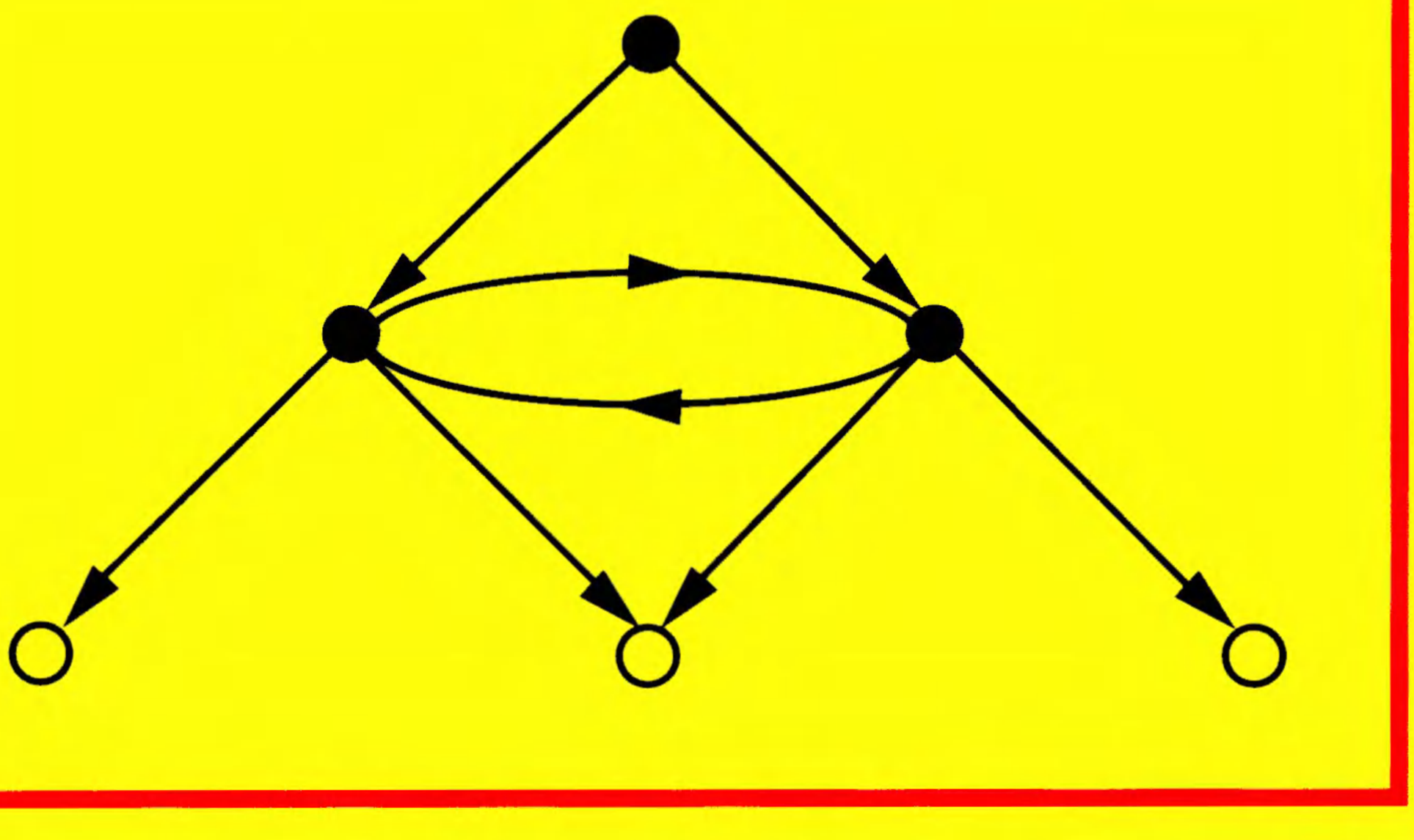
Undergraduate Texts in Mathematics

Keith Devlin

The Joy of Sets

Fundamentals of Contemporary Set Theory

Second Edition



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Fundamentals of Contemporary Set Theory

Second Edition

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Preface

This book provides an account of those parts of contemporary set theory of direct relevance to other areas of pure mathematics. The intended reader is either an advanced-level mathematics undergraduate, a beginning graduate student in mathematics, or an accomplished mathematician who desires or needs some familiarity with modern set theory. The book is written in a fairly easy-going style, with minimal formalism.

In Chapter 1, the basic principles of set theory are developed in a 'naive' manner. Here the notions of 'set', 'union', 'intersection', 'power set', 'relation', 'function', etc., are defined and discussed. One assumption in writing Chapter 1 has been that, whereas the reader may have met all of these concepts before and be familiar with their usage, she[1] may not have considered the various notions as forming part of the continuous development of a pure subject (namely, set theory). Consequently, the presentation is at the same time rigorous and fast.

Chapter 2 develops the theory of sets proper. Starting with the naive set theory of Chapter 1, I begin by asking the question 'What is a set?' Attempts to give a rigorous answer lead naturally to the axioms of set theory introduced by Zermelo and Fraenkel, which is the system taken as basic in this book. (Zermelo–Fraenkel set theory is in fact the system now accepted in 'contemporary set theory'.) Great emphasis is placed on the evolution of the axioms as 'inevitable' results of an analysis of a highly intuitive notion. For, although set theory has to be developed as an axiomatic theory, occupying as it does a well-established foundational position in mathematics, the axioms themselves must be 'natural'; otherwise everything would reduce to a meaningless game with prescribed rules. After developing the axioms, I go on to discuss the recursion principle—which plays a central role in the development of set theory but is nevertheless still widely misunderstood and rarely appreciated fully—and the Axiom of Choice, where I prove all of the usual variants, such as Zorn's Lemma.

[1] I use both 'he' and 'she' as gender-neutral pronouns interchangeably throughout the book.

Chapter 3 deals with the two basic number systems, the ordinal numbers, and the cardinal numbers. The arithmetics of both systems are developed sufficiently to allow for most applications outside set theory.

In Chapter 4, I delve into the subject set theory itself. Since contemporary set theory is a very large subject, this foray is of necessity very restricted. I have two aims in including it. First, it provides good examples of the previous theory. And second, it gives the reader some idea of the flavor of at least some parts of pure set theory.

Chapter 5 presents a modification of Zermelo–Fraenkel set theory. The Zermelo–Fraenkel system has a major defect as a foundational subject. Many easily formulated problems cannot be solved in the system. The Axiom of Constructibility is an axiom that, when added to the Zermelo–Fraenkel system, eliminates most, if not all, of these undecidable problems.

In Chapter 6, I give an account of the method by which one can *prove* within the Zermelo–Fraenkel system that various statements are themselves *not provable* in that system.

Chapters 5 and 6 are nonrigorous. My aim is to *explain* rather than *develop*. They are included because of their relevance to other areas of mathematics. A detailed investigation of these topics would double the length of this book at the very least and as such is the realm of the set-theorist, though I would, of course, be delighted to think that any of my readers would be encouraged to go further into these matters.

Finally, in Chapter 7, I present an introductory account of an alternative conception of set theory that has proved useful in computer science (and elsewhere), the non-well-founded set theory of Peter Aczel.

Chapters 1 through 3 contain numerous easy exercises. In Chapters 1 and 2, they are formally designated as 'Exercises' and are intended for solution as the reader proceeds. The aim is to provide enough material to help the student understand fully the concepts that are introduced. In Chapter 3, the exercises take the form of simple proofs of basic lemmas, which are left to the reader to provide. Again, the aim is to assist the reader's comprehension.

