_1 effective, having no common components, and with A supported in the strict normal crossing divisor Z. Since $\pi : U \to T$ is smooth, it follows that $U_1 := Y_1 \cap U$ is a smooth dense open subscheme of Y_1. Let $r : T_1 \to T$, $j_1 : U_1 \to Y_1$ be the inclusions.

By resolution of singularities, there is a projective birational morphism $g : \tilde{Y} \to Y$ which is an isomorphism over U, such that $g^*(\bar{Y})$ and the exceptional divisor of g are strict normal crossing divisors on $\tilde{Y}$. As above, we may replace Y with $\tilde{Y}$ and L_j with $g^* L_j$. Changing notation, we may assume that Y_1 is smooth.

Since $L_1 = O_Y(Y_1 + A)$, and A is supported on Z, it suffices to show that the class of $[Y, O_Y(Y_1), L_2, \ldots, L_r]$ is in $i_*\Omega_*(Z)$. Letting $i_1 : Y_1 \to Y$ be the inclusion, we have

$$[Y, O_Y(Y_1), \ldots, L_r] = i_{1*}[Y_1, i_1^* L_2, \ldots, i_1^* L_r].$$

Since $\pi_{|Y_1} : Y_1 \to T_1$ is smooth and equi-dimensional on the dense open subscheme U_1, and since the restriction of $i_1^* L_j$ to Y_1 is $\pi_{|Y_1}^*(r^* M_j)$, we may use induction to conclude that $(Y_1, i_1^* L_2, \ldots, i_1^* L_r)$ is in the image of $\Omega_*(Y_1 \cap Z)$, completing the proof in case k is infinite.

If k is finite, the same argument works: at the point at which we need to find a section of the very ample line bundle M_1 with smooth divisor, we enlarge k as in remark 2.3.8 and use lemma 2.3.5 to descend. $\square$

Theorem 3.2.7. *Suppose that k admits resolution of singularities. Let X be a finite type k-scheme, $i : Z \to X$ a closed subscheme and $j : U \to X$ the open complement. Then the sequence*

$$\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U) \to 0,$$

is exact.

Proof. Step I. We first prove that the map

$$j^* : \mathcal{Z}_*(X) \to \mathcal{Z}_*(U)$$

is surjective. Let $[f : Y \to U, L_1, \ldots, L_r]$ be a cobordism cycle on U. As $f : Y \to U$ is a projective morphism and Y is smooth and quasi-projective over k, there exists a closed immersion $Y \to U \times \mathbb{P}^N$ for some N, with f being the projection on U. Let $\bar{Y}$ be the closure of Y in $X \times \mathbb{P}^N$. Applying resolution of singularities, there is a projective birational morphism $\mu : \tilde{Y} \to \bar{Y}$ which is an isomorphism over Y, such that $\tilde{Y}$ is smooth (and quasi-projective as well). Thus $\tilde{f} := p_1 \circ \mu : \tilde{Y} \to X$ lifts f. Moreover, as the restriction map $\mathrm{Pic}(\tilde{Y}) \to \mathrm{Pic}(Y)$ is onto, one can extend the line bundles $L_1, \ldots, L_r$ on Y to line bundles $\tilde{L}_1, \ldots, \tilde{L}_r$ on $\tilde{Y}$. The cobordism cycle $[\tilde{f} : \tilde{Y} \to X, \tilde{L}_1, \ldots, \tilde{L}_r]$ on X clearly lifts $[f : Y \to U, L_1, \ldots, L_r]$, thus proving the surjectivity. In particular this implies the surjectivity of the maps $j^* : \underline{\mathcal{Z}}_*(X) \to \underline{\mathcal{Z}}_*(U)$, $j^* : \underline{\Omega}_*(X) \to \underline{\Omega}_*(U)$ and $j^* : \Omega_*(X) \to \Omega_*(U)$.

Step II. We prove that the map

$$\ker(\mathcal{Z}_*(X) \to \Omega_*(X)) \xrightarrow{j^*} \ker(\mathcal{Z}_*(U) \to \Omega_*(U))$$

is surjective. Since each of the maps

$$\mathcal{Z}_*(X) \to \underline{\mathcal{Z}}_*(X) \to \underline{\Omega}_*(X) \to \Omega_*(X)$$

are surjective, an easy diagram chase shows it is sufficient to prove that the maps

$$\ker(\underline{\mathcal{Z}}_*(X) \to \underline{\Omega}_*(X)) \xrightarrow{j^*} \ker(\underline{\mathcal{Z}}_*(U) \to \underline{\Omega}_*(U))$$

and

$$\ker(\underline{\Omega}_*(X) \to \Omega_*(X)) \xrightarrow{j^*} \ker(\underline{\Omega}_*(U) \to \Omega_*(U))$$

are surjective, and that

$$j^*(\ker(\mathcal{Z}_*(X) \to \Omega_*(X))) \supset \ker(\mathcal{Z}_*(U) \to \underline{\mathcal{Z}}_*(U)).$$

By lemma 2.4.7, $\ker(\underline{\mathcal{Z}}_*(U) \to \underline{\Omega}_*(U)) = \langle R_*^{Sect} \rangle(U)$ is the subgroup of $\underline{\mathcal{Z}}_*(U)$ generated by elements of the form:

$$[Y \to U, L_1, \ldots, L_r] - [D \to U, i^*(L_1), \ldots, i^*(L_{r-1})]$$

with $r > 0$, $[Y \to U, L_1, \ldots, L_r]$ a standard cobordism cycle on U, and $i : D \to Y$ the closed immersion of a smooth divisor in Y such that $L_r \cong O_Y(D)$. By step I, one may find a standard cobordism cycle $[\tilde{f} : \tilde{Y} \to X, \tilde{L}_1, \ldots, \tilde{L}_r]$ on X lifting $[Y \to U, L_1, \ldots, L_r]$. Let $\tilde{i} : \tilde{D} \to \tilde{Y}$ be the closure of D in $\tilde{Y}$.

Applying resolution of singularities (to $\tilde{D} \subset \tilde{Y}$), there is a projective birational morphism $\mu : \tilde{Y}' \to \tilde{Y}$, such that μ is an isomorphism outside of $\tilde{D} \setminus D$, and such that the proper transform $\mu^{-1}[\tilde{D}]$ is smooth. Replacing $\tilde{Y}$ with $\tilde{Y}'$, and $\tilde{L}_i$ with $\mu^*\tilde{L}_j$ and changing notation, we may assume that the closure $\tilde{D}$ of D is smooth. Since $L_r \cong O_Y(D)$, we may take $\tilde{L}_r = O_Y(\tilde{D})$. Thus the element

$$[\tilde{Y} \to X, \tilde{L}_1, \ldots, \tilde{L}_r] - [\tilde{D} \to X, \tilde{i}^*(\tilde{L}_1), \ldots, \tilde{i}^*(\tilde{L}_{r-1})]$$

is an element of $\langle R_*^{Sect}\rangle(X)$ lifting the given element of $\langle R_*^{Sect}\rangle(U)$.

We now show that

$$\ker(\underline{\Omega}_*(X) \to \Omega_*(X)) \xrightarrow{j^*} \ker(\underline{\Omega}_*(U) \to \Omega_*(U))$$

is surjective. By proposition 2.5.15, we know that $\ker(\underline{\Omega}_*(U) \to \Omega_*(U))$ is generated as a group by the elements of the form

$$f_* \circ \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r)\big([F(L, M)] - [L \otimes M]\big),$$

where $f : Y \to U$ is a projective morphism with Y irreducible and smooth, $(L_1, \ldots, L_r, L, M)$ are line bundles on Y, and $F(u, v)$ is a fixed choice of a lifting of $F_{\mathbb{L}}(u, v)$ to $\mathcal{M}^+(k)[[u, v]]$ (see 2.5.13). Again by step I, we may lift such an element to the element

$$\tilde{f}_* \circ \tilde{c}_1(\tilde{L}_1) \circ \cdots \circ \tilde{c}_1(\tilde{L}_r)\big([F(\tilde{L}, \tilde{M})] - [\tilde{L} \otimes \tilde{M}]\big),$$

which obviously lies in $\ker(\underline{\Omega}_*(X) \to \Omega_*(X))$.

Finally, we show that

$$j^*(\ker(\mathcal{Z}_*(X) \to \Omega_*(X))) \supset \ker(\mathcal{Z}_*(U) \to \underline{\mathcal{Z}}_*(U)).$$

Indeed, lemma 2.4.2 shows that $\ker(\mathcal{Z}_*(U) \to \underline{\mathcal{Z}}_*(U))$ is generated by elements of the form

$$x := [Y \to U, \pi^* M_1, \ldots, \pi^* M_r, L_{r+1}, \ldots, L_m],$$

where $\pi : Y \to T$ is a smooth equi-dimensional morphism to a smooth quasi-projective k-scheme T of dimension $< r$, and $M_1, \ldots, M_r$ are line bundles on T. By step I, we can lift x to an element $\tau := [\tilde{Y} \to X, \tilde{L}_1, \ldots, \tilde{L}_m]$ of $\mathcal{Z}_*(X)$. By lemma 3.2.6, there is an element η in $\mathcal{Z}_*(Z)$ such that $\tau - i_*(\eta)$ is in $\ker(\mathcal{Z}_*(X) \to \Omega_*(X))$. Thus $\tau - i_*(\eta)$ is a lifting of x to $\ker(\mathcal{Z}_*(X) \to \Omega_*(X))$. This completes step II.

Step III. The kernel of $j^* : \Omega_*(X) \to \Omega_*(U)$ is generated by differences

$$[f : Y \to X, L_1, \ldots, L_r] - [f' : Y' \to X, L'_1, \ldots, L'_r]$$

of standard cobordism cycles which agree on U. Indeed, take $x \in \mathcal{Z}_*(X)$ whose class in $\Omega_*(X)$ lies in the kernel of $j^* : \Omega_*(X) \to \Omega_*(U)$. By step II, we may modify x by an element in $\ker(\mathcal{Z}_*(X) \to \Omega_*(X))$, so that $j^*x = 0$ in $\mathcal{Z}_*(U)$. Since $\mathcal{Z}_*(U)$ and $\mathcal{Z}_*(X)$ are the free abelian groups on the standard cobordism cycles, it follows that x can be expressed in $\mathcal{Z}_*(X)$ as a sum of differences of standard cobordism cycles on X which agree on U, as required.

Step IV. We finish by proving that the differences $[f : Y \to X, L_1, \ldots, L_r]$ $-[f' : Y' \to X, L'_1, \ldots, L'_r]$ of standard cobordism cycles which agree on U lie in the image of $\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X)$. Let α denote $[f : Y \to X, L_1, \ldots, L_r]$ and α' denote $[f' : Y' \to X, L'_1, \ldots, L'_r]$.

If $Y \times_X U$ and $Y' \times_X U$ are both empty, then clearly $[f : Y \to X, L_1, \ldots, L_r]$ and $[f' : Y' \to X, L'_1, \ldots, L'_r]$ both are in the image of i_*.

If $Y \times_X U$ and $Y' \times_X U$ are (both) non-empty, then choose a U-isomorphism $\phi : Y \times_X U \to Y' \times_X U$. Let $\Gamma \subset Y \times_X Y'$ be the closure of the graph of ϕ. Resolving the singularities of Γ, we have a Y'' in $\mathbf{Sm}_k$, with projective morphisms $\mu : Y'' \to Y$, $\mu' : Y'' \to Y'$ which are isomorphisms over U, and with $f \circ \mu = f' \circ \mu'$. We may also assume that $(f\mu)^{-1}(X \setminus U)$ is a normal crossing divisor on Y''. Let $\beta = [Y'' \to X, \mu^* L_1, \ldots, \mu^* L_r]$ and $\beta' = [Y'' \to X, \mu'^* L'_1, \ldots, \mu'^* L'_r]$, so

$$\alpha - \alpha' = (\alpha - \beta) - (\alpha' - \beta') - (\beta' - \beta).$$

It suffices to prove that $\alpha - \beta$, $\alpha' - \beta'$ and $\beta' - \beta$ lie in the image of $\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X)$.

As $(f\mu)^{-1}(X \setminus U)$ is a normal crossing divisor on Y'', we may apply proposition 3.2.4 to $Y'' \to Y$: if $F = Y \setminus Y_U$, there is an $\eta \in \Omega_*(F)$ such that $[\mu] - [\mathrm{Id}_Y] = i_{F*}(\eta)$ in $\Omega_*(Y)$. Applying the $\tilde{c}_1(L_i)$ and pushing forward by f shows that $\alpha - \beta$ lies in the image of $\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X)$. This reasoning shows the same holds for $\alpha' - \beta'$.

It remains to show that $\beta' - \beta$ lies in the image of $\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X)$ as well.

Since f_* maps $\Omega_*((f\mu)^{-1}(Z))$ to $\Omega_*(Z)$, we may replace X with Y''. Changing notation, we may assume that X is smooth and quasi-projective over k and that Z is a strict normal crossing divisor on X. Since $j^* L_1 \cong j^* L'_1$, it follows from lemma 3.2.5 that $\beta' - \beta$ is in $i_*(\Omega_*(Z))$.

This finishes the proof of theorem 3.2.7. □

3.3 Transversality

The main task of this section is to prove

Proposition 3.3.1. *Let k be a field. Let W be in $\mathbf{Sm}_k$ and let $i : Z \to W$ be a smooth closed subscheme. Then $\Omega_*(W)$ is generated by standard cobordism cycles of the form $[f : Y \to W]$ with f transverse to i in $\mathbf{Sm}_k$.*

We need a series of technical results before we begin the proof. Let $Y := (Y_{ij})$, $0 \le i \le r$, $1 \le j \le r$ be the generic $r+1 \times r$ matrix and let $B =$

Spec $k[Y_{ij}]$. Let $X = (X_0, \ldots, X_r)$ and let $Z \subset B \times \mathbb{P}^r$ be the subscheme defined by the matrix equation $XY = 0$, with projection $\mu : Z \to B$. We let $B_i \subset B$ be the reduced closed subscheme defined by $\mathrm{rnk} Y \le i$, $i = 0, \ldots, r-1$ and let $Z_i = \mu^{-1}(B_i)$.

Remark 3.3.2. B_i is equal to the closed subscheme $\mathcal{B}_i$ of B defined by the ideal of $i+1 \times i+1$ minors of (Y_{ij}). Indeed, this is easy to verify in a neighborhood of a matrix of rank equal to i and it is also evident that B_i is the support of $\mathcal{B}_i$. By results of [8], $\mathcal{B}_i$ is Cohen-Macaulay; since $\mathcal{B}_i = B_i$ at the generic point, $\mathcal{B}_i$ is reduced, hence $\mathcal{B}_i = B_i$.

Form the sequence of blow-ups

$$B^r \xrightarrow{\tau_r} \ldots \xrightarrow{\tau_2} B^1 \xrightarrow{\tau_1} B^0 = B \tag{3.4}$$

where $\tau_{i+1} : B^{i+1} \to B^i$ is the blow-up of B^i along the proper transform of B_i. Form the sequence of blow-ups

$$Z^r \xrightarrow{\tilde{\tau}_r} \ldots \xrightarrow{\tilde{\tau}_2} Z^1 \xrightarrow{\tilde{\tau}_1} Z^0 = Z \tag{3.5}$$

similarly, replacing B with Z and B_i with Z_i. The map $\mu^0 := \mu$ induces the map $\mu^i : Z^i \to B^i$, forming a big commutative diagram of B-schemes. Let $\langle B_i \rangle \subset B^i$ denote the proper transform of B_i and let $E^{i+1} \subset B^{i+1}$ denote the exceptional divisor $\tau_{i+1}^{-1}(\langle B_i \rangle)$. Similarly, let $\langle Z_i \rangle \subset Z^i$ denote the proper transform of Z_i and let $\tilde{E}^{i+1} \subset Z^{i+1}$ denote the exceptional divisor $\tilde{\tau}_{i+1}^{-1}(\langle Z_i \rangle)$

Lemma 3.3.3. *1. $B_i \setminus B_{i-1}$ and $Z_i \setminus Z_{i-1}$ are smooth for all i.*
2. Z is smooth and irreducible.
3. B^i is smooth and $\langle B_i \rangle \subset B^i$ is smooth; Z^i is smooth and $\langle Z_i \rangle \subset Z^i$ is smooth.
4. The maps $\tau_i : E^i \cap \langle B_i \rangle \to \langle B_{i-1} \rangle$ and $\mu^{i-1} : \langle Z_{i-1} \rangle \to \langle B_{i-1} \rangle$ are smooth for all $i = 1, \ldots, r$.
5. the maps μ^i are all birational and μ^r and $\tilde{\tau}_r$ are isomorphisms.

Proof. (1) is some elementary linear algebra, which we leave to the reader. For the remainder of the assertions, we proceed by induction on r, starting with $r = 1$. In this case $B = \mathbb{A}^2$, and Z is the blow-up of $\mathbb{A}^2$ at the origin. Since $B = B_1$, $B_0 = (0, 0)$, the result follows.

Assume the result for $r-1$, with $r \ge 2$. B_0 is the 0-matrix, and $Z_0 \subset Z$ is the subscheme defined by the ideal generated by the Y_{ij}. Thus $Z_0 = B_0 \times \mathbb{P}^r$, so B_0 and Z_0 are both smooth.

Z is covered by the affine open subschemes $X_i = 1$, $i = 0, \ldots, r$; we consider the open subscheme U defined by $X_0 = 1$. Let $\hat{Y}$ be the square submatrix (Y_{ij}), $1 \le i \le r$, $1 \le j \le r$ of Y, and let Y_0 and $\hat{X}$ be the row vectors (Y_{0j}), (X_j), $1 \le j \le r$, respectively. U is the closed subscheme of $B \times \mathbb{A}^r$ defined by the matrix equation $Y_0 = -\hat{X}\hat{Y}$. Clearly the coordinates

Y_{ij}, X_j, $1 \le i, j \le r$ give an isomorphism of U with $\mathbb{A}^{r^2+r}$, hence Z is smooth and irreducible, proving (2). This also shows that Z^1 and B^1 are smooth.

Let $\bar{B} = \mathrm{Proj}_k k[Y_{ij}] = \mathbb{P}^{r(r+1)-1}$ and let $\bar{Z} \subset \bar{B} \times \mathbb{P}^r$ be the closed subscheme defined by the bihomogeneous matrix equation $XY = 0$, with morphism $\bar{\mu} : \bar{Z} \to \bar{B}$. The projection from $0 \in B$ defines morphisms $\pi : B^1 \to \bar{B}$, $\tilde{\pi} : Z^1 \to \bar{Z}$, which make B^1 and Z^1 into $\mathbb{P}^1$-bundles over $\bar{B}$ and $\bar{Z}$ respectively. Defining closed subschemes $\bar{B}_i \subset \bar{B}$ and $\bar{Z}_i \subset \bar{Z}$ by the rank conditions as above, we see that $\pi^{-1}(\bar{B}_i)$ is the proper transform of B_i to B^1 and $\tilde{\pi}^{-1}(\bar{Z}_i)$ is the proper transform of Z_i to Z^1.

Let $V_{ij} \subset \bar{B}$ be the open subscheme for which $Y_{ij} \neq 0$ for some ij, $0 \le i \le r$, $1 \le j \le r$, and let $U_{ij} = \bar{\mu}^{-1}(V_{ij})$. Since $\bar{B} = \cup_{ij} V_{ij}$, we have $\bar{Z} = \cup_{ij} U_{ij}$. We prove (3)-(5) by induction on r. Noting that have already shown that $Z_0 \to B_0$ is smooth, it suffices to prove the evident analogs of (3)-(5) for the restriction of $\bar{\mu}$ to $U_{ij} \to V_{ij}$, with Z_k, B_k replaced by $U_{ij} \cap \bar{Z}_k$ and $V_{ij} \cap \bar{B}_k$, respectively. It suffices to consider the case $ij = 01$.

As above, U_{01} is given by the matrix equation $XY = 0$, except that $Y_{01} = 1$. Let $B' = \mathrm{Spec}\, k[\{Y'_{ij} \mid 0 \le i \le r-1, 1 \le j \le r-1\}]$, and let $Z' \subset B' \times \mathbb{P}^{r-1}$ be the solution of the matrix equation $X'Y' = 0$, $X' = (X_0, \ldots, X_{r-1})$ the generic row vector of length r. Multiplying (Y_{ij}) (with $Y_{01} = 1$) on the right by the appropriate elementary matrix to make the first row of Y the unit row vector $(1, 0, \ldots, 0)$ transforms the equation $XY = 0$ to an equivalent system of the form

$$X \begin{pmatrix} 1 & 0 \\ A & Y'' \end{pmatrix} = 0;$$

one sees thereby that there are smooth morphisms $q : V_{01} \to B'$, $\tilde{q} : U_{01} \to Z'$ intertwining $U_{01} \to V_{01}$ and $Z' \to B'$, with

$$q^{-1}(B'_i) = V_{01} \cap \bar{B}_{i+1}$$
$$\tilde{q}^{-1}(V'_i) = U_{01} \cap \bar{V}_{i+1}$$

The induction hypothesis then implies the properties we need for $U_{01} \to V_{01}$. $\square$

In what follows, we say that a morphism $f : Y \to X$ is a *locally closed immersion* if f factors as $Y \xrightarrow{g} U \xrightarrow{j} X$, with g a closed immersion and j an open immersion.

We consider the following situation: Let X be a smooth irreducible affine variety over k, $Y \subset X$ a smooth irreducible closed subvariety of codimension $r+1$. We are given a closed subset C of Y and a smooth closed subscheme $D \subset X$, intersecting $Y \setminus C$ transversely. We are given as well a finite dimensional k-subspace $\mathcal{V}$ of $\Gamma(X, \mathcal{O}_X)$ such that, for all sufficiently large N, there is an open subset $\mathcal{U}_N \subset \mathcal{V}^N$ such that each element $f = (f_1, \ldots, f_N) \in \mathcal{U}_N$ defines a locally closed immersion $f : Y \to \mathbb{A}^N$. Finally, we suppose that Y is a complete intersection, defined by elements $x_0, \ldots, x_r \in \Gamma(X, \mathcal{O}_X)$.

Take $f_{ij} \in \mathcal{V}$, $0 \leq i \leq r$, $1 \leq j \leq r$, and let $F_j = \sum_{i=0}^{r} x_i f_{ij}$. Let $T \subset X$ be the closed subscheme defined by $F_1 = \ldots = F_r = 0$; clearly $Y \subset T$. Let T_{sing} denote the non-smooth locus of T and let $\bar{f}_{ij}$ denote the restriction of f_{ij} to Y.

Let $\mu : \tilde{T} \to T$ be the blow-up of T along Y, let $\tilde{Y} = \mu^{-1}(Y)$ and let $\mu : \tilde{Y} \to Y$ be the restriction of μ. Let $Y_i \subset Y$ be the subscheme defined by the vanishing of the $i+1 \times i+1$ minors of $\bar{f}_{ij}$ and let $\tilde{Y}_i = \mu^{-1}(Y_i)$. Define $\tau_{i+1} : Y^{(i+1)} \to Y^{(i)}$ inductively by setting $Y^{(0)} = Y$ and letting τ_{i+1} be the blow-up $Y^{(i)}$ along the proper transform of Y_i. Replacing Y with $\tilde{Y}$, we have the blow-ups $\tilde{\tau}_{i+1} : \tilde{Y}^{(i+1)} \to \tilde{Y}^{(i)}$ defined similarly.

Lemma 3.3.4. *Suppose k is infinite and that $r(r+1) \geq 2\dim_k Y + 1$. Suppose that there is a non-empty open subset $\mathcal{V}_0^{r(r+1)}$ of $\mathcal{V}^{r(r+1)}$ such that, for all $f \in \mathcal{V}_0^{r(r+1)}$, we have*

(a) $T \setminus Y$ is smooth; T is irreducible and of codimension r in X
(b) $T \setminus Y$ intersects D transversely.

Suppose in addition

(c) for each point $y \in Y$, there is an $f \in \mathcal{V}$ with $f(y) \neq 0$.

Then for all (f_{ij}) in a non-empty open subset of $\mathcal{V}_0^{r(r+1)}$:

1. *$\tilde{T}$ and $\tilde{Y}$ are smooth and irreducible*
2. *For each irreducible component C' of C, $\dim \mu^{-1}(C') = \dim C'$.*
3. *the restrictions $\mu : \tilde{T} \setminus C \to X$ and $\mu : \tilde{Y} \setminus C \to X$ are transverse to the inclusion $D \to X$.*
4. *$\mu : \tilde{Y} \to Y$ is birational, the schemes $Y^{(i)}$ and $\tilde{Y}^{(i)}$ are smooth, the proper transforms of Y_i to $Y^{(i)}$ and of $\tilde{Y}_i$ to $\tilde{Y}^{(i)}$ are smooth and the map $\tilde{Y}^{(r)} \to Y^{(r)}$ induced by μ is an isomorphism.*

Proof. By (a), T is Cohen-Macaulay and generically reduced, hence reduced. Let $p : \tilde{X} \to X$ be the blow-up of X along Y; we have the canonical closed immersion $\tilde{T} \to \tilde{X}$ identifying $\tilde{T}$ with the proper transform $p^{-1}[T]$. $\tilde{X}$ is the closed subscheme of $X \times \mathbb{P}^r$ defined by the equations $x_i X_j = x_j X_i$ for $0 \leq i, j \leq r$ (with $X_0, \ldots, X_r$ the homogeneous coordinates on $\mathbb{P}^r$). Let $E \subset \tilde{X}$ be the exceptional divisor of p, then $E \cap p^{-1}[T]$ is the exceptional divisor $\tilde{Y}$ of μ.

We now prove (1). $\tilde{T} \to T$ is an isomorphism over every point x such that Y is a Cartier divisor on T; since Y has codimension one in T, this is the case for all $x \in T \setminus T_{\mathrm{sing}}$.

Let $U \subset \tilde{X}$ be the open subscheme $X_0 \neq 0$. This gives us affine coordinates $u_i := X_i/X_0$, $i = 1, \ldots, r$, $u_0 = x_0$; with respect to these coordinates, we have

$$p^* F_j = u_0(p^* f_{0j} + \sum_{i=1}^{r} u_i p^* f_{ij})$$

on U. Define functions $p^*[F_j]$ on U by

$$p^*[F_j] := p^*f_{0j} + \sum_{i=1}^{r} u_i p^* f_{ij}.$$

We claim that $p^{-1}[T] \cap U$ is the subscheme T' of U defined by $p^*[F_j] = 0$, $j = 1, \ldots, r$.

To see this, it is clear that $T' \setminus E = (p^{-1}[T] \cap U) \setminus E$ and that $T' \supset p^{-1}[T] \cap U$. Since T is reduced, so is $p^{-1}[T]$, hence to show that $T' = p^{-1}[T] \cap U$, it suffices to show that $T' \cap E = p^{-1}[T] \cap U \cap E$.

For this, note that, the restriction of p to $p : E \cap U \to Y$ identifies $E \cap U$ with the affine space $\mathbb{A}^r_Y$, with coordinates $u_1, \ldots, u_r$. By our assumption on $\mathcal{V}$, for $N >> 0$, all $f := (f_1, \ldots, f_N)$ in an open subset of $\mathcal{V}^N$ give a locally closed immersion $f : Y \to \mathbb{A}^N$. By our assumption (c), if we are given a point $y \in Y$, for a general choice of $f \in \mathcal{V}$ the functions $u_1 \cdot f \circ p, \ldots, u_r \cdot f \circ p$ give a Y-isomorphism $E \cap U \to \mathbb{A}^r_Y$ over a neighborhood of y. Let $\mathcal{W}$ be the k-vector space of functions on $E \cap U$ generated by the functions of the form $f \circ p, u_1 \cdot f \circ p, \ldots, u_r \cdot f \circ p$ for $f \in \mathcal{V}$. Then for N sufficiently large, a general element $g \in \mathcal{W}^N$ gives a locally closed immersion $g : E \cap U \to \mathbb{A}^N$. Thus, by Bertini's theorem, if we take elements $G_1, \ldots, G_r$ of the form

$$G_j = \sum_{i=1}^{N} g_{ij},\ j = 1, \ldots, r, g_{ij} \in \mathcal{W},$$

the subscheme of $E \cap U$ defined by $G_1 = \ldots, G_r = 0$ is smooth and irreducible of codimension r. But each G_j can be rewritten as a linear combination of the form

$$G_j = f_{0j} \circ p + \sum_{i=1}^{r} u_i \cdot f_{ij} \circ p$$

with the $f_{ij} \in \mathcal{V}$. Thus, for a general choice of $(f_{ij}) \in \mathcal{V}^{r(r+1)}$, the subscheme $T' \cap E$ of $E \cap U$ is smooth, irreducible and has codimension r on $E \cap U$; since $p^{-1}[T] \cap E$ also has codimension r on E, we must have $T' \cap E = p^{-1}[T] \cap U \cap E$ as desired.

This also shows that $\tilde{Y} \cap U = \tilde{T} \cap U \cap E$ is smooth for general (f_{ij}); since our choice of special coordinate X_0 was arbitrary, $\tilde{Y} = \tilde{T} \cap E$ is smooth for general (f_{ij}). But since $p^{-1}[T] \cap U = T'$ is a complete intersection on U, $p^{-1}[T]$ is Cohen-Macaulay. Since $p^{-1}[T] \setminus E$ and $p^{-1}[T] \cap E$ are both smooth, this implies $p^{-1}[T]$ is smooth, proving (1).

For (2), noting that $E \to Y$ is a $\mathbb{P}^r$-bundle, $p^{-1}(C')$ has dimension $\dim C' + r$. Using Bertini's theorem as above, we find that $p^{-1}(C') \cap p^{-1}[T]$ has dimension $\dim C'$ for general (f_{ij}), proving (2).

To show (3), $T \setminus Y$ intersects D transversely for general (f_{ij}) by assumption. As in the argument for (1), it suffices to show that $\mu : \mu^{-1}(Y \setminus C) \to X$ is transverse to $D \to X$. By assumption $D \cap Y \setminus C$ is smooth of the proper dimension. As in the proof of (2), $p^{-1}(D \cap Y \setminus C) \to D \cap Y \setminus C$ is a smooth map of relative dimension r; the argument for (1) using Bertini's theorem shows that $p^{-1}[T] \cap p^{-1}(D \cap Y \setminus C) \cap U$ is smooth and has the same dimension as $D \cap Y \setminus C$, which proves (3).

Finally, for (4), we apply lemma 3.3.3; we use the notation of that lemma. The matrix $(\bar{f}_{ij})$ defines a morphism

$$\bar{f} : Y \to B := \operatorname{Spec} k[\{Y_{ij} \mid 0 \le i \le r, 1 \le j \le r\}].$$

Since we are assuming $r(r+1) \ge 2d + 1$, the Whitney embedding theorem implies that, for all $(\bar{f}_{ij})$ in an open subset of $\mathcal{V}^{r(r+1)}$, the map $\bar{f}$ is a locally closed immersion, in particular, $\bar{f}$ is unramified. We want to show that the morphism $Z \to B$ pulls back via $\bar{f}$ to the morphism $\tilde{Y} \to Y$, and that the blow-up sequences (3.4) and (3.5) pull back via $\bar{f}$ to sequences of blow-ups with smooth center. Assuming the first statement, the smoothness condition of lemma 3.3.3(4) implies that it suffices that the map $\bar{f}$ be transverse to the inclusions $B_i \setminus B_{i-1} \to B$ for $i = 0, \ldots, r$. By Kleiman's transversality theorem [15, Theorem 10], this will be the case if we change $\bar{f}$ by a sufficiently general affine linear transformation of the affine space B, which implies that $\bar{f}$ satisfies the necessary transversality condition for all f in an open subset of $\mathcal{V}^{r(r+1)}$.

We finish the proof by showing that $\tilde{Y} \to Y$ is the pull-back of $Z \to B$ for general $\bar{f}$. $\tilde{T} \subset T \times \mathbb{P}^r$ is the closed subscheme defined by the kernel of the surjection

$$\big(k[x_0 \ldots, x_N]/(F_1, \ldots, F_r)\big)[T_0, \ldots, T_r] \to k[x_0 \ldots, x_N]/(F_1, \ldots, F_r)$$

sending T_i to x_i, $i = 0, \ldots, r$. Note that $k[x_0 \ldots, x_N]$ is a polynomial algebra over k, since $x_0, \ldots, x_N$ is a regular sequence by assumption.

Clearly the elements $\sum_i T_i f_{ij}$ are in this kernel; the local computations of $p^{-1}[T]$ we made above show that these elements generate the kernel. As $Y \subset T$ is defined by the ideal $(x_0, \ldots, x_r)$, this shows that $\tilde{Y} = \mu^{-1}(Y)$ is the subscheme of $Y \times \mathbb{P}^r$ defined by the matrix equation $(T_0, \ldots, T_r) \cdot (\bar{f}_{ij}) = 0$, i.e., $\tilde{Y} \cong Z \times_B Y$ as a Y-scheme, as desired. □

We now consider a global version of the situation. Let $X \subset \mathbb{P}^M$ be a smooth quasi-projective scheme, $Y \subset X$ a smooth closed subscheme of codimension $r+1$ defined by an ideal-sheaf $\mathcal{I}$. We suppose $r(r+1) \ge 2\dim_k Y + 1$. Let D be a smooth closed subscheme of X, and let $\bar{Y} \subset \bar{X}$ be the closures of Y and X in $\mathbb{P}^M$.

Let $\mathcal{O}_X(1)$ be the restriction of $\mathcal{O}(1)$ to X. Choose an $n \ge 1$ such that $\mathcal{I}_{\bar{Y}}(n)$ is generated by global sections on $\bar{X}$ and choose generating global sections $s_1, \ldots, s_m \in H^0(\bar{X}, \mathcal{I}(n))$. Fix a degree $d \ge 1$ and let $(f_{ij}) \in H^0(X, \mathcal{O}_X(d))^{mr}$ be an $m \times r$ matrix of elements of $H^0(X, \mathcal{O}_X(d))$. Set

$$F_j := \sum_{i=1}^{m} s_i f_{ij}; \quad j = 1, \ldots, r,$$

and let $T \subset X$ be the closed subscheme defined by $F_1 = \ldots, F_m = 0$. Then $Y \subset T$; let $\mu : \tilde{T} \to T$ be the blow-up of T along Y.

Lemma 3.3.5. *Suppose k is infinite. Let $C \subset Y$ be a closed subset of Y such that $Y \setminus C$ intersects D transversely. Let $\mathcal{V}$ be the image of $H^0(\mathbb{P}^M, \mathcal{O}(d))$ in $H^0(X, \mathcal{O}_X(d))$. Then for all (f_{ij}) in an open subset of $\mathcal{V}^{mr}$*

1. *$\tilde{T}$ is smooth, $\mu^{-1}(Y)$ is smooth, $\dim \tilde{T} = \dim Y + 1$ and $\mu : \mu^{-1}(Y) \to Y$ is birational. In addition, the map $\mu : \mu^{-1}(Y) \to Y$ factors as*

$$\begin{tikzcd}
\mu^{-1}(Y) \arrow[r, equal] \arrow[d, "\mu"'] & \tilde{Y}^0 & \tilde{Y}^1 \arrow[l, "\tilde{\tau}_1"'] & \cdots \arrow[l, "\tilde{\tau}_2"'] & \tilde{Y}^r \arrow[l, "\tilde{\tau}_r"'] \arrow[d, equal] \\
Y \arrow[r, equal] & Y^0 & Y_1 \arrow[l, "\tau_1"] & \cdots \arrow[l, "\tau_2"] & Y^r \arrow[l, "\tau_r"]
\end{tikzcd}$$

where all the Y^i and $\tilde{Y}^i$ are smooth, and the maps τ_i and $\tilde{\tau}_i'$ are all blow-ups along smooth subschemes.

2. *$\dim \mu^{-1}(C') = \dim C'$ for each irreducible component C' of C.*
3. *$\mu : \tilde{T} \setminus \mu^{-1}(C) \to X$ and $\mu : \mu^{-1}(Y \setminus C) \to X$ are transverse to the inclusion $D \to X$.*

Proof. We may assume that X is irreducible. We claim that the vector space $\mathcal{W}$ of sections of $\mathcal{I}(n+d)$ of the form $s_i f$, $f \in \mathcal{V}$, $i = 1, \ldots, m$, define a locally closed immersion of $X \setminus Y$ in a projective space. To see this, let $\bar{X}^* \to \bar{X}$ be the blow-up of $\bar{X}$ along $\bar{Y}$. The generators $s_1, \ldots, s_m$ of $\mathcal{I}_{\bar{Y}}(n)$ define a closed embedding of $\bar{X}^*$ in $\bar{X} \times \mathbb{P}^{m-1}$, and the map $X \setminus Y$ given by $\mathcal{W}$ is just the composition

$$X \setminus Y \subset \bar{X}^* \subset \bar{X} \times \mathbb{P}^{m-1} \xrightarrow{i} \mathbb{P}^M \times \mathbb{P}^{m-1} \xrightarrow{S} \mathbb{P}^{M(m-1)-1}$$

where i is the closed immersion induced by $\bar{X} \subset \mathbb{P}^M$ and S is the Segre embedding. This proves our claim.

It follows therefore from Bertini's theorem that $T \setminus Y$ is smooth and irreducible and $T \cap D \setminus Y$ is smooth for all f_{ij} in an open subset of $\mathcal{V}^{r(r+1)}$. In particular let X' be an affine subset of X contained in the open subscheme $gf \neq 0$ for some $f \in V$ and some $g \in H^0(X, \mathcal{O}_X(n))$, and such that $Y \cap X'$ is a complete intersection defined by some collection $s_{i_0}/g, \ldots, s_{i_r}/g$, then the functions $(1/f)\mathcal{V}$ on X' satisfy all the hypotheses for lemma 3.3.4. Also, we can rewrite the F_j on X' as functions

$$F_j' := \sum_{k=0}^{r} \frac{1}{gf} s_{i_k} f_{kj}'; \quad j = 1, \ldots, r.$$

Let $\bar{f}_{kj}'$ be the restriction of f_{kj}'/f to $Y \cap X'$, and let $(Y \cap X')_i$ be the closed subscheme defined by the vanishing of the $i+1 \times i+1$ minors of $(\bar{f}_{kj}')$. It is clear that the $(Y \cap X')_i$ for varying X' patch together to form a closed subscheme Y_i of Y.

The data consisting of: $Y \cap X' \subset X'$, $C \cap X'$ and $D \cap X' \subset X'$, generators $\{s_{i_0}/g, \ldots, s_{i_r}/g\}$ of $\mathcal{I}(X')$, vector space of functions $(1/f)\mathcal{V}$ and subschemes $(Y \cap X')_i$ thus gives a special case of the data used in lemma 3.3.4. An open condition on the matrix (f'_{ij}/f) will be implied by an open condition on the matrix (f_{ij}). Covering X by finitely many affine open subsets of the type described above, we see that a finite intersection of the resulting open subsets of $\mathcal{V}^{r(r+1)}$ gives the desired open condition on the matrix (f_{ij}). □

Lemma 3.3.6. *Let B be a quasi-projective variety over k, $E \to B$ a vector bundle. Then there is a smooth projective variety H over k, a vector bundle $E_H \to H$ and a morphism $f : B \to H$ such that f^*E_H is isomorphic to E.*

Proof. This is proved in [10]; we give the proof here for the reader's convenience. B is quasi-projective, let $i : B \to \mathbb{P}^N$ be a locally closed immersion. $E(n)$ is generated by global sections on B for some $n \geq 0$; choosing M global generators gives a morphism $s : B \to \mathrm{Gr}(r, M)$ such that $E \cong s^*U_{r,M}$, where $U_{r,M} \to \mathrm{Gr}(r, M)$ is the universal rank r quotient bundle. Taking $H = \mathbb{P}^N \times \mathrm{Gr}(r, M)$, $E_H = p_1^*O(-n) \otimes p_2^*U_{r,M}$ and $f = (i, s)$ yields the desired result. □

Proof (of proposition 3.3.1). We first reduce to the case of an infinite base field. If $K \supset k$ is a finite separable field extension, and if $f : Y \to W \times_k K := W_K$ is transverse to the base-extension $i_K : Z_K \to W_K$, then $p_1 \circ f : Y \to W$ is transverse to i. Thus, using lemma 2.3.5 and the standard trick of taking extensions of k of relatively prime degrees allows us to assume the base field is infinite.

By lemma 2.5.11, $\Omega_*(W)$ is generated by the cobordism cycles of the form $[f : Y \to W]$. We proceed by induction on $\dim_k Y$, starting with the trivial case $\dim = -1$.

Assume the result for all $[Y \to W]$ with $\dim_k Y < r$, and for all $W \in \mathbf{Sm}_k$. We first note that this implies the result for $[f : Y \to W]$ with $\dim_k Y = r$, such that f factors as

$$Y \xrightarrow{p} B \xrightarrow{g} W$$

with $p : Y \to B$ a projective space bundle $\mathbb{P}(E) \to B$ for some vector bundle E of rank $r \geq 2$, and $g : B \to W$ a projective morphism in $\mathbf{Sm}_k$. Indeed, by lemma 3.3.6, we have a morphism $s : B \to H$, with H smooth and projective over k, and a vector bundle $E_H \to H$ such that $s^*E_H \cong E$. Let $h = (g, s) : B \to W \times H$. Then h is projective; since $\dim_k B < r$, we may write $[h] = \sum_i n_i[f_i : B_i \to W \times H]$ in $\Omega_*(W \times H)$ with each f_i transverse to $i \times \mathrm{Id} : Z \times H \to W \times H$. Let $q : \mathbb{P} \to W \times H$ be the projective space bundle $\mathbb{P}(p_2^*E_H)$. Then by taking the smooth pull-back by q, we have

$$[\tilde{h} : h^*\mathbb{P} \to \mathbb{P}] = \sum_i n_i[\tilde{f}_i : f_i^*\mathbb{P} \to \mathbb{P}]$$

in $\Omega_*(\mathbb{P})$. Pushing forward to W by $p_1 \circ q$ writes $[Y \to W]$ as a sum of maps $[f_i^*\mathbb{P} \to W]$ which are transverse to $i : Z \to W$.

Similarly, the induction assumption implies the result for the decomposable elements $\Omega_r(W) \cap \mathbb{L}_{*\geq 1}\Omega_*(W)$ in $\Omega_r(W)$.

This allows us to make the following reduction:

Claim. Let $[f : Y \to W]$ be a cobordism cycle with $\dim_k Y = r$, and let $\mu : \tilde{Y} \to Y$ be a projective birational map in $\mathbf{Sm}_k$. Suppose that the map μ admits a weak factorization: there is a commutative diagram of projective Y-schemes, with each $Y^i \in \mathbf{Sm}_k$,

$$\begin{tikzcd}
& Y^1 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & Y^n \arrow[dl] \arrow[dr] & \\
\tilde{Y} = Y^0 \arrow[rrrrrr, bend right, "\mu"'] & & Y^2 & \cdots & Y^{n-1} & & Y^{n+1} = Y
\end{tikzcd}$$

such that each slanted arrow is the blow-up of the base along a smooth subscheme. Then the result is true for $[f \circ \mu : \tilde{Y} \to W]$ if and only if the result is true for $[f : Y \to W]$.

Proof (of claim). Let $q : Y_1 \to Y_2$ be the blow-up of $Y_2 \in \mathbf{Sm}_k$ along some smooth closed subscheme F of Y_2, and let $f : Y_2 \to W$ be a projective morphism. Let ν be the conormal sheaf of $F \to Y_2$ and suppose Y_2 has dimension r. The blow-up formula of proposition 2.5.2 implies that, modulo decomposable elements in $\Omega_r(W)$, we have

$$[f \circ q : Y_1 \to W] + [\mathbb{P}(\nu \oplus O_F) \to W] = [f : Y_2 \to W].$$

This together with the consequences of the induction hypothesis we have already mentioned implies that the result is true for $[f \circ q : Y_1 \to W]$ if and only if it is true for $[f : Y_2 \to W]$. The existence of the weak factorization of μ thus yields the claim. □

Now take a cobordism cycle $[Y \to W]$ with $\dim_k Y = r$. Since f is projective, there is a closed immersion $Y \to \mathbb{P}^n \times W$ such that f is the restriction of the projection. By increasing n, we may assume that Y has codimension r on $\mathbb{P}^n \times W$ with $r(r+1) \geq 2\dim_k Y + 1$. Let $Y_0 \subset Y$ be the singular locus with respect to $Z \to W$, that is, the minimal closed subset C of Y such that the $f : Y \setminus C \to W$ is transverse to $Z \to W$. We proceed by induction on $\dim Y_0$ to show that the result holds true for $[Y \to W]$.

W is quasi-projective, hence so is $\mathbb{P}^n \times W$. We may thus apply lemma 3.3.5 with $X = \mathbb{P}^n \times W$, with $C = Y_0$ and with D the smooth closed subscheme $\mathbb{P}^n \times Z$. In particular by lemma 3.3.5(5), $\mu^{-1}(Y) \to Y$ admits a weak factorization, so by the claim, it suffices to prove the result for $[f \circ \mu : \mu^{-1}(Y) \to W]$; by the lemma, $f \circ \mu$ is transverse to i away from a closed subset Y_0' of the same dimension as Y_0. In addition, $\mu^{-1}(Y)$ is a divisor on a k-scheme $\tilde{T} \in \mathbf{Sm}_k$ and $f \circ \mu$ extends to the projective morphism $p_2 \circ \mu : \tilde{T} \to W$, which is transverse to i away from Y_0'. Changing notation, we may assume that Y is a divisor on some $T \in \mathbf{Sm}_k$ such that the morphism $f : Y \to W$ extends to a projective morphism $F : T \to W$ and that $F : T \setminus Y_0 \to W$ is transverse to $i : Z \to W$.

We can find very ample line bundles L and M on T such that $O_T(Y) = L \otimes M^{-1}$. Using the formal group law, and working modulo decomposable elements, we see that it suffices to prove the result for the divisor Y' of a general section of a very ample line bundle, say L, on T, with map $Y' \to T \xrightarrow{F} W$. Now, since the non-transverse locus of $F : T \to W$ is contained in Y_0, if Y' is the divisor of a sufficiently general section of L, then by Bertini's theorem, Y' is smooth, $\dim(Y' \cap Y_0) = \dim Y_0 - 1$, and $Y' \setminus Y_0 \to W$ is transverse to i. By induction, we may write $[Y' \to W]$ as a sum of cobordism cycles of the desired form, completing the proof. □

3.4 Homotopy invariance

In this section, we show that, for a finite type k-scheme X, the smooth pull-back $p^* : \Omega_*(X) \to \Omega_{*+1}(X \times \mathbb{A}^1)$ is an isomorphism. We assume that k is a field admitting resolution of singularities; in particular, k is perfect.

Proposition 3.4.1. *Let X be a finite type k-scheme, and let $p : X \times \mathbb{A}^1 \to X$ be the projection. Then $p^* : \Omega_*(X) \to \Omega_{*+1}(X \times \mathbb{A}^1)$ is surjective.*

Proof. If X is a finite type k-scheme, then $\Omega_*(X) = \Omega_*(X_{\mathrm{red}})$, so we may assume that X is reduced. Since k is perfect, X has a filtration by reduced closed subschemes

$$\emptyset = X_0 \subset X_1 \subset \ldots \subset X_N = X$$

such that $X_i \setminus X_{i-1}$ is in $\mathbf{Sm}_k$. Using noetherian induction, we may assume the result is true for X_{N-1}. Letting $U = X_N \setminus X_{N-1}$, the commutative diagram of localization sequences

$$\begin{tikzcd}
\Omega_*(X_{N-1}) \arrow[r] \arrow[d] & \Omega_*(X) \arrow[r] \arrow[d] & \Omega_*(U) \arrow[r] \arrow[d] & 0 \\
\Omega_{*+1}(X_{N-1} \times \mathbb{A}^1) \arrow[r] & \Omega_{*+1}(X \times \mathbb{A}^1) \arrow[r] & \Omega_{*+1}(U \times \mathbb{A}^1) \arrow[r] & 0
\end{tikzcd}$$

and a diagram chase reduces us to the case of $X \in \mathbf{Sm}_k$.

Take X in $\mathbf{Sm}_k$. Using lemma 2.3.5 and the standard trick of taking extensions of k of relatively prime degrees, we reduce to the case of an infinite field k.

By resolution of singularities, there is a smooth projective k-scheme $\bar{X}$ containing X as an open subscheme. Since $\Omega_*(\bar{X} \times \mathbb{A}^1) \to \Omega_*(X \times \mathbb{A}^1)$ is surjective, it suffices to prove the result for $\bar{X}$. Changing notation, we may assume that X is projective.

By proposition 3.3.1, it suffices to show that cobordism cycles of the form $[f : Y \to X \times \mathbb{A}^1]$ such that $f^{-1}(X \times 0)$ is smooth and codimension one on Y are in the image of p^*. If $f : Y \to X \times \mathbb{A}^1$ is such a projective morphism, then, as X is projective, $p_2 f : Y \to \mathbb{A}^1$ is smooth over a neighborhood of 0 in $\mathbb{A}^1$. Let $m : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ be the multiplication map $m(x, y) = xy$. The map m is flat, and is smooth over $\mathbb{A}^1 \setminus \{0\}$. Since Y is smooth over a neighborhood of 0, it follows that $Y_m := Y \times_{\mathbb{A}^1} (\mathbb{A}^1 \times \mathbb{A}^1)$ is in $\mathbf{Sm}_k$. Let $g : Y_m \to X \times \mathbb{A}^1 \times \mathbb{A}^1$ be the projection. Then $g^{-1}(X \times \mathbb{A}^1 \times 0) = f^{-1}(X \times 0) \times \mathbb{A}^1$ and $g^{-1}(X \times \mathbb{A}^1 \times 1) = Y$. It is easy to check that $g : g^{-1}(X \times \mathbb{A}^1 \times 0) \to X \times \mathbb{A}^1$ is isomorphic to $p^*(f : f^{-1}(X \times 0) \to X)$.

By resolution of singularities, we may extend $g : Y_m \to X \times \mathbb{A}^1 \times \mathbb{A}^1$ to a projective morphism $\tilde{g} : \tilde{Y}_m \to X \times \mathbb{A}^1 \times \mathbb{P}^1$, with $\tilde{Y}_m$ smooth over k. By remark 2.4.8, $\tilde{g}$ gives a cobordism between f and something in the image of p^*. □

Theorem 3.4.2. *Let k be a field admitting resolution of singularities. Let X be a finite type k-scheme. Then $p^* : \Omega_*(X) \to \Omega_{*+N}(X \times \mathbb{A}^N)$ is an isomorphism for all N.*

Proof. As for proposition 3.4.1, we may assume that k is infinite. It suffices to prove the case $N = 1$. Having proved surjectivity in proposition 3.4.1, it suffices to prove the injectivity of p^*. Let $i_\infty : X \to X \times \mathbb{P}^1$ be the inclusion $i_\infty(x) = (x, (1:0))$. Consider the localization sequence

$$\Omega_*(X) \xrightarrow{i_{\infty *}} \Omega_*(X \times \mathbb{P}^1) \xrightarrow{j^*} \Omega_*(X \times \mathbb{A}^1) \to 0.$$

Let $q : X \times \mathbb{P}^1 \to X$ be the projection, and let $\psi : \Omega_*(X \times \mathbb{P}^1) \to \Omega_{*-1}(X)$ be the map

$$\psi(\eta) = q_*(\tilde{c}_1(\mathcal{O}(1))(\eta)).$$

If $\eta = i_{\infty *}(\tau)$ for some τ in $\Omega_*(X)$, then

$$\begin{aligned} \tilde{c}_1(\mathcal{O}(1))(\eta) &= i_{\infty *}(\tilde{c}_1(i_\infty^*(\mathcal{O}(1)))(\tau)) \\ &= i_{\infty *}(\tilde{c}_1(\mathcal{O}_X)(\tau)) \\ &= 0. \end{aligned}$$

Thus, $\psi \circ i_{\infty *} = 0$, and ψ descends to a well-defined homomorphism

$$\bar{\psi} : \Omega_*(X \times \mathbb{A}^1) \to \Omega_{*-1}(X).$$

On the other hand, for τ in $\Omega_*(X)$ of the form $[f : Y \to X]$,

$$\begin{aligned}\psi \circ q^*(\tau) &= q_*(\tilde{c}_1(\mathcal{O}(1))([f \times \mathrm{Id} : Y \times \mathbb{P}^1 \to X \times \mathbb{P}^1]) \\ &= q_*[i_\infty \circ f : Y \to X \times \mathbb{P}^1] \\ &= [f : Y \to X] = \tau,\end{aligned}$$

since $(f \times \mathrm{Id})^*(\mathcal{O}(1))$ has the section X_0 with smooth divisor $Y \times \infty$. Since classes of the form $[f : Y \to X]$ generate $\Omega_*(X)$ by lemma 2.5.11, it follows that $\psi \circ q^* = \mathrm{Id}$, hence $\bar{\psi} \circ p^* = \mathrm{Id}$, and p^* is injective. □

3.5 The projective bundle formula

The proof of the projective bundle formula follows roughly the same pattern as Grothendieck's argument for the Chow groups [1]: one first handles the case of the trivial bundle by localization on $\mathbb{P}^n$ with respect to the cellular structure, and then one passes to the general case by localization on the base. The situation is complicated however by the fact that the pushforward of the powers $c_1(\mathcal{O}(1))^i$ are non-zero for all i in the relevant range.

To simplify the notation, we pass freely between the category of line bundles and the category of invertible sheaves, writing for instance $\tilde{c}_1(\mathcal{L})$ for the endomorphism $\tilde{c}_1(L)$, if L is a line bundle with sheaf of sections $\mathcal{L}$.

3.5.1 Support conditions

Let X be a finite type k-scheme, and $i : F \to X$ a closed subset; give F the reduced scheme structure. Let $\Omega_*^F(X) \subset \Omega_*(X)$ denote the image of $i_* : \Omega_*(F) \to \Omega_*(X)$.

Lemma 3.5.1. *Let F be a closed subset of X.*

1. *Let $f(u_1, \ldots, u_n)$ be a power series with $\Omega_*(k)$ coefficients. Choose invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_m$ on X. Then $\Omega_*^F(X)$ is stable under the endomorphism $f(\tilde{c}_1(\mathcal{L}_1), \ldots, \tilde{c}_1(\mathcal{L}_m))$ of $\Omega_*(X)$.*
2. *Let $p : Y \to X$ be a smooth quasi-projective morphism, $q : X \to Z$ a projective morphism. Then p^* maps $\Omega_*^F(X)$ to $\Omega_*^{p^{-1}(F)}(Y)$, and q_* maps $\Omega_*^F(X)$ to $\Omega_*^{q(F)}(X)$.*

Proof. Since Ω_* is an oriented Borel-Moore functor, (1) and (2) follow directly from the axioms given in definition 2.1.2: (1) follows from the projection formula (A3), and (2) follows from functoriality of projective push-forward (D1), and the compatibility of smooth pull-back and projective push-forward (A2). □

3.5.2 Projective bundles

Let X be a k-scheme of finite type and let $p : \mathcal{E} \to X$ be a vector bundle of rank $n+1$, giving the $\mathbb{P}^n$-bundle $q : \mathbb{P}(\mathcal{E}) \to X$, with canonical quotient invertible sheaf $\mathcal{O}(1)$. Write ξ for the operator $\tilde{c}_1(\mathcal{O}(1))$. We have the group homomorphisms

$$\phi_j := \xi^j \circ q^* : \Omega_{*-n+j}(X) \to \Omega_*(\mathbb{P}(\mathcal{E}));$$

let

$$\Phi_{X,\mathcal{E}} : \oplus_{j=0}^n \Omega_{*-n+j}(X) \to \Omega_*(\mathbb{P}(\mathcal{E}))$$

be the sum of the ϕ_j.

In the case of a trivial bundle $\mathcal{E} = \mathcal{O}_X^{n+1}$, we have

$$\mathbb{P}(\mathcal{E}) = \mathrm{Proj}_{\mathcal{O}_X} \mathcal{O}_X[X_0, \ldots, X_n] = \mathbb{P}^n_X,$$

and $\mathcal{O}(1)$ is the invertible sheaf with $q_*\mathcal{O}(1)$ the $\mathcal{O}_X$-module generated by $X_0, \ldots, X_n$. Let $i_m : \mathbb{P}^m_X \to \mathbb{P}^n_X$ be the subscheme defined by $X_{m+1} = \ldots = X_n = 0$, and let $q_m : \mathbb{P}^m_X \to X$ be the projection.

The following is an elementary computation, using the relations $\mathcal{R}_*^{Sect}$:

Lemma 3.5.2. *For $\mathcal{E} = \mathcal{O}_X^{n+1}$, we have $\xi^{n-m} \circ q^*(\eta) = i_{m*}(q_m^*(\eta))$, and $q_*(\xi^{n-m} \circ q^*(\eta)) = [\mathbb{P}^m_k] \cdot \eta$. Also $\xi^{n+1} = 0$.*

3.5.3 Some operators

We proceed to define $\mathbb{Z}$-linear combinations of composable expressions in ξ, q_* and q^*, which we write as $\psi_0, \ldots, \psi_n$. Evaluating the expression ψ_j as an operator will define a graded map $\psi_j : \Omega_*(\mathbb{P}(\mathcal{E})) \to \Omega_{*-n+j}(X)$, and, in case $\mathcal{E} = \mathcal{O}_X^{n+1}$, we will have

$$\psi_i \circ (\xi^j \circ q^*) = \begin{cases} \mathrm{Id}_{\Omega_*(X)} & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases} \tag{3.6}$$

We define the ψ_i inductively, starting with $\psi_0 := q_* \circ \xi^n$. Suppose we have defined ψ_i for $i = 0, \ldots, m-1$, $1 \leq m \leq n$. Let

$$\psi_m = q_* \circ \big(\xi^{n-m} \circ \big(\mathrm{Id} - \sum_{j=0}^{m-1} \xi^j \circ q^* \circ \psi_j\big)\big).$$

It follows directly from lemma 3.5.2 that ψ_m satisfies the conditions (3.6), and the induction continues. We let Ψ be the formal expression $\prod_{j=0}^n \psi_j$; having chosen a finite type k-scheme X and a rank $n+1$ bundle $\mathcal{E}$ on X, the expression Ψ determines the homomorphism

$$\Psi_{X,\mathcal{E}} : \Omega_*(\mathbb{P}(\mathcal{E})) \to \oplus_{j=0}^n \Omega_{*-n+j}(X),$$

natural in the pair $(X, \mathcal{E})$.

Lemma 3.5.3. *Suppose $\mathcal{E} = \mathcal{O}_X^{n+1}$. Then $\Psi_{X,\mathcal{E}} \circ \Phi_{X,\mathcal{E}} = \mathrm{Id}$.*

Proof. This follows directly from the identities (3.6). □

3.5.4 The main result

Theorem 3.5.4. *Let k be a field admitting resolution of singularities. Let X be a k-scheme of finite type, $\mathcal{E}$ a rank $n+1$ vector bundle on X. Then*

$$\Phi_{X,\mathcal{E}} : \oplus_{j=0}^{n} \Omega_{*-n+j}(X) \to \Omega_*(\mathbb{P}(\mathcal{E}))$$

is an isomorphism.

Proof. We first consider the case of the trivial bundle $\mathcal{E} = \mathcal{O}_X^{n+1}$. We have shown the injectivity of Φ in lemma 3.5.3. We show surjectivity by induction on n, the case $n = 0$ being trivial.

We have the inclusion $i_{n-1} : \mathbb{P}_X^{n-1} \to \mathbb{P}_X^n$; let $j : \mathbb{A}_X^n \to \mathbb{P}_X^n$ be the open complement.

From lemma 3.5.2, we have the commutative diagram, where the first row is the evident inclusion,

$$\begin{tikzcd}
\oplus_{j=0}^{n-1} \Omega_{*-n+1+j}(X) \arrow[r] \arrow[d, "\Phi_{X,\mathcal{O}_X^n}"'] & \oplus_{j=0}^{n} \Omega_{*-n+j}(X) \arrow[d, "\Phi_{X,\mathcal{O}_X^{n+1}}"] \\
\Omega_*(\mathbb{P}_X^{n-1}) \arrow[r, "i_{n-1*}"'] & \Omega_*(\mathbb{P}_X^n).
\end{tikzcd}$$

The image of $\oplus_{j=1}^{n} \Omega_{*-n+j}(X)$ under $\Phi_{X,\mathcal{O}_X^{n+1}}$ therefore contains the image of i_{n-1*}. On the other hand, we have

$$j^* \circ \phi_i = j^* \circ \xi^i \circ q^* = \tilde{c}_1(j^*\mathcal{O}(1))^i \circ p^*$$

for all i. Since $j^*\mathcal{O}(1) \cong \mathcal{O}_{\mathbb{A}_X^n}$, and $\tilde{c}_1(\mathcal{O}_{\mathbb{A}_X^n}) = 0$ by lemma 2.3.6, we have $j^* \circ \phi_i = 0$ for $i > 0$. Thus, using the localization sequence

$$\Omega_*(\mathbb{P}_X^{n-1}) \xrightarrow{i_{n-1*}} \Omega_*(\mathbb{P}_X^n) \xrightarrow{j^*} \Omega_*(\mathbb{A}_X^n) \to 0,$$

we see that $\Phi_{X,\mathcal{O}_X^{n+1}}$ is surjective if

$$p^* = j^* \circ \phi_0 : \Omega_{*-n}(X) \to \Omega_*(\mathbb{A}_X^n)$$

is. The surjectivity of p^* follows from the homotopy theorem 3.4.2.

We now pass to the general case. Choose a filtration of X by closed subschemes

$$\emptyset = X_{N+1} \subset X_N \subset \ldots \subset X_1 \subset X_0 = X$$

such that the restriction of $\mathcal{E}$ to $X_m \setminus X_{m+1}$ is trivial. To simplify the text, we omit the mention of the appropriate restriction of $\mathcal{E}$ in the notation for Φ and Ψ; similarly, we write simply $\mathcal{E}$ for the restriction of $\mathcal{E}$ to the various locally closed subsets $X_m \setminus X_{m+1}$ or $X \setminus X_m$, etc. As $\mathcal{E}$ is trivial on $X_m \setminus X_{m+1}$, we write $\mathbb{P}^n_{X_m \setminus X_{m+1}}$ for $\mathbb{P}_{X_m \setminus X_{m+1}}(\mathcal{E})$.

Assume by induction that $\Phi_{X\setminus X_m}$ is an isomorphism. By the case of the trivial bundle, $\Phi_{X_m\setminus X_{m+1}}$ is an isomorphism.

It follows from lemma 3.5.1 that the maps $\Phi_{X_m\setminus X_{m+1}}$ and $\Psi_{X_m\setminus X_{m+1}}$ descend to maps on the images

$$\Phi^{X_m\setminus X_{m+1}} : \oplus_{j=0}^n \Omega_{*-n+j}^{X_m\setminus X_{m+1}}(X\setminus X_{m+1}) \to \Omega_*^{\mathbb{P}^n_{X_m\setminus X_{m+1}}}(\mathbb{P}_{X\setminus X_{m+1}}(\mathcal{E}))$$
$$\Psi^{X_m\setminus X_{m+1}} : \Omega_*^{\mathbb{P}^n_{X_m\setminus X_{m+1}}}(\mathbb{P}_{X\setminus X_{m+1}}(\mathcal{E})) \to \oplus_{j=0}^n \Omega_{*-n+j}^{X_m\setminus X_{m+1}}(X\setminus X_{m+1})$$

with $\Psi^{X_m\setminus X_{m+1}} \circ \Phi^{X_m\setminus X_{m+1}} = \mathrm{Id}$. Thus $\Phi^{X_m\setminus X_{m+1}}$ is an isomorphism.

The localization sequences for the inclusions $X_m\setminus X_{m+1} \to X\setminus X_{m+1}$ and $\mathbb{P}^n_{X_m\setminus X_{m+1}} \to \mathbb{P}^n_{X\setminus X_{m+1}}(\mathcal{E})$ give us the commutative diagram with exact columns

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\oplus_{j=0}^n \Omega_{*-n+j}^{X_m\setminus X_{m+1}}(X\setminus X_{m+1}) \arrow[r, "\Phi^{X_m\setminus X_{m+1}}"] \arrow[d, "i_*"] & \Omega_*^{\mathbb{P}^n_{X_m\setminus X_{m+1}}}(\mathbb{P}_{X\setminus X_{m+1}}(\mathcal{E})) \arrow[d, "i_*"] \\
\oplus_{j=0}^n \Omega_{*-n+j}(X\setminus X_{m+1}) \arrow[r, "\Phi_{X\setminus X_{m+1}}"] \arrow[d, "j^*"] & \Omega_*(\mathbb{P}_{X\setminus X_{m+1}}(\mathcal{E})) \arrow[d, "j^*"] \\
\oplus_{j=0}^n \Omega_{*-n+j}(X\setminus X_m) \arrow[r, "\Phi_{X\setminus X_m}"] \arrow[d] & \Omega_*(\mathbb{P}_{X\setminus X_m}(\mathcal{E})) \arrow[d] \\
0 & 0.
\end{tikzcd}$$

As the top and bottom horizontal maps are isomorphisms, $\Phi_{X\setminus X_{m+1}}$ is an isomorphism, and the induction continues. □

3.6 The extended homotopy property

The projective bundle formula allows us to extend the homotopy property (theorem 3.4.2) to the case of an affine-space bundle.

Lemma 3.6.1. *Let $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ be an exact sequence of locally free coherent sheaves on a finite type k-scheme X, with $\mathcal{L}$ an invertible sheaf, let $i : \mathbb{P}(\mathcal{F}) \to \mathbb{P}(\mathcal{E})$ be the associated closed immersion of projective bundles over X. Let $q : \mathbb{P}(\mathcal{E}) \to X$, $\bar{q} : \mathbb{P}(\mathcal{F}) \to X$ be the projections. Then*

$$i_* \circ \bar{q}^* = \tilde{c}_1(q^*\mathcal{L}^{-1} \otimes \mathcal{O}(1)_{\mathcal{E}}) \circ q^*.$$

Proof. The subscheme $\mathbb{P}(\mathcal{F})$ of $\mathbb{P}(\mathcal{E})$ is defined by the vanishing of the map $\rho : q^*\mathcal{L} \to \mathcal{O}(1)_{\mathcal{E}}$ induced by the inclusion $\mathcal{L} \to \mathcal{E}$; letting s be the section of $q^*\mathcal{L}^{-1} \otimes \mathcal{O}(1)_{\mathcal{E}}$ induced by ρ, $\mathbb{P}(\mathcal{F})$ is defined by $s = 0$, so

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mathbb{P}(\mathcal{F})) \cong q^*\mathcal{L}^{-1} \otimes \mathcal{O}(1)_{\mathcal{E}}.$$

Let $f : Y \to X$ be a projective morphism with $Y \in \mathbf{Sm}_k$. The map f induces the map $\tilde{f} : \mathbb{P}(f^*\mathcal{E}) \to \mathbb{P}(\mathcal{E})$, and the inclusion $i_Y : \mathbb{P}(f^*\mathcal{F}) \to \mathbb{P}(f^*\mathcal{E})$. Then $q^*([f])$ is represented by $\tilde{f}$, and $(i_*\bar{q}^*)([f])$ is represented by $\tilde{f} \circ i_Y$. By the above computation and the relations defining Ω_*, we have

$$[\mathbb{P}(f^*\mathcal{F}) \to \mathbb{P}(f^*\mathcal{E})] = [\tilde{f}^*(q^*\mathcal{L}^{-1} \otimes \mathcal{O}(1)_{\mathcal{E}})].$$

Applying $\tilde{f}_*$, and using the definition of $\tilde{c}_1$, we find

$$\begin{aligned}
(i_*\bar{q}^*)([f]) &= [\tilde{f} \circ i_Y] \\
&= \tilde{f}_*([\mathbb{P}(f^*\mathcal{F}) \to \mathbb{P}(f^*\mathcal{E})]) \\
&= \tilde{c}_1(q^*\mathcal{L}^{-1} \otimes \mathcal{O}(1)_{\mathcal{E}})([\tilde{f}]) \\
&= \tilde{c}_1(q^*\mathcal{L}^{-1} \otimes \mathcal{O}(1)_{\mathcal{E}})(q^*([f])).
\end{aligned}$$

□

Lemma 3.6.2. *Let $0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F} \to 0$ be an exact sequence of vector bundles on a finite type k-scheme X, let $i : \mathbb{P}(\mathcal{F}) \to \mathbb{P}(\mathcal{E})$ be the associated closed immersion of projective bundles over X and let $q : \mathbb{P}(\mathcal{E}) \to X$, $\bar{q} : \mathbb{P}(\mathcal{F}) \to X$ be the projections. Let $\xi = \tilde{c}_1(\mathcal{O}(1)_{\mathcal{E}})$ and let $\bar{\xi} = \tilde{c}_1(\mathcal{O}(1)_{\mathcal{F}})$. Then*

$$i_* \circ (\bar{\xi}^j \circ \bar{q}^*) = \xi^{j+1} \circ q^*$$

for all $j \geq 0$.

Proof. For $j = 0$, this is just a special case of lemma 3.6.1. In general, we apply the projection formula:

$$i_* \circ (\bar{\xi}^j \circ \bar{q}^*) = i_* \circ (i^*\xi^j \circ \bar{q}^*) = \xi^j \circ i_* \circ \bar{q}^*,$$

which reduces us to the case $j = 0$. □

Theorem 3.6.3. *Let k be a field admitting resolution of singularities, let $p : F \to X$ be a rank n vector bundle over a k-scheme of finite type X, and let $\tilde{p} : V \to X$ be a principal homogeneous space for F (i.e., an affine-space bundle over X). Then $\tilde{p}^* : \Omega_*(X) \to \Omega_{*+n}(V)$ is an isomorphism.*

Proof. Let $\mathcal{F}$ be the sheaf of sections of F, and $\mathcal{F}^\vee$ the dual. V is classified by an element $v \in H^1(X, \mathcal{F}) = \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F})$; explicitly, if

$$0 \to \mathcal{F} \to \mathcal{E} \xrightarrow{\pi} \mathcal{O}_X \to 0$$

represents v, then $\mathcal{V} := \pi^{-1}(1)$ is the sheaf of sections of V.

Let

$$0 \to \mathcal{O}_X \to \mathcal{E}^\vee \to \mathcal{F}^\vee \to 0$$

be the extension dual to v, giving the projective bundles $q : \mathbb{P}(\mathcal{E}^\vee) \to X$, $\bar{q} : \mathbb{P}(\mathcal{F}^\vee) \to X$, and the inclusion $i : \mathbb{P}(\mathcal{F}^\vee) \to \mathbb{P}(\mathcal{E}^\vee)$. The complement of $\mathbb{P}(\mathcal{F}^\vee)$ in $\mathbb{P}(\mathcal{E}^\vee)$ is thus isomorphic to V, as an X-scheme.

This yields the localization sequence

$$\Omega_*(\mathbb{P}(\mathcal{F}^\vee)) \xrightarrow{i_*} \Omega_*(\mathbb{P}(\mathcal{E}^\vee)) \xrightarrow{j^*} \Omega_*(V) \to 0. \tag{3.7}$$

Let

$$\begin{aligned}
&\kappa : \oplus_{j=0}^{n-1} \Omega_{*-n+1-j}(X) \to \oplus_{j=0}^{n} \Omega_{*-n-j}(X) \\
&\pi : \oplus_{j=0}^{n} \Omega_{*-n-j}(X) \to \Omega_*(X)
\end{aligned}$$

be the evident inclusion and projection, giving the exact sequence

$$0 \to \oplus_{j=0}^{n-1} \Omega_{*-n+1-j}(X) \xrightarrow{\kappa} \oplus_{j=0}^{n} \Omega_{*-n-j}(X) \xrightarrow{\pi} \Omega_*(X) \to 0 \tag{3.8}$$

It follows from lemma 3.6.2 that $i_* \circ \Phi_{X,\mathcal{F}^\vee} = \Phi_{X,\mathcal{E}^\vee} \circ \kappa$. Since $\mathbb{P}(\mathcal{F}^\vee)$ is the subscheme of $\mathbb{P}(\mathcal{E}^\vee)$ defined by the vanishing of the composition $\mathcal{O} \to \mathcal{E}^\vee \to \mathcal{O}(1)$, it follows that $j^*\mathcal{O}(1) = \mathcal{O}$. Thus, $j^* \circ \Phi_{X,\mathcal{E}^\vee} = p^* \circ \pi$. Therefore, $(\Phi_{X,\mathcal{F}^\vee}, \Phi_{X,\mathcal{E}^\vee}, p^*)$ defines a map of sequences (3.8) $\to$ (3.7). Since $\Phi_{X,\mathcal{F}^\vee}$ and $\Phi_{X,\mathcal{E}^\vee}$ are isomorphisms by theorem 3.5.4, it follows that p^* is an isomorphism. □

Remark 3.6.4. The method used above can also be applied to prove the analogue of theorem 3.6.3 for Chow groups as well as for the K_0 functor.