At the end of each of Chapters 1 through 3, there is also a small selection of problems. These are more challenging than the exercises and constitute digressions from, or extensions of, the main development. In some instances the reader may need to seek assistance in order to do these problems.

This book is a greatly expanded second edition of my earlier *Fundamentals of Contemporary Set Theory*, published by Springer-Verlag in 1979. In addition to the various changes I have made to my original account, I could not resist a change in title, relegating the title of the first edition to a subtitle for the second, thereby enabling me to join the growing ranks of *Joy*

books, which began many years ago with *The Joy of Cooking*, achieved worldwide fame, and a certain notoriety, with *The Joy of Sex*, and more recently moved into the mathematical world with *The Joy of* TEX. (This is by no means an exhaustive list.)

The basis for the first edition was a series of lectures I gave at the University of Bonn, Germany, in the years 1975 and 1976. Chapter 7 is entirely new; its inclusion reflects the changing nature of set theory, as a foundational subject influenced by potential applications. Apart from this addition, the remainder of the account is largely as in the first edition, apart from some stylistic changes and the correction of some minor errors.

I wrote this new edition during the spring of 1992. At that time, I was the Carter Professor of Mathematics at Colby College, in Maine. The manuscript was prepared on an Apple Macintosh IIcx computer running the TEXTURES implementation of TEX together with LATEX. I started with an electronic version of the first edition produced during the summer of 1990 by Mehmet Darmar, a Colby mathematics graduate of the Class of 1990, supported by a Colby College faculty assistant summer stipend. Mehmet first created an electronic version of the original book using an optical character reader, and then massaged it into a LATEX document I could work on. The final manuscript was carefully combed for errors by my Colby students Stuart Pitrat and Amy Richters.

KEITH DEVLIN

Contents

1

Naive Set Theory

Zermelo–Fraenkel set theory, which forms the main topic of the book, is a rigorous theory, based on a precise set of axioms. However, it is possible to develop the theory of sets considerably without any knowledge of those axioms. Indeed, the axioms can only be fully understood after the theory *has* been investigated to some extent. This state of affairs is to be expected. The concept of a 'set of objects' is a very intuitive one, and, with care, considerable, sound progress may be made on the basis of this intuition alone. Then, by analyzing the nature of the 'set' concept on the basis of that initial progress, the axioms may be 'discovered' in a perfectly natural manner.

Following standard practice, I refer to the initial, intuitive development as 'naive set theory'. A more descriptive, though less concise, title would be 'set theory from the naive viewpoint'. Once the axioms have been introduced, this account of 'naive set theory' can be re-read, without any changes being necessary, as the elementary development of *axiomatic* set theory.

1.1 What is a Set?

In naive set theory we assume the existence of some given domain of 'objects', out of which we may build sets. Just what these objects are is of no interest to us. Our only concern is the behavior of the 'set' concept. This is, of course, a very common situation in mathematics. For example, in algebra, when we discuss a group, we are (usually) not interested in what the elements of the group are, but rather in the way the group operation acts upon those elements. When we come to develop our set theory axiomatically we shall, in fact, remove this assumption of an initial domain, since *everything* will then be a set; but that comes much later.

In set theory, there is really only one fundamental notion:

The ability to regard any collection of objects as a single entity (i.e. as a set).

It is by asking ourselves what may and what may not determine 'a collection' that we shall arrive at the axioms of set theory. For the present, we regard the two words 'set' and 'collection (of objects)' as synonymous and understood.

If a is an object and x is a set, we write

$$a \in x$$

to mean that a is an *element* of (or member of) x , and

$$a \notin x$$

to mean that a is not an element of x.

In set theory, perhaps more than in any other branch of mathematics, it is vital to set up a collection of symbolic abbreviations for various logical concepts. Because the basic assumptions of set theory are absolutely minimal, all but the most trivial assertions about sets tend to be logically complex, and a good system of abbreviations helps to make otherwise complex statements readable. For instance, the symbol $\in$ has already been introduced to abbreviate the phrase *'is an element of'*. I also make considerable use of the following (standard) logical symbols:

$\rightarrow$	abbreviates	'implies'
$\leftrightarrow$	abbreviates	'if and only if'
$\neg$	abbreviates	'not'
$\wedge$	abbreviates	'and'
$\vee$	abbreviates	'or'
$\forall$	abbreviates	'for all'
$\exists$	abbreviates	'there exists'.

Note that in the case of 'or' we adopt the usual, mathematical interpretation, whereby $\phi \vee \psi$ means that either ϕ is true or ψ is true, or else both ϕ and ψ are true, where ϕ, ψ denote any assertions in any language.

The above logical notions are not totally independent, of course. For instance, for any statements, we have

$$\begin{aligned}
\phi \leftrightarrow \psi \quad &\text{is the same as} \quad (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi) \\
\phi \rightarrow \psi \quad &\text{is the same as} \quad (\neg\phi) \vee \psi \\
\phi \vee \psi \quad &\text{is the same as} \quad \neg((\neg\phi) \wedge (\neg\psi)) \\
\exists x\phi \quad &\text{is the same as} \quad \neg((\forall x)(\neg\phi))
\end{aligned}$$

where the phrase 'is the same as' means that the two expressions are logically equivalent.

Exercise 1.1.1. *Let $\phi \dot{\vee} \psi$ mean that exactly one of ϕ, ψ is true. Express $\phi \dot{\vee} \psi$ in terms of the symbols introduced above.*

Let us return now to the notion of a set. Since a set is the same as a collection of objects, a set will be uniquely determined once we know what its elements are. In symbols, this fact can be expressed as follows:

$$x = y \leftrightarrow \forall a[(a \in x) \leftrightarrow (a \in y)].$$

This principle will, in fact, form one of our axioms of set theory: the *Axiom of Extensionality.*

If x, y are sets, we say x is a subset of y if and only if every element of x is an element of y, and write

$$x \subseteq y$$

in this case. In symbols, this definition reads[1]

$$(x \subseteq y) \leftrightarrow \forall a[(a \in x) \rightarrow (a \in y)].$$

We write

$$x \subset y$$

in case x is a subset of y and x is not equal to y; thus:

$$(x \subset y) \leftrightarrow (x \subseteq y) \wedge (x \neq y)$$

where, as usual, we write $x \neq y$ instead of $\neg(x = y)$, just as we did with $\in$. Clearly we have

$$(x = y) \leftrightarrow [(x \subseteq y) \wedge (y \subseteq x)].$$

Exercise 1.1.2. *Check the above assertion by replacing the subset symbol by its definition given above, and reducing the resulting formula logically to the axiom of extensionality. Is the above statement an equivalent formulation of the axiom of extensionality?*

[1]The reader should attain the facility of 'reading' symbolic expressions such as this as soon as possible. In more complex situations the symbolic form can be by far the most intelligible one.

1.2 Operations on Sets

There are a number of simple operations that can be performed on sets, forming new sets from given sets. I consider below the most common of these.