4

Algebraic cobordism and the Lazard ring

We are now in position to exploit the deeper structures of algebraic cobordism. We first axiomatize these structures in section 4.1 with the definition of an oriented Borel-Moore weak homology theory; the results of chapter 3 immediately imply that Ω_* is the universal such theory. Following Grothendieck [11] and Quillen [30], we develop the theory of Chern classes of vector bundles with values in an oriented Borel-Moore weak homology theory, and adapt the "twisting" construction of Quillen to our setting.

In section 4.2, we show that algebraic K-theory is the universal oriented theory with multiplicative periodic formal group law and characterize the Chow ring as the universally additive theory in section 4.5.

The central result of this chapter is the identification of $\Omega_*(k)$ with the Lazard ring (theorem 4.3.7). In addition to our standing resolution of singularities assumption, the proof uses the weak factorization conjecture, established in characteristic zero in [2] and [37]. We give in section 4.4 an axiomatic treatment of Rost's generalized degree formula; the main results of this chapter and chapter 3 enable the application of the degree formulas to Ω_*.

To simplify the discussion at various points, we will use some general results on oriented cohomology theories, oriented weak cohomology theories and oriented Borel-Moore weak homology theories which will be proven in chapter 5. These results are independent of the results discussed in this chapter, and their use will not result in any circular arguments.

4.1 Weak homology and Chern classes

In this section, we introduce the notion of an oriented Borel-Moore weak homology theory. This is, as the terminology suggests, stronger than an oriented Borel-Moore functor and weaker than an oriented Borel-Moore homology theory, which will be introduced in in chapter 5. What we have proven in chapter 3 immediately implies that algebraic cobordism is such a theory, assuming that k admits resolution of singularities.

We then develop the theory of Chern classes of vector bundles for oriented Borel-Moore weak homology theories. We deduce several results: comparison of algebraic cobordism with K-theory, definition of the Landweber-Novikov operations in algebraic cobordism and a comparison of rational algebraic cobordism with rational Chow groups.

Throughout this section, $\mathcal{V}$ will be an admissible subcategory of $\mathbf{Sch}_k$.

4.1.1 Axioms for weak homology

We introduce three axioms concerning an oriented Borel-Moore functor A_* on $\mathcal{V}$:

(PB). Given a rank n vector bundle $E \to X$ on $X \in \mathcal{V}$ with sheaf of sections $\mathcal{E}$, let $q : \mathbb{P}(\mathcal{E}) \to X$ denote the projective bundle, and let $O(1)_E \to \mathbb{P}(\mathcal{E})$ be the canonical quotient line bundle of q^*E. For each $i \in \{0, \dots, n\}$, let

$$\xi^{(i)} : A_{*+i-n}(X) \to A_*(\mathbb{P}(\mathcal{E}))$$

be the composition

$$A_{*+i-n}(X) \xrightarrow{q^*} A_{*+i}(\mathbb{P}(\mathcal{E})) \xrightarrow{\tilde{c}_1(O(1)_E)^i} A_*(\mathbb{P}(E)).$$

Then the homomorphism

$$\Sigma_{i=0}^{n-1} \xi^{(i)} : \oplus_{i=0}^{n-1} A_{*+i-n}(X) \to A_*(\mathbb{P}(\mathcal{E}))$$

is an isomorphism.

(EH). Let $E \to X$ be a vector bundle of rank r over $X \in \mathcal{V}$, and let $p : V \to X$ be an E-torsor. Then $p^* : A_*(X) \to A_{*+r}(V)$ is an isomorphism.

(Loc). Let $L \to X$ be a line bundle on $X \in \mathcal{V}$. Suppose that L admits a section $s : X \to L$ such that s is transverse in $\mathcal{V}$ to the zero-section of L; let $i : D \to X$ be the subscheme defined by s. Then the image of $\tilde{c}_1 : A_*(X) \to A_{*-1}(X)$ is contained in the image of $i_* : A_{*-1}(D) \to A_{*-1}(X)$.

If (PB) holds, we shall say that A_* satisfies the projective bundle formula.

Remark 4.1.1. The axiom (Loc) is used to prove the Whitney product formula for the total Chern class (see proposition 4.1.15 below). Applying (Loc) to the case of a nowhere vanishing section of a trivial bundle (with $D = \emptyset$), we see that (Loc) implies the property

(Triv). Let $L \to X$ be the trivial bundle. Then $\tilde{c}_1(L)$ is the zero map.

On the other hand, in the presence of (Triv), (Loc) is a consequence of the axiom

(Loc′). Let $i : D \to X$ be a closed embedding in $\mathcal{V}$ with open complement $j : U \to X$. Then the sequence

$$A_*(D) \xrightarrow{i_*} A_*(X) \xrightarrow{j^*} A_*(U)$$

is exact.

Indeed, if $L \to X$ admits a section s with zero-locus contained in D, then clearly $j^*L \to U$ is the trivial bundle. By (Triv), $\tilde{c}_1(j^*L) = 0$; since $\tilde{c}_1(j^*L)(j^*\eta) = j^*(\tilde{c}_1(L)(\eta))$, it follows from (Loc′) that the image of $\tilde{c}_1(L)$ is contained in the image of i_*.

The axiom (Loc′) appears in many discussions of oriented (co)homology throughout the literature, for instance, in the work of Panin [27].

Remark 4.1.2 (The splitting principle). Let A_* be an oriented Borel-Moore functor with product on an admissible $\mathcal{V}$. Suppose in addition that A_* satisfies the axioms (PB) and (EH). Take $X \in \mathcal{V}$ and a vector bundle $E \to X$. Then there is a smooth morphism (of relative dimension say d) $f : X' \to X$ such that $f^* : A_*(X) \to A_{*+d}(X')$ is injective and $f^*(E)$ is a sum of line bundles. Indeed, let $q : \mathcal{F}l(E) \to X$ be the full flag variety. Since q factors as a tower of projective space bundles $\mathbb{P}(E_i)$ for appropriate vector bundles E_i, the projective bundle formula (PB) implies that q^* is injective. Let $p : \mathcal{S}pl(q^*E) \to \mathcal{F}l(E)$ be the bundle of splittings of the universal flag on q^*E. Then p is a torsor for the vector bundle of Hom's of the quotient line bundles in the universal flag, hence by (EH) the map p^* is an isomorphism. As the universal flag on q^*E clearly splits after pull-back by p, the composition $q \circ p : \mathcal{S}pl(q^*E) \to X$ does the job.

Similarly, if $0 \to E' \to E \to E'' \to 0$ is a short exact sequence of vector bundles on X, then there is a smooth morphism $f : X' \to X$ such that $f^* : A_*(X) \to A_{*+d}(X')$ is injective, $f^*(E')$ and $f^*(E'')$ are sums of line bundles, and the sequence $0 \to f^*E' \to f^*E \to f^*E'' \to 0$ splits.

In what follows A_* denotes an oriented Borel-Moore functor with product on an admissible $\mathcal{V}$, which moreover satisfies axioms (Sect), (PB) and (EH), unless specific mention to the contrary is made. We will freely use the associated weak cohomological functor $Y \to A^*(Y)$ for smooth k-schemes Y.

4.1.2 The axiom (Nilp)

The following lemma establishes axiom (Nilp) of remark 2.2.3 for A_*.

Lemma 4.1.3. *Let A_* be an oriented Borel-Moore functor with product on $\mathbf{Sm}_k$, which moreover satisfies axioms (Sect) and (EH). Let Y be in $\mathbf{Sm}_k$. Then there is an integer N_Y such that for any family $(L_1, \ldots, L_n)$ of line bundles on Y with $n > N_Y$, one has*

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n)(1_Y) = 0 \in A_*(Y)$$

Proof. Using Jouanolou's trick [14] we see there is a vector bundle $E \to Y$ of rank r, and a torsor $\pi : T \to Y$ under $E \to Y$ such that T is an affine (and smooth) scheme. Set $N_Y := \dim(T) = \dim(Y) + r$. We claim this integer satisfies the property. By the homotopy axiom (EH) for A_* we see that $\pi^* : A_*(Y) \to A_{*+d}(T)$ is an isomorphism. Thus it suffices to show that $\tilde{c}_1(\pi^* L_1) \circ \cdots \circ \tilde{c}_1(\pi^* L_n)(1_T) = 0 \in A_*(T)$. Because T is affine each $\pi^*(L_i)$ is very ample, hence lemma 2.3.9 implies the result. □

4.1.3 Functorialities

Let $i : \mathbb{P}^{n-r} \to \mathbb{P}^n$ be a linear embedding and let X be in $\mathcal{V}$. We define a pull-back map

$$(\mathrm{Id}_X \times i)^* : A_*(X \times_k \mathbb{P}^n) \to A_{*-r}(X \times_k \mathbb{P}^{n-r}),$$

that enjoys the usual functorialities.

For this, let $\mathbb{P}^{r-1} \to \mathbb{P}^n$ be a linearly embedding projective subspace, with $\mathbb{P}^{r-1} \cap \mathbb{P}^{n-r} = \emptyset$, let $j : U \to \mathbb{P}^n$ be the inclusion of the complement $\mathbb{P}^n \setminus \mathbb{P}^{r-1}$, and let $\pi : U \to \mathbb{P}^{n-r}$ be the linear projection with center $\mathbb{P}^{r-1}$. The projection π makes U into a rank r vector bundle over $\mathbb{P}^{n-r}$; in particular, the pull-back map

$$(\mathrm{Id}_X \times \pi)^* : A_{*-r}(X \times \mathbb{P}^{n-r}) \to A_*(X \times U)$$

is defined and is an isomorphism. We then set $(\mathrm{Id}_X \times i)^* := (\mathrm{Id}_X \times \pi^*)^{-1} \circ j^*$.

The evident extension of this procedure allows one to define the pull-back (where $r = \sum r_j$)

$$A_*(X \times \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_s}) \xrightarrow{(\mathrm{Id}_X \times i_1 \times \ldots \times i_s)^*} A_{*-r}(X \times \mathbb{P}^{n_1 - r_1} \times \ldots \times \mathbb{P}^{n_s - r_s}),$$

given linear embeddings $i_j : \mathbb{P}^{n_j - r_j} \to \mathbb{P}^{n_j}$, $j = 1, \ldots, s$.

The following list of properties is easily checked; we leave the details to the reader:

1. $(\mathrm{Id}_X \times i)^*$ is independent of the choice of complementary subspace $\mathbb{P}^{r-1} \to \mathbb{P}^n$.
2. For a composite of linear embeddings $i_0 : \mathbb{P}^{n_0} \to \mathbb{P}^{n_1}$, $i_1 : \mathbb{P}^{n_1} \to \mathbb{P}^{n_2}$, we have
$$(\mathrm{Id}_X \times i_0)^* \circ (\mathrm{Id}_X \times i_1)^* = (\mathrm{Id}_X \times (i_1 \circ i_0))^*.$$
3. Let $i : \mathbb{P}^{n-r} \to \mathbb{P}^n$ be a linear embedding, $L \to X \times \mathbb{P}^n$ a line bundle, and $\eta \in A_*(X \times \mathbb{P}^n)$. Then
$$i^*(\tilde{c}_1(L)(\eta)) = \tilde{c}_1(i^* L)(i^* \eta).$$
4. $(\mathrm{Id}_X \times i)^*(a \times b) = a \times i^*(b)$, for $a \in A_*(X)$, $b \in A_*(\mathbb{P}^n)$.
5. Given linear embeddings $i_j : \mathbb{P}^{n_j - r_j} \to \mathbb{P}^{n_j}$, $j = 1, \ldots, s$, the pull-back $(\mathrm{Id}_X \times i_1 \times \ldots \times i_s)^*$ is the composition of the pull-backs corresponding to the s individual linear embeddings, taken in any order.

4.1.4 The formal group law

Lemma 4.1.4. *Let $n \geq 0$ be an integer.*

(1) For $X \in \mathcal{V}$, each element $\eta \in A_*(X \times \mathbb{P}^n)$ is a sum

$$\eta = \sum_{i=0}^{n} \eta_i \cdot [\mathbb{P}^{n-i} \to \mathbb{P}^n],$$

with the η_i elements of $A_*(X)$ uniquely determined by η.

(2) $A_*(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r})$ is the free $A_*(k)$-module on the classes

$$[\mathbb{P}^{n_1-i_1} \to \mathbb{P}^{n_1}] \times \ldots \times [\mathbb{P}^{n_r-i_r} \to \mathbb{P}^{n_r}]; \quad 0 \leq i_j \leq n_j, j = 1, \ldots, r,$$

where for $i \in \{0, \ldots, m\}$, $\mathbb{P}^{m-i} \to \mathbb{P}^m$ denotes (any choice of) a linearly embedded projective space of dimension $m - i$.

Proof. (2) follows from (1) by induction on r. To prove (1), take $\eta \in A_*(X \times \mathbb{P}^n)$. Writing $X \times \mathbb{P}^n$ as $\mathbb{P}(\mathcal{O}_X^{n+1})$, we have the canonical quotient line bundle $\gamma_{X,n} := O(1)$ on $X \times \mathbb{P}^n$. From axiom (PB) applied to the trivial vector bundle of rank $n+1$ over X , there are uniquely determined elements $\eta_i \in A_*(X)$ with

$$\eta = \sum_{i=0}^{n} \tilde{c}_1(\gamma_{X,n})^i(p_1^*\eta_i).$$

Let $\pi_n : \mathbb{P}^n \to \operatorname{Spec} k$ be the structure morphism. Since $\gamma_{X,n} = p_2^*\gamma_n$, and $p_1^*\eta_i = \eta_i \cdot \pi_n^*(1)$, we have

$$\tilde{c}_1(\gamma_{X,n})^i(p_1^*\eta_i) = \eta_i \cdot \tilde{c}_1(\gamma_n)^i(\pi_n^*(1)).$$

Thus, it suffice to show that

$$\tilde{c}_1(\gamma_n)^i(\pi_n^*1) = [\mathbb{P}^{n-i} \to \mathbb{P}^n].$$

For $i = 1$, this follows from the axiom (Sect), together with the fact that each hyperplane $\mathbb{P}^{n-1} \to \mathbb{P}^n$ is defined by a transverse section of γ_n. For $i > 1$, let $\iota : \mathbb{P}^{n-i+1} \to \mathbb{P}^n$ be a linear embedding. By induction on i we have

$$\begin{aligned}
\tilde{c}_1(\gamma_n)^i(\pi_n^*1) &= \tilde{c}_1(\gamma_n)\big(\tilde{c}_1(\gamma_n)^{i-1}(\pi^*1)\big) \\
&= \tilde{c}_1(\gamma_n)\big([\mathbb{P}^{n-i+1} \to \mathbb{P}^n]\big) \\
&= \tilde{c}_1(\gamma_n)\big(\iota_*(\pi_{n-i+1}^*(1))\big) \\
&= \iota_*(\tilde{c}_1(\gamma_{n-i+1})(\pi_{n-i+1}^*(1))) \\
&= \iota_*([\mathbb{P}^{n-i} \to \mathbb{P}^{n-i+1}] \\
&= [\mathbb{P}^{n-i} \to \mathbb{P}^n],
\end{aligned}$$

and the induction goes through. □

Remark 4.1.5. From this lemma we easily deduce that $\tilde{c}_1(\gamma_n)^{n+1} = 0$. Indeed, using the axiom (Sect), one has $\tilde{c}_1(\gamma_n)([\mathbb{P}^i \to \mathbb{P}^n]) = [\mathbb{P}^{i-1} \to \mathbb{P}^n]$ unless $i = 0$ in which case $\tilde{c}_1(\gamma_n)([\mathbb{P}^0 \to \mathbb{P}^n]) = 0$.

Remark 4.1.6. Let $\pi : \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r} \to \operatorname{Spec} k$ be the structure morphism. The proof of the lemma yields the formula:

$$[\mathbb{P}^{n_1-i_1} \subset \mathbb{P}^{n_1}] \times \ldots \times [\mathbb{P}^{n_r-i_r} \subset \mathbb{P}^{n_r}] = \tilde{c}_1(p_1^*\gamma_{n_1})^{i_1} \circ \ldots \circ \tilde{c}_1(p_1^*\gamma_{n_r})^{i_r}(\pi^*(1)).$$

It is also easy to check that, given a linear embedding $i : \mathbb{P}^{n-r} \to \mathbb{P}^n$, we have

$$i^*([\mathbb{P}^s \to \mathbb{P}^n]) = \begin{cases} [\mathbb{P}^{s-r} \to \mathbb{P}^{n-r}] & \text{for } r \leq s \\ 0 & \text{for } r > s. \end{cases}$$

Indeed, if $r > s$, we can take the $\mathbb{P}^s$ inside the complementary linear space $\mathbb{P}^r$, and if $r \leq s$, we can take the $\mathbb{P}^s$ to be the closure of $\pi^*(\mathbb{P}^{s-r})$, for some linearly embedded $\mathbb{P}^{s-r} \to \mathbb{P}^r$.

Remark 4.1.7. Let X be in $\mathcal{V}$, and let $i : X \to X \times \mathbb{P}^n$ be the closed embedding $i(x) = x \times \mathbb{P}^0$, where $\mathbb{P}^0$ is a chosen k-rational point of $\mathbb{P}^n$. Let $j : \mathbb{P}^{n-1} \to \mathbb{P}^n$ be a linear embedding. It follows directly from lemma 4.1.4 and remark 4.1.6 that the sequence

$$0 \to A_*(X) \xrightarrow{i_*} A_*(X \times \mathbb{P}^n) \xrightarrow{j^*} A_*(X \times \mathbb{P}^{n-1}) \to 0$$

is exact. Additionally, one has $i_*(\eta) = \eta \cdot [\mathbb{P}^0 \to \mathbb{P}^n]$.

Corollary 4.1.8. *There is a unique power series*

$$F_{\mathbb{A}}(u, v) = \sum_{i,j} a_{i,j}\, u^i\, v^j \in A_*(k)[[u, v]]$$

with $a_{i,j} \in A_{i+j-1}(k)$, such that, for any integers $n > 0$ and $m > 0$ we have in the endomorphism ring of $A_*(\mathbb{P}^n \times \mathbb{P}^m)$:

$$F_{\mathbb{A}}(\tilde{c}_1(pr_1^*(\gamma_n)), \tilde{c}_1(pr_2^*(\gamma_m))) = \tilde{c}_1(pr_1^*(\gamma_n) \otimes pr_2^*(\gamma_m)). \tag{4.1}$$

Moreover, $(A_*(k), F_{\mathbb{A}}(u, v))$ is a commutative formal group law.

Proof. Let $n > 0$ and $m > 0$ be integers and consider the line bundle $O_{n,m}(a, b) := pr_1^*(\gamma_n^{\otimes a}) \otimes pr_2^*(\gamma_m^{\otimes b})$ over $\mathbb{P}^n \times \mathbb{P}^m$. Let $\pi_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \operatorname{Spec} k$ be the structure morphism. We write $1_{n,m}$ for $\pi_{n,m}^*(1_{\operatorname{Spec} k})$.

By lemma 4.1.4, we can write

$$\tilde{c}_1(O_{n,m}(1, 1))(1_{n,m}) = \sum_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}} a_{i,j}^{n,m} [\mathbb{P}^{n-i} \to \mathbb{P}^n] \times [\mathbb{P}^{m-j} \to \mathbb{P}^m],$$

for unique elements $a_{i,j}^{n,m} \in A_{i+j-1}(k)$. We first check that the $a_{i,j}^{n,m}$ are independent of (n, m). Indeed, if $n \leq N$, $m \leq M$, we have linear embeddings

$i_1 : \mathbb{P}^n \to \mathbb{P}^N$, $i_2 : \mathbb{P}^m \to \mathbb{P}^M$. By checking on basis elements and using the properties of $(i_1 \times i_2)^*$ described above, we

$$\begin{aligned}\tilde{c}_1(O_{n,m}(1,1))(1_{n,m}) &= \tilde{c}_1((i_1 \times i_2)^* O_{N,M}(1,1))((i_1 \times i_2)^* 1_{N,M}) \\ &= (i_1 \times i_2)^*(\tilde{c}_1(O_{N,M}(1,1))(1_{N,M})) \\ &= \sum_{\substack{0 \le i \le N \\ 0 \le j \le M}} a_{i,j}^{N,M} \cdot i_1^*[\mathbb{P}^{N-i} \to \mathbb{P}^N] \times i_2^*[\mathbb{P}^{M-j} \to \mathbb{P}^M] \\ &= \sum_{\substack{0 \le i \le n \\ 0 \le j \le m}} a_{i,j}^{N,M} \cdot [\mathbb{P}^{n-i} \to \mathbb{P}^n] \times [\mathbb{P}^{m-j} \to \mathbb{P}^m].\end{aligned}$$

Equating coefficients with respect to our basis for $A_*(\mathbb{P}^n \times \mathbb{P}^m)$ over $A_*(k)$ yields $a_{i,j}^{n,m} = a_{i,j}^{N,M}$ for $0 \le i \le n$, $0 \le j \le m$, as desired. We may therefore write $a_{i,j}$ for $a_{i,j}^{n,m}$.

Next, we claim that

$$\tilde{c}_1(O_{n,m}(1,1)) = \sum_{\substack{0 \le i \le n \\ 0 \le j \le m}} a_{i,j} \tilde{c}_1(O_{n,m}(1,0))^i \circ \tilde{c}_1(O_{n,m}(0,1))^j$$

as endomorphisms of $A_*(\mathbb{P}^n \times \mathbb{P}^m)$. As both sides are $A_*(k)$-linear, we need only check on the basis elements $[\mathbb{P}^r \to \mathbb{P}^n][\mathbb{P}^s \to \mathbb{P}^m]$. Letting $i_1 : \mathbb{P}^r \to \mathbb{P}^n$ and $i_2 : \mathbb{P}^s \to \mathbb{P}^m$ be linear embeddings, we have

$$\begin{aligned}&\tilde{c}_1(O_{n,m}(1,1))([\mathbb{P}^r \to \mathbb{P}^n][\mathbb{P}^s \to \mathbb{P}^m]) \\ &\quad = \tilde{c}_1(O_{n,m}(1,1))((i_1 \times i_2)_*(1_{r,s})) \\ &\quad = (i_1 \times i_2)_*(\tilde{c}_1(O_{r,s}(1,1))(1_{r,s})) \\ &\quad = \sum_{\substack{0 \le i \le r \\ 0 \le j \le s}} a_{i,j} \cdot [\mathbb{P}^{r-i} \to \mathbb{P}^n][\mathbb{P}^{s-j} \to \mathbb{P}^m].\end{aligned}$$

Similarly

$$\begin{aligned}&\tilde{c}_1(O_{n,m}(1,0))^i \circ \tilde{c}_1(O_{n,m}(0,1))^j([\mathbb{P}^r \to \mathbb{P}^n][\mathbb{P}^s \to \mathbb{P}^m]) \\ &\quad = (i_1 \times i_2)_*(\tilde{c}_1(O_{r,s}(1,0))^i \circ \tilde{c}_1(O_{r,s}(0,1))^j(1_{r,s}) \\ &\quad = \begin{cases} [\mathbb{P}^{r-i} \to \mathbb{P}^n][\mathbb{P}^{s-j} \to \mathbb{P}^m] & \text{if } i \le r, j \le s \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

This verifies our claim.

Setting $F_{\mathbb{A}}(u,v) = \sum_{ij} a_{ij} u^i v^j$ and noting that $\tilde{c}_1(\gamma_n)^{n+1} = 0$ (remark 4.1.5) completes the proof of formula (4.1). We now show that $F_{\mathbb{A}}$ defines a formal group law.

Taking $n = 0$ or $m = 0$, and using (4.1), we see that $a_{i0} = 0 = a_{0j}$ for $i, j \ge 2$, and that $a_{10} = a_{01} = 1$. Similarly, using lemma 4.1.4, the isomorphism

$\tau^* O_{n,m}(1,1) \cong O_{m,n}(1,1)$, where $\tau : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^m$ is the exchange of factors, yields the commutativity $F_A(u,v) = F_A(v,u)$.

For associativity, let $O_{n,m,\ell}(a,b,c)$ be the line bundle $p_1^*\gamma_n^{\otimes a} \otimes p_2^*\gamma_m^{\otimes b} \otimes p_3^*\gamma_\ell^{\otimes c}$ on $\mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^\ell$. The same argument as above gives us a unique power series $G_A(u,v,w) = \sum a_{ijk} u^i v^j w^k$ with

$$G_A(\tilde{c}_1(O_{n,m,\ell}(1,0,0)), \tilde{c}_1(O_{n,m,\ell}(0,1,0)), \tilde{c}_1(O_{n,m,\ell}(0,0,1))) \\ = \tilde{c}_1(O_{n,m,\ell}(1,1,1)).$$

We claim that

$$F_A(u, F_A(v,w)) = G_A(u,v,w) = F_A(F_A(u,v),w).$$

It suffices to prove the first equality. To simplify the notation, we write $O(a,b,c)$ for $O_{n,m,\ell}(a,b,c)$.

Arguing as in the proof of formula (4.1) reduces us to showing

$$F_A(\tilde{c}_1(O(1,0,0)), F_A(\tilde{c}_1(O(0,1,0)), \tilde{c}_1(O(0,0,1))))(1_{n,m,\ell}) \\ = G_A(\tilde{c}_1(O(1,0,0)), \tilde{c}_1(O(0,1,0)), \tilde{c}_1(O(0,0,1)))(1_{n,m,\ell})$$

for all n, m, ℓ, where $1_{n,m,\ell}$ is the pull-back of $1 \in A_*(k)$ to $A_*(\mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^\ell)$ via the structure morphism. For $\ell = 0$, this identity follows directly from formula (4.1); we proceed by induction on ℓ. If we write

$$F_A(u, F_A(v,w)) = \sum_{ijk} a'_{ijk} u^i v^j w^k,$$

our induction hypothesis, along with the exact sequence of remark 4.1.7, implies that

$$F_A(\tilde{c}_1(O(1,0,0)), F_A(\tilde{c}_1(O(0,1,0)), \tilde{c}_1(O(0,0,1))))(1_{n,m,\ell}) \\ - G_A(\tilde{c}_1(O(1,0,0)), \tilde{c}_1(O(0,1,0)), \tilde{c}_1(O(0,0,1)))(1_{n,m,\ell}) \\ = \sum_{ij} (a'_{ij0} - a_{ij0})[\mathbb{P}^i \to \mathbb{P}^n][\mathbb{P}^j \times \mathbb{P}^0 \to \mathbb{P}^m \times \mathbb{P}^\ell].$$

Let $\iota : \mathbb{P}^m \times \mathbb{P}^\ell \to \mathbb{P}^N$ be the Segre embedding ($N = (m+1)(\ell+1) - 1$). Since $\iota_*([\mathbb{P}^j \times \mathbb{P}^0 \to \mathbb{P}^m \times \mathbb{P}^\ell]) = [\mathbb{P}^j \to \mathbb{P}^N]$, $(\mathrm{Id} \times \iota)_* : A_*(\mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^\ell) \to A_*(\mathbb{P}^n \times \mathbb{P}^N)$ is injective on the $A_*(k)$ submodule generated by the classes $[\mathbb{P}^i \to \mathbb{P}^n][\mathbb{P}^j \times \mathbb{P}^0 \to \mathbb{P}^m \times \mathbb{P}^\ell]$. Thus we need only check our identity after pushing forward to $A_*(\mathbb{P}^n \times \mathbb{P}^N)$.

By smooth functoriality, we have

$$F_A(\tilde{c}_1(O(0,1,0), \tilde{c}_1(O(0,0,1)) = \tilde{c}_1(O(0,1,1)),$$

so

$$F_A(\tilde{c}_1(O(1,0,0)), F_A(\tilde{c}_1(O(0,1,0)), \tilde{c}_1(O(0,0,1))))(1_{n,m,\ell}) \\ = F_A(\tilde{c}_1(O(1,0,0)), \tilde{c}_1(O(0,1,1))(1_{n,m,\ell}).$$

Similarly,

$$G_A(\tilde{c}_1(O(1,0,0)), \tilde{c}_1(O(0,1,0)), \tilde{c}_1(O(0,0,1)))(1_{n,m,\ell}) \\ = \tilde{c}_1(O(1,1,1))(1_{n,m,\ell}).$$

Since $O_{n,m,\ell}(a,1,1) = (\mathrm{Id} \times \iota)^*(O_{n,N}(a,1))$, we have

$$\begin{aligned} &(\mathrm{Id} \times \iota)_*(F_A(\tilde{c}_1(O(1,0,0)), \tilde{c}_1(O(0,1,1))(1_{n,m,\ell})) \\ &\qquad = F_A(\tilde{c}_1(O_{n,N}(1,0), \tilde{c}_1(O_{n,N}(0,1))((\mathrm{Id} \times \iota)_*(1_{n,m,\ell})), \\ &(\mathrm{Id} \times \iota)_*(\tilde{c}_1(O(1,1,1))(1_{n,m,\ell})) \\ &\qquad = \tilde{c}_1(O_{n,N}(1,1))((\mathrm{Id} \times \iota)_*(1_{n,m,\ell})). \end{aligned}$$

Thus, we need only check that

$$F_A(\tilde{c}_1(O_{n,N}(1,0), \tilde{c}_1(O_{n,N}(0,1))((\mathrm{Id} \times \iota)_*(1_{n,m,\ell})) \\ = \tilde{c}_1(O_{n,N}(1,1))((\mathrm{Id} \times \iota)_*(1_{n,m,\ell})).$$

This this follows from our formula (4.1). □

4.1.5 Oriented Borel-Moore weak homology theories

Definition 4.1.9. *Let $\mathcal{V}$ be an admissible subcategory of* $\mathbf{Sch}_k$. *An* oriented Borel-Moore weak homology theory on $\mathcal{V}$ *is an oriented Borel-Moore functor on $\mathcal{V}$ with product, A_*, which satisfies the axiom (Sect), the axioms (PB), (EH) and (Loc) of §4.1.1, as well as the axiom (FGL) for the formal group law $F_A(u,v)$ given by corollary 4.1.8.*

Remarks 4.1.10. (1) One observes that, because of lemma 4.1.3, A_* satisfies the axiom (Nilp), so that one can make sense of the axiom (FGL). By theorem 2.3.13, we see that A_* satisfies axiom (Dim), so A_* is of geometric type on $\mathcal{V}$.

(2) Conversely, suppose we have an oriented Borel-Moore $\mathbb{L}_*$-functor A_* on $\mathcal{V}$ of geometric type, satisfying the axioms (PB), (EH) and (Loc). Then A_* satisfies (FGL) for the formal group law $F(u,v)$ given by the homomorphism $\mathbb{L}_* \to A_*(k)$ defining the $\mathbb{L}_*$-structure. By the uniqueness in corollary 4.1.8, F is equal to the formal group law F_A given by that corollary. Thus, an oriented Borel-Moore weak homology theory on $\mathcal{V}$ is the same thing as an oriented Borel-Moore $\mathbb{L}_*$-functor on $\mathcal{V}$, of geometric type, and satisfying the axioms (PB), (EH) and (Loc).

Theorem 4.1.11. *Let k be a field admitting resolution of singularities and let $\mathcal{V}$ be an admissible subcategory of* $\mathbf{Sch}_k$. *Then algebraic cobordism*

$$X \mapsto \Omega_*(X)$$

is an oriented Borel-Moore weak homology theory on $\mathcal{V}$. Moreover, Ω_* is the universal oriented Borel-Moore weak homology theory on $\mathcal{V}$.

Proof. This follows directly from the results proven in chapter 3. □

Remark 4.1.12. We call a formal group law F on a ring R *multiplicative* if $F(u,v) = u + v - buv$ for some element $b \in R$, *additive* if $F(u,v) = u + v$. We say a multiplicative formal group on R is *periodic* if b is invertible in R. Thus, the universal multiplicative formal group is classified by the homomorphism $\mathbb{L} \to \mathbb{Z}[\beta]$ sending a_{11} to $-\beta$, and a_{ij} to 0 for $(i,j) \neq (1,1)$, where β is an indeterminant. Similarly, the universal periodic multiplicative formal group is given by $\mathbb{L} \to \mathbb{Z}[\beta, \beta^{-1}]$, and the universal additive formal group is $\mathbb{L} \to \mathbb{Z}$, sending all a_{ij} to zero. From theorem 4.1.11, it follows that $\Omega_* \otimes_{\mathbb{L}} \mathbb{Z}[\beta]$, $\Omega_* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}]$ and $\Omega_* \otimes_{\mathbb{L}} \mathbb{Z}$ are respectively the universal multiplicative, periodic multiplicative and additive oriented Borel-Moore weak homology theories on $\mathbf{Sm}_k$ and on $\mathbf{Sch}_k$.

The Chow groups

$$X \mapsto \mathrm{CH}_*(X)$$

and the G-theory functor

$$X \mapsto G_0(X)[\beta, \beta^{-1}]$$

are oriented Borel-Moore weak homology theories on $\mathbf{Sch}_k$ as well. CH_* is additive and $G_0[\beta, \beta^{-1}]$ is multiplicative and periodic.

4.1.6 Oriented weak cohomology theories

We consider the cohomological version of an oriented Borel-Moore weak homology theory, and relate this to oriented cohomology.

Definition 4.1.13. *An* oriented weak cohomology theory *on* $\mathbf{Sm}_k$ *is an oriented cohomological functor with product A^* such that the associated oriented Borel-Moore functor A_* is an oriented Borel-Moore weak homology theory on* $\mathbf{Sm}_k$.

For instance, any oriented cohomology theory on $\mathbf{Sm}_k$ in the sense of the introduction defines an oriented weak cohomology theory on $\mathbf{Sm}_k$. For a detailed proof of this, see proposition 5.2.4 below.

Remark 4.1.14. The essential difference between the notion of oriented weak cohomology theory on $\mathbf{Sm}_k$ and that of oriented cohomology theory A^* on $\mathbf{Sm}_k$ is the existence of a pull-back $f^* : A^*(X) \to A^*(Y)$ for each morphism $f : Y \to X$ between smooth k-schemes For instance, one can recover the ring structure on $A^*(X)$ as the external product $A^*(X) \otimes A^*(X) \to A^*(X \times X)$ followed by pull back along the diagonal: $\Delta^* : A^*(X \times X) \to A^*(X)$.

4.1.7 Chern classes and Conner-Floyd Chern classes

In this section A_* will be an oriented Borel-Moore weak homology theory on an admissible subcategory $\mathcal{V}$.

Let $E \to X$ be a vector bundle of rank n over $X \in \mathcal{V}$ with sheaf of sections $\mathcal{E}$. One observes that the isomorphism in (PB) implies the existence of unique homomorphisms

$$\tilde{c}_i(E) : A_*(X) \to A_{*-i}(X)$$

for $i \in \{0, \dots, n\}$, with $\tilde{c}_0(E) = 1$, and satisfying the equation (as a homomorphism $A_*(X) \to A_{*-n}(\mathbb{P}(\mathcal{E}))$):

$$\sum_{i=0}^{n} (-1)^i \tilde{c}_1(O(1)_E)^{n-i} \circ q^* \circ \tilde{c}_i(E) = 0.$$

The homomorphism $\tilde{c}_i(E)$ is called the i-th Chern class operator of E.

We will use a modification of Grothendieck's arguments from [11] to verify the expected properties of the Chern class operators:

Proposition 4.1.15. *Let A_* be an oriented Borel-Moore weak homology theory on $\mathcal{V}$. Then the Chern class operators satisfy the following properties:*

(0) Given vector bundles $E \to X$ and $F \to X$ on $X \in \mathcal{V}$ one has

$$\tilde{c}_i(E) \circ \tilde{c}_j(F) = \tilde{c}_j(F) \circ \tilde{c}_i(E)$$

for all i, j.

(1) For any line bundle L, $\tilde{c}_1(L)$ agrees with the one given in the structure of an oriented Borel-Moore weak homology theory for A_*.

(2) For any smooth equi-dimensional morphism $Y \to X$ in $\mathcal{V}$ and any vector bundle $E \to X$ over X, one has

$$\tilde{c}_i(f^*E) \circ f^* = f^* \circ \tilde{c}_i(E).$$

(3) If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles over X, then for each integer $n \geq 0$ one has the following equation in $End(A_*(X))$:

$$\tilde{c}_n(E) = \sum_{i=0}^{n} \tilde{c}_i(E') \, \tilde{c}_{n-i}(E'').$$

(4) For any projective morphism $Y \to X$ in $\mathcal{V}$, and any vector bundle $E \to X$ over X one has

$$f_* \circ \tilde{c}_i(f^*E) = \tilde{c}_i(E) \circ f_*.$$

Moreover the Chern class operators are characterized by (0-3).

Remark 4.1.16. We will see below (proposition 5.2.4) that each oriented cohomology theory on $\mathbf{Sm}_k$ defines an oriented weak cohomology theory on $\mathbf{Sm}_k$. Given an oriented cohomology theory A^* on $\mathbf{Sm}_k$, the procedure we use here to construct Chern classes can be applied *mutatis mutandis* to define Chern classes $c_i(E) \in A^i(X)$ of a vector bundle E on $X \in \mathbf{Sm}_k$. The relationship between the two approaches is that for a vector bundle E on a smooth scheme X one has

$$c_i(E) = \tilde{c}_i(E)(1_X).$$

Conversely, one recovers the Chern class operators constructed here,

$$\tilde{c}_i(E) : A^*(X) \to A^{*+i}(X),$$

as the cup product by $c_i(E)$. For a general oriented weak cohomology theory, we don't have the (internal) cup product in the structure, so this formula would have no meaning.

As an example, the axiom (Triv): $\tilde{c}_1(O_X) = 0$ would follow from the previous formula and the obvious fact that $c_1(O_X) = 0$. But in general it doesn't hold.

The key to applying Grothendieck's arguments for the properties of the Chern classes in the case of an oriented Borel-Moore weak homology theory is given by the following two lemmas.

Lemma 4.1.17. *Let A_* be an oriented Borel-Moore functor on $\mathcal{V}$ satisfying the axiom (Loc). Let X be in $\mathcal{V}$, and let $D_1, \ldots, D_n$ be effective Cartier divisors on X such that for each set of distinct indices $1 \leq i_1, \ldots, i_r \leq n$, the intersection $D_{i_1} \cap (\cap_{j=2}^r D_{i_j})$ is transverse in $\mathcal{V}$, i.e., the cartesian square*

$$\begin{tikzcd}
\cap_{j=1}^r D_{i_j} \arrow[r] \arrow[d] & \cap_{j=2}^r D_{i_j} \arrow[d] \\
D_{i_1} \arrow[r] & X
\end{tikzcd}$$

is transverse in $\mathcal{V}$. Suppose in addition that $\cap_{i=1}^n D_i = \emptyset$. Then

$$\prod_{i=1}^n \tilde{c}_1(\mathcal{O}_X(D_i)) = 0$$

as an operator on $A_*(X)$.

Proof. Let $\tilde{i}^j : D_1 \cap \ldots \cap D_j \to D_1 \cap \ldots \cap D_{j-1}$, $i^j : D_1 \cap \ldots \cap D_j \to X$ be the inclusions. By our assumption, i^j is a closed embedding in $\mathcal{V}$. Note that the canonical section s_j of $i^{j-1*}\mathcal{O}_X(D_j)$ has divisor $D_1 \cap \ldots \cap D_j$, hence s_j is transverse in $\mathcal{V}$ to the zero-section of $i^{j-1*}\mathcal{O}_X(D_j)$. By (Loc), it follows that $\tilde{c}_1(i^{j-1*}\mathcal{O}_X(D_j))$ maps $A_*(D_1 \cap \ldots \cap D_{j-1})$ to $\tilde{i}^j_*(A_{*-1}(D_1 \cap \ldots \cap D_j))$. Since

$i_*^{j-1}\circ\tilde{c}_1(i^{j-1*}\mathcal{O}_X(D_j)) = \tilde{c}_1(\mathcal{O}_X(D_j))\circ i_*^{j-1}$, it follows that $\tilde{c}_1(\mathcal{O}_X(D_j))$ maps $i_*^{j-1}(A_*(D_1\cap\ldots\cap D_{j-1}))$ to $i_*^j(A_*(D_1\cap\ldots\cap D_j))$, and hence by induction

$$[\prod_{i=1}^{j}\tilde{c}_1(\mathcal{O}_X(D_i))](A_*(X)) \subset i_*^j(A_*(D_1\cap\ldots\cap D_j)).$$

for all j. Since $A_*(\emptyset)=0$, this proves the result. □

Lemma 4.1.18. *Let A_* be an oriented Borel-Moore weak homology theory on $\mathcal{V}$. Let X be in $\mathcal{V}$, $D_1,\ldots,D_n$ effective Cartier divisors on X satisfying the transversality condition of lemma 4.1.17, and with $\cap_{i=1}^n D_i=\emptyset$. Let $L_1,\ldots,L_n$ be line bundles on X, and let $M_i = L_i\otimes\mathcal{O}_X(D_i)$, $i=1,\ldots,n$. Then*

$$\prod_{i=1}^{n}(\tilde{c}_1(M_i)-\tilde{c}_1(L_i)) = 0$$

as an operator on $A_*(X)$.

Proof. We use the formal group law $F_A(u,v) = u+v+\sum_{p,q\geq 1} a_{pq}u^pv^q$. $M_i = L_i\otimes\mathcal{O}_X(D_i)$, so

$$\begin{aligned}\tilde{c}_1(M_i) &= F_A(\tilde{c}_1(L_i),\tilde{c}_1(\mathcal{O}_X(D_i))\\ &= \tilde{c}_1(L_i)+\tilde{c}_1(\mathcal{O}_X(D_i))+\sum_{p,q\geq 1} a_{pq}\tilde{c}_1(L_i)^p\tilde{c}_1(\mathcal{O}_X(D_i))^q.\end{aligned}$$

Letting $g(u,v) = 1+\sum_{p,q\geq 1} a_{pq}u^pv^{q-1}$, we thus have

$$\tilde{c}_1(M_i)-\tilde{c}_1(L_i) = \tilde{c}_1(\mathcal{O}_X(D_i))\cdot g(\tilde{c}_1(L_i),\tilde{c}_1(\mathcal{O}_X(D_i))).$$

Since the operators $\tilde{c}_1(?)$ all commute, the result follows from lemma 4.1.17. □

We can now verify a weak version of the Whitney product formula.

Lemma 4.1.19. *Let A_* be an oriented Borel-Moore weak homology theory on $\mathcal{V}$ and let X be in $\mathcal{V}$. Let $L_1,\ldots,L_n$ be line bundles on X and let $E=\oplus_{i=1}^n L_i$. Then $\tilde{c}_p(E)$ is the pth elementary symmetric polynomial in the (commuting) first Chern class operators $\tilde{c}_1(L_1),\ldots,\tilde{c}_1(L_n)$.*

Proof. Let $q:\mathbb{P}\to X$ be the projective space bundle

$$\mathbb{P} := \mathrm{Proj}_{\mathcal{O}_X}(\mathrm{Sym}^*(\oplus_{i=1}^n L_i)),$$

with tautological quotient $q^*(\oplus_{i=1}^n L_i)\to\mathcal{O}(1)$. The composition $q^*L_i\to\oplus_{i=1}^n q^*L_i\to\mathcal{O}(1)$ defines the section $s_i:\mathcal{O}_{\mathbb{P}}\to\mathcal{O}(1)\otimes q^*L_i^{-1}$; let D_i be the divisor of s_i. If the line bundles are all trivialized on some open $U\subset X$, then the 1-sections in $L_1,\ldots,L_n$ defines the homogeneous coordinates $X_1,\ldots,X_n$ on

$\mathbb{P} = \mathbb{P}^{n-1}_U$, and D_i is the divisor of X_i. Thus the D_i are all Cartier divisors. In addition, for each collection of distinct indices $1 \le i_1, \ldots, i_r \le n$, the intersection $\cap_{j=1}^r D_{i_j}$ has codimension r on $\mathbb{P}$ and is smooth over X. Thus $D_1, \ldots, D_n$ satisfy the transversality conditions of lemma 4.1.17 and $\cap_{i=1}^n D_i = \emptyset$. As $\mathcal{O}(1) \cong q^* L_i \otimes \mathcal{O}_X(D_i)$, it follows by lemma 4.1.18 that

$$\prod_{i=1}^n (\tilde{c}_1(\mathcal{O}(1)) - \tilde{c}_1(q^* L_i)) = 0.$$

From the uniqueness of the relation defining the Chern class $\tilde{c}_p(\oplus_{i=1}^n L_i)$, this implies that $\tilde{c}_p(\oplus_{i=1}^n L_i)$ is the pth symmetric function in the operators $\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_n)$, exactly as desired. □

Proof (of proposition 4.1.15). The naturality (2) follows from the naturality of $\tilde{c}_1(L)$, the naturality of the tautological quotient line bundle $\mathcal{O}(1)$ and the projective bundle formula (PB). The normalization (1) follows from lemma 4.1.19 with $n = 1$.

For the proof of (0), the splitting principle (remark 4.1.2), together with the naturality (2), implies that we may assume that F and E are direct sums of line bundles. In this case, (0) follows from lemma 4.1.19 and the commutativity of the first Chern classes of line bundles.

For (3), we may assume by the splitting principle that $E = E' \oplus E''$ and E' and E'' are both sums of line bundles. The result in this case follows directly from lemma 4.1.19. Conversely, (3) implies that $\tilde{c}_p(\oplus_i L_i)$ is the pth elementary symmetric function in the $\tilde{c}_1(L_i)$, hence the fact that (1-3) characterizes the $\tilde{c}_i$ follows from the splitting principle.

To prove (4), set $n = \mathrm{rnk} E$. We have the transverse cartesian diagram

$$\begin{tikzcd} \mathbb{P}(f^*E) \arrow[r, "f'"] \arrow[d, "q'"'] & \mathbb{P}(E) \arrow[d, "q"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

Take $y \in A_*(Y)$. Applying f'_* to the defining relation

$$0 = \sum_{i=0}^n (-1)^i (\tilde{c}_1(O(1)_{f^*E})^{n-i} \circ q'^* \circ \tilde{c}_i(f^*E))(y)$$

gives

$$
\begin{aligned}
0 &= \sum_{i=0}^{n}(-1)^i f'_*[(\tilde{c}_1(f'^*O(1)_E)^{n-i} \circ q'^* \circ \tilde{c}_i(f^*E))(y)] \\
&= \sum_{i=0}^{n}(-1)^i (\tilde{c}_1(O(1)_E)^{n-i} \circ (f'_* q'^*) \circ \tilde{c}_i(f^*E))(y) \\
&= \sum_{i=0}^{n}(-1)^i (\tilde{c}_1(O(1)_E)^{n-i} \circ q^*)(f_*(\tilde{c}_i(f^*E)(y)))
\end{aligned}
$$

Since $f_*(\tilde{c}_0(f^*E)(y)) = f_*(y)$, the defining relation for the $\tilde{c}_i(E)$ gives

$$f_*(\tilde{c}_i(f^*E))(y)) = \tilde{c}_i(E))(f_*(y))$$

as desired. □

4.1.8 Todd classes

Let A_* be an oriented Borel-Moore weak homology theory on $\mathcal{V}$ and $\tau = (\tau_i) \in \Pi_{i=0}^{\infty} A_i(k)$, with $\tau_0 = 1$. Define the *inverse Todd class operator* of a line bundle $L \to X$ to be the operator on $A_*(X)$ given by the infinite sum

$$\widetilde{\mathrm{Td}}_\tau^{-1}(L) = \sum_{i=0}^{\infty} \tilde{c}_1(L)^i \, \tau_i.$$

Note that the axiom (Dim) implies that $\widetilde{\mathrm{Td}}_\tau^{-1}(L)$ is a well-defined degree 0 endomorphism of $A_*(X)$.

Proposition 4.1.20. *Let A_* be an oriented Borel-Moore weak homology theory on $\mathcal{V}$ and $\tau = (\tau_i) \in \Pi_{i=0}^{\infty} A_i(k)$, with $\tau_0 = 1$. Then one can define in a unique way for each $X \in \mathcal{V}$ and each vector bundle E on X an endomorphism (of degree zero)*

$$\widetilde{\mathrm{Td}}_\tau^{-1}(E) : A_*(X) \to A_*(X)$$

such that the following holds:

(0) Given vector bundles $E \to X$ and $F \to X$ one has

$$\widetilde{\mathrm{Td}}_\tau^{-1}(E) \circ \widetilde{\mathrm{Td}}_\tau^{-1}(F) = \widetilde{\mathrm{Td}}_\tau^{-1}(F) \circ \widetilde{\mathrm{Td}}_\tau^{-1}(E).$$

(1) For a line bundle L one has:

$$\widetilde{\mathrm{Td}}_\tau^{-1}(L) = \sum_{i=0}^{\infty} \tilde{c}_1(L)^i \, \tau_i.$$

(2) For any smooth equi-dimensional morphism $Y \to X \in \mathcal{V}$, and any vector bundle $E \to X$ over X one has

$$\widetilde{\mathrm{Td}}_\tau^{-1}(f^*E) \circ f^* = f^* \circ \widetilde{\mathrm{Td}}_\tau^{-1}(E).$$

(3) If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles over X, then one has:

$$\widetilde{\mathrm{Td}}_\tau^{-1}(E) = \widetilde{\mathrm{Td}}_\tau^{-1}(E') \circ \widetilde{\mathrm{Td}}_\tau^{-1}(E'').$$

(4) For any projective morphism $Y \to X \in \mathcal{V}$, and any vector bundle $E \to X$ over X one has

$$f_* \circ \widetilde{\mathrm{Td}}_\tau^{-1}(f^*E) = \widetilde{\mathrm{Td}}_\tau^{-1}(E) \circ f_*.$$

$\widetilde{\mathrm{Td}}^{-1}(E)$ *is called the* inverse Todd class operator *of E.*

Proof. This is just an exercise with symmetric functions, together with the splitting principle and the properties of the Chern class operators listed in proposition 4.1.15. Consider the power series $f(t) := \sum_{i=0}^{\infty} \tau_i t^i \in A_*(k)[[t]]$, where we give t degree -1. Clearly the product $f(t_1) \cdot \ldots \cdot f(t_n)$ is a sum

$$f(t_1) \cdot \ldots \cdot f(t_n) = \sum_{i=0}^{\infty} P_i(t_1, \ldots, t_n)$$

with $P_i \in A_*(k)[t_1, \ldots, t_n]$ a symmetric polynomial of total degree $-i$ in $t_1, \ldots, t_n$, where we give t_i degree -1 and $A_*(k)$ degree 0. If we use the actual degrees in $A_*(k)$, then each P_i has total degree 0. Rewriting P_i as a polynomial in the elementary symmetric polynomials $\sigma_1(t_*), \ldots, \sigma_i(t_*)$ gives polynomials $Q_i \in A_*(k)[\sigma_1, \ldots, \sigma_i]$ with

$$P_i(t_1, \ldots, t_n) = Q_{i,n}(\sigma_1(t_*), \ldots, \sigma_i(t_*))$$

It is easy to check that the $Q_{i,n}$ for $n \geq i$ are independent of n, and that $Q_{i,n}$ is homogeneous of degree $-i$, where we give σ_m degree $-m$ and $A_*(k)$ degree 0; if we use the actual degrees in $A_*(k)$, then $Q_{i,n}$ has total degree 0. Set $Q_i := Q_{i,i}$. For a vector bundle $E \to X$, we set

$$\widetilde{\mathrm{Td}}_\tau^{-1}(E) = \sum_{i=0}^{\infty} Q_i(\tilde{c}_1(E), \ldots, \tilde{c}_i(E)),$$

where $\tilde{c}_i(E)$ is defined to be the zero endomorphism for $i > \mathrm{rank}(E)$.

By construction, (3) is valid for $E = E' \oplus E''$, with E' and E'' a direct sum of line bundles. Applying the splitting principle and proposition 4.1.15 shows that the properties (0)-(4) are satisfied in general. $\square$

Remark 4.1.21. Observe that $\widetilde{\mathrm{Td}}_\tau^{-1}(L)$ is an automorphism because the power series

$$\sum_{i=0}^{\infty} \tau_i u^i \in A_*(k)[[u]]$$

has leading term 1. For each vector bundle E, $\widetilde{\mathrm{Td}}_\tau^{-1}(E)$ is an automorphism as well: the formal inverse to the power series $\sum_i Q_i(\sigma_1, \ldots, \sigma_n)$ exists in the power series ring $A_*(k)[[\sigma_1, \sigma_2, \ldots]]$ and gives the inverse to $\widetilde{\mathrm{Td}}_\tau^{-1}(E)$ by substituting $\tilde{c}_i(E)$ for σ_i.

By (3) the assignment $E \mapsto \tilde{c}_\tau(E)$ descends to a group homomorphism

$$\widetilde{\mathrm{Td}}_\tau^{-1} : K^0(X) \to Aut(A_*(X)),$$

making $A_*(X)$ a $\mathbb{Z}[K^0(X)]$-module.

Example 4.1.22. Universal example. Let $\mathbb{Z}[\mathbf{t}] := \mathbb{Z}[t_1, \ldots, t_n, \ldots]$ be the graded ring of polynomials with integral coefficients on variables t_i, $i > 0$, of degree i. We apply the above construction to the oriented Borel-Moore weak homology theory $X \mapsto A_*(X)[\mathbf{t}] := A_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{t}]$ obtained from A_* by extension of scalars. We take for the family $\tau = (\tau_i)_{i \in \mathbb{N}}$ the "universal" one given by $\tau_i = t_i$. For any line bundle L on X, we write $\widetilde{\mathrm{Td}}_{\mathbf{t}}^{-1}(L)$ for the automorphism

$$\widetilde{\mathrm{Td}}_\tau^{-1}(L) = \sum_{i=0}^{\infty} \tilde{c}_1(L)^i t_i \; : A_*(X)[\mathbf{t}] \to A_*(X)[\mathbf{t}]$$

and, for each vector bundle E over X, we denote by $\widetilde{\mathrm{Td}}_{\mathbf{t}}^{-1}(E)$ the automorphism $\widetilde{\mathrm{Td}}_\tau^{-1}(E)$. We may expand $\widetilde{\mathrm{Td}}_{\mathbf{t}}^{-1}(E)$ as:

$$\widetilde{\mathrm{Td}}_{\mathbf{t}}^{-1}(E) = \sum_{I=(n_1, \ldots, n_r, \ldots)} \tilde{c}_{(n_1, \ldots, n_r, \ldots)}(E) t_1^{n_1} \ldots t_r^{n_r} \ldots$$

The $\tilde{c}_I := \tilde{c}_{(n_1, \ldots, n_r, \ldots)}(E)$ are the *Conner-Floyd Chern class endomorphisms*

$$\tilde{c}_I : A_*(X) \to A_{*-\sum_i i n_i}(X).$$

We recover for instance the ith Chern class $\tilde{c}_i(E)$ as the coefficient of t_1^i.

Now, to give $\tau = (\tau_i) \in \Pi_{i=0}^{\infty} A_i(k)$, with $\tau_0 = 1$, is the same as to give a morphism

$$\vartheta_\tau : A_*[\mathbf{t}] \to A_*, t_i \mapsto \tau_i.$$

If we consider $A_*(k)$ as an $A_*(k)[\mathbf{t}]$-algebra via ϑ_τ, this induces an isomorphism

$$A_*(X)[\mathbf{t}] \otimes_{A_*(k)[\mathbf{t}]} A_*(k) \cong A_*(X)$$

which maps $\widetilde{\mathrm{Td}}_{\mathbf{t}}^{-1}(E)$ to $\widetilde{\mathrm{Td}}_\tau^{-1}(E)$.

4.1.9 Twisting a Borel-Moore weak homology theory

The ideas in this section come from Quillen's paper [30].

Let A_* be a Borel-Moore weak homology theory on $\mathcal{V}$ and $\tau = (\tau_i) \in \Pi_{i=0}^{\infty} A_i(k)$, with $\tau_0 = 1$. We construct a new Borel-Moore weak homology theory on $\mathcal{V}$, denoted by $A_*^{(\tau)}$, as follows. $A_*^{(\tau)}(X) = A_*(X)$ and the push-forward maps are unchanged: $f_*^{(\tau)} = f_*$.

For any smooth equi-dimensional morphism $f : Y \to X$ we have the bundle of *vertical tangent vectors* T_f , defined as the dual of the bundle with sheaf of sections the relative Kähler differentials $\Omega^1_{Y/X}$. The *virtual normal bundle of* f, N_f, is the element of $K^0(Y)$ defined by $N_f := -T_f$.

We define

$$f^*_{(\tau)} := \widetilde{\mathrm{Td}}_\tau^{-1}(N_f) \circ f^*,$$

and for any line bundle L over X we set

$$\tilde{c}_1^{(\tau)}(L) := \widetilde{\mathrm{Td}}_\tau^{-1}(L) \circ \tilde{c}_1(L)$$

The properties of $\widetilde{\mathrm{Td}}_\tau^{-1}$ described in proposition 4.1.20, together with the fact that $\widetilde{\mathrm{Td}}_\tau^{-1}(L)$ is an automorphism of $A_*(X)$ (remark 4.1.21) shows that this data does in fact define a Borel-Moore weak homology theory on $\mathcal{V}$.

The definition of $\tilde{c}_1^{(\tau)}(L)$ can be rewritten as

$$\tilde{c}_1^{(\tau)}(L) = \lambda_{(\tau)}(\tilde{c}_1(L))$$

where $\lambda_{(\tau)}(u) = \sum_{i\geq 0} \tau_i \cdot u^{i+1} \in A_*(k)[[u]]$. Since $\lambda_{(\tau)}(u) \equiv u \mod u^2$, there is a unique power series $\lambda^{-1}_{(\tau)}(u)$ such that $\lambda^{-1}_{(\tau)}(\lambda_{(\tau)}(u)) = u$. As the the formal group law $F_A^{(\tau)}$ associated to $A_*^{(\tau)}$ is determined by the equations

$$F_A^{(\tau)}(\tilde{c}_1^{(\tau)}(L), \tilde{c}_1^{(\tau)}(M)) = \tilde{c}_1^{(\tau)}(L \otimes M),$$

it follows that $F_A^{(\tau)}(\lambda_{(\tau)}(u), \lambda_{(\tau)}(v)) = \lambda_{(\tau)}(F_A(u,v))$, hence

$$F_A^{(\tau)}(u,v) = \lambda_{(\tau)}\,(\,F_A(\,\lambda^{-1}_{(\tau)}(u), \lambda^{-1}_{(\tau)}(v)\,)\,). \tag{4.2}$$

If we restrict attention to $\mathbf{Sm}_k$ we can do something slightly different. Fix a sequence $\tau = \prod_i \tau_i$ as above. For a vector bundle $E \to X$, define the *Todd class* of E, $\widetilde{\mathrm{Td}}_\tau(E)$, by

$$\widetilde{\mathrm{Td}}_\tau(E) := \widetilde{\mathrm{Td}}_\tau^{-1}(-E) = (\widetilde{\mathrm{Td}}_\tau^{-1}(E))^{-1}$$

Alternatively, let $f_\tau(t) = \sum_{i=0}^\infty \tau_i t^i$ and let $\tau^{(-1)} \in \prod_i A_i(k)$ be the sequence with

$$f_{\tau^{(-1)}}(t) = \frac{1}{f_\tau(t)}$$

Then $\widetilde{\mathrm{Td}}_\tau(E) = \widetilde{\mathrm{Td}}^{-1}_{\tau^{(-1)}}(E)$. Clearly the automorphisms $\widetilde{\mathrm{Td}}_\tau(E)$ have the formal properties given in proposition 4.1.20, with the normalization changed to $\widetilde{\mathrm{Td}}_\tau(L) = \sum_i \tau_i^{(-1)} \tilde{c}_1(L)^i$.

For $X \in \mathbf{Sm}_k$, let T_X denote the tangent bundle of X. For a morphism $f: Y \to X$ in $\mathbf{Sm}_k$, we have the *virtual tangent bundle* $T_f \in K^0(Y)$:

$$T_f = [T_Y] - [f^*T_X] \in K^0(Y).$$

In case f is smooth, the exact sequence $0 \to f^*\Omega^1_X \to \Omega^1_Y \to \Omega^1_{Y/X} \to 0$ shows that our two definitions of T_f agree as classes in K^0. We define the virtual normal bundle of f, N_f, by $N_f := -T_f$. In case $f: Y \to \operatorname{Spec} k$ is the structure morphism, we write N_Y for N_f, and call N_Y the *virtual normal bundle of* Y.

Define a Borel-Moore weak homology theory A^τ_* on $\mathbf{Sm}_k$, with $A^\tau_*(X) = A_*(X)$ for $X \in \mathbf{Sm}_k$, and with the pull-backs unchanged ($f^*_\tau = f^*$). For a projective morphism $f: Y \to X$, we set

$$f^\tau_* := f_* \circ \widetilde{\mathrm{Td}}_\tau(T_f),$$

and for a line bundle L over X we set

$$\tilde{c}^\tau_1(L) := \tilde{c}_1(L) \circ \widetilde{\mathrm{Td}}_\tau(-L) = \tilde{c}_1(L) \circ \widetilde{\mathrm{Td}}_\tau^{-1}(L) \tag{4.3}$$

One easily checks that this does indeed define a Borel-Moore weak homology theory on $\mathbf{Sm}_k$.

Of course, we could have used the inverse Todd class of the virtual normal bundle, $\widetilde{\mathrm{Td}}^{-1}(N_f)$, in the definition of A^τ_*, but the classical Riemann-Roch theorem uses $\widetilde{\mathrm{Td}}(T_f)$, so we took this opportunity to introduce the inverse notation.

Let $\lambda_\tau(u) = \lambda_{(\tau)}(u) = \sum_{i=0}^\infty \tau_i u^{i+1}$, so $\tilde{c}^\tau_1(L) = \lambda_\tau(\tilde{c}_1(L)) = \tilde{c}^{(\tau)}_1(L)$. Set[1]

$$\vartheta_\tau(u) = \frac{u}{\lambda_\tau(u)}.$$

The identity (4.3) gives the expression for $\widetilde{\mathrm{Td}}_\tau(L)$ as

$$\widetilde{\mathrm{Td}}_\tau(L) = \vartheta_\tau(\tilde{c}_1(L)). \tag{4.4}$$

Lemma 4.1.23. *Let X be in $\mathbf{Sm}_k$, with tangent bundle T_X. Then the automorphism*

$$\widetilde{\mathrm{Td}}_\tau^{-1}(T_X) : A^{(\tau)}_*(X) \xrightarrow{\sim} A^\tau_*(X)$$

determines an isomorphism of Borel-Moore weak homology theories on $\mathbf{Sm}_k$. *In particular the two theories have the same formal group law.*

Example 4.1.24. Let us consider the Borel-Moore weak homology theory on $\mathbf{Sch}_k$

$$X \mapsto \mathrm{CH}_*(X) \otimes \mathbb{Q}[\beta, \beta^{-1}]$$

[1] Compare with [27, §5].

obtained from CH_* by the extension of scalars $\mathbb{Z} \subset \mathbb{Q}[\beta, \beta^{-1}]$ (where β has degree one). Note that the associated weak cohomology theory CH^* is actually an oriented cohomology theory; in particular, we have cup products. We apply our construction for the family τ given by

$$\lambda_\tau(u) = \frac{1 - \mathrm{e}^{-\beta u}}{\beta}.$$

This gives

$$\vartheta_\tau(u) = \frac{\beta u}{1 - \mathrm{e}^{-\beta u}} = \sum_i \tau_i t^i,$$

and thus $\widetilde{\mathrm{Td}}_\tau(L)$ is cup product with the classical Todd class[2], where $td(L)$ for a line bundle L is defined as:

$$td(L) := \vartheta(c_1(L)).$$

More generally

$$f_*^\tau(x) = f_*(x \cup td(T_f)),$$

where $td(E)$ is the classical Todd class of a vector bundle E. This new theory is denoted $X \mapsto \mathrm{CH}_*(X) \otimes \mathbb{Q}[\beta, \beta^{-1}]^{td}$. The formula

$$\begin{aligned} 1 - \mathrm{e}^{-(u+v)} &= 1 - \mathrm{e}^{-u} \cdot \mathrm{e}^{-v} \\ &= (1 - \mathrm{e}^{-u}) + (1 - \mathrm{e}^{-v}) - (1 - \mathrm{e}^{-u})(1 - \mathrm{e}^{-v}) \end{aligned}$$

implies that the formal group law for this theory is the multiplicative one:

$$F_m(u, v) = u + v - \beta u v.$$

Example 4.1.25. Assume k admits resolution of singularities. Following Quillen [30], the Landweber-Novikov operations are obtained as follows. We consider the Borel-Moore weak homology theory on $\mathbf{Sm}_k$.

$$X \mapsto \Omega_*(X)[\mathbf{t}].$$

Via the process described above, we modify this into $\Omega_*(X)[\mathbf{t}]^{\mathbf{t}}$, using the family $\tau = \mathbf{t} = (t_i)$. By the universality of algebraic cobordism, we have a canonical morphism

$$\vartheta^{LN} : \Omega_* \to \Omega_*[\mathbf{t}]^{\mathbf{t}},$$

which we then expand as

$$\vartheta^{LN} = \sum_I S_I \, t^I,$$

where I runs over all finite sequences $(n_1, \ldots, n_r)$ (of arbitrary length), and $t^I = t_1^{n_1} \ldots t_r^{n_r}$. The natural transformations

$$S_{(n_1,\ldots,n_r)} : \Omega_* \to \Omega_{*-(\sum n_i \cdot i)}$$

are called the *Landweber-Novikov operations.*

[2] slightly modified by the introduction of β, which changes the classical Todd class in CH_* to a purely degree zero element in $\mathrm{CH}_*[\beta, \beta^{-1}]$

Example 4.1.26. Still assuming that k admits resolution of singularities, we have the oriented Borel-Moore weak homology theory CH_* on $\mathbf{Sch}_k$. We consider the other twisting

$$X \mapsto \mathrm{CH}_*(X)[\mathbf{t}]^{(\mathbf{t})},$$

which is also a weak Borel-Moore theory on $\mathbf{Sch}_k$.

By universality of Ω_* we get a canonical morphism

$$\vartheta^{CF} : \Omega_* \to \mathrm{CH}_*[\mathbf{t}]^{(\mathbf{t})}.$$

We shall prove later on that this morphism is an isomorphism after $\otimes\mathbb{Q}$.

At this point we can prove something weaker as follows.

Definition 4.1.27. *Assume that k admits resolution of singularities. We denote by Ω_*^{ad} the Borel-Moore homology weak theory*

$$X \mapsto \Omega_*^{ad}(X) := \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}.$$

As above, we have the twisted theory $\Omega_*^{ad}[\mathbf{t}]^{(\mathbf{t})}$ and the canonical morphism

$$\vartheta : \Omega_* \to \Omega_*^{ad}[\mathbf{t}]^{(\mathbf{t})} \tag{4.5}$$

of oriented Borel-Moore weak homology theories on $\mathbf{Sch}_k$.

From equation 4.2 we know that the formal group law F on $\Omega_*^{ad}[\mathbf{t}]^{(\mathbf{t})}$ is given by

$$F(u,v) = \lambda_{(\mathbf{t})}\,(\,F_a(\,\lambda_{(\mathbf{t})}^{-1}(u), \lambda_{(\mathbf{t})}^{-1}(v)\,)\,) = \lambda_{(\mathbf{t})}\,(\,\lambda_{(\mathbf{t})}^{-1}(u) + \lambda_{(\mathbf{t})}^{-1}(v)\,). \tag{4.6}$$

with $\lambda_{(\mathbf{t})}(u) = \sum_{i=0}^{\infty} t_i u^{i+1}$.

Theorem 4.1.28. *Let k be a field admitting resolution of singularities. Then the morphism* (4.5)*induces an isomorphism:*

$$\vartheta \otimes \mathbb{Q} : \Omega_* \otimes \mathbb{Q} \to \Omega_*^{ad}[\mathbf{t}]^{(\mathbf{t})} \otimes \mathbb{Q}.$$

The idea behind this theorem is the following classical lemma:

Lemma 4.1.29. *Let R be a commutative $\mathbb{Q}$-algebra and let $F(u,v) \in R[[u,v]]$ be a commutative formal group law of rank one over R. Then there exists a unique power series $\ell_F(u) = \sum_i \tau_i\, u^{i+1} \in R[[u]]$ such that $\tau_0 = 1$ and satisfying*

$$\ell_F(F(u,v)) = \ell_F(u) + \ell_F(v).$$

This power series is called the *logarithm of F*. Thus to give a commutative formal group law of rank one over a $\mathbb{Q}$-algebra is exactly the same as to give its logarithm. Conversely, given a $\mathbb{Q}$-algebra R and elements $\tau_i \in R$, $i = 1, 2, \ldots$, letting $\ell(u) = \sum_{i=1}^{\infty} \tau_i u^{i+1}$ and setting

$$F_\ell(u,v) = \ell^{-1}(\ell(u) + \ell(v)),$$

F_ℓ defines a formal group law over R with logarithm $\ell(u)$.

Thus, if the logarithm of the universal formal group law on $\mathbb{L}_* \otimes \mathbb{Q}$ is denoted by $\ell_{\mathbb{L}_*}(u) = \sum_i b_i \cdot u^{i+1} \in \mathbb{L}_* \otimes \mathbb{Q}[[u]]$ the expression of the b_i as elements of $\mathbb{L}_* \otimes \mathbb{Q}$ determine a canonical graded ring homomorphism

$$\alpha : \mathbb{Q}[b_1, \ldots, b_n, \ldots] \to \mathbb{L}_* \otimes \mathbb{Q},$$

where the ring on the left is the polynomial ring on generators b_i of degree i.

By construction, α represents the natural transformation $F(u,v) \mapsto \ell_F(u)$; concretely, given a graded ring homomorphism $\psi : \mathbb{L}_* \otimes \mathbb{Q} \to R$, the induced formal group law $F_R := \psi(F_{\mathbb{L}})$ has logarithm $\psi \circ \alpha(\ell_{\mathbb{L}_*})$. Lemma 4.1.29 and the universality of $(F_{\mathbb{L}}, \mathbb{L})$ clearly imply that α is an isomorphism.

Let $\mathbb{Q}[t_1, \ldots, t_n, \ldots]$ be the polynomial ring with t_i having degree i. Let $\lambda_{(\mathbf{t})}(u) = \sum_i t_i u^{i+1}$, and let $\lambda^{-1}_{(\mathbf{t})}(u)$ be the inverse power series, i.e. $\lambda_{(\mathbf{t})}(\lambda^{-1}_{(\mathbf{t})}(u)) = u$. Let

$$\rho : \mathbb{Q}[b_1, \ldots, b_n, \ldots] \to \mathbb{Q}[t_1, \ldots, t_n, \ldots]$$

be the homomorphism with $\rho(\ell_{\mathbb{L}_*}) = \lambda^{-1}_{(\mathbf{t})}$. Since we can clearly express t_i in terms of $\rho(b_1) \ldots, \rho(b_i)$, ρ is an isomorphism. The map

$$\gamma := \rho \circ \alpha^{-1} : \mathbb{L}_* \otimes \mathbb{Q} \to \mathbb{Q}[t_1, \ldots, t_n, \ldots]$$

is therefore also an isomorphism.

Thus γ yields a formal group law on $\mathbb{Q}[t_1, \ldots, t_n, \ldots]$ with logarithm $\lambda^{-1}_{(\mathbf{t})}(u)$; the formal group law given by γ is therefore

$$\lambda_{(\mathbf{t})}\left(\, \lambda^{-1}_{(\mathbf{t})}(u) + \lambda^{-1}_{(\mathbf{t})}(v) \,\right)$$

(compare with (4.6)).

Our proof of theorem 4.1.28 is directly inspired by the construction of the isomorphism $\mathbb{L} \otimes \mathbb{Q} \cong \mathbb{Q}[t_1, t_2, \ldots]$.

Proof (of theorem 4.1.28). Let $\ell(u) \in \Omega_*(k) \otimes \mathbb{Q}[[u]]$ denote the logarithm of $F_\Omega(u,v)$. Writing $\ell(u) = \sum_{i \geq 0} b_i \cdot u^{i+1}$, we can twist rational algebraic cobordism using this family $\mathbf{b} = \{b_i\}_i$ to get the Borel-Moore weak homology theory on $\mathbf{Sch}_k$

$$X \mapsto \Omega_*(X) \otimes \mathbb{Q}^{(\mathbf{b})}.$$

By formula (4.2), the formal group law on $\Omega_* \otimes \mathbb{Q}^{(\mathbf{b})}$ is the additive one. Thus the canonical morphism of Borel-Moore weak homology theories on $\mathbf{Sch}_k$, $\Omega_* \to \Omega_* \otimes \mathbb{Q}^{(\mathbf{b})}$ given by universality of Ω_* factors through the morphism $\Omega_* \to \Omega_*^{ad}$, inducing a canonical morphism

$$\phi : \Omega_*^{ad} \to \Omega_* \otimes \mathbb{Q}^{(\mathbf{b})}.$$

Let $\lambda(u)$ denote $\ell^{-1}(u)$; define $\tau_i \in \Omega_i(k)$, $i = 1, 2, \ldots$, by the equation:

$$\lambda(u) = \sum_{i \geq 0} \tau_i \cdot u^{i+1}.$$

Using the universal construction of example 4.1.22, we may extend ϕ to

$$\Phi : \Omega_*^{ad}[\mathbf{t}] \to \Omega_* \otimes \mathbb{Q}^{(\mathbf{b})},$$

so that t_i is mapped to τ_i. By twisting the morphism Φ both at the source and the target we get a morphism

$$\Omega_*^{ad}[\mathbf{t}]^{(\mathbf{t})} \to (\Omega_* \otimes \mathbb{Q}^{(\mathbf{b})})^{(\tau)}.$$

In fact, $(\Omega_* \otimes \mathbb{Q}^{(\mathbf{b})})^{(\tau)} = \Omega_* \otimes \mathbb{Q}$, because, as λ is the inverse to ℓ, twisting by τ is the inverse to twisting by b. One easily checks that this morphism is an inverse to $\vartheta \otimes \mathbb{Q}$. Indeed, the composition

$$\Omega_* \otimes \mathbb{Q} \to \Omega_*^{ad}[\mathbf{t}]^{(\mathbf{t})} \otimes \mathbb{Q} \to \Omega_* \otimes \mathbb{Q}$$

is the identity by the universality of Ω_*. Thus $\vartheta \otimes \mathbb{Q}$ is a monomorphism.

To prove that $\vartheta \otimes \mathbb{Q}$ is an epimorphism, it follows from (4.6) that the group law on $\Omega_*^{ad}[\mathbf{t}]^{(\mathbf{t})}$ has logarithm $\lambda_{(\mathbf{t})}^{-1}(u)$. From our discussion following lemma 4.1.29, the homomorphism $\phi : \mathbb{L}_* \to \Omega_*^{ad}(k)[\mathbf{t}]^{(\mathbf{t})}$ classifying the group law on $\Omega_*^{ad}(k)[\mathbf{t}]^{(\mathbf{t})}$ induces an isomorphism $\mathbb{L}_* \otimes \mathbb{Q} \to \mathbb{Q}[\mathbf{t}]$, where $\Omega_*^{ad}(k) \to \mathbb{Q}$ is the ring homomorphism sending $\Omega_{\geq 1}^{ad}(k)$ to zero (we know that $\Omega_0^{ad}(k) = \mathbb{Z}$ by theorem 2.5.12). Note that ϕ factors through ϑ via the canonical map $\mathbb{L}_* \to \Omega_*(k)$, and $\mathbb{L}_{\geq 1}$ clearly goes to zero under the map $\Omega_* \to \Omega_*^{ad}$. Since Ω_*^{ad} is graded and concentrated in degrees ≥ 0, one can use an induction on the degree to reduce to showing that $\Omega_* \to \Omega_*^{ad}$ is an epimorphism, which is clear from the definition of Ω_*^{ad}. □

Remark 4.1.30. As a corollary we see that

$$\vartheta : \Omega_*(k) \to \Omega_*^{ad}(k)[\mathbf{t}]^{(\mathbf{t})}$$

induces an isomorphism after $\otimes \mathbb{Q}$, proving that $\Omega_*(k) \otimes \mathbb{Q}$ is a polynomial algebra on $\Omega_*^{ad}(k) \otimes \mathbb{Q}$.

4.2 Algebraic cobordism and K-theory

In this section, we give a proof of theorem 1.2.3 and corollary 1.2.18.

4.2.1 Projective bundles

In this section A_* denotes an oriented Borel-Moore functor of geometric type (definition 2.2.1) on $\mathbf{Sm}_k$, for which the formal group law is multiplicative. Thus if $b = [\mathbb{P}^1]_A \in A^{-1}$ one has $F_A(u,v) = u + v - buv$. However, we never assume that A_* is periodic so that the results below still hold if $b = 0$ (so that A_* additive is permitted)! Throughout this section, we will drop the index $_A$ in the notation $[-]_A$.

Clearly the power series $\chi_m(u) = [-1]_m(u)$ is given by

$$[-1]_m(u) = \frac{-u}{1 - b \cdot u}.$$

Write $u \cdot g(u) = [-1]_m(u)$ so that $g(u) = \frac{-1}{1-b \cdot u}$. We obviously have:

$$[-1]_m(u) \cdot g([-1]_m(u)) = u \tag{4.7}$$

(because $[-1]_m([-1]_m(u)) = u$) proving that $g([-1]_m(u)) = b \cdot u - 1$.

Proposition 4.2.1. *Let $i : Z \to X$ be a closed immersion of codimension c between smooth k-schemes, X being of dimension d. Let $X_Z \to X$ be the blow-up of X at Z, η_i the conormal sheaf of i. Then one has the following equality in $A_d(X)$:*

$$[X_Z \to X] = [\mathrm{Id}_X] + b \cdot i_*[\mathbb{P}(\eta_i) \to Z] - i_*[\mathbb{P}(\eta_i \oplus \mathcal{O}_Z) \to Z]$$

Proof. Let $q : \mathbb{P} := \mathbb{P}(\eta_i \oplus \mathcal{O}_Z) \to Z$ be the structure morphism. We apply proposition 2.5.2. The identity (4.7) yields the identities in $A_*(\mathbb{P}(\nu_i \oplus \mathcal{O}))$:

$$g([O(-1)]) = g([-1]_m([O(1)])) = b \cdot [O(1)] - 1.$$

As the closed subscheme $\mathbb{P}(\eta_i)$ of $\mathbb{P}(\eta_i \oplus \mathcal{O}_Z)$ is defined by the vanishing of the section $\mathcal{O}_{\mathbb{P}} \to q^*(\eta_i \oplus \mathcal{O}_Z) \to \mathcal{O}(1)$, we have $[O(1)] = [\mathbb{P}(\nu_i) \to \mathbb{P}(\eta_i \oplus \mathcal{O}_Z)]$ and the formula follows. □

Take $X \in \mathbf{Sm}_k$. If $D \subset X$ is a divisor on X and $\mathcal{F}$ is a coherent sheaf on X we denote by $\mathcal{F}(D)$ the tensor product $\mathcal{F} \otimes \mathcal{O}_X(D)$.

Lemma 4.2.2. *Let $\mathcal{L}$ be an invertible sheaf on $X \in \mathbf{Sm}_k$, $\mathcal{E} \to X$ a locally free sheaf of rank $r \geq 0$ and let $i : D \to X$ be a smooth closed subscheme of X of codimension one. Then one has the following formula in $A_*(X)$:*

$$\begin{aligned}[\mathbb{P}(\mathcal{L}(D) \oplus \mathcal{E}) \to X] &= [\mathbb{P}(\mathcal{L} \oplus \mathcal{E})) \to X] \\ &+ i_*[\mathbb{P}(i^*(\mathcal{L}(D) \oplus \mathcal{L} \oplus \mathcal{E})) \to D] - b \cdot i_*[\mathbb{P}(i^*(\mathcal{L} \oplus \mathcal{E})) \to D]\end{aligned}$$

Proof. Consider the X-scheme $q : Y := \mathbb{P}(\mathcal{L}(D) \oplus \mathcal{L} \oplus \mathcal{E}) \to X$. The projections

$$\mathcal{L}(D) \oplus \mathcal{L} \oplus \mathcal{E} \to \mathcal{L}(D) \oplus \mathcal{E},$$
$$\mathcal{L}(D) \oplus \mathcal{L} \oplus \mathcal{E} \to \mathcal{L} \oplus \mathcal{E}$$

determine the closed embeddings

$$i_0 : \tilde{D}_0 := \mathbb{P}(\mathcal{L}(D) \oplus \mathcal{E}) \to Y,$$
$$i_1 : \tilde{D}_1 := \mathbb{P}(\mathcal{L} \oplus \mathcal{E}) \to Y.$$

Note that $q^*D = \mathbb{P}(i^*(\mathcal{L}(D) \oplus \mathcal{L} \oplus \mathcal{E}))$ as an X-scheme.

As $\tilde{D}_0$ is defined by the vanishing of the composition

$$q^*\mathcal{L} \to q^*\mathcal{L}(D) \oplus q^*\mathcal{L} \oplus q^*\mathcal{E} \to \mathcal{O}(1)$$

it follows that $\mathcal{O}_Y(\tilde{D}_0) \cong (q^*\mathcal{L}^\vee)(1)$. Similarly,

$$\mathcal{O}_Y(\tilde{D}_1) \cong q^*(\mathcal{L}(D)^\vee)(1),$$
$$\mathcal{O}_Y(q^*D) \cong q^*\mathcal{O}_X(D),$$

giving an isomorphism

$$\mathcal{O}_Y(q^*D + \tilde{D}_1) \cong q^*\mathcal{O}_X(D) \otimes \mathcal{O}_Y(\tilde{D}_1) \cong \mathcal{O}_Y(\tilde{D}_0).$$

This yields the identity of endomorphisms:

$$\begin{aligned}\tilde{c}_1(\mathcal{O}_Y(\tilde{D}_0)) &= \tilde{c}_1(q^*\mathcal{O}_X(D) \otimes \mathcal{O}_Y(\tilde{D}_1)) \\ &= \tilde{c}_1(q^*\mathcal{O}_X(D)) + \tilde{c}_1(\mathcal{O}_Y(\tilde{D}_1)) \\ &\qquad - b \cdot \tilde{c}_1(q^*\mathcal{O}_X(D)) \circ \tilde{c}_1(\mathcal{O}_Y(\tilde{D}_1)).\end{aligned}$$

Let $\tilde{i} : q^{-1}D \to Y$ be the inclusion. Applying the above identity to 1_Y and using the axiom (Sect) gives

$$\begin{aligned}[\tilde{D}_0 \to Y] &= [q^*D \to Y] + [\tilde{D}_1 \to Y] - b \cdot [q^*D \cdot \tilde{D}_1 \to Y] \\ &= \tilde{i}_*[\mathbb{P}(i^*(\mathcal{L}(D) \oplus \mathcal{L} \oplus \mathcal{E})) \to q^{-1}(D)] + [\mathbb{P}(\mathcal{L} \oplus \mathcal{E}) \to Y] \\ &\qquad - b \cdot \tilde{i}_*[\mathbb{P}(i^*(\mathcal{L} \oplus \mathcal{E})) \to q^{-1}(D)].\end{aligned}$$

Pushing forward to X and D gives the desired formula. □

Lemma 4.2.3. *Let $\mathcal{E}$ be a direct sum of $n+1$ invertible sheaves on some $X \in \mathbf{Sm}_k$. Then in $A_*(X)$,*

$$[\mathbb{P}(\mathcal{E}) \to X] = b^n \cdot 1_X.$$

In particular, one has $[\mathbb{P}^n] = b^n$ in $A_*(k)$.

Proof. We proceed by induction on $\dim_k X$. We first consider the case $\dim X = 0$, $X = \operatorname{Spec} F$ for F a finite extension field of k. Since $\mathbb{P}^n_F = \mathbb{P}^n_k \times_k F$, we need only consider the case $F = k$, i.e., we must show that $[\mathbb{P}^N] = b^N$. We proceed by induction on N.

For $N = 0$ there is nothing to prove, and for $N = 1$ the result follows from the equation $[\mathbb{P}^1] = b$ in remark 2.5.6.

Take $N \geq 2$. We apply lemma 4.2.2 with $X = \mathbb{P}^N$, D a linearly embedded $\mathbb{P}^{N-1}$, $\mathcal{L} = \mathcal{O}$, and $\mathcal{E} = 0$. Pushing forward the identity of classes in $A_*(X)$ to $A_*(k)$ yields

$$[\mathbb{P}_{\mathbb{P}^N}(\mathcal{O}(1))] = [\mathbb{P}_{\mathbb{P}^N}(\mathcal{O})] + [\mathbb{P}_{\mathbb{P}^{N-1}}(\mathcal{O}(1) \oplus \mathcal{O})] - b \cdot [\mathbb{P}_{\mathbb{P}^{N-1}}(\mathcal{O})].$$

Since $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E} \otimes \mathcal{L})$ for $\mathcal{E}$ a locally free sheaf and $\mathcal{L}$ an invertible sheaf, this gives

$$[\mathbb{P}_{\mathbb{P}^{N-1}}(\mathcal{O}(1) \oplus \mathcal{O})] = b \cdot [\mathbb{P}_{\mathbb{P}^{N-1}}(\mathcal{O})] = b \cdot [\mathbb{P}^{N-1}] = b^N,$$

the last identity using our induction hypothesis. Now suppose we have shown that $[\mathbb{P}_{\mathbb{P}^{N-r}}(\mathcal{O} \oplus \mathcal{O}(1)^r)] = b^N$ for some r with $1 \leq r < N$. We may assume by induction on N that $[\mathbb{P}_{\mathbb{P}^{M-r}}(\mathcal{O} \oplus \mathcal{O}(1)^r)] = b^M$ for all $M < N$ and $1 \leq r \leq M$. Apply lemma 4.2.2 with $X = \mathbb{P}^{N-r}$, D a linearly embedded $\mathbb{P}^{N-r-1}$, $\mathcal{L} = \mathcal{O}$, and $\mathcal{E} = \mathcal{O}(1)^r$. Pushing forward to $A_*(k)$ gives

$$\begin{aligned}
b^N &= [\mathbb{P}_{\mathbb{P}^{N-r}}(\mathcal{O} \oplus \mathcal{O}(1)^r)] \\
&= [\mathbb{P}_{\mathbb{P}^{N-r}}(\mathcal{O}(1) \oplus \mathcal{O}(1)^r)] - [\mathbb{P}_{\mathbb{P}^{N-r-1}}(\mathcal{O} \oplus \mathcal{O}(1)^{r+1})] \\
&\qquad\qquad + b \cdot [\mathbb{P}_{\mathbb{P}^{N-r-1}}(\mathcal{O} \oplus \mathcal{O}(1)^r)] \\
&= [\mathbb{P}^r \times \mathbb{P}^{N-r}] - [\mathbb{P}_{\mathbb{P}^{N-r-1}}(\mathcal{O} \oplus \mathcal{O}(1)^{r+1})] + b \cdot b^{N-1} \\
&= [\mathbb{P}^r][\mathbb{P}^{N-r}] + b^N - [\mathbb{P}_{\mathbb{P}^{N-r-1}}(\mathcal{O} \oplus \mathcal{O}(1)^{r+1})].
\end{aligned}$$

By induction, $[\mathbb{P}^r] = b^r$ and $[\mathbb{P}^{N-r}] = b^{N-r}$, giving

$$[\mathbb{P}_{\mathbb{P}^{N-r-1}}(\mathcal{O} \oplus \mathcal{O}(1)^{r+1})] = b^N.$$

Taking $r = N-1$ gives us $[\mathbb{P}^N] = b^N$, as desired.

Now let $\mathcal{L}_0, \ldots, \mathcal{L}_r$ be invertible sheaves on $X \in \mathbf{Sm}_k$. We prove that

$$[\mathbb{P}(\mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_r) \to X] = b^r[X]$$

by induction on $n = \dim_k(X)$, the case $n = 0$ having been settled above. Assume $n > 0$ and that we have proven the above formula when the dimension of the base is $< n$.

Let $\mathcal{E}$ be the sum $\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$. Given a smooth divisor $i : D \to X$ in X, lemma 4.2.2 gives

$$\begin{aligned}
&[\mathbb{P}(\mathcal{L}_0 \oplus \mathcal{E}) \to X] = [\mathbb{P}(\mathcal{L}_0(D) \oplus \mathcal{E}) \to X] \\
&\qquad - i_*[\mathbb{P}(i^*(\mathcal{L}_0(D) \oplus \mathcal{L}_0 \oplus \mathcal{E})) \to D] + b \cdot i_*[\mathbb{P}(i^*(\mathcal{L}_0(D) \oplus \mathcal{E})) \to D].
\end{aligned}$$

Since $\dim D < n$, we may use our inductive hypothesis, giving

$$\begin{aligned}
[\mathbb{P}(\mathcal{L}_0 \oplus \mathcal{E}) \to X] &= [\mathbb{P}(\mathcal{L}_0(D) \oplus \mathcal{E}) \to X] - i_* b^r[\mathrm{Id}_D] + b \cdot i_* b^{r-1}[\mathrm{Id}_D] \\
&= [\mathbb{P}(\mathcal{L}_0 \oplus \mathcal{E}) \to X].
\end{aligned}$$

Doing the same for each $\mathcal{L}_i$, it follows that, given smooth divisors $D_0, \ldots, D_r$ on X, we have

$$[\mathbb{P}(\oplus_{i=0}^r \mathcal{L}_i) \to X] = [\mathbb{P}(\oplus_{i=0}^r \mathcal{L}_i(D_i)) \to X].$$

Since X is quasi-projective, there is a very ample invertible sheaf $\mathcal{L}$ on X such that $\mathcal{L} \otimes \mathcal{L}_i^\vee$ is very ample for each i. Replacing $\mathcal{L}$ with a high tensor power if necessary, we may assume that, for each i, the divisor D_i of some section of $\mathcal{L} \otimes \mathcal{L}_i^\vee$ is smooth. Then $\mathcal{L}_i(D_i) \cong \mathcal{L}$ for each i, so

$$\begin{aligned}
[\mathbb{P}(\oplus_{i=0}^r \mathcal{L}_i) \to X] &= [\mathbb{P}(\oplus_{i=0}^r \mathcal{L}_i(D_i)) \to X] \\
&= [\mathbb{P}(\oplus_{i=0}^r \mathcal{L}) \to X] \\
&= [p_2 : \mathbb{P}^r \times X \to X] \\
&= b^r[\mathrm{Id}X].
\end{aligned}$$

□

Remark 4.2.4. If we assume that A_* satisfies the projective bundle formula (PB) and extended homotopy property (EH) on $\mathbf{Sm}_k$, the previous lemma implies that for *any* locally free sheaf $\mathcal{E}$ on X of rank $r+1$ one has

$$[\mathbb{P}(\mathcal{E}) \to X] = b^r \cdot 1_X$$

Indeed, the splitting principle and axiom (A3) reduces to the case in which E is a direct sum of line bundles.

We also get the following simplification of proposition 4.2.1:

Corollary 4.2.5. *Let A_* be an oriented Borel-Moore functor of geometric type for which the formal group law is multiplicative. Suppose in addition that A_* satisfies (PB) and (EH). Let $Z \subset X$ be a smooth closed subscheme of some X in $\mathbf{Sm}_k$. Let $X_Z \to X$ be the blow-up of X at Z. Then one has*

$$[X_Z \to X]_A = [\mathrm{Id}_X]_A$$

in $A_*(X)$. In particular, when X is smooth and projective over k, one has

$$[X_Z]_A = [X]_A$$

in $A_*(k)$.

Proof. The result follows from proposition 4.2.1 and remark 4.2.4. □

Remark 4.2.6. The universal oriented Borel-Moore weak homology theory on $\mathbf{Sm}_k$ having multiplicative formal group law is of course given by:

$$X \mapsto \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta]$$

Since an oriented Borel-Moore weak homology theory on $\mathbf{Sm}_k$ satisfies (PB) and (EH), $\Omega_* \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta]$ thus satisfies corollary 4.2.5 above. It is tempting to make the following:

Conjecture 4.2.7. *Let k be a field. Then the oriented Borel-Moore functor of geometric type*

$$X \mapsto \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta]$$

is the universal oriented Borel-Moore functor of geometric type which has "birational invariance" in the following sense: given a birational projective morphism $f : Y \to X$ then $f_* 1_Y = 1_X$.

In fact we shall establish conjecture 4.2.7 over a field of characteristic zero in §4.3.3 below.

4.2.2 The Chern character

In this section, we let A^* be a multiplicative and periodic oriented weak cohomology theory on $\mathbf{Sm}_k$. We construct the Chern character ch_A as a transformation $ch_A : K^0[\beta, \beta^{-1}] \to A^*$, compatible with smooth-pullback, external products and first Chern class operators.

We have the canonical morphism of weak cohomology theories

$$\vartheta_A : \Omega^* \to A^*;$$

let $\bar{A}^*(X)$ be the $A^*(k)$-submodule of $A^*(X)$ generated by $\vartheta_A(\Omega^*(X))$. Clearly $X \mapsto \bar{A}^*(X)$ defines a sub-weak cohomology theory of A^*.

Now, for $X \in \mathbf{Sm}_k$, and E a vector bundle of rank r on X with sheaf of sections $\mathcal{E}$, define the *Chern character operator* $\widetilde{ch}_A(\mathcal{E} \otimes \beta^n) : \bar{A}^*(X) \to \bar{A}^*(X)$ by

$$\widetilde{ch}_A(\mathcal{E} \otimes \beta^n) = (\mathrm{rank}(\mathcal{E}) - b \cdot \tilde{c}_1^A(E^\vee))b^n.$$

We set $ch_A(\mathcal{E} \cdot \beta^n) := \widetilde{ch}_A(\mathcal{E} \cdot \beta^n)(1_X)$. Since $\tilde{c}_1^A$ is additive in exact sequences, we have the homomorphism

$$\widetilde{ch}_A : K^0(X)[\beta, \beta^{-1}] \to \mathrm{End}_{A^*(k)}(\bar{A}^*(X)).$$

Let L and M be line bundles on some $X \in \mathbf{Sm}_k$ with respective sheaves of sections $\mathcal{L}$ and $\mathcal{M}$. Using the group law for A, we have the identity of endomorphisms of $\bar{A}^*(X)$:

$$\begin{aligned}
\widetilde{ch}_A(\mathcal{L} \otimes \mathcal{M})) &= 1 - \tilde{c}_1^A(L^\vee \otimes M^\vee)b \\
&= 1 - \tilde{c}_1^A(L^\vee)b - \tilde{c}_1^A(M^\vee)b + \tilde{c}_1^A(L^\vee)\tilde{c}_1^A(M^\vee)b^2 \\
&= \widetilde{ch}_A(\mathcal{L})\widetilde{ch}_A(\mathcal{M}).
\end{aligned}$$

Similarly, we have

$$\widetilde{ch}_A(\tilde{c}_1^K(L)(\mathcal{M})) = \tilde{c}_1^A(L) \circ \widetilde{ch}_A(\mathcal{M}).$$

By the splitting principle, this shows that

$$\widetilde{ch}_A(\mathcal{E} \otimes \mathcal{F}) = \widetilde{ch}_A(\mathcal{E})\widetilde{ch}_A(\mathcal{F}); \quad \widetilde{ch}_A(\tilde{c}_1^K(L)(\mathcal{E})) = \tilde{c}_1^A(L)\widetilde{ch}_A(\mathcal{E})$$

for all locally free sheaves $\mathcal{E}$ and $\mathcal{F}$, and line bundles L, that is, $\widetilde{ch}_A$ is a ring homomorphism, and intertwines the operations of first Chern classes of line bundles.

The splitting principle also shows that $\widetilde{ch}_A$ is compatible with smooth pull-back: for $f : Y \to X$ a smooth morphism in $\mathbf{Sm}_k$,

$$f^*(\widetilde{ch}_A(x)(y)) = \widetilde{ch}_A(f^*x)(f^*y)$$

for $x \in K^0(X)[\beta, \beta^{-1}]$, $y \in A^*(X)$, and satisfies the projection formula for a projective morphism $f : Y \to X$ in $\mathbf{Sm}_k$:

$$f_*(\widetilde{ch}_A(f^*x)(y)) = \widetilde{ch}_A(x)(f_*y)$$

for $x \in K^0(X)[\beta, \beta^{-1}]$, $y \in A^*(Y)$.

Evaluating these formulas at 1_X shows that ch_A is compatible with smooth pull-backs, with the Chern class operators for line bundles and with external products, as desired.

4.2.3 The Riemann-Roch theorem for a multiplicative theory

Now we prove that ch_A commutes with projective push-forwards. We follow the argument used in Fulton's proof of Grothendieck-Riemann-Roch for closed embedding as in [9, §15.2]: first handle the case of a projective space bundle with a section, then reduce to this case by deformation to the normal bundle.

Lemma 4.2.8. *Let Z be in $\mathbf{Sm}_k$, $\mathcal{E}$ a locally free coherent sheaf on Z, $q : \mathbb{P} \to Z$ the projective bundle $\mathbb{P}(\mathcal{E} \oplus \mathcal{O}_Z)$. Let $s : Z \to \mathbb{P}$ be the section defined by the projection $\mathcal{E} \oplus \mathcal{O}_X \to \mathcal{O}_X$. Take $x \in K^0(X)[\beta, \beta^{-1}]$. Then in $A^*(\mathbb{P})$ we have*

$$ch_A(s_*(x)) = s_*(ch_A(x)).$$

Proof. We first reduce to the case $x = 1_Z$. It clearly suffices to prove the lemma for $x = [\mathcal{F}]$, $\mathcal{F}$ a locally free coherent sheaf on Z. Since ch_A and s_* commute with smooth pull-back, we may use the splitting principle to reduce to the case $\mathcal{F} = \oplus_i \mathcal{L}_i$, $\mathcal{L}_i$ invertible sheaves; the additivity of ch_A and s_* reduce us to the case $\mathcal{F} = \mathcal{L}$, the invertible sheaf of sections of a line bundle L. Since $\tilde{c}_1^K(L)(1_Z) = 1 - [\mathcal{L}^\vee]$, it suffices to handle instead the case $x = \tilde{c}_1^K(L)(1_Z)$.

Assuming the result for $x = 1_Z$, we use the compatibility of ch_A with $\tilde{c}_1(?)$ and the projection formula (definition 2.1.2(A3)):

$$\begin{aligned}
ch_A(s_*(\tilde{c}_1^K(L)(1_Z))) &= ch_A(s_*(\tilde{c}_1^K(s^*q^*L)(1_Z))) \\
&= ch_A(\tilde{c}_1^K(q^*L)(s_*(1_Z))) \\
&= \tilde{c}_1^A(q^*L)(ch_A(s_*(1_Z))) \\
&= \tilde{c}_1^A(q^*L)(s_*(ch_A(1_Z))) \\
&= s_*(\tilde{c}_1^A(L)(ch_A(1_Z))) \\
&= s_*(ch_A(\tilde{c}_1^K(L)(1_Z))),
\end{aligned}$$

completing the reduction.

We now prove the case $x = 1_Z$. As above, we may use the splitting principle to reduce to the case $\mathcal{E} = \oplus_{i=1}^n \mathcal{L}_i$, with the $\mathcal{L}_i$ invertible sheaves on Z. Let $\mathcal{L}_0 = \mathcal{O}_Z$ and let D_i be the subscheme of $\mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X)$ given by the projection

$$\mathcal{E} \oplus \mathcal{O}_Z \to \oplus_{j \neq i} \mathcal{L}_j,$$

$i = 1, \ldots, n$. D_i is thus the subscheme of zeros of the composition $q^*(\mathcal{L}_i) \to q^*(\mathcal{E} \oplus \mathcal{O}_Z) \to \mathcal{O}(1)$, hence is the Cartier divisor of a section of $\mathcal{O}(1) \otimes q^*(\mathcal{L}_i)^\vee$. D_i is smooth and $s(Z) = \cap_{i=1}^n D_i$, so repeated applications of (Sect) give

$$s_*(1_Z^A) = \prod_{i=1}^n \tilde{c}_1^A(\mathcal{O}(1) \otimes q^*(\mathcal{L}_i)^\vee)(1_{\mathbb{P}}^A).$$

for all oriented weak cohomology theories A, including $K^0[\beta, \beta^{-1}]$. Since ch_A commutes with the first Chern class operators and sends $1_{\mathbb{P}}^K$ to $1_{\mathbb{P}}^A$, the proof is complete. □

In short, the Todd class of a vector bundle in a multiplicative periodic theory is trivial.

Proposition 4.2.9. *Let $f : Y \to X$ be a projective morphism in* $\mathbf{Sm}_k$ *and take $x \in K^0(Y)[\beta, \beta^{-1}]$. Then*

$$f_*(ch_A(x)) = ch_A(f_*(x))$$

Proof. It sufficient to check this for the projection $\mathbb{P}^n \times X \to X$ to a smooth k-scheme X and for a closed immersion $Z \to X$ between smooth k-schemes.

We first consider the case of the projection $p_2 : \mathbb{P}^n \times X \to X$ for some $n > 0$. Using the projective bundle formula and the compatibility of ch_A with the external product and the first Chern class operators, we need only consider the case of the push forward along $\pi : \mathbb{P}^n \to \operatorname{Spec} k$, and show that

$$\pi_*^A(ch_A(1_{\mathbb{P}^n}^K)) = ch_A(\pi_*^K(1_{\mathbb{P}^n})).$$

Since $ch_A(1_{\mathbb{P}^n}^K) = 1_{\mathbb{P}^n}^A$ this is the same as showing

$$[\mathbb{P}^n]_A = ch_A([\mathbb{P}^n]_K).$$

But by lemma 4.2.3, $[\mathbb{P}^n]_K = \beta^n \in K^0(k)[\beta, \beta^{-1}]$ and $[\mathbb{P}^n]_A = b^n \in A^*(k)$; as $ch_A(\beta) = b$, this verifies our formula.

We proceed to the case of a closed immersion $i : Z \to X$, using the deformation to the normal bundle to reduce to the case handled in lemma 4.2.8. Although the proof is just a slight modification of that used in [9, *loc. cit.*], we give the details here for the reader's convenience. We will use some computations from chapter 5, but these do not rely on the Riemann-Roch theorem, and the reader will easily check that the argument is not circular.

Let $i : Z \to X$ be a closed immersion in $\mathbf{Sm}_k$. We have the deformation diagram (2.4) from section 2.5; we repeat the relevant portion here for the reader's convenience:

$$
\begin{tikzcd}
Z \times 1 \arrow[rr, "i"] \arrow[d, "i_1^Z"'] & & Y_1 \arrow[r, equal] \arrow[d, "i_1"'] & X \times 1 \arrow[d, hook] & \\
Z \times \mathbb{P}^1 \arrow[rr, "\tilde{i}"] & & Y \arrow[r, "\pi"'] & X \times \mathbb{P}^1 \arrow[r, "p_1"'] & X \\
& X_Z \arrow[rr, "p", crossing over] \arrow[dr] & & X \times 0 \arrow[u, hook] & \\
Z \times 0 \arrow[uu, "i_0^Z"] \arrow[dr, "s"'] & & Y_0 \arrow[uu] \arrow[ur, "\pi_0"'] & & \\
& \mathbb{P} \arrow[ur] & & &
\end{tikzcd}
$$

We let $i_{\mathbb{P}} : \mathbb{P} \to Y$, $i_0 : Y_0 \to Y$, $i_{X_Z} : X_Z \to Y$ be the inclusions.

Take $x \in K^0(Z)$, giving the element $p_1^* x \in K^0(Z \times \mathbb{P}^1)$ and let $\tilde{x} := \tilde{i}_*(p_1^* x) \in K^0(Y)$. Since K^0 is an oriented cohomology theory on $\mathbf{Sm}_k$, we have

$$
\begin{aligned}
i_1^*(\tilde{x}) &= i_* x \in K^0(X)[\beta, \beta^{-1}] \\
i_{\mathbb{P}}^*(\tilde{x}) &= s_* x \in K^0(\mathbb{P})[\beta, \beta^{-1}]
\end{aligned}
$$

Noting that $X_Z \cap Z \times \mathbb{P}^1 = \emptyset$, it follows from lemma 5.1.11 and proposition 5.2.1 that

$$\tilde{c}_1^K(O_Y(X_Z))(\tilde{x}) = i_{X_Z *}(i_{X_Z}^*(\tilde{x})) = 0.$$

hence

$$
\begin{aligned}
[\widetilde{ch}_A(\tilde{x}) \circ \tilde{c}_1^A(O_Y(X_Z))](1_Y) &= [\tilde{c}_1^A(O_Y(X_Z)) \circ \widetilde{ch}_A(\tilde{x})](1_Y) \\
&= \widetilde{ch}_A(\tilde{c}_1^K(O_Y(X_Z))(\tilde{x}))(1_Y) \\
&= 0.
\end{aligned}
$$

Thus

$$\begin{aligned}
i_{1*}(ch_A(i_*x)) &= i_{1*}(\widetilde{ch}_A(i_1^*(\tilde{x}))(1_{Y_1})) \\
&= \widetilde{ch}_A(\tilde{x})(i_{1*}(1_{Y_1})) \\
&= \widetilde{ch}_A(\tilde{x}) \circ \tilde{c}_1^A(O_Y(Y_1))(1_Y) \\
&= \widetilde{ch}_A(\tilde{x}) \circ \tilde{c}_1^A(O_Y(Y_0))(1_Y) \\
&= \widetilde{ch}_A(\tilde{x}) \circ \big(\tilde{c}_1^K(O_Y(\mathbb{P})) + \tilde{c}_1^K(O_Y(X_Z)) \\
&\qquad\qquad - b\tilde{c}_1^K(O_Y(\mathbb{P}))\tilde{c}_1^K(O_Y(X_Z))\big)(1_Y) \\
&= \widetilde{ch}_A(\tilde{x}) \circ \tilde{c}_1^K(O_Y(\mathbb{P}))(1_Y) \\
&= \widetilde{ch}_A(\tilde{x})(i_{\mathbb{P}*}(1_{\mathbb{P}})) \\
&= i_{\mathbb{P}*}(\widetilde{ch}_A(i_{\mathbb{P}}^*(\tilde{x}))(1_{\mathbb{P}})) \\
&= i_{\mathbb{P}*}(ch_A(s_*x)).
\end{aligned}$$

Let $i_0^Z, i_1^Z : Z \to Z \times \mathbb{P}^1$ be the 0- and 1-sections. For any element $y \in A^*(Z)$, we have $p_1^* y \in A^*(Z \times \mathbb{P}^1)$ and it follows from lemma 2.3.7 that

$$i_{1*}^Z(y) = \tilde{c}_1^A(\mathcal{O}_{Z\times\mathbb{P}^1}(1))(p_1^* y) = i_{0*}^Z(y).$$

Thus

$$\begin{aligned}
i_{1*}(i_*(ch_A(x))) &= \tilde{i}_*(i_{1*}^Z(ch_A(x))) \\
&= \tilde{i}_*(i_{0*}^Z(ch_A(x))) \\
&= i_{\mathbb{P}*}(s_*(ch_A(x))).
\end{aligned}$$

Now apply the push-forward along $p := p_1 \circ \pi : Y \to X$ to all our identities. Using lemma 4.2.8, we have

$$\begin{aligned}
ch_A(i_*(x)) &= p_*(i_{1*}(ch_A(i_*(x)))) \\
&= p_*(i_{\mathbb{P}*}(ch_A(s_*(x)))) \\
&= p_*(i_{\mathbb{P}*}(s_*(ch_A(x)))) \\
&= p_*(i_{1*}(i_*(ch_A(x)))) \\
&= i_*(ch_A(x))
\end{aligned}$$

This completes the proof of the proposition. □

4.2.4 Universal property of K-theory

We can now prove the following version of theorem 1.2.3 involving oriented weak cohomology theories on $\mathbf{Sm}_k$ (rather than genuine oriented cohomology theories):

Theorem 4.2.10. *Let A^* be an oriented weak cohomology theory on $\mathbf{Sm}_k$. Assume that A^* is multiplicative and periodic. Then there exists one and only one morphism of oriented weak cohomology theories*

$$ch_A : K^0[\beta, \beta^{-1}] \to A^*$$

Proof. For a vector bundle E on $X \in \mathbf{Sm}_k$, we write $c_i^A(E)$ for $\tilde{c}_i^A(E)(1_X)$. We first observe that ch_A has to map $\beta = [\mathbb{P}^1]_K$ to $b = [\mathbb{P}^1]_A$. Moreover, as for $X \in \mathbf{Sm}_k$ and any line bundle L over X with sheaf of sections $\mathcal{L}$, one has in $K^0(X)$

$$[\mathcal{L}] = 1 - (1 - [(\mathcal{L}^\vee)^\vee] = 1 - c_1^K(L^\vee) \cdot \beta,$$

one must have

$$ch_A([\mathcal{L}]) = ch_A(1 - c_1^K(L^\vee) \cdot \beta) = 1 - c_1^A(L^\vee) \cdot b.$$

Using the splitting principle, this establishes uniqueness.

Let A_* be the oriented Borel-Moore weak homology theory on $\mathbf{Sm}_k$ corresponding to A^*. We have already shown that the Chern character $ch_A : K^0[\beta, \beta^{-1}] \to A_*$ constructed above is a natural transformation of oriented Borel-Moore weak homology theories, which completes the proof. □

Remark 4.2.11. The classical Grothendieck-Riemann-Roch theorem can be easily deduced from theorem 4.2.10. Indeed, consider the oriented weak cohomology theory

$$X \mapsto \mathrm{CH}^*(X) \otimes \mathbb{Q}[\beta, \beta^{-1}]^{td}$$

constructed in example 4.1.24. Now by theorem 4.2.10 there exists one (and only one) morphism

$$\vartheta : K^0[\beta, \beta^{-1}] \to \mathrm{CH}^* \otimes \mathbb{Q}[\beta, \beta^{-1}]^{td}$$

of oriented cohomology theories. One then checks by the splitting principle that ϑ is equal in degree 0 to the Chern character

$$ch : K^0(X) \to \mathrm{CH}(X) \otimes \mathbb{Q},$$

where $\mathrm{CH}(X)$ denotes the ungraded Chow ring. The explicit formula for the push-forward maps in $\mathrm{CH}^* \otimes \mathbb{Q}[\beta, \beta^{-1}]^{td}$ yields the Grothendieck-Riemann-Roch theorem.

Corollary 4.2.12. *Let k be a field admitting resolution of singularities. Then for any smooth k-scheme X the natural homomorphism:*

$$\Omega^*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta, \beta^{-1}] \to K^0(X)[\beta, \beta^{-1}]$$

is an isomorphism.

Proof. By remark 4.1.12, $X \mapsto \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta, \beta^{-1}]$ is the universal oriented multiplicative periodic Borel-Moore weak homology theory (either on $\mathbf{Sch}_k$ or on $\mathbf{Sm}_k$). Theorem 4.2.10 implies on the other hand that $X \mapsto K^0(X)[\beta, \beta^{-1}]$ is also the universal oriented multiplicative periodic Borel-Moore weak homology theory on $\mathbf{Sm}_k$, whence the result. □

Remark 4.2.13. We do not know whether or not the canonical homomorphism

$$\Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta, \beta^{-1}] \to G_0(X)[\beta, \beta^{-1}]$$

is an isomorphism for all finite type k-schemes X. What follows from the previous theorem is that

$$\Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta, \beta^{-1}] \to K^{0BM}(X)[\beta, \beta^{-1}]$$

is an isomorphism for any finite type k-scheme X. The remaining problem is to decide whether or not

$$K^{0BM}(X)[\beta, \beta^{-1}] \to G_0(X)[\beta, \beta^{-1}]$$

is an isomorphism for any finite type k-scheme X, assuming k admits the resolution of singularities.

Theorem 1.2.3 follows easily from theorem 4.2.10:

Proof (of theorem 1.2.3). We use the fact (proposition 5.2.4) that an oriented cohomology theory on $\mathbf{Sm}_k$ gives rise to an oriented weak cohomology theory on $\mathbf{Sm}_k$ by restricting the pull-back maps to smooth morphisms, and setting $\tilde{c}_1(L)(x) = c_1(L) \cup x$.

By theorem 4.2.10, there is a unique natural transformation

$$ch_A : K^0[\beta, \beta^{-1}] \to A^*$$

of oriented weak cohomology theories on $\mathbf{Sm}_k$. We claim that ch_A commutes with all pull-back maps, not just pull-back for smooth morphisms. Indeed, the pull-backs in K-theory come from pulling back vector bundles, and the result follows from the naturality of Chern classes. Since ch_A is compatible with external products, this implies that ch_A is a ring homomorphism; ch_A is thus a natural transformation of oriented cohomology theories on $\mathbf{Sm}_k$, completing the proof. □

4.3 The cobordism ring of a point

In this section, we compute $\Omega_*(k)$ by showing that the canonical homomorphism

$$\Phi : \mathbb{L}_* \to \Omega_*(k)$$

is an isomorphism over any field admitting resolution of singularities and weak factorization (at present, this limits the result to fields of characteristic zero).

Our strategy to prove that $\mathbb{L}_* = \Omega_*(k)$ is to show first that Φ is always injective, for any field. Then to prove surjectivity it is sufficient to prove that the augmentation $\Omega_*(k) \to \mathbb{Z}$ induces an isomorphism $\Omega_*(k) \otimes_{\mathbb{L}*} \mathbb{Z} \cong \mathbb{Z}$. To do this we proceed inductively in the dimension.

4.3.1 The canonical splitting

In this section we prove that Φ is always a monomorphism, over any field. The first method is only valid when the field admits a complex embedding.

Lemma 4.3.1. *Let $\sigma : k \to \mathbb{C}$ be a complex embedding. Then sending a smooth projective k-scheme X to the complex cobordism class of the compact complex manifold $X_\sigma(\mathbb{C})$ descends to a homomorphism*

$$\psi : \Omega_*(k) \to MU_{2*},$$

which is independent of the choice of σ. In addition, composing ψ with Φ induces Quillen's isomorphism

$$\Phi_{MU} : \mathbb{L}_* \cong MU_{2*}.$$

In particular, Φ is injective.

Proof. An embedding σ gives a functor from the category of smooth k-schemes to the category of complex manifolds, which we denote by $X \mapsto X_\sigma(\mathbb{C})$. The functor $X \mapsto MU^{2*}(X_\sigma(\mathbb{C}))$ defines by [30] an oriented cohomology theory on $\mathbf{Sm}_k$; by proposition 5.2.4 this functor is also an oriented weak cohomology theory on $\mathbf{Sm}_k$ in the sense of definition 4.1.13. The universality of algebraic cobordism gives us a canonical map of oriented weak cohomology theories $\vartheta_{MU,\sigma}(X) : \Omega^*(X) \to MU^{2*}(X_\sigma(\mathbb{C}))$. In particular, $\vartheta_{MU,\sigma}(k)$ yields the ring homomorphism $\psi : \Omega_*(k) \to MU_{2*}$.

By construction, the composition $\mathbb{L}_* \to \Omega_*(k) \to MU_{2*}$ is the canonical homomorphism Φ_{MU}, which, by Quillen's result [30], is an isomorphism.

We conclude by showing that ψ doesn't depends on σ. Indeed, $\Omega_*(k)$ is generated by classes $[X]$ of smooth projective varieties over k. In addition, by Milnor [22], the class $[X_\sigma(\mathbb{C})] \in MU_{2*}$ depends only on the Chern numbers of X. But the latter can be computed algebraically and are thus independent of the choice of complex embedding. In other words, given two complex embedding σ and τ, the varieties $X_\sigma(\mathbb{C})$ and $X_\tau(\mathbb{C})$ are automatically cobordant. □

Another way to prove the injectivity, over any field, is as follows. We use the morphism

$$\vartheta^{CF} : \Omega_* \to \mathrm{CH}_*[\mathbf{t}]^{(\mathbf{t})}$$

defined in example 4.1.26. Its associated formal group law is by construction the power series

$$F_{\mathbf{t}}(u, v) = \lambda(\lambda^{-1}(u) + \lambda^{-1}(v)),$$

where

$$\lambda(u) = \sum_{i \geq 0} t_i u^{i+1} \in \mathbb{Z}[t_1, \ldots, t_n, \ldots][[u]],$$

with the convention that $t_0 = 1$. Here $\lambda^{-1}(u)$ is the inverse power series, satisfying $\lambda(\lambda^{-1}(u)) = u$. The statement of the following lemma is obvious, by construction, except for the injectivity assertion.

Lemma 4.3.2. *Let k be a field. Then the composition of the homomorphism*

$$\Omega_*(k) \to \mathbb{Z}[t_1, \ldots, t_n, \ldots]$$

induced by ϑ^{CF} and of

$$\Phi : \mathbb{L}_* \to \Omega_*(k)$$

is the canonical monomorphism

$$\mathbb{L}_* \to \mathbb{Z}[t_1, \ldots, t_n, \ldots]$$

classifying the formal group law $F_{\mathbf{t}}$.

For the injectivity in lemma 4.3.2, we refer the reader to [3, 30]. We can thus deduce:

Corollary 4.3.3. *Let k be any field. Then*

$$\Phi : \mathbb{L}_* \to \Omega_*(k)$$

is a monomorphism.

Remark 4.3.4. One can check that the homomorphism

$$\Omega_*(k) \to \mathbb{Z}[t_1, \ldots, t_n, \ldots]$$

sends the class $[X]$ of a smooth projective k-scheme X of dimension d to the sum

$$chern(X) := \sum_{\alpha_1, \ldots, \alpha_d} \langle c_1(N_X)^{\alpha_1} \ldots c_d(N_X)^{\alpha_d}, [X] \rangle$$

where the α_i's are non-negative integers, $N_X = -T_X \in K^0(X)$ is the virtual normal bundle[3] of X and $\langle x, [X] \rangle \in \mathbb{Z}$ denotes the degree of a class $x \in \mathrm{CH}^*(X)$ (which is zero unless x has codimension d).

Thus this homomorphism is just "computing all the Chern numbers" of X. A. Merkurjev has proven in [24] that for an arbitrary base-field k, all these $chern(X) \in \mathbb{Z}[t_1, \ldots, t_n, \ldots] =: \mathbb{Z}[\mathbf{t}]$ indeed lie in the (image of the) Lazard ring in $\mathbb{Z}[\mathbf{t}]$. This provides in fact a ring homomorphism

$$\Psi : \Omega_*(k) \to \mathbb{L}_*$$

which is left inverse to Φ, over any field. In the sequel, however, we will not use this fact, only corollary 4.3.3.

[3] Voevodsky has proven that N_X can always be represented as a difference $V - O_X^n$ for some vector bundle V over X [35].

4.3.2 The main theorem

Throughout this section we assume that k admits resolution of singularities and weak factorization.

Remark 4.3.5. For a field of characteristic zero, the weak factorization property has been proved in [2] and [37]. Of course, resolution of singularities follows for a field of characteristic zero by Hironaka [13]. Thus all the results in this section are valid for k a field of characteristic zero.

We are now prepared to begin the proof of the surjectivity of $\Phi : \mathbb{L}_* \to \Omega_*(k)$ which will thus finish the proof of theorem 1.2.7, restated as theorem 4.3.7 below. In the sequel, we denote by $\Omega_*^{ad}(k)$ the ring $\Omega_*(k) \otimes_{\mathbb{L}_*} \mathbb{Z}$.

We first show that the class of W in $\Omega_*^{ad}(k)$ is a birational invariant.

Proposition 4.3.6. *Let W and W' be smooth projective varieties over k. Suppose that W and W' are birationally isomorphic. Then $[W'] = [W]$ in $\Omega_*^{ad}(k)$.*

Proof. The proof uses weak factorization and resolution of singularities in an essential way. By resolution of singularities, we may assume there is a birational morphism $W \to W'$. By weak factorization, there is a sequence of birational morphisms

$$W = W_0 \leftarrow Y_0 \to W_1 \leftarrow \ldots \leftarrow Y_n \to W_n = W'$$

each of which is the blow-up along a smooth center. This reduces us to the case $W' = W_F \to W$, the blow-up of W along a smooth center F. We conclude by using corollary 4.2.5. □

Theorem 4.3.7. *Let k be a field admitting resolution of singularities and weak factorization. Then the natural map $\Phi : \mathbb{L}_* \to \Omega_*(k)$ is an isomorphism.*

Proof. By corollary 4.3.3, we need only to show that Φ is surjective. Noting that $\mathbb{L}_*$ and $\Omega_*(k)$ are graded rings, and are zero in negative degrees, that

$$\Omega_*^{ad}(k) = \Omega_*(k)/\mathbb{L}_{*<0}\Omega_*(k)$$

and that $\mathbb{L}_0 = \mathbb{Z}$, it suffices to show that $\Omega_*^{ad}(k) \cong \mathbb{Z}$. By theorem 2.5.12, the degree map $\Omega_0(k) \to \mathbb{Z}$ is an isomorphism. We now show that $\Omega_n^{ad}(k) = 0$ for all $n > 0$.

Let Y be a smooth irreducible projective variety of dimension n over k. Embed Y in a $\mathbb{P}^N$, and take a general linear projection of Y to a $\bar{Y} \subset \mathbb{P}^{n+1}$, with $Y \to \bar{Y}$ finite and birational. Let $\mu : S \to \mathbb{P}^{n+1}$ be a sequence of blow-ups with smooth centers lying over $\bar{Y}_{\text{sing}}$ such that $\mu^*(\bar{Y})$ is a strict normal crossing divisor. Write

$$\mu^*(\bar{Y}) = \tilde{Y} + \sum_i n_i E_i,$$

where $\tilde{Y}$ is the proper transform of $\bar{Y}$, and the E_i are components of the exceptional divisor of μ.

Since $\tilde{Y} \to \bar{Y}$ and $Y \to \bar{Y}$ are birational isomorphisms, $\tilde{Y}$ is birationally isomorphic to Y. Thus, by proposition 4.3.6, we have $[\tilde{Y}] = [Y]$ in $\Omega_n^{ad}(k)$. Write μ as a composition of blow-ups of S_i along the smooth center F_i:

$$S = S_0 \xrightarrow{\mu_1} S_1 \xrightarrow{\mu_2} \dots \xrightarrow{\mu_r} S_r = \mathbb{P}^{n+1}.$$

Let $\bar{E}_i \subset S_{i-1}$ be the exceptional divisor $\mu_i^{-1}(F_i)$; $\bar{E}_i$ is the projective bundle $\mathbb{P}(\mathcal{N}_i) \to F_i$, where $\mathcal{N}_i$ is the conormal sheaf of F_i in S_i. Reordering the E_i, the map $S \to S_{i-1}$ restricts to a birational morphism $E_i \to \bar{E}_i$. By remark 4.2.4 and proposition 4.3.6, it follows that $[E_i] = 0$ in $\Omega_n^{ad}(k)$.

Suppose $\bar{Y}$ has degree d in $\mathbb{P}^{n+1}$. Let $D \subset \mathbb{P}^{n+1}$ be the divisor of a general section of $\mathcal{O}(d)$. Then both D and μ^*D are smooth and irreducible, and $\mu : \mu^*D \to D$ is birational. Thus $[\mu^*D] = [D]$ in $\Omega_n^{ad}(k)$ by proposition 4.3.6. On the other hand, μ^*D is linearly equivalent to $\mu^*(\bar{Y})$, hence $[\mu^*D \to S] = [\mu^*\bar{Y} \to S]$ in $\Omega_n(S)$. Pushing forward to $\operatorname{Spec} k$ and using remark 3.1.7, we find

$$[D] = [\mu^*D] = [\mu^*\bar{Y}] = [\tilde{Y}] = [Y] \in \Omega_n^{ad}(k)$$

Furthermore, D is linearly equivalent to d hyperplanes in $\mathbb{P}^{n+1}$, so by remark 3.1.7 and lemma 4.2.3,

$$[D] = d[\mathbb{P}^n] = 0 \in \Omega_n^{ad}(k),$$

completing the proof. □

Remark 4.3.8. It is reasonable to make the

Conjecture. *Let k be a field. Then*

$$\Phi(k) : \mathbb{L}_* \to \Omega_*(k)$$

is an isomorphism.

We have proven this conjecture in characteristic zero and we know that the map $\Phi(k)$ is always injective. Moreover, theorem 2.5.12 shows that this conjecture is true in degree 0 over any field. Furthermore, one can check that our previous proof can be carried out for smooth curves over a field, proving the conjecture in degree 1 as well.

In fact, the group $\Omega_1(k)$ is the free abelian group generated by $[\mathbb{P}^1]$. Moreover, let C be a smooth projective curve over k, g its genus (the genus of its extension to an algebraic closure of k). Then

$$[C] = (1 - g) \cdot [\mathbb{P}^1]$$

in $\Omega_1(k)$.

Finally, it seems reasonable to suppose that once one assumes that k admits resolution of singularities, the weak factorization theorem can be proved using the methods of [2] and [37], in which case theorem 4.3.7 would be valid over a field admitting resolution of singularities.

4.3.3 Birationally invariant theories

In this section we prove the following theorem establishing conjecture 4.2.7 for fields admitting resolution of singularities and weak factorization.

Theorem 4.3.9. *Let k be a field admitting resolution of singularities and weak factorization. Then the oriented Borel-Moore functor of geometric type*

$$X \mapsto \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta]$$

is the universal oriented Borel-Moore functor of geometric type which has "birational invariance" in the following sense: given a birational projective morphism $f : Y \to X$ between smooth irreducible varieties, then $f_* 1_Y = 1_X$.

We start by proving the following:

Proposition 4.3.10. *Let $I \in \mathbb{L}_*$ denote the ideal generated by elements of the form $[W] - [W']$, with W and W' be smooth projective varieties over k which are birationally isomorphic. Then I is the kernel of the map*

$$\Omega_*(k) = \mathbb{L}_* \to \mathbb{Z}[\beta]$$

classifying the multiplicative formal group law.

Proof. Let W be a smooth projective irreducible k-variety. The image of $[W]$ in $\mathbb{Z}[\beta]$ will be denoted by $[W]_\beta$ in the sequel.

First, if W and W' are smooth projective varieties over k which are birationally isomorphic then by resolution of singularities and weak factorization there is a sequence of birational morphisms

$$W = W_0 \leftarrow Y_0 \to W_1 \leftarrow \ldots \leftarrow Y_n \to W_n = W'$$

each of which is the blow-up along a smooth center. By corollary 4.2.5, we conclude that $[W]_\beta = [W']_\beta$ proving that I is contained in the kernel Ker of $\Omega_*(k) = \mathbb{L}_* \to \mathbb{Z}[\beta]$.

To prove the converse inclusion $Ker \subset I$ we proceed by induction on degree. If we write the universal formal group law as $F_{\mathbb{L}}(u, v) = u + v + \sum_{i,j\geq 1} a_{ij} u^i v^j$, then the a_{ij} are generators for $\mathbb{L}_*$ as a $\mathbb{Z}$-algebra and the homomorphism $\mathbb{L}_* \to \mathbb{Z}[\beta]$ sends a_{11} to β and a_{ij} to zero if $(i,j) \neq (1,1)$. The Ker is thus the ideal generated by the a_{ij} with $(i,j) \neq (1,1)$. Fix $n, m \geq 1$ and let us assume by induction that $a_{ij} \in I$ for $i + j < n + m$, $(i,j) \neq (1,1)$. By remark 2.5.8 we see that

$$\begin{aligned} a_{nm} \equiv [H_{n,m}] - [\mathbb{P}^n \times \mathbb{P}^{m-1}] \\ - [\mathbb{P}^{n-1} \times \mathbb{P}^m] + [\mathbb{P}^1 \times \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}] \mod I. \end{aligned}$$

As $\mathbb{P}^n \times \mathbb{P}^{m-1}$, $\mathbb{P}^{n-1} \times \mathbb{P}^m$ and $\mathbb{P}^1 \times \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ are all birational to $\mathbb{P}^{n+m-1}$, the result follows from corollary 4.2.5 and the lemma below. □

Lemma 4.3.11. *For each pair (n, m) of positive integers, the Milnor hypersurface $H_{n,m} \subset \mathbb{P}^n \times \mathbb{P}^m$ is birational to $\mathbb{P}^{n+m-1}$.*

Proof. Assume $n \le m$. Then the projection $H_{n,m} \to \mathbb{P}^n$ realizes $H_{n,m}$ as a $\mathbb{P}^{m-1}$-bundle over $\mathbb{P}^n$. More precisely, if $X_0, \ldots, X_n$ and $Y_0, \ldots, Y_m$ are standard coordinates on $\mathbb{P}^n$ and $\mathbb{P}^m$, one can use the section $\sum_{i=0}^n X_i Y_i$ of $O(1,1)$ to define $H_{n,m}$, and then $H_{n,m} = \mathrm{Proj}_{\mathbb{P}^n}(E)$, where E is the kernel of the surjection $O_{\mathbb{P}^n}^{m+1} \to O_{\mathbb{P}^n}(1)$ with matrix $(X_0, ..., X_n, 0, \ldots, 0)$. Thus $H_{n,m}$ is birational to $\mathbb{P}^{n+m-1}$. □

Proof (of theorem 4.3.9). Let A_* be an oriented Borel-Moore functor of geometric type which has birational invariance. By birational invariance of A_*, we see that the map $\mathbb{L}_* = \Omega_*(k) \to A_*(k)$ vanishes on the ideal I considered in proposition 4.3.10. From that proposition, the map $\mathbb{L}_* = \Omega_*(k) \to A_*(k)$ factors through $\mathbb{L}_* \to \mathbb{Z}[\beta]$. Thus the map

$$\Omega_*(X) \to A_*(X)$$

given by the universality of Ω_* induces a canonical morphism

$$\Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta] \to A_*(X).$$

Thus it remains only to prove that the theory $X \mapsto \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta]$ itself has birational invariance. Given a birational projective morphism $f : Y \to X$ between smooth irreducible varieties, then by weak factorization there exists a finite sequence of blow-up and blow-down with smooth centers in the category of X-schemes starting with Y and ending with X. So we reduce to proving $f_* 1_Y = 1_X$ when f is a blow-up with smooth center, and this clearly follows from corollary 4.2.5. □

4.4 Degree formulas

We use algebraic cobordism to give a proof of Rost's degree formula, and various other related formulas; these are all consequences of the *generalized degree formula*. The generalized degree formula follows from some of the basic structural properties of algebraic cobordism; for this reason, we give a treatment of the general degree formula for an oriented Borel-Moore weak homology theory sharing these properties.

4.4.1 The degree homomorphism

Let A_* be an oriented Borel-Moore weak homology theory on $\mathbf{Sch}_k$. For a finitely generated field extension $k \subset F$, one denotes by $A_*(F/k)$ the colimit over the category of all models[4] X of F over k of the groups

[4] A model of F over k is an integral finite type k-scheme X together with an isomorphism between the field F and the field of functions of X.

$A_{*+\deg.tr(F/k)}(X)$. We observe that the smooth pull-backs in A_* make the assignment $F \mapsto A_*(F/k)$ covariantly functorial with respect to separable field extensions: given a separable extension $\phi : F \subset L$ of fields which are finitely generated over k, one has a homomorphism $\phi^* : A_*(F/k) \to A_*(L/k)$, functorial in towers.

For instance, given an integral k-scheme X with function field $F = k(X)$, then $A_*(F/k)$ can be identified with the colimit

$$\mathrm{colim}_{U \subset X} A_{*+\dim_k X}(U)$$

where U ranges over the set of non-empty open subsets of X. Letting $i : \eta \to X$ denotes the generic point of X, we denote by

$$i^* : A_{*+\dim_k X}(X) \to A_*(F/k)$$

the canonical homomorphism.

Definition 4.4.1. *The oriented Borel-Moore weak homology theory A_* is said to be* generically constant *if for each finitely generated separable field extension $k \subset F$ the canonical morphism $A_*(k) \to A_*(F/k)$ is an isomorphism.*

For instance, the Chow group functor has this property; recall that $\mathrm{CH}_*(F/k) = \mathbb{Z}$ placed in degree zero, for any finitely generated field extension $k \subset F$. The K-theory functor $K^0[\beta, \beta^{-1}]$ has this property as well. We now proceed to prove that algebraic cobordism also satisfies this property in characteristic zero; the proof will crucially rely on theorem 4.3.7.

Let $k \subset F$ be a finitely generated field extension of characteristic zero. We define a ring homomorphism

$$\Omega_*(F/k) \to \Omega_*(F)$$

as follows: The group $\Omega_*(F/k)$ is generated by classes of the form $[f : Y \to X]$, with $f : Y \to X$ a projective morphism, Y smooth and irreducible, and X integral with field of function F. Let η be the generic point of X. Then, since the characteristic is zero, the generic fiber Y_η of f, which is a projective F-scheme, is also a smooth F-scheme. The assignment $[Y \to X] \mapsto [Y_\eta]$ then induces the desired homomorphism. Indeed, it is easy to check that the kernel of $\mathcal{M}(X)^+ \to \Omega_*(X)$ maps to zero, and that the resulting homomorphism $\Omega_*(X) \to \Omega_*(F)$ is natural on the category of models of F over k, hence descends to the direct limit $\Omega_*(F/k)$.

Lemma 4.4.2. *Let k be a field of characteristic zero, then for a finitely generated field extension $k \subset F$ the homomorphism*

$$\Omega_*(F/k) \to \Omega_*(F)$$

is an isomorphism

Proof. Let X denote an integral model for F. Since each $f : Y \to \eta$ in $\mathcal{M}(\eta)$ is projective over η, it is clear that i^* induces an isomorphism

$$i^* : \lim_{\overrightarrow{U}} \mathcal{M}(U) \to \mathcal{M}(F).$$

If $Y \to \eta$ is in $\mathcal{M}(\eta)$, then each invertible sheaf $\mathcal{L}$ on Y is the restriction of an invertible sheaf $\tilde{\mathcal{L}}$ on $\tilde{Y}$ for some open U and some $\tilde{f} : \tilde{Y} \to U$ in $\mathcal{M}(U)$ inducing $Y \to \eta$. Similarly, each smooth morphism $q : Y \to Z$ and and each smooth divisor $i : D \to Y$ over η extend to a smooth morphism $\tilde{q} : \tilde{Y} \to \tilde{Z}$ and a smooth divisor $\tilde{i} : \tilde{D} \to \tilde{Y}$ over some $U \subset X$. Considering the maps

$$i^* : \lim_{\overrightarrow{U}} \underline{\mathcal{Z}}_{*+\dim_k X}(U) \to \Omega_*(F)$$

and

$$i^* : \lim_{\overrightarrow{U}} \underline{\Omega}_{*+\dim_k X}(U) \to \Omega_*(F)$$

shows relations for $\Omega_*(F)$ lift to relations in the limit, hence

$$i^* : \lim_{\overrightarrow{U}} \Omega_{*+\dim_k X}(U) \to \Omega_*(F)$$

is an isomorphism. □

Corollary 4.4.3. *In characteristic zero, algebraic cobordism is generically constant.*

Proof. One easily checks that the diagram

$$\begin{tikzcd}
\Omega_*(k) \arrow[rr, "\pi^*"] & & \Omega_*(F) \\
& \mathbb{L}_* \arrow[ul, "\Phi_k"] \arrow[ur, "\Phi_F"'] &
\end{tikzcd}$$

commutes, where π^* is the base-change map. By theorem 4.3.7 the maps Φ_k and Φ_F are isomorphisms, hence π^* is an isomorphism. Using lemma 4.4.2 completes the proof. □

Definition 4.4.4. *Let A_* be a generically constant oriented Borel-Moore weak homology theory on* $\mathbf{Sch}_k$. *Let X be a reduced irreducible finite type k-scheme. Let η be the generic point of X, with inclusion $i_\eta : \eta \to X$, and let $p_\eta : \eta \to \operatorname{Spec} k$ be the structure morphism. The map $p_\eta^* : A_*(k) \to A_*(k(\eta)/k)$ is an isomorphism by assumption, hence we have the homomorphism* $\deg : A_*(X) \to A_{*-\dim_k X}(k)$ *defined by* $\deg = (p_\eta^*)^{-1} \circ i_\eta^*$.

More generally, if X is a reduced finite type k-scheme with irreducible components $X_1, \ldots, X_r$, we have the homomorphisms

$\deg_i : A_*(X) \to A_{*-\dim_k X_i}(k)$, *$i = 1, \ldots, r$, defined by*

$$\deg_i := (p_{\eta_i}^*)^{-1} \circ i_{\eta_i}^*$$

where η_i is the generic point of X_i.

On classes $[f : Y \to X]_A$ with f separable of relative dimension zero, the degree homomorphism is just the classical notion of degree:

Lemma 4.4.5. *Let A_* be an oriented Borel-Moore weak homology theory on $\mathbf{Sm}_k$ and let $f : Y \to X$ be a projective morphism between smooth irreducible k-schemes of the same dimension; if f is dominant, we assume in addition that f is separable. Then one has the equality*

$$\deg([Y \to X]) = [k(Y) : k(X)] \cdot 1_X$$

in $A_*(k(X)/k)$.

Proof. The proof is basically the same as that of theorem 2.5.12, taking into account that one may replace X by any of its non-empty open subsets, so one may assume that $X = \operatorname{Spec} R$ is affine and that f corresponds to an elementary étale algebra $R[T]/P$. □

4.4.2 The generalized degree formula

Definition 4.4.6. *Let A_* be an oriented Borel-Moore weak homology theory on $\mathbf{Sch}_k$. We say that A_* has the localization property if for any closed immersion $i : Z \to X$ with $j : U \subset X$ the complementary open immersion, the sequence:*

$$A_*(Z) \xrightarrow{i_*} A_*(X) \xrightarrow{j^*} A_*(U) \to 0$$

is exact.

For instance, the Chow group functor and the K-theory functor have this property over any field. Algebraic cobordism has the localization property as well, at least assuming resolution of singularities over k, by theorem 3.2.7.

Let X be a finite type k-scheme, Z an integral closed subscheme. We write $\operatorname{codim}_X Z > 0$ to mean that Z contains no generic point of X.

Theorem 4.4.7 (Generalized degree formula). *Let k be a field. Let A_* be an oriented Borel-Moore weak homology theory on $\mathbf{Sch}_k$. Assume that A_* is generically constant and has the localization property.*

Let X be a reduced finite type k-scheme. Assume that, for each closed integral subscheme $Z \subset X$, we are given a projective birational morphism $\tilde{Z} \to Z$ with $\tilde{Z}$ in $\mathbf{Sm}_k$. Then the $A_*(k)$-module $A_*(X)$ is generated by the classes $[\tilde{Z} \to X]$.

More precisely, let $X_1, \ldots, X_r$ be the irreducible components of X. Let α be an element of $A_*(X)$. Then, for each integral subscheme $Z \subset X$ with $\operatorname{codim}_X Z > 0$, there is an element $\omega_Z \in A_{*-\dim_k Z}(k)$, all but finitely many being zero, such that

$$\alpha - \sum_{i=1}^{r} \deg_i(\alpha) \cdot [\tilde{X}_i \to X] = \sum_{Z,\ \operatorname{codim}_X Z > 0} \omega_Z \cdot [\tilde{Z} \to X].$$

One should observe that we don't use the resolution of singularities in the proof. The problem of course is to find a desingularization $\tilde{Z} \to Z$ of each closed integral subscheme of X. See theorem 4.4.9 below for a variant of theorem 4.4.7 which uses de Jong's theorem [6].

Proof (of theorem 4.4.7). We proceed by noetherian induction. For U_i an open subscheme of X_i, we let $\tilde{U}_i$ denote the inverse image of U_i by $\tilde{X}_i \to X$.

Let $\alpha \in A_*(X)$. Then the element

$$\alpha - \sum_{i=1}^{r} \deg_i(\alpha) \cdot [\tilde{X}_i \to X]$$

vanishes upon applying each of the homomorphisms $\deg_i$. Thus there is for each i an open subscheme $j_i : U_i \to X$, containing the generic point of X_i and disjoint from $\cup_{j \neq i} X_j$, such that $j_i^* \alpha = \deg_i(\alpha) \cdot [\tilde{U}_i \to U_i]$ in $\Omega_*(U_i)$.

Thus, letting $U = \cup_{i=1}^r U_i$, with inclusion $j : U \to X$, we have

$$j^*\big(\alpha - \sum_{i=1}^{r} \deg_i(\alpha) \cdot [\tilde{X}_i \to X]\big) = 0$$

in $\Omega_*(U)$. Let $W = X \setminus U$ with closed immersion $i : W \to X$. By the localization property of A_*, there is an element α^1 of $A_*(W)$ such that

$$\alpha = \sum_{i=1}^{r} \deg_i(\alpha) \cdot [\tilde{X}_i \to X] + i_*(\alpha^1)$$

Each closed integral subscheme $Z \subset W$ is also a closed integral subscheme in X thus we have our projective birational morphisms $\tilde{Z} \to Z$ with $\tilde{Z} \in \mathbf{Sm}_k$. We then apply the inductive hypothesis to W together with the given family of projective morphisms $\tilde{Z} \to W$, where Z ranges over all the closed integral subscheme $Z \subset W$. We get an expression of our class $\alpha^1 \in A_*(W)$ as

$$\alpha^1 = \sum_{Z \subset W} \omega_Z \cdot [\tilde{Z} \to W]$$

Together with our previous expression we thus get

$$\begin{aligned} \alpha &= \sum_{i=1}^{r} \deg_i(\alpha) \cdot [\tilde{X}_i \to X] + i_*(\alpha^1) \\ &= \sum_{i=1}^{r} \deg_i(\alpha) \cdot [\tilde{X}_i \to X] + \sum_{Z \subset W} \omega_Z \cdot [\tilde{Z} \to X], \end{aligned}$$

proving the theorem. □

Corollary 4.4.8. *With the assumptions as in theorem 4.4.7, suppose in addition that each X_i is in $\mathbf{Sm}_k$.*

(1) Let $f : Y \to X$ be a projective morphism with Y in $\mathbf{Sm}_k$. Then, for each integral subscheme $Z \subset X$ with $\mathrm{codim}_X Z > 0$, there is an element $\omega_Z \in A_{*-\dim_k Z}(k)$, all but finitely many being zero, such that

$$[f : Y \to X] - \sum_i \deg_i(f) \cdot [X_i \to X] = \sum_{Z,\ \mathrm{codim}_X Z>0} \omega_Z \cdot [\tilde{Z} \to X]$$

(2) Suppose that X is irreducible. Let $f : Y \to X$ be a projective birational morphism with Y in $\mathbf{Sm}_k$. Then, for each integral subscheme $Z \subset X$ with $\mathrm{codim}_X Z > 0$, there is an element $\omega_Z \in A_{*-\dim_k Z}(k)$, all but finitely many being zero, such that

$$[f : Y \to X] = [\mathrm{Id}_X] + \sum_{Z,\ \mathrm{codim}_X Z>0} \omega_Z \cdot [\tilde{Z} \to X]$$

Proof. Since X_i is in $\mathbf{Sm}_k$, we may take $\tilde{X}_i \to X$ to be $X_i \to X$. The first assertion then follows from theorem 4.4.7. The second follows from the first, noting that $\deg f = 1 \in A_*(k)$ if f is birational (see lemma 4.4.5). □

Theorem 4.4.9 (Rational generalized degree formula). *Let k be perfect field. Let A_* be an oriented Borel-Moore weak homology theory on $\mathbf{Sch}_k$ such that $A_*(k)$ is a $\mathbb{Q}$-algebra. Assume A_* is generically constant and that it satisfies the localization property.*

Let X be a reduced finite type k-scheme. For each closed integral subscheme $Z \subset X$ choose a projective morphism $f_Z : \tilde{Z} \to Z$ with $\tilde{Z}$ smooth over k and f_Z generically étale[5].