If x and y are sets, the *union* of x and y is the set consisting of the members of x together with the members of y, and is denoted by

$$x \cup y.$$

Thus, in symbols, we have

$$(z = x \cup y) \leftrightarrow \forall a[(a \in z) \leftrightarrow (a \in x \vee a \in y)].$$

In the above, in order to avoid proliferation of brackets, I have adopted the convention that the symbol $\in$ predominates over logical symbols. This convention, and a similar one for $=$, will be adhered to throughout. An alternative way of denoting the above definition is

$$(a \in x \cup y) \leftrightarrow (a \in x \vee a \in y).$$

Using this last formulation, it is easy to show that the union operation on sets is both commutative and associative; thus

$$x \cup y = y \cup x,$$

$$x \cup (y \cup z) = (x \cup y) \cup z.$$

The beginner should check these and any similar assertions made in this chapter.

The *intersection* of sets x and y is the set consisting of those objects that are members of both x and y, and is denoted by

$$x \cap y.$$

Thus

$$(a \in x \cap y) \leftrightarrow (a \in x \wedge a \in y).$$

The intersection operation is also commutative and associative.

The (*set-theoretic*) *difference* of sets x and y is the set consisting of those elements of x that are not elements of y, and is denoted by

$$x - y.$$

Thus

$$(a \in x - y) \leftrightarrow (a \in x \wedge a \notin y).$$

Care should be exercised with the difference operation at first. Notice that $x - y$ is always defined and is always a subset of x, regardless of whether y is a subset of x or not.

Exercise 1.2.1. *Prove the following assertions directly from the definitions. The drawing of 'Venn diagrams' is forbidden; this is an exercise in the manipulation of logical formalisms.*

(i) $x \cup x = x \quad ; \quad x \cap x = x$;

(ii) $x \subseteq x \cup y \quad ; \quad x \cap y \subseteq x$;

(iii) $[(x \subseteq z) \wedge (y \subseteq z)] \rightarrow [x \cup y \subseteq z]$;

(iv) $[(z \subseteq x) \wedge (z \subseteq y)] \rightarrow [z \subseteq x \cap y]$;

(v) $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$;

(vi) $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$;

(vii) $(x \subseteq y) \leftrightarrow (x \cap y = x) \leftrightarrow (x \cup y = y)$.

Exercise 1.2.2. *Let x, y be subsets of a set z. Prove the following assertions:*

(i) $z - (z - x) = x$;

(ii) $(x \subseteq y) \leftrightarrow [(z - y) \subseteq (z - x)]$;

(iii) $x \cup (z - x) = z$;

(iv) $z - (x \cup y) = (z - x) \cap (z - y)$;

(v) $z - (x \cap y) = (z - x) \cup (z - y)$.

Exercise 1.2.3. *Prove that for any sets x, y,*

$$x - y = x - (x \cap y).$$

In set theory, it is convenient to regard the collection of no objects as a set, the *empty* (or *null*) *set*. This set is usually denoted by the symbol $\emptyset$, a derivation from a Scandinavian letter.

Exercise 1.2.4. *Prove, from the axiom of extensionality, that there is only one empty set. (This requires a sound mastery of the elementary logical concepts introduced earlier.)*

Two sets x and y are said to be *disjoint* if they have no members in common; in symbols,

$$x \cap y = \emptyset.$$

Exercise 1.2.5. *Prove the following*:

(i) $x - \emptyset = x$;

(ii) $x - x = \emptyset$;

(iii) $x \cap (y - x) = \emptyset$;

(iv) $\emptyset \subseteq x$.

1.3 Notation for Sets

Suppose we wish to provide an accurate description of a set x. How can we do this? Well, if the set concerned is finite, we can enumerate its members: if x consists of the objects $a_1, \ldots, a_n$, we can denote x by

$$\{a_1, \ldots, a_n\}.$$

Thus, the statement

$$x = \{a_1, \ldots, a_n\}$$

should be read as 'x is the set whose elements are $a_1, \ldots, a_n$'. For example, the *singleton* of a is the set

$$\{a\}$$

and the *doubleton* of a, b is the set

$$\{a, b\}.$$

In the case of infinite sets, we sometimes write

$$\{a_1, a_2, a_3, \ldots\}$$

to denote the set whose elements are precisely

$$a_1, a_2, a_3, \ldots .$$

An alternative notation is possible in the case where the set concerned is defined by some *property* P: if x is the set of all those a for which $P(a)$ holds, we may write

$$x = \{a \mid P(a)\}.$$

Thus, for example, the set of all real numbers may be denoted by

$$\{a \mid a \text{ is a real number}\}.$$

Exercise 1.3.1. *Prove the following equalities*:

(i) $x \cup y = \{a \mid a \in x \vee a \in y\}$;

(ii) $x \cap y = \{a \mid a \in x \wedge a \in y\}$;

(iii) $x - y = \{a \mid a \in x \wedge a \notin y\}$.

1.4 Sets of Sets

So far, I have been tacitly distinguishing between sets and objects. Admittedly, I did not restrict in any way the choice of initial objects — they could themselves be sets; but I did distinguish these initial objects from the sets of those objects that we could form. However, as I said at the beginning, the main idea in set theory is that any collection of objects can be regarded as a single entity (i.e. a set). Thus we are entitled to build sets out of entities that are themselves sets. Commencing with some given domain of objects then, we can first build sets of those objects, then sets of sets of objects, then sets of sets of sets of objects, and so on. Indeed, we can make more complicated sets, some of whose elements are basic objects, and some of which are sets of basic objects, etc.

For example, we can *define* the ordered pair of two objects a, b by

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

According to this definition, (a, b) is a set: it is a set of sets of objects.

Exercise 1.4.1. *Show that the above definition does define an ordered-pair operation; i.e. prove that for any* a, b, a', b'

$$(a, b) = (a', b') \leftrightarrow (a = a' \wedge b = b').$$

(*Don't forget the case* $a = b$.)

The inverse operations $(-)_0, (-)_1$ to the ordered pair are defined thus: if $x = (a, b)$, then $(x)_0 = a$ and $(x)_1 = b$. If x is not an ordered pair, $(x)_0$ and $(x)_1$ are undefined.

The *n-tuple* $(a_1, \ldots, a_n)$ may now be defined iteratively, thus

$$(a_1, \ldots, a_n) = ((a_1, \ldots, a_{n-1}), a_n).$$

It is clear that

$$(a_1, \ldots, a_n) = (a'_1, \ldots, a'_n) \quad \text{if and only if} \quad a_1 = a'_1 \wedge \ \ldots \ \wedge a_n = a'_n.$$

The inverse operations to the n-tuple are defined in the obvious way, so that if $x = (a_0, \ldots, a_{n-1})$, then $(x)_0^n = a_0, \ldots, (x)_{n-1}^n = a_{n-1}$.

Of course, it is not important how an ordered-pair operation is defined. What counts is its behavior. Thus, the property described in Exercise 1.4.1 is the only requirement we have of an ordered pair. In naive set theory, we could just take (a, b) as a basic, undefined operation from pairs of objects to objects. But when we come to axiomatic set theory a definition of the ordered pair operation in terms of sets, such as the one above, will be necessary. Though there are other definitions, the one given is the most common, and it is the one I shall use throughout this book.