Then the $A_*(k)$-module $A_*(X)$ is generated by the classes $[\tilde{Z} \to X]$. More precisely, let $X_1, \ldots, X_r$ be the irreducible components of X, and let α be an element of $A_*(X)$. Then, for each $Z \subset X$ with $\mathrm{codim}_X Z > 0$, there is an element $\omega_Z \in A_{*-\dim_k Z}(k)$, all but finitely many being zero, such that

$$\alpha - \sum_{i=1}^{r} \deg_i(\alpha) \cdot [\tilde{X}_i \to X] = \sum_{Z,\ \mathrm{codim}_X Z>0} \omega_Z \cdot [\tilde{Z} \to X].$$

Remark 4.4.10. Under the assumptions of theorem 4.4.9, the analog of corollary 4.4.8 is also valid. In particular, if X is in $\mathbf{Sm}_k$ and is irreducible, then, given $\alpha \in A_*(X)$, there exists, for each $Z \subset X$ of codimension > 0, elements $\omega_Z \in A_*(k)$, all but a finite number being zero, such that

$$\alpha - \deg(\alpha) \cdot [\mathrm{Id}_X] = \sum_{Z,\ \mathrm{codim}_X Z>0} \omega_Z \cdot [\tilde{Z} \to X]$$

[5] This is possible by de Jong's theorem.

The proof of theorem 4.4.9 is exactly the same as for theorem 4.4.7, using lemma 4.4.5.

Remark 4.4.11. Let A_* denote an oriented Borel-Moore weak homology theory which is generically constant and satisfies the localization property. Then theorem 4.4.7 implies that the natural map

$$\begin{aligned} A_*(k) \otimes_{\mathbb{Z}} \mathcal{M}(X)_*^+ &\to A_*(X) \\ \omega \otimes [Y \to X]) &\mapsto \omega \cdot [Y \to X]_A \end{aligned}$$

is surjective. Thus the morphism $\Omega_*(X) \otimes_Z A_*(k) \to A_*(X)$ must be surjective as well.

In particular, if we assume further that the ring $A_*(k)$ is generated as a group by classes $[X]$ of smooth projective varieties over k, then it follows that $\Omega_*(X) \to A_*(X)$ is surjective.

We also get:

Corollary 4.4.12. *Let k be a field admitting resolution of singularities. Let A_* be an oriented Borel-Moore weak homology theory on* $\mathbf{Sch}_k$. *Assume A_* is generically constant and that it satisfies the localization property. Then for any reduced finite type k-scheme X, the $A_*(k)$-module $A_*(X)$ is generated over $A_*(k)$ by the classes of degree $\in \{0, \ldots, \dim(X)\}$.*

In addition, suppose that X has pure dimension d over k. Let $X_1, \ldots, X_n$ be the irreducible components of X, and let $\tilde{X}_i \to X_i$ be a projective birational morphism with $\tilde{X}_i$ smooth. Then $A_*(X)$ is generated over $A_*(k)$ by the classes $[\tilde{X}_i \to X] \in A_d(X)$ and classes of degree $\in \{0, \ldots, \dim(X) - 1\}$.

Let A_* be an oriented Borel-Moore weak homology theory on $\mathbf{Sch}_k$ and let X be a reduced finite type k-scheme with irreducible components $X_1,\ldots,X_s$. Let $d_i = \dim_k X_i$ and let $\iota_i : X_i \to X$ be the inclusion. We let $\tilde{A}_*(X)$ denote the kernel of the total degree homomorphism $\sum_i \deg_i : A_*(X) \to \oplus_i A_{*-d_i}(k(X_i)/k)$. Assume that A_* is generically constant and that each X_i is in $\mathbf{Sm}_k$. We have the map $p_i^* : A_{*-d_i}(k) \to A_*(X)$ defined by sending α to $\alpha \cdot [X_i \to X]_A$. The composition

$$A_*(k) \xrightarrow{p_i^*} A_{*+d_i}(X) \to A_*(k(X_i)/k)$$

is an isomorphism, so, via the maps p_i^*, one gets a direct sum decomposition as $A_*(k)$-modules

$$A_*(X) = \tilde{A}_*(X) \oplus \oplus_{i=1}^s A_{*-d_i}(k)$$

even if each X_i has no rational k-point!

Corollary 4.4.13. *Let k be a field admitting resolution of singularities. Let A_* be an oriented Borel-Moore weak homology theory on* $\mathbf{Sch}_k$ *that is generically constant and satisfies the localization property. Let X be in* $\mathbf{Sch}_k$. *Choose* *for each integral closed subvariety Z of X a projective birational morphism $\tilde{Z} \to Z$ with $\tilde{Z}$ in $\mathbf{Sm}_k$. Then the $A_*(k)$-module $\tilde{A}_*(X)$ is generated by the classes $[\tilde{Z} \to X]_A$, as $Z \subset X$ runs over all irreducible closed subsets which contain no generic point of X.*

This is clear from theorem 4.4.7.

Given an oriented Borel-Moore weak homology theory A_* and a projective k-scheme X of pure dimension $d > 0$, denote by $M(X) \subset A_*(k)$ the ideal generated by classes $[Y]_A \in A_*(k)$ of smooth projective k-schemes Y of dimension $\dim_k(Y) < d$ for which there exists a (projective) morphism $Y \to X$ over k.

Theorem 4.4.14. *Let k be a field admitting resolution of singularities. Let A_* be an oriented Borel-Moore weak homology theory on $\mathbf{Sch}_k$ that is generically constant and satisfies the localization property. Then for any pure dimensional reduced projective k-scheme X, the ideal $M(X)$ is the image of $\tilde{A}_*(X)$ under the push-forward $\pi_* : A_*(X) \to A_*(k)$ associated to $\pi : X \to \operatorname{Spec} k$.*

This easily follows from corollary 4.4.13.

4.4.3 Consequences for Ω_*

Let k be a field of characteristic zero. As Ω_* is generically constant and satisfies the localization property on $\mathbf{Sch}_k$, all the results of §4.4.2 are valid for Ω_*. In addition, since push-forward by the inclusion $X_{\mathrm{red}} \to X$ induces an isomorphism $\Omega_*(X_{\mathrm{red}}) \to \Omega_*(X)$, the definitions and results of §4.4.2 extend to non-reduced schemes by passing from X to X_{red}.

Rost has described a number of so-called "degree formulas" which relate the degree of a map $f : Y \to X$ of smooth projective varieties, the Segre numbers of X and Y, and the degrees of zero-cycles on X. As pointed out in [23], these all follow from a formula in the cobordism of X, called the generalized degree formula.

Given a smooth projective irreducible k-scheme X of dimension $d > 0$, Rost introduces the ideal $M(X) \subset \mathbb{L}_* = \Omega_*(\operatorname{Spec} k)$ generated by classes $[Y] \in \mathbb{L}_*$ of smooth projective k-schemes Y of dimension $\dim_k(Y) < d$ for which there exists a morphism $Y \to X$ over k. We extend this definition to projective X which are pure dimension d over k by taking $A_* = \Omega_*$ and using the definition given in the previous section §4.4.2.

We now recall the statement of theorem 1.2.15:

Theorem 4.4.15. *Let k be a field of characteristic zero. For a morphism $f : Y \to X$ between smooth projective irreducible k-schemes one has*

$$[Y] - \deg(f) \cdot [X] \in M(X)$$

This is an immediate consequence of the theorem 4.4.7 applied to algebraic cobordism, by pushing forward the identity in theorem 4.4.7 from $\Omega_*(X)$ to $\Omega_*(k)$ Note also that, by theorem 2.5.12, $\deg(f)$ is as it usually is, i.e., the degree of the field extension $k(Y)/k(X)$ if f is surjective. and zero if not.

We also prove:

Theorem 4.4.16. *For $\pi : X \to \operatorname{Spec} k$ projective and pure dimensional, the ideal $M(X) \subset \mathbb{L}_*$ is the image of $\tilde{\Omega}_*(X)$ by the push-forward*

$$\pi_* : \Omega_*(X) \to \Omega_*(k).$$

This an immediate consequence of theorem 4.4.14.

Theorem 4.4.17. *Let k be a field of characteristic zero. Let X be a smooth projective k-variety.*

1. *The ideal $M(X)$ is a birational invariant of X.*
2. *The class of X modulo $M(X)$:*

$$[X] \in \mathbb{L}^*/M(X)$$

is a birational invariant of X as well.

Proof. For (1), if $f : Y \to X$ is a birational morphism, then clearly $M(Y) \subset M(X)$. For the converse inclusion, we claim that, for each point $z \in X$, there is a point $z' \in Y$ with $f(z') = z$ and with $f^* : k(z) \to k(z')$ an isomorphism. Assuming this to be the case, we see that, for each irreducible closed subvariety Z of X, there is an irreducible closed subvariety Z' of Y with $f(Z') = Z$ and with $f : Z' \to Z$ birational. Letting $\tilde{Z} \to Z'$ be a resolution of singularities of Z', we may use $\tilde{Z} \to Z$ as the chosen resolution of singularities in theorem 4.4.7. In particular, by corollary 4.4.13, $\tilde{\Omega}_*(X)$ is the $\Omega_*(k)$-submodule of $\Omega_*(X)$ generated by the classes $[\tilde{Z} \to X]$, with Z a proper subvariety of X. Since $[\tilde{Z} \to X]$ lifts to the element $[\tilde{Z} \to Y]$ of $\tilde{\Omega}_*(Y)$, the result follows from theorem 4.4.14.

To prove our claim, there is a sequence of blow-ups of X with smooth centers $Y' \to X$ which dominates $Y \to X$. It clearly suffices to prove the claim for $Y' \to X$, which in turn reduces the problem to the case of a single blow-up $f : Y \to X$, with smooth center C. Let z be in X. If z is not in C, then we may take z' to be the single point $f^{-1}(z)$. If z is in C, then the fiber $f^{-1}(z)$ is a projective space $\mathbb{P}^r_{k(z)}$ for some r. We may then take z' to be any $k(z)$-rational point of $\mathbb{P}^r_{k(z)}$. This completes the proof of (1).

(2) is an immediate consequence of theorem 4.4.15 with $\deg(f) = 1$. $\square$

Corollary 4.4.18. *Let X be a pure dimensional projective k-scheme. Then the ideal $M(X)$ in $\Omega_*(k)$ is stable under the action of Landweber-Novikov operations.*

Proof. By example 4.1.25, the Landweber-Novikov operations define a morphism of weak oriented Borel-Moore homology theories:

$$\vartheta^{LN} = \sum_I S_I t^I : \Omega_* \to \Omega_*[\mathbf{t}]^{\mathbf{t}}$$

which in particular commutes with push-forward. With the help of corollary 4.4.13, this implies that $\tilde{\Omega}_*(X)$ is stable under the Landweber-Novikov operations. Noting these facts, the result follows directly from theorem 4.4.16. □

A. Merkurjev has given a proof of corollary 4.4.18 over any field in [24].

4.4.4 Rost's degree formulas

Rost's degree formula (corollary 1.2.17), and the higher degree formulas discussed in [4] are simple consequences of the generalized degree formula corollary 4.4.8 (for algebraic cobordism) and properties of the relevant characteristic classes.

Let $P = P(x_1, \ldots, x_d)$ be a weighted-homogeneous polynomial of degree d (where we give x_i degree i) with coefficients in some commutative ring R. If X is a smooth projective variety over k of dimension d, define

$$P(X) := \deg P(c_1, \ldots, c_d)(N_X) \in \mathrm{CH}_0(k) \otimes R \cong R.$$

where $N_X \in K^0(X)$ is the virtual normal bundle of X. Since P is obvious additive under disjoint union, we have the homomorphism $\hat{P} : \mathcal{M}^+(k) \to R$.

Lemma 4.4.19. *$\hat{P} : \mathcal{M}^+(k) \to R$ descends uniquely to an R-linear map*

$$P : \Omega_d(k) \otimes_{\mathbb{Z}} R \to R.$$

Proof. Since $\mathcal{M}^+(k) \to \Omega_*(k)$ is surjective, P is unique. To show existence, we have the natural transformation $\vartheta^{CF} : \Omega_* \to \mathrm{CH}_*[\mathbf{t}]^{(\mathbf{t})}$ defined in example 4.1.26. To a monomial $t_1^{n_1} \cdot \ldots \cdot t_r^{n_r}$ of weighted degree d ($\deg(t_i) = i$), we associate the symmetric function in the Chern roots

$$\sum \xi_1^{a_1} \xi_2^{a_2} \cdots$$

where exactly n_j of the a_i's are j, for $j = 1, \ldots, r$, and the remaining a_j are zero. In this way, we have an isomorphism ϕ of the degree d part of $\mathbb{Z}[t_1, \ldots]$ with the degree d symmetric functions in $\xi_1, \xi_2, \ldots$. Additionally, writing

$$c_{\mathbf{t}} = \sum t_1^{n_1} t_2^{n_2} \cdots t_r^{n_r} c_{n_1,\ldots,n_r},$$

we see that $c_{n_1,\ldots,n_r} = F(c_1, \ldots, c_n)$, where F is the polynomial with $\mathbb{Z}$-coefficients expressing $\phi(t_1^{n_1} t_2^{n_2} \ldots t_r^{n_r})$ in terms of the elementary symmetric functions in the ξ_i.

On the other hand, from the definition of $\mathrm{CH}_*[\mathbf{t}]^{(\mathbf{t})}$, if $p : X \to \operatorname{Spec} k$ is smooth and projective, we have (in $\mathrm{CH}_*[\mathbf{t}]^{(\mathbf{t})}$)

$$p_*(1_X) = p_*(p^*(1)) = \deg c_{\mathbf{t}}(N_X) \in \mathbb{Z}[t_1, t_2, \ldots].$$

In particular, sending X to $\deg c_{n_1,\ldots,n_r}(N_X)$ descends to a homomorphism $\Omega_d(k) \to \mathbb{Z}$, where $d = \sum_i in_i$. Writing $P(c_1, \ldots, c_d)$ as an R-linear combination of the $c_{n_1,\ldots,n_r}$ yields the result. □

Let S_d be the polynomial in the Chern classes which corresponds to the symmetric function in the Chern roots $\sum_i \xi_i^d$, that is, $S_d = \phi(t_d)$. Clearly

$$S_d(c_1, \ldots, c_d)(E \oplus F) = S_d(c_1, \ldots, c_d)(E) + S_d(c_1, \ldots, c_d)(F)$$

for vector bundles E and F, from which it easily follows that

$$S_d(X \times Y) = 0 \tag{4.8}$$

if $\dim_k X > 0$ and $\dim_k Y > 0$, $\dim_k X + \dim_k Y = d$. Furthermore, it follows from lemma 7.9 (iii) (page 67) and corollary 7.14 (page 73) of [3, Part II], together with theorem 4.3.7 and lemma 4.3.2, that the image

$$\vartheta^{CF}(\Omega_*(k)) \subset \mathbb{Z}[t_1, t_2 \ldots]$$

is the subring generated by $a_d t_d$, $d = 1, 2 \ldots$, where

$$a_d = \begin{cases} p & \text{if } d = p^n - 1 \text{ for some prime } p \text{ and some } n \geq 1 \\ 1 & \text{otherwise.} \end{cases}$$

Since $S_d = \phi(t_d)$, it follows from our description of ϑ^{CF} in the proof of lemma 4.4.19 that

If $d = p^n - 1$ for some prime p and some integer $n \geq 1$, then

$$S_d(\Omega_d(k)) = p\mathbb{Z} \subset \mathbb{Z}. \tag{4.9}$$

Setting $s_d := S_d/p$, this yields

Lemma 4.4.20. *Let $d = p^n - 1$ for some prime p and some integer $n \geq 1$. Then*

$$s_d : \Omega_d(k) \to \mathbb{Z}$$

induces an isomorphism of the indecomposable quotient $Q\Omega_d(k)$ with $\mathbb{Z}$.

Proof. From [3, *loc. cit.*], the Lazard ring $\mathbb{L}_*$ is a polynomial ring on generators $y_1, y_2, \ldots$, with $\deg y_i = i$. As $\mathbb{L}_* \cong \Omega_*(k)$, it follows that the indecomposable quotient $Q\Omega_*(k)$ is $\mathbb{Z}$ in each degree. Furthermore S_d vanishes on products and $s_d : \Omega_d(k) \to \mathbb{Z}$ is surjective, completing the proof. □

In addition to the integral classes s_d, for integers $d = p^n - 1$ and $r \geq 1$ there are mod p characteristic classes $t_{d,r}$. The classes are defined as follows: Let $\mathbb{F}_p[v_n]$ be the graded polynomial ring with v_n in degree $p^n - 1$, and let $\psi : \mathbb{L}_* \to \mathbb{F}_p[v_n]$ be a graded ring homomorphism such that the resulting group law F over $\mathbb{F}_p[v_n]$ has *height* n, that is, the power series $[p]_F(u)$ has lowest degree term au^{p^n}, with $a \neq 0$. Identifying $\mathbb{L}_*$ with $\Omega_*(k)$ using theorem 4.3.7, we have the homomorphism $\psi_{p,n} : \Omega_*(k) \to \mathbb{F}_p[v_n]$. For $\eta \in \Omega_{rd}(k)$, there is a unique element $t_{d,r}(\eta) \in \mathbb{F}_p$ with $\psi_{p,n}(\eta) = t_{d,r}(\eta)v_n^r$. Clearly

$$t_{d,r} : \Omega_{rd}(k) \to \mathbb{F}_p$$

is a homomorphism. Even though the $t_{d,r}$ may depend on the choice of ψ, we usually omit this choice from the notation.

Example 4.4.21. Let V_p be the polynomial ring $\mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ away from p, and v_i has degree $p^i - 1$. By [31, Theorem A2.1.25, page 364], there is a split surjective graded ring homomorphism

$$\Phi_p : \mathbb{L}^* \otimes \mathbb{Z}_{(p)} \to V_p$$

which classifies the formal group laws which are *p-typical* [31, Appendix A.2]. The map Φ_p is not unique, but there are two commonly used choices of Φ_p, one due to Hazewinkel and one due to Araki, which agree mod p. Composing Φ_p with the quotient map

$$V_p \to V_p/(p, v_1, \ldots, v_{n-1}, v_{n+1}, \ldots) \cong \mathbb{F}_p[v_n]$$

defines a surjective graded ring homomorphism

$$\psi_{p,n} : \mathbb{L}_* \to \mathbb{F}_p[v_n].$$

It follows from the formula [31, A2.2.4, page 370] that the group law defined by $\psi_{p,n}$ has height n.

In particular, there do exist height n formal group laws over $\mathbb{F}_p[v_n]$, so our definition of the $t_{d,r}$ is not empty.

Proposition 4.4.22. *Let p be a prime, $n \geq 1$ an integer and let $d = p^n - 1$. Let $\psi : \mathbb{L}_* \to \mathbb{F}_p[v_n]$ be a graded ring homomorphism giving a height n formal group over $\mathbb{F}_p[v_n]$. Then s_d and the $t_{d,r}$ have the following properties:*

1. *The homomorphisms $t_{d,r} : \Omega_{dr}(k) \to \mathbb{F}_p$ are non-zero for all $r \geq 1$.*
2. *If $X_1, \ldots, X_s$ are smooth projective varieties with $\sum_i \dim X_i = rd$, then $t_{d,r}(\prod_i X_i) = 0$ unless $d | \dim X_i$ for each i.*
3. *There is an element $u \in \mathbb{F}_p^\times$ such that*

$$s_d = ut_{d,1} \mod p.$$

Proof. (1) is the same as asserting the surjectivity of ψ. As ψ is a homomorphism of graded rings, ψ is surjective if and only if $t_{d,1}$ is non-zero. To see that $t_{d,1}$ is non-zero, write the universal group law as $F_{\mathbb{L}}(u,v) = u + v + \sum_{ij\geq 1} a_{ij}u^iv^j$, with $a_{ij} \in \mathbb{L}_{i+j-1}$. Then the group law F over $\mathbb{F}_p[v_n]$ coming from ψ is

$$F(u,v) = u + v + v_n \cdot \sum_{i+j=p^n} t_{d,1}(a_{ij})u^iv^j \mod (u,v)^{p^n+1}.$$

Since F has height n, it must be the case that $F(u,v) - u - v$ is non-zero modulo $(u,v)^{p^n+1}$, hence $t_{d,1}(a_{ij}) \neq 0$ for some i, j, proving (1).

Since ψ is a ring homomorphism, we have

$$t_{d,r}(X \times Y) = \sum_{i=0}^{r} t_{d,i}(X)t_{d,r-i}(Y),$$

where we set $t_{d,0} = 1$. Since $t_{d,i}(X) = 0$ unless $\dim_k(X) = di$, (2) follows easily.

For (3), we have already seen that $t_{d,1} : \Omega_d(k) \to \mathbb{F}_p$ is surjective. By (2), $t_{d,1}$ factors through the mod p indecomposable quotient $Q\Omega_d(k)/p$. By lemma 4.4.20, s_d gives an isomorphism of $Q\Omega_d(k)/p$ with $\mathbb{F}_p$, hence $t_{d,1} : Q\Omega_d(k)/p \to \mathbb{F}_p$ is an isomorphism as well, proving (3). □

We can now prove the main results of this section.

Theorem 4.4.23. *Let $f : Y \to X$ be a morphism of smooth projective k-schemes of dimension d, with $d = p^n - 1$ for some prime p. Then there is a zero-cycle η on X such that*

$$s_d(Y) - (\deg f)s_d(X) = \deg(\eta).$$

Theorem 4.4.24. *Let $f : Y \to X$ be a morphism of smooth projective k-schemes of dimension rd, $d = p^n - 1$ for some prime p. Suppose that X admits a sequence of surjective morphisms*

$$X = X_0 \to X_1 \to \ldots \to X_{r-1} \to X_r = \operatorname{Spec} k$$

such that:

1. *Each X_i is in $\mathbf{Sm}_k$ and $\dim_k X_i = d(r-i)$.*
2. *Let η be a zero-cycle on $X_i \times_{X_{i+1}} \operatorname{Spec} k(X_{i+1})$. Then $p | \deg(\eta)$.*

Then

$$t_{d,r}(Y) = \deg(f)t_{d,r}(X).$$

Remark 4.4.25. By proposition 4.4.22, theorem 4.4.24 for $r = 1$ follows from theorem 4.4.23.

Proof. For theorem 4.4.23, corollary 4.4.8 gives the identity

$$[Y \to X] - (\deg f)[X = X] = \sum_{i=1}^{m} \omega_i[\tilde{Z}_i \to X]$$

in $\Omega_*(X)$ with $\omega_i \in \Omega_{*>0}(k)$. Write

$$\omega_i = \sum_j n_{ij}[Y_{ij}]$$

with Y_{ij} smooth projective over k and $\dim_k Y_{ij} > 0$. Pushing forward to $\Omega_*(k)$ gives the identity

$$[Y] - (\deg f)[X] = \sum_{i,j} n_{ij}[Y_{ij} \times \tilde{Z}_i]$$

in $\Omega_*(k)$. Apply s_d to this identity. Since s_d vanishes on non-trivial products, and since $\tilde{Z}_j \to Z_j$ is an isomorphism if $\dim_k Z_j = 0$, we have

$$s_d(Y) - \deg(f)s_d(X) = \sum_{\dim Y_{ij} = d} n_{ij}s_d(Y_{ij})\deg(z_i)$$

for closed points z_i of X. Since $s_d(Y_{ij}) = m_{ij}$ for suitable integers m_{ij}, we have

$$s_d(Y) - \deg(f)s_d(X) = \deg(\sum_{ij} m_{ij}n_{ij}z_i),$$

proving theorem 4.4.23.

For theorem 4.4.24, we have as before

$$[Y \to X] - (\deg f)[X = X] = \sum_{i=1}^{m} \omega_i[\tilde{Z}_i \to X]$$

in $\Omega_*(X)$. We push forward to $\Omega_*(X_2)$, and decompose each $\tilde{Z}_j \to X_2$ using corollary 4.4.8, giving

$$[\tilde{Z}_j \to X_2] = \omega_{0j}[X_2 = X_2] + \sum_i n_{ij}\omega_{ij}[\tilde{Z}_{ij} \to X_2]$$

in $\Omega_*(X_2)$. Iterating, we have the identity in $\Omega_*(k)$

$$[Y] - (\deg f)[X] = \sum_{I=(i_0,\ldots,i_r)} n_I[\prod_{j=0}^{r} Y_{Ij}],$$

where the Y_{Ij} are smooth projective k-schemes. In addition, the conditions on the tower imply that $p|n_I$ for each product $\prod_{j=0}^{r} Y_{Ij}$ such that $d|\dim Y_{Ij}$ for all j. Apply $t_{d,r}$ to this identity yields $t_{d,r}(Y) = \deg(f)t_{d,r}(X)$. □

Remark 4.4.26. Let k be any field. Let $f : Y \to X$ be a morphism between smooth projective varieties of dimension $d > 0$. Then there always exists a 0-cycle on X with integral coefficients $\Sigma_\alpha n_\alpha \cdot z_\alpha$ (where the z_α are closed points in X) and satisfying

$$S_d(Y) - \deg(f) \cdot S_d(X) = \Sigma_\alpha n_\alpha [\kappa(z_\alpha) : k]. \tag{4.10}$$

In characteristic zero this easily follows from the above considerations but this can proven over any field as follows. One considers the oriented Borel-Moore weak homology theory given by $X \mapsto \mathrm{CH}_*(X)[\mathbf{t}]^{\mathbf{t}}$ and constructed in §4.1.9. It is then obvious that the class $[f : Y \to X]$ can be written

$$[f : Y \to X] = \deg(f) \cdot [\mathrm{Id}_X] + \Sigma_\alpha \omega_\alpha \cdot [Z_\alpha \subset X]$$

with $\omega_\alpha \in \mathbb{Z}[\mathbf{t}]$ and $\mathrm{codim}_X(Z) > 0$. Pushing this forward to $\mathrm{CH}_*(k)[\mathbf{t}]^{\mathbf{t}} = \mathbb{Z}[\mathbf{t}]$ gives

$$[Y] - \deg(f) \cdot [X] = \Sigma_\alpha \omega_\alpha \cdot (\pi_X)_*^\nu [Z_\alpha]$$

from which the formula (4.10) follows by taking S_d. However one cannot deduce the more subtle corollary 1.2.17, because it is not true in general that, if $d = p^n - 1$ for some prime number p and $n > 0$, that $S_d(\omega)$ is divisible by p for $\omega \in \mathbb{Z}[\mathbf{t}]$, even though this holds for the elements in $\mathbb{L}_* \subset \mathbb{Z}[\mathbf{t}]$. Thus the difficulty is that, if one uses only the theory CH^*, one doesn't know that the ω_Z lie in $\mathbb{L}_*$.

4.5 Comparison with the Chow groups

In this section we prove theorem 1.2.19, which we restate:

Theorem 4.5.1. *Let k be a field of characteristic zero. Then the canonical morphism*

$$\Omega_* \otimes_{\mathbb{L}_*} \mathbb{Z} \to \mathrm{CH}_*$$

is an isomorphism of Borel-Moore weak homology theories on $\mathbf{Sch}_k$.

As the theory $\Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}$ is evidently the universal ordinary oriented Borel-Moore weak homology theory on $\mathbf{Sch}_k$, we can reformulate this theorem by saying that CH_* is the universal ordinary oriented Borel-Moore weak homology theory on $\mathbf{Sch}_k$. It is reasonable to conjecture that this statement still holds over any field.

To prove the theorem, we construct an explicit inverse morphism $\mathrm{CH}_* \to \Omega_* \otimes_{\mathbb{L}_*} \mathbb{Z}$. Throughout this section, k will be a field of characteristic zero.

4.5.1 The map $Z_*(X) \to \Omega_* \otimes_{\mathbb{L}_*} \mathbb{Z}$

Given a finite type k-scheme X, we denote by $Z_*(X)$ the free abelian group on the set of closed integral subschemes $Z \subset X$, graded by the dimension of Z.

Lemma 4.5.2. *Let $\pi : \tilde{Z} \to Z$ be a projective birational morphism with Z and $\tilde{Z}$ smooth over k. Then the class of the projective morphism $\tilde{Z} \to Z$:*

$$[\tilde{Z} \to Z] = \pi_* 1_{\tilde{Z}} \in \Omega_*(Z) \otimes_{\mathbb{L}_*} \mathbb{Z}$$

equals 1_Z.

Proof. This follows from theorem 4.3.7 and corollary 4.4.8 because the classes ω involved, being of positive degree, vanish in $\Omega_*(Z) \otimes_{\mathbb{L}_*} \mathbb{Z}$. □

Let X denote a finite type k-scheme, and let $Z \subset X$ be a closed integral subscheme of X.

Lemma 4.5.3. *Let $\tilde{Z} \to Z$ be a projective birational morphism with $\tilde{Z}$ smooth over k. Then the class of the projective morphism $\tilde{Z} \to X$:*

$$[\tilde{Z} \to X] \in \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}$$

depends only on Z. We denote this class by $[Z \subset X]$.

Proof. Indeed, let $f_1 : \tilde{Z}_1 \to Z$ and $f_2 : \tilde{Z}_2 \to Z$ be projective birational morphisms with $\tilde{Z}_1$ and $\tilde{Z}_2$ smooth. Letting $\tilde{Z}_3 \to \tilde{Z}_1 \times \tilde{Z}_2$ be a resolution of singularities of the closure of the graph of the birational map $f_2^{-1} f_1 : \tilde{Z}_1 \to \tilde{Z}_2$, the projective morphisms $\pi_1 : \tilde{Z}_3 \to \tilde{Z}_1$ and $\pi_2 : \tilde{Z}_3 \to \tilde{Z}_2$ are birational. We deduce from lemma 4.5.2 that, in $\Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}$, one has:

$$\begin{aligned}[\tilde{Z}_1 \to X] &= f_{1*}(1_{\tilde{Z}_1}) = f_{1*}(\pi_{1*}(1_{\tilde{Z}_3})) = (f_1 \circ \pi_1)_*(1_{\tilde{Z}_3}) \\ &= (f_2 \circ \pi_2)_*(1_{\tilde{Z}_3}) = f_{2*}(\pi_{2*}(1_{\tilde{Z}_3})) = f_{2*}(1_{\tilde{Z}_2}) = [\tilde{Z}_2 \to X],\end{aligned}$$

thus establishing the lemma. □

Definition 4.5.4. *For a finite type k-scheme X, we denote by*

$$\begin{aligned}\phi : Z_*(X) &\to \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z} \\ Z \subset X &\mapsto [Z \subset X]\end{aligned}$$

the induced group homomorphism.

It is clear that the composition

$$Z_*(X) \to \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z} \to \mathrm{CH}_*(X)$$

is the canonical morphism.

Moreover we observe that $\phi : Z_*(X) \to \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}$ is an epimorphism: this follows easily from theorem 4.3.7 and theorem 4.4.7. Finally, it follows from theorem 2.5.12 and theorem 4.4.7 that ϕ is compatible with projective push-forwards.

Thus to finish the proof of theorem 4.5.1, it suffices to prove:

Lemma 4.5.5. *Let X be a finite type k-scheme, let $W \subset X$ be an integral closed subscheme, and let $f \in k(W)^*$ be a rational function on W with divisor* $\mathrm{div}(f) \in Z_*(X)$*. Then one has*

$$\phi(\mathrm{div}(f)) = 0 \in \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}$$

Indeed, by definition, $\mathrm{CH}_*(X)$ is the quotient of $Z_*(X)$ by the subgroup generated by the cycles of the form $\mathrm{div}(f)$. Thus lemma 4.5.5 implies that ϕ induces a homomorphism $\mathrm{CH}_*(X) \to \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}$ which is surjective and right inverse to $\Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z} \to \mathrm{CH}_*(X)$; ϕ is thus an isomorphism.

Proof (of lemma 4.5.5).
Let $W \subset X$ an integral closed subscheme $W \subset X$ and f a non-zero rational function on W. By resolution of singularities, there is a projective birational morphism $\pi : \tilde{W} \to W$ such that $\tilde{W}$ is in $\mathbf{Sm}_k$ and such that the induced rational function $\tilde{f}$ on $\tilde{W}$ defines a morphism $\tilde{f} : \tilde{W} \to \mathbb{P}^1$. We may also assume that $\mathrm{div}\tilde{f}$ is a strict normal crossing divisor on $\tilde{W}$; write $\mathrm{div}\tilde{f} = D_0 - D_\infty$, with D_0 and D_∞ effective and having no common components. In particular, the strict normal crossing divisors D_0 and D_∞ have classes $[D_0 \to \tilde{W}]$, $[D_\infty \to \tilde{W}]$ in $\Omega_*(\tilde{W})$.

From our explicit formula for the class of a normal crossing divisor, and the isomorphism $\Omega_*^{ad}(k) \cong \mathbb{Z}$ (theorem 4.3.7), it follows that

$$\phi(D_0) = [D_0 \to \tilde{W}]_{\Omega^{ad}}; \quad \phi(D_\infty) = [D_\infty \to \tilde{W}]_{\Omega^{ad}}.$$

Thus

$$\phi(\mathrm{div} f) = \phi(\pi_*(D_0 - D_\infty)) = \pi_*([D_0 \to \tilde{W}]_{\Omega^{ad}} - [D_\infty \to \tilde{W}]_{\Omega^{ad}}).$$

In addition, as $\mathcal{O}_{\tilde{W}}(D_0) \cong \tilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\tilde{W}}(D_\infty)$, we have

$$[D_0 \to \tilde{W}] = \tilde{c}_1(\tilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1))(1_{\tilde{W}}) = [D_\infty \to \tilde{W}],$$

by proposition 3.1.9. This completes the proof of the lemma. □

Remark 4.5.6. Let $X \in \mathbf{Sch}_k$. We denote by $I_*(X) \subset \Omega_*(X)$ the kernel of $\Omega_*(X) \to \mathrm{CH}_*(X)$. Then theorem 4.5.1 implies that

$$I_*(X) = \mathbb{L}_{\geq 1} \cdot \Omega_*(X).$$

4.5.2 A filtration of algebraic cobordism

Let X be a finite type k-scheme and let $n \geq 0$ be an integer. We define the graded subgroup

$$F^{(n)}\Omega_*(X) \subset \Omega_*(X)$$

to be the one generated by classes $[f : Y \to X]$ with Y smooth, irreducible and $\dim(Y) - \dim f(Y) \geq n$. We observe that this is a sub-$\Omega_*(k)$-module of $\Omega_*(X)$. For $X = \mathrm{Spec}\, k$ one has $F^{(n)}\Omega_*(k) = \Omega_{* \geq n}(k)$, the subgroup of elements of degree $\geq n$. In characteristic zero, we moreover know that $\mathbb{L}_* \cong \Omega_*(k)$. By the result of the previous section, we have $F^{(1)}\Omega_*(X) = I_*(X)$.

Theorem 4.5.7. *Let X be a finite type k-scheme and let $n \geq 0$ be an integer. Then one has*
$$F^{(n)}\Omega_*(X) = \mathbb{L}_{\geq n} \cdot \Omega_*(X)$$

Proof. This follows easily from theorem 4.4.7. □

The associated bigraded abelian group $\oplus_n F^{(n)}\Omega_*(X)/F^{(n+1)}\Omega_*(X)$ is denoted by $Gr_*\Omega_*(X)$. For $X = \operatorname{Spec} k$ it is canonically isomorphic to $\Omega_*(k)$ via the obvious isomorphism: $F^{(n)}\Omega_*(k)/F^{(n+1)}\Omega_*(k) = \Omega_n(k)$.

As
$$Gr_*\Omega_*(X) \cong \oplus_n \mathbb{L}_{\geq n} \cdot \Omega_*(X)/\mathbb{L}_{\geq n+1} \cdot \Omega_*(X)$$
and $\mathrm{CH}_*(X) = Gr_0\Omega_*(X)$, the multiplication map $\mathbb{L}_* \otimes \Omega_*(X) \to \Omega_*(X)$ induces the canonical homomorphism of bigraded abelian groups
$$\Phi_X : \mathbb{L}_* \otimes \mathrm{CH}_*(X) \to Gr_*\Omega_*(X).$$

Corollary 4.5.8. *For any finite type k-scheme X, the map*
$$\Phi_X : \mathbb{L}_* \otimes \mathrm{CH}_*(X) \to Gr_*\Omega_*(X)$$
is an epimorphism of $\mathbb{L}_*$-modules.

Remark 4.5.9. Using theorem 4.1.28 one can show that $\Phi_X \otimes \mathbb{Q}$ is an isomorphism.

4.5.3 Some computations

Lemma 4.5.10. *Let X be a finite type k-scheme. Then the group $F^{(1)}\Omega_0(X)$ vanishes, and the canonical morphism*
$$\Omega_0(X) \to \mathrm{CH}_0(X)$$
is an isomorphism.

Proof. This is an immediate consequence of corollary 4.5.8. Indeed, the homomorphism
$$\Phi_X : \mathbb{L}_* \otimes \mathrm{CH}_*(X) \to Gr_*(\Omega_*(X))$$
induces in bi-degree $(0,0)$ an epimorphism:
$$\Phi_X : \mathbb{L}_0 \otimes \mathrm{CH}_0(X) \to Gr_0(\Omega_0(X)) = \Omega_0(X)$$
which is left inverse to the canonical morphism $\Omega_0(X) \to \mathrm{CH}_0(X)$. □

Now we are going to study $\Omega_1(X)$. By the theorem 4.5.7 we have an exact sequence of abelian groups
$$L_1 \otimes \Omega_0(X) \to \Omega_1(X) \to \mathrm{CH}_1(X) \to 0$$
We recall that L_1 is a free abelian group on the class $[\mathbb{P}^1]$ so that the left hand side is isomorphic to $\mathbb{Z} \otimes \Omega_0(X) = \Omega_0(X)$.

Lemma 4.5.11. *Let X be a smooth k-scheme. Then the composition*

$$\Omega_0(X) = \mathbb{Z} \otimes \Omega_0(X) \to \Omega_1(X) \to K^0(X)$$

is the canonical homomorphism $\mathrm{CH}_0(X) = \Omega_0(X) \to K^0(X)$ *which maps a* 0*-cycle to the class of its associated* $\mathcal{O}_X$*-module..*

This is easy to prove and is left to the reader.

For a finite type k-scheme X, we have the reduced K^0 of X, $\tilde{K}^0(X)$, defined as the kernel of the rank map $K^0(X) \to H^0(X_{\mathrm{Zar}}, \mathbb{Z})$.

Corollary 4.5.12. *(1) Let X be a smooth k-scheme of dimension* 1*. Then the homomorphism $\Omega_1(X) \to K^0(X)$ is an isomorphism.*

(2) Let X be a smooth k-scheme of dimension 2*. Then the homomorphism $\Omega_1(X) \to K^0(X)$ is an monomorphism which identifies $\Omega_1(X)$ with $\tilde{K}^0(X)$.*

Proof. (1) By lemma 4.5.11, we know that the homomorphism $\Omega_1(X) \to K^0(X)$ induces a map from the exact sequence

$$\mathrm{CH}_0(X) \to \Omega_1(X) \to \mathrm{CH}_1(X) \to 0$$

to the short exact sequence

$$0 \to \mathrm{CH}_0(X) \to K^0(X) \to \mathrm{CH}_1(X) \to 0,$$

and is thus an isomorphism.

(2) We have the homomorphism $\det : \tilde{K}^0(X) \to \mathrm{CH}_1(X)$ induced by assigning to a vector bundle E of rank n the isomorphism class of its maximal exterior power $\Lambda^n(E) \in \mathrm{CH}_1(X) = \mathrm{Pic}(X)$. For a smooth surface, it follows from the Grothendieck-Riemann-Roch theorem that we have a short exact sequence

$$0 \to \mathrm{CH}_0(X) \to \tilde{K}^0(X) \xrightarrow{\det} \mathrm{CH}_1(X) \to 0.$$

Noting that $\Omega_1(X) \to K^0(X)$ lands in $\tilde{K}^0(X)$ for reasons of dimension, we then argue as in (1) to conclude the proof. □

5

Oriented Borel-Moore homology

The main task that remains is the extension of the pull-back morphisms $f^* : \Omega^*(X) \to \Omega^*(Y)$ from smooth quasi-projective morphisms to arbitrary morphisms $f : Y \to X$ in $\mathbf{Sm}_k$. In fact, we will work in a more general context, giving pull-back morphisms $f^* : \Omega^*(X) \to \Omega^*(Y)$ for each *local complete intersection morphism* $f : Y \to X$ in $\mathbf{Sch}_k$. The proper context for this construction is that of an *oriented Borel-Moore homology theory* on $\mathbf{Sch}_k$. As we shall see in §5.1, this notion simultaneously extends both that of an oriented Borel-Moore weak homology theory on $\mathbf{Sch}_k$ and that of an oriented cohomology theory on $\mathbf{Sm}_k$.

5.1 Oriented Borel-Moore homology theories

We begin with the definition of an oriented Borel-Moore homology theory and a discussion of the basic structures arising from such a theory.

5.1.1 Admissible subcategories and l.c.i. morphisms

Let S be a noetherian separated scheme. We remind the reader that $\mathbf{Sch}_S$ denotes the category of finite type separated S-schemes and $\mathbf{Sm}_S$ the full subcategory of smooth quasi-projective S-schemes.

Recall that a closed immersion $i : Z \to X$ is called a *regular embedding* if the ideal sheaf of Z in X is locally generated by a regular sequence. Also, a local complete intersection morphism in $\mathbf{Sch}_S$, an *l.c.i. morphism* for short, is a morphism $f : X \to Y$ of flat finite type S-schemes which admits a factorization as $f = q \circ i$, where $i : X \to P$ is a regular embedding and $q : P \to Y$ is a smooth, quasi-projective morphism.

Let $\mathcal{V} \subset \mathbf{Sch}_S$ be an admissible subcategory (1.1). We note that each admissible subcategory of $\mathbf{Sch}_S$ contains $\mathbf{Sm}_S$. We let $\mathcal{V}'$ denote the subcategory of $\mathcal{V}$ consisting of only projective morphisms.

We sometimes require additional objects in $\mathcal{V}$, as described by the modified version of (1.1)(2):

(2)′ (a) If $Y \to X$ is an l.c.i. morphism with $X \in \mathcal{V}$, then $Y \in \mathcal{V}$.
(b) Let $i : Z \to X$ be a regular embedding in $\mathcal{V}$. Then the blow-up of X along Z is in $\mathcal{V}$.

(5.1)

We will refer to a full subcategory $\mathcal{V}$ of $\mathbf{Sch}_S$ satisfying (1.1)(1)-(4) and (2)′ as an *l.c.i.-closed* admissible subcategory of $\mathbf{Sch}_S$.

Remark 5.1.1. Let $i : Z \to X$ be a regular embedding in $\mathbf{Sch}_S$, let $\mu : X_Z \to X$ be the blow-up of X along Z and let $\mathcal{I}$ be the ideal sheaf of Z. Suppose that X has a line bundle L such that $\mathcal{I} \otimes L$ is generated by global sections; this will be the case if for instance X is a quasi-projective R-scheme for some commutative noetherian ring R. Then μ is a projective l.c.i. morphism (see lemma 7.2.3(2) below).

Thus, in case S is a quasi-projective R-scheme for some commutative noetherian ring R, and $\mathcal{V}$ is an admissible subcategory of $\mathbf{Sch}_S$ consisting of quasi-projective S-schemes, then $\mathcal{V}$ is l.c.i.-closed if and only if $\mathcal{V}$ satisfies (2)′(a) above.

Remarks 5.1.2. (1) Our notion of an l.c.i. morphism $f : X \to Y$ may differ somewhat from other texts, as we require that the smooth morphism in the factorization be quasi-projective, and that X and Y be flat over S.

(2) For the basic properties of regular embeddings and l.c.i. morphisms, we refer the reader to [9, Appendix B.7]. For example, if $f : X \to Y$ is an l.c.i. morphism, and if we have any factorization of f as $q \circ i$, where $i : X \to P$ is a closed embedding and $q : P \to Y$ is smooth, then i is automatically a regular embedding. In particular, if $f : X \to Y$ is a quasi-projective morphism of flat finite type S-schemes, then the condition that f be an l.c.i. morphism is local on X (for the Zariski topology). Also, the composition of two regular embeddings is a regular embedding.

(3) If $f : X \to Y$ and $g : Y \to Z$ are l.c.i. morphisms, then so is $g \circ f : X \to Z$. Indeed, factor f and g as $f = q_1 i_1$, $g = q_2 i_2$, with $i_1 : X \to P_1$, $i_2 : Y \to P_2$ regular embeddings, and $q_1 : P_1 \to Y$, $q_2 : P_2 \to Z$ smooth and quasi-projective. Since q_1 is quasi-projective, we can factor q_1 as a (locally closed) immersion $i : P_1 \to Y \times_S \mathbb{P}^N$ followed by the smooth projection $p_1 : Y \times_S \mathbb{P}^N \to Y$. Since $Y \to P_2$ is a closed immersion, there is an open subscheme $U \subset P_2 \times_S \mathbb{P}^N$ containing $(i_2 \times \mathrm{Id})(i(P_1))$ such that the composition

$$P_1 \xrightarrow{i} Y \times_{P_2} U \xrightarrow{p_2} U$$

is a closed immersion. By (2), we may replace P_1 with $Y \times_{P_2} U$, giving the commutative diagram

$$\begin{tikzcd} X \arrow[r, "i\circ i_1"] \arrow[dr, "f"'] & Y \times_{P_2} U \arrow[r, "p_2"] \arrow[d, "p_1"] & U \arrow[d, "q"] \\ & Y \arrow[r, "i_2"] \arrow[dr, "g"'] & P_2 \arrow[d, "q_2"] \\ & & Z \end{tikzcd}$$

with both $i \circ i_1$ and $p_2 \circ i \circ i_1$ regular embeddings, q smooth and quasi-projective, and the square cartesian. This gives the desired factorization $g \circ f = (q_2 q) \circ (p_2 i i_1)$.

Similarly, if $f_i : X_i \to Y_i$, $i = 1, 2$ are l.c.i. morphisms, then the product $f_1 \times f_2 : X_1 \times_S X_2 \to Y_1 \times_S Y_2$ is also an l.c.i. morphism. This follows from the fact that a flat pull-back of a regular embedding is a regular embedding.

(4) We call a finite type S-scheme $p : X \to S$ an *l.c.i. S-scheme* if p is an l.c.i. morphism, and let $\mathbf{Lci}_S$ denote the full subcategory of $\mathbf{Sch}_S$ with objects the l.c.i. S-schemes. From our above remarks, $\mathbf{Lci}_S$ satisfies the conditions (1.1)(1)-(4) and (5.1), i.e., $\mathbf{Lci}_S$ is an l.c.i.-closed admissible subcategory of $\mathbf{Sch}_S$. Clearly every l.c.i.-closed admissible subcategory of $\mathbf{Sch}_S$ contains $\mathbf{Lci}_S$.

(5) Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in $\mathbf{Sch}_S$. Suppose that $f : X \to Z$ is an l.c.i. morphism and that f and g are Tor-independent. Then the projection $X \times_Z Y \to Y$ is an l.c.i. morphism. Similarly, let f be an l.c.i. morphism in $\mathbf{Lci}_S$, and g a morphism in $\mathbf{Lci}_S$. If f and g are Tor-independent (i.e. transverse in $\mathbf{Sch}_S$), then $X \times_Z Y$ is in $\mathbf{Lci}_S$, hence f and g are transverse in $\mathbf{Lci}_S$.

5.1.2 Oriented Borel-Moore homology

We introduce the notion of an oriented Borel-Moore homology theory.

Definition 5.1.3. *Let $\mathcal{V}$ be an admissible subcategory of* $\mathbf{Sch}_S$. *An* oriented Borel-Moore homology theory *A on $\mathcal{V}$ is given by*

(D1). An additive functor

$$A_* : \mathcal{V}' \to \mathbf{Ab}_*\,,\ X \mapsto A_*(X).$$

(D2). For each l.c.i. morphism $f : Y \to X$ in $\mathcal{V}$ of relative dimension d, a homomorphism of graded groups

$$f^* : A_*(X) \to A_{*+d}(Y).$$

(D3). An element $1 \in A_0(S)$ and, for each pair (X, Y) of objects in $\mathcal{V}$, a bilinear graded pairing:

$$A_*(X) \otimes A_*(Y) \to A_*(X \times_S Y)$$
$$u \otimes v \mapsto u \times v,$$

called the external product, *which is associative, commutative and admits 1 as unit element.*

These satisfy

(BM1). One has $Id_X^* = Id_{A_*(X)}$ for any $X \in \mathcal{V}$. Moreover, given composable l.c.i. morphisms $f : Y \to X$ and $g : Z \to Y$ in $\mathcal{V}$ of pure relative dimension, one has $(f \circ g)^* = g^* \circ f^*$.

(BM2). Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in $\mathcal{V}$. Suppose that f and g are transverse in $\mathcal{V}$, that f is projective and that g is an l.c.i. morphism, giving the cartesian square

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z. \end{tikzcd}$$

Note that f' is projective and g' is an l.c.i. morphism. Then $g^* f_* = f'_* g'^*$.

(BM3). Let $f : X' \to X$ in $\mathcal{V}$ and $g : Y' \to Y$ be morphisms in $\mathcal{V}$. If f and g are projective, then for $u' \in A_*(X')$ and $v' \in A_*(Y')$ one has

$$(f \times g)_*(u' \times v') = f_*(u') \times g_*(v').$$

If f and g are l.c.i. morphisms, then for $u \in A_*(X)$ and $v \in A_*(Y)$ one has

$$(f \times g)^*(u \times v) = f^*(u) \times g^*(v)$$

(PB). For $L \to Y$ a line bundle on $Y \in \mathcal{V}$ with zero-section $s : Y \to L$, define the operator

$$\tilde{c}_1(L) : A_*(Y) \to A_{*-1}(Y)$$

by $\tilde{c}_1(L)(\eta) = s^*(s_*(\eta))$. Let $\mathcal{E}$ be a rank $n+1$ locally free coherent sheaf on $X \in \mathcal{V}$, with projective bundle $q : \mathbb{P}(\mathcal{E}) \to X$. For $i = 0, \ldots, n$, let

$$\xi^{(i)} : A_{*+i-n}(X) \to A_*(\mathbb{P}(\mathcal{E}))$$

be the composition of $q^* : A_{*+i-n}(X) \to A_{*+i}(\mathbb{P}(\mathcal{E}))$ with $\tilde{c}_1(O(1)_{\mathcal{E}})^i : A_{*+i}(\mathbb{P}(\mathcal{E})) \to A_*(\mathbb{P}(\mathcal{E}))$. Then the homomorphism

$$\Sigma_{i=0}^n \xi^{(i)} : \oplus_{i=0}^n A_{*+i-n}(X) \to A_*(\mathbb{P}(\mathcal{E}))$$

is an isomorphism.

(EH). Let $E \to X$ be a vector bundle of rank r over $X \in \mathbf{Sch}_k$, and let $p : V \to X$ be an E-torsor. Then $p^* : A_*(X) \to A_{*+r}(V)$ is an isomorphism.

(CD). For integers $r, N > 0$, let $W = \mathbb{P}^N \times_S \ldots \times_S \mathbb{P}^N$ (r factors), and let $p_i : W \to \mathbb{P}^N$ be the ith projection. Let $X_0, \ldots, X_N$ be the standard homogeneous coordinates on $\mathbb{P}^N$, let $n_1, \ldots, n_r$ be non-negative integers, and let $i : E \to W$ be the subscheme defined by $\prod_{i=1}^r p_i^*(X_N)^{n_i} = 0$. Suppose that E is in $\mathcal{V}$. Then $i_* : A_*(E) \to A_*(W)$ is injective.

Example 5.1.4. The *locally finite* singular homology and étale homology theories studied in [7] are examples of oriented Borel-Moore homology theories.

Example 5.1.5. The Chow groups functor $X \mapsto CH_*(X)$ on $\mathbf{Sch}'_k$. One can check, using the projective push-forwards, the pull-backs and their properties described in [9], that CH_* is indeed endowed with such a structure. In fact, there is one and only one structure of oriented Borel-Moore homology theory on CH_* whose underlying structure gives the usual one. This follows rather easily from [9].

Example 5.1.6. The functor $X \mapsto G_0(X)[\beta, \beta^{-1}]$ can be shown as well to admit a unique structure of oriented Borel-Moore homology theory on $\mathcal{V}$ whose underlying structure is the usual one.

Remark 5.1.7. The axiom (CD) may appear at first glance to be somewhat unnatural, but it is implied by a cellular decomposition property enjoyed by many examples of oriented Borel-Moore homology theories, namely:

(CD′). Let E be a scheme in $\mathcal{V}$. Suppose that the reduced subscheme E_{red} has a filtration by reduced closed subschemes

$$\emptyset = E_0 \subset E_1 \subset \ldots \subset E_N = E_{\mathrm{red}}$$

such that

1. $E_i \setminus E_{i-1}$ is a disjoint union of its irreducible components.
2. Each irreducible component E_{ij}^0 of $E_i \setminus E_{i-1}$ is an affine space $\mathbb{A}_S^{N_{ij}}$.
3. Let E_{ij} be the closure of E_{ij}^0 in E_i. Then E_{ij} is smooth over S.

Then the evident map $\coprod_{i,j} E_{ij} \to E$ induces a surjection

$$\oplus_{i,j} A_*(E_{ij}) \to A_*(E).$$

We will verify in §5.2.4 that (CD′) implies (CD). Since we will only need this special consequence of the cellular decomposition property, we list only the property (CD) as an axiom, even though it may seem less natural than the axiom (CD′).

We will be mostly interested in the sequel in the case S is the spectrum of a field and $\mathcal{V}$ is the category of finite type k-schemes. However, many of the problems we have considered in the previous chapters for $\mathcal{V} = \mathbf{Sch}_k$ have interesting generalizations for other choices of $\mathcal{V}$. One can easily develop a general theory of Chern classes of vector bundles as in §4.1. Taking for instance $\mathcal{V}$ to be the category of finite type S-schemes which are regular, one has the oriented Borel-Moore homology theory

$$X \mapsto K^0(X)[\beta, \beta^{-1}].$$

Then, for a given multiplicative and periodic theory A^*, the assignment $E \mapsto \mathrm{rank}(E) - c_1^A(E^\vee)$ gives a natural transformation $ch_A : K^0[\beta, \beta^{-1}] \to A^*$, which makes $K^0[\beta, \beta^{-1}]$ the universal multiplicative and periodic oriented Borel-Moore homology theory. This raises the question of whether the analogues of theorems 1.2.2 and 1.2.6 are still valid in the general situation. When $S = \mathrm{Spec}\,(k)$ for a field k which is not perfect, for example, then one can take for $\mathcal{V}$ one of the three following categories: that of all finite type k-schemes, that of all regular finite type k-schemes, that of all smooth finite type k-schemes, and we do not know if the analogues of theorems 1.2.2 and 1.2.6 remain true in these cases.

Another interesting example is the category $\mathbf{Lci}_S$. This may be viewed as the largest full subcategory of $\mathbf{Sch}_S$ for which each object $\pi_X : X \to S$ has an obvious *fundamental class* $1_X := \pi_X^*(1)$[1]. As we shall see in §7.4, from the point of view of algebraic cobordism an l.c.i. S-scheme is essentially the same as a smooth S-scheme. For instance, one has a virtual normal bundle for an l.c.i. S-scheme, and a reasonable theory of Chern "numbers" for those which are projective over S.

5.1.3 Fundamental classes

Let A_* be an oriented Borel-Moore homology theory on an admissible subcategory $\mathcal{V}$ of $\mathbf{Sch}_S$.

Definition 5.1.8. *Let $p_X : X \to S$ be an l.c.i. scheme over S, $X \in \mathcal{V}$. Define the* fundamental class *of X, $1_X \in A_*(X)$, by $1_X := p_X^*(1)$, where $1 \in A_0(S)$ is the unit element.*

For $f : Y \to X$ a projective morphism in $\mathcal{V}$, with $Y \in \mathbf{Lci}_S$, we write $[f : Y \to X]_A$ for $f_*(1_Y)$. We omit the A in the notation if the context makes the meaning clear.

Remark 5.1.9. Take $S = \mathrm{Spec}\, k$, k a field. $X \to \mathrm{Spec}\, k$ is an l.c.i. morphism if and only X is a local complete intersection closed subscheme of a smooth quasi-projective P over k. In particular, an l.c.i. k-scheme X is a Cohen-Macaulay scheme, so X is unmixed (has no embedded components) and is locally equi-dimensional over k. Thus, for such a scheme, we may use cohomological notation: $A^n(X) = A_{d-n}(X)$ if X is connected of dimension d over

[1] It is not clear if $\mathbf{Lci}_S$ is in fact the largest full subcategory of $\mathbf{Sch}_S$ for which one can define natural fundamental classes, but one can show by examples (see [17]) that it is impossible to define natural fundamental classes on all of $\mathbf{Sch}_S$, even for $S = \mathrm{Spec}\, k$, k a field.

k, and then extend to locally equi-dimensional X by taking the direct sum over the connected components. In particular, the fundamental class 1_X is in $A^0(X)$.

Remark 5.1.10. Let $f : Y \to X$ be an l.c.i. morphism of l.c.i. S-schemes. Then $f^*(1_X) = 1_Y$.

The axioms for an oriented Borel-Moore homology theory yield a simple description of the first Chern class operator $\tilde{c}_1(L)$ in case L admits a section which is a non-zero divisor.

Lemma 5.1.11. *Let A be an oriented Borel-Moore homology theory on some admissible subcategory $\mathcal{V}$ of $\mathbf{Sch}_S$. Take $X \in \mathcal{V}$, $L \to X$ a line bundle with sheaf of sections $\mathcal{L}$. Let $s : X \to L$ a section such that the induced map $\times s : \mathcal{O}_X \to \mathcal{L}$ is injective, and let $i : D \to X$ be the Cartier divisor defined by $s = 0$. Suppose that D is in $\mathcal{V}$. Then*

$$\tilde{c}_1(L) = i_* i^*.$$

Proof. Since D is a Cartier divisor on X, i is a regular embedding, so i^* is defined. Let $s_0 : X \to L$ be the zero section. Then both s and s_0 are regular embeddings. We first show that

$$s_0^* = s^* : A_*(L) \to A_{*-1}(X).$$

Indeed, we have the map $s(t) : X \times \mathbb{A}^1 \to L$ defined by $s(t) = ts + (1-t)s_0$, where $\mathbb{A}^1 = \operatorname{Spec} k[t]$. The map

$$\times s(t) : \mathcal{O}_{X \times \mathbb{A}^1} \to p_1^* \mathcal{L}$$

is the same as $\times t \cdot s$, hence injective. Thus $(s(t), \mathrm{Id}_{\mathbb{A}^1}) : X \times \mathbb{A}^1 \to L \times \mathbb{A}^1$ is a regular embedding, hence $s(t) : X \times \mathbb{A}^1 \to L$ is an l.c.i. morphism.

Letting $i_0, i_1 : X \to X \times \mathbb{A}^1$ be the sections with value 0,1, respectively, it follows from the homotopy property (EH) that $i_0^* = i_1^*$, hence

$$s_0^* = i_0^* s(t)^* = i_1^* s(t)^* = s^*,$$

as claimed.

Now consider the cartesian square

$$\begin{tikzcd} D \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "s_0"] \\ X \arrow[r, "s"'] & L \end{tikzcd}$$

By our assumption that $\times s$ is injective, this square is Tor-independent; since D is in $\mathcal{V}$, s_0 and s are transverse in $\mathcal{V}$. Clearly s_0 is projective and s is an l.c.i. -morphism, hence by (BM2)

$$s^* s_{0*} = i_* i^*.$$

Since $s^* = s_0^*$, this shows that $i_* i^* = \tilde{c}_1(L)$, as desired. □

Proposition 5.1.12. *Let S be a Cohen-Macaulay scheme, $\mathcal{V}$ an l.c.i. closed admissible subcategory of $\mathbf{Sch}_S$, and A_* an oriented Borel-Moore homology theory on $\mathcal{V}$. Let X be an l.c.i. scheme over S, $p : L \to X$ a line bundle. Let $s : X \to L$ be a section of L such that the subscheme $i : E \to X$ defined by $s = 0$ has pure codimension one on X. Then*

$$i_*(1_E) = \tilde{c}_1(L)(1_X)$$

in $A^1(X)$.

Proof. Since X is an l.c.i. scheme over S, X is also a Cohen-Macaulay scheme. Thus s is a (local) non-zero divisor, hence E is also an l.c.i. scheme and the inclusion i is a regular embedding. In particular, $1_E = i^*(1_X)$. Also, since s is a non-zero divisor, we may apply lemma 5.1.11, giving

$$\tilde{c}_1(L)(1_X) = i_*(i^*(1_X)) = i_*(1_E),$$

as desired. □

5.1.4 Operational structure

Let A_* be an oriented Borel-Moore homology theory on some admissible subcategory $\mathcal{V}$ of $\mathbf{Sch}_S$. For Y in $\mathbf{Sm}_S$ of pure dimension d over S, let $A^n(Y) := A_{d-n}(Y)$; we extend to arbitrary $Y \in \mathbf{Sm}_S$ by additivity over the connected components of Y. Since Y is smooth over S, the diagonal $\delta_Y : Y \to Y \times_S Y$ is a regular embedding, so we may define a product on $A^*(Y)$ by

$$a \cup_Y b := \delta_Y^*(a \times b).$$

This makes $A^*(Y)$ into a commutative graded ring with unit 1_Y, natural with respect to pull-back in $\mathbf{Sm}/S$.

Similarly, for each morphism $f : Z \to Y$ in $\mathcal{V}$, the graph

$$(\mathrm{Id}_Z, f) : Z \to Z \times_S Y$$

is a regular embedding, and we may make $A_*(Z)$ into a graded $A^{-*}(Y)$-module by

$$a \cap_f b := (\mathrm{Id}_Z, f)^*(b \times a)$$

for $a \in A^*(Y)$, $b \in A_*(Z)$.

The following properties are easy to verify (see the proof of proposition 5.2.1 below for details on the projection formula (4)):

(5.2)

1. For $Y \in \mathbf{Sm}/S$,

$$a \cap_{\mathrm{Id}_Y} b = a \cup_Y b$$

for all $a \in A_*(Y)$, $b \in A^*(Y)$.

2. For a composition $W \xrightarrow{g} Z \xrightarrow{f} Y$ in $\mathcal{V}$ with Y in $\mathbf{Sm}/S$ and g an l.c.i. morphism, we have
$$a \cap_{fg} g^*(b) = g^*(a \cap_f b)$$
for $a \in A^*(Y)$, $b \in A_*(Z)$.
3. For a composition $W \xrightarrow{g} Z \xrightarrow{f} Y$ in $\mathcal{V}$ with $f : Z \to Y$ a morphism in $\mathbf{Sm}/S$, we have
$$a \cap_{fg} b = f^*(a) \cap_g b$$
for $a \in A^*(Y)$, $b \in A_*(W)$.
4. For a composition $W \xrightarrow{g} Z \xrightarrow{f} Y$ in $\mathcal{V}$ with g projective and Y in $\mathbf{Sm}/S$, we have
$$g_*(a \cap_{fg} b) = a \cap_f g_*(b)$$
for $a \in A^*(Y)$, $b \in A_*(W)$.
5. For $f : Z \to Y$ a morphism in $\mathcal{V}$ with Y in $\mathbf{Sm}/S$, and $L \to Y$ a line bundle, we have
$$\tilde{c}_1(f^*L)(b) = \tilde{c}_1(L)(1_Y) \cap_f b$$
for all $b \in A_*(Z)$.

5.2 Other oriented theories

We relate the notion of oriented Borel-Moore homology to oriented cohomology and oriented Borel-Moore weak homology.

5.2.1 Oriented cohomology

Let $\mathcal{V}$ be an admissible subcategory of $\mathbf{Sch}_S$. We recall from definition 1.1.2 the notion of an oriented cohomology theory on $\mathcal{V}$. In general, an oriented cohomology theory on $\mathcal{V}$ is not equivalent to an oriented Borel-Moore homology theory on $\mathcal{V}$, but, as we shall see in this section, for $\mathcal{V} = \mathbf{Sm}_S$, the two notions coincide.

Indeed, let A_* be an oriented Borel-Moore homology theory on $\mathbf{Sm}_S$, and let A^* be the theory A with cohomological grading. The cup product on $A^*(X)$ was defined in section 5.1.4. Conversely, if A^* is a cohomology theory on $\mathbf{Sm}_S$, let A_* be A with homological grading, $A_n(X) = A^{\dim_S X - n}(X)$. Define the external product of $\alpha \in A_*(X)$, $\beta \in A_*(Y)$ by $\alpha \times \beta := p_1^*(\alpha) \cup p_2^*(\beta) \in A^*(X \times_S Y)$.

Proposition 5.2.1. *The operations $A_* \mapsto A^*$, $A^* \mapsto A_*$ give equivalences of the category of oriented Borel-Moore homology theories on $\mathbf{Sm}_S$ with the category of oriented cohomology theories on $\mathbf{Sm}_S$.*

Proof. Suppose we are given an oriented Borel-Moore homology theory A_* on $\mathbf{Sm}_S$. Since the external product is unital, commutative and associative, the same is true for the cup product, where the unit is $1_X := p_X^*(1)$, $p_X : X \to S$ being the structure morphism. Noting that each morphism $X \to Y$ in $\mathbf{Sm}_S$ is an l.c.i. morphism, axiom (BM1) defines A^* as a functor from $\mathbf{Sm}_S$ to graded groups, and axiom (BM3) shows that the cup product is functorial. Thus, A^* is a commutative ring valued functor on $\mathbf{Sm}_S$.

To show that A^* is an oriented cohomology theory on $\mathbf{Sm}_S$, we need to show that

1. If $f : Y \to X$ is a projective morphism in $\mathbf{Sm}_S$ of relative dimension d, then the push-forward $f_* : A^*(Y) \to A^{*-d}(X)$ is $A^*(X)$-linear (the projection formula).
2. For a line bundle $p : L \to X$ on $X \in \mathbf{Sm}_S$, the Chern class endomorphism $\tilde{c}_1(L) : A^*(X) \to A^{*+1}(X)$ is given by cup product with $c_1(L)$.

For (1), we have the commutative diagram, in which the square is cartesian:

$$\begin{tikzcd}
 & Y \times_S Y \arrow[dr, "\mathrm{Id}\times f"] & \\
Y \arrow[ur, "\delta_Y"] \arrow[rr, "(\mathrm{Id},f)"'] \arrow[d, "f"'] & & Y \times_S X \arrow[d, "f\times \mathrm{Id}"] \\
X \arrow[rr, "\delta_X"'] & & X \times_S X.
\end{tikzcd}$$

In addition, the maps $f \times \mathrm{Id}$ and δ_X are transverse in $\mathbf{Sm}_S$. Using axioms (BM1), (BM2) and (BM3), we have

$$\begin{aligned}
f_*(\alpha \cup f^*(\beta)) &= f_*(\delta_Y^*(\alpha \times f^*(\beta))) \\
&= f_*(\delta_Y^* \circ (\mathrm{Id} \times f)^*(\alpha \times \beta)) \\
&= f_* \circ (\mathrm{Id}, f)^*(\alpha \times \beta) \\
&= \delta_X^* \circ (f \times \mathrm{Id})_*(\alpha \times \beta) \\
&= \delta_X^*(f_*(\alpha) \times \beta) \\
&= f_*(\alpha) \cup \beta.
\end{aligned}$$

For (2), let $s : X \to L$ be the zero section. Then $c_1(L) = s^*(s_*(1_X))$ by definition, while for $\eta \in A^*(X)$,

$$\begin{aligned}
\tilde{c}_1(L)(\eta) &:= s^*(s_*(\eta)) \\
&= s^*(s_*(1_X \cup s^*(p^*\eta))) \\
&= s^*(s_*(1_X) \cup p^*\eta) \\
&= s^*(s_*(1_X)) \cup \eta \\
&= c_1(L) \cup \eta.
\end{aligned}$$

Similarly, given an oriented cohomology theory A^* on $\mathbf{Sm}_S$, the functor A_* on the projective morphisms of $\mathbf{Sm}_S$ evidently satisfies all the axioms of an oriented Borel-Moore homology theory, with the possible exception of (BM2) for push-forward and the axiom (CD). For the axiom (BM2), it suffices to show that $(f\times\mathrm{Id})_*(\alpha\times\beta)=f_*(\alpha)\times\beta$ for a projective morphism $f:X'\to X$ in $\mathbf{Sm}_S$; this follows easily from the projection formula and (A2):

$$\begin{aligned}(f\times\mathrm{Id})_*(\alpha\times\beta)&=(f\times\mathrm{Id})_*(p_1^*(\alpha)\cup p_2^*(\beta))\\&=(f\times\mathrm{Id})_*(p_1^*(\alpha)\cup(f\times\mathrm{Id})^*(p_2^*(\beta)))\\&=(f\times\mathrm{Id})_*(p_1^*(\alpha))\cup p_2^*(\beta)\qquad\text{(projection formula)}\\&=p_1^*(f_*(\alpha))\cup p_2^*(\beta)\qquad\text{(A2)}\\&=f_*(\alpha)\times\beta\end{aligned}$$

For the axiom (CD), since $\mathcal{V}=\mathbf{Sm}_S$, the only choice for E (up to permuting the factors in $W:=(\mathbb{P}^N)^r$) is $E=\mathbb{P}^{N-1}\times\mathbb{P}^N\ldots\times\mathbb{P}^N$. By repeated applications of the projective bundle formula, $A^*(W)$ is the free $A^*(S)$-module on the classes $\xi_1^{n_1}\cdot\ldots\cdot\xi_r^{n_r}$, $0\le n_i\le N$, where $\xi_i=c_1(p_i^*(O(1)))$, and $p_i:W\to\mathbb{P}^N$ is the ith projection. Similarly, $A^*(E)$ is the free $A^*(S)$-module on the classes $\bar\xi_1^{n_1}\cdot\ldots\cdot\bar\xi_r^{n_r}$, $0\le n_1\le N-1$, $0\le n_i\le N$, $i=2,\ldots,r$, where $\bar\xi_i$ is the restriction of ξ_i to E. Let $i:E\to W$ be the inclusion, and let 1_E be the unit in $A^*(E)$. By lemma 5.1.11, we have

$$\xi_1:=c_1(p_1^*(O(1)))=\tilde c_1(p_1^*(O(1)))(1_W)=i_*(i^*(1_W))=i_*(1_E).$$

Using the projection formula we have

$$\begin{aligned}i_*(\bar\xi_1^{n_1}\cdot\ldots\cdot\bar\xi_r^{n_r})&=i_*(i^*(\xi_1^{n_1}\cdot\ldots\cdot\xi_r^{n_r})\cup 1_E)\\&=\xi_1^{n_1}\cdot\ldots\cdot\xi_r^{n_r}\cup i_*(1_E)\\&=\xi_1^{n_1+1}\cdot\ldots\cdot\xi_r^{n_r}\end{aligned}$$

Thus $i_*:A^*(E)\to A^{*+1}(W)$ is injective, verifying (CD). □

Remark 5.2.2. We have actually proved a bit more in proposition 5.2.1, in that we never used the axiom (CD) in showing that an oriented Borel-Moore homology theory gives rise to a cohomology theory. In particular, for $\mathcal{V}=\mathbf{Sm}_S$, the axiom (CD) is a consequence of the other axioms.

5.2.2 Cohomology and weak cohomology

In this section, k is an arbitrary field. We proceed to show that any oriented cohomology theory on $\mathbf{Sm}_k$ defines an oriented weak cohomology theory on $\mathbf{Sm}_k$. Consider the following property (taken from definition 2.2.1):

(Sect*). Let $L\to Y$ be a line bundle on some $Y\in\mathbf{Sm}_S$, $s:Y\to L$ a section transverse to the zero-section (in $\mathbf{Sm}_S$) and $i:D\to Y$ the closed immersion of the zero subscheme of s. Then

$$c_1(L)=i_*(1_D).$$

Lemma 5.2.3. *Let A^* be an oriented cohomology theory on $\mathbf{Sm}_S$. The property (Sect*) holds for A^*.*

Proof. Let A_* be the oriented Borel-Moore homology theory on $\mathbf{Sm}_S$ given by proposition 5.2.1. Since $s : Y \to L$ is transverse to the zero-section s_0, we have

$$\mathrm{Tor}_i^{\mathcal{O}_L}(\mathcal{O}_{s(Y)}, \mathcal{O}_{s_0(Y)}) = 0$$

for $i > 0$; this easily implies that $\times s : \mathcal{O}_Y \to \mathcal{L}$ is injective, where $\mathcal{L}$ is the sheaf of sections of L.

By lemma 5.1.11, we have

$$c_1(L) := \tilde{c}_1(L)(1_Y) = i_*(i^*(1_Y)) = i_*(1_D)$$

□

We now specialize to $S = \operatorname{Spec} k$.

Proposition 5.2.4. *Let k be a field and let A^* be an oriented cohomology theory on $\mathbf{Sm}_k$. Then A^* defines an an oriented weak cohomology theory on $\mathbf{Sm}_k$ with first Chern class operator $\tilde{c}_1(L)$ given by*

$$\tilde{c}_1(L)(\eta) = c_1(L) \cup \eta.$$

Proof. Let A^* be an oriented cohomology theory on $\mathbf{Sm}_k$. Changing to homological notation, we have by proposition 5.2.1 the oriented Borel-Moore homology theory A_* with $\tilde{c}_1(L)(\eta) = c_1(L) \cup \eta$. In particular, A_* is an oriented Borel-Moore functor with product, satisfying the axioms (PB) and (EH) of §4.1.1. We have just shown in lemma 5.2.3 that A^* satisfies (Sect*), hence A_* satisfies (Sect). Thus by lemma 4.1.3, A_* satisfies the axiom (Nilp) of remark 2.2.3.

Next, we show that A_* satisfies the axiom (Loc) of §4.1.1. For this, take $X \in \mathbf{Sm}_k$ with line bundle L and a smooth closed subscheme $i : D \to X$ such that L admits a section s, transverse to the zero-section, with zero-locus contained in D; we may suppose that D is the divisor of s. By (Sect*), $\tilde{c}_1(L)$ is cup product with $i_*(1_D)$, so for $x \in A^*(X)$, we have

$$\tilde{c}_1(L)(x) = i_*(1_D) \cup x = i_*(i^*x),$$

verifying (Loc) for A_*.

Let $F_A(u, v) \in A_*(k)[[u, v]]$ be the formal group law given by corollary 4.1.8. To complete the proof, we need only show that A_* satisfies the axiom (FGL) for this group law. Since $\tilde{c}_1(L)(\eta) = c_1(L) \cup \eta$, we need to show that, for line bundles L, M on $X \in \mathbf{Sm}_k$,

$$F_A(c_1(L), c_1(M)) = c_1(L \otimes M)$$

in $A^1(X)$. Since X is quasi-projective over k, we may use Jouanolou's trick (and axiom (EH)) to replace X with a smooth affine scheme over k. Thus we may assume that L and M are globally generated on X, and so there are morphisms $f : X \to \mathbb{P}^n$, $g : X \to \mathbb{P}^m$ with $L \cong f^*(\gamma_n)$, $M \cong g^*(\gamma_m)$. The naturality of c_1 reduces us to the case $X = \mathbb{P}^n \times \mathbb{P}^m$, $L = p_1^*(\gamma_n)$, $M = p_2^*(\gamma_m)$, which follows from corollary 4.1.8. □

Remark 5.2.5. Proposition 5.2.4 implies lemma 1.1.3, clearing up that bit of unfinished business.

5.2.3 Weak homology theories

Fix a field k. In this section $\mathcal{V}$ will be an admissible subcategory of $\mathbf{Sch}_k$. We relate the notions of oriented Borel-Moore homology and oriented Borel-Moore weak homology (see definition 4.1.9).

Proposition 5.2.6. *Let A_* be an oriented Borel-Moore homology theory on $\mathcal{V}$. By restricting the pull-back maps f^* to smooth quasi-projective morphisms $f : Y \to X$ in $\mathcal{V}$ having pure relative dimension, A_* defines an oriented Borel-Moore weak homology theory on $\mathcal{V}$, also denoted A_*.*

Proof. We need to show (see remark 4.1.10(2)):

1. Let L be a line bundle on some $X \in \mathcal{V}$. If $f : Y \to X$ is a smooth morphism in $\mathcal{V}$, then $\tilde{c}_1(f^*L) \circ f^* = f^* \circ \tilde{c}_1(L)$. If $f : Y \to X$ is a projective morphism in $\mathcal{V}$, then $f_* \circ \tilde{c}_1(f^*L) = \tilde{c}_1(L) \circ f_*$.
2. If L and M are line bundles on $X \in \mathcal{V}$, then $\tilde{c}_1(L) \circ \tilde{c}_1(M) = \tilde{c}_1(M) \circ \tilde{c}_1(L)$.
3. Let X and Y be in $\mathcal{V}$, and $L \to X$ be a line bundle on X. For $\alpha \in A_*(X)$, $\beta \in A_*(Y)$, we have
$$\tilde{c}_1(L)(\alpha) \times \beta = \tilde{c}_1(p_1^*L)(\alpha \times \beta),$$
where $p_1 : X \times_k Y \to X$ is the projection.
4. A_* has the structure of an oriented Borel-Moore $\mathbb{L}_*$-functor and the axioms (Sect), (FGL) and (Loc) of definition 2.2.1 are valid for A_*.

The property (1) follows easily from the functoriality of smooth pull-back (BM1) and projective push-forward, plus axiom (BM2) applied to the transverse cartesian diagram

$$\begin{tikzcd} Y \arrow[r, "f^*s"] \arrow[d, "f"'] & f^*L \arrow[d, "f_L"] \\ X \arrow[r, "s"'] & L. \end{tikzcd}$$

For (2), we have the transverse cartesian diagram

$$\begin{tikzcd} X \arrow[r, "s_L"] \arrow[d, "s_M"'] & L \arrow[d, "\tilde{s}_M"] \\ M \arrow[r, "\tilde{s}_L"'] & L \oplus M. \end{tikzcd}$$

Applying (BM2) and the functoriality of smooth pull-back and projective push-forward, we have

$$\begin{aligned}\tilde{c}_1(M) \circ \tilde{c}_1(L) &= s_M^* s_{M*} s_L^* s_{L*} \\ &= s_M^* \tilde{s}_L^* \tilde{s}_{M*} s_{L*} \\ &= s_L^* \tilde{s}_M^* \tilde{s}_{L*} s_{M*} \\ &= s_L^* s_{L*} s_M^* s_{M*} \\ &= \tilde{c}_1(L) \circ \tilde{c}_1(M).\end{aligned}$$

(3) is an easy consequence of (BM3).

For (4), we note that A_* satisfies the axiom (Sect): Let $p : Y \to \operatorname{Spec} k$ be in $\mathbf{Sm}_k$ and let $1_Y = p^*(1)$. Let $L \to Y$ be a line bundle on some $Y \in \mathbf{Sm}_k$, $s : Y \to L$ a section transverse to the zero-section (in $\mathbf{Sm}_k$) and $i : Z \to Y$ the closed immersion of the zero subscheme of s. Then

$$\tilde{c}_1(L)(1_Y) = i_*(1_Z).$$

Indeed, by proposition 5.2.1, A_* defines an oriented cohomology theory A^* on $\mathbf{Sm}_k$, and by lemma 5.2.3, (Sect*) holds for A^*. Since $\tilde{c}_1(L)(\eta) = c_1(L) \cup \eta$ for $L \to X$ a line bundle on $X \in \mathbf{Sm}_k$ and for $\eta \in A^*(X)$ (see the proof of proposition 5.2.1), it follows that (Sect) holds for A_*. We note that, by lemma 4.1.3, A_* satisfies the axiom (Nilp).

To give the $\mathbb{L}_*$-structure and to verify (FGL), we need to show: there is a power series $F_A(u, v) \in A_*(k)[[u, v]]$ such that, given line bundles L and M on $Y \in \mathbf{Sm}_k$, we have

$$F_A(\tilde{c}_1(L), \tilde{c}_1(M))(1_Y) = \tilde{c}_1(L \otimes M)(1_Y).$$

Indeed, by the axiom (Nilp), the left-hand side of this equation makes sense; the commutativity and associativity of tensor product, together with the vanishing of $\tilde{c}_1(O_Y)(1_Y)$, imply that F_A is a formal group law, which yields the $\mathbb{L}_*$-structure.

Since, by proposition 5.2.1, the oriented Borel-Moore homology theory A_*, restricted to $\mathbf{Sm}_k$, defines an oriented cohomology theory on $\mathbf{Sm}_k$, the existence of F_A satisfying axiom (FGL) follows from lemma 1.1.3 and proposition 5.2.4.

Finally, to prove (Loc), take X in $\mathcal{V}$, $L \to X$ a line bundle, and s a section of L such that s is transverse in $\mathcal{V}$ to the zero-section of L. Let $i : D \to X$ be the zero-subscheme of s. In particular, s and the zero-section are Tor-independent, which implies that s is a non-zero divisor on X. Also, D is in $\mathcal{V}$. We may therefore apply lemma 5.1.11 to yield the identity

$$\tilde{c}_1(L)(x) = i_*(i^*(x))$$

for all $x \in A_*(X)$, proving (Loc). □

Remark 5.2.7. One consequence of propositions 5.2.1 and 5.2.6 is that an oriented cohomology theory A^* on $\mathbf{Sm}_k$ gives rise to an oriented Borel-Moore weak homology theory A_* on $\mathbf{Sm}_k$.

Definition 5.2.8. *Given an oriented Borel-Moore homology theory A_* on some admissible $\mathcal{V} \subset \mathbf{Sch}_k$, the oriented Borel-Moore weak homology theory it defines is called the* underlying *one. Similarly, if A^* is an oriented cohomology theory on $\mathbf{Sm}_k$, the oriented Borel-Moore weak homology theory A_* it defines is called the underlying one.*

Conversely, given an oriented Borel-Moore weak homology theory A_* on $\mathcal{V}$, we say that A_* admits a structure of an oriented Borel-Moore homology theory if there is such a theory $\tilde{A}_*$ whose underlying oriented Borel-Moore weak homology theory is A_*. For $\mathcal{V} = \mathbf{Sm}_k$, we say that A_* admits a structure of an oriented cohomology theory if there is such a theory $\tilde{A}^*$ whose underlying oriented Borel-Moore weak homology theory is A_*.

Remark 5.2.9 (Extended nilpotence and formal group law axioms). Let A_* be an oriented Borel-Moore homology theory. The axioms (Nilp) and (FGL) refer only to identities satisfied by the first Chern class operators after evaluation on a fundamental class 1_Y for $Y \in \mathbf{Sm}_S$, so it is natural to ask if these identities are satisfied as operators on all of $A_*(X)$. In fact, this is the case, at least for S affine, and X quasi-projective over S, or if X is itself affine (without restriction on S); the operator version of (Dim) is likewise satisfied for all affine X if $S = \operatorname{Spec} k$.

Indeed, we can use Jouanolou's trick to replace X with an affine scheme X' admitting a smooth quasi-projective morphism $p : X' \to X$ such that $p^* : A_*(X) \to A_{*+r}(X')$ is an isomorphism. Thus, for a line bundle L on X, there is a morphism $f : X' \to \mathbb{P}^n$ for some n such that $p^*L \cong f^*(O(1))$; in fact, we may take n to be the Krull dimension $\dim X'$ of X'. Using the properties (5.2) of the $A^{-*}(\mathbb{P}^n)$-module structure on $A_*(X')$ given by f, we have

$$p^*(\tilde{c}_1(L)(x)) = \tilde{c}_1(p^*L)(p^*(x)) = c_1(O(1)) \cap_f p^*(x)$$

for all $x \in A_*(X)$, where $c_1(O(1)) := \tilde{c}_1(O(1))(1_{\mathbb{P}^n})$. Since $\tilde{c}_1(O(1))^{n+1}(1_{\mathbb{P}^n}) = 0$, we see that $\tilde{c}_1(L)^{n+1} = 0$. In particular, if X is itself affine, we have the identity

$$\tilde{c}_1(L)^{\dim X+1} = 0$$

for any line bundle L on X; since $\dim X = \dim_k X$ for X of finite type over a field k, this verifies the operator version of (Dim) in this case.

The identity

$$F_A(\tilde{c}_1(L), \tilde{c}_1(M)) = \tilde{c}_1(L \otimes M)$$

for L, M line bundles on X is proved similarly.

5.2.4 The axiom (CD)

We conclude this section by showing that the axiom (CD′) of remark 5.1.7 implies the axiom (CD) of definition 5.1.3, and that the axiom (CD′) is implied by a certain localization property. In this section, S is a noetherian separated scheme and $\mathcal{V}$ is an admissible subcategory of $\mathbf{Sch}_S$.

Lemma 5.2.10. *Suppose $\mathcal{V}$ contains $\mathbf{Lci}_S$. Suppose we are given the data from definition 5.1.3 (D1)-(D3) of an oriented Borel-Moore homology theory A_* on $\mathcal{V}$, satisfying the axioms of definition 5.1.3, with the possible exception of the axiom (CD), and suppose that A_* satisfies the axiom (CD′) of remark 5.1.7. Then A_* satisfies the axiom (CD).*

Proof. Let E be as in axiom (CD); Note that E is in $\mathbf{Lci}_S$, hence in $\mathcal{V}$. We may suppose that $n_1, \ldots, n_m$ are non-zero and $n_{m+1}, \ldots, n_r$ are all zero.

Let $X_0, \ldots, X_N$ be the standard homogeneous coordinates on $\mathbb{P}^N$, and let $X_{j,i} = p_i^*(X_j)$, where $p_i : W = (\mathbb{P}^N)^r \to \mathbb{P}^N$ is the ith projection.

We may thus write E_{red} as a strict normal crossing divisor $E_{\mathrm{red}} = \sum_{i=1}^m E_i$, with
$$E_i = \mathbb{P}^N \times \ldots \times \mathbb{P}^{N-1} \times \ldots \times \mathbb{P}^N \subset (\mathbb{P}^N)^r,$$
with the $\mathbb{P}^{N-1}$ the linearly imbedded subspace of $\mathbb{P}^N$ defined by $X_{N,i} = 0$, $i = 1, \ldots, m$.

By proposition 5.2.1 and remark 5.2.2, A_* defines an oriented cohomology theory A^* on $\mathbf{Sm}_S$; we have also shown in the proof of proposition 5.2.1 that
$$\tilde{c}_1(L)(\eta) = c_1(L) \cup \eta$$
for $L \to Y$ a line bundle on $Y \in \mathbf{Sm}_S$ and $\eta \in A_*(Y) = A^*(Y)$. By lemma 5.2.3, the property (Sect*) is valid for A^*. Using these two properties, together with repeated applications of the axiom (PB), we see that $A_*(\mathbb{P}^{m_1} \times \ldots \times \mathbb{P}^{m_r})$ is a free $A_*(S)$-module with basis the classes $\hat{i}_*(1_{\mathbb{P}^{a_1} \times \ldots \times \mathbb{P}^{a_r}})$, where $0 \le a_j \le n_j$ for $1 \le j \le r$, and where
$$\hat{i} : \mathbb{P}^{a_1} \times \ldots \times \mathbb{P}^{a_r} \to \mathbb{P}^{m_1} \times \ldots \times \mathbb{P}^{m_r}$$
is the subscheme defined by the vanishing of $m_j - a_j$ coordinates $X_{l,j}$, $j = 1, \ldots, r$. It follows from the axiom (Sect*) that the class $\hat{i}_*(1_{\mathbb{P}^{a_1} \times \ldots \times \mathbb{P}^{a_r}})$ is independent of the particular $m_j - a_j$ coordinates chosen.

Using the standard cellular decomposition of each E_i, we have a filtration of E_{red} satisfying the conditions of axiom (CD′), and with each of the "closed cells" of the form of an embedded product $\mathbb{P}^{a_1} \times \ldots \times \mathbb{P}^{a_r}$ in some E_i, defined by the vanishing of coordinates $X_{l,j}$ as above. If a given product $\mathbb{P}^{a_1} \times \ldots \times \mathbb{P}^{a_r}$ should occur twice, say as a cell C in E_i and another cell C' in $E_{i'}$, then both $a_i \le N-1$ and $a_{i'} \le N-1$, so there is a cell C'' of the same type in $E_i \cap E_{i'}$. Thus $i_{C*}(1_C) = i_{C''*}(1_{C''}) = i_{C'*}(1_{C'})$ in $A_*(E)$, where $i_C : C \to E$, $i_{C'}$, $i_{C''}$ are the inclusions. Therefore, by axiom (CD′) and our description of $A_*(E_i)$

above, $A_*(E)$ is generated as an $A_*(S)$-module by the classes $\tilde{i}_*(1_{\mathbb{P}^{a_1}\times\ldots\times\mathbb{P}^{a_r}})$, where

$$\tilde{i}:\mathbb{P}^{a_1}\times\ldots\times\mathbb{P}^{a_r}\to E$$

is the composition of a map $\hat{i}$ followed by the inclusion $E_i\to E$, and the indices $(a_1,\ldots,a_r)$ run over all r-tuples with $0\le a_j\le N$, $j=1,\ldots,r$, with at least one index $a_i\le N-1$ for some $i\le m$. Comparing this with the description of $A_*((\mathbb{P}^N)^r)$ as a free $A_*(S)$-module with basis $\hat{i}_*(1_{\mathbb{P}^{a_1}\times\ldots\times\mathbb{P}^{a_r}})$, $0\le a_i\le N$, $i=1,\ldots,r$, we see that the $A_*(S)$-module generators for $A_*(E)$ described above are actually an $A_*(S)$-basis for $A_*(E)$, and therefore $A_*(E)$ is a summand of $A_*((\mathbb{P}^N)^r)$, which verifies the axiom (CD). □

Lemma 5.2.11. *Suppose we are given the data from definition 5.1.3 (D1)-(D3) of an oriented Borel-Moore homology theory A_* on $\mathbf{Sch}_S$, satisfying the axioms of definition 5.1.3, with the possible exception of the axiom (CD). Suppose that A_* satisfies the weak localization property (Loc′) (see remark 4.1.1) for $\mathcal{V}=\mathbf{Sch}_S$:*

Let $i:Z\to X$ be a closed immersion of finite type k-schemes with complement $j:U\to X$. Then the sequence

$$A_*(Z)\xrightarrow{i_*}A_*(X)\xrightarrow{j^*}A_*(U)$$

is exact.

Then A_* satisfies the axiom (CD′) of remark 5.1.7.

Proof. First take $Z=E_{\rm red}$. Then U is the empty scheme, hence $A_*(X_{\rm red})\to A_*(X)$ is surjective. Thus, it suffices to prove the axiom (CD′) for reduced E.

We proceed by noetherian induction. Take a filtration of E by closed subschemes

$$\emptyset=E_0\subset E_1\subset\ldots\subset E_N=E$$

satisfying the conditions of axiom (CD′). By induction, the map

$$\oplus_{i\le N-1,j}A_*(E_{ij})\to A_*(E_{N-1})$$

is surjective, where E_{ij} is the closure of the irreducible component E^0_{ij} of E_i, $i=1,\ldots,N-1$.

Let $E^0_{N1},\ldots,E^0_{Nm}$ be the irreducible components of $E_N\setminus E_{N-1}$, and let E_{Nj} be the closure of E^0_{Nj}. By assumption, each E^0_{Nj} is an affine space $\mathbb{A}^{N_j}$ over S and each E_{Nj} is smooth and quasi-projective over S. Let $\bar{p}:E_{Nj}\to S$ and $p:E^0_{Nj}\to S$ be the structure morphisms, and let $f:E^0_{Nj}\to E_{Nj}$ be the inclusion. We have the commutative diagram

$$\begin{tikzcd} A_*(E_{Nj}) \arrow[r, "f^*"] & A_*(E^0_{Nj}) \\ & A_{*-N_j}(S) \arrow[ul, "\bar{p}^*"] \arrow[u, "p^*"'] \end{tikzcd}$$

Since $p^* : A_{*-N_j}(S) \to A_*(E^0_{Nj})$ is an isomorphism by the homotopy property (EH), it follows that $f^* : A_*(E_{Nj}) \to A_*(E^0_{Nj})$ is surjective. Letting $i : E_{Nj} \to E$ and $\tilde{f} : E^0_{Nj} \to E$ be the inclusions, we have $\tilde{f}^* i_* = f^*$, by axiom (BM2). Thus the restriction map

$$A_*(E) \to A_*(E \setminus E_N) = \oplus_{j=1}^m A_*(E^0_{Nj})$$

is surjective.

Adding these surjectivities to our weak localization property, we have the exact sequences

$$A_*(E_{N-1}) \to A_*(E_N) \to \oplus_{j=1}^m A_*(E^0_{Nj}) \to 0$$
$$A_*(E_{Nj} \cap E_{N-1}) \to A_*(E_{Nj}) \to A_*(E^0_{Nj}) \to 0$$

By an elementary diagram chase the map

$$\oplus_{i,j} A_*(E_{ij}) \to A_*(E_N)$$

is surjective, and the induction goes through. $\square$

6

Functoriality

In this chapter, we construct pull-back maps in Ω_* for l.c.i. morphisms, giving Ω_* the structure of an oriented Borel-Moore homology theory. The basic idea is to use Fulton's method: first define the operation of intersection with a Cartier divisor, then use the deformation to the normal bundle to extend this operation to a pull-back map for a regular embedding. The case of a general l.c.i. morphism f is handled by factoring f as the composition of a regular embedding with a smooth morphism.

Since the generators of Ω_* are built out of smooth k-schemes, we need to modify Fulton's program. As it is difficult in algebraic geometry to make two maps transverse, we use instead resolution of singularities to modify a given Cartier divisor to a strict normal crossing divisor. This leads us to the notion of "refined cobordism" with respect to a fixed Cartier divisor D on some X, by restricting the maps $f : Y \to X$ to those for which either f^*D is a strict normal crossing divisor, or $f(Y)$ is contained in D. Fulton's program works well for the refined groups $\Omega_*(X)_D$, and we are left with proving a "moving lemma", namely, that the map $\Omega_*(X)_D \to \Omega_*(X)$ is an isomorphism. With this extra complication resolved, Fulton's method goes through to yield the desired l.c.i. pull-back.

We conclude in section 6.6 with a discussion of refined pull-back and refined intersection product, following again the ideas and methods of Fulton [9, section 6].

6.1 Refined cobordism

We begin by constructing pull-back maps for the inclusion of a divisor $i : D \to X$. In fact, a more flexible notion, due to Fulton [9], is that of a *pseudo-divisor*. In this section, we define a refined version $\Omega_*(X)_D$ of the cobordism group $\Omega_*(X)$ when one has the extra data of a pseudo-divisor D on X and we define intersection with a pseudo-divisor D, $D(-) : \Omega_*(X)_D \to \Omega_*(|D|)$, in the following section.

We go on to prove the required properties of this product, most notably the commutativity of intersection for two pseudo-divisors. For this, we will require the auxiliary construction of groups $\Omega_*(X)_{D|D'}$, which we also give in this section. We conclude the construction in §6.4 by proving the moving lemma, showing that the homomorphism $\Omega_*(X)_D \to \Omega_*(X)$ is an isomorphism.

6.1.1 Pseudo-divisors

Let X be a finite type k-scheme. Following Fulton [9], a *pseudo-divisor* D on X is a triple $D := (Z, \mathcal{L}, s)$, where $Z \subset X$ is a closed subset, $\mathcal{L}$ is an invertible sheaf on X, and s is a section of $\mathcal{L}$ on X, such that the subscheme $s = 0$ has support contained in Z; we identify triples $(Z, \mathcal{L}, s)$, $(Z, \mathcal{L}', s')$ if there is an isomorphism $\phi : \mathcal{L} \to \mathcal{L}'$ with $s' = \phi(s)$. In particular, having fixed $\mathcal{L}$, the section s is determined exactly up to a global unit on X. If we have a morphism $f : Y \to X$, we define $f^*(Z, \mathcal{L}, s) := (f^{-1}(Z), f^*\mathcal{L}, f^*s)$; clearly $(fg)^*(D) = g^*(f^*D)$ for a pseudo-divisor D. Also, an effective Cartier divisor D on X uniquely determines a pseudo-divisor $(|D|, \mathcal{O}_X(D), s_D)$, where $s_D : \mathcal{O}_X \to \mathcal{O}_X(D)$ is the canonical section and $|D|$ is the support of D.

We call Z the *support* of a pseudo-divisor $D := (Z, \mathcal{L}, s)$, and write $Z = |D|$. Similarly, we call s the *defining equation* of D, and write $s = \mathrm{eq.}(D)$. We let $\mathrm{div}(D)$ denote the subscheme $s = 0$, and write $\mathcal{O}_X(D)$ for $\mathcal{L}$. If X is in $\mathbf{Sm}_k$, if $|D| = |\mathrm{div} D|$ and if this subset has codimension one on X, then we identify D with the Cartier divisor $\mathrm{div} D$.

The zero pseudo-divisor is $(\emptyset, \mathcal{O}_X, 1)$. If we have pseudo-divisors $D = (Z, \mathcal{L}, s)$ and $D' = (Z', \mathcal{L}', s')$, define $D + D' = (Z \cup Z', \mathcal{L} \otimes \mathcal{L}', s \otimes s')$.

Remark 6.1.1. The main advantage of pseudo-divisors over Cartier divisors is that one has the functorial pull-back $D \mapsto f^*(D)$ for D a pseudo-divisor on some X and $f : Y \to X$ an arbitrary morphism; this is of course not the case for Cartier divisors. Even though we are mainly interested in defining the operation of intersection with a Cartier divisor, we will need the added flexibility of pseudo-divisors to handle the composition of two intersections, especially when the two divisors involved have a common component.

6.1.2 The group $\Omega_*(X)_D$

Let X be a finite type k-scheme and D a pseudo-divisor on X. We define the series of groups

$$\mathcal{Z}_*(X)_D \to \underline{\mathcal{Z}}_*(X)_D \to \underline{\Omega}_*(X)_D \to \mathbb{L}_* \otimes \underline{\Omega}_*(X)_D \to \Omega_*(X)_D$$

analogous to the sequence

$$\mathcal{Z}_*(X) \to \underline{\mathcal{Z}}_*(X) \to \underline{\Omega}_*(X) \to \mathbb{L}_* \otimes \underline{\Omega}_*(X) \to \Omega_*(X)$$

used to define $\Omega_*(X)$ in chapter 2.

Let $E = \sum_{i=1}^{r} n_i E_i$ be an effective strict normal crossing on a scheme $W \in \mathbf{Sm}_k$ with irreducible components $E_1, \ldots, E_r$. For each $J \subset \{1, \ldots, r\}$, we have the face

$$E_J := \cap_{j \in J} E_j,$$

which is smooth over k and has codimension $|J|$ on W. We call E *reduced* if all the $n_i = 1$.

We recall from §2.1.1 that $\mathcal{M}(X)$ is the set of isomorphism classes of projective morphisms $f : Y \to X$, with Y in $\mathbf{Sm}_k$ (where "isomorphism" means isomorphism over X). $\mathcal{M}(X)$ is a monoid under disjoint union; we have the group completion $\mathcal{M}^+(X)$, which is the free abelian group on the isomorphism classes $f : Y \to X$ in $\mathcal{M}(X)$ with Y irreducible.

Let $\mathcal{M}(X)_D$ be the submonoid of $\mathcal{M}(X)$ generated by $f : Y \to X$, with Y irreducible, and with either $f(Y) \subset |D|$, or with $\mathrm{div} f^* D$ a strict normal crossing divisor on Y. We let $\mathcal{M}^+(X)_D$ be the group completion of $\mathcal{M}(X)_D$; $\mathcal{M}^+(X)_D$ is clearly a subgroup of $\mathcal{M}^+(X)$.

Recall from definition 2.1.6 the notion of a cobordism cycle over X, and the free abelian group $\mathcal{Z}_*(X)$ on the isomorphism classes of cobordism cycles, graded by giving $(f : Y \to X, L_1, \ldots, L_r)$ degree $\dim_k Y - r$. We let $\mathcal{Z}_*(X)_D$ be the subgroup of $\mathcal{Z}_*(X)$ generated by the cobordism cycles $(f : Y \to X, L_1, \ldots, L_r)$ with $Y \to X$ in $\mathcal{M}(X)_D$.

Definition 6.1.2. *Let X be in $\mathbf{Sch}_k$ and let D be a pseudo-divisor on X. Let $\langle \mathcal{R}_*^{Dim} \rangle (X)_D$ be the subgroup of $\mathcal{Z}_*(X)_D$ generated by cobordism cycles of the form*

$$(f : Y \to X, \pi^*(L_1), \ldots, \pi^*(L_r), M_1, \ldots, M_s),$$

where $\pi : Y \to Z$ is a smooth quasi-projective morphism, Z is in $\mathbf{Sm}_k$, $L_1, \ldots, L_r$ are line bundles on Z and $r > \dim_k Z$. We set

$$\underline{\mathcal{Z}}_*(X)_D := \mathcal{Z}_*(X)_D / \langle \mathcal{R}_*^{Dim} \rangle (X)_D.$$

Just as for $\mathcal{M}(X)$, $\mathcal{Z}_*(X)$ and $\underline{\mathcal{Z}}_*(X)$, we have functoriality for smooth quasi-projective morphisms of relative dimension d, $f : X' \to X$:

$$f^* : \mathcal{M}(X)_D \to \mathcal{M}(X')_{f^*(D)},$$
$$f^* : \mathcal{Z}_*(X)_D \to \mathcal{Z}_{*+d}(X')_{f^*(D)},$$
$$f^* : \underline{\mathcal{Z}}_*(X)_D \to \underline{\mathcal{Z}}_{*+d}(X')_{f^*(D)},$$

and push-forward maps for projective morphisms $f : X' \to X$:

$$f_* : \mathcal{M}(X')_{f^*(D)} \to \mathcal{M}(X)_D,$$
$$f_* : \mathcal{Z}_*(X')_{f^*(D)} \to \mathcal{Z}_*(X)_D,$$
$$f_* : \underline{\mathcal{Z}}_*(X')_{f^*(D)} \to \underline{\mathcal{Z}}_*(X)_D.$$

Also, for $L \to X$ a line bundle on X, we have the Chern class endomorphism

$$\tilde{c}_1(L) : \mathcal{Z}_*(X)_D \to \mathcal{Z}_{*-1}(X)_D,$$

defined as for $\mathcal{Z}_*(X)$

$$\tilde{c}_1(L)((f : Y \to X, L_1, \ldots, L_r)) := (f : Y \to X, L_1, \ldots, L_r, f^*L).$$

This descends to the locally nilpotent endomorphism

$$\tilde{c}_1(L) : \underline{\mathcal{Z}}_*(X)_D \to \underline{\mathcal{Z}}_{*-1}(X)_D.$$

The operation of product over k defines external products

$$\times : \mathcal{Z}_*(X)_D \otimes \mathcal{Z}_*(X')_{D'} \to \mathcal{Z}_*(X \times_k X')_{p_1^*D + p_2^*D'},$$

which descend to $\underline{\mathcal{Z}}_*(-)_-$, and have all the compatibilities with f_*, f^* and $\tilde{c}_1(L)$ as for $\mathcal{Z}_*(-)$ and $\underline{\mathcal{Z}}_*(-)$. All these operations are compatible with the corresponding ones defined for $\mathcal{Z}_*(-)$ and $\underline{\mathcal{Z}}_*(-)$, via the natural maps $\mathcal{Z}_*(X)_D \to \mathcal{Z}_*(X)$ and $\underline{\mathcal{Z}}_*(X)_D \to \underline{\mathcal{Z}}_*(X)$.

6.1.3 Good position

We define various notions of "good position" of a divisor E with respect to a pseudo-divisor D.

Definition 6.1.3. *Let $f : W \to X$ be in $\mathcal{M}(X)$, with W irreducible, and let E be a strict normal crossing divisor on W. Let D be a pseudo-divisor on X. We say that*

1. *E is* in good position *with respect to D if, for each face E_J of E, the composition $E_J \to W \to X$ is in $\mathcal{M}(X)_D$.*
2. *E is* in very good position *with respect to D if either $f(W) \subset |D|$, or, if not, $E + \mathrm{div} f^*D$ is a strict normal crossing divisor on W.*
3. *E is in* general position *with respect to D if E is in very good position with respect to D and in addition, in case $f(W) \not\subset |D|$, E and $\mathrm{div} f^*D$ have no common components.*

We extend these notions to W not necessarily irreducible by imposing the appropriate condition on each component of W.

Remarks 6.1.4. (1) It is easy to see that, for $f : W \to X$ in $\mathcal{M}(X)$ with strict normal crossing divisor E, if E is in very good position, it is in good position with respect to D.

(2) If $f : W \to X$ is in $\mathcal{M}(X)_D$, and $\mathcal{L}$ is a very ample invertible sheaf on W, it follows from the Bertini theorem that, for a general section s of $\mathcal{L}$, the divisor of s is in general position with respect to D.

Definition 6.1.5. *Let X be in $\mathbf{Sch}_k$ and let D be a pseudo-divisor on X. Let $\langle\mathcal{R}_*^{Sect}\rangle(X)_D$ be the subgroup of $\underline{\mathcal{Z}}_*(X)_D$ generated by elements of the form*

$$[f : Y \to X, L_1, \ldots, L_r] - [f \circ i : Z \to X, i^*(L_1), \ldots, i^*(L_{r-1})],$$

with $r > 0$, $[f : Y \to X, L_1, \ldots, L_r]$ a cobordism cycle in $\mathcal{Z}_*(X)_D$ and $i : Z \to Y$ the closed immersion of the subscheme defined by the vanishing of a transverse section $s : Y \to L_r$, such that Z is in very good position with respect to D.

We set

$$\underline{\Omega}_*(X)_D := \underline{\mathcal{Z}}_*(X)_D/\langle\mathcal{R}_*^{Sect}\rangle(X)_D.$$

We have the evident natural map $\underline{\Omega}_*(X)_D \to \underline{\Omega}_*(X)$. The operations f^*, f_* and $\tilde{c}_1(L)$ descend to the quotient $\underline{\Omega}_*(-)_-$ of $\underline{\mathcal{Z}}_*(-)_-$, and are compatible with the corresponding operations on $\underline{\Omega}_*(-)$ via the natural maps $\underline{\Omega}_*(X)_D \to \underline{\Omega}_*(X)$. Similarly, the external products for $\underline{\mathcal{Z}}_*(-)_-$ descend to $\underline{\Omega}_*(-)_-$, and these external products are compatible with the external products on $\underline{\Omega}_*(-)$.

As in §1.1, we have the universal formal group law $(F_{\mathbb{L}}, \mathbb{L}_*)$ with coefficient ring $\mathbb{L}_*$ the Lazard ring. If $T_1, T_2 : B \to B$ are commuting locally nilpotent operators on an abelian group B, and $F(u,v) = \sum_{i,j} a_{ij}u^iv^j$ is a power series with $\mathbb{L}_*$-coefficients, we have the well-defined $\mathbb{L}_*$-linear operator $F(T_1,T_2) : \mathbb{L}_* \otimes B \to \mathbb{L}_* \otimes B$ defined by

$$F(T_1,T_2)(a \otimes b) := \sum_{i,j} aa_{ij} \otimes (T_1^i \circ T_2^j)(b).$$

Definition 6.1.6. *For X in $\mathbf{Sch}_k$, let $\langle\mathbb{L}_*\mathcal{R}_*^{FGL}\rangle(X)_D$ be the $\mathbb{L}_*$-submodule of $\mathbb{L}_* \otimes \underline{\Omega}_*(X)_D$ generated by elements of the form*

$$(\mathrm{Id} \otimes f_*)\big(F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))(\eta) - \tilde{c}_1(L \otimes M)(\eta)\big),$$

where $f : Y \to X$ is in $\mathcal{M}(X)_D$, L and M are line bundles on Y, and η is in $\underline{\Omega}_*(Y)_{f^*D}$. We set

$$\Omega_*(X)_D := \mathbb{L}_* \otimes \underline{\Omega}_*(X)_D/\langle\mathbb{L}_*\mathcal{R}_*^{FGL}\rangle(X)_D.$$

The natural transformation $\underline{\Omega}_*(X)_D \to \underline{\Omega}_*(X)$ descends to a natural $\mathbb{L}_*$-linear transformation $\Omega_*(X)_D \to \Omega_*(X)$. The structures we have defined for $\underline{\Omega}_*(-)_-$: f^*, f_*, $\tilde{c}_1(L)$ and external products, all descend to $\Omega_*(-)_-$, and are compatible with the corresponding structures on $\Omega_*(-)$, via the natural transformation $\Omega_*(X)_D \to \Omega_*(X)$.

If $f : Y \to X$ is an X-scheme, and D is a pseudo-divisor on X, we will often write $\Omega_*(Y)_D$ for $\Omega_*(Y)_{f^*(D)}$, and similarly for $\underline{\Omega}_*(Y)_D$, etc.

6.1.4 Refined divisor classes

The operators

$$\tilde{c}_1(L) : \Omega_*(X)_D \to \Omega_{*-1}(X)_D$$

are locally nilpotent and commute with one another, thus, if we have line bundles $L_1, \ldots, L_r$ on X, and a power series $F(u_1, \ldots, u_r)$ with $\mathbb{L}_*$-coefficients, we have the $\mathbb{L}_*$-linear endomorphism

$$F(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_r)) : \Omega_*(X)_D \to \Omega_{*-1}(X)_D.$$

If $f : W \to X$ is in $\mathcal{M}(X)_D$, we have the element $1_W^D = [\mathrm{Id} : W \to W] \in \Omega_*(W)_D$. Given line bundles $L_1, \ldots, L_r$ on W we set

$$[F(L_1, \ldots, L_r)]_D := F(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_r))(1_X^D) \in \Omega_*(W)_D.$$

We recall some notation from §3.1.1. Let $n_1, \ldots, n_r$ be positive integers. We have the power series with $\mathbb{L}_*$-coefficients $F^{n_1,\ldots,n_r}$ giving the sum in the universal group law $(F_{\mathbb{L}}, \mathbb{L}_*)$:

$$F^{n_1,\ldots,n_r}(u_1, \ldots, u_r) = n_1 \cdot_F u_1 +_F \ldots +_F n_r \cdot_F u_r.$$

We have as well the canonical decomposition

$$F^{n_1,\ldots,n_r}(u_1, \ldots, u_r) = \sum_{||J||=1} u^J F_J^{n_1,\ldots,n_r}(u_1, \ldots, u_r),$$

defining the power series $F_J^{n_1,\ldots,n_r}$.

If $E = \sum_{i=1}^r n_i E_i$ is a strict normal crossing divisor on a scheme $W \in \mathbf{Sm}_k$, with support $|E| := \cup_{i=1}^r E_i$, and irreducible components $E_1, \ldots, E_r$, we have defined the divisor class $[E \to |E|]$ of $\Omega_*(|E|)$ by the formula

$$[E \to |E|] := \sum_{J,\ ||J||=1} \iota_*^J([F_J^{n_1,\ldots,n_r}(O_W(E_1)^J, \ldots, O_W(E_r)^J)]).$$

Suppose now that we have $f : W \to X$ in $\mathcal{M}(X)$, and a strict normal crossing divisor E on W, such that E is in good position with respect to D. Write $E = \sum_{i=1}^m n_i E_i$, with the E_i irreducible.

Since the subscheme E^J is in good position with respect to D for each J, the morphism $f \circ \iota^J : E^J \to X$ is in $\mathcal{M}(X)_D$, so we have the class

$$[F_J^{n_1,\ldots,n_r}(O_W(E_1)^J, \ldots, O_W(E_r)^J)]_D \in \Omega_*(E^J)_D,$$

giving the *refined divisor class*

$$[E \to |E|]_D := \sum_J \iota_*^J [F_J^{n_1,\ldots,n_r}(O_W(E_1)^J, \ldots, O_W(E_r)^J)]_D$$

in $\Omega_*(|E|)_D$.

The properties of $\tilde{c}_1(L)$, $[E \to |E|]$ and $\Omega_*(X)$ proved in §2.3 and §3.1 carry over without change to $\tilde{c}_1(L)$ (acting on $\Omega_*(X)_D$), $[E \to |E|]_D$ and $\Omega_*(X)_D$.

6.1.5 A further refinement

In order to discuss issues of functoriality, it will be necessary to make an extension of the above construction.

Definition 6.1.7. *Let X be a finite type k-scheme, with pseudo-divisors D, D'. Let $i : |D| \to X$ be the inclusion. We let $\mathcal{M}(X)_{D|D'}$ be the submonoid of $\mathcal{M}(X)_D$ generated by those $f : Y \to X$, with Y irreducible, such that*

1. *If $f(Y) \subset |D|$, then $f : Y \to |D|$ is in $\mathcal{M}(|D|)_{D'}$.*
2. *If $f(Y) \not\subset |D|$, then, for each face F of $\mathrm{div} f^*D$, the map $f : F \to |D|$ is in $\mathcal{M}(|D|)_{D'}$, i.e., $\mathrm{div} f^*D$ is in good position with respect to D'.*

Definition 6.1.8. *Let $f : W \to X$ be in $\mathcal{M}(X)$, and let E be a strict normal crossing divisor on W. We say that E is* in good position with respect to $D|D'$ *if for each face E^J of E, the composition*

$$E^J \xrightarrow{i^J} W \xrightarrow{f} X$$

is in $\mathcal{M}(X)_{D|D'}$.

For $f : W \to X$ in $\mathcal{M}(X)_{D|D'}$ with W irreducible, we say that E is in very good position with respect to $D|D'$ *if E is in very good position with respect to D and*

1. *if $f(W) \subset |D|$, then E is in very good position with respect to D'*
2. *if $f(W) \not\subset |D|$, then for each face F of $\mathrm{div} f^*D$ not contained in E, the normal crossing divisor $F \cdot E$ on F is in very good position with respect to D'.*

We say that E is in general position with respect to $D|D'$ *if E is in very good position with respect to $D|D'$ and in general position with respect to D.*

We extended these notions to reducible W by requiring the appropriate condition on each component of W.

Remark 6.1.9. Suppose that $W \to X$ is in $\mathcal{M}(X)_{D|D'}$. If E is a divisor on W, in very good position with respect to $D|D'$, and E^J is a face of E, then E^J is in $\mathcal{M}(X)_{D|D'}$. Indeed, if $f(W) \subset |D|$, this is evident, and if $f(W) \not\subset |D|$, then each face F_E of the normal crossing divisor $E^J \cdot \mathrm{div} f^*D$ on E^J is of the form $E^J \cdot F$ for F a face of $\mathrm{div} f^*D$. Thus F_E is a face of the normal crossing divisor $F \cdot E$ on F, and hence $F_E \to |D|$ is in $\mathcal{M}(|D|)_{D'}$. This shows that $E^J \to X$ is in $\mathcal{M}(X)_{D|D'}$, as claimed. In other words, if E is in very good position with respect to $D|D'$, then E is in good position with respect to $D|D'$.

Making the evident modifications to the constructions of the previous section, we have the sequence of abelian groups

$$\mathcal{Z}_*(X)_{D|D'} \to \underline{\mathcal{Z}}_*(X)_{D|D'} \to \underline{\Omega}_*(X)_{D|D'} \to \Omega_*(X)_{D|D'}.$$

Explicitly, $\mathcal{Z}_*(X)_{D|D'}$ is the subgroup of $\mathcal{Z}_*(X)$ generated by the cobordism cycles $(Y \to X, L_1, \ldots, L_r)$ with $Y \to X$ in $\mathcal{M}(X)_{D|D'}$.

Let $\langle \mathcal{R}_*^{Dim} \rangle (X)_{D|D'}$ be the subgroup generated by cobordism cycles of the form

$$(f : Y \to X, \pi^*(L_1), \ldots, \pi^*(L_r), M_1, \ldots, M_s),$$

where $\pi : Y \to Z$ is a smooth quasi-projective morphism, Z is in $\mathbf{Sm}_k$, $L_1, \ldots, L_r$ are line bundles on Z and $r > \dim_k Z$. We let $\underline{\mathcal{Z}}_*(X)_{D|D'}$ be the quotient group $\mathcal{Z}_*(X)_{D|D'} / \langle \mathcal{R}_*^{Dim} \rangle (X)_{D|D'}$. $\underline{\Omega}_*(X)_{D|D'}$ is the quotient of $\underline{\mathcal{Z}}_*(X)_{D|D'}$ by the subgroup $\langle \mathcal{R}_*^{Sect} \rangle (X)_{D|D'}$ generated by elements of the form

$$(f : Y \to X, L_1, \ldots, L_r) - (f \circ i : Z \to X, i^*(L_1), \ldots, i^*(L_{r-1})),$$

with $r > 0$, $(f : Y \to X, L_1, \ldots, L_r)$ a cobordism cycle in $\mathcal{Z}_*(X)_{D|D'}$ and $i : Z \to Y$ the closed immersion of the subscheme defined by the vanishing of a transverse section $s : Y \to L_r$, such that Z is in very good position with respect to $D|D'$. $\Omega_*(X)_{D|D'}$ is the quotient of $\mathbb{L}_* \otimes \underline{\Omega}_*(X)_{D|D'}$ by the $\mathbb{L}_*$-submodule $\langle \mathbb{L}_* \mathcal{R}_*^{FGL} \rangle (X)_{D|D'}$ generated by elements of the form

$$(\mathrm{Id} \otimes f_*)\big(F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))(\eta) - \tilde{c}_1(L \otimes M)(\eta)\big),$$

where $f : Y \to X$ is in $\mathcal{M}(X)_{D|D'}$, L and M are line bundles on Y, and η is in $\underline{\Omega}_*(Y)_{f^*D|f^*D'}$.

Forgetting D' defines in an evident manner the natural transformations

$$\mathcal{Z}_*(X)_{D|D'} \to \mathcal{Z}_*(X)_D, \ \underline{\mathcal{Z}}_*(X)_{D|D'} \to \underline{\mathcal{Z}}_*(X)_D,$$

etc. The operations f^*, f_*, $\tilde{c}_1(L)$ and external products defined on $\mathcal{Z}_*(X)_D$, $\underline{\mathcal{Z}}_*(X)_D$, etc. have their evident refinements to $\mathcal{Z}_*(X)_{D|D'}$, $\underline{\mathcal{Z}}_*(X)_{D|D'}$, $\underline{\Omega}_*(X)_{D|D'}$ and $\Omega_*(X)_{D|D'}$, satisfying the same structural relations, and compatible with the operations on $\mathcal{Z}_*(X)_D$, $\underline{\mathcal{Z}}_*(X)_D$, etc.

In particular, the operations $\tilde{c}_1(L)$ on $\underline{\mathcal{Z}}_*(X)_{D|D'}$ and $\underline{\Omega}_*(X)_{D|D'}$ are locally nilpotent and commute with one another, so the definition of $\Omega_*(X)_{D|D'}$ makes sense. Also, as for one pseudo-divisor, if we have an X-scheme $f : Y \to X$, we write $\mathcal{M}(Y)_{D|D'}$, $\Omega_*(Y)_{D|D'}$, etc., for $\mathcal{M}(Y)_{f^*D|f^*D'}$, $\Omega_*(Y)_{f^*D|f^*D'}$, etc.

For $f : W \to X$ in $\mathcal{M}(X)_{D|D'}$, the identity map $\mathrm{Id}_W : W \to W$ is in $\mathcal{M}(W)_{D|D'}$, giving the identity class $1_W^{D|D'} \in \Omega_*(W)_{D|D'}$. Thus, if $F(u_1, \ldots, u_r)$ is a power series with $\mathbb{L}_*$-coefficients, and $L_1, \ldots, L_r$ are line bundles on W, we have the class

$$[F(L_1, \ldots, L_r)]_{D|D'} := F(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_r))(1_W^{D|D'}) \in \Omega_*(W)_{D|D'}.$$

Therefore, given $f : W \to X$ in $\mathcal{M}(X)$ and a strict normal crossing divisor $E = \sum_{i=1}^r n_i E_i$ on W which is in good position with respect to $D|D'$, we have the class of E, $[E \to |E|]_{D|D'} \in \Omega_*(|E|)_{D|D'}$, defined by

$$[E \to |E|]_{D|D'} := \sum_{J,\ ||J||=1} \iota_*^J([F_J^{n_1,\ldots,n_r}(O_W(E_1)^J, \ldots, O_W(E_r)^J)]_{D|D'}).$$

We recover the groups $\mathcal{M}(X)_D$, $\underline{\Omega}_*(X)_D$, etc., as a special case by taking D' to be the 0 pseudo-divisor $(\emptyset, \mathcal{O}_X, 1)$. Similarly, we recover $\mathcal{M}(X)_{D'}$, $\underline{\Omega}_*(X)_{D'}$, etc., by taking $D = (X, \mathcal{O}_X, 0)$.

As above, the properties of $\underline{\Omega}_*(X)$, $\tilde{\Omega}_*(X)$, $\Omega_*(X)$, $\tilde{c}_1(L)$ and $[E \to |E|]$ discussed in §2.3 and §3.1 carry over without change to $\underline{\Omega}_*(X)_{D|D'}$, $\tilde{\Omega}_*(X)_{D|D'}$, $\Omega_*(X)_{D|D'}$, $\tilde{c}_1(L)$ and $[E \to |E|]_{D|D'}$. Also, as in §3.1, if E is a strict normal crossing divisor on some W, and $f : W \to X$ is in $\mathcal{M}(X)_{D|D'}$, we let $[E \to W]_{D|D'}$ denote $i_*([E \to |E|]_{D|D'})$, where $i : |E| \to W$ is the inclusion.

Remark 6.1.10. Let $f : Y \to X$ be in $\mathcal{M}(X)_{D|D'}$, with Y irreducible, and suppose that $f(Y) \not\subset |D|$. Then $\mathrm{div} f^*D$ is a strict normal crossing divisor, in good position with respect to D', so if E is an effective divisor on Y with $f(|E|) \subset |D|$, then E is in good position with respect to D', hence the class $[E \to |E|]_{D'}$ in $\Omega_*(|E|)_{D'}$ is defined.

Remark 6.1.11. Among the results of §3.1 that extend to $\Omega_*(X)_{D|D'}$, we note the analog of proposition 3.1.9: Let E be a strict normal crossing divisor on some $W \in \mathbf{Sm}_k$. Let D, D' be pseudo-divisors on W. Suppose that E is in good position with respect to $D|D'$. Then

$$[E \to W]_{D|D'} = [O_W(E)]_{D|D'}.$$

The proof is exactly the same as for proposition 3.1.9.

6.1.6 Some properties of $\Omega_*(X)_D$

In this section, we assume that k admits resolution of singularities.

Lemma 6.1.12. *Let X be in $\mathbf{Sch}_k$, and let D be a pseudo-divisor on X. Let $f : Y \to X$ be in $\mathcal{M}(X)_D$, let $i : F \to Y$ be a closed subscheme of Y and let $\mu : W \to Y$ be a projective birational morphism. Suppose that the exceptional divisor E of μ is in very good position with respect to D and that $\mu(|E|) \subset F$. Then there is an element $\alpha \in \Omega_*(F)_D$ with*

$$[f \circ \mu : W \to X]_D = [f : Y \to X]_D + (f \circ i)_*(\alpha) \in \Omega_*(X)_D.$$

Proof. Since E is in very good position with respect to D, E is a strict normal crossing divisor on W. We may replace X with Y, so Id_Y is in $\mathcal{M}(Y)_D$, and we may suppose that Y is irreducible. If $|D| = Y$, then $\Omega_*(Y)_D = \Omega_*(Y)$, $\Omega_*(F)_D = \Omega_*(F)$, and the result is proposition 3.2.4. Suppose $|D| \neq Y$. Then D is a strict normal crossing divisor on Y. Since E is in very good position with respect to D, $E + \mu^*D$ is a strict normal crossing divisor on W; in particular, $\mu : W \to Y$ is in $\mathcal{M}(Y)_D$, so $[\mu : W \to Y]_D$ is defined.

We consider the deformation to the normal bundle, as in §3.2.1. Let $Z \subset Y$ be a closed subscheme supported in F such that $W \to Y$ is the blow-up of Y along Z, and let $\rho' : T' \to Y \times \mathbb{P}^1$ be the blow-up of $Y \times \mathbb{P}^1$ along $Z \times 0$. Without loss of generality, we may assume that $F = Z_{\mathrm{red}}$. Let $\langle F \times \mathbb{P}^1 \rangle \subset T'$ be the proper transform of $F \times \mathbb{P}^1$, let $\langle Y \times 0 \rangle$ be the proper transform of $Y \times 0$, and let $\hat{E}'$ be the exceptional divisor of ρ'. Let $U = T' \setminus \langle F \times \mathbb{P}^1 \rangle \cap \hat{E}'$. Then (by lemma 3.2.1) U is smooth over k, U contains $\langle Y \times 0 \rangle$ and the induced morphism $\langle Y \times 0 \rangle \to Y$ is isomorphic to $W \to Y$. In addition (see the proof of *loc. cit.*) $U \to Y$ is locally isomorphic to $\mu \circ p_1 : W \times \mathbb{A}^1 \to Y$, and, in these coordinates, $\hat{E}' \cap U \to Y$ is locally isomorphic to $\mu \circ p_1 : E \times \mathbb{A}^1 \to Y$ and $\langle Y \times 0 \rangle \subset U$ is the subscheme $W \times 0$. Thus, $(\rho'^*(D \times \mathbb{P}^1) + \hat{E}' + \langle Y \times 0 \rangle) \cap U$ is a strict normal crossing divisor on U. Therefore, by resolution of singularities, there is a projective birational morphism $\pi : T \to T'$ which an isomorphism over U, such that the induced morphism $\rho : T \to Y \times \mathbb{P}^1$ satisfies:

1. $T \to Y \times \mathbb{P}^1$ is in $\mathcal{M}(Y \times \mathbb{P}^1)_D$,
2. Letting $\hat{E} \subset T$ be the exceptional divisor of ρ, $\rho^*(D \times \mathbb{P}^1) + \hat{E} + \langle Y \times 0 \rangle$ is a strict normal crossing divisor on T

Since $\rho : T \to Y \times \mathbb{P}^1$ is an isomorphism away from $Y \times 0$, $\rho^{-1}(Y \times 1)$ is isomorphic to Y. By (1) and (2) above, we have the classes $[\rho^*(Y \times 1) \to T]_D$ and $[\hat{E} + \langle Y \times 0 \rangle \to T]_D$. Since $\hat{E} + \langle Y \times 0 \rangle = \rho^*(Y \times 0)$, we have

$$[\rho^*(Y \times 1) \to T]_D = [(p_2 \circ \rho)^*(\mathcal{O}_{\mathbb{P}^1}(1))]_D = [\hat{E} + \langle Y \times 0 \rangle \to T]_D \qquad (6.1)$$

by remark 6.1.11. On the other hand, by definition of the divisor class $[\hat{E} + \langle Y \times 0 \rangle \to |\hat{E} + \langle Y \times 0 \rangle|]_D$, there is a class $\beta \in \Omega_*(|\hat{E}|)_D$ such that

$$[\hat{E} + \langle Y \times 0 \rangle \to T]_D = [\langle Y \times 0 \rangle \to T]_D + \hat{i}_*(\beta), \qquad (6.2)$$

where $\hat{i} : |\hat{E}| \to T$ is the inclusion.

Let $p : T \to Y$ be $p_1 \circ \mu$, and let $p^F : |\hat{E}| \to F$ be the map induced by p. Let $\alpha = p^F_*(\beta)$. Applying the push-forward p_* to the identity (6.2) and using (6.1) yields

$$[\mathrm{Id}_Y]_D = [f : W \to Y]_D + i_*(\alpha),$$

as desired. $\square$

Lemma 6.1.13. *Let $f : X \to Z$ be a morphism in* $\mathbf{Sch}_k$ *with Z in* $\mathbf{Sm}_k$*, and let $L_1, \ldots, L_r$ be line bundles on Z with $r > \dim_k Z$. Let D be a pseudo-divisor on X. Then the operator $\tilde{c}_1(f^*L_1) \circ \ldots \circ \tilde{c}_1(f^*L_r)$ vanishes on $\Omega_*(X)_D$.*

Proof. We proceed by induction on $\dim_k Z$. Since the operators $\tilde{c}_1(L)$ are $\mathbb{L}_*$-linear and commute with each other, it suffices to show that the operator in question vanishes on elements $g : Y \to X$ of $\mathcal{M}(X)_D$. Using the projection formula reduces us to the case $g = \mathrm{Id}_Y$, that is, it suffices to show that

$$(\mathrm{Id}_Y, f^*L_1, \ldots, f^*L_r) = 0 \text{ in } \Omega_*(Y)_D,$$

assuming Id_Y is in $\mathcal{M}(Y)_D$.

We may assume that Y is irreducible. Thus, either $Y \subset |D|$, or D is a strict normal crossing divisor on Y. We give the proof in the second case; the proof in the first case is essentially the same, but easier, and is left to the reader.

Using the formal group law, we reduce to the case of very ample line bundles L_i (see the proof of lemma 3.2.6).

If $\dim_k Z = 0$, all the line bundles are trivial, hence have a nowhere vanishing section. We may then use the relations in $\langle \mathcal{R}^{Sect}_* \rangle(Y)_{D|D'}$ (for the empty divisor) to conclude that $(\mathrm{Id}_Y, f^*L_1, \ldots, f^*L_r) = 0$.

Suppose now that $\dim_k Z > 0$. Let s be a section of L_r, chosen so that f^*s is not identically zero on Y. We may also assume that $s = 0$ is a smooth divisor $\bar{i} : \bar{Z} \to Z$ on Z. Let H be the divisor of f^*s with inclusion $i : |H| \to Y$, and let $\bar{f} : |H| \to \bar{Z}$ be the induced morphism.

By resolution of singularities, there is a projective birational morphism $\mu : W \to Y$ such that $\mu^*(H + D)$ is a strict normal crossing divisor on W, and with μ an isomorphism over $Y \setminus |H|$. By lemma 6.1.12, there is a class $\alpha \in \Omega_*(H)_D$ with

$$[W \to Y]_D = [\mathrm{Id}_Y]_D + i_*(\alpha).$$

Since

$$\tilde{c}_1(f^*L_1) \circ \ldots \tilde{c}_1(f^*L_r)(i_*(\alpha)) = i_*(\tilde{c}_1(\bar{f}^*(\bar{i}^*L_1)) \circ \ldots \tilde{c}_1(\bar{f}^*(\bar{i}^*L_r))(\alpha)),$$

our induction hypothesis implies that $\tilde{c}_1(f^*L_1)\circ\ldots\tilde{c}_1(f^*L_r)(i_*(\alpha)) = 0$. Thus, it suffices to show that $\tilde{c}_1(f^*L_1) \circ \ldots \tilde{c}_1(f^*L_r)([W \to Y]_D) = 0$. As this element is just the push-forward of $(W, (f\mu)^*L_1, \ldots, (f\mu)^*L_r)$ by μ, we may replace Y with W; changing notation, we may assume that $H + D$ is a strict normal crossing divisor on Y.

By remark 6.1.11, we have the identity in $\Omega_*(Y)_D$

$$[H \to Y]_D = [O_Y(H)]_D = [f^*L_r]_D.$$

Thus

$$\begin{aligned}
&(\mathrm{Id}_Y, f^*L_1, \ldots, f^*L_r) \\
&\qquad = \tilde{c}_1(f^*L_1) \circ \ldots \circ \tilde{c}_1(f^*L_{r-1})([H \to Y]_D) \\
&\qquad = i_*(\tilde{c}_1(\bar{f}^*(\bar{i}^*L_1)) \circ \ldots \circ \tilde{c}_1(\bar{f}^*(\bar{i}^*L_r))([H \to |H|]_D)).
\end{aligned}$$

As this last element is zero by our induction hypothesis, the lemma is proved. □

6.2 Intersection with a pseudo-divisor

In this section and for the remainder of this chapter, we assume that k admits resolution of singularities, unless explicitly stated otherwise.

6.2.1 The intersection map on cobordism cycles

If L is a line bundle on a k-scheme X with sheaf of sections $\mathcal{L}$, we write $\tilde{c}_1(\mathcal{L})$ for $\tilde{c}_1(L)$.

Let $D = (|D|, \mathcal{O}_X(D), s)$ be a pseudo-divisor on X, let $f : Y \to X$ be in $\mathcal{M}(X)_D$ with Y irreducible, and consider a cobordism cycle $\eta := (Y \to X, L_1, \ldots, L_r)$ in $\mathcal{Z}_*(X)_D$. We define the element $D(\eta) \in \Omega_*(|D|)$ as follows: If $f(Y) \subset |D|$, let $f^D : Y \to |D|$ be the morphism induced by f. We have the element $\tilde{c}_1(f^*\mathcal{O}_X(D))(\eta)$ in $\Omega_*(Y)$; we then define

$$D(\eta) := f^D_*(\tilde{c}_1(f^*\mathcal{O}_X(D))(\eta)) \in \Omega_*(|D|).$$

If $f(Y) \not\subset |D|$, then $\tilde{D} := \mathrm{div} f^*D$ is a strict normal crossing divisor on Y. We let $f^D : |\tilde{D}| \to |D|$ be the restriction of f, L_i^D the restriction of L_i to $|\tilde{D}|$, and define

$$D(\eta) := f^D_*(\tilde{c}_1(L_1^D) \circ \ldots \circ \tilde{c}_1(L_r^D)([\tilde{D} \to |\tilde{D}|])) \in \Omega_*(|D|).$$

We extend this operation to a homomorphism $D(-) : \mathcal{Z}_*(X)_D \to \Omega_*(|D|)$ by linearity.

Suppose we have a second pseudo-divisor D' on X. We refine the above construction to give, for each $\eta \in \mathcal{Z}_*(X)_{D|D'}$, a class $D(\eta)_{D'}$ in $\Omega_*(|D|)_{D'}$. For this, take $f : Y \to X$ in $\mathcal{M}(X)_{D|D'}$ with Y irreducible, and consider $\eta = (Y \to X, L_1, \ldots, L_r)$. If $f(Y) \subset |D|$, then $f^D : Y \to |D|$ is in $\mathcal{M}(|D|)_{D'}$. We may thus set

$$D(\eta)_{D'} := f^D_*(\tilde{c}_1(f^*\mathcal{O}_X(D))(\eta)),$$

giving a well-defined class in $\Omega_*(|D|)_{D'}$. If $f(Y) \not\subset |D|$, then $\tilde{D} := \mathrm{div} f^*D$ is a strict normal crossing divisor contained in $f^{-1}(|D'|)$. The condition that f is in $\mathcal{M}(X)_{D|D'}$ implies that $\tilde{D}$ is in good position with respect to D', hence the refined divisor class $[\tilde{D} \to |\tilde{D}|]_{D'} \in \Omega_*(|\tilde{D}|)_{D'}$ is defined. Letting $f^D : |\tilde{D}| \to |D|$ be the map induced by f, we then set

$$D(\eta)_{D'} := f^D_*(\tilde{c}_1(L_1) \circ \ldots \tilde{c}_1(L_r)([\tilde{D} \to |\tilde{D}|]_{D'}))$$

in $\Omega_*(|D|)_{D'}$.

Extending by linearity gives $D(\eta)_{D'}$ in $\Omega_*(|D|)_{D'}$ for each $\eta \in \mathcal{Z}_*(X)_{D|D'}$. If we take $D' = 0$, then we recover the definitions for $D(f)$ given above. We sometimes omit the subscript D' from the notation if the context makes the meaning clear.

The next two results follow directly from the definitions:

Lemma 6.2.1. *Let X be a finite type k-scheme, with pseudo-divisors D, D'. Let $g : X' \to X$ be a morphism of finite type, and let $g_D : |g^*D| \to |D|$ be the restriction of g.*

1. *Suppose that g is projective and let η be in $\mathcal{Z}_*(X')_{D|D'}$. Then $g_*\eta$ is in $\mathcal{Z}_*(X)_{D|D'}$, and*
$$g_{D*}(g^*D(\eta)_{D'}) = D(g_*\eta)_{D'}.$$

2. *Suppose that g is smooth and quasi-projective. Take $\eta \in \mathcal{Z}_*(X)_D$. Then $g^*\eta$ is in $\mathcal{Z}_*(X')_D$, g_D is smooth and quasi-projective, and*

$$g_D^*(D(\eta)_{D'}) = (g^*D)(g^*\eta)_{D'}.$$

Lemma 6.2.2. *Let X be a finite type k-scheme, with pseudo-divisors D, D'. Let η be in $\mathcal{Z}_*(X)_{D|D'}$, let L be a line bundle on X and let L^D be the restriction of L to D. Then*

$$\tilde{c}_1(L^D)(D(\eta)_{D'}) = D(\tilde{c}_1(L)(\eta))_{D'}.$$

We extend the operation $D(-)_{D'}$ to $\mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D|D'}$ by $\mathbb{L}_*$-linearity. More generally, let $F(u_1, \ldots, u_r)$ be a power series with $\mathbb{L}_*$-coefficients, let $L_1, \ldots, L_r$ be line bundles on X, and let $f : Y \to X$ be in $\mathcal{M}(X)_{D|D'}$. Letting F_N denote the truncation of F after total degree N, we have, for all $N \geq \dim_k Y$ and all $m \geq 0$

$$\begin{aligned}
&D(F_N(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_m))([f]))_{D'} \\
&\qquad = f_{D*}(f^*D(F_N(\tilde{c}_1(f^*L_1), \ldots, \tilde{c}_1(f^*L_m))(1_Y^{D|D'}))) \\
&\qquad = f_{D*}(F_N(\tilde{c}_1(f^*L_1), \ldots, \tilde{c}_1(f^*L_m))(f^*D(1_Y^{D|D'})) \\
&\qquad = f_{D*}(F_{N+m}(\tilde{c}_1(f^*L_1), \ldots, \tilde{c}_1(f^*L_m))(f^*D(1_Y^{D|D'})) \\
&\qquad = D(F_{N+m}(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_m))([f]))_{D'}.
\end{aligned}$$

Thus, for $\eta \in \mathcal{Z}_*(X)_{D|D'}$, we may set

$$D(F(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_m))(\eta))_{D'} := \lim_{N\to\infty} D(F_N(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_m))(\eta))_{D'},$$

as the limit is eventually constant.

With this definition, we may extend lemma 6.2.2 to power series in the Chern class operators:

Lemma 6.2.3. *Let η be in $\mathcal{Z}_*(X)_{D|D'}$, let $f^D : |f^*D| \to |D|$ be the restriction of f and let $i : |f^*D| \to Y$ be the inclusion. Let $F(u_1, \ldots, u_r)$ be a power series with $\mathbb{L}_*$-coefficients, and let $L_1, \ldots, L_r$ be line bundles on Y. Then*

$$D(f_*([F(L_1, \ldots, L_r)]_{D|D'})) = f_*^D(F(\tilde{c}_1(i^*L_1), \ldots, \tilde{c}_1(i^*L_r))(f^*D)(1_Y^{D|D'})).$$

The next result requires a bit more work.

Lemma 6.2.4. *Let $f : W \to X$ be in $\mathcal{M}(X)_{D|D'}$ and let $Y \to W$ be a smooth codimension one closed subscheme of W. Suppose that Y is in general position with respect to $D|D'$. Suppose further that W is irreducible and $f(W) \not\subset |D|$. Let $f^D : |f^*D| \to |D|$ be the morphism induced by f and let $i : |f^*D| \to W$ be the inclusion. Then*

$$D([Y \to X])_{D'} = f_*^D(\tilde{c}_1(i^*O_W(Y))([\mathrm{div} f^*D \to |f^*D|]_{D'})).$$

Proof. Write $\tilde{D}$ for $\mathrm{div} f^*D$. Let $i_Y : Y \to W$ denote the inclusion. Since Y is in general position with respect to $D|D'$, $Y \to X$ is in $\mathcal{M}(X)_{D|D'}$, $i_Y^*(\tilde{D})$ is a strict normal crossing divisor on Y and is in good position with respect to D'. Furthermore, $D([Y \to X])_{D'}$ is by definition $[i_Y^*(\tilde{D}) \to |D|]_{D'}$.

Write $\tilde{D} = \sum_{i=1}^m n_i \tilde{D}_i$, with each $\tilde{D}_i$ irreducible. We may write $[\tilde{D} \to |\tilde{D}|]_{D'}$ as a sum over the faces $\tilde{D}^J$ of $\tilde{D}$,

$$[\tilde{D} \to |\tilde{D}|]_{D'} = \sum_J \iota_*^J([F_J^{n_1,\dots,n_m}(L_1^J, \dots, L_m^J)]_{D'}),$$

where $\iota^J : \tilde{D}^J \to |\tilde{D}|$ is the inclusion, $L_i = O_W(\tilde{D}_i)$ and L_i^J is the restriction of L_i to $\tilde{D}^J$.

Since Y is in general position with respect to D, it follows that the intersection $Y^J := Y \cap \tilde{D}^J$ is transverse; since Y is in very good position with respect to $D|D'$, the smooth codimension one subscheme $\iota^{YJ} : Y^J \to |\tilde{D}|$ of $\tilde{D}^J$ is in very good position with respect to D', for each index J. Thus, the relations in $\langle \mathcal{R}_*^{Sect} \rangle(|\tilde{D}|)_{D'}$ imply that

$$\tilde{c}_1(i^* O_W(Y))([\tilde{D} \to |\tilde{D}|]_{D'}) = \sum_J \iota_*^{YJ}([F_J^{n_1,\dots,n_m}(L_1^{YJ}, \dots, L_m^{YJ})]_{D'}),$$

where $\iota^{YJ} : Y^J \to |\tilde{D}|$ is the inclusion, and L_i^{YJ} is the restriction of L_i to Y^J. Letting $\bar{i} : |\tilde{D}| \cap Y \to |\tilde{D}|$ be the inclusion, the right-hand side above is clearly the same as the class $\bar{i}_*([i_Y^*(\tilde{D}) \to |\tilde{D}| \cap Y]_{D'})$. Pushing this identity forward via f^D gives

$$\begin{aligned} D([Y \to X])_{D'} &= [i_Y^*(\tilde{D}) \to |D|]_{D'} \\ &= f_*^D([i_Y^*(\tilde{D}) \to |\tilde{D}|]_{D'}) \\ &= f_*^D(\tilde{c}_1(i^* O_W(Y))([\tilde{D} \to |\tilde{D}|]_{D'})). \end{aligned}$$

□

6.2.2 Descent to $\Omega_*(X)_{D|D'}$

Let X be a finite type k-scheme with pseudo-divisors D and D'. We proceed to show that intersection with a pseudo-divisor D descends to a homomorphism $D(-)_{D'} : \Omega_*(X)_{D|D'} \to \Omega_{*-1}(|D|)_{D'}$.

Lemma 6.2.5. *Let $f : Y \to X$ a projective morphism in $\mathcal{M}(X)_{D|D'}$, and let $L_1, \dots, L_r$ be line bundles on Y with $r \geq \dim_k Y$. Then*

$$D((Y \to X, L_1, \dots, L_r))_{D'} = 0$$

in $\Omega_{*-1}(|D|)_{D'}$.