If x is any set, the collection of *all* subsets of x is a well-defined collection of objects and, hence, may itself be regarded as an entity (i.e. set). It is called the *power set* of x, denoted by $\mathcal{P}(x)$. Thus

$$\mathcal{P}(x) = \{y \mid y \subseteq x\}.$$

Suppose now that x is a set of sets of objects. The *union* of x is the set of all elements of all elements of x, and is denoted by $\bigcup x$. Thus

$$\bigcup x = \{a \mid \exists y(y \in x \wedge a \in y)\}.$$

Extending our logical notation by writing

$$(\exists y \in x)$$

to mean 'there exists a y in x such that', this may be re-written as

$$\bigcup x = \{a \mid (\exists y \in x)(a \in y)\}.$$

The *intersection* of x is the set of all objects that are elements of all elements of x, and is denoted by $\bigcap x$. Thus

$$\bigcap x = \{a \mid \forall y(y \in x \rightarrow a \in y)\}.$$

Or, more succinctly,

$$\bigcap x = \{a \mid (\forall y \in x)(a \in y)\}$$

where $(\forall y \in x)$ means 'for all y in x'.

If $x = \{y_i \mid i \in I\}$ (so I is some indexing set for the elements of x), we often write

$$\bigcup_{i \in I} y_i$$

for $\bigcup x$ and

$$\bigcap_{i \in I} y_i$$

for $\bigcap x$. This ties in with our earlier notation to some extent, since we clearly have, for any sets x, y,

$$x \cup y = \bigcup \{x, y\}, \quad x \cap y = \bigcap \{x, y\}.$$

Exercise 1.4.2.

(i) *What are* $\bigcup \{x\}$ *and* $\bigcap \{x\}$?

(ii) *What are* $\bigcup \emptyset$ *and* $\bigcap \emptyset$?

Verify your answers.

Exercise 1.4.3. *Prove that if* $\{x_i \mid i \in I\}$ *is a family of sets, then*

(i) $\bigcup_{i \in I} x_i = \{a \mid (\exists i \in I)(a \in x_i)\}$;

(ii) $\bigcap_{i \in I} x_i = \{a \mid (\forall i \in I)(a \in x_i)\}$.

Exercise 1.4.4. *Prove the following:*

(i) $(\forall i \in I)(x_i \subseteq y) \rightarrow (\bigcup_{i \in I} x_i \subseteq y)$;

(ii) $(\forall i \in I)(y \subseteq x_i) \rightarrow (y \subseteq \bigcap_{i \in I} x_i)$;

(iii) $\bigcup_{i \in I}(x_i \cup y_i) = (\bigcup_{i \in I} x_i) \cup (\bigcup_{i \in I} y_i)$;

(iv) $\bigcap_{i \in I}(x_i \cap y_i) = (\bigcap_{i \in I} x_i) \cap (\bigcap_{i \in I} y_i)$;

(v) $\bigcup_{i \in I}(x_i \cap y) = (\bigcup_{i \in I} x_i) \cap y$;

(vi) $\bigcap_{i \in I}(x_i \cup y) = (\bigcap_{i \in I} x_i) \cup y$.

Exercise 1.4.5. *Let* $\{x_i \mid i \in I\}$ *be a family of subsets of* z. *Prove:*

(i) $z - \bigcup_{i \in I} x_i = \bigcap_{i \in I}(z - x_i)$;

(ii) $z - \bigcap_{i \in I} x_i = \bigcup_{i \in I}(z - x_i)$.

1.5 Relations

If x, y are sets, the *cartesian product* of x and y is defined to be the set

$$x \times y = \{(a, b) \mid a \in x \wedge b \in y\}.$$

More generally, if $x_1, \ldots, x_n$ are sets, we define their cartesian product by

$$x_1 \times \ldots \times x_n = \{(a_1, \ldots, a_n) \mid a_1 \in x_1 \wedge \ \ldots \ \wedge a_n \in x_n\}.$$

A *unary relation* on a set x is defined to be a subset of x. An n-ary relation on x, for $n > 1$, is a subset of the n-fold cartesian product $x \times \ldots \times x$.

Notice that an n-ary relation on x is a unary relation on the n-fold product $x \times \ldots \times x$.

These formal definitions provide a concrete realization within set theory of the intuitive concept of a relation.

However, as is often the case in set theory, having seen how a concept may be defined set-theoretically, we revert at once to the more familiar notation. For example, if P is some property that applies to pairs of elements of a set x, we often speak of 'the binary relation P on x', though strictly speaking, the relation concerned is the *set*

$$\{(a, b) \mid a \in x \wedge b \in x \wedge P(x, y)\}.$$

Also common is the tacit identification of such a property P with the relation it defines, so that $P(a, b)$ and $(a, b) \in P$ mean the same.

Similarly, going in the opposite direction, if R is some binary relation on a set x, I often write $R(a, b)$ instead of $(a, b) \in R$. Indeed, in the specific case of binary relations, I sometimes go even further, writing aRb instead of $R(a, b)$. In the case of ordering relations, this notation is, of course, very common: we rarely write $< (a, b)$ or $(a, b) \in <$, though from a set-theoretic point of view, both could be said to be more accurate than the more common notation $a < b$.

Binary relations play a particularly important role in set theory and, indeed, in mathematics as a whole. The rest of this section is devoted to a rapid review of binary relations.

There are several properties that apply to binary relations. Let R denote

any binary relation on a set x. We say:

R	is *reflexive*	if	$(\forall a \in x)(aRa)$;
R	is *symmetric*	if	$(\forall a, b \in x)(aRb \rightarrow bRa)$;
R	is *antisymmetric*	if	$(\forall a, b \in x)[(aRb \wedge a \neq b) \rightarrow \neg(bRa)]$;
R	is *connected*	if	$(\forall a, b \in x)[(a \neq b) \rightarrow (aRb \vee bRa)]$;
R	is *transitive*	if	$(\forall a, b, c \in x)[(aRb \wedge bRc) \rightarrow (aRc)]$.

Notice the obvious use of the repeated quantifier in the above, writing, for example, $(\forall a, b \in x)$ instead of the more cumbersome $(\forall a \in x)(\forall b \in x)$.

Exercise 1.5.1. *Which of the above properties are satisfied by the membership relation $\in$ on a set x ?*

A binary relation on a set is said to be an *equivalence relation* just in case it is reflexive, symmetric, and transitive. If R is an equivalence relation on a set x, the *equivalence class* of an element a of x under the equivalence relation R is defined to be the set

$$[a] = [a]_R = \{b \in x \mid aRb\}.$$

Exercise 1.5.2. *Let R be an equivalence relation on a set x. Then R partitions x into a collection of disjoint equivalence classes.*

Examples of equivalence relations pervade the whole of contemporary pure mathematics. So too do examples of our next concept, that of an ordering relation.

A *partial ordering* of a set x is a binary relation on x which is reflexive, antisymmetric, and transitive. Usually (but not always), partial orderings are denoted by the symbol $\leq$.

A *partially ordered set*, or *poset*, consists of a set x together with a partial ordering $\leq$ of x. More formally, we define the poset to be the ordered pair $(x, \leq)$.

Let $(x, \leq)$ be a poset, and let $y \subseteq x$. An element a of y is a *minimal element* of y if and only if there is no b in y such that $b < a$, where, as usual, we write $b < a$ to denote $b \leq a \wedge b \neq a$.

A poset $(x, \leq)$ is said to be *well-founded* if every nonempty subset of x has a minimal element. (Equivalently, we often say that the ordering relation $\leq$ is well-founded.)

Lemma 1.5.1 Let $(x, \leq)$ be a poset. $(x, \leq)$ is well-founded if and only if there is no sequence $\{a_n\}_{n=0}^{\infty}$ of elements of x such that $a_{n+1} < a_n$ for all n, i.e. no sequence $\{a_n\}_{n=0}^{\infty}$ such that $a_0 > a_1 > a_2 > \ldots$.