Proof. We may suppose Y to be irreducible. Write $\tilde{D}$ for f^*D, and let $f^D : |\mathrm{div}\tilde{D}| \to |D|$ be the restriction of f. Using lemmas 6.2.1 and 6.2.2, we have

$$D((f : Y \to X, L_1, \ldots, L_r))_{D'} = f_*^D(\tilde{c}_1(L_1) \circ \ldots \circ \tilde{c}_1(L_r)(\tilde{D}(1_Y^{D|D'}))).$$

If $f(Y) \subset |D|$, then $\tilde{D}(1_Y^{D|D'}) = \tilde{c}_1(O_Y(\tilde{D}))(1_Y^{D|D'})$, and thus

$$\tilde{c}_1(L_1) \circ \ldots \circ \tilde{c}_1(L_r)(\tilde{D}(1_Y^{D|D'})) = (\mathrm{Id}_Y, L_1, \ldots, L_r, O_Y(\tilde{D})) = 0$$

in $\underline{\mathcal{Z}}_*(Y)_{D'}$. If $f(Y) \not\subset |D|$, then $\tilde{D}(1_Y^{D|D'})$ is a sum of terms of the form

$$a \cdot \iota_*^J((\tilde{D}^J, M_1, \ldots, M_s)),$$

with $a \in \Omega_*(k)$, $\iota^J : \tilde{D}^J \to Y$ the inclusion of a face of $\tilde{D}^J$, and the M_i line bundles on $\tilde{D}^J$. Thus $\tilde{c}_1(L_1) \circ \ldots \circ \tilde{c}_1(L_r)(\tilde{D}(1_Y^{D|D'}))$ is a sum of terms of the form

$$a \cdot \iota_*^J((\tilde{D}^J, M_1, \ldots, M_s, \iota^{J*}L_1, \ldots, \iota^{J*}L_r))$$

Since each face $\tilde{D}^J$ has dimension $< \dim_k Y$, the terms

$$(\tilde{D}^J, M_1, \ldots, M_s, \iota^{J*}L_1, \ldots, \iota^{J*}L_r)$$

vanish in $\Omega_*(\tilde{D}^J)_{D'}$ (use the relations $\langle \mathcal{R}_*^{Dim} \rangle (\tilde{D}^J)_{D'}$), whence the result. □

Lemma 6.2.6. *Take X in $\mathbf{Sm}_k$, D' a pseudo-divisor on X. Let Z_1, Z_2 be smooth disjoint divisors on X. in good position with respect to D'.*

1. *Then*
$$\tilde{c}_1(O_X(Z_1)) \circ \tilde{c}_1(O_X(Z_2))(1_X^{D'}) = 0$$
in $\Omega_*(X)_{D'}$.
2. *Let D be a strict normal crossing divisor on X, in good position with respect to D', $i : |D| \to X$ the inclusion. Suppose that Z_1 and Z_2 are both in very good position with respect to D and in good position with respect to $D|D'$. Then*
$$\tilde{c}_1(i^*O_X(Z_1)) \circ \tilde{c}_1(i^*O_X(Z_2))([D \to |D|]_{D'}) = 0$$
in $\Omega_*(|D|)_{D'}$.

Proof. For each J, write D^J as a disjoint union of irreducible components,

$$D^J = \amalg_j D_j^J$$

and let $\iota_j^J : D_j^J \to |D|$, $i_j^J : D_j^J \to X$ be the inclusions. Write $D = \sum_{i=1}^r n_i D_i$, and let $\eta_{J,j} = \iota_{j*}^J(\tilde{c}_1(i_j^{J*}O_X(Z_1)) \circ \tilde{c}_1(i_j^{J*}O_X(Z_2))(1_{D_j^J}^{D'}))$. Then

$$\tilde{c}_1(i^*O_X(Z_1)) \circ \tilde{c}_1(i^*O_X(Z_2))([D \to |D|]_{D'})$$
$$= \sum_{J,j} F_J^{n_1,\ldots,n_r}(\tilde{c}_1(i^*O_X(D_1)), \ldots, \tilde{c}_1(i^*O_X(D_r)))(\eta_{J,j}).$$

If D_j^J is not contained in $Z_1 \cup Z_2$, then $Z_1 \cap D_j^J$ and $Z_2 \cap D_j^J$ are smooth disjoint divisors on D_j^J, in good position with respect to D'. Thus (2) follows from (1) in this case. If D_j^J is contained in say Z_1, then $D_j^J \cap Z_2 = \emptyset$, and $\tilde{c}_1(i_j^{J*}O_X(Z_2))(1_{D_j^J}^{D'})) = 0$, using the relations $\langle \mathcal{R}_*^{Sect} \rangle (D^J)_{D|D'}$ in the case of an empty divisor, so (2) follows in this case as well.

For (1), let $i_j : Z_j \to X$ be the inclusion. We have

$$\begin{aligned}\tilde{c}_1(O_X(Z_1)) \circ \tilde{c}_1(O_X(Z_2))(1_X^{D'}) &= \tilde{c}_1(i_2^* O_X(Z_1))(1_{Z_2}^{D'}) \\ &= \tilde{c}_1(O_{Z_2}))(1_{Z_2}^{D'}) \\ &= 0\end{aligned}$$

using $\langle \mathcal{R}_*^{Sect} \rangle (Z_2)_{D'}$, again in the case of an empty divisor. □

Finally, we need a strengthening of lemma 6.2.4.

Lemma 6.2.7. *Let W be in $\mathbf{Sm}_k$ and irreducible. Let $i_Y : Y \to W$ be an irreducible codimension one closed subscheme, smooth over k. Let D, D' be pseudo-divisors on W such that $W \neq |D|$ and let $i_D : |D| \to W$ be the inclusion. Suppose that $Y + D$ is a strict normal crossing divisor on W, in very good position with respect to D'. Then*

$$D([Y \to W])_{D'} = \tilde{c}_1(i_D^* O_W(Y))([D \to |D|]_{D'}) \tag{6.3}$$

in $\Omega_*(|D|)_{D'}$.

Proof. The condition that $Y + D$ is a strict normal crossing divisor, in very good position with respect to D' implies that $i_Y : Y \to W$ is in $\mathcal{M}(W)_{D|D'}$, and that D is in good position with respect to D'. Thus, all the terms in (6.3) are defined.

In case Y is not a component of D, Y is in general position with respect to $D|D'$, so the result follows from lemma 6.2.4.

In what follows, we use the following notation to keep track of where the various cycle classes are located: Let $L_1, \ldots, L_r$ be line bundles on some $T \in \mathbf{Sm}_k$, and $F(u_1, \ldots, u_r) \in \Omega_*(k)[[u_1, \ldots, u_r]]$ a power series. We write $[T; F(L_1, \ldots, L_r)]$ for $[F(L_1, \ldots, L_r)] \in \Omega_*(T)$.

Now suppose that Y is a component of D. Write $D = \sum_{i=1}^m n_i D_i$, with $Y = D_1$. Let $\iota_Y : Y \to |D|$, $\eta^J : D^J \cap Y \to Y$, $\tau^J : D^J \cap Y \to D^J$, $i_Y^J : D^J \cap Y \to W$, $\iota^J : D^J \to |D|$ and $i^J : D^J \to W$ be the inclusions. Since

$$n_1 \cdot_F u_1 +_F + \ldots +_F n_m \cdot_F u_m = F^{n_1,\ldots,n_m}(u_1, \ldots, u_m)$$
$$= \sum_J u^J F_J^{n_1,\ldots,n_m}(u_1, \ldots, u_m),$$

we have

$$\tilde{c}_1(O_W(D)) = \sum_J \tilde{c}_1(O_W(D_*))^J F_J(\tilde{c}_1(O_W(D_1)), \ldots, \tilde{c}_1(O_W(D_m))),$$

where $F_J := F_J^{n_1,\ldots,n_m}$. Also, if $J = (j_1, \ldots, j_m)$, then

$$\tilde{c}_1(O_W(D_*))^J = \tilde{c}_1(O_W(D_1))^{j_1} \circ \ldots \circ \tilde{c}_1(O_W(D_m))^{j_m}.$$

Since we therefore have

$$\begin{aligned} D([Y \to W])_{D'} &= \tilde{c}_1(i_D^* O_W(D))([Y \to |D|]) \\ &= \sum_J \iota_{Y*}\big(\tilde{c}_1(i_Y^* O_W(D_*))^J([Y; F_J(i_Y^* O_W(D_1), \ldots, i_Y^* O_W(D_m))]_{D'})\big); \\ \tilde{c}_1(i_D^* O_W(Y))&([D \to |D|]_{D'}) \\ &= \sum_J \iota_*^J \big(\tilde{c}_1(i^{J*} O_W(Y))([D^J; F_J(i^{J*} O_W(D_1), \ldots, i^{J*} O_W(D_m))]_{D'})\big), \end{aligned}$$

it suffices to prove that

$$\begin{aligned} &\iota_*^J\big(\tilde{c}_1(i^{J*} O_W(Y))([D^J; F_J(i^{J*} O_W(D_1), \ldots, i^{J*} O_W(D_m))]_{D'})\big) \\ &\quad = \iota_{Y*}\big(\tilde{c}_1(i_Y^* O_W(D_*))^J([Y; F_J(i_Y^* O_W(D_1), \ldots, i_Y^* O_W(D_m))]_{D'})\big) \end{aligned} \tag{6.4}$$

in $\Omega_*(|D|)_{D'}$, for each index J.

Suppose that D^J is not contained in Y. Since $Y + D$ is a strict normal crossing divisor, the intersection $Y \cap D^J$ is transverse. By symmetry, we may assume that $J = (1, \ldots, 1, 0, \ldots, 0)$, with say s 1's. Applying the relations $\langle \mathcal{R}_*^{Sect} \rangle_{D'}$ repeatedly, we see that

$$\begin{aligned} &\tilde{c}_1(i_Y^* O_W(D_*))^J\big([Y; F_J(i_Y^* O_W(D_1), \ldots, i_Y^* O_W(D_m))]_{D'}\big) \\ &\qquad\qquad = \eta_*^J[Y \cap D^J; F_J(i_Y^{J*} O_W(D_1), \ldots, i_Y^{J*} O_W(D_m))]_{D'}. \end{aligned}$$

Applying the same relations to the divisor $Y \cap D^J$ on D^J, we have

$$\begin{aligned} &\tau_*^J[Y \cap D^J; F_J(i_Y^{J*} O_W(D_1), \ldots, i_Y^{J*} O_W(D_m))]_{D'} \\ &\qquad\qquad = \tilde{c}_1(i^{J*} O_W(Y))\big([Y; F_J(i^{J*} O_W(D_1), \ldots, i^{J*} O_W(D_m))]_{D'}\big). \end{aligned}$$

Since $\iota_*^J \circ \tau_*^J = \iota_{Y*} \circ \eta_*^J$, these two identities yield the equality (6.4) in this case.

In case D^J is contained in $Y = D_1$, then $J = (1, j_2, \ldots, j_m)$. Letting $J' = (0, j_2, \ldots, j_m)$, we have $D^J = Y \cap D^{J'}$, and $D^{J'}$ is not contained in Y. Suppose that $J' \neq (0, \ldots, 0)$. By the same argument as above, we have

$$\begin{aligned} &\iota_*^{J'}\big(\tilde{c}_1(i^{J'*} O_W(Y))([D^{J'}; F_J(i^{J'*} O_W(D_1), \ldots, i^{J'*} O_W(D_m))]_{D'})\big) \\ &\qquad = \iota_{Y*}\big(\tilde{c}_1(i_Y^* O_W(D_*))^{J'}([Y; F_J(i_Y^* O_W(D_1), \ldots, i_Y^* O_W(D_m))]_{D'})\big) \end{aligned}$$

in $\Omega_*(|D|)_{D'}$. Letting $\bar{i}: D^J \to D^{J'}$ be the inclusion, the relations $\langle \mathcal{R}_*^{Sect} \rangle_{D'}$ yield the identity

$$\begin{aligned}\tilde{c}_1(i^{J'*}O_W(Y))([D^{J'}; F_J(i^{J'*}O_W(D_1), \ldots, i^{J'*}O_W(D_m))]_{D'}) \\ = \bar{i}_*[D^J; F_J(i^{J*}O_W(D_1), \ldots, i^{J*}O_W(D_m))]_{D'}\end{aligned}$$

in $\Omega_*(D^{J'})_{D'}$. Thus,

$$\begin{aligned}&\iota_*^J\big(\tilde{c}_1(i^{J*}O_W(Y))([D^J; F_J(i^{J*}O_W(D_1), \ldots)]_{D'})\big) \\ &\quad = \iota_*^{J'}\big(\tilde{c}_1(i^{J'*}O_W(Y))^2([D^{J'}; F_J(i^{J'*}O_W(D_1), \ldots)]_{D'})\big) \\ &\quad = \tilde{c}_1(i_D^*O_W(Y))\big(\iota_*^{J'}\big(\tilde{c}_1(i^{J'*}O_W(Y))([D^{J'}; F_J(i^{J'*}O_W(D_1), \ldots)]_{D'})\big)\big) \\ &\quad = \tilde{c}_1(i_D^*O_W(Y))\big(\iota_{Y*}(\tilde{c}_1(i_Y^*O_W(D_*))^{J'}([Y; F_J(i_Y^*O_W(D_1), \ldots)]_{D'}))\big) \\ &\quad = \iota_{Y*}\big((\tilde{c}_1(i_Y^*O_W(D_*))^J([Y; F_J(i_Y^*O_W(D_1), \ldots)]_{D'})\big),\end{aligned}$$

verifying (6.4). If $J' = (0, \ldots, 0)$, then $J = (1, 0, \ldots, 0)$, $D^J = D_1 = Y$ and $H_J(u_1, \ldots, u_m) = n_1$. Thus in $\Omega_*(Y)_{D'}$, we have

$$\begin{aligned}&[Y; F_J(i_Y^*O_W(D_1), \ldots, i_Y^*O_W(D_1))]_{D'} \\ &\qquad = n_1 \cdot 1_Y^{D'} \\ &\qquad = [D^J; F_J(i^{J*}O_W(D_1), \ldots, i^{J*}O_W(D_m))]_{D'}.\end{aligned}$$

Similarly, $\tilde{c}_1(i^{J*}O_W(D_*))^J = \tilde{c}_1(i_Y^*O_W(Y))$, which yields (6.4). This finishes the proof. □

We now show that $D(-)_{D'}$ descends to $\Omega_*(X)_{D|D'}$ in a series of steps.

Step 1: The descent to $\underline{\mathcal{Z}}_*(X)_{D|D'}$. Let $\pi: Y \to Z$ be a smooth morphism with Z and Y in $\mathbf{Sm}_k$, $L_1, \ldots, L_r$ line bundles on Z with $r > \dim_k Z$, and $f: Y \to X$ a projective morphism in $\mathcal{M}(X)_{D|D'}$, with Y irreducible. Using lemmas 6.2.1 and 6.2.2, it suffices to show that $(f^*D)(\mathrm{Id}_Y, \pi^*L_1, \ldots, \pi^*L_r)_{D'} = 0$ in $\Omega_*(|f^*D|)_{D'}$. Changing notation, we may assume that $X = Y$, and that either D is a strict normal crossing divisor on Y, or $|D| = Y$. We need to show that $D(\mathrm{Id}_Y, \pi^*L_1, \ldots, \pi^*L_r)_{D'} = 0$ in $\Omega_*(|D|)_{D'}$.

If $|D| = Y$, then

$$\begin{aligned}D(\mathrm{Id}_Y, \pi^*L_1, \ldots, \pi^*L_r)_{D'} &= \tilde{c}_1(O_Y(D))(\mathrm{Id}_Y, \pi^*L_1, \ldots, \pi^*L_r) \\ &= (\mathrm{Id}_Y, \pi^*L_1, \ldots, \pi^*L_r, O_Y(D)),\end{aligned}$$

which is zero in $\Omega_*(|D|)_{D'} = \Omega_*(Y)_{D'}$ by the relations $\langle \mathcal{R}_*^{Dim} \rangle (Y)_{D'}$.

If D is a strict normal crossing divisor on Y, with inclusion $i: |D| \to Y$, then

$$D(\mathrm{Id}_Y, \pi^*L_1, \ldots, \pi^*L_r)_{D'} = \tilde{c}_1((\pi \circ i)^*L_1) \circ \ldots \circ \tilde{c}_1((\pi \circ i)^*L_r)([D \to |D|]_{D'}).$$

To see that this class vanishes, apply lemma 6.1.13 to $\pi \circ i : |D| \to Z$.

Step 2: The descent to $\underline{\Omega}_*(X)_{D|D'}$. Let $f : Y \to X$ be in $\mathcal{M}(X)_{D|D'}$, and let $Z \to Y$ be a codimension one smooth closed subscheme in very good position with respect to $D|D'$. Let $i : |D| \to Y$ be the inclusion. We may suppose that Y is irreducible. As in step 1, we reduce to the case $X = Y$. Also, it suffices to show that

$$D([O_Y(Z)]_{D|D'})_{D'} = D([Z \to Y])_{D'}.$$

Write the divisor Z as a sum $Z = \sum_{i=1}^r Z_i$. Since Z is smooth, we have $Z_i \cap Z_j = \emptyset$ for $i \neq j$. Let $Z' = \sum_{i=2}^r Z_i$. Using lemma 6.2.2 and the relations $\langle \mathcal{R}_*^{FGL} \rangle (|D|)_{D'}$, we have

$$\begin{aligned} D([O_Y(Z)]_{D|D'})_{D'} &= D(\tilde{c}_1(O_Y(Z))(1_Y^{D|D'}))_{D'} \\ &= \tilde{c}_1(i^* O_Y(Z))(D(1_Y^{D|D'})_{D'}) \\ &= F_{\mathbb{L}}(\tilde{c}_1(O_Y(Z_1)), \tilde{c}_1(O_Y(Z')))(D(1_Y^{D|D'})_{D'}). \end{aligned}$$

Suppose that $Y \neq |D|$, so $\operatorname{div} D$ is a strict normal crossing divisor in good position with respect to D'. Since $F_{\mathbb{L}}(u,v) = u + v + \sum_{i,j \geq 1} a_{ij} u^i v^j$, it follows from lemma 6.2.6 that

$$\begin{aligned} &F_{\mathbb{L}}(\tilde{c}_1(i^* O_Y(Z_1)), \tilde{c}_1(i^* O_Y(Z')))(D(1_Y^{D|D'})_{D'}) \\ &\qquad = \tilde{c}_1(i^* O_Y(Z_1))(D(1_Y^{D|D'})_{D'}) + \tilde{c}_1(i^* O_Y(Z'))(D(1_Y^{D|D'})_{D'}) \\ &\qquad = D([O_Y(Z_1)]_{D|D'})_{D'} + D([O_Y(Z')]_{D|D'})_{D'}. \end{aligned}$$

Thus, by induction, we have

$$D([O_Y(Z)]_{D|D'})_{D'} = \sum_{i=1}^r D([O_Y(Z_i)]_{D|D'})_{D'}.$$

If $Y = |D|$, then $D(1_Y^{D|D'})_{D'} = \tilde{c}_1(O_Y(D))(1_Y^{D'})$, and

$$\tilde{c}_1(O_Y(Z_i)) \circ \tilde{c}_1(O_Y(Z_j)(1_Y^{D'}) = 0$$

by lemma 6.2.6(1). Since $[Z \to Y]_{D'} = \sum_i [Z_i \to Y]_{D'}$, we reduce to the case of an irreducible Z.

If $Z \not\subset |D|$, then Z is in general position with respect to D. Using lemma 6.2.2 and lemma 6.2.4, we have

$$\begin{aligned} D([O_Y(Z)]_{D|D'})_{D'} &= \tilde{c}_1(i^* O_Y(Z))(D(1_Y^{D|D'})) \\ &= \tilde{c}_1(i^* O_Y(Z))([D \to |D|]_{D'}) \\ &= D([Z \to Y])_{D'}, \end{aligned}$$

as desired. If $Z \subset |D|$, then the same argument, using lemma 6.2.7 in place of lemma 6.2.4, yields the desired identity.

Step 3: The descent to $\Omega_*(X)_{D|D'}$. Let $f : Y \to X$ be in $\mathcal{M}(X)_{D|D'}$ and let L and M be line bundles on Y. It suffices to show that

$$D([F_{\mathbb{L}}(L, M)])_{D'} = D([L \otimes M])_{D'}.$$

For this, let $\tilde{D} = f^*D$, let $f^D : |\tilde{D}| \to |D|$ be the morphism induced by f and let $i : |\tilde{D}| \to Y$ be the inclusion. Using the relations $\langle \mathcal{R}_*^{FGL} \rangle(|f^*D|)_{D'}$ and lemma 6.2.3, we have

$$\begin{aligned} D([F_{\mathbb{L}}(L, M)])_{D'} &= D(F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))(1_Y^{D|D'}))_{D'} \\ &= f_*^D(F_{\mathbb{L}}(\tilde{c}_1(i^*L), \tilde{c}_1(i^*M))(\tilde{D}(1_Y^{D|D'})) \\ &= f_*^D(\tilde{c}_1(i^*L \otimes i^*M)(\tilde{D}(1_Y^{D|D'})) \\ &= D([L \otimes M])_{D'}, \end{aligned}$$

which finishes the descent to $\Omega_*(X)_{D|D'}$.

6.3 Intersection with a pseudo-divisor II

We establish two basic properties of the operation $D(-)_{D'}$.

6.3.1 Commutativity

The first important property is the commutativity of intersection. We begin with some preliminary results.

Lemma 6.3.1. *Let D and D' be pseudo-divisors on some finite type k-scheme X, and let $i : |D| \to X$ be the inclusion. Then, for $\eta \in \Omega_*(X)_{D|D'}$, $i_*(D(\eta)_{D'}) = \tilde{c}_1(O_X(D))(\eta)$ in $\Omega_*(X)_{D'}$.*

Proof. It suffices to consider the case of $\eta = [f : Y \to X]$ for some $f \in \mathcal{M}(X)_{D|D'}$, with Y irreducible. If $f(Y) \subset |D|$, then

$$D(f) = \tilde{c}_1(i^*O_X(D))([Y \to |D|]_{D'}),$$

from which the desired formula follows directly. If $f(Y) \not\subset |D|$, then $D(f) = f_*^D([f^*D \to |f^*D|]_{D'})$, where $f^D : |f^*D| \to |D|$ is the map induced by f. By remark 6.1.11, we have $[f^*D \to Y]_{D'} = \tilde{c}_1(O_Y(f^*D))(1_Y^{D'})$. Thus

$$\begin{aligned} i_*(D(f)) &= f_*([f^*D \to Y]_{D'}) \\ &= f_*(\tilde{c}_1(f^*O_X(D))(1_Y^{D'})) \\ &= \tilde{c}_1(O_X(D))(f). \end{aligned}$$

□

Write $F_{\mathbb{L}}(u,v) = u + v + uvF_{11}(u,v)$. Let $G_{11}(u,v) = vF_{11}(u,v)$.

Lemma 6.3.2. *Let D, D' be pseudo-divisors on $W \in \mathbf{Sm}_k$. Suppose that D is a strict normal crossing divisor, in good position with respect to D'. Write $D = D_0 + D_1$, with $D_0 > 0$, $D_1 > 0$ and D_1 smooth. Let $i : |D| \to W$ be the inclusion.*

1. *We have the identity*

$$\begin{aligned}[D \to |D|]_{D'} = [D_0 \to |D|]_{D'} &+ [D_1 \to |D|]_{D'} \\ &+ G_{11}(\tilde{c}_1(i^*O_W(D_1)), \tilde{c}_1(i^*O_W(D_0)))([D_1 \to |D|]_{D'}).\end{aligned}$$

2. *Let $f : Y \to W$ be in $\mathcal{M}(W)_{D|D'} \cap \mathcal{M}(W)_{D_0|D'} \cap \mathcal{M}(W)_{D_1|D'}$. Let $i_0^Y : |f^*D_0| \to |f^*D|$, $i_1^Y : |f^*D_1| \to |f^*D|$ be the inclusions, and let $\bar{f} : |f^*D_1| \to W$ be the induced morphism. Suppose that Y is irreducible and either $f(Y) \subset |D_1|$ or f^*D_1 is a smooth divisor on Y. Then*

$$\begin{aligned}(f^*D)(1_Y^{D|D'})_{D'} = i^Y_{0*}((f^*D_0)(1_Y^{D_0|D'})_{D'}) + i^Y_{1*}((f^*D_1)(1_Y^{D_1|D'})_{D'}) +& \\ i^Y_{1*}\big(G_{11}(\tilde{c}_1(\bar{f}^*O_W(D_1)), \tilde{c}_1(\bar{f}^*O_W(D_0)))((f^*D_1)(1_Y^{D_1|D'})_{D'})\big)&\end{aligned}$$

Proof. For the second assertion, suppose first that $f(Y) \subset |D|$. Let $i : |D| \to W$ be the inclusion. Then by definition

$$(f^*D)(1_Y^{D|D'})_{D'} = \tilde{c}_1(O_Y(f^*D))(1_Y^{D'}).$$

Since $D = D_0 + D_1$, we have

$$\begin{aligned}\tilde{c}_1(O_Y(f^*D)) &= F_{\mathbb{L}}(\tilde{c}_1(O_Y(f^*D_1)), \tilde{c}_1(O_Y(f^*D_0)) \\ &= \tilde{c}_1(O_Y(f^*D_1)) + \tilde{c}_1(O_Y(f^*D_0)) \\ &\quad + \tilde{c}_1(O_Y(f^*D_1))G_{11}(\tilde{c}_1(O_Y(f^*D_1)), \tilde{c}_1(O_Y(f^*D_0)).\end{aligned}$$

Let $i_j^Y : |f^*D_j| \to Y$ be the inclusion, $j = 0, 1$. Then for $j = 0, 1$, it follows from lemma 6.3.1 that $i^Y_{j*}((f^*D_j)(1_Y^{D|D'})_{D'}) = \tilde{c}_1(O_Y(f^*D_j))(1_Y^{D'})$. Thus

$$\begin{aligned}(f^*D)(1_Y^{D|D'})_{D'} = i^Y_{0*}((f^*D_0)(1_Y^{D|D'})_{D'}) &+ i^Y_{1*}((f^*D_1)(1_Y^{D|D'})_{D'}) \\ &+ \tilde{c}_1(O_Y(f^*D_1)) \circ G_{11}(\tilde{c}_1(f^*O_W(D_1)), \tilde{c}_1(f^*O_W(D_0))(1_Y^{D'}).\end{aligned}$$

Using lemma 6.3.1 again, we have

$$\begin{aligned}&\tilde{c}_1(f^*O_W(D_1)) \circ G_{11}(\tilde{c}_1(f^*O_W(D_1)), \tilde{c}_1(f^*O_W(D_0))(1_Y^{D'}) \\ &= G_{11}(\tilde{c}_1(f^*O_W(D_1)), \tilde{c}_1(f^*O_W(D_0)))\big(\tilde{c}_1(O_Y(f^*D_1))(1_Y^{D'})\big) \\ &= i^Y_{1*}\big(G_{11}(\tilde{c}_1(\bar{f}^*O_W(D_1)), \tilde{c}_1(\bar{f}^*O_W(D_0)))((f^*D_1)(1_Y^{D|D'})_{D'})\big).\end{aligned}$$

This verifies the second assertion in this case.

If $f(Y) \not\subset |D|$, the second assertion is a consequence of the first. Indeed, in this case, f^*D is a strict normal crossing divisor on Y, in good position with respect to D', and thus $(f^*D)(1_Y^{D|D'})$ is $[f^*D \to |f^*D|]_{D'}$. Thus, applying the first assertion to $f^*D = f^*D_0 + f^*D_1$, we have

$$\begin{aligned}
&(f^*D)(1_Y^{D|D'}) = [f^*D \to |f^*D|]_{D'}\\
&\quad = i^Y_{0*}[f^*D_0 \to |f^*D_0|]_{D'} + i^Y_{1*}[f^*D_1 \to |f^*D_1|]_{D'}\\
&\qquad + i^Y_{1*}\big(G_{11}(\tilde{c}_1^{D'}(i_1^*O_Y(f^*D_1)), \tilde{c}_1(i_1^*O_Y(f^*D_0)))([f^*D_1 \to |f^*D_1|]_{D'})\big)\\
&\quad = i^Y_{0*}(D_0(1_Y^{D_0|D'})_{D'}) + i^Y_{1*}(D_1(1_Y^{D_1|D'})_{D'})\\
&\qquad + i^Y_{1*}\big(G_{11}(\tilde{c}_1(\bar{f}^*O_W(D_1)), \tilde{c}_1(\bar{f}^*O_W(D_0)))(D_1(1_Y^{D_1|D'})_{D'})\big).
\end{aligned}$$

To prove (1), we first reduce to the case of irreducible D_1. Write $D_1 = E_1 + D_{11}$, with $E_1 > 0$ and D_{11} irreducible. Since D_{11} and E_1 are disjoint, $\tilde{c}_1(i^*_{D_{11}}O_W(E_1))([D_{11} \to |D_{11}|]_{D'}) = 0$. Thus

$$\begin{aligned}
&G_{11}(\tilde{c}_1(O_W(D_{11})), \tilde{c}_1(O_W(D_0 + E_1))([D_{11} \to |D_{11}|]_{D'})\\
&\qquad = G_{11}(\tilde{c}_1(O_W(D_{11})), F_{\mathbb{L}}(\tilde{c}_1(O_W(D_0), \tilde{c}_1(E_1)))([D_{11} \to |D_{11}|]_{D'})\\
&\qquad = G_{11}(\tilde{c}_1(O_W(D_{11})), \tilde{c}_1(O_W(D_0))([D_{11} \to |D_{11}|]_{D'})
\end{aligned}$$

(we omit the pull-back of the line bundles here and for the remainder of the argument to simplify the notation). Similarly,

$$\begin{aligned}
&G_{11}(\tilde{c}_1(O_W(E_1 + D_{11})), \tilde{c}_1(O_W(D_0)))([D_1 \to |D|]_{D'})\\
&\quad = G_{11}(\tilde{c}_1(O_W(E_1)) + \tilde{c}_1(O_W(D_{11})), \tilde{c}_1(O_W(D_0)))([D_1 \to |D|]_{D'})\\
&\quad = G_{11}(\tilde{c}_1(O_W(E_1), \tilde{c}_1(O_W(D_0)))([E_1 \to |D|]_{D'})\\
&\qquad\qquad + G_{11}(\tilde{c}_1(O_W(D_{11}), \tilde{c}_1(O_W(D_0)))([D_{11} \to |D|]_{D'}),
\end{aligned}$$

using lemma 6.2.6. With these formulas, one easily shows that (1) for the decompositions $E = D_0 + E_1$, $D = E + D_{11}$ and $D_1 = E_1 + D_{11}$ implies (1) for $D = D_0 + D_1$.

We now assume D_1 irreducible. Write $D = \sum_{i=1}^m n_i D_i$, with the D_i irreducible. For each face D^J of D properly contained in D_1, let $i_1^J : D^J \to D_1$ be the inclusion.

Let F_n denote the n-fold sum in the formal group $(F_{\mathbb{L}}, \mathbb{L}_*)$. The identity $F_{\mathbb{L}}(u_1, F_{n-1}(u_2, \ldots, u_n)) = F_n(u_1, \ldots, u_n)$ gives us the identity

$$\sum_J u^J F_J^{n_1,\ldots,n_m}(u_1, \ldots, u_m) = u_1 + V + u_1 V F_{11}(u_1, V)$$

where

$$V = \begin{cases} F^{n_1-1,\ldots,n_m}(u_1, \ldots, u_m) & \text{if } n_1 > 1, \\ F^{n_2,\ldots,n_m}(u_2, \ldots, u_m) & \text{if } n_1 = 1. \end{cases}$$

Thus

$$V = \begin{cases} \sum_{J'} u^{J'} F_{J'}^{n_1-1,n_2,\ldots,n_m}(u_1, u_2, \ldots, u_m) & \text{if } n_1 > 1, \\ \sum_{J'} u^{J'} F_{J'}^{n_2,\ldots,n_m}(u_2, \ldots, u_m) & \text{if } n_1 = 1, \end{cases}$$

where $\sum_{J'}$ is over all faces of D_0.

We write $F^{0,n_2,\ldots,n_m}_{(0,j_2,\ldots,j_m)}(u_1, u_2, \ldots, u_m)$ for $F^{n_2,\ldots,n_m}_{(j_2,\ldots,j_m)}(u_2, \ldots, u_m)$ and set $F^{0,n_2,\ldots,n_m}_{(j_1,\ldots,j_m)} = 0$ if $j_1 \neq 0$.

Write $u_1 V F_{11}(u_1, V) = u_1 G_{11}(u_1, V)$ as the sum

$$u_1 V F_{11}(u_1, V) = \sum_K u^K F'_K(u_1, \ldots, u_m),$$

where the sum is over all indices $K = (k_1, \ldots, k_m)$ with $0 \leq k_i \leq 1$, $k_1 = 1$ (if $n_1 = 1$ $F'_K = 0$ unless $\sum_i k_i \geq 2$). Note that

$$G_{11}(u_1, V) = \sum_K u^{K-1} F'_K \tag{6.5}$$

where $K - 1 := (0, k_2, \ldots, k_n)$. For each index $K = (1, k_2, \ldots, k_n)$, we have

$$F_K^{n_1,\ldots,n_m}(u_1, \ldots, u_m) = F_K^{n_1-1,n_2,\ldots,n_m}(u_1, u_2, \ldots, u_m) + F'_K(u_1, \ldots, u_m).$$

Referring to the definition of $[D \to |D|]_{D'}$, and noting that

$$V(\tilde{c}_1(O(D_1)), \ldots, \tilde{c}_1(O(D_m))) = \tilde{c}_1(O(D_0)),$$

we thus need to show

$$\begin{aligned} &\sum_K{}' \iota_*^K F'_K(\tilde{c}_1(O(D_1)), \ldots, \tilde{c}_1(O(D_m)))(1_{D^K}^{D'}) \\ &\qquad = G_{11}(\tilde{c}_1(O(D_1)), \tilde{c}_1(O(D_0)))([D_1 \to |D|]_{D'}), \end{aligned} \tag{6.6}$$

where $\sum'_K$ means the sum over all $K = (1, k_2, \ldots, k_n)$ if $n_1 > 1$, and with the added restriction that $\sum_i k_i \geq 2$ if $n_1 = 1$.

For each $K = (1, k_2, \ldots, k_m)$, let

$$\tilde{c}_1(O(D_*))^{K-1} = \tilde{c}_1(O(D_2))^{k_2} \circ \cdots \circ \tilde{c}_1(O(D_m))^{k_m}.$$

By repeated applications of the relations $\langle \mathcal{R}_*^{Sect} \rangle_{D'}$, we have

$$\begin{aligned} &i_{1*}^K(F'_K(\tilde{c}_1(O(D_1)), \ldots, \tilde{c}_1(O(D_m)))(1_{D^K}^{D'})) \\ &\qquad = \left(F'_K(\tilde{c}_1(O(D_1)), \ldots, \tilde{c}_1(O(D_m))) \circ \tilde{c}_1(O(D_*))^{K-1}\right)(1_{D_1}^{D'}). \end{aligned}$$

Thus, as $\iota_*^K = i_{1*} \circ i_{1*}^K$, the relation (6.5) implies

$$\begin{aligned} &\sum_K{}' \iota_*^K \left(F'_K(\tilde{c}_1(O(D_1)), \ldots, \tilde{c}_1(O(D_m)))(1_{D^K}^{D'})\right) \\ &\quad = \sum_K{}' i_{1*}(\tilde{c}_1(O(D_*))^{K-1} F'_K(\tilde{c}_1(O(D_1)), \ldots, \tilde{c}_1(O(D_m)))(1_{D_1}^{D'})) \\ &\quad = i_{1*}(G_{11}(\tilde{c}_1(O(D_1)), V(\tilde{c}_1(O(D_1)), \ldots, \tilde{c}_1(O(D_m))))(1_{D_1}^{D'})) \\ &\quad = G_{11}(\tilde{c}_1(O(D_1)), \tilde{c}_1(O(D_0)))([D_1 \to |D|]_{D'}). \end{aligned}$$

This verifies the identity (6.6), completing the proof. □

Proposition 6.3.3 (Commutativity). *Let D'' be a pseudo-divisor on a finite type k-scheme X. Let $f : T \to X$ be in $\mathcal{M}(X)$, and let D, D' be pseudo-divisors on T. Suppose that*

1. *$D + D'$ is a strict normal crossing divisor on T.*
2. *D is in good position with respect to $D'|D''$.*
3. *D' is in good position with respect to $D|D''$.*

Let $i_D : |D| \to T$ and $i_{D'} : |D'| \to T$ be the inclusions. Then

$$(i_{D'}^* D)([D' \to |D'|]_{D|D''})_{D''} = (i_D^* D')([D \to |D|]_{D'|D''})_{D''}$$

in $\Omega_*(|D| \cap |D'|)_{D''}$.

Proof. Write $D = \sum_i n_i D_i$ with each D_i irreducible, and similarly $D' = \sum_j n'_j D'_j$. We show more generally that

$$(i_{E'}^* E)([E' \to |E'|]_{D|D''})_{D''} = (i_E^* E')([E \to |E|]_{D'|D''})_{D''}$$

in $\Omega_*(|E| \cap |E'|)$ for all divisors $0 \le E \le D$, $0 \le E' \le D'$; if $E = \sum_i m_i D_i$ and $E' = \sum_j m'_j D'_j$, we may proceed by induction on $m := \sum_i m_i$ and $m' := \sum_i m'_i$.

Suppose $m = m' = 1$; we may suppose $E = D_1$ and $E' = D'_1$. If $D_1 \neq D'_1$, then D_1 and D'_1 intersect transversely and $(i_{D_1}^* D'_1)([D_1 \to |D_1|]_{D'|D''})_{D''}$ is the element $1_{D_1 \cap D'_1}^{D''}$ in $\Omega_*(|D_1| \cap |D'_1|)_{D''}$, as is $(i_{D'_1}^* D_1)([D'_1 \to |D'_1|]_{D|D''})_{D''}$. If $D_1 = D'_1$, then $D_1([D'_1 \to |D'_1|]_{D|D''})_{D''}$ and $D'_1([D_1 \to |D_1|]_{D'|D''})_{D''}$ are obviously the same in $\Omega_*(|D_1| \cap |D'_1|)_{D''}$.

In the general case, we may assume that D_1 is a component of E. Let $E_0 = E - D_1$. By symmetry, it suffices to induct on m and assume the result for the pairs E_0, E' and D_1, E'. Thus

$$(i_{E'}^* E_0)([E' \to |E'|]_{D|f^*D''}) = (i_{E_0}^* E')([E_0 \to |E_0|]_{D'|D''})_{D''}$$

in $\Omega_*(|E_0| \cap |E'|)_{D''}$ and

$$(i_{E'}^* D_1)([E' \to |E'|]_{D|D''})_{D''} = (i_{D_1}^* E')([D_1 \to |D_1|]_{D'|D''})_{D''}$$

in $\Omega_*(|D_1| \cap |E'|)_{D''}$.

The divisor class $[E' \to |E'|]_{D|D''}$ is an $\Omega_*(k)$-linear combination of maps of the form $g : Y \to |E'|$, with $g(Y) \subset |D_1|$, or $g^*(D_1)$ smooth, so we may apply lemma 6.3.2(2) to give

$$\begin{aligned}(i_{E'}^* E)([E' \to |E'|]_{D|D''})_{D''} = \\ i_{0*}\big((i_{E'}^* E_0)([E' \to |E'|]_{D|D''})_{D''}\big) + i_{1*}\big((i_{E'}^* D_1)([E' \to |E'|]_{D|D''})_{D''}\big) \\ + i_{1*}\big(G_{11}(\tilde{c}_1(O_W(D_1)), \tilde{c}_1(O_W(E_0)))((i_{E'}^* D_1)([E' \to |E'|]_{D|D''})_{D''}\big),\end{aligned}$$

where $i_0 : |E_0| \cap |E'| \to |E| \cap |E'|$, $i_1 : |D_1| \cap |E'| \to |E| \cap |E'|$ are the inclusions. Using our induction hypothesis, together with lemma 6.2.3, we have the identity in $\Omega_*(|E| \cap |E'|)_{D''}$:

$$\begin{aligned}
&(i^*_{E'}E)([E' \to |E'|]_{D|D''})_{D''} \\
&\quad = (i^*_E E')([E_0 \to |E|]_{D'|D''})_{D''} + (i^*_E E')([D_1 \to |E|]_{D'|D''})_{D''} \\
&\qquad + G_{11}(\tilde{c}_1(O_W(D_1)), \tilde{c}_1(O_W(E_0)))((i^*_E E')([D_1 \to |E|]_{D'|D''})_{D''}) \\
&\quad = (i^*_E E')([E_0 \to |E|]_{D'|D''})_{D''} + (i^*_E E')([D_1 \to |E|]_{D'|D''})_{D''} \\
&\qquad + (i^*_E E')(G_{11}(\tilde{c}_1(O_W(D_1)), \tilde{c}_1(O_W(E_0)))([D_1 \to |E|]_{D'|D''}))_{D''} \\
&\quad = (i^*_E E')\big([E_0 \to |E|]_{D'|D''} + [D_1 \to |E|]_{D'|D''} \\
&\qquad + G_{11}(\tilde{c}_1(O_W(D_1)), \tilde{c}_1(O_W(E_0)))([D_1 \to |E|]_{D'|D''})\big)_{D''} \\
&\quad = (i^*_E E')([E \to |E|]_{D'|D''})_{D''},
\end{aligned}$$

the last identity following from lemma 6.3.2(1). □

6.3.2 Linear equivalent pseudo-divisors

We show how to relate the operations $D_0(-)_D$ and $D_1(-)_D$ for linearly equivalent pseudo-divisors D_0 and D_1.

Proposition 6.3.4. *Let W be in $\mathbf{Sm}_k$, with pseudo-divisors D_0, D_1 and D, such that $O_W(D_0) \cong O_W(D_1)$. Let $i_j : |D_j| \to W$ be the inclusion, $j = 0, 1$.*

1. *Let η be in $\mathcal{Z}_*(W)_{D_0|D} \cap \mathcal{Z}_*(W)_{D_1|D}$. Then*
$$i_{0*}(D_0(\eta)_D) = i_{1*}(D_1(\eta)_D)$$
in $\Omega_*(W)_D$.
2. *Let E be a strict normal crossing divisor on W, in good position with respect to $D_j|D$, $j = 0, 1$. Then*
$$i_{0*}(D_0([E \to |E|]_{D_0|D})_D) = i_{1*}(D_1([E \to |E|]_{D_1|D})_D)$$
in $\Omega_*(W)_D$.

Proof. Clearly (2) is a special case of (1). To prove (1), the operations $D_j(-)$ commute with the Chern class operators $\tilde{c}_1(L)$. Thus, it suffices to prove (1) for the case of $f : Y \to W$ in $\mathcal{M}(W)_{D_j|D}$, $j = 0, 1$. Using lemma 6.2.1, we reduce to the case $Y = W$, $f = \mathrm{Id}$. But by lemma 6.3.1

$$i_{0*}(D_0(1_W^{D_0|D})_D) = [O_W(D_0)]_D = [O_W(D_1)]_D = i_{1*}(D_1(1_W^{D_1|D})_D).$$

□

6.4 A moving lemma

The next step is to show that the canonical map $\Omega_*(X)_D \to \Omega_*(X)$ is an isomorphism.

6.4.1 Distinguished liftings

Given a finite type k-scheme X with a pseudo-divisor D, we give a method for lifting elements of $\mathcal{Z}_*(X)$ to $\Omega_*(X)_D$.

Lemma 6.4.1. *Let Y be in $\mathbf{Sm}_k$ and let $\tilde{D}$ be an effective divisor on Y. Then there is a projective birational morphism $\rho : W \to Y \times \mathbb{P}^1$, with $W \in \mathbf{Sm}_k$, such that*

1. *The fundamental locus of ρ is contained in $|\tilde{D}| \times 0$.*
2. *The proper transforms to W of $Y \times 0$ and $|\tilde{D}| \times \mathbb{P}^1$, denoted $\langle Y \times 0\rangle$ and $\langle|\tilde{D}| \times \mathbb{P}^1\rangle$ respectively, are disjoint.*
3. *$\langle Y \times 0\rangle$ is smooth. Letting E be the exceptional divisor of ρ, $\langle Y \times 0\rangle + E$ is a strict normal crossing divisor on W.*

(6.7)

Proof. We may assume that Y is irreducible. To construct such a ρ, first blow up $Y \times \mathbb{P}^1$ along the reduced subscheme $|\tilde{D}| \times 0$, forming the projective birational morphism

$$\rho_1 : W_1 \to Y \times \mathbb{P}^1.$$

Let $U = Y \times \mathbb{P}^1 \setminus |\tilde{D}| \times 0$. Since $|\tilde{D}| \times 0 = Y \times 0 \cap |\tilde{D}| \times \mathbb{P}^1$, the proper transforms of $Y \times 0$ and $|\tilde{D}| \times \mathbb{P}^1$ to W_1 are disjoint. By Hironaka [13, Main Theorem I*, pg.132], we may resolve the singularities of W_1 by a projective birational morphism $W_2 \to W_1$ which is an isomorphism over U. Let $\rho_2 : W_2 \to Y \times \mathbb{P}^1$ be the induced morphism, let E_2 be the exceptional divisor of ρ_2, and consider the Cartier divisor $D' := \rho_2^*(Y \times 0) + E_2$. Clearly $D' \cap \rho_2^{-1}(U)$ is a strict normal crossing divisor; by corollary A.3, there is a projective birational morphism $\mu : W \to W_2$, with $W \in \mathbf{Sm}_k$, such that μ is an isomorphism over $\rho_2^{-1}(U)$, and with $\mu^* D' + E'$ a strict normal crossing divisor on W, where E' is the exceptional divisor of μ. Letting $\rho : W \to Y \times \mathbb{P}^1$ be the induced morphism, it is clear that ρ has all the necessary properties. □

Let X be a finite type k-scheme, and D a pseudo-divisor on X. Let $f : Y \to X$ be in $\mathcal{M}(X)$, with Y irreducible. Suppose that $f(Y) \not\subset |D|$. Then $\tilde{D} := \mathrm{div} f^* D$ is an effective Cartier divisor on Y. Take a projective birational morphism $\rho : W \to Y \times \mathbb{P}^1$ satisfying the conditions (6.7). We claim that $\rho^*(Y \times 0)$ is in good position with respect to D. Indeed, E is supported in $|\rho^* p_1^* f^* D|$, and $\mathrm{div} \rho^* p_1^* f^* D$ is supported in $|E| \cup \langle|p_1^* f^* D|\rangle$. Since $\langle|p_1^* f^* D|\rangle$ is disjoint from $\langle Y \times 0\rangle$ and $\langle Y \times 0\rangle + E$ is a strict normal crossing divisor,

$\langle Y \times 0 \rangle + E$ is in good position with respect to D. Since $\rho^*(Y \times 0)$ has the same support as $\langle Y \times 0 \rangle + E$, $\rho^*(Y \times 0)$ is in good position with respect to D, as claimed.

Note that $\rho^*(Y \times 0)$ is linearly equivalent to $\rho^*(Y \times 1) \cong Y$, so $[f] = (p_1 \circ \rho)_*([\rho^*(Y \times 0) \to W])$ in $\Omega_*(X)$. Thus $(f \circ p_1 \circ \rho)_*([\rho^*(Y \times 0) \to W]_D)$ gives a lifting of f to an element of $\Omega_*(X)_D$.

Definition 6.4.2. *(1) Given an element $f : Y \to X$ of $\mathcal{M}(X)$ with Y irreducible, $f(Y) \not\subset |D|$, and a birational morphism $\rho : W \to Y \times \mathbb{P}^1$ satisfying the conditions* (6.7)*, we call the element*

$$(f \circ p_1 \circ \rho)_*([\rho^*(Y \times 0) \to W]_D)$$

of $\Omega_*(X)_D$ a distinguished lifting *of $f \in \mathcal{M}(X)$. If $f(Y) \subset |D|$, a distinguished lifting of f is just $[f] \in \Omega_*(X)_D$.*

(2) Let $\eta = (f : Y \to X, L_1, \dots, L_r)$ be a cobordism cycle on X, with Y irreducible. Suppose that $f(Y) \not\subset |D|$. Choose $\rho : W \to Y \times \mathbb{P}^1$ as in (1), and let $\tilde{L}_i = (p_1\rho)^* L_i$. We call the element

$$(f \circ p_1 \circ \rho)_*\big(\tilde{c}_1(\tilde{L}_1) \circ \dots \circ \tilde{c}_1(\tilde{L}_r)([\rho^*(Y \times 0) \to W]_D)\big)$$

of $\Omega_*(X)_D$ a distinguished lifting of η. If $f(Y) \subset |D|$, then $[\eta] \in \Omega_*(X)_D$ is a distinguished lifting of η. We extend this notion to arbitrary elements of $\mathcal{Z}_*(X)$ by linearity.

Remark 6.4.3. The comment immediately preceding definition 6.4.2 justifies our terminology: if $\eta_D \in \Omega_*(X)_D$ is a distinguished lifting of $\eta \in \mathcal{Z}_*(X)$, then η_D and η both map to the same element in $\Omega_*(X)$.

Remark 6.4.4. Using the notation of definition 6.4.2(2), we can write a distinguished lifting of $(f : Y \to X, L_1, \dots, L_r)$ as

$$f_*\big(\tilde{c}_1(L_1) \circ \dots \circ \tilde{c}_1(L_r)((p_1 \circ \rho)_*([\rho^*(Y \times 0) \to W]_D))\big),$$

noting that $(p_1 \circ \rho)_*([\rho^*(Y \times 0) \to W]_D)$ is in $\Omega_*(Y)_{f^*D}$.

Remark 6.4.5. Let T be a smooth projective k-scheme, and let $f : Y \to X$ be in $\mathcal{M}(X)$. If $\mu : W \to Y \times \mathbb{P}^1$ satisfies the conditions (6.7) for f^*D on Y, then clearly $\mathrm{Id}_T \times \mu : T \times W \to T \times Y \times \mathbb{P}^1$ satisfies the conditions (6.7) for the divisor $(f \circ p_2)^*D$ on $T \times Y$. From this, it easily follows that, if $\tilde{\eta}$ is a distinguished lifting of some $\eta \in \mathcal{Z}_*(X)$, and α is in $\mathcal{Z}_*(k)$, then $\alpha\tilde{\eta}$ is a distinguished lifting of $\alpha\eta$.

Lemma 6.4.6. *Let η be in $\mathcal{Z}_*(X)$, and let η_1, η_2 be distinguished liftings of η. Then $\eta_1 = \eta_2$ in $\Omega_*(X)_D$.*

Proof. First of all, we may assume that η is a cobordism cycle $(f : Y \to X, L_1, \ldots, L_r)$, with Y irreducible. Next, it follows from the formula in remark 6.4.4 that, if τ is a distinguished lifting of $(f, L_1, \ldots, L_r)$, then there is a distinguished lifting $\tilde{\mathrm{Id}} \in \Omega_*(Y)_{f^*D}$ of the cobordism cycle $\mathrm{Id}_Y \in \Omega_*(Y)$ with

$$\tau = f_*(\tilde{c}_1(L_1) \circ \ldots \circ \tilde{c}_1(L_r)(\tilde{\mathrm{Id}}))$$

Thus, it suffices to consider the case of $X \in \mathbf{Sm}_k$, and to show that two distinguished liftings of $\mathrm{Id}_X \in \mathcal{M}(X)$ agree in $\Omega_*(X)_D$.

We may assume that X is irreducible. If $|D| = X$, then $\mathcal{Z}_*(X) = \mathcal{Z}_*(X)_D$, $\Omega_*(X) = \Omega_*(X)_D$, and the unique distinguished lifting of Id_X is the class of Id_X in $\Omega_*(X)_D$. Thus, we may assume that D is a Cartier divisor on X. Let η_1 and η_2 be two distinguished liftings of Id_X.

Suppose η_i is constructed via a birational morphism $\rho_i : W_i \to X \times \mathbb{P}^1$ satisfying (6.7), for $i = 1, 2$. Let Z_i be a subscheme of $X \times \mathbb{P}^1$, supported in $|D| \times 0$, such that W_i is the blow-up of $X \times \mathbb{P}^1$ along Z_i, $i = 1, 2$.

Let $T_1 \to X \times \mathbb{P}^1 \times \mathbb{P}^1$ be the blow-up along $Z_1 \times \mathbb{P}^1$, and let $\langle Z_2 \rangle$ denote the proper transform of $p_{13}^*(Z_2)$ to T_1. Let $T_2 \to T_1$ be the blow-up of T_1 along $\langle Z_2 \rangle$, with structure morphism $\phi : T_2 \to X \times \mathbb{P}^1 \times \mathbb{P}^1$.

We claim there is a blow-up of T_2 at a closed subscheme Z supported over $|D| \times 0 \times 0$, $T \to T_2$, such that T is smooth over k, and such that the divisor $X \times \mathbb{P}^1 \times 0 + X \times 0 \times \mathbb{P}^1$ pulls back to a strict normal crossing divisor on T, in good position with respect to D. To see this, let U be the open subscheme $T_2 \setminus \phi^{-1}(|D| \times 0 \times 0)$. We note that U is smooth and the pull-back of $X \times \mathbb{P}^1 \times 0 + X \times 0 \times \mathbb{P}^1$ to U is a strict normal crossing divisor, and the proper transform of $X \times \mathbb{P}^1 \times 0 + X \times 0 \times \mathbb{P}^1$ is disjoint from the proper transform of $D \times \mathbb{P}^1 \times \mathbb{P}^1$, after restricting to U. Arguing as in the proof of lemma 6.4.1, we construct a projective birational morphism $T \to T_2$, with T smooth over k and isomorphic to T_2 over U, such that $X \times \mathbb{P}^1 \times 0 + X \times 0 \times \mathbb{P}^1$ pulls back to a strict normal crossing divisor and the proper transform of $X \times \mathbb{P}^1 \times 0 + X \times 0 \times \mathbb{P}^1$ is disjoint from the proper transform of $D \times \mathbb{P}^1 \times \mathbb{P}^1$. This verifies our claim. We let $p : T \to X \times \mathbb{P}^1 \times \mathbb{P}^1$ be the induced morphism, and let $q : T \to X$ be p followed by the projection $p_X : X \times \mathbb{P}^1 \times \mathbb{P}^1 \to X$.

Clearly $p^{-1}(X \times \mathbb{P}^1 \times 1)$ is isomorphic to W_1, and $p^{-1}(X \times 1 \times \mathbb{P}^1)$ is isomorphic to W_2, as schemes over $X \times \mathbb{P}^1$. Let $D_0 = X \times \mathbb{P}^1 \times 0$ and $D'_0 = X \times 0 \times \mathbb{P}^1$, $D_1 = X \times \mathbb{P}^1 \times 1$ and $D'_1 = X \times 1 \times \mathbb{P}^1$. By our construction, we have the classes $[p^*D_0 \to T]_{D'_i|D}$, $[p^*D'_0 \to T]_{D_i|D}$, $i = 0, 1$.

Let $i_j : |p^*D_j| \to T$, $i'_j : |p^*D'_j| \to T$, $j = 0, 1$, be the inclusions. We first note that

$$\eta_1 = (qi_1)_*\big((p^*D_1)([p^*D'_0 \to T]_{D_1|D})_D\big).$$

Indeed, by proposition 6.3.3,

$$i_{1*}\big((p^*D_1)([p^*D'_0 \to T]_{D_1|D})_D\big) = i'_{0*}\big((p^*D'_0)([p^*D_1 \to T]_{D'_0|D})_D\big).$$

Let $j_1 : X \times 0 \to X \times \mathbb{P}^1$ be the inclusion. As $p^{-1}(X \times \mathbb{P}^1 \times 1)$ is isomorphic to W_1 over $X \times \mathbb{P}^1$, we have

$$(qi_0')_*\big((p^*D_0')([p^*D_1 \to T]_{D_0'|D})_D\big) = (p_1 j_1)_*\big((X \times 0)(W_1 \to X \times \mathbb{P}^1)_D\big)$$
$$= \eta_1$$

Similarly,

$$\eta_2 = (qi_1')_*\big((p^*D_1')([p^*D_0 \to T]_{D_1'|D})_D\big).$$

From proposition 6.3.4, we have

$$i_{0*}\big((p^*D_0)([p^*D_0' \to T]_{D_0|D})_D\big) = i_{1*}\big((p^*D_1)([p^*D_0' \to T]_{D_1|D})_D\big)$$
$$i_{0*}'\big((p^*D_0')([p^*D_0 \to T]_{D_0'|D})_D\big) = i_{1*}'\big((p^*D_1')([p^*D_0 \to T]_{D_1'|D})_D\big)$$

in $\Omega_*(T)_D$. By proposition 6.3.3, we have

$$i_{0*}'\big((p^*D_0')([p^*D_0 \to T]_{D_0'|D})_D\big) = i_{0*}\big((p^*D_0)([p^*D_0' \to T]_{D_0|D})_D\big)$$

in $\Omega_*(T)_D$. Pushing forward to X, we have

$$\eta_1 = (qi_1)_*\big((p^*D_1)([p^*D_0' \to T]_{D_1|D})_D\big)$$
$$= (qi_1')_*\big((p^*D_1')([p^*D_0 \to T]_{D_1'|D})_D\big)$$
$$= \eta_2,$$

as desired. □

Remark 6.4.7. Via this result, we may speak of *the* distinguished lifting of an element of $\mathcal{Z}_*(X)$ to $\Omega_*(X)_D$. We have the following properties of the distinguished lifting:

1. Sending $\eta \in \mathcal{Z}_*(X)$ to its distinguished lifting $\tilde{\eta}$ defines a $\mathcal{Z}_*(k)$-linear homomorphism $\mathcal{Z}_*(X) \to \Omega_*(X)_D$, lifting the canonical map $\mathcal{Z}_*(X) \to \Omega_*(X)$.
2. Given $f : X' \to X$ projective, if $\tilde{\eta} \in \Omega_*(X')_{f^*D}$ is the distinguished lifting of $\eta \in \mathcal{Z}_*(X')$, then $f_*\tilde{\eta} \in \Omega_*(X)_D$ is the distinguished lifting of $f_*\eta$.
3. If $L_1, \ldots, L_r$ are line bundles on X, and $\tilde{\eta}$ is the distinguished lifting of $\eta \in \mathcal{Z}_*(X)$, then $\tilde{c}_1(L_1) \circ \ldots \circ \tilde{c}_1(L_r)(\tilde{\eta})$ is the distinguished lifting of $\tilde{c}_1(L_1) \circ \ldots \circ \tilde{c}_1(L_r)(\tilde{\eta})$.

The $\mathcal{Z}_*(k)$-linearity in (1) follows from remark 6.4.5, and the lifting property follows from remark 6.4.3. These last two properties follow from the formula in remark 6.4.4.

We extend the distinguished lifting to $\mathbb{L}_* \otimes \mathcal{Z}_*(X)$ by $\mathbb{L}_*$-linearity.

Lemma 6.4.8. *Let F be in $\mathbb{L}_*[[u_1, \ldots, u_r]]$, let $f : W \to X$ be in $\mathcal{M}(X)$, and let $L_1, \ldots, L_r$ be line bundles on W. Take an element $\eta \in \mathcal{Z}_*(W)$ and let $\tilde{\eta}$ be the distinguished lifting of η to $\Omega_*(W)_{f^*D}$. Let F_N denote the truncation of F after total degree N. Then $f_*(F(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_r))(\tilde{\eta}))$ is the distinguished lifting of $f_*(F_N(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_r))(\eta))$ for all N sufficiently large.*

Proof. This follows from remark 6.4.7. □

Remark 6.4.9. As an application, consider the case of a strict normal crossing divisor $E = \sum_{j=1}^{m} n_j E_j$ on $Y \in \mathbf{Sm}_k$, with inclusion $i : |E| \to Y$, and a projective map $f : Y \to X$. The class $[E \to |E|] \in \Omega_*(|E|)$ is

$$\sum_J \iota_*^J F_J^{n_1,\ldots,n_m}(\tilde{c}_1(O_Y(E_1)),\ldots,\tilde{c}_1(O_Y(E_m)))(\mathrm{Id}_{E^J}). \tag{6.8}$$

where $\iota^J : E^J \to |E|$ is the inclusion. Let $\widetilde{[\mathrm{Id}_{E^J}]}_D$ be the distinguished lifting of Id_{E^J}, and let $\widetilde{[E \to |E|]}$ be a lifting of $[E \to |E|]$ to $\mathcal{Z}_*(|E|)$ defined by truncating $F_J^{n_1,\ldots,n_m}(u_1,\ldots,u_m)$ after total degree N, for some $N > \dim_k E^J$. By remark 6.4.7 and lemma 6.4.8, the element

$$\widetilde{[E \to |E|]}_D := \sum_J \iota_*^J F_J^{n_1,\ldots,n_m}(\tilde{c}_1(O_Y(E_1)),\ldots,\tilde{c}_1(O_Y(E_m)))(\widetilde{[\mathrm{Id}_{E^J}]}_D).$$

of $\Omega_*(|E|)_D$ is the distinguished lifting of $\widetilde{[E \to |E|]}$.

Lemma 6.4.10. *Let η be in $\mathcal{Z}_*(X)_D$, and let $\tilde{\eta}$ be the distinguished lifting of the image of η in $\mathcal{Z}_*(X)$. Then $\tilde{\eta}$ is the image of η in $\Omega_*(X)_D$ under the canonical homomorphism $\mathcal{Z}_*(X)_D \to \Omega_*(X)_D$.*

Proof. From the relations described in remark 6.4.7, we reduce to the case of $\eta = \mathrm{Id}_X \in \mathcal{M}(X)_D$, with X irreducible and in $\mathbf{Sm}_k$. The case $|D| = X$ is evident, so assume that D is a strict normal crossing divisor on X. Let $\mu : W \to X \times \mathbb{P}^1$ be a projective birational morphism such that the conditions (6.7) are satisfied, and let $g = p_1 \circ \mu$. Then the divisors $\mu^*(X\times 1)$ and $\mu^*(X\times 0)$ are both in good position with respect to D. As both are divisors of sections of $\mu^*(O_{X\times\mathbb{P}^1}(1))$, we have

$$g_*([\mu^*(X \times 1) \to W]_D) = g_*([\mu^*(X \times 0) \to W]_D)$$

in $\Omega_*(X)_D$, by remark 6.1.11. As $g_*([\mu^*(X \times 1) \to W]_D) = \mathrm{Id}_X$ and $g_*([\mu^*(X \times 0) \to W]_D)$ is the distinguished lifting of Id_X, the result follows. □

6.4.2 Lifting divisor classes

We need some information on the distinguished lifting of divisor classes before proving the main moving lemma.

Let $f : Y \to X$ be in $\mathcal{M}(X)$, and let $i : S \to Y$ be a smooth Cartier divisor on Y. Let D be a pseudo-divisor on X. Suppose that $f(Y) \not\subset |D|$, and that Y is irreducible. Let $\tilde{D} = \mathrm{div} f^* D$.

We apply the construction of §6.4.1: blow up $Y \times \mathbb{P}^1$ along a subscheme contained in $|\tilde{D}| \times 0$, forming the scheme $\rho : T \to Y \times \mathbb{P}^1$, satisfying the conditions (6.7). Note that $S \times \mathbb{P}^1 + Y \times 0$ is a strict normal crossing divisor on $Y \times \mathbb{P}^1$. By corollary A.3, we may blow up T along smooth centers over $|\tilde{D}| \times 0$, forming the scheme $\tau : \tilde{T} \to Y \times \mathbb{P}^1$ which satisfies the conditions (6.7) and in addition has the property that $\tau^*(S \times \mathbb{P}^1 + Y \times 0)$ is a strict normal crossing divisor. Finally, letting E be the exceptional divisor of τ, we may suppose that

$$|\tau^{-1}(D \times 0)| = |E|.$$

We let $\langle Y \times 0\rangle$ denote the proper transform of $Y \times 0$ and $\langle S \times \mathbb{P}^1\rangle$ the proper transform of $S \times \mathbb{P}^1$. Let $\tilde{S} = \tau^*(S \times \mathbb{P}^1)$, $\tilde{Y} = \tau^*(Y \times 0)$.

Lemma 6.4.11.

1. *$\tilde{S} + \tilde{Y}$ is in good position with respect to $\tilde{Y}|D$.*
2. *Let $q : |\tilde{Y}| \cap |\tilde{S}| \to X$ be the composition of the inclusion into $\tilde{Y}$ with $f \circ \tau$. Then $q_*(\tilde{Y}([\tilde{S} \to |\tilde{S}|]_{\tilde{Y}|D})_D) \in \Omega_*(X)_D$ is the distinguished lifting of $f_*([S \to Y])$.*

Proof. (1) follows directly from the construction: Since $\tilde{S}+\tilde{Y}$ is a strict normal crossing divisor, $\tilde{S} + \tilde{Y}$ is in $\mathcal{M}(\tilde{T})_{\tilde{Y}}$. Since

$$|\tau^*(S \times \mathbb{P}^1 + Y \times 0)| = |\langle Y \times 0\rangle + E + \tilde{S}|$$

and $|\tau^{-1}(D \times 0)| = |E|$, it follows that, for each face F of $\tilde{S} + \tilde{Y}$, $F \cap \tilde{Y}$ is in good position with respect to D.

For (2), we first claim that

$$\tilde{Y}([\tilde{S} \to |\tilde{S}|]_{\tilde{Y}|D})_D = \tilde{Y}([\langle S \times \mathbb{P}^1\rangle \to |\tilde{S}|]_{\tilde{Y}|D})_D \tag{6.9}$$

in $\Omega_*(|\tilde{Y}| \cap |\tilde{S}|)_D$. Indeed we may write

$$\tilde{S} = \langle S \times \mathbb{P}^1\rangle + A,$$

where A is an effective divisor, supported in $|\tilde{S}| \cap |E|$. Since the exceptional divisor E is supported in $|\tilde{Y}|$, A is supported in $|\tilde{Y}|$. From the above decomposition of $\tilde{S}$, we have

$$[\tilde{S} \to |\tilde{S}|]_{\tilde{Y}|D} = [\langle S \times \mathbb{P}^1\rangle \to |\tilde{S}|]_{\tilde{Y}|D} + i_*\alpha,$$

where α is a class in $\Omega_*(|A|)_{\tilde{Y}|D}$, and $i : |A| \to |\tilde{S}|$ is the inclusion. Then

$$\tilde{Y}([\tilde{S} \to |\tilde{S}|]_{\tilde{Y}|D})_D = \tilde{Y}([\langle S \times \mathbb{P}^1\rangle \to |\tilde{S}|]_{\tilde{Y}|D})_D + i_*\tilde{c}_1(i^*O_{\tilde{T}}(\tilde{Y}))(\alpha).$$

But $\tilde{Y} = \tau^*(Y \times 0)$, hence $O_{\tilde{T}}(\tilde{Y}) \cong O_{\tilde{T}}$ in a neighborhood of $|A|$. Thus $\tilde{c}_1(i^*O_{\tilde{T}}(\tilde{Y}))(\alpha) = 0$, proving our claim.

We are thus reduced to showing that $q_*(\tilde{Y}([\langle S \times \mathbb{P}^1 \to |\tilde{S}|\rangle]_{\tilde{Y}|D}))_D$ is the distinguished lifting of $f_*([S \to Y])$. For this, the fact that $\tilde{S} + E$ is a strict normal crossing divisor, that $\tau^{-1}(D \times 0) = |E|$ and that τ satisfies the conditions (6.7) for D imply that the restriction of τ,

$$\tau_S : \langle S \times \mathbb{P}^1 \rangle \to S \times \mathbb{P}^1$$

satisfies the conditions (6.7) for the pull-back of D to S. Thus, the class

$$[\tau_S^*(S \times 0) \to X]$$

is the distinguished lifting of $[S \to X]$. Since

$$\tau_S^*(S \times 0) = \tilde{Y} \cap \langle S \times \mathbb{P}^1 \rangle$$

as divisors on $\langle S \times \mathbb{P}^1 \rangle$, and since

$$[\tilde{Y} \cap \langle S \times \mathbb{P}^1 \rangle \to |\tilde{S}| \cap |\tilde{Y}|]_D = \tilde{Y}[\langle S \times \mathbb{P}^1 \rangle \to |\tilde{S}|]_{\tilde{Y}|D|}),$$

the proof is complete. □

6.4.3 The proof of the moving lemma

We are now ready to prove the main result of this section.

Theorem 6.4.12. *Let X be a finite type k-scheme, and D a pseudo-divisor on X. Then the canonical map $\vartheta_X : \Omega_*(X)_D \to \Omega_*(X)$ is an isomorphism.*

Proof. As noted in remark 6.4.7, taking the distinguished lifting defines a $\mathcal{Z}_*(k)$-linear homomorphism $\phi : \mathcal{Z}_*(X) \to \Omega_*(X)_D$, with $\vartheta_X(\phi(\eta))$ the image of η in $\Omega_*(X)$. In fact, by lemma 6.4.10, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{Z}_*(X)_D \arrow[r] \arrow[d, "\tilde{\vartheta}_X"'] & \Omega_*(X)_D \arrow[d, "\vartheta_X"] \\
\mathcal{Z}_*(X) \arrow[r] \arrow[ur, "\phi"'] & \Omega_*(X)
\end{tikzcd}$$

where $\tilde{\vartheta}_X$ is the canonical map, and the horizontal arrows are the canonical maps.

Since $\mathcal{Z}_*(X) \to \Omega_*(X)$ is surjective (lemma 2.5.9), the surjectivity of ϑ_X follows. To show that ϑ_X is injective, it suffices to show that the distinguished lifting homomorphism ϕ descends to an $\Omega_*(k)$-linear homomorphism $\bar{\phi} : \Omega_*(X) \to \Omega_*(X)_D$. Indeed, $\Omega_*(X)_D$ is generated as an $\mathbb{L}_*$-module by the image of $\mathcal{Z}_*(X)_D$, and thus $\Omega_*(X)_D$ is generated as an $\Omega_*(k)$-module by the image of $\mathcal{Z}_*(X)_D$. Therefore, $\bar{\phi}$ is surjective, and $\vartheta_X \circ \bar{\phi} = \mathrm{Id}$, hence ϑ_X is injective. We proceed to show that ϕ descends to $\bar{\phi}$.

First, we show that ϕ descends to $\phi_1 : \underline{\mathcal{Z}}_*(X) \to \Omega_*(X)_D$. For this, take a generator $\eta := (f : Y \to X, \pi^*L_1, \ldots, \pi^*L_r, M_1, \ldots, M_s)$ of $\langle \mathcal{R}_*^{Dim} \rangle(X)$ (see lemma 2.4.2), which is the kernel of $\mathcal{Z}_*(X) \to \underline{\mathcal{Z}}_*(X)$. Here $f : Y \to X$ is in $\mathcal{M}(X)$, $\pi : Y \to Z$ is a smooth morphism to some $Z \in \mathbf{Sm}_k$, $L_1, \ldots, L_r$ are line bundles on Z, $M_1, \ldots, M_s$ are line bundles on Y, and $r > \dim_k Z$. Let $\mu : W \to Y \times \mathbb{P}^1$ be a projective birational morphism used to construct a distinguished lifting of $f : Y \to X$. Let $g = f \circ p_1 \circ \mu$, $\tau = \pi \circ p_1 \circ \mu$ and $\rho = p_1 \circ \mu$. Then the distinguished lifting of η is

$$g_*\big(\tilde{c}_1(\tau^* L_1) \circ \ldots \circ \tilde{c}_1(\tau^* L_r) \circ \tilde{c}_1(\rho^* M_1) \circ \ldots \\ \ldots \circ \tilde{c}_1(\rho^* M_s)([\mu^*(Y \times 0) \to W]_D)\big).$$

By lemma 6.1.13, the operator $\tilde{c}_1(\tau^* L_1) \circ \ldots \circ \tilde{c}_1(\tau^* L_r)$ is zero on $\Omega_*(W)_D$, so the distinguished lifting of η is zero, as desired.

Next, we check that ϕ_1 descends to $\phi_2 : \underline{\Omega}_*(X) \to \Omega_*(X)_D$. Since ϕ intertwines the operators $\tilde{c}_1(L)$ on $\mathcal{Z}_*(X)$ and $\Omega(X)_D$, it suffices by lemma 2.4.7 to check that ϕ_1 vanishes on elements of the form

$$[f : Y \to X, O_Y(S)] - [f \circ i : S \to X],$$

where f is in $\mathcal{M}(X)$, and $i : S \to Y$ is the inclusion of a smooth divisor.

Let $\eta \in \Omega_*(Y)_{f^*D}$ be the distinguished lifting of Id_Y. By lemma 6.4.8, $f_*(\tilde{c}_1(O_Y(S))(\eta))$ is the distinguished lifting of $(f : Y \to X, O_Y(S)) = f_*(\tilde{c}_1(O_Y(S))(\mathrm{Id}_Y))$.