Proof: Suppose $(x, \leq)$ is not well-founded. Let $y \subseteq x$ have no minimal element. Let $a_0 \in y$. Since a_0 is not minimal in y, we can find $a_1 \in y$, $a_1 < a_0$. Again, a_1 is not minimal in y, so we can find $a_2 \in y$, $a_2 < a_1$. Proceeding inductively, we obtain a sequence $a_0 > a_1 > a_2 > \ldots$.

Now suppose there is a sequence $a_0 > a_1 > a_2 > \ldots$. Let y be the set $\{a_0, a_1, a_2, \ldots\}$. Clearly, y has no minimal member. □

The subset relation $\subseteq$ on the power set, $\mathcal{P}(x)$, of a set x clearly constitutes a partial ordering of $\mathcal{P}(x)$. Indeed, the subset relation on *any* collection of sets is a partial ordering of that collection. In fact, up to isomorphism, the subset relation is the *only* partial ordering there is, as I prove next.

Theorem 1.5.2 Let $(x, \leq)$ be a poset. Then there is a set y of subsets of x such that $(x, \leq) \cong (y, \subseteq)$.

Proof: For each $a \in x$, let $z_a = \{b \in x \mid b \leq a\}$, and let $y = \{z_a \mid a \in x\}$. Define a map π from x to y by $\pi(a) = z_a$. Clearly π is a bijection. Moreover, $a_1 \leq a_2 \leftrightarrow z_{a_1} \subseteq z_{a_2}$, so π is an isomorphism between $(x, \leq)$ and $(y, \subseteq)$. □

A *total ordering* (or *linear ordering*) of a set x is a connected, partial ordering of x. A *totally ordered set* (or *toset*) is a pair $(x, \leq)$ such that $\leq$ is a total ordering of the set x.

A *well-ordering* of a set x is a well-founded, total ordering of x. A *well-ordered set* (or *woset*) is a pair $(x, \leq)$ such that $\leq$ is a well-ordering of x. The concept of a well-ordering is central in set theory, as we see presently.

1.6 Functions

We all know, more or less, what a function is. Indeed, in Section 1.5 we have already made use of functions in stating and proving Theorem 1.5.2. But there we followed the usual mathematical practice of using the function concept without worrying too much about what a function really is. In this section we give a formal, set-theoretic definition of the function concept.

Let R be an $(n+1)$-ary relation on a set x. The *domain* of R is defined to be the set

$$\mathrm{dom}(R) = \{a \mid \exists b[(a, b) \in R]\}.$$

The *range* of R is defined to be the set

$$\text{ran}(R) = \{b \mid \exists a[(a, b) \in R]\}.$$

If $n = 1$, so that R is a binary relation, then it is clear what is meant by these definitions: elements of R are ordered pairs, $\text{dom}(R)$ is the set of first components of members of R, and $\text{ran}(R)$ the set of second components. But what if $n > 1$? In this case, any member of R will be an $(n+1)$-tuple. But what is an $(n+1)$-tuple? Well, by definition, an $(n+1)$-tuple, c, has the form (a, b) where a is an n-tuple and b is an object in x. Thus, even if $n > 1$, the elements of R will still be *ordered pairs*, only now the domain of R will consist not of elements of x but elements of the n-fold product $x \times \ldots \times x$. So in all cases, $\text{dom}(R)$ is the set of first components of members of R and $\text{ran}(R)$ is the set of second components.

Although the notions of domain and range for an arbitrary relation are quite common in more advanced parts of set theory, chances are that the reader is not used to these concepts. But when we define the notion of a function as a special sort of relation, as we do below, you will see at once that the above definitions coincide with what one usually means by the 'domain' and 'range' of a function.

An n-ary *function* on a set x is an $(n+1)$-ary relation, R, on x such that for every $a \in \text{dom}(R)$ there is exactly one $b \in \text{ran}(R)$ such that $(a, b) \in R$.

As usual, if R is an n-ary function on x and $a_1, \ldots, a_n, b \in x$, we write

$$R(a_1, \ldots, a_n) = b$$

instead of

$$(a_1, \ldots, a_n, b) \in R.$$

Exercise 1.6.1. *Comment on the assertion that a set-theorist is a person for whom all functions are unary. (This is a serious exercise, and concerns a subtle point which often causes problems for the beginner.)*

I write

$$f : x \to y$$

to denote that f is a function such that $\text{dom}(f) = x$ and $\text{ran}(f) \subseteq y$.

Notice that if $f : x \to y$, then $f \subseteq x \times y$.

A *constant function* from a set x to a set y is a function of the form

$$f = \{(a, k) \mid a \in \text{dom}(f)\}$$

where k is a fixed member of y.

The *identity function* on x is the unary function defined by

$$\mathrm{id}_x = \{(a, a) \mid a \in x\}.$$

If $f : x \to y$ and $g : y \to z$, we define $g \circ f : x \to z$ by

$$g \circ f(a) = g(f(a))$$

for all $a \in x$.

Exercise 1.6.2. *Express $g \circ f$ as a set of ordered pairs.*

Let $f : x \to y$. If $u \subseteq x$, we define the *image* of u under f to be the set

$$f[u] = \{f(a) \mid a \in u\};$$

and if $v \subseteq y$, we define the *preimage* of v under f to be the set

$$f^{-1}[v] = \{a \in x \mid f(a) \in v\}.$$

Exercise 1.6.3. *Let $f : x \to y$, and let $v_i \in y$, for $i \in I$. Prove that:*

(i) $f^{-1}\bigcup_{i\in I}[v_i] = \bigcup_{i\in I} f^{-1}[v_i]$;

(ii) $f^{-1}\bigcap_{i\in I}[v_i] = \bigcap_{i\in I} f^{-1}[v_i]$;

(iii) $f^{-1}[v_i - v_j] = f^{-1}[v_i] - f^{-1}[v_j]$.

If $f : x \to y$ and $u \subseteq x$, we define the *restriction* of f to u by

$$f \quad u = \{(a, f(a)) \mid a \in u\}.$$

Notice that $f \quad u$ is a function, with domain u.

Exercise 1.6.4. *Prove that if $f : x \to y$ and $u \subseteq x$, then*

(i) $f[u] = \mathrm{ran}(f \quad u)$;

(ii) $f \quad u = f \cap (u \times ran(f))$.

Let $f : x \to y$. We say f is *injective* (or *one-one*) if and only if

$$a \neq b \to f(a) \neq f(b).$$

We say f is *surjective* (or *onto*) (relative to the given set y) if and only if

$$f[x] = y.$$

We say f is *bijective* if and only if it is both injective and surjective. In this last case we often write $f : x \leftrightarrow y$.

If $f : x \to y$ is bijective, then f has a unique inverse function, f^{-1}, defined by

$$f^{-1} = \{(b, a) \mid (a, b) \in f\}.$$

Thus, $f^{-1} : y \to x$, $f^{-1} \circ f = \mathrm{id}_x$, and $f \circ f^{-1} = \mathrm{id}_y$.

Notice that whenever $f : x \to y$ and $v \subseteq y$, then the *set* $f^{-1}[v]$ is defined, regardless of whether f is bijective (and hence has an inverse function) or not. If, in fact, f is bijective, so that f^{-1} exists, then the two possible interpretations of $f^{-1}[v]$ clearly coincide. Thus, our choice of notation should cause no problems.

Having defined the notion of a function now, we may give a very general definition of a 'cartesian product' of an arbitrary (possibly infinite) family of sets.

Let $x_i, i \in I$, be a family of sets. The *cartesian product* of the family $\{x_i \mid i \in I\}$ is defined to be the set

$$\prod_{i \in I} x_i = \{f \mid (f : I \to \bigcup_{i \in I} x_i) \wedge (\forall i \in I)(f(i) \in x_i)\}.$$

If $x_i = x$ for all $i \in I$, we write x^I instead of $\prod_{i \in I} x_i$.

Now, in case I is finite, the above identity provides us with a second definition of 'cartesian product', quite different from the first. However, though formally different, the two notions of finite cartesian product are clearly closely related, and either definition of product may be used. In general, we use the original definition for finite products, using the notation $x_1 \times \ldots \times x_n$, and the above definition for infinite (or arbitrary) products, writing $\prod_{i \in I} x_i$.

Exercise 1.6.5. *What set is the cartesian product* $x^{\{1\}}$?

Exercise 1.6.6. *The ordered-pair operation* (a, b) *defines a binary function on sets. The inverse functions to the function are defined as follows: if* $w = (a, b)$, *then* $(w)_0 = a$ *and* $(w)_1 = b$.

Prove that if w *is an ordered pair, then*

(i) $(w)_0 = \bigcup\bigcap w$;

(ii) $(w)_1 = \begin{cases} \bigcup[\bigcup w - \bigcap w] & , \text{ if } \bigcup w \neq \bigcap w \\ \bigcup\bigcup w & , \text{ if } \bigcup w = \bigcap w \end{cases}$

To avoid unnecessary complication, I have not bothered to specify the set on which the above functions are defined. This is, of course, common mathematical practice when one is only interested in the behavior of the functions concerned.

1.7 Well-Orderings and Ordinals

I promised earlier that well-orderings would return, and here they come. I start out by explaining why well-orderings play an important role in set theory.

You are doubtless familiar with the principle of mathematical induction in proving results about the positive integers. Indeed, this method is not restricted to proving results about the positive integers but will work for any set that may be enumerated as a sequence $\{a_n\}_{n=0}^{\infty}$ indexed by the positive integers. Now what makes the induction method work is the fact that the positive integers are well-ordered. There is, after all, no real possibility of ever proving, case by case, that some property $P(n)$ holds for *every* positive integer n. But since the positive integers are well-ordered, if $P(n)$ were ever to fail, it would fail at a *least* n, and then we would have $P(n-1)$ true but $P(n)$ false, and it is precisely this situation that we exclude in our 'induction proof'.

Is it possible to extend this powerful method of proof to cover transfinite sets that are not enumerable as an integer-indexed sequence? Well, a natural place to start looking for an answer is to see if we can extend the positive integers into the transfinite, to obtain a system of numbers suitable for enumerating any set, however large. To do this we adopt more or less the same method that a small child uses when learning the number concept. The child first learns to count collections, by enumerating them in a linear way, and then, after repeating this process many times, abstracts from it the concept of 'natural number'. This is just what we will do, only in a more formal manner. Of course, since we are going to allow infinite collections, we shall not be doing any actual 'counting', but the concept of a well-ordering will provide the mathematical counterpart to this.

Recall that a well-ordering of a set x is a total ordering of x that is well-founded. Now, according to our previous definition, a partial ordering of a set x is well-founded if and only if every nonempty subset y of x has a minimal element (i.e. an element of y having no predecessor in y). But in the case of total orderings, an element of a subset y of x will be minimal if and only if it is the unique smallest member of y. Thus an alternative definition of a well-ordering of a set x is a total ordering of x such that every nonempty subset of x has a (unique) smallest member. This formulation

enables us to prove:

Theorem 1.7.1 [Induction on a Well-Ordering] Let $(X, \leq)$ be a woset. Let E be a subset of X such that:

(i) the smallest element of X is a member of E;

(ii) for any $x \in X$, if $\forall y[y < x \rightarrow y \in E]$, then $x \in E$.

Then $E = X$.

Proof: Suppose $E \neq X$. Let x be the smallest member of the nonempty set $X - E$. Then, by (i), x is not the smallest member of X. But by choice of x, we have $y < x \rightarrow y \in E$. Hence, by (ii), $x \in E$, a contradiction. □

Notice the notation adopted above. I used capital letters to denote sets and lower-case letters to denote their elements. This is a very common notational convention, which I shall often adopt. Of course, it is really only helpful in simple situations; once there are sets of sets floating about it becomes rather confusing.

Theorem 1.7.1 allows us to prove results by induction on a well-founded set, but it does not provide us with a system of transfinite numbers for 'counting'. For that we need to isolate just what it is that all wosets have in common. So we commence by comparing wosets.

Let $(X, \leq)$, $(X', \leq')$ be wosets. A function $f : X \rightarrow X'$ is an *order isomorphism* if and only if f is bijective and

$$x < y \rightarrow f(x) <' f(y).$$

I write $f : X \cong X'$ in this case. (As usual, I adopt the convention of writing X in place of $(X, \leq)$, etc., it being clear from the context that X is a set with a well-ordering here.)

Theorem 1.7.2 Let $(X, \leq)$ be a woset, $Y \subseteq X$, $f : X \cong Y$. Then for all $x \in X$, $x \leq f(x)$.

Proof: Let $E = \{x \in X \mid f(x) < x\}$. We must prove that $E = \emptyset$. Suppose otherwise. Then E has a smallest member, x_0. Since $x_0 \in E$, it follows that $f(x_0) < x_0$. Let $x_1 = f(x_0)$. Since $x_1 < x_0$, applying f gives $f(x_1) < f(x_0)$. Thus $f(x_1) < x_1$. Thus $x_1 \in E$.

But $x_1 < x_0$, so this contradicts the choice of x_0 as the least member of E, and the proof is complete. □

Theorem 1.7.3 Let $(X, \leq)$, $(X', \leq')$ be wosets. If $(X, \leq) \cong (X', \leq')$, there is exactly one order-isomorphism $f : X \cong X'$.

Proof: Let $f : X \to X'$, $g : X \to X'$. Set $h = f^{-1} \circ g$. It is easily seen that $h : X \cong X$. So, by Theorem 1.7.2, $x \leq h(x)$ for all $x \in X$. So, applying f, we see that for any $x \in X, f(x) \leq f(h(x)) = g(x)$. Similarly, $g(x) \leq f(x)$ for any $x \in X$. Thus $f = g$, and the proof is complete. □

It should be noticed that the above result does not hold for any tosets; well-ordering is essential. For example, let $\mathcal{Z}$ be the set of all integers, $\leq$ the usual ordering on $\mathcal{Z}$. For any integer m, the mapping $f_m : \mathcal{Z} \to \mathcal{Z}$ defined by $f_m(n) = n+m$ is an order-isomorphism, and $m \neq m'$ implies $f_m \neq f_{m'}$.

Notice also that if $m < 0$, then $f_m(n) < n$ for all n, so this example also shows that Theorem 1.7.2 requires well-ordering as well.

Let $(X, \leq)$ be a woset, $a \in X$. By the *segment* X_a of X determined by a we mean the set

$$X_a = \{x \in X \mid x < a\}.$$

Theorem 1.7.4 Let $(X, \leq)$ be a woset. There is no isomorphism of X onto a segment of X.