On the other hand, let $\rho : W \to Y \times \mathbb{P}^1$ be a blow-up used to define the distinguished lifting η, so η is represented by $(p_1 \circ \rho)_*([\rho^*(Y \times 0) \to W]_D)$. Blowing up W further, and changing notation, we may assume that $\rho^*(S \times \mathbb{P}^1)$ is a normal crossing divisor on W, in good position with respect to $Y \times 0|f^*D$. By lemma 6.4.11,

$$f_*\big((Y \times 0)((p_1 \circ \rho)_*[\rho^*(S \times \mathbb{P}^1) \to W]_{Y \times 0|D})_{f^*D}\big)$$

is the distinguished lifting of $f_*[S \to Y]$ to $\Omega_*(X)_D$. Let $\tilde{i}_S : |\rho^*(S \times \mathbb{P}^1)| \to W$ and $\tilde{i}_0 : |\rho^*(Y \times 0)| \to W$ be the inclusions. By lemma 6.3.1 and proposition 6.3.3, we have

$$\begin{aligned}
&f_*\big((Y \times 0)((p_1 \circ \rho)_*[\rho^*(S \times \mathbb{P}^1) \to W]_{Y \times 0|f^*D})\big) \\
&\quad = (f \circ p_1 \circ \rho)_*(\tilde{i}_{0*}(\rho^*(Y \times 0)([\rho^*(S \times \mathbb{P}^1) \to W]_{Y \times 0|D}))) \\
&\quad = (f \circ p_1 \circ \rho)_*\big(\tilde{i}_{S*}(\rho^*(S \times \mathbb{P}^1)([\rho^*(Y \times 0) \to W]_{S \times \mathbb{P}^1|D}))\big) \\
&\quad = (f \circ p_1 \circ \rho)_*\big(\tilde{c}_1((p_1 \circ \rho)^* O_Y(S))([\rho^*(Y \times 0) \to W]_D)\big) \\
&\quad = f_*(\tilde{c}_1(O_Y(S))(\eta)).
\end{aligned}$$

Thus $f_*[S \to Y]$ and $f_*(\tilde{c}_1(O_Y(S))(1_Y))$ have the same distinguished lifting to $\Omega_*(X)_D$, that is $\phi_1([f : Y \to X, O_Y(S)] - [f \circ i : S \to X]) = 0$, as desired.

Finally, we check that ϕ_2 descends to $\bar{\phi} : \Omega_*(X) \to \Omega_*(X)_D$. For this we use the description of the kernel of the surjection $\underline{\Omega}_*(X) \to \Omega_*(X)$ given by definition 2.5.13 and proposition 2.5.15. To describe this kernel, we consider the power series $F_\Omega(u,v) = u+v+\sum_{i,j\geq 1} a_{ij}u^iv^j$ giving the formal group law on $\Omega_*(k)$. We choose liftings $\alpha_{ij} \in \mathcal{M}^+(k)$ of $a_{ij} \in \Omega_*(k)$, and let $F(u,v) = u + v + \sum_{i,j\geq 1} \alpha_{ij}u^iv^j$ be the resulting lifting of F_Ω. Then the kernel of $\underline{\Omega}_*(X) \to \Omega_*(X)$ is generated by elements of the form

$$f_*\big(\tilde{c}_1(L_1) \circ \ldots \circ \tilde{c}_1(L_r)([F(L, M)] - [L \otimes M])\big),$$

where $f : Y \to X$ is in $\mathcal{M}(X)$, and $L_1, \ldots, L_r, L$ and M are line bundles on Y. Given such an element, it suffices to show that

$$\phi_2([F(L,M)] - [L \otimes M]) = 0$$

in $\Omega_*(Y)_D$, since ϕ_2 is compatible with f_*, and with the Chern class operators $\tilde{c}_1(L_i)$. Now, $[F(L,M)] = F(\tilde{c}_1(L), \tilde{c}_1(M))(\mathrm{Id}_Y)$, and $[L \otimes M] = \tilde{c}_1(L \otimes M)(\mathrm{Id}_Y)$. Thus, if $\eta \in \Omega_*(Y)_D$ is the distinguished lifting of Id_Y, it follows from the definition of distinguished liftings that

$$\phi_2([F(L,M)]) = F(\tilde{c}_1(L), \tilde{c}_1(M))(\eta),$$
$$\phi_2([L \otimes M]) = \tilde{c}_1(L \otimes M)(\eta).$$

Since $F(u,v)$ and $F_{\mathbb{L}}(u,v)$ both have image $F_\Omega(u,v)$ in $\Omega_*(k)[[u,v]]$, it follows that $F(\tilde{c}_1(L), \tilde{c}_1(M))(\eta) = F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))(\eta)$. Thus, using the relations $\langle \mathbb{L}_*\mathcal{R}_*^{FGL}\rangle(Y)_{f^*D}$, we see that $\phi_2([F(L,M)] - [L \otimes M]) = 0$, which completes the descent and the proof of the theorem. □

6.5 Pull-back for l.c.i. morphisms

Starting with the pull-back for a divisor, defined using the results of the previous sections, we use the deformation to the normal bundle to define the Gysin morphism for a regular embedding of k-schemes. Combined with smooth pull-back, this gives us functorial pull-back maps for l.c.i. morphisms of k-schemes, in particular, for arbitrary morphisms in $\mathbf{Sm}_k$.

6.5.1 Pull-back for divisors

The results of §6.2-§6.4 allow us to make the following definition:

Definition 6.5.1. *Let D be a pseudo-divisor on a finite type k-scheme X. We define the operation of "pull-back by D",*

$$i_D^* : \Omega_*(X) \to \Omega_{*-1}(|D|),$$

to be the composition

$$\Omega_*(X) \xrightarrow{\vartheta_X^{-1}} \Omega_*(X)_D \xrightarrow{D(-)} \Omega_{*-1}(|D|).$$

6.5.2 The Gysin morphism

Let $i_Z : Z \to X$ be a regular embedding in $\mathbf{Sch}_k$ of codimension $d > 0$. We proceed to define the map $i_Z^* : \Omega_*(X) \to \Omega_{*-d}(Z)$.

We form the deformation diagram for i, being the Fulton-MacPherson deformation diagram (2.4) with the proper transform of $X \times 0$ removed: Let $\mu : M \to X \times \mathbb{P}^1$ be the blow-up of $X \times \mathbb{P}^1$ along $Z \times 0$, let $\langle X \times 0\rangle$ and $\langle Z \times \mathbb{P}^1\rangle$ denote the proper transforms of $X \times 0$ and $Z \times \mathbb{P}^1$ respectively, let E be the exceptional divisor of μ, let $N = E \setminus \langle X \times 0\rangle$ and $U = M \setminus \langle X \times 0\rangle$. Let $j : U \setminus N \to U$, and $i_N : N \to U$ be the inclusions. This yields the commutative diagram

$$\begin{tikzcd}
N \arrow[rrr, "i_N"] \arrow[dd, "\mu_N"'] & & & U \arrow[dd, "\mu"] & U \setminus N \arrow[l, "j"'] \\
& \langle Z \times \mathbb{P}^1\rangle_0 \arrow[ul, "s"] \arrow[r, "\tilde{i}_0"'] \arrow[dl, equal] & \langle Z \times \mathbb{P}^1\rangle \arrow[urr, "\tilde{i}"'] \arrow[d, equal] & & X \arrow[ul, "i_1^X"'] \arrow[u] \arrow[d, equal] \\
Z \times 0 \arrow[rr, "i_0"'] & & Z \times \mathbb{P}^1 \arrow[r, "i"'] & X \times \mathbb{P}^1 & X \times 1 \arrow[l]
\end{tikzcd} \tag{6.10}$$

The equalities are isomorphisms induced by μ, μ_N is the morphism induced by μ and $\langle Z \times \mathbb{P}^1\rangle_0$ is the fiber of $\langle Z \times \mathbb{P}^1\rangle$ over $0 \in \mathbb{P}^1$. We also have the identity $\langle Z \times \mathbb{P}^1\rangle_0 = \langle Z \times \mathbb{P}^1\rangle \cap N$, which gives the map s.

Letting ν be the conormal sheaf of Z in X, $\mu_E : E \to Z \times 0 = Z$ is the projective bundle $\mathbb{P}(\nu \oplus \mathcal{O}_Z) \to Z$, and $\mu_N : N \to Z$ is the vector bundle $\mathrm{Spec}\,(\mathrm{Sym}^*(\nu)) \to Z$, i.e., the normal bundle $N_Z X$ of Z in X. Note that μ restricts to an isomorphism

$$\mu : U \setminus N \xrightarrow{\sim} X \times (\mathbb{P}^1 \setminus 0).$$

When we need to indicate explicitly X and Z in the notation, we write $M_Z X$ for M, $U_Z X$ for U, etc.

Lemma 6.5.2. *The map $i_N^* \circ i_{N*} : \Omega_{*+1}(N) \to \Omega_*(N)$ is the zero map*

Proof. If $f : T \to N$ is in $\mathcal{M}(N)$, then $i_N \circ f : T \to U$ is clearly in $\mathcal{M}(U)_N$, and $N(i_N \circ f) = f_*(\tilde{c}_1(f^* O_U(N))(\mathrm{Id}_T))$. But on U, the divisor N is linearly equivalent to $\mu^*(X \times 1)$, which is disjoint from N. Thus $f^* O_U(N) \cong O_T$ and hence $\tilde{c}_1(f^* O_U(N))(\mathrm{Id}_T) = 0$. Thus $i_N^* i_{N*}(f) = N(i_N \circ f) = 0$, as desired. $\square$

We have the diagram

$$\begin{tikzcd}
\Omega_*(X) \arrow[r, "p_1^*"] & \Omega_{*+1}(X \times (\mathbb{P}^1 \setminus 0)) \arrow[d, "\mu^*", "\sim"'] & \\
& \Omega_{*+1}(U \setminus N) & \Omega_{*+1}(U) \arrow[l, "j^*"'] \arrow[d, "i_N^*"'] \\
& \Omega_{*-d}(Z) \arrow[r, "\mu_N^*"'] & \Omega_*(N).
\end{tikzcd} \tag{6.11}$$

We call (6.11) the *zigzag diagram* for the regular embedding i_Z. We have as well the exact localization sequence (theorem 3.2.7)

$$\Omega_{*+1}(N) \xrightarrow{i_{N*}} \Omega_{*+1}(U) \xrightarrow{j^*} \Omega_{*+1}(U \setminus N) \to 0;$$

by lemma 6.5.2, the composition

$$\Omega_{*+1}(X \times (\mathbb{P}^1 \setminus 0)) \xrightarrow{\mu^*} \Omega_{*+1}(U \setminus N) \xrightarrow{(j^*)^{-1}} \Omega_{*+1}(U) \xrightarrow{i_N^*} \Omega_*(N)$$

gives a well-defined homomorphism $\psi_{Y,Z} : \Omega_{*+1}(X \times (\mathbb{P}^1 \setminus 0)) \to \Omega_*(N)$.

Since the map $\mu_N : N \to Z$ makes N into a vector bundle over Z, it follows from the homotopy property for algebraic cobordism (theorem 3.6.3) that the smooth pull-back

$$\mu_N^* : \Omega_{*-d}(Z) \to \Omega_*(N)$$

is an isomorphism.

Definition 6.5.3. *Let $i_Z : Z \to X$ be a regular embedding of codimension d in $\mathbf{Sch}_k$. The* Gysin morphism *$i_Z^* : \Omega_*(X) \to \Omega_{*-d}(Z)$ is defined as the composition*

$$\Omega_*(X) \xrightarrow{p_1^*} \Omega_{*+1}(X \times (\mathbb{P}^1 \setminus 0)) \xrightarrow{\psi_{Y,Z}} \Omega_*(N_{Z/X}) \xrightarrow{(\mu_N^*)^{-1}} \Omega_{*-d}(Z).$$

6.5.3 Properties of the Gysin morphism

Let $f : Y \to X$, $g : Z \to X$ be morphisms in $\mathbf{Sch}_k$. Recall that f and g are called *Tor-independent* if $\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z) = 0$ for $i > 0$.

Proposition 6.5.4. *Let $f : Y \to X$, $g : Z \to X$ be Tor-independent morphisms in $\mathbf{Sch}_k$, giving the cartesian diagram*

$$\begin{tikzcd}
Y \times_X Z \arrow[r, "f'"] \arrow[d, "g'"'] & Z \arrow[d, "g"] \\
Y \arrow[r, "f"'] & X
\end{tikzcd}$$

Suppose that g is a regular embedding.

1. *If f is projective, then $g^* \circ f_* = f'_* \circ g'^*$.*
2. *If f is smooth and quasi-projective, then $f'^* \circ g^* = g'^* \circ f^*$.*

Proof. Since f and g are Tor-independent, the map g' is a regular embedding, so g'^* is defined. Also, if we apply the functor $Y \times_X -$ to zigzag diagram (6.11) for the regular embedding $g : Z \to X$, we arrive at the zigzag diagram for the regular embedding $g' : Y \times_X Z \to Y$. Calling the first diagram $\mathcal{D}(g)$ and the second $\mathcal{D}(g')$, the projection $p_2 : Y \times_X ? \to ?$ gives the map of diagrams $p_{2*} : \mathcal{D}(g') \to \mathcal{D}(g)$ in case f is projective, and the map of diagrams $p_2^* : \mathcal{D}(g) \to \mathcal{D}(g')$ if f is smooth and quasi-projective (with shift in the grading).

We note that the diagrams $p_{2*} : \mathcal{D}(g') \to \mathcal{D}(g)$ and $p_2^* : \mathcal{D}(g) \to \mathcal{D}(g')$ are commutative. Indeed, this follows from the following facts:

1. projective push-forward commutes with smooth pull-back in cartesian diagrams (definition 2.1.2(A3)).
2. pull-back by a divisor satisfies a projection formula with respect to projective push-forward (lemma 6.2.1(1)).
3. pull-back by a divisor commutes with smooth pull-back (lemma 6.2.1(2)).

This proves the proposition. □

Corollary 6.5.5. *Let Z and X be in $\mathbf{Sch}_k$.*

1. *Let $i : Z \to X$ be a regular embedding, and let $f : Y \to X$ be in $\mathcal{M}(X)$. Suppose f and i are Tor-independent and that $Y \times_X Z$ is in $\mathbf{Sm}_k$. Then $i^*(f)$ is represented by $p_2 : Y \times_X Z \to Z$.*
2. *Let $p : X \to Z$ be a smooth morphism with a section $i : Z \to X$. Then i is a regular embedding and $i^* \circ p^* = \mathrm{Id}$ on $\Omega_*(Z)$.*
3. *Let $p : X \to Z$ be a rank n vector bundle over Z, and let $i : Z \to X$ be a section. Then i is a regular embedding and i^* is the inverse of p^*.*

Proof. For (3), the fact that p^* is an isomorphism if $p : X \to Z$ is a vector bundle over Z, theorem 3.6.3 shows that (2) implies (3). Also, (2) follows from (1), since $p^*(f : Y \to Z)$ is represented by $p_2 : Y \times_Z X \to X$. It remains to prove (1).

By proposition 6.5.4, we have $i^* f_*(\eta) = f'_* i'^*(\eta)$ for $\eta \in \Omega_*(Y)$, where $i' : Y \times_X Z \to Y$, $f' : Y \times_X Z \to Z$ are the projections. Thus, it suffices to show that $i'^*(\mathrm{Id}_Y) = \mathrm{Id}_{Y \times_X Z}$. Denoting $Y \times_X Z$ by Z', we form the deformation diagram (6.11) for i', letting $\mu : M \to Y \times \mathbb{P}^1$ be the blow-up of $Y \times \mathbb{P}^1$ along $Z' \times 0$ with exceptional divisor E, $N := E \cap U$, $i_N : N \to U$ the inclusion of the normal bundle $N_{Z'}Y$, $j : U \setminus N \to U$ the open complement of N, and $\mu_N : N \to Z'$ the projection.

Clearly $j^*(\mathrm{Id}_U) = p_1^*(\mathrm{Id}_Y)$ in $\Omega_{*+1}(U \setminus N)$, hence

$$i'^*(\mathrm{Id}_Y) = (\mu_{Z'}^*)^{-1}(i_N^*(\mathrm{Id}_U)) = (\mu_Z^*)^{-1}(\mathrm{Id}_N).$$

Since $\mu_{Z'}^*(\mathrm{Id}_{Z'}) = \mathrm{Id}_N$, we have $i'^*(\mathrm{Id}_Y) = \mathrm{Id}_{Z'}$, as desired. □

We check that our two definitions of pull-back for a Cartier divisor agree.

Lemma 6.5.6. *Let $i : Z \to X$ be a regular embedding, with* $\mathrm{codim}_X Z = 1$. *Then, for $f : Y \to X$ in $\mathcal{M}(X)_Z$, we have $Z(f) = i^*(f)$ in $\Omega^*(Z)$, so $i^* = i_Z^*$.*

Proof. Form the deformation diagram (6.10) for i; we have the blow-up $\mu : M \to X \times \mathbb{P}^1$ of $X \times \mathbb{P}^1$ along $Z \times 0$.

Let $\rho : \tilde{T} \to Y \times \mathbb{P}^1$ be a blow-up of $Y \times \mathbb{P}^1$ along smooth centers lying over $f^{-1}(Z) \times 0$ so that the rational map $\mu^{-1} \circ (f \times \mathrm{Id}) : Y \times \mathbb{P}^1 \to M$ defines a morphism $\tilde{\phi} : \tilde{T} \to M$. Blowing up further over $Y \times 0$, if necessary, we may assume that $\tilde{\phi}$ is in $\mathcal{M}(M)_{E|\langle Z \times \mathbb{P}^1 \rangle}$.

Let $\phi : T \to U$ be the pull-back of $\tilde{\phi}$ via the open immersion $U \to M$. Clearly

$$\rho : T \times_U U \setminus N \to Y \times (\mathbb{P}^1 \setminus 0)$$

is an isomorphism. Thus, by definition of i^*, we have

$$i^*(f) = (\mu_N^*)^{-1}(i_N^*(\phi)).$$

Let $i_0 : Z \to N$ be the zero section, giving the divisor $i_0(Z)$ on N. Since $i_0(Z)(\mu_N^* g) = g$ for $g \in \Omega^*(Z)$, the map $i_0(Z)(-) : \Omega^*(N) \to \Omega^*(Z)$ is inverse to μ^*. Thus

$$i^*(f) = i_0(Z)(i_N^*(\phi)) = i_0(Z)(N(\phi)) = i_0(Z)(\mu^*(X \times 0)(\phi)).$$

Let $i_1 : Z \to X \times 1$ be the evident inclusion.

Write D for $\langle Z \times \mathbb{P}^1 \rangle$, and let $i_D : |D| \to M$, $\iota_0 : |\mu^*(X \times 0)| \to M$ and $\iota_1 : |\mu^*(X \times 1)| \to M$ be the inclusions. Since $\langle Z \times \mathbb{P}^1 \rangle \cong Z \times \mathbb{P}^1$, we have the projection $p_1 : |D| \to Z$. Noting that $i_0(Z) = \iota_0^*(D)$ as Cartier divisors, and using the projection formula (lemma 6.2.1(1)) and proposition 6.3.4, we have

$$\begin{aligned} i^*(f) &= i_0(Z)(\mu^*(X \times 0)(\phi)) = (\iota_0^* D)(\mu^*(X \times 0)(\phi)) \\ &= p_{1*} \circ i_{0*}\big((\iota_0^* D)(\mu^*(X \times 0)(\phi))\big) \\ &= p_{1*}\big(D(\iota_{0*}(\mu^*(X \times 0)(\phi))\big) = p_{1*}\big(D(\iota_{1*}(\mu^*(X \times 1)(\phi))\big) \\ &= i_1(Z)(\mu^*(X \times 1)(\phi)) = Z(f). \end{aligned}$$

□

Lemma 6.5.7. *Let $i : Z \to X$ be a regular embedding, let $p : Y \to X$ be a smooth quasi-projective morphism, and let $s : Z \to Y$ be a section of Y over Z. Then $s^* \circ p^* = i^*$.*

Proof. Form the deformation diagram for the inclusion $s : Z \to Y$, letting $\mu_s : M_s \to Y \times \mathbb{P}^1$ be the blow-up of $Y \times \mathbb{P}^1$ along $s(Z) \times 0$, $U_s := M_s \setminus \langle Y \times 0 \rangle$, and $j_{N_s} : N_s \to U_s$ the inclusion of the normal bundle $N_{s(Z)/Y}$. Similarly, let

$\mu_i : M_i \to X \times \mathbb{P}^1$ be the blow-up of $X \times \mathbb{P}^1$ along $i(Z) \times 0$; we change s to i in the notation.

Checking in local coordinates, one sees that the projection $p : Y \to X$ extends to a smooth morphism $\tilde{p} : U_s \to U_i$, inducing the natural map $Np : N_Z Y \to N_Z X$ from N_s to N_i. Take an element $\eta \in \Omega^*(X)$, and let $\tilde{\eta}_i$ be a lifting of $p_1^*\eta$ to $\Omega^*(U_i)$. Then $\tilde{\eta}_s := \tilde{p}^*\tilde{\eta}_i$ is a lifting of $p_1^* p^*\eta$ to $\Omega^*(U_s)$. Since $\tilde{p}$ is smooth, we have

$$i_{N_s}^*(\tilde{\eta}_s) = Np^* i_{N_i}^*(\tilde{\eta}_i).$$

Letting $p_s : N_s \to Z$, $p_i : N_i \to Z$ be the projections, we have $p_s^* = Np^* \circ p_i^*$, hence, by proposition 6.5.4,

$$s^*(p^*\eta) = (p_s^*)^{-1}(i_{N_s}^*(\tilde{\eta}_s)) = (p_i^*)^{-1}(i_{N_i}^*\tilde{\eta}_i) = i^*\eta.$$

□

Theorem 6.5.8. *Let $i : Z \to Z'$, $i' : Z' \to X$ be regular embeddings. Then $(i' \circ i)^* = i^* \circ i'^*$.*

Proof. Form the deformation diagram (6.10) for $i' : Z' \to X$ by blowing up $X \times \mathbb{P}^1$ along $Z' \times 0$, giving the birational morphism $\mu : M \to X \times \mathbb{P}^1$ with exceptional divisor E, let $U := M \setminus \langle X \times 0 \rangle$, let $i_N : N \to U$ be the inclusion of the normal bundle $N_{Z'}X$ and $\mu_N : N \to Z'$ the projection.

Restricting to Z, we have the closed subscheme $\langle Z \times \mathbb{P}^1 \rangle$ of $\langle Z' \times \mathbb{P}^1 \rangle$, which is isomorphic to $Z \times \mathbb{P}^1$ via μ, and the restriction of s defines a section $s_0 : Z \to N$ over Z, with $s_0(Z) = N \cap \langle Z \times \mathbb{P}^1 \rangle$. Letting $s_1 : Z \to X \times 1 \subset M$ be the map $s_1(z) = (i'(i(z)), 1)$, we similarly have $s_1(Z) = X \times 1 \cap \langle Z \times \mathbb{P}^1 \rangle$.

Let η be an element of $\Omega^*(U)_{N+X\times 1}$. We claim that

$$s_0^*(N(\eta)) = s_1^*((X \times 1)(\eta)) \tag{6.12}$$

in $\Omega^*(Z)$. Indeed, form the deformation diagram for the inclusion $i'' : \langle Z \times \mathbb{P}^1 \rangle \to U$: Let $\phi : T \to U \times \mathbb{P}^1$ be the blow-up of $U \times \mathbb{P}^1$ along $\langle Z \times \mathbb{P}^1 \rangle \times 0$, let $j^Z : U^Z \to T$ be open subscheme $T \setminus \langle U \times 0 \rangle$, let $i_{N^Z} : N^Z \to U^Z$ be the inclusion of the normal bundle of $N_{\langle Z \times \mathbb{P}^1 \rangle / U}$. The structure morphism $M \to \mathbb{P}^1$ induces the morphism $\tau : U^Z \to \mathbb{P}^1$. Via τ the deformation diagram for i_{N^Z} is a diagram of schemes over $\mathbb{P}^1$, where the fiber over $0 \in \mathbb{P}^1$ is the deformation diagram for the inclusion $s_0 : Z \to N$, and the fiber over $1 \in \mathbb{P}^1$ is the deformation diagram for the inclusion $s_1 : Z \to X \times 1$. Let N_0^Z, U_0^Z, etc., denote the fibers over 0, and N_1^Z, U_1^Z, etc. the fibers over 1. Let $\rho_0 : N_0^Z \to Z$, $\rho_1 : N_1^Z \to Z$ and $\rho : N^Z \to Z \times \mathbb{P}^1$ denote the projections. Take an element $\tilde{\eta} \in \Omega^*(U^Z)_{U_0^Z + U_1^Z | N^Z}$ lifting $p_1^*(\eta)$. Then $U_1^Z(\tilde{\eta})$ lifts $(p_1^*(X \times 1))(\eta)$ and $U_0^Z(\tilde{\eta})$ lifts $(p_1^* N)(\eta)$, so

$$\begin{aligned} \rho_0^*(s_0^*(N(\eta))) &= N_0^Z(U_0^Z(\tilde{\eta})) \\ \rho_1^*(s_1^*((X \times 1)(\eta))) &= N_1^Z(U_1^Z(\tilde{\eta})). \end{aligned} \tag{6.13}$$

Here, we consider N_0^Z as a pseudo-divisor on U_0^Z and N_1^Z as a pseudo-divisor on U_1^Z.

Let $i_0 : U_0^Z \to U_Z$, $i_0^N : N_0^Z \to N^Z$ be the inclusions. We have $N_0^Z = i_0^* N^Z$ (as pseudo-divisors), hence by lemma 6.2.1(1)

$$i_{0*}^N(N_0^Z(U_0^Z(\tilde{\eta}))) = i_{0*}^N(i_0^* N^Z(U_0^Z(\tilde{\eta}))) = N^Z(i_{0*}(U_0^Z(\tilde{\eta}))).$$

Similarly, letting $i_1^N : N_1^Z \to N^Z$ be the inclusion, we have

$$i_{1*}^N(N_1^Z(U_1^Z(\tilde{\eta}))) = N^Z(i_{1*}(U_1^Z(\tilde{\eta}))).$$

Also, by proposition 6.3.4, we have $i_{0*}(U_0^Z(\tilde{\eta})) = i_{1*}(U_1^Z(\tilde{\eta}))$ in $\Omega^*(U^Z)$. Thus, in $\Omega^*(N^Z)$ we have

$$\begin{aligned} i_{0*}^N(N_0^Z(U_0^Z(\tilde{\eta}))) &= N^Z(i_{0*}(U_0^Z(\tilde{\eta}))) \\ &= N^Z(i_{1*}(U_1^Z(\tilde{\eta}))) \\ &= i_{1*}^N(N_1^Z(U_1^Z(\tilde{\eta}))) \end{aligned} \tag{6.14}$$

Let $i_0^Z : Z \to Z \times \mathbb{P}^1$, $i_1^Z : Z \to Z \times \mathbb{P}^1$ be the 0 and 1 sections, respectively. Since $\rho^* : \Omega^*(Z \times \mathbb{P}^1) \to \Omega^*(N^Z)$ is an isomorphism by the homotopy property theorem 3.6.3, (6.13) and (6.14) imply $i_{0*}^Z(s_0^*(N(\eta))) = i_{1*}^Z(s_1^*((X \times 1)(\eta)))$. Projecting to $\Omega_*(Z)$ by p_{1*} shows that $s_0^*(N(\eta)) = s_1^*((X \times 1)(\eta))$ in $\Omega^*(Z)$, as claimed.

Now take an element x of $\Omega^*(X)$, and let η be a lifting of $p_1^* x$ to $\Omega^*(U)_{N+X\times 1}$. Then $(X \times 1)(\eta) = x$, so $s_1^*((X \times 1)(\eta)) = (i \circ i')^*(x)$. On the other hand, letting $\mu_{Z'} : N \to Z'$ be the projection, we have $N(\eta) = \mu_{Z'}^* i'^*(x)$. By lemma 6.5.7, we have

$$s_0^*(N(\eta)) = s_0^*(\mu_{Z'}^* i'^*(x)) = i^*(i'^*(x)),$$

hence $i^*(i'^*(x)) = (i' \circ i)^*(x)$. □

6.5.4 L.c.i. pull-back

Let $f : X \to Y$ be an l.c.i. morphism in $\mathbf{Sch}_k$. By definition, there is a factorization $f = q \circ i$, with $i : X \to P$ a regular embedding and $q : P \to Y$ a smooth quasi-projective morphism.

Lemma 6.5.9. *Let $f : X \to Y$ be an l.c.i. morphism. If we have factorizations $f = q_1 \circ i_1 = q_2 \circ i_2$, with $i_j : X \to P_j$ regular embeddings and $q_j : P_j \to Y$ smooth and quasi-projective. then*

$$i_1^* \circ q_1^* = i_2^* \circ q_2^*.$$

Proof. Form the diagonal embedding $(i_1, i_2) : X \to P_1 \times_Y P_2$. By remark 5.1.2(2), (i_1, i_2) is a regular embedding. Form the cartesian diagram $(j = 1, 2)$

$$\begin{tikzcd} P^j \arrow[r, "i'_j"] \arrow[d, "q_j"'] & P_1 \times_Y P_2 \arrow[d, "p_j"] \\ X \arrow[r, "i_j"'] & P_j \end{tikzcd}$$

Then $p_j : P_1 \times_Y P_2 \to P_j$ is smooth and quasi-projective, so by proposition 6.5.4(2), we have

$$q_j^* \circ i_j^* = i_j'^* \circ p_j^*; \quad j = 1, 2.$$

Also, the map (i_1, i_2) induces a section $s_j : X \to P^j$ to q_j. Applying lemma 6.5.7 and theorem 6.5.8 gives

$$\begin{aligned} i_j^* &= s_j^* \circ q_j^* \circ i_j^* \\ &= s_j^* \circ i_j'^* \circ p_j^* \\ &= (i_1, i_2)^* \circ p_j^*. \end{aligned}$$

Let $q : P_1 \times_Y P_2 \to Y$ be the map $q_1 p_1 = q_2 p_2$. Using the functoriality of smooth pull-back, we have

$$\begin{aligned} i_j^* \circ q_j^* &= (i_1, i_2)^* \circ p_j^* \circ q_j^* \\ &= (i_1, i_2)^* \circ q^*. \end{aligned}$$

□

We may therefore make the following definition:

Definition 6.5.10. *Let $f : X \to Y$ be an l.c.i. morphism in* $\mathbf{Sch}_k$ *of relative dimension d. Define $f^* : \Omega_*(Y) \to \Omega_{*+d}(X)$ as $i^* \circ q^*$, where $f = q \circ i$ is a factorization of f with i a regular embedding and q smooth and quasi-projective.*

Theorem 6.5.11. *Let $f_1 : X \to Y$, $f_2 : Y \to Z$ be l.c.i. morphisms in* $\mathbf{Sch}_k$*. Then $(f_2 \circ f_1)^* = f_1^* \circ f_2^*$.*

Proof. As in remark 5.1.2, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[dr, "f_1"'] & P_1 \arrow[r, "i"] \arrow[d, "q_1"] & P \arrow[d, "q"] \\ & Y \arrow[r, "i_2"'] \arrow[dr, "f_2"'] & P_2 \arrow[d, "q_2"] \\ & & Z, \end{tikzcd}$$

with q_1, q_2 and q smooth and quasi-projective, i_1, i_2 and i regular embeddings, and the square cartesian. Using the functoriality of smooth pull-back, proposition 6.5.4(2) and theorem 6.5.8, we have

$$\begin{aligned}(f_2 \circ f_1)^* &= (ii_1)^* \circ (q_2 q)^* \\ &= i_1^* \circ i^* \circ q^* \circ q_2^* \\ &= i_1^* \circ q_1^* \circ i_2^* \circ q_2^* \\ &= f_1^* \circ f_2^*\end{aligned}$$

□

Theorem 6.5.12. *Let $f : X \to Z$, $g : Y \to Z$ be Tor-independent morphisms in $\mathbf{Sch}_k$, giving the cartesian diagram*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z. \end{tikzcd}$$

Suppose that f is an l.c.i. morphism and that g projective. Then

$$f^* \circ g_* = p_{1*} \circ p_2^*.$$

Proof. Since f and g are Tor-independent, p_2 is an l.c.i. morphism, so the statement makes sense.

Write $f = q \circ i$, with $q : P \to Z$ smooth and quasi-projective, and $i : X \to P$ a regular embedding. This gives us the diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "i\times\mathrm{Id}"] \arrow[d, "p_1"'] & P \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "i"'] & P \arrow[r, "q"'] & Z. \end{tikzcd}$$

with both squares cartesian. Using the functoriality of projective push-forward and theorem 6.5.11, it suffices to prove the case of f a regular embedding, or f a smooth quasi-projective morphism. The first case is proposition 6.5.4(1), the second follows easily from the definition of algebraic cobordism given in §2.4.

□

Proposition 6.5.13. *Let $f_i : X_i \to Y_i$, $i = 1, 2$ be l.c.i. morphisms in $\mathbf{Sch}_k$. Then for $\eta_i \in \Omega_*(Y_i)$, $i = 1, 2$, we have*

$$(f_1 \times f_2)^*(\eta_1 \times \eta_2) = f_1^*(\eta_1) \times f_2^*(\eta_2).$$

Proof. We first note that $f_1 \times f_2 : X_1 \times_k X_2 \to Y_1 \times Y_2$ is indeed an l.c.i. morphism: if $X_2 = Y_2$ and $f_2 = \mathrm{Id}$, this is clear, and we have the factorization $f_1 \times f_2 = (f_1 \times \mathrm{Id}) \circ (\mathrm{Id} \times f_2)$. Similarly, it suffices to prove the case $X_2 = Y_2$, $f_2 = \mathrm{Id}$.

We may assume η_2 is a cobordism cycle $(g : Z \to Y_2, L_1, \dots, L_r)$. Since both smooth pull-back and the Gysin morphism are compatible with the Chern class operators $\tilde{c}_1(L)$, we may assume that $r = 0$.

Note that $\eta_2 = g_*(\mathrm{Id}_Z)$, so $\eta_1 \times \eta_2 = (\mathrm{Id} \times g)_*(\eta_1 \times \mathrm{Id}_Z)$. Similarly, $f_1^*(\eta_1) \times \eta_2 = (\mathrm{Id} \times g)_*(f_1^*(\eta_1) \times \mathrm{Id}_{Z_2})$. Thus, using theorem 6.5.12, we may replace Y_2 with Z and η_2 with Id_Z, so it suffices to prove the result with $Y_2 \in \mathbf{Sm}_k$ and $\eta_2 = \mathrm{Id}_{Y_2}$.

In this case $\eta_1 \times \eta_2 = p^*(\eta_1)$, where $p : Y_1 \times Y_2 \to Y_1$ is the projection. Similarly, $f_1^*(\eta_1) \times \eta_2 = q^*(f_1^*(\eta_1))$, where $q : X_1 \times Y_2 \to X_1$ is the projection. Thus we need to show

$$(f_1 \times \mathrm{Id})^*(p^*(\eta_1)) = q^*(f_1^*(\eta_1)).$$

This follows from the functoriality of l.c.i. pull-back, theorem 6.5.11. □

6.6 Refined pull-back and refined intersections

Let $g : Y \to X$ be a morphism in $\mathbf{Sch}_k$, $f : Z \to X$ an l.c.i. morphism in $\mathbf{Sch}_k$ of relative codimension d and $W := Z \times_X Y$ the fiber product. Following Fulton [9], we define the *refined pull-back*

$$f^! : \Omega_*(Y) \to \Omega_{*-d}(W).$$

The construction and the proofs of all the basic properties of $f^!$ are taken from Fulton [9, Chapter 6] with minor modifications; we will therefore be somewhat sketchy with the arguments, concentrating on the main ideas and the places where the arguments need modification.

In this section, we will abuse notation slightly: if $i : Z \to Y$ is a regular embedding of codimension one (i.e., Z is a Cartier divisor on Y), and α is in $\Omega_*(Y)$ we write $Z(\alpha)$ for $i_Z^*(\alpha) \in \Omega_{*-1}(Z)$.

We call a diagram in $\mathbf{Sch}_k$ consisting of squares a *fiber diagram* if all squares commute and are cartesian.

We remind the reader that we are assuming throughout this section that k is a field admitting resolution of singularities.

6.6.1 Normal cone and normal bundle

For $i' : W \to Y$ a closed immersion with ideal sheaf $\mathcal{I}$, we have the *normal cone* of i', namely, the W-scheme

$$C_W Y := \mathrm{Spec}_{\mathcal{O}_W}(\oplus_{n=0}^{\infty} \mathcal{I}^n / \mathcal{I}^{n+1}).$$

If $\mu : M_W Y \to Y \times \mathbb{P}^1$ is the blow-up of $Y \times \mathbb{P}^1$ along W with exceptional divisor E_W, then

$$C_W Y \cong E_W \setminus \langle Y \times 0 \rangle;$$

in particular, $C_W Y$ is a Cartier divisor on $U_W Y := M_W Y \setminus \langle Y \times 0 \rangle$, and we have the following generalization of the deformation diagram (6.10)

$$\begin{tikzcd}
C_W Y \arrow[rrr, "i_C"] \arrow[dd, "\mu_C"'] & & & U_W Y \arrow[dd, "\mu"] & U_W Y \setminus C_W Y \arrow[l, "j_W"'] \\
& \langle W \times \mathbb{P}^1 \rangle_0 \arrow[ul, "s"] \arrow[r, "\tilde{i}_0"'] \arrow[dl, equal] & \langle W \times \mathbb{P}^1 \rangle \arrow[urr, "\tilde{i}"] \arrow[d, equal] & & Y \arrow[ul, "i_1^Y"] \arrow[u] \arrow[d, equal] \\
W \times 0 \arrow[rr, "i_0"'] & & W \times \mathbb{P}^1 \arrow[r, "i"'] & Y \times \mathbb{P}^1 & Y \times 1 \arrow[l]
\end{tikzcd} \tag{6.15}$$

In case i' is a regular embedding, then $C_W Y$ is just the normal bundle $N_W Y$, and the diagram (6.15) is just (6.10) with $i' : W \to Y$ replacing $i : Z \to X$.

Now suppose we are given a cartesian square in $\mathbf{Sch}_k$:

$$\begin{tikzcd}
W \arrow[r, "i'"] \arrow[d, "q"'] & Y \arrow[d, "g"] \\
Z \arrow[r, "i"'] & X
\end{tikzcd} \tag{6.16}$$

with i a regular embedding. We have the surjections $q^*\mathcal{I}_Z^n \to \mathcal{I}_W^n$, which give the closed immersion

$$i_{W/Z} : C_W Y \to q^* N_Z X.$$

Let $\mu_{q^*N} : q^* N_Z X \to W$ be the structure morphism. The cartesian square (6.16) defines a morphism Ψ of the diagram (6.15) to the diagram (6.10).

6.6.2 Refined Gysin morphism

Suppose we have a cartesian square (6.16) with i a regular embedding of pure codimension d. The facts that allow us to define the pull-back for i have their direct analogs for $i' : W \to Y$:

1. The sequence

$$\Omega_{*+1}(C_W Y) \xrightarrow{i_{C*}} \Omega_{*+1}(U_W Y) \xrightarrow{j_W^*} \Omega_{*+1}(U_W Y \setminus C_W Y) \to 0$$

   is exact.
2. $C_W Y$ is a Cartier divisor on $U_W Y$ and $i_C^* \circ i_{C*} = 0$.
3. $\mu_{q^*N} : q^* N_Z X \to W$ induces an isomorphism

$$\mu_{q^*N}^* : \Omega_{*-d}(W) \to \Omega_*(q^* N_Z X).$$

The proofs are exactly the same as in §6.5, and are left to the reader.

We thus have the *extended zig-zag diagram*:

$$\begin{tikzcd}
\Omega_*(Y) \arrow[r, "p_1^*"] & \Omega_{*+1}(Y \times (\mathbb{P}^1 \setminus 0)) \arrow[d, "\mu^*", "\sim"'] & \\
& \Omega_{*+1}(U_W Y \setminus C_W Y) & \Omega_{*+1}(U_W Y) \arrow[l, "j_W^*"'] \arrow[d, "i_C^*"] \\
\Omega_{*-d}(W) \arrow[r, "\mu_{q^* N}^*"'] & \Omega_*(q^* N_Z X) & \Omega_*(C_W Y) \arrow[l, "i_{W/Z*}"]
\end{tikzcd} \tag{6.17}$$

and the composition

$$\Omega_*(Y) \xrightarrow{\mu^* \circ p_1^*} \Omega_{*+1}(U_W Y \setminus C_W Y)$$
$$\xrightarrow{i_C^* \circ (j_W^*)^{-1}} \Omega_*(C_W Y) \xrightarrow{(\mu_{q^* N}^*)^{-1} \circ i_{W/Z*}} \Omega_{*-d}(W)$$

gives a well-defined homomorphism

$$i_{i'}^! : \Omega_*(Y) \to \Omega_{*-d}(W).$$

We sometimes write $i^!$ for $i_{i'}^!$ when the context makes the meaning clear.

Remark 6.6.1. In the situation considered above, the zero-section $s : W \to q^* N$ is a regular embedding, so $s^* : \Omega_*(q^* N) \to \Omega_{*-d}(W)$ is defined, and is in fact the inverse to $\mu_{q^* N}^*$ (corollary 6.5.5(3)). Thus for a given $\alpha \in \Omega_*(Y)$ we can describe $i_{i'}^!(\alpha)$ as follows: choose an element $\tilde{\alpha} \in \Omega_{*+1}(U_W Y)$ with $j_W^*(\tilde{\alpha}) = \mu^* \circ p_1^*(\alpha)$. Then

$$i_{i'}^!(\alpha) = s^*\big(i_{W/Z*}(C_W Y(\tilde{\alpha}))\big).$$

6.6.3 Properties of the refined Gysin morphism

The results and proofs of §6.5 on the Gysin morphisms i^* extend with only minor changes to the refined Gysin morphisms $i^!$. We give a sketch of the arguments, indicating the points one needs to change. In what follows, $i : Z \to X$ will be a regular embedding of codimension d and

$$\begin{tikzcd}
W \arrow[r, "i'"] \arrow[d, "q"'] & Y \arrow[d, "g"] \\
Z \arrow[r, "i"'] & X
\end{tikzcd}$$

is a cartesian square.

Lemma 6.6.2. *Suppose we have a fiber diagram*

$$\begin{tikzcd} W' \arrow[r, "i''"] \arrow[d, "q'"'] & Y' \arrow[d, "g'"] \\ W \arrow[r, "i'"] \arrow[d, "q"'] & Y \arrow[d, "g"] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$

Suppose that i and g are Tor-independent. Then i' is a regular embedding and $i^!_{i''} = i'^!_{i''}$. As a particular case ($g' = \mathrm{Id}$) we have $i^!_{i'} = i'^*$.

Proof. Since i and g are Tor-independent, i' is a regular embedding, and $q^*N_ZX \cong N_WY \cong C_WY$. The result follows directly from this. □

The proof of the following result is essentially the same as that of proposition 6.5.4 and is left to the reader.

Proposition 6.6.3. *Let $g' : Y' \to Y$, be a morphism in $\mathbf{Sch}_k$, giving the fiber diagram*

$$\begin{tikzcd} W' \arrow[r, "i''"] \arrow[d, "q'"'] & Y' \arrow[d, "g'"] \\ W \arrow[r, "i'"] \arrow[d, "q"'] & Y \arrow[d, "g"] \\ Z \arrow[r, "i"'] & X. \end{tikzcd}$$

1. *If g' is projective, then $i^!_{i'} \circ g'_* = q'_* \circ i^!_{i''}$.*
2. *If g' is smooth and quasi-projective, then $i^!_{i''} \circ g'^* = q'^* \circ i^!_{i'}$.*

Lemma 6.6.4. *Let*

$$\begin{tikzcd} W \arrow[r, "i'"] \arrow[d, "q"'] & Y \arrow[d, "g"] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$

be a cartesian diagram in $\mathbf{Sch}_k$, with i a regular embedding of codimension one, i.e., Z is a Cartier divisor on X. Then we have the pseudo-divisor g^*Z on Y with support W and

$$i^!(\alpha) = g^*Z(\alpha) \in \Omega_{*-1}(W)$$

for all $\alpha \in \Omega_*(Y)$.

Proof. Let $h : \tilde{Y} \to Y$ be a projective morphism and suppose there is an element $\tilde{\alpha} \in \Omega_*(\tilde{Y})$ with $h_*(\tilde{\alpha}) = \alpha$. By proposition 6.6.3, we may replace (Y, α) with $(\tilde{Y}, \tilde{\alpha})$. Since $\Omega_*(Y)$ is generated by elements $h_*(1_{\tilde{Y}})$ with $\tilde{Y}$ irreducible and in $\mathbf{Sm}_k$, we may assume that Y is irreducible and in $\mathbf{Sm}_k$ and that $\alpha = 1_Y$. Let $s : Y \to g^*N_ZX$ be the zero-section.

If $g(Y) \subset i(Z)$, then $C_WY = Y = W$, $i_{W/Z} = s$. Using the notation of remark 6.6.1 we may take $\tilde{\alpha} = 1_{U_WY}$, and we have

$$i^!(1_Y) = s^*(s_*(Y(1_{U_WY}))) = s^*(s_*(1_Y)) = \tilde{c}_1(g^*N_ZX)(1_Y).$$

Since $N_ZX = O_X(Z)$, this agrees with $g^*Z(1_Y)$, proving the result in this case.

If $g(Y)$ is not contained in Z, then $W = g^*Z$ is a Cartier divisor on Y. Thus g and i are Tor-independent and by lemma 6.6.2

$$i^!(1_Y) = i'^*(1_Y).$$

By lemma 6.5.6, $i'^*(1_Y) = g^*Z(1_Y)$, which completes the proof. $\square$

Theorem 6.6.5. *Let $i_2 : Z_2 \to Z_1$, $i_1 : Z_1 \to X$ be regular embeddings, giving the fiber diagram*

$$\begin{tikzcd} W_2 \arrow[r, "i'_2"] \arrow[d, "q_2"'] & W_1 \arrow[r, "i'_1"] \arrow[d, "q_1"'] & Y \arrow[d, "g"] \\ Z_2 \arrow[r, "i_2"'] & Z_1 \arrow[r, "i_1"'] & X \end{tikzcd}$$

Then $(i'_1 \circ i_2)^! = i_2^! \circ i_1^!$.

Proof. We first reduce to the case in which X is a vector bundle $E \to Z_1$ with i_1 the zero-section, $Y = q_1^*E$ and i'_1 is the zero-section.

For this, we form the deformation diagram over $\mathbb{P}^1 \times \mathbb{P}^1$ as in the proof of theorem 6.5.8: We start with the deformation diagram $U_{Z_1}X \to \mathbb{P}^1$ for i_1, and then form the deformation diagram $U_{Z_2 \times \mathbb{P}^1}U_{Z_1} \to \mathbb{P}^1$ for $Z_2 \times \mathbb{P}^1 \to U_{Z_1}X$. Forming the deformation diagram $U_{W_1}Y$ for i'_1 gives us the cartesian diagram

$$\begin{tikzcd} W_2 \times \mathbb{P}^1 \arrow[r, "\tilde{i}'_2"] \arrow[d] & W_1 \times \mathbb{P}^1 \arrow[r, "\tilde{i}'_1"] \arrow[d] & U_{W_1}Y \arrow[d, "\tilde{g}"] \\ Z_2 \times \mathbb{P}^1 \arrow[r, "\tilde{i}_2"'] & Z_1 \times \mathbb{P}^1 \arrow[r, "\tilde{i}_1"'] & U_{Z_1}X. \end{tikzcd} \tag{6.18}$$

Forming the deformation diagrams for $\tilde{i}'_1 \circ \tilde{i}'_2$ and $\tilde{i}_1 \circ \tilde{i}_1$ yields the cartesian diagram

$$\begin{tikzcd} W_2 \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r] \arrow[d] & U_{W_2 \times \mathbb{P}^1}U_{W_1}Y \arrow[d, "\tilde{g}"] \\ Z_2 \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r] & U_{Z_2 \times \mathbb{P}^1}U_{Z_1}X \end{tikzcd} \tag{6.19}$$

with a projection to $\mathbb{P}^1 \times \mathbb{P}^1$. The order of the product is chosen so that the fiber of (6.19) over $\mathbb{P}^1 \times 1$ is the diagram (6.18) (with the middle column omitted) and the fiber over $\mathbb{P}^1 \times 0$ is

$$\begin{tikzcd} W_2 \times \mathbb{P}^1 \arrow[r] \arrow[d, "Q"'] & C_{W_2\times\mathbb{P}^1} U_{W_1} Y \arrow[d, "\tilde{g}"] \\ Z_2 \times \mathbb{P}^1 \arrow[r] & N_{Z_2\times\mathbb{P}^1} U_{Z_1} X \end{tikzcd} \tag{6.20}$$

The fiber of (6.18) over $0 \in \mathbb{P}^1$ is

$$\begin{tikzcd} W_2 \arrow[r, "i_2'"] \arrow[d] & W_1 \arrow[r, "s'"] \arrow[d] & C_{W_1} Y \arrow[d, "\tilde{g}"] \\ Z_2 \arrow[r, "i_2"'] & Z_1 \arrow[r, "s"'] & N_{Z_1} X. \end{tikzcd}$$

and the fiber of (6.18) over $1 \in \mathbb{P}^1$ is the original diagram. Arguing as in the proof of theorem 6.5.8 proves the following statement: for $\alpha \in \Omega_{*+1}(U_{W_1} Y)$,

$$s^!(C_{W_1} Y(\alpha)) = i_1^!(X \times 1(\alpha))$$
$$(s \circ i_2)^!(C_{W_1} Y(\alpha)) = (i_1 \circ i_2)^!(X \times 1(\alpha))$$

This reduces us to proving the result with X replaced by $N_{Z_1} X$, and Y replaced by $C_{W_1} Y$:

$$\begin{tikzcd} W_2 \arrow[r, "i_2'"] \arrow[d, "q_2"'] & W_1 \arrow[r, "s'"] \arrow[d, "q_1"'] & C_{W_1} Y \arrow[d, "g'"] \\ Z_2 \arrow[r, "i_2"'] & Z_1 \arrow[r, "s"'] & N_{Z_1} X \end{tikzcd}$$

with s and s' the zero-sections. Letting $f : C_{W_1} Y \to q_1^* N_{Z_1} X$ be the canonical closed immersion, we have the fiber diagram

$$\begin{tikzcd} W_2 \arrow[r, "i_2'"] \arrow[d, equal] & W_1 \arrow[r, "s'"] \arrow[d, equal] & C_{W_1} Y \arrow[d, "f"] \\ W_2 \arrow[r, "i_2'"] \arrow[d, "q_2"'] & W_1 \arrow[r, "s''"] \arrow[d, "q_1"'] & q_1^* N_{Z_1} X \arrow[d, "g''"] \\ Z_2 \arrow[r, "i_2"'] & Z_1 \arrow[r, "s"'] & N_{Z_1} X, \end{tikzcd}$$

with s'' the zero-section. Using proposition 6.6.3(1), this reduces us to the case $Y = q_1^* N_{Z_1} X$, i_1 the zero-section, as desired.

In this case, we have the isomorphism (see [9, proof of theorem 6.5])

$$C_{W_2}Y \cong C_{W_2}W_1 \times_{W_2} (i_2 \circ q_2)^* N_{Z_1}X,$$

and similarly $U_{W_2}Y \cong U_{W_2}W_1 \times_{W_1} q_1^* N_{Z_1}X$.

Since the square

$$\begin{tikzcd}
W_1 \arrow[r, "i_1'"] \arrow[d] & Y \arrow[d] \\
Z_1 \arrow[r, "i_1"'] & X
\end{tikzcd}$$

is Tor-independent, $i^! = i_1'^*$, hence $i_1^! : \Omega_*(Y) \to \Omega_{*-d}(W_1)$ is the inverse to $p^* : \Omega_{*-d}(W_1) \to \Omega_*(q_1^* N_{Z_1}X)$.

We have the fiber diagram

$$\begin{tikzcd}
(i_1 \circ q_2)^* N_{Z_1}X \arrow[r, "i'"] \arrow[d, "q"'] & Y \arrow[d, "p"] \\
W_2 \arrow[r, "i_2'"'] \arrow[d] & W_1 \arrow[d] \\
Z_2 \arrow[r, "i_2"'] & Z_1
\end{tikzcd}$$

which identifies $(i_1 \circ i_2)^! : \Omega_*(Y) \to \Omega_{*-d-e}(W_2)$ with the composition of

$$(i_2)^!_{i'} : \Omega_*(Y) \to \Omega_{*-e}((i_1 \circ q_2)^* N_{Z_1}X)$$

followed by the inverse of the pull-back isomorphism

$$q^* : \Omega_{*-d-e}(W_2) \to \Omega_{*-e}((i_1 \circ q_2)^* N_{Z_1}X).$$

Since $q^* \circ (i_2)^!_{i_2'} = (i_2)^!_{i'} \circ p^*$, we have

$$(i_1 \circ i_2)^! = (q^*)^{-1} \circ (i_2)^!_{i'} = (i_2)^!_{i_2'} \circ (p^*)^{-1} = i_2^! \circ i_1^!,$$

as desired. □

6.6.4 Refined pull-back for l.c.i. morphisms

The same program we used for extending the Gysin morphism i^* for a regular embedding to the pull-back f^* for an l.c.i. morphism gives an extension of the refined Gysin morphism $i^!$ to a refined pull-back $f^!$ for an l.c.i. morphism. Indeed, let

$$
\begin{tikzcd}
W \arrow[r, "f'"] \arrow[d, "q"'] & Y \arrow[d, "g"] \\
Z \arrow[r, "f"'] & X
\end{tikzcd}
\tag{6.21}
$$

be a cartesian square in $\mathbf{Sch}_k$, with f an l.c.i. morphism of relative codimension d. Factor f as $f = p \circ i$, with $i : Z \to P$ a regular embedding of codimension $d + e$ and $p : P \to X$ smooth and quasi-projective of relative dimension e. This gives us the fiber diagram

$$
\begin{tikzcd}
W \arrow[r, "i'"] \arrow[d, "q"'] & T \arrow[r, "p'"] \arrow[d] & Y \arrow[d, "g"] \\
Z \arrow[r, "i"'] & P \arrow[r, "p"'] & X.
\end{tikzcd}
$$

Define

$$f^!_{f'} : \Omega_*(Y) \to \Omega_{*-d}(W)$$

to be the composition

$$\Omega_*(Y) \xrightarrow{p'^*} \Omega_{*+e}(T) \xrightarrow{i^!_{i'}} \Omega_{*-d}W).$$

The basic properties of $f^!$ are essentially the same as for f^*. We list these results in the following omnibus theorem; the proofs are also essentially the same as for f^*, and we leave the details to the reader.

Theorem 6.6.6. *(1) Let*

$$
\begin{tikzcd}
W \arrow[r, "f'"] \arrow[d, "q"'] & Y \arrow[d, "g"] \\
Z \arrow[r, "f"'] & X
\end{tikzcd}
$$

be a cartesian square in $\mathbf{Sch}_k$*, with f an l.c.i. morphism of relative codimension d.*

(a) $f^!_{f'} : \Omega_*(Y) \to \Omega_{*-d}(W)$ is independent of the choice of factorization of f.
(b) If f and g are Tor-independent, then $f^!_{f'} = f'^*$.
(c) Let E be a vector bundle on Y. Then $f^!_{f'} \circ \tilde{c}_n(E) = \tilde{c}_n(f'^*E) \circ f^!_{f'}$

(2) Let

$$\begin{tikzcd} W' \arrow[r, "f''"] \arrow[d, "q'"'] & Y' \arrow[d, "g'"] \\ W \arrow[r, "f'"] \arrow[d, "q"'] & Y \arrow[d, "g"] \\ Z \arrow[r, "f"'] & X \end{tikzcd}$$

a fiber diagram in $\mathbf{Sch}_k$*, with* f *an l.c.i. morphism.*

(a) if g' *is projective, then* $f^!_{f'} \circ g'_* = q'_* \circ f^!_{f''}$.
(b) if g' *is smooth and quasi-projective, then* $q'^* \circ f^!_{f'} = f^!_{f''} \circ g'^*$.

(3) Let

$$\begin{tikzcd} W_2 \arrow[r, "f'_2"] \arrow[d, "q_2"'] & W_1 \arrow[r, "f'_1"] \arrow[d, "q_1"'] & Y \arrow[d, "g"] \\ Z_2 \arrow[r, "f_2"'] & Z_1 \arrow[r, "f_1"'] & X \end{tikzcd}$$

be a fiber diagram in $\mathbf{Sch}_k$*, with* f_1 *and* f_2 *l.c.i. morphisms. Then* $(f_1 \circ f_2)^! = f_2^! \circ f_1^!$.

(4) For $i = 1, 2$*, let*

$$\begin{tikzcd} W_i \arrow[r, "f'_i"] \arrow[d, "q_i"'] & Y_i \arrow[d, "g_i"] \\ Z_i \arrow[r, "f_i"'] & X_i \end{tikzcd}$$

be a cartesian square in $\mathbf{Sch}_k$*, with* f_i *an l.c.i. morphism of relative codimension* d_i*. For* $\alpha_i \in \Omega_*(Y_i)$*, we have*

$$(f_1 \times f_2)^!(\alpha_1 \times \alpha_2) = f_1^!(\alpha_1) \times f_2^!(\alpha_2).$$

6.6.5 Excess intersection formula

We show how Fulton's excess intersection formula [9, Theorem 6.3] extends to algebraic cobordism.

Lemma 6.6.7. *Let* $p : E \to X$ *be a vector bundle of rank* d *on some* $X \in \mathbf{Sch}_k$*. Suppose that* E *has a section* $s : X \to E$ *such that the zero-subscheme of* s*,* $i : Z \to X$ *is a regularly embedded closed subscheme of codimension* d*. Then*

$$\tilde{c}_d(E) = i_* \circ i^*.$$

Proof. The case $d = 1$ is lemma 5.1.11. We may use the splitting principle (remark 4.1.2) to reduce to the case $E = \oplus_{i=1}^{d} L_i$, with the L_i line bundles on X; the Whitney product formula gives us the identity

$$\tilde{c}_d(E) = \prod_{i=1}^{d} \tilde{c}_1(L_i).$$

The homotopy property allows us to prove the result for any section s such that the zero-subscheme of s is a regularly embedded subscheme, as in the proof of lemma 5.1.11. We may use the homotopy property again to pull back to E and replace s with the tautological section of p^*E. In this case, the section of $\oplus_{i=1}^{r} q^*L_i$ induced by the tautological section q^*E and the surjection $q^*E \to \oplus_{i=1}^{r} q^*L_i$ has zero-subscheme which is a codimension r regularly embedded closed subscheme, so we may use induction and the case $d = 1$ to finish the result. □

Lemma 6.6.8. *Let $0 \to E' \xrightarrow{i} E \to E'' \to 0$ be an exact sequence of vector bundles on some $X \in \mathbf{Sch}_k$, let d be the rank of E''. Let $s : X \to E$, $s' : X \to E'$ be the zero-sections. Then*

$$s^* \circ i_* = \tilde{c}_d(E'') \circ s'^*.$$

Proof. Let $p : E \to X$ be the structure morphism. The closed subscheme $i(E')$ is the zero-subscheme of canonical section of p^*E'' induced by the surjection $E \to E''$. Also, $s = i \circ s'$, so

$$\begin{aligned} s^* \circ i_* &= s'^* \circ i^* \circ i_* \\ &= s'^* \circ \tilde{c}_d(p^*E'') \\ &= \tilde{c}_d(E'') \circ s'^*, \end{aligned}$$

using lemma 6.6.7. □

We can now state and prove the excess intersection formula; we take the formulation and proof from Fulton [9, Theorem 6.3].

Start with a fiber diagram in $\mathbf{Sch}_k$:

$$\begin{array}{ccc} W & \xrightarrow{i''} & Y \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle g'} \\ Z' & \xrightarrow{i'} & X' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[i]{} & X \end{array}$$

such that i and i' are both regular embeddings, with i of codimension d and i' of codimension d'. Write $N := N_Z X$, $N' := N_{Z'} X'$. The bottom square gives the inclusion $i_{Z'/Z} : N' \to q^*N$; we define E to be the quotient bundle q^*N/N'. E is of rank $e := d - d'$. E is called the *excess normal bundle* for (i, i').

Theorem 6.6.9 (Excess intersection formula).

$$i^!_{i''} = \tilde{c}_e(q'^*E) \circ i'^!_{i''}.$$

Proof. Form the deformation diagram (6.15) for $i'' : W \to Y$. For $\alpha \in \Omega_*(Y)$, let $\tilde{\alpha} \in \Omega_{*+1}(U_W Y)$ be an element restricting to $p_1^*\alpha$ on $U_W Y \setminus C_W Y$. We have the closed immersions

$$C_W Y \xrightarrow{i_{W/Z'}} q'^*N'$$

$$C_W Y \xrightarrow{i_{W/Z}} (q \circ q')^*N$$

$$N_{Z'}X' \xrightarrow{i_{Z'/Z}} q^*N$$

with

$$i_{W/Z} = q'^*(i_{Z'/Z}) \circ i_{W'/Z'}.$$

Let $s : W \to (q \circ q')^*N$, $s' : W \to q'^*N'$ be the zero sections. Then

$$\begin{aligned} i'^!(\alpha) &= s'^*\big(i_{W'/Z'*}(C_W Y(\tilde{\alpha}))\big) \\ i^!(\alpha) &= s^*\big(i_{W/Z*}(C_W Y(\tilde{\alpha}))\big) \\ &= s^* \circ (q'^*(i_{Z'/Z}))_*\big(i_{W'/Z'*}(C_W Y(\tilde{\alpha}))\big) \end{aligned}$$

and the result thus follows from lemma 6.6.8. □

6.6.6 Commutativity of refined pull-backs

We consider a fiber diagram

$$\begin{array}{ccccc} W' & \xrightarrow{f''} & Y' & \longrightarrow & Z' \\ \downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{f'} & Y & \longrightarrow & X' \\ {\scriptstyle q}\downarrow & & \downarrow & & \\ Z & \xrightarrow[f]{} & X & & \end{array}$$

with f and g l.c.i. morphisms of relative codimension d and e, respectively.

Theorem 6.6.10. $f^!g^! = g^!f^! : \Omega_*(Y) \to \Omega_{*-d-e}(W')$.

Proof. The result is easy to prove if either f or g is smooth and quasi-projective; we may therefore assume that f and g are both regular embeddings; in this case all the maps in the diagram are closed immersions.

Take $\alpha \in \Omega_*(Y)$. Suppose we have a projective morphism $h : \tilde{Y} \to Y$ and an element $\tilde{\alpha} \in \Omega_*(\tilde{Y})$. Using proposition 6.6.3, we may replace Y with $\tilde{Y}$ and α with $\tilde{\alpha}$. Since $\Omega_*(Y)$ is generated by elements of the form $h_*(1_{\tilde{Y}})$ with $\tilde{Y}$ irreducible and in $\mathbf{Sm}_k$, we may assume that Y is irreducible and in $\mathbf{Sm}_k$. We proceed by induction on $\dim_k Y$.

If now $h : \tilde{Y} \to Y$ is a blow-up of Y along a smooth center F, the localization theorem for $i : F \to Y$ and $h^{-1}(F) \to \tilde{Y}$ imply that the map

$$\Omega_*(\tilde{Y}) \oplus \Omega_*(F) \xrightarrow{(h_*, i_*)} \Omega_*(Y)$$

is surjective. By our induction hypothesis, we may replace Y with $\tilde{Y}$ and assume that $\alpha = 1_Y$. In particular, we may replace Y with any sequence of blow-ups of Y along smooth centers, so we may assume that either W is a Cartier divisor on Y or that $W = Y$. Similarly, we may assume that either Y' is a Cartier divisor on Y or that $Y' = Y$. In case both W and Y' are Cartier divisors on Y, we may blow-up further and assume that $W + Y'$ is a strict normal crossing divisor on Y. In particular, both f' and p are regular embeddings, so we may apply the excess intersection theorem 6.6.9. Let E be the excess normal bundle for (f, f') and F the excess bundle for (g, p).

Suppose that $W = Y$. Then $f^!_{f'} = \tilde{c}_d(E)$, $f^!_{f''} = \tilde{c}_d(p^*E)$ and the excess intersection formula gives

$$\begin{aligned} g^! f^! &= g^! \circ \tilde{c}_d(E) \\ &= \tilde{c}_d(p^*E) \circ g^!_p \\ &= f^! g^! \end{aligned}$$

Similarly, $f^! g^! = g^! f^!$ if $Y' = Y$. Thus we may assume that W and Y' are both Cartier divisors on Y.

In this case, W and Y' determine pseudo-divisors p^*W on Y' and f'^*Y' on W, respectively. By lemma 6.6.2 and lemma 6.6.4, we have

$$\begin{aligned} g^! f^!(\alpha) &= f'^*Y'(W(\alpha)) \\ f^! g^!(\alpha) &= p^*W(Y'(\alpha)). \end{aligned}$$

Since $f'^*Y'(W(\alpha)) = p^*W(Y'(\alpha))$ by proposition 6.3.3, the proof is complete. ∎

6.6.7 Refined intersections

Take $Z \in \mathbf{Sm}_k$ of dimension d over k, so the diagonal $\delta : Z \to Z \times_k Z$ is a regular embedding of codimension d. Suppose we have $X, Y \in \mathbf{Sch}_k$ and projective morphisms $f : X \to Z$, $g : Y \to Z$. This gives the cartesian diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\delta'"] \arrow[d, "(f,g)"'] & X \times_k Y \arrow[d, "f \times g"] \\ Z \arrow[r, "\delta"'] & Z \times_k Z \end{tikzcd}$$

For $\alpha \in \Omega_n(X)$, $\beta \in \Omega_m(Y)$, define

$$\alpha \cdot_Z \beta \in \Omega_{n+m-d}(X \times_Z Y)$$

by

$$\alpha \cdot_Z \beta := \delta^!(\alpha \times \beta).$$

The following are easy consequences of the properties of refined pull-back:

1. Let $\tau : Y \times_Z X \to X \times_Z Y$ be the symmetry. Then

$$\tau^*(\alpha \cdot_Z \beta) = \beta \cdot_Z \alpha.$$

If we have $h : W \to Z$ and $\gamma \in \Omega_*(W)$, then

$$(\alpha \cdot_Z \beta) \cdot_Z \gamma = \alpha \cdot_Z (\beta \cdot_Z \gamma).$$

2. Given projective morphisms $f' : X' \to X$, $g' : Y' \to Y$ and $\alpha' \in \Omega_*(X')$, $\beta' \in \Omega_*(Y')$, then

$$(f', g')_*(\alpha' \cdot_Z \beta') = f'_*(\alpha) \cdot_Z g'_*(\beta).$$

In particular,

$$(f, g)_*(\alpha \cdot_Z \beta) = f_*(\alpha) \cup g_*(\beta).$$

3. Let $p : Z' \to Z$ be a morphism in $\mathbf{Sm}_k$. Then

$$p^!(\alpha) \cdot_{Z'} p^!(\beta) = p^!(\alpha \cdot_Z \beta).$$

7

The universality of algebraic cobordism

In this chapter, we complete our program, finishing the proofs of the results stated in chapter 1. We also extend a number of our results on oriented cohomology of smooth schemes to oriented Borel-Moore homology of l.c.i. schemes.

7.1 Statement of results

The main aim of this chapter is to establish the following results:

Theorem 7.1.1. *Let k be a field admitting resolution of singularities. Let $\mathcal{V}$ be an l.c.i.-closed admissible subcategory of* $\mathbf{Sch}_k$. *Then the oriented Borel-Moore weak homology theory*

$$X \mapsto \Omega_*(X)$$

on $\mathcal{V}$ admits one and only one structure of an oriented Borel-Moore homology theory on $\mathcal{V}$, which we still denote Ω_*.

Theorem 7.1.2. *Let k be a field admitting resolution of singularities. Then the oriented Borel-Moore weak homology theory Ω_* on* $\mathbf{Sm}_k$ *admits one and only one structure of an oriented cohomology theory Ω^* on* $\mathbf{Sm}_k$.

We will also show

Theorem 7.1.3. *Assume k admits resolution of singularities.*
(1) Let $\mathcal{V}$ be an l.c.i.-closed admissible subcategory of $\mathbf{Sch}_k$. *Then algebraic cobordism, $X \mapsto \Omega_*(X)$, is the universal oriented Borel-Moore homology theory on $\mathcal{V}$.*
(2) Algebraic cobordism, considered as an oriented cohomology theory on $\mathbf{Sm}_k$, *is the universal oriented cohomology theory on* $\mathbf{Sm}_k$.

These results complete the proof of theorem 1.2.6. Similarly, we have so far proved theorem 1.2.18 and theorem 1.2.19 only for the underlying Borel-Moore weak homology theories (see theorem 4.2.10 and theorem 4.5.1); theorem 7.1.3(2) completes the proofs of these results for the respective oriented cohomology theories on $\mathbf{Sm}_k$. For the sake of completeness, we state these results here, in their full generality:

Let $\mathbb{L}^* \to \mathbb{Z}$ and $\mathbb{L}^* \to \mathbb{Z}[\beta, \beta^{-1}]$ be the homomorphisms classifying the additive formal group law $F_a(u, v) = u + v$ and the multiplicative periodic formal group law $F_m(u, v) = u + v - \beta uv$.

Theorem 7.1.4. *Assume k admits resolution of singularities.*
(1) The canonical morphism $\Omega^* \to K^0[\beta, \beta^{-1}]$ of oriented cohomology theories on $\mathbf{Sm}_k$ induces an isomorphism of such theories

$$\Omega_* \otimes_{\mathbb{L}^*} \mathbb{Z}[\beta, \beta^{-1}] \to K^0[\beta, \beta^{-1}].$$

(2) Suppose that k has characteristic zero. Then the canonical morphism $\Omega_* \to \mathrm{CH}_*$ of oriented Borel-Moore homology theories on $\mathbf{Sch}_k$ induces an isomorphism of such theories

$$\Omega_* \otimes_{\mathbb{L}^*} \mathbb{Z} \to \mathrm{CH}_*.$$

In particular, this isomorphism restricts to an isomorphism of oriented cohomology theories on $\mathbf{Sm}_k$.

Theorem 1.2.2 is a direct consequence of theorem 7.1.3(2) and theorem 7.1.4(2). Indeed, theorem 7.1.3(2) shows that $\Omega_* \otimes_{\mathbb{L}^*} \mathbb{Z}$ is the universal ordinary oriented cohomology theory on $\mathbf{Sm}_k$, and thus the isomorphism $\Omega_* \otimes_{\mathbb{L}^*} \mathbb{Z} \to \mathrm{CH}^*$ of theorem 7.1.4 yields theorem 1.2.2.

Some of the degree formulas of §4.4 can be improved, if one considers a Borel-Moore homology theory instead of a weak homology theory. For example, one can replace "smooth" with "l.c.i." in theorem 4.4.7, yielding the following result:

Theorem 7.1.5. *Let k be a field. Let A_* be an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$. Assume A_* is generically constant and has the localization property.*

Let X be a reduced finite type k-scheme. Assume that, for each closed integral subscheme $Z \subset X$, we are given a projective birational morphism $\tilde{Z} \to Z$ with $\tilde{Z}$ reduced and in $\mathbf{Lci}_k$. Let $X_1, \ldots, X_r$ be the irreducible components of X, and let α be in $A_*(X)$. Then for each integral closed subscheme $Z \subset X$ with $\mathrm{codim}_X Z > 0$, there is an element $\omega_Z \in A_{*-\dim_k Z}(k)$, all but finitely many being zero, such that

$$\alpha - \sum_{i=1}^{r} \deg_i(\alpha) \cdot [\tilde{X}_i \to X] = \sum_{Z,\ \mathrm{codim}_X Z > 0} \omega_Z [\tilde{Z} \to X].$$

Similarly, corollary 4.4.8 can be modified as follows:

Corollary 7.1.6. *With the assumptions as in theorem 7.1.5, suppose that each irreducible component X_i is in $\mathbf{Lci}_k$.*

1. *Let $f : Y \to X$ be a projective morphism with Y in $\mathbf{Lci}_k$. Then for each integral closed subscheme $Z \subset X$ with $\mathrm{codim}_X Z > 0$, there is an element $\omega_Z \in A_{*-\dim_k Z}(k)$, all but finitely many being zero, such that*

$$[f : Y \to X] - \sum_i \deg_i(f) \cdot [X_i \to X] = \sum_{Z,\ \mathrm{codim}_X Z > 0} \omega_Z [\tilde{Z} \to X].$$

2. *Let $f : Y \to X$ be a projective birational morphism, with Y in $\mathbf{Lci}_k$. Then, for each integral closed subscheme $Z \subset X$ with $\mathrm{codim}_X Z > 0$, there is an element $\omega_Z \in A_{*-\dim_k Z}(k)$, all but finitely many being zero, such that*

$$[f : Y \to X] = \sum_i [X_i \to X] + \sum_{Z,\ \mathrm{codim}_X Z > 0} \omega_Z \cdot [\tilde{Z} \to X].$$

Here, if Y is an l.c.i. k-scheme and $f : Y \to X$ is a projective morphism, we write $\deg_i(f)$ for $\deg_i([f : Y \to X])$.