Proof: Suppose $f : X \cong X_a$. By Theorem 1.7.2, $x \leq f(x)$ for all x in X. In particular, therefore, $a \leq f(a)$. But $\text{ran}(f) = X_a$, so $f(a) \in X_a$, giving $f(a) < a$, a contradiction. □

Notice that well-ordering is required for Theorem 1.7.4. For example, let $\mathcal{Z}^-$ denote the nonpositive integers, and define $f : \mathcal{Z}^- \cong \mathcal{Z}_0^-$ by $f(n) = n - 1$.

Theorem 1.7.5 Let $(X, \leq)$ be a woset, $A = \{X_a \mid a \in x\}$. Then

$$(X, \leq) \cong (A, \subseteq).$$

Proof: Define $f : X \cong A$ by $f(a) = X_a$. □

An *ordinal* is defined to be a woset $(X, \leq)$ such that $X_a = a$ for all a in X. (I am not making any claims about the existence of such sets at the moment.)

Exercise 1.7.1. *Suppose $(X, \leq)$ is an ordinal. What is the first member of X? Well, if x_0 is the first member of X, then $X_{x_0} = \emptyset$, so as $(X, \leq)$ is an ordinal, $x_0 = X_{x_0} = \emptyset$. Now what is the second member, x_1, of X? In general, what is the n'th member of X ? What can you guess about both the existence and uniqueness of ordinals?*

Let $(X, \leq)$ be an ordinal. Then, for x, y in X, we have

$$x < y \text{ if and only if } X_x \subset X_y \text{ if and only if } x \subset y.$$

The first equivalence here holds for any woset, the second holds because, if X is an ordinal, $X_x = x$ and $X_y = y$.

Thus the ordering of an ordinal X is the subset relation. In other words, when we specify an ordinal, we do not have to say what the ordering is; it must be the subset relation.

Theorem 1.7.6 Let X be an ordinal. If $a \in X$, then X_a is an ordinal.

Proof: Let $b \in X_a$. Then

$$\begin{aligned} (X_a)_b &= \{x \in X_a \mid x < b\} &&= \{x \in X \mid x < a \wedge x < b\} \\ &= \{x \in X \mid x < b\} &&= X_b = b \end{aligned}$$

and the theorem follows. □

Theorem 1.7.7 Let X be an ordinal. Let $Y \subset X$. If Y is an ordinal, then $Y = X_a$ for some $a \in X$.

Proof: Let a be the smallest element of $X - Y$. Thus $X_a \subseteq Y$. Now let $b \in Y$. Then $Y_b = b = X_b$, so if $a < b$, then $a \in X_b$; so $a \in Y_b$, and hence $a \in Y$, which is not the case. Thus $b \leq a$. But $b \neq a$, since $b \in Y$. Hence $b < a$. Thus $b \in X_a$. This proves that $Y \subseteq X_a$. Hence $Y = X_a$. □

Theorem 1.7.8 If X, Y are ordinals, then $X \cap Y$ is an ordinal.

Proof: Let $a \in X \cap Y$. Then $X_a = a = Y_a$, i.e.

$$\{x \in X \mid x < a\} = a = \{y \in Y \mid y < a\}.$$

Hence

$$a = \{z \in X \cap Y \mid z < a\} = (X \cap Y)_a$$

and the proof is complete. □

Theorem 1.7.9 Let X, Y be ordinals. If $X \neq Y$, then one is a segment of the other.

Proof: If $X \subset Y$ or $Y \subset X$, we are done by Theorem 1.7.7. So suppose otherwise. Thus $X \cap Y \subset X$ and $X \cap Y \subset Y$. Now, by Theorem 1.7.8, $X \cap Y$ is an ordinal, so by Theorem 1.7.7, $X \cap Y = X_a$ for some $a \in X$ and $X \cap Y = Y_b$ for some $b \in Y$. Then

$$a = X_a = X \cap Y = Y_b = b.$$

But $a \in X, b \in Y$. Thus $a = b \in X \cap Y$. But $X \cap Y = X_a$, so

$$x \in X \cap Y \rightarrow x < a.$$

In particular, $a < a$, and we have a contradiction. □

Theorem 1.7.10 If X, Y are isomorphic ordinals, then $X = Y$.

Proof: Let $f : X \cong Y$. We prove that $f = \mathrm{id}_X$. Set

$$E = \{x \in X \mid f(x) \neq x\}.$$

We must prove that $E = \emptyset$. Suppose otherwise, and let a be the smallest member of E. Then $x < a \rightarrow f(x) = x$, so $X_a = Y_{f(a)}$. But then $a = X_a = Y_{f(a)} = f(a)$, contrary to $a \in E$. □

Theorem 1.7.11 Let $(X, \leq)$ be a woset such that for each $a \in X$, X_a is isomorphic to an ordinal. Then X is isomorphic to an ordinal.

Proof: For each $a \in X$, let $g_a : X_a \cong Z(a)$ be an isomorphism of X_a onto an ordinal $Z(a)$. By Theorems 1.7.10 and 1.7.3, both $Z(a)$ and g_a are unique. Hence this defines a function Z on X. Let W be its range.[2] That is,

$$W = \{Z(a) \mid a \in X\}.$$

Define $f : X \rightarrow W$ by

$$f(a) = Z(a).$$

Claim: If $x, y \in X$, then $x < y \rightarrow Z(x) \subset Z(y)$.

Proof of claim: Let $x, y \in X$, $x < y$. Then

(1) $$g_x : X_x \cong Z(x).$$

[2]When we come to describe the axioms of set theory, the reader will be able to see that what we are actually doing here is applying the Axiom of Replacement. So this step is, in fact, one of the deeper steps in our present development. If the reader finds this footnote confusing, it just demonstrates what a natural principle the Axiom of Replacement is.

Also, since

$$\begin{aligned} X_x &= \{z \in X \mid z < x\} \\ &= \{z \in X \mid z < y \wedge z < x\} \\ &= \{z \in X_y \mid z < x\} \\ &= (X_y)_x, \end{aligned}$$

we have

$$(g_y \quad X_x) : X_x \cong (Z(y))_{g_y(x)}. \tag{2}$$

Now, $Z(y)$ is an ordinal, so by Theorem 1.7.6, $(Z(y))_{g_y(x)}$ is an ordinal. But by (1) and (2), $Z(x) \cong (Z(y))_{g_y(x)}$. Hence by Theorem 1.7.10,

$$Z(x) = (Z(y))_{g_y(x)}. \tag{3}$$

Thus, in particular, $Z(x) \subset Z(y)$. The claim is proved.

By the claim, f is a bijection of X onto W. Also by the claim, f is an order isomorphism of X onto the poset $(W, \subseteq)$. Thus, in particular, W is well-ordered by $\subseteq$. We finish the proof by showing that W is an ordinal.

Let $y \in X$. Since $Z(y)$ is an ordinal, we have

$$x < y \to (Z(y))_{g_y(x)} = g_y(x).$$

So by (3),

$$x < y \to Z(x) = g_y(x). \tag{4}$$

Hence,

$$\begin{aligned} W_{z(y)} &= \{Z(x) \mid Z(x) \subset Z(y)\} \\ &= \{Z(x) \mid x < y\} \\ &= \{g_y(x) \mid x < y\} \\ &= g_y[X_y] \\ &= Z(y). \end{aligned}$$

Thus, as $Z(y)$ was an arbitrary member of W (since y was an arbitrary member of X), W is an ordinal. □

Exercise 1.7.2. *During the course of the above proof, I emphasized one point by a footnote. From the point of view of naive set theory, there is no problem: the proof is a sound mathematical argument. But when we*

come to axiomatize set theory we shall want to state explicitly all procedures which may be used to construct sets. Try to formulate, in a precise manner, the construction principle we used at the crucial part of the proof of Theorem 1.7.11. (The footnote may be of some assistance here.)