For example, if k is perfect, X is in $\mathbf{Sm}_k$ and has dimension at most four over k, then, as one has resolution of singularities for finite type k-schemes of dimension at most two, and since each codimension one subvariety of X is an l.c.i. k-scheme, we may apply these two results without having resolution of singularities for arbitrary finite type k-schemes.

Also, suppose as in corollary 7.1.6 that X is a reduced finite type k-scheme such that the irreducible components $X_1, \ldots, X_s$ are all in $\mathbf{Lci}_k$. Let $d_i = \dim_k X_i$. We have the map $p_i^* : A_{*-d_i}(k) \to A_*(X)$ defined by

$$p_i(\alpha) = \alpha \cdot [X_i \to X].$$

Letting $\tilde{A}_*(X)$ be the kernel of the total degree map

$$\prod_i \deg_i : A_*(X) \to \oplus_{i=1}^s A_{*-d_i}(k),$$

the maps p_i define the splitting

$$A_*(X) = \tilde{A}_*(X) \oplus \oplus_{i=1}^s A_{*-d_i}(k). \tag{7.1}$$

7.2 Pull-back in Borel-Moore homology theories

As algebraic cobordism is already universal as a Borel-Moore weak homology theory, the main point we need to resolve is the universal nature of the pull-back maps we have defined for Ω_*.

7.2.1 Divisor classes

Let A_* be an oriented Borel-Moore homology theory on $\mathcal{V}$. Restricting A_* to $\mathbf{Sm}_k$, we have by proposition 5.2.1 the associated oriented cohomology theory A^* on $\mathbf{Sm}_k$. In particular, there is a formal group law $(F_A, A^*(k))$, $F_A(u,v) \in A^*(k)[[u,v]]$, such that, for X in $\mathbf{Sm}_k$ and for line bundles L, M on X, we have $F_A(c_1(L), c_1(M)) = c_1(L \otimes M)$. Considering A_* as an oriented Borel-Moore homology theory, this gives the identity of endomorphisms of $A_*(X)$

$$F_A(\tilde{c}_1(L), \tilde{c}_1(M)) = \tilde{c}_1(L \otimes M).$$

Also, since $(F_A, A_*(k))$ is a formal group law, there is a canonical ring homomorphism $\phi_A : \mathbb{L}_* \to A_*(k)$ with $\phi_A(F_{\mathbb{L}}) = F_A$.

Thus, given positive integers $n_1, \ldots, n_r$, we may form the power series with $A_*(k)$-coefficients

$$F^{n_1,\ldots,n_r}(u_1,\ldots,u_r)_A := n_1 \cdot_{F_A} u_1 +_{F_A} \ldots +_{F_A} n_r \cdot_{F_A} u_r,$$

and $F_J^{n_1,\ldots,n_r}(u_1,\ldots,u_r)_A$ with

$$F^{n_1,\ldots,n_r}(u_1,\ldots,u_r)_A = \sum_{J,\ ||J||\leq 1} u^J F_J^{n_1,\ldots,n_r}(u_1,\ldots,u_r)_A,$$

by the methods of §3.1.1 (see also §6.1.4).

If $E = \sum_{i=1}^r n_i E_i$ is a strict normal crossing divisor on some $W \in \mathbf{Sm}_k$, let $i : |E| \to W$ denote the closed subscheme (not necessarily reduced) defined by E. We have the inclusions of the faces $\iota^J : E^J \to |E|$, and we may define the class $[E \to |E|]_A \in A_*(|E|)$ by

$$[E \to |E|]_A := \sum_{J,\ ||J||\leq 1} \iota_*^J([F_J^{n_1,\ldots,n_r}(\iota^{J*}O_W(E_1), \ldots, \iota^{J*}O_W(E_r))]_A),$$

where $[F_J^{n_1,\ldots,n_r}(\iota^{J*}O_W(E_1), \ldots, \iota^{J*}O_W(E_r))]_A$ denotes the element

$$F_J^{n_1,\ldots,n_r}(\tilde{c}_1(\iota^{J*}O_W(E_1)), \ldots, \tilde{c}_1(\iota^{J*}O_W(E_r)))(1_{E^J}) \in A_*(E^J).$$

If $i : |E| \to W$ is the inclusion we write $[E \to W]_A$ for $i_*([E \to |E|]_A)$.

Lemma 7.2.1. $[E \to W]_A = \tilde{c}_1(O_W(E))(1_W)$.

Proof. For $A_* = \Omega_*$, this is proved in proposition 3.1.9. One can either repeat the proof, replacing Ω_* with A_* throughout, or use the universality of Ω_* as an oriented Borel-Moore homology weak theory (theorem 4.1.11): if $\vartheta_A : \Omega_* \to A_*$ is the natural transformation of oriented Borel-Moore homology weak theories on $\mathbf{Sm}_k$, $\vartheta_A(f : Y \to X) = f_*(1_Y^A)$, then it is easy to check that $\vartheta_A([E \to W]_\Omega) = [E \to W]_A$. As ϑ_A intertwines the respective Chern class operators, the formula for Ω_* implies the formula for A_*. □

Proposition 7.2.2. *Let A_* be an oriented Borel-Moore homology theory on $\mathcal{V}$. Let W be in $\mathbf{Sm}_k$, and let E be a strict normal crossing divisor on W. Then*

$$[E \to |E|]_A = 1_{|E|}$$

in $A_*(|E|)$.

Proof. Write $E = \sum_{i=1}^r n_i E_i$, with E_i irreducible. Note that $|E|$ is an l.c.i. scheme over k, so we have a fundamental class $1_{|E|}$. Using Jouanolou's trick and the homotopy invariance of A_*, we may assume that W is an affine scheme. Thus, the line bundles $O_W(E_i)$ are all very ample, hence there are morphisms $f_i : W \to \mathbb{P}^N$ (for N sufficiently large) with $E_i = f_i^*(X_N = 0)$, $X_0, \ldots, X_N$ being the standard homogeneous coordinates on $\mathbb{P}^N$. Let $f = (f_1, \ldots, f_r) : W \to (\mathbb{P}^N)^r := \tilde{W}$, and let $\tilde{E}_i$ be the subscheme $p_i^*(X_N = 0)$, where $p_i : (\mathbb{P}^N)^r \to \mathbb{P}^N$ is the projection on the ith factor. Let $\tilde{E} = \sum_{i=1}^r n_i \tilde{E}_i$. Then $f^{-1}(|\tilde{E}|) = |E|$ and $f^*(\tilde{E}) = E$. Letting $f_E : |E| \to |\tilde{E}|$ be the restriction of f, f_E is the Tor-independent pull-back of the l.c.i. morphism f by the regular embedding $|\tilde{E}| \to (\mathbb{P}^N)^r$, hence f_E is an l.c.i. morphism. It follows from the functoriality of the fundamental class that $f_E^*(1_{|\tilde{E}|}) = 1_{|E|}$. Similarly, $1_{E^J} = f_E^*(1_{\tilde{E}^J})$ and

$$\begin{aligned}[F_J^{n_1,\ldots,n_r}(\iota^{J*}O_W(E_1), \ldots, \iota^{J*}O_W(E_r))]_A \\ = f_E^*([F_J^{n_1,\ldots,n_r}(\iota^{J*}O_{\tilde{W}}(\tilde{E}_1), \ldots, \iota^{J*}O_{\tilde{W}}(\tilde{E}_r))]_A).\end{aligned}$$

Thus, $f_E^*([\tilde{E} \to |\tilde{E}|]_A) = [E \to |E|]_A$. Therefore, it suffices to prove the result for $W = (\mathbb{P}^N)^r$, $E = \sum_{i=1}^r n_i p_i^*(X_N = 0)$. In this case, applying the axiom (CD), the map $i_* : A_*(|E|) \to A_*(W)$ is injective, where $i : |E| \to W$ is the inclusion, so it suffices to show that

$$i_*([E \to |E|]_A) = i_*(1_{|E|}).$$

By proposition 5.1.12 and lemma 7.2.1, both sides are $\tilde{c}_1(O_W(E))(1_W)$, whence the result. □

7.2.2 The deformation diagram revisited

In this section, we show that the method used to define the Gysin morphism for algebraic cobordism is compatible with the pull-back in a Borel-Moore homology theory A_*.

Let $i : Z \to X$ be a regular embedding in $\mathcal{V}$. We refer to the deformation diagram (6.10) and retain the notations involving that diagram.

It follows from the next lemma and the condition (5.1) that all the schemes in the above diagram are in $\mathcal{V}$.

Lemma 7.2.3. *The map $\tilde{i} : \langle Z \times \mathbb{P}^1 \rangle \to U$ is a regular embedding.*

Proof. The assertion is local on X, so we may assume that X is affine, $X = \operatorname{Spec} R$, and that Z is a complete intersection, defined by a regular sequence $f_0, \ldots, f_N$. We may also replace $\mathbb{P}^1$ with $\mathbb{A}^1 := \mathbb{P}^1 \setminus \infty$.

Let $k[y_0, \ldots, y_N]$ be a polynomial ring, and consider the k-algebra homomorphism $\psi : k[y_0, \ldots, y_N] \to R$ defined by sending y_i to f_i. Since $f_0, \ldots, f_N$ is a regular sequence, ψ is a flat extension, at least after inverting some element $z \in R$ outside $(f_0, \ldots, f_N)$. Since a flat extension of a regular embedding is still a regular embedding, we see that it suffices to prove the result for $X = \operatorname{Spec} k[y_0, \ldots, y_N]$ and Z the subscheme defined by the maximal ideal $(y_0, \ldots, y_N)$. Letting t be the standard coordinate on $\mathbb{A}^1$, $Z \times 0$ is a complete intersection in $X \times \mathbb{A}^1$, defined by the regular sequence $y_0, \ldots, y_N, t$.

We therefore have the identification of W with the subscheme of $X \times \mathbb{A}^1 \times \mathbb{P}^{N+1}$ defined by the equations

$$\begin{aligned} X_i y_j - X_j y_i &= 0, \ 0 \le i, j \le N \\ X_i t - X_{N+1} y_i &= 0, \ 0 \le i \le N, \end{aligned}$$

where $X_0, \ldots, X_{N+1}$ are standard homogeneous coordinates on $\mathbb{P}^{N+1}$. Also, we have noted in lemma 3.2.1 that $\langle X \times 0 \rangle$ is the subscheme defined by $X_{N+1} = 0$. This gives the description of U as the subscheme of $\mathbb{A}^N \times \mathbb{A}^1 \times \mathbb{A}^{N+1} = \operatorname{Spec} k[y_0, \ldots, y_N, t, x_0 \ldots, x_N]$ defined by the equations

$$\begin{aligned} x_i y_j - x_j y_i &= 0, \ 0 \le i, j \le N \\ x_i t - y_i &= 0, \ 0 \le i \le N. \end{aligned}$$

Clearly, this gives an isomorphism $U \cong \operatorname{Spec} k[t, x_0, \ldots, x_N]$, and identifies $\langle Z \times \mathbb{A}^1 \rangle$ with the subscheme defined by the ideal $(x_0, \ldots, x_N)$, showing that $\langle Z \times \mathbb{A}^1 \rangle \to U$ is a regular embedding. □

Proposition 7.2.4. *Let A_* be an oriented Borel-Moore homology theory on $\mathcal{V}$, and let $i_Z : Z \to X$ be a regular embedding in $\mathcal{V}$. Form the diagram* (6.10) *and take $\eta \in A_*(X)$. Suppose there is an element $\tilde{\eta}$ of $A_{*+1}(U)$ such that $i_1^{X*}(\tilde{\eta}) = \eta$. Then*

$$q^*(i_Z^*(\eta)) = i_V^*(\tilde{\eta}).$$

Proof. Let $\tilde{i}_0 : Z \to \langle Z \times \mathbb{P}^1 \rangle$ be the inclusion of the fiber over $0 \in \mathbb{P}^1$, and let $\tilde{i}_1 : Z \to \langle Z \times \mathbb{P}^1 \rangle$ be the inclusions of the fiber over $1 \in \mathbb{P}^1$. Let $\tau = \tilde{i}^*(\tilde{\eta})$. Then $\tilde{i}_1^*(\tau) = i_Z^*(i_1^{X*}(\tilde{\eta})) = i_Z^*(\eta)$. By the homotopy property (EH) for A_*, we have $\tilde{i}_0^*(\tau) = \tilde{i}_1^*(\tau) = i_Z^*(\eta)$. But $\tilde{i}_0^*(\tau) = s^*(i_V^*(\tilde{\eta}))$ and $q^* s^* = \mathrm{Id}$, so

$$\begin{aligned} q^*(i_Z^*(\eta)) &= q^* \tilde{i}_0^*(\tau) \\ &= q^* s^*(i_V^*(\tilde{\eta})) \\ &= i_V^*(\tilde{\eta}). \end{aligned}$$

□

We can also use the deformation diagram to compute i_Z^* of a special type of element.

Lemma 7.2.5. *Let $i : Z \to X$ be a regular codimension one embedding, and let $f : Y \to X$ be a projective morphism of finite type k-schemes. Suppose that $f(Y) \subset Z$ and that Y is l.c.i. over k. Then*

$$i^*(f_*(1_Y)) = \tilde{c}_1(i_Z^* O_X(Z))(f_*^Z(1_Y)),$$

where $f^Z : Y \to Z$ is the map induced by f.

Proof. We use the diagram (6.10). The map f^Z gives the map $f^Z \times \mathrm{Id} : Y \times \mathbb{P}^1 \to Z \times \mathbb{P}^1$. Mapping $Z \times \mathbb{P}^1$ to U by identifying with $\langle Z \times \mathbb{P}^1 \rangle$, we have the map $\tilde{f} : Y \times \mathbb{P}^1 \to U$. Clearly $\tilde{f}_*(1_{Y \times \mathbb{P}^1})$ is an element $\tilde{\eta}$ of $A_*(U)$ with $i_1^{X*}(\tilde{\eta}) = f_*(1_Y)$. By proposition 7.2.4, we have

$$q^*(i_Z^*(f_*(1_Y))) = i_V^*(\tilde{\eta}).$$

Since $1_{Y \times \mathbb{P}^1} = p_1^*(1_Y)$, we have

$$i_V^*(\tilde{\eta}) = s_*(f_*(1_Y)).$$

Since s^* and q^* are inverse, we thus have

$$i_Z^*(f_*(1_Y)) = s^* s_*(f_*(1_Y)),$$

and the right-hand side is $\tilde{c}_1(i_Z^* O_X(Z))(f_*^Z(1_Y))$, by definition. □

7.3 Universality

We are now ready to prove the main universality results.

Proof (of theorem 7.1.1 and theorem 7.1.2). It follows from theorem 4.1.11 that Ω_* has the structure of an oriented Borel-Moore weak homology theory on $\mathcal{V}$ and is in fact the universal such theory. From the results of §6.5, we have pull-back maps $f^* : \Omega_*(X) \to \Omega_{*+d}(Y)$ for each l.c.i. morphism $f : Y \to X$ in $\mathbf{Sch}_k$, satisfying the axioms (BM1), (BM2) and (BM3). The axioms (PB) and (EH) are already valid for an oriented Borel-Moore weak homology theory, so we need only verify the axiom (CD). This follows from lemma 5.2.11, as Ω_* satisfies the required localization property by theorem 1.2.8. Thus, Ω_* defines an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$. By proposition 5.2.1, the restriction of the oriented Borel-Moore homology theory Ω_* to $\mathbf{Sm}_k$ defines an extension of the oriented Borel-Moore weak homology theory Ω_* on $\mathbf{Sm}_k$ to an oriented cohomology theory Ω_* on $\mathbf{Sm}_k$.

The uniqueness of the extension of Ω_* to an an oriented Borel-Moore homology theory on $\mathcal{V}$ follows from the universality of Ω_* as a Borel-Moore homology theory on $\mathcal{V}$ (theorem 7.1.3, which we prove below). Indeed, suppose the oriented Borel-Moore weak homology theory Ω_* on $\mathbf{Sch}_k$ has a second extension $\hat{\Omega}_*$ to an oriented Borel-Moore homology theory on $\mathcal{V}$, with pull-back maps $\hat{f}^*$ for each l.c.i. morphism $f : Y \to X$ in $\mathbf{Sch}_k$. By the universality of Ω_* as an oriented Borel-Moore homology theory, there is a unique natural transformation of oriented Borel-Moore homology theories on $\mathcal{V}$, $\vartheta : \Omega_* \to \hat{\Omega}_*$. But the underlying oriented Borel-Moore weak homology theory Ω_* on $\mathcal{V}$ is also universal, and both Ω_* and $\hat{\Omega}_*$ agree as weak homology theories, so ϑ is the identity transformation, forcing $f^* = \hat{f}^*$ for all l.c.i. morphisms f. The uniqueness of the extension of Ω_* to an oriented cohomology theory on $\mathbf{Sm}_k$ is proved the same way. $\square$

Proof (of theorem 7.1.3). (1) Let $\mathcal{V}$ be an l.c.i. closed admissible subcategory of $\mathbf{Sch}_k$, and let A_* be an oriented Borel-Moore homology theory on $\mathcal{V}$. By theorem 4.1.11, Ω_* is the universal oriented Borel-Moore weak homology theory on $\mathbf{Sch}_k$, so there is a unique natural transformation

$$\vartheta_A : \Omega_* \to A_*$$

of the underlying oriented Borel-Moore weak homology theories, i.e., ϑ_A is compatible with projective push-forward, smooth pull-back, Chern class operators and external products. It thus suffices to show that ϑ_A is compatible with pull-backs for l.c.i. morphisms in $\mathcal{V}$. We use the notation f^*_A, f^A_*, etc., to indicate which theory we are using.

As ϑ_A is already compatible with smooth pull-back, we need only check compatibility with respect to regular embeddings $i_Z : Z \to X$ in $\mathcal{V}$.

It suffices to check that, for $f : Y \to X$ in $\mathcal{M}(X)$, we have

$$i^*_{ZA}(\vartheta_A([f : Y \to X])) = \vartheta_A(i^*_{Z\Omega}([f : Y \to X])).$$

Note that, as $p : Y \to \operatorname{Spec} k$ is smooth and quasi-projective, we have

$$\vartheta_A(1^\Omega_Y) = \vartheta_A(p^*_\Omega(1_\Omega)) = p^*_A(1_A) = 1^A_Y.$$

Thus

$$\begin{aligned}\vartheta_A([f : Y \to X]) &= \vartheta_A(f^\Omega_*(1^\Omega_Y)) \\ &= f^A_*(1^A_Y).\end{aligned}$$

Therefore, we need to show

$$i^*_{ZA}(f^A_*(1^A_Y)) = \vartheta_A(i^*_{Z\Omega}(f^\Omega_*(1^\Omega_Y))).$$

We first reduce to the case of a codimension one regular embedding by using the deformation diagram (6.10); we retain the notations surrounding that diagram. Let $\eta = f^A_*(1^A_Y)$ and let $\eta_\Omega = f^\Omega_*(1^\Omega_Y)$.

As the map $\mu : U \to X \times \mathbb{P}^1$ is an isomorphism over $X \times \mathbb{P}^1 \setminus Z \times 0$, we can lift $p_1^*\eta_\Omega \in \Omega_{*+1}(X \times \mathbb{P}^1)$ to an element $\eta'_\Omega \in \Omega_{*+1}(U \setminus V)$. In particular, $i_1^{X*}(\eta'_\Omega) = \eta_\Omega$. Since the (smooth) restriction map

$$j^* : \Omega_{*+1}(U) \to \Omega_{*+1}(U \setminus V)$$

is surjective, we can lift η'_Ω to an element $\tilde{\eta}_\Omega \in \Omega_{*+1}(U)$. Let $\tilde{\eta} = \vartheta_A(\tilde{\eta}_\Omega)$. Then, as $i_{1A}^{X*} \circ p_{1A}^* = \mathrm{Id}$, we have

$$i_{1A}^{X*}(\tilde{\eta}) = \eta.$$

We may therefore apply proposition 7.2.4, giving

$$q_A^*(i_{ZA}^*(\vartheta_A(f_*^\Omega(1_Y^\Omega)))) = i_{VA}^*(\vartheta_A(\tilde{\eta}_\Omega)).$$

On the Ω side, we similarly have

$$q_\Omega^*(i_{Z\Omega}^*(f_*^\Omega(1_Y^\Omega))) = i_{V\Omega}^*(\tilde{\eta}_\Omega).$$

Since q is smooth, and q_A^*, q_Ω^* are isomorphisms, this reduces us to the case of a codimension one regular embedding $i_Z : |Z| \to X$, where Z is a Cartier divisor on X.

If $f(Y) \subset |Z|$, let $f^Z : Y \to |Z|$ be the map induced by f. By lemma 7.2.5, we have

$$\begin{aligned} i_{ZA}^*(f_*^A(1_Y^A)) &= \tilde{c}_1^A(i_Z^* O_X(Z))(f_*^{ZA}(1_Y^A)) \\ i_{Z\Omega}^*(f_*^\Omega(1_Y^\Omega)) &= \tilde{c}_1^\Omega(i_Z^* O_X(Z))(f_*^{Z\Omega}(1_Y^\Omega)). \end{aligned}$$

Since ϑ_A is compatible with Chern class operators, push-forward and units 1_Y for $Y \in \mathbf{Sm}_k$, we have the desired compatibility in this case. Thus, we have

$$i_{ZA}^*(\vartheta_A(i_{Z*}^\Omega(\rho))) = \vartheta_A(i_{Z\Omega}^*(i_{Z*}^\Omega(\rho)))$$

for all $\rho \in \Omega_*(Z)$.

If $f(Y) \not\subset |Z|$, then there is a projective birational map $\tau : Y' \to Y$, which is an isomorphism over $X \setminus |Z|$, such that $(f\tau)^*(Z)$ is a strict normal crossing divisor on Y'. As $f - (f\tau)$ vanishes in $\Omega_*(X \setminus |Z|)$, it follows from the localization sequence

$$\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \to \Omega_*(X \setminus |Z|) \to 0$$

that $f = (f\tau) + i_*(\rho)$ for some $\rho \in \Omega_*(|Z|)$, Thus, we may assume that f^*Z is a strict normal crossing divisor on Y with associated codimension one subscheme $i_{f^*Z} : |f^*Z| \to Y$.

Let $f^Z : |f^*Z| \to |Z|$ be the map induced by f. Since Y is smooth, the maps $f : Y \to X$ and $i_Z : |Z| \to X$ are Tor-independent, hence transverse in $\mathbf{Sch}_k$. But, as the diagram

$$\begin{tikzcd} {|f^*Z|} \arrow[r, "i_{f^*Z}"] \arrow[d, "f^Z"'] & Y \arrow[d, "f"] \\ {|Z|} \arrow[r, "i_Z"'] & X \end{tikzcd}$$

is cartesian, it follows from axiom (BM2) that, for both the theory Ω_* and the theory A_*, we have

$$\begin{aligned} i_Z^*(f_*(1_Y)) &= f_*^Z(i_{f^*Z}^*(1_Y)) \\ &= f_*^Z(1_{|f^*Z|}). \end{aligned}$$

By proposition 7.2.2, we have (in both theories)

$$1_{|f^*Z|} = [f^*Z \to |f^*Z|].$$

Thus

$$\begin{aligned} i_{ZA}^*(f_*^A(1_Y^A)) &= f_*^{ZA}([f^*Z \to |f^*Z|]_A), \\ i_{Z\Omega}^*(f_*^\Omega(1_Y^\Omega)) &= f_*^{Z\Omega}([f^*Z \to |f^*Z|]_\Omega). \end{aligned} \tag{7.2}$$

Let E be a Cartier divisor on some $W \in \mathbf{Sm}_k$. Since the divisor class $[E \to |E|]_A \in A_*(|E|)$ depends only on the weak homology theory underlying A_*, we have

$$\vartheta_A([E \to |E|]_\Omega) = [E \to |E|]_A$$

This, together with (7.2), shows that

$$\begin{aligned} i_{ZA}^*(f_*^A(1_Y^A)) &= f_*^{ZA}(\vartheta_A([f^*Z \to |f^*Z|]_\Omega)) \\ &= \vartheta_A(f_*^{Z\Omega}([f^*Z \to |f^*Z|]_\Omega)) \\ &= \vartheta_A(i_{Z\Omega}^*(f_*^\Omega(1_Y^\Omega))). \end{aligned}$$

This completes the proof of the universality of Ω_* as an oriented Borel-Moore homology theory on $\mathcal{V}$.

For (2), we use proposition 5.2.1 to reduce the proof to showing that Ω_* is the universal oriented Borel-Moore homology theory on $\mathbf{Sm}_k$; for this, we will use proposition 3.3.1 as the essential point.

Let A_* be an oriented Borel-Moore homology theory on $\mathbf{Sm}_k$. As above, we have the unique natural transformation $\vartheta_A : \Omega_* \to A_*$ of weak homology theories on $\mathbf{Sm}_k$, and we need to show that ϑ_A intertwines the pull-back maps g_Ω^* and g_A^* for each morphism $g : X' \to X$ in $\mathbf{Sm}_k$.

It suffices to prove the result for g a regular embedding $i : Z \to X$, and for elements of $\Omega_*(X)$ of the form $[f : Y \to X]$, $f \in \mathcal{M}(X)$. By proposition 3.3.1, we need only consider maps f which are transverse to i in $\mathbf{Sm}_k$. Form the cartesian diagram

$$\begin{tikzcd} \tilde{Y} \arrow[r, "\tilde{i}"] \arrow[d, "\tilde{f}"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$

Using axiom (BM3), we have

$$\begin{aligned} i^*_\Omega(f : Y \to X) &= i^*_\Omega(f^\Omega_*(1^\Omega_Y)) \\ &= \tilde{f}^\Omega_*(\tilde{i}^*_\Omega(1^\Omega_Y)) \\ &= \tilde{f}^\Omega_*(1_{\tilde{Y}}). \end{aligned}$$

Similarly, as $\vartheta_A(f : Y \to X) = f^A_*(1^A_Y)$, we have

$$i^*_A(\vartheta_A(f)) = \tilde{f}^A_*(1^A_{\tilde{Y}}).$$

Thus, as ϑ_A is compatible with push-forward and units, we have

$$\vartheta_A(i^*_\Omega(f)) = i^*_A(\vartheta_A(f)),$$

as desired. □

7.4 Some applications

Having extended Ω_* to an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$ (assuming k admits resolution of singularities), we are able to extend some of the main applications of the theory from smooth varieties to l.c.i. schemes over k. For most of this section, k will be a field of characteristic zero, although the results of §7.4.1 and §7.4.2 are valid over an arbitrary field.

7.4.1 Chern classes and Conner-Floyd classes

Let A_* be an oriented Borel-Moore weak homology theory on an admissible $\mathcal{V} \subset \mathbf{Sch}_k$. In proposition 4.1.15 we showed how to define Chern class operators $\tilde{c}_i(E) : A_*(X) \to A_{*-i}(X)$, $i = 0, \ldots, n$, for each rank n vector bundle $E \to X$, $X \in \mathcal{V}$, satisfying the standard properties. Also, given a sequence $\tau = (\tau_i) \in \prod_{i=0}^\infty A_i(k)$ with $\tau_0 = 1$, we defined in proposition 4.1.20, for each vector bundle $E \to X$, $X \in \mathcal{V}$, a degree zero endomorphism $\widetilde{\mathrm{Td}}_\tau^{-1} : A_*(X) \to A_*(X)$, with properties listed in that lemma.

Now let A_* be an oriented Borel-Moore homology theory on an admissible $\mathcal{V} \subset \mathbf{Sch}_k$. Exactly the same construction gives the Chern class operators $\tilde{c}_i(E) : A_*(X) \to A_{*-i}(X)$ for $E \to X$ a vector bundle on $X \in \mathcal{V}$, and, given a sequence τ as above, the degree-zero endomorphisms $\widetilde{\mathrm{Td}}_\tau^{-1}(E)$ of $A_*(X)$. These satisfy exactly the same properties as in the case of the weak homology theory, with the added property of functoriality with respect to l.c.i. pull-back. For the sake of precision, we list these properties here:

Lemma 7.4.1. *Let A_* be an oriented Borel-Moore homology theory on $\mathcal{V}$. Then the Chern classes satisfy the following properties:*

(0) Given vector bundles $E \to X$ and $F \to X$ on $X \in \mathcal{V}$ one has

$$\tilde{c}_i(E) \circ \tilde{c}_j(F) = \tilde{c}_j(F) \circ \tilde{c}_i(E)$$

for any (i, j).

(1) For any line bundle L, $\tilde{c}_1(L)$ agrees with the one given in axiom (PB) applied to A_*.

(2) For any l.c.i. morphism $Y \to X \in \mathcal{V}$, and any vector bundle $E \to X$ over X one has

$$\tilde{c}_i(f^*E) \circ f^* = f^* \circ \tilde{c}_i(E).$$

(3) If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles over X, then for each integer $n \geq 0$ one has the following equation in $End(A_*(X))$:

$$\tilde{c}_n(E) = \sum_{i=0}^{n} \tilde{c}_i(E')\, \tilde{c}_{n-i}(E'').$$

(4) For any projective morphism $Y \to X$ in $\mathcal{V}$ and any vector bundle $E \to X$ over X, one has

$$f_* \circ \tilde{c}_i(f^*E) = \tilde{c}_i(E) \circ f_*.$$

Moreover, the Chern class operators are characterized by the properties (0)-(3).

Lemma 7.4.2. *Let A_* be an oriented Borel-Moore homology theory on $\mathcal{V}$ and let $\tau = (\tau_i) \in \Pi_{i=0}^{\infty} A_i(k)$, with $\tau_0 = 1$. Then one can define in a unique way, for each $X \in \mathcal{V}$ and each vector bundle E on X, an endomorphism (of degree zero)*

$$\widetilde{\mathrm{Td}}_{\tau}^{-1}(E) : A_*(X) \to A_*(X)$$

such that the following holds:

(0) Given vector bundles $E \to X$ and $F \to X$ one has

$$\widetilde{\mathrm{Td}}_{\tau}^{-1}(E) \circ \widetilde{\mathrm{Td}}_{\tau}^{-1}(F) = \widetilde{\mathrm{Td}}_{\tau}^{-1}(F) \circ \widetilde{\mathrm{Td}}_{\tau}^{-1}(E).$$

(1) For a line bundle L one has:

$$\widetilde{\mathrm{Td}}_{\tau}^{-1}(L) = \sum_{i=0}^{\infty} \tilde{c}_1(L)^i\, \tau_i.$$

(2) For any l.c.i. morphism $Y \to X$ in $\mathcal{V}$, and any vector bundle $E \to X$ over X one has

$$\widetilde{\mathrm{Td}}_{\tau}^{-1}(f^*E) \circ f^* = f^* \circ \widetilde{\mathrm{Td}}_{\tau}^{-1}(E).$$

(3) *If* $0 \to E' \to E \to E'' \to 0$ *is an exact sequence of vector bundles over* X, *then one has:*
$$\widetilde{\mathrm{Td}}_\tau^{-1}(E) = \widetilde{\mathrm{Td}}_\tau^{-1}(E') \circ \widetilde{\mathrm{Td}}_\tau^{-1}(E'').$$

(4) *For any projective morphism* $Y \to X$ *in* $\mathcal{V}$ *and any vector bundle* $E \to X$ *over* X, *one has*
$$f_* \circ \widetilde{\mathrm{Td}}_\tau^{-1}(f^*E) = \widetilde{\mathrm{Td}}_\tau^{-1}(E) \circ f_*.$$

Remark 7.4.3. If $E \to X$ is a vector bundle on an l.c.i. k-scheme X in $\mathcal{V}$, we may evaluate the operators $\tilde{c}_i(E)$ or $\widetilde{\mathrm{Td}}_\tau^{-1}(E)$ on 1_X, yielding the Chern classes $c_i(E) \in A^i(X)$ and the total inverse Todd class (or Conner-Floyd class) $\widetilde{\mathrm{Td}}_\tau^{-1}(E) \in A^0(X)$.

It follows directly from the two lemmas above that the Chern class operators $\tilde{c}_i(E)$ and the inverse Todd class operator $\widetilde{\mathrm{Td}}_\tau^{-1}(E)$ depend only on the class of E in $K^0(X)$.

As in example 4.1.22, we consider the "universal" example:. Let A_* be a Borel-Moore homology theory on $\mathcal{V}$, let $\mathbb{Z}[\mathbf{t}] := \mathbb{Z}[t_1, t_2, \ldots, t_n, \ldots]$, with t_i having degree i, and consider the Borel-Moore homology theory $X \mapsto A_*(X)[\mathbf{t}] := A_*(X) \otimes \mathbb{Z}[\mathbf{t}]$. Let $\mathbf{t}$ be the family $(1, t_1, t_2, \ldots)$. For each line bundle $L \to X$, $X \in \mathcal{V}$, we thus have the automorphism
$$\widetilde{\mathrm{Td}}_{\mathbf{t}}^{-1}(L) = \sum_{i=0}^{\infty} \tilde{c}_1(L)^i t_i : A_*(X)[\mathbf{t}] \to A_*(X)[\mathbf{t}]$$
and for each vector bundle $E \to X$ the automorphism $\widetilde{\mathrm{Td}}_{\mathbf{t}}^{-1}(E)$, which we expand as
$$\widetilde{\mathrm{Td}}_{\mathbf{t}}^{-1}(E) = \sum_{(n_1, n_2, \ldots)} \tilde{c}_{n_1, n_2, \ldots, n_r}(E) t_1^{n_1} \cdot \ldots \cdot t_r^{n_r}.$$

As in *loc. cit.*, we call the endomorphisms $\tilde{c}_{n_1, n_2, \ldots, n_r}(E)$ the *Conner-Floyd Chern class endomorphisms* for E. For X an l.c.i. k-scheme, we have as well the Conner-Floyd classes
$$c_{n_1, n_2, \ldots, n_r}(E) := \tilde{c}_{n_1, n_2, \ldots, n_r}(E)(1_X) \in A_{d - \sum_i i n_i}(X),$$
$d = \dim_k X$.

We write $\tilde{c}^A_{n_1, n_2, \ldots, n_r}(E)$ or $c^A_{n_1, n_2, \ldots, n_r}(E)$ if we need to specify A.

7.4.2 Twisting a Borel-Moore homology theory

We extend the twisting constructions of §4.1.9 from oriented Borel-Moore weak homology theories to oriented Borel-Moore homology theories, and from smooth k-schemes to l.c.i. k-schemes.

Let $f : Y \to X$ be an l.c.i. morphism. Choose a factorization of f as $f = qi$, with $i : Y \to P$ a regular embedding and $q : P \to X$ a smooth morphism. We have the *relative tangent bundle* $T_q \to P$, defined as the vector bundle whose dual has sheaf of sections the relative differentials $\Omega^1_{Y/X}$. Letting $\mathcal{I}$ be the ideal sheaf of Y in P, we let $N_i \to Y$ be the bundle whose dual has sheaf of sections $\mathcal{I}/\mathcal{I}^2$. We let $[N_f] \in K^0(Y)$ be the class $N_i - i^*T_q$. It is easy to see that $[N_f]$ is independent of the choice of the factorization of f.

If $X = \operatorname{Spec} k$, we write $[N_Y]$ for $[N_f]$, and set $[T_Y] := -[N_Y]$. If $f : Y \to X$ is an arbitrary morphism of l.c.i. k-schemes, define the *virtual tangent bundle* of f by $[T_f] := [T_Y] - f^*[T_X] \in K^0(Y)$.

Given a Borel-Moore homology theory A_* on $\mathcal{V}$, and a family τ as in the previous section §7.4.1, we may twist A_* by τ, forming the Borel-Moore homology theory $A_*^{(\tau)}$ with the same push-forward maps as A_*, and with

$$f^*_{(\tau)} = \widetilde{\mathrm{Td}}_\tau^{-1}([N_f]) \circ f^*$$

for an l.c.i. morphism f. For a line bundle $L \to X$, one has the Chern class operator

$$\tilde{c}_1^{(\tau)}(L) = \widetilde{\mathrm{Td}}_\tau^{-1}(L) \circ \tilde{c}_1(L).$$

One easily checks that this does define an oriented Borel-Moore homology theory on $\mathcal{V}$.

We can extend the second construction given in 4.1.9 from $\mathcal{V} \subset \mathbf{Sm}_k$ to $\mathcal{V} \subset \mathbf{Lci}$. We have the Todd class operator

$$\widetilde{\mathrm{Td}}_\tau(E) := \widetilde{\mathrm{Td}}_\tau^{-1}(-[E]) = (\widetilde{\mathrm{Td}}_\tau^{-1}(E))^{-1}.$$

Define the Borel-Moore homology theory A_*^τ on $\mathcal{V}$ as having the same l.c.i. pull-backs as A_*, with push-forward

$$f_*^\tau := f_* \circ \widetilde{\mathrm{Td}}_\tau([T_f])$$

for $f : Y \to X$ projective. The Chern class operators are given by

$$\tilde{c}_1^\tau(L) = \tilde{c}_1(L) \circ \widetilde{\mathrm{Td}}_\tau(-L) = \tilde{c}_1^{(\tau)}(L)$$

for each line bundle $L \to X$, $X \in \mathcal{V}$.

As in lemma 4.1.23, the isomorphisms

$$\widetilde{\mathrm{Td}}_\tau^{-1}([T_X]) : A_*^{(\tau)}(X) \xrightarrow{\sim} A_*^\tau(X)$$

determine an isomorphism of oriented Borel-Moore homology theories $A_*^{(\tau)} \to A_*^\tau$ for $\mathcal{V} \subset \mathbf{Lci}$.

7.4.3 Operations on Ω_*

For the remainder of this section, we assume that k admits resolution of singularities.

Take $\mathcal{V} = \mathbf{Lci}_k$, and consider the universal twisting $\Omega_*[\mathbf{t}]^{\mathbf{t}}$ of Ω_*. By the universality of Ω_*, we have a canonical transformation of Borel-Moore homology theories

$$\vartheta^{LN} : \Omega_* \to \Omega_*[\mathbf{t}]^{\mathbf{t}},$$

which we expand as

$$\vartheta^{LN} = \sum_{I=(n_1,\ldots,n_r,\ldots)} S_I t^I.$$

The individual terms $S_{(n_1,\ldots,n_r)} : \Omega_* \to \Omega_{*-n}$, $n = \sum_i in_i$, are the Landweber-Novikov operations defined in example 4.1.25; our extension of ϑ^{LN} to the setting of Borel-Moore homology theories has just verified that the Landweber-Novikov operations are natural with respect to l.c.i. pull-back.

Taking $\mathcal{V} = \mathbf{Sch}_k$ and using the other twisting $\Omega_*[\mathbf{t}]^{(\mathbf{t})}$, we have the natural transformation $\vartheta^{LN'} : \Omega_* \to \Omega_*[\mathbf{t}]^{(\mathbf{t})}$ of homology theories on $\mathbf{Sch}_k$. Using the canonical transformation $\Omega_* \to \mathrm{CH}_*$, we have the natural transformation

$$\vartheta^{CF} : \Omega_* \to \mathrm{CH}_*[\mathbf{t}]^{(\mathbf{t})}.$$

Expanding ϑ^{CF} as

$$\vartheta^{CF} = \sum_I c_I^{CF} t^I$$

defines the transformations $c^{CF}_{n_1,\ldots,n_r} : \Omega_* \to \mathrm{CH}_{*-n}$, $n = \sum_i in_i$.

Explicitly, we have

Lemma 7.4.4. *Let $\pi : Y \to \operatorname{Spec} k$ be an l.c.i. k-scheme. For $f : Y \to X$ a projective morphism with $X \in \mathbf{Sch}_k$, we have*

$$c^{CF}_{n_1,\ldots,n_r}([f : Y \to X]) = f_*^{\mathrm{CH}}(c^{\mathrm{CH}}_{n_1,\ldots,n_r}([N_Y])).$$

Proof. Note that $1_Y = \pi^*(1)$ by definition, and $[f : Y \to X] = f_*(1_Y)$. Thus, since ϑ^{CF} is a natural transformation of oriented Borel-Moore homology theories, we have

$$\begin{aligned}
\vartheta^{CF}([f : Y \to X]) &= \vartheta^{CF}(f_*^{\Omega}(\pi_{\Omega}^*(1))) \\
&= f_*^{\mathrm{CH}_*[\mathbf{t}]^{(\mathbf{t})}}(\pi^*_{\mathrm{CH}_*[\mathbf{t}]^{(\mathbf{t})}}(1)) \\
&= \sum_I f_*^{\mathrm{CH}}\big(\tilde{c}_I^{\mathrm{CH}}([N_Y])(\pi^*_{\mathrm{CH}}(1))\big)t^I \\
&= \sum_I f_*^{\mathrm{CH}}(c_I^{\mathrm{CH}}([N_Y]))t^I.
\end{aligned}$$

Equating the coefficients of $t_1^{n_1} \cdot \ldots \cdot t_r^{n_r}$ yields the result. $\square$

The same proof shows that, for l.c.i. k-schemes X and Y and a projective morphism $f : Y \to X$, we have

$$S_{n_1,\ldots,n_r}([f : Y \to X]) = f_*^{\Omega}(c^{\Omega}_{n_1,\ldots,n_r}([N_f])) \in \Omega_*(X). \tag{7.3}$$

Let $P(x_1, \ldots, x_d)$ be a degree d weighted-homogeneous polynomial with coefficients in a commutative ring R, where we give x_i degree i. Let $\pi : X \to \operatorname{Spec} k$ be a projective l.c.i. k-scheme of dimension d over k. Define $P(X) \in R$ by

$$P(X) := \deg(P(c_1^{\mathrm{CH}}, \ldots, c_d^{\mathrm{CH}})([N_X])).$$

Here $\deg : \mathrm{CH}_0(X) \otimes R \to R$ is map induced by the composition of the push-forward $\pi_* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(k)$ followed by the canonical isomorphism $\mathrm{CH}_0(k) \cong \mathbb{Z}$.

Using lemma 7.4.4, the same proof as for lemma 4.4.19 yields the following result.

Proposition 7.4.5. *Let $P(x_1, \ldots, x_d)$ be a degree d weighted-homogeneous polynomial with coefficients in a commutative ring R, with x_i having degree i. There is a unique homomorphism*

$$P_\Omega : \Omega_d \to R$$

with $P_\Omega(\pi_*(1_X)) = P(X)$ for each projective l.c.i. k-scheme $\pi : X \to \operatorname{Spec} k$ of dimension d over k.

7.4.4 Degree formulas for Ω_*

For the remainder of this section, k will be a field of characteristic zero. We can now extend many of the results of §4.4.3 and §4.4.4 from smooth k-schemes to l.c.i. k-schemes.

Let $\pi : X \to \operatorname{Spec} k$ be a projective l.c.i. k-scheme of dimension d over k. We write $[X] \in \Omega_d(k)$ for $\pi_*(1_X)$; this agrees with the notation we have used for X smooth and projective over k.

For X a projective purely d dimensional k-scheme, we have the ideal $M(X)$ of $\Omega_*(k)$, defined as the ideal generated by the classes $[Y] \in \Omega_*(k)$, for those Y smooth and projective over k, of dimension $< d$, for which there is a morphism $f : Y \to X$ over k (see §4.4.2). More generally, if X is locally equi-dimensional over k, define $M(X)$ as the ideal generated by the $M(X_i)$, as X_i runs over the connected components of X.

With these notations, theorem 4.4.15 generalizes to

Theorem 7.4.6. *Let k be a field of characteristic zero. Let X and Y be reduced projective l.c.i. k-schemes and let $f : Y \to X$ be a morphism. Suppose that X is irreducible. Then one has*

$$[Y] - \deg(f) \cdot [X] \in M(X).$$

The proof is exactly the same as for theorem 4.4.15.

Remark 7.4.7. Let X be a finite type k-scheme of dimension d over k, and let η be an arbitrary element of $\Omega_*(X)$. It follows from the generalized degree formula (theorem 4.4.7) applied to $A_* = \Omega_*$ that $\Omega_*(X)$ is generated as an $\Omega_*(k)$-module by classes of the form $f : Y \to X \in \mathcal{M}(X)$, with $\dim_k Y \leq d$. Thus $M(X)$ is the ideal in $\Omega_*(k)$ generated by the classes $[Y]$, where Y is a projective l.c.i. k-scheme of dimension $< d$ for which there is a morphism $f : Y \to X$.

Recall from §4.4.3 the $\mathbb{Z}$-valued characteristic class s_d and the $\mathbb{F}_p$-valued characteristic classes $t_{d,r}$, $d = p^n - 1$. By lemma 7.4.4 and proposition 7.4.5, these characteristic classes extend uniquely to projective l.c.i. k-schemes Y by the formulas

$$s_d(Y) := \frac{1}{p} \deg(S_d(c_1, \ldots, c_d)([N_Y])),$$
$$t_{d,r}(Y) := \deg(t_{d,r}(c_1, \ldots, c_{dr})([N_Y])),$$

and these functions uniquely define homomorphisms

$$s_d^\Omega : \Omega_d(k) \to \mathbb{Z},$$
$$t_{d,r}^\Omega : \Omega_{dr}(k) \to \mathbb{F}_p,$$

with $s_d(Y) = s_d^\Omega([Y])$ and $t_{d,r}(Y) = t_{d,r}^\Omega([Y])$ for all projective l.c.i. k-schemes Y.

Noting these remarks, the proofs of theorem 4.4.23 and theorem 4.4.24 yield the following extensions of these results to l.c.i. schemes:

Theorem 7.4.8. *Let $f : Y \to X$ be a morphism of reduced projective l.c.i. k-schemes, both of dimension d, with X irreducible. Suppose that $d = p^n - 1$ for some prime p and some integer $n > 0$. Then there is a zero-cycle η on X such that*

$$s_d(Y) - (\deg f)s_d(X) = \deg(\eta).$$

Theorem 7.4.9. *Let $f : Y \to X$ be a morphism of reduced projective l.c.i. k-schemes, with X irreducible. Suppose both X and Y have dimension rd over k, where $r > 0$ is an integer and $d = p^n - 1$ for some prime p and some integer $n > 0$. Suppose in addition that X admits a sequence of surjective morphisms to reduced finite type k-schemes*

$$X = X_0 \to X_1 \to \ldots \to X_{r-1} \to X_r = \operatorname{Spec} k$$

such that:

1. *Each X_i is in* $\mathbf{Lci}_k$ *and $\dim_k X_i = d(r - i)$.*
2. *Let η be a zero-cycle on $X_i \times_{X_{i+1}} \operatorname{Spec} k(X_{i+1})$. Then $p | \deg(\eta)$.*

Then

$$t_{d,r}(Y) = \deg(f)t_{d,r}(X).$$

A

Resolution of singularities

In this appendix, we make precise the statements we use from resolution of singularities. We first recall a version of the well-known theorem of Hironaka [13].

For a rational map $f : Y \to X$ of reduced k-schemes of finite type, let $\mathrm{Sing}\, f$ denote the closed subset of Y of points at which Y is not smooth over k, or at which f is not a morphism.

Theorem A.1. *Let k be a field of characteristic zero, and let $f : Y \to X$ be a rational map of reduced k-schemes of finite type. Then there is a projective birational morphism $\mu : Y' \to Y$ such that*

1. *Y' is smooth over k.*
2. *The induced birational map $f \circ \mu : Y' \to X$ is a morphism.*
3. *The morphism μ can be factored as a sequence of blow-ups of Y along smooth centers lying over* $\mathrm{Sing}\, f$.

For instance, taking $f = \mathrm{Id}_Y : Y \to Y$, one has the result that there is a sequence of blow-ups of Y along smooth centers lying over $\mathrm{Sing}\, Y$ which resolves the singularities of Y.

Throughout this paper, we have used the phrase "k admits resolution of singularities" to mean that, at the least, the conclusion of theorem A.1 is valid for varieties over k. There are two additional results that we will require to be valid when we say that k admits resolution of singularities. For a field of characteristic zero these results may be found in [32]; we give a short proof here for the reader's convenience. We are indebted to D. Cutkosky for supplying the arguments given below.

Theorem A.2. *Let X be a smooth quasi-projective variety over a field k of characteristic zero, let D be a strict normal crossing divisor on X, and let S be a reduced and irreducible codimension one subscheme of X. Let V be an open subset of X containing each generic point of $|D+S|$ such that $V \cap (D+S)$ is a strict normal crossing divisor on V. Then there is a sequence of blow-ups of smooth centers lying over $X \setminus V$,*

$$X' = X_r \to \ldots \to X_0 = X$$

such that, letting E_j be the exceptional divisor of $X_j \to X$, and D_j, S_j the proper transforms of D, S to X_j:

(1) $E_j + D_j$ is a strict normal crossing divisor for all j,

(2) $E_r + D_r + S_r$ is a strict normal crossing divisor on X'.

Proof. This is a special case of [13, Theorem $\mathrm{I}_2^{N,n}$, pg. 170], where, in the notation of that result, we take $N = \dim_k X$, $n = N - 1$, and $(\mathfrak{R}_I^{N,N-1}, U)$ is the resolution datum $((|D|; X; S), V)$. □

Corollary A.3. *Let X be a quasi-projective variety over a field k of characteristic zero, and let D and D' be effective divisors on X, with D' a strict normal crossing divisor on X. Let U be a smooth open subscheme of X, containing each generic point of $|D + D'|$, such that $(D + D') \cap U$ is a strict normal crossing divisor on U.*

(1) There is a sequence of blow-ups of smooth centers lying over $X \setminus U$

$$X_r \to \ldots \to X_0 = X$$

such that, letting E be the exceptional divisor of $\mu : X_r \to X$ and D_r, D'_r the proper transform of D to X_r, $E + D_r + D'_r$, is a strict normal crossing divisor on X_r.

(2) If X is smooth, we may take U such that $X \setminus U \subset |D + D'|$. In this case, E is supported in $\mu^*(D + D')$, and we may take the sequence of blow-ups so that, letting E_j be the exceptional divisor of $X_j \to X$ and D'_j the proper transform of D' to X_j, $E_j + D'_j$ is a strict normal crossing divisor on X_j for all j.

Proof. We note that $|D'|$ is contained in the smooth locus of X. By resolving the singularities of X via a sequence of blow-ups of smooth centers lying over X_{sing} [13, Main Theorem I*, pg.132], and taking the proper transforms of D and D', we reduce to the case of a smooth X. We may assume that D is reduced; write $D = \sum_{i=1}^m D^i$ with the D^i irreducible. We proceed by induction on m.

Let $D^* = \sum_{i=1}^{m-1} D^i$. Assuming the result for $m - 1$, we have a sequence of blow-ups as above such that $E_r + D_r^* + D'_r$ is a strict normal crossing divisor, and $E_j + D'_j$ is a strict normal crossing divisor for all j. Replacing X with X_r, D with the proper transform of D^m, and D' with $E_r + D_r^* + D'_r$, we reduce to the case $m = 1$, which is theorem A.2. □

We conclude with the statement of the result on weak factorization that we will need.

Conjecture A.4 (Weak factorization). *Let $\mu : \tilde{Y} \to Y$ be a projective birational morphism in $\mathbf{Sm}_k$. Then there is a commutative diagram of projective Y-schemes, with each $Y^i \in \mathbf{Sm}_k$,*

$$\begin{tikzcd}
& Y^1 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & Y^n \arrow[dl] \arrow[dr] & \\
\tilde{Y} = Y^0 \arrow[rrrrrr, bend right, "\mu"'] & & Y^2 & \cdots & Y^{n-1} & & Y^{n+1} = Y
\end{tikzcd}$$

such that each slanted arrow is the blow-up of the base along a smooth (over k) subscheme.

This weak factorization conjecture is consequence of the main result of [2] or [37] in case k has characteristic zero; in case the conjecture is true for a given field k, we will say that k *admits weak factorization.*

References

1. Séminaire Chevalley: Classification de groupes de Lie, **t. 1**, (1956-1958)
2. Abramovich, D., Karu, K. , Matsuki, K., Wlodarczyk, J.: Torification and factorization of birational maps. J. Amer. Math. Soc. **15** no. 3, 531–572 (2002)
3. Adams, J.F.: Stable homotopy and generalised homology. Chicago Lectures in Mathematics. University of Chicago Press, Chicago-London, 1974.
4. Borghesi, S.: Algebraic Morava K-theories and the higher degree formula. Preprint. http://www.math.uiuc.edu/K-theory/0412/ (2000)
5. Conner, P.E., Floyd, E.E.: The relation of cobordism to K-theories. Lecture Notes in Mathematics, **28** Springer-Verlag, Berlin-New York (1966)
6. de Jong, A.J.: Smoothness, semi-stability and alterations. Publ. Math. Inst. Hautes Études Sci., **83** (1996)
7. Douady, A., Verdier, J.-L.: Séminaire de géométrie analytique.
8. Eagon, J., Hochster, M.: Cohen-Macaulay rings, invariant theory and the generic perfection of determinantal loci, Amer.J.Math. **93**, 1020–1058 (1971)
9. Fulton, W.: Intersection theory. Ergebnisse der Mathematik und ihrer Grenzgebiete, (3) **2**, Springer-Verlag, Berlin-New York, (1984)
10. Fulton, W.: Rational equivalence on singular varieties. Publ. Math. Inst. Hautes Études Sci., **45**, 147–167 (1975)
11. Grothendieck, A.: La théorie des classes de Chern, Bull. Soc. Math. France, **86**, 137–154 (1958)
12. Hartshorne, R.: Algebraic Geometry. Graduate Texts in Math., **52**, Springer-Verlag, Berlin-New York, 1977.
13. Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. Ann. of Math., (2) **79**, 109–203 (1964) ; ibid. (2) **79**, 205–326 (1964)
14. Jouanolou, J.-P.: Une suite exacte de Mayer-Vietoris en K-théorie algébrique. In: Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972). 293–316. Lecture Notes in Math., **341**, Springer, Berlin (1973)
15. Kleiman, S.: The transversality of a general translate. Comp. Math., **28** fasc. 3, 287–297 (1974)
16. Lazard, M.: Sur les groupes de Lie formels à un paramètre. Bull. Soc. Math. France, **83**, 251–274 (1955)

17. Levine, M.: Fundamental classes in algebraic cobordism. K-Theory, **30** (2), 129–135 (2003)
18. Levine, M., Morel, F.: Algebraic Cobordism I. Preprint. http://www.math.uiuc.edu/K-theory/0547/ (2002)
19. Levine, M.: Algebraic Cobordism II. Preprint. http://www.math.uiuc.edu/K-theory/0577/ (2002)
20. Levine, M., Morel, F.: Cobordisme algébrique I. C.R. Acad. Sci. Paris, **332** Série I, 723–728 (2001)
21. Levine, M., Morel, F.: Cobordisme algébrique II. C.R. Acad. Sci. Paris, **332** Série I, 815–820 (2001)
22. Milnor, J.: On the cobordism ring Ω^* and a complex analogue. I. Amer. J. Math., **82**, 505–521 (1960)
23. Merkurjev, A.: Steenrod operations and degree formulas. J. Reine Angew. Math., **565**, 13–26 (2003)
24. Merkurjev, A.: Algebraic oriented cohomology theories. In: Algebraic number theory and algebraic geometry, 171–193, Contemp. Math., **300**, Amer. Math. Soc., Providence, RI (2002)
25. Murthy, P.: Zero cycles and projective modules. Ann. of Math., (2) **140**, 405–434 (1994)
26. Novikov, S.: Algebraic topological methods from the point of view of cobordism theory, Izv. Akad. Nausk SSSR, **31**, 855-951 (1967) [Math. USSR-Izv., **1**, 827–913 (1967)]
27. Panin, I.: Oriented cohomology theories of algebraic varieties. Special issue in honor of Hyman Bass on his seventieth birthday. Part III. K-Theory, **30** no. 3, 265–314 (2003)
28. Panin, I.: Riemann-Roch theorems for oriented cohomology. In: Axiomatic, enriched and motivic homotopy theory, 261–333, NATO Sci. Ser. II Math. Phys. Chem., **131**, Kluwer Acad. Publ., Dordrecht (2004)
29. Quillen, D.G.: On the formal group laws of unoriented and complex cobordism theory. Bull. Amer. Math. Soc., **75**, 1293–1298 (1969)
30. Quillen, D.G.: Elementary proofs of some results of cobordism theory using Steenrod operations. Advances in Math., **7**, 29–56 (1971)
31. Ravenel, D.C.: Complex cobordism and stable homotopy groups of spheres. Pure and Applied Mathematics, **121**, Academic Press, Inc. (1986)
32. Szabó, E.: Divisorial log terminal singularities. J. Math. Sci. Univ. Tokyo, **1** no. 3, 631–639 (1994)
33. Thom, R.: Quelques propriétés globales des variétés différentiables. Comment. Math. Helv., **28**, 17–86 (1954)
34. Totaro, B.: The Chow ring of a classifying space. In: Algebraic K-theory (Seattle, WA, 1997), 249–281, Proc. Sympos. Pure Math., **67**, Amer. Math. Soc., Providence, RI (1999)
35. Voevodsky, V.: Motivic cohomology with $\mathbb{Z}/2$-coefficients. Publ. Math. Inst. Hautes Études Sci., **98**, 59–104 (2003)
36. Voevodsky, V.: $\mathbb{A}^1$-homotopy theory. Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998). Doc. Math., Extra Vol. I, 579–604 (1998)
37. Wlodarczyk, J.: Toroidal varieties and the weak factorization theorem, Invent. Math., **154** no. 2, 223–331 (2003)

Index

Glossary of Notation

$\mathbf{Sch}_S$: separated schemes of finite type over S IX
$\mathbf{Sm}_S$: smooth quasi-projective schemes over S IX
π_X: structure morphism IX
$\mathcal{O}_X$: structure sheaf IX
O_X: trivial line bundle IX
$\mathcal{O}_X(D)$: invertible sheaf determined by a divisor D IX
$O_X(D)$: line bundle associated to $\mathcal{O}_X(D)$ IX
$\mathcal{O}_X(E)$: sheaf of sections of a vector bundle E IX
$\mathbb{P}(\mathcal{E})$: projective space bundle X
O_X^n: trivial rank n bundle X
γ_n: tautological quotient line bundle on $\mathbb{P}^n$ X
$\dim_S(Z,z)$: dimension of Z over S at $z \in Z$ 1
$\mathbf{R}^*$: category of commutative, graded rings with unit 2
$\mathbb{L}$: Lazard ring 4
$F_{\mathbb{L}}$: universal formal group law 4
Φ_A 5
CH^*: Chow ring 5
$K^0(X)$: Grothendieck group of locally free coherent sheaves on X 5
$K^0(X)[\beta,\beta^{-1}]$ 5
Ω^*: algebraic cobordism 7
$M(X)$ 10
N_d: Newton polynomial 10
$S_d(X)$: Segre class 11
$s_d(X)$ 11
$\mathcal{V}'$: subcategory of projective morphisms in $\mathcal{V}$ 17
$\mathbf{Ab}_*$: category of graded abelian groups 17
$\tilde{c}_1$: first Chern class operator 18
A_*^{BM} 19
$\mathcal{M}(X)$ 19
$\mathcal{M}_*^+(X)$ 19
$\mathcal{Z}_*(X)$: cobordism cycles on X 21
$[f : Y \to X, L_1, \ldots, L_r]$: class of a cobordism cycle 21
$\bar{A}_*$ 24
$[f : Y \to X, L_1, \ldots, L_r]_A$: standard cycle 24
$H_*/\mathcal{R}_*$: quotient Borel-Moore functor 25
$\langle \mathcal{R}_* \rangle$: oriented Borel-Moore sub-functor generated by $\mathcal{R}_*$ 25
$G_0[\beta,\beta^{-1}]$ 28
$[F(L_1,\ldots,L_r)]$ 36
$\underline{\mathcal{Z}}_*$ 36
$\mathcal{R}_*^{Dim}$ 36
$\mathcal{R}_*^{Sect}$ 37
$\underline{\Omega}_*$: algebraic pre-cobordism 37
$\mathcal{R}_*^{FGL}$ 38
Ω_*: algebraic cobordism 38
ϑ_A 39
$H_{n,m}$: Milnor hypersurface 44
$F^{n_1,\ldots,n_m}$ 52
$||J||$ 52
$F_J^{n_1,\ldots,n_m}$ 52
E_I: face of a strict normal crossing divisor E 53

$|E|$: support of a divisor E 53
$[E \to |E|]$: cobordism class of a strict normal crossing divisor E 53
$\langle W \times 0 \rangle$ 56
$\langle T \times \mathbb{P}^1 \rangle$ 56
Ω_*^F 77
$\tilde{c}_i$: Chern class operator 93
$\widetilde{\mathrm{Td}}_\tau^{-1}$: inverse Todd class operator 97
$\tilde{c}_I$: Conner-Floyd Chern class operator 99
$A_*^{(\tau)}$: twisted theory (first form) 100
T_f: tangent bundle of a smooth f 100
N_f: virtual normal bundle 100
$\tilde{c}_1^{(\tau)}$: twisted first Chern class operator 100
$\widetilde{\mathrm{Td}}_\tau$: Todd class operator 100
T_f: virtual tangent bundle 101
A_*^τ: twisted theory (second form) 101
Ω_*^{ad}: algebraic cobordism made additive 103
$\mathbf{Lci}_S$: category of local complete intersection schemes over S 143
$[f : Y \to X]_A$ 146
$|D|$: support of a pseudo-divisor D 160
$\mathcal{M}(X)_D$ 161
$\mathcal{M}^+(X)_D$ 161
$\mathcal{Z}_*(X)_D$ 161
$\langle \mathcal{R}_*^{Dim} \rangle (X)_D$ 161
$\underline{\mathcal{Z}}_*(X)_D$ 161
$\langle \mathcal{R}_*^{Sect} \rangle (X)_D$ 163
$\underline{\Omega}_*(X)_D$ 163
$\langle \mathbb{L}_* \mathcal{R}_*^{FGL} \rangle (X)_D$ 163
$\Omega_*(X)_D$ 163
$[F(L_1, \ldots, L_r)]_D$ 164
$[E \to |E|]_D$: refined divisor class 164
$\mathcal{M}(X)_{D|D'}$ 165
$\mathcal{Z}_*(X)_{D|D'}$ 166
$\langle \mathcal{R}_*^{Dim} \rangle (X)_{D|D'}$ 166
$\underline{\mathcal{Z}}_*(X)_{D|D'}$ 166
$\underline{\Omega}_*(X)_{D|D'}$ 166
$\langle \mathcal{R}_*^{Sect} \rangle (X)_{D|D'}$ 166
$\Omega_*(X)_{D|D'}$ 166
$\langle \mathbb{L}_* \mathcal{R}_*^{FGL} \rangle (X)_{D|D'}$ 166
$[E \to |E|]_{D|D'}$: refined divisor class 167
$D(-)$: intersection homomorphism for a pseudo-divisor D 170
$D(-)_{D'}$: refined intersection homomorphism 170
$C_W Y$: normal cone of W in Y 201
$i^!_{i'}$: refined Gysin morphism 203
$i^!$: refined Gysin morphism 203
$f^!_{f'}$: refined pull-back 208
$f^!$: refined pull-back 208
$[E \to |E|]_A$: divisor class in $A^*(|E|)$ 218

Springer Monographs in Mathematics

This series publishes advanced monographs giving well-written presentations of the "state-of-the-art" in fields of mathematical research that have acquired the maturity needed for such a treatment. They are sufficiently self-contained to be accessible to more than just the intimate specialists of the subject, and sufficiently comprehensive to remain valuable references for many years. Besides the current state of knowledge in its field, an SMM volume should also describe its relevance to and interaction with neighbouring fields of mathematics, and give pointers to future directions of research.

Abhyankar, S.S. **Resolution of Singularities of Embedded Algebraic Surfaces** 2nd enlarged ed. 1998
Alexandrov, A.D. **Convex Polyhedra** 2005
Andrievskii, V.V.; Blatt, H.-P. **Discrepancy of Signed Measures and Polynomial Approximation** 2002
Angell, T.S.; Kirsch, A. **Optimization Methods in Electromagnetic Radiation** 2004
Ara, P.; Mathieu, M. **Local Multipliers of C*-Algebras** 2003
Armitage, D.H.; Gardiner, S.J. **Classical Potential Theory** 2001
Arnold, L. **Random Dynamical Systems** corr. 2nd printing 2003 (1st ed. 1998)
Arveson, W. **Noncommutative Dynamics and E-Semigroups** 2003
Aubin, T. **Some Nonlinear Problems in Riemannian Geometry** 1998
Auslender, A.; Teboulle, M. **Asymptotic Cones and Functions in Optimization and Variational Inequalities** 2003
Banagl, M. **Topological Invariants of Stratified Spaces** 2006
Banasiak, J.; Arlotti, L. **Perturbations of Positive Semigroups with Applications** 2006
Bang-Jensen, J.; Gutin, G. **Digraphs** 2001
Baues, H.-J. **Combinatorial Foundation of Homology and Homotopy** 1999
Böttcher, A.; Silbermann, B. **Analysis of Toeplitz Operators** 2006
Brown, K.S. **Buildings** 3rd printing 2000 (1st ed. 1998)
Chang, K.-C. **Methods in Nonlinear Analysis** 2005
Cherry, W.; Ye, Z. **Nevanlinna's Theory of Value Distribution** 2001
Ching, W.K. **Iterative Methods for Queuing and Manufacturing Systems** 2001
Chudinovich, I. **Variational and Potential Methods for a Class of Linear Hyperbolic Evolutionary Processes** 2005
Coates, J.; Sujatha, R. **Cyclotomic Fields and Zeta Values** 2006
Crabb, M.C.; James, I.M. **Fibrewise Homotopy Theory** 1998
Dineen, S. **Complex Analysis on Infinite Dimensional Spaces** 1999
Dugundji, J.; Granas, A. **Fixed Point Theory** 2003
Ebbinghaus, H.-D.; Flum, J. **Finite Model Theory** 2006
Edmunds, D.E.; Evans, W.D. **Hardy Operators, Function Spaces and Embeddings** 2004
Elstrodt, J.; Grunewald, F.; Mennicke, J. **Groups Acting on Hyperbolic Space** 1998
Engler, A.J.; Prestel, A. **Valued Fields** 2005
Fadell, E.R.; Husseini, S.Y. **Geometry and Topology of Configuration Spaces** 2001
Fedorov, Yu. N.; Kozlov, V.V. **A Memoir on Integrable Systems** 2001
Flenner, H.; O'Carroll, L.; Vogel, W. **Joins and Intersections** 1999
Gelfand, S.I.; Manin, Y.I. **Methods of Homological Algebra** 2nd ed. 2003 (1st ed. 1996)
Griess, R.L. Jr. **Twelve Sporadic Groups** 1998
Gras, G. **Class Field Theory** corr. 2nd printing 2005
Greuel, G.-M.; Lossen, C.; Shustin, E. **Introduction to Singularities and Deformations** 2007
Hida, H. ***p*-Adic Automorphic Forms on Shimura Varieties** 2004
Ischebeck, E; Rao, R.A. **Ideals and Reality** 2005
Ivrii, V. **Microlocal Analysis and Precise Spectral Asymptotics** 1998
Jakimovski, A.; Sharma, A.; Szabados, J. **Walsh Equiconvergence of Complex Interpolating Polynomials** 2006
Jech, T. **Set Theory** (3rd revised edition 2002)
Jorgenson, J.; Lang, S. **Spherical Inversion on SLn (R)** 2001
Kanamori, A. **The Higher Infinite** corr. 2nd printing 2005 (2nd ed. 2003)
Kanovei, V. **Nonstandard Analysis, Axiomatically** 2005

Khoshnevisan, D. **Multiparameter Processes** 2002
Koch, H. **Galois Theory of p-Extensions** 2002
Komornik, V. **Fourier Series in Control Theory** 2005
Kozlov, V.; Maz'ya, V. **Differential Equations with Operator Coefficients** 1999
Lam, T.Y. **Serre's Problem on Projective Modules** 2006
Landsman, N.P. **Mathematical Topics between Classical & Quantum Mechanics** 1998
Leach, J.A.; Needham, D.J. **Matched Asymptotic Expansions in Reaction-Diffusion Theory** 2004
Lebedev, L.P.; Vorovich, I.I. **Functional Analysis in Mechanics** 2002
Lemmermeyer, F. **Reciprocity Laws: From Euler to Eisenstein** 2000
Malle, G.; Matzat, B.H. **Inverse Galois Theory** 1999
Mardesic, S. **Strong Shape and Homology** 2000
Margulis, G.A. **On Some Aspects of the Theory of Anosov Systems** 2004
Miyake, T. **Modular Forms** 2006
Murdock, J. **Normal Forms and Unfoldings for Local Dynamical Systems** 2002
Narkiewicz, W. **Elementary and Analytic Theory of Algebraic Numbers** 3rd ed. 2004
Narkiewicz, W. **The Development of Prime Number Theory** 2000
Neittaanmaki, P.; Sprekels, J.; Tiba, D. **Optimization of Elliptic Systems. Theory and Applications** 2006
Onishchik, A.L. **Projective and Cayley–Klein Geometries** 2006
Parker, C.; Rowley, P. **Symplectic Amalgams** 2002
Peller, V. (Ed.) **Hankel Operators and Their Applications** 2003
Prestel, A.; Delzell, C.N. **Positive Polynomials** 2001
Puig, L. **Blocks of Finite Groups** 2002
Ranicki, A. **High-dimensional Knot Theory** 1998
Ribenboim, P. **The Theory of Classical Valuations** 1999
Rowe, E.G.P. **Geometrical Physics in Minkowski Spacetime** 2001
Rudyak, Y.B. **On Thorn Spectra, Orientability and Cobordism** 1998
Ryan, R.A. **Introduction to Tensor Products of Banach Spaces** 2002
Saranen, J.; Vainikko, G. **Periodic Integral and Pseudodifferential Equations with Numerical Approximation** 2002
Schneider, P. **Nonarchimedean Functional Analysis** 2002
Serre, J-P. **Complex Semisimple Lie Algebras** 2001 (reprint of first ed. 1987)
Serre, J-P. **Galois Cohomology** corr. 2nd printing 2002 (1st ed. 1997)
Serre, J-P. **Local Algebra** 2000
Serre, J-P. **Trees** corr. 2nd printing 2003 (1st ed. 1980)
Smirnov, E. **Hausdorff Spectra in Functional Analysis** 2002
Springer, T.A.; Veldkamp, F.D. **Octonions, Jordan Algebras, and Exceptional Groups** 2000
Székelyhidi, L. **Discrete Spectral Synthesis and Its Applications** 2006
Sznitman, A.-S. **Brownian Motion, Obstacles and Random Media** 1998
Taira, K. **Semigroups, Boundary Value Problems and Markov Processes** 2003
Talagrand, M. **The Generic Chaining** 2005
Tauvel, P.; Yu, R.W.T. **Lie Algebras and Algebraic Groups** 2005
Tits, J.; Weiss, R.M. **Moufang Polygons** 2002
Uchiyama, A. **Hardy Spaces on the Euclidean Space** 2001
Üstünel, A.-S.; Zakai, M. **Transformation of Measure on Wiener Space** 2000
Vasconcelos, W. **Integral Closure. Rees Algebras, Multiplicities, Algorithms** 2005
Yang, Y. **Solitons in Field Theory and Nonlinear Analysis** 2001
Zieschang, P.-H. **Theory of Association Schemes** 2005