Theorem 1.7.12 Every woset is isomorphic to a unique ordinal.

Proof: The uniqueness assertion follows from Theorem 1.7.10. We prove existence.

Let $(X, \leq)$ be a woset. By Theorem 1.7.11, it suffices to prove that for every $a \in X$, X_a is isomorphic to an ordinal. Let

$$E = \{a \in X \mid X_a \text{ is not isomorphic to an ordinal}\}.$$

We show that $E = \emptyset$. Suppose otherwise. Let a be the smallest element of E. Thus, if $x < a, X_x$ is isomorphic to an ordinal. But for $x < a, X_x = (X_a)_x$. Hence every segment of X_a is isomorphic to an ordinal. Hence by Theorem 1.7.11, X_a is isomorphic to an ordinal, contrary to $a \in E$. □

If $(X, \leq)$ is a woset, I shall denote by $\mathrm{Ord}(X)$ the unique ordinal isomorphic to X. Clearly, if X, Y are wosets, we shall have $X \cong Y$ if and only if $\mathrm{Ord}(X) = \mathrm{Ord}(Y)$. Since the ordinals have a certain uniqueness property (in the sense of Theorem 1.7.10), this means that we may use the ordinals as a yardstick for 'measuring' the 'length' of any woset: $\mathrm{Ord}(X)$ being the 'length' of the woset X.

But just how reasonable is it to take the ordinals, as defined above, as a system of 'numbers', which is what I am now proposing? Well, by Theorem 1.7.9, the ordinals are totally ordered by $\subset$. In fact, Theorem 1.7.9 tells us more: if X, Y are ordinals, then

$$\begin{aligned} X \subset Y \quad &\text{if and only if} \quad X = Y_a \quad (\text{for some } a \in Y) \\ &\text{if and only if} \quad X = a \quad (\text{since } Y_a = a) \\ &\text{if and only if} \quad X \in Y. \end{aligned}$$

Thus the ordering $\subset$ on ordinals and the ordering $\in$ on ordinals are identical. This implies also that the ordinals are well-ordered by $\subset$, or, equivalently, by $\in$. To see this, we make use of Lemma 1.5.1. Suppose the ordinals were not well-ordered by $\subset$. Then we could find a sequence $\{X(n)\}_{n=0}^{\infty}$ of ordinals such that

$$X(0) \supset X(1) \supset X(2) \supset \dots .$$

Now, for all $n > 0, X(n) \subset X(0)$, so $X(n) \in X(0)$. Thus $\{X(n+1)\}_{n=0}^{\infty}$ is a decreasing (under $\subset$) sequence of members of $X(0)$. But since $X(0)$ is an ordinal, it is well-ordered by $\subset$, so we have a contradiction.

From the above, it would seem, therefore, that the ordinals constitute an eminently reasonable number system, suitable for 'measuring' the 'length' of any woset.

It is common in contemporary set theory to reserve lower-case Greek letters $\alpha, \beta, \gamma, \ldots$ to denote ordinals. (Since the ordering of an ordinal is always $\subset$, there is, of course, no need to specify the ordering each time. But it should be remembered that an ordinal is, strictly speaking, a *well-ordered* set.) It is also customary to denote the order relation between ordinals by

$$\alpha < \beta$$

instead of the two equivalent forms

$$\alpha \subset \beta \ , \ \alpha \in \beta,$$

though the latter is also quite common.

Since the ordinals will 'measure' any woset, they will certainly measure any finite woset. But so too will the positive integers. So do we have some duplication here? Well no, because in mathematics one (almost) never bothers to *define* the integers as specific objects. As a result of our development of ordinals, we obtain, *gratis*, a neat definition of the natural numbers as specific sets; namely, the finite ordinals.

What do the ordinals look like as sets? Well, if α is an ordinal, then by definition we will have

$$\alpha = \{\beta \mid \beta < \alpha\}.$$

That is, an ordinal is the set of all smaller ordinals.

In the case of the first ordinal, there is no smaller ordinal, of course. Hence the first ordinal must be the empty set, $\emptyset$ (regarded as a well-ordered set). Let us denote this ordinal by the symbol 0. Thus, by definition, ignoring the well-ordering as usual,

$$0 = \emptyset.$$

What is the second ordinal? Well, it has to be the set of all smaller ordinals, so if we denote the second ordinal by 1, we must have

$$1 = \{0\}.$$

The third ordinal, which we denote by 2, is

$$2 = \{0, 1\}.$$

The pattern is now clear. We have

$$\begin{aligned} 3 &= \{0,1,2\}, \\ 4 &= \{0,1,2,3\}, \end{aligned}$$

and in general,

$$n = \{0,1,2,\ldots,n-1\}.$$

Notice that the ordinal n is a set with exactly n elements, making the finite ordinals ideal for 'measuring' finite sets. Notice also that if α, β are distinct finite ordinals, then one must be a segment of the other and, hence, an element of the other.

What will be the first infinite ordinal? Clearly, it must be the set (ordered by inclusion)

$$\{0,1,2,\ldots,n,n+1,\ldots\}.$$

We denote this ordinal by ω. And the next? Clearly

$$\{0,1,2,\ldots,n,n+1,\ldots,\omega\}.$$

In general, if α is an ordinal, the next ordinal will be

$$\alpha \cup \{\alpha\}.$$

It is customary to denote the first ordinal after α by $\alpha+1$, the (*ordinal*) *successor* of α. Thus

$$\alpha + 1 = \alpha \cup \{\alpha\}.$$

If, as in the case of ω above,

$$0,1,2,\ldots,\omega,\omega+1,\ldots,\alpha,\alpha+1,\ldots$$

is a listing of some initial segment of the well-ordered collection of ordinals having no greatest member, then the next ordinal will be the set

$$\{0,1,2,\ldots,\omega,\omega+1,\ldots,\alpha,\alpha+1,\ldots\}.$$

Since such an ordinal will have no greatest member, it cannot be the successor of any ordinal. Such an ordinal is called a *limit ordinal*. For example, ω is a limit ordinal. An ordinal that is the successor of some ordinal is called a *successor ordinal*.

A *sequence* is a function whose domain is an ordinal. If f is a sequence and $\mathrm{dom}(f) = \alpha$, we say f is an *α-sequence*. If $f(\xi) = x_\xi$ for all $\xi < \alpha$, we often write

$$\langle x_\xi \mid \xi < \alpha \rangle$$

in place of f. Then, for $\beta < \alpha$,

$$\langle x_\xi \mid \xi < \beta \rangle$$

denotes $f \quad \beta$. This clearly gives a precise meaning to what we generally think of as a (transfinite, perhaps) sequence. The 'sequences' of elementary analysis are just the special case of ω-sequences, of course; so

$$\{a_n\}_{n=0}^{\infty} = \langle a_n \mid n < \omega \rangle.$$

Exercise 1.7.3. *I have already introduced the notation $\alpha + 1$ for the next ordinal after α. Let us denote by $\alpha + n$ the n-th ordinal after α, where n is any natural number. Show that if α is any ordinal, either α is a limit ordinal or else there is a limit ordinal β and a natural number n such that $\alpha = \beta + n$. (Hint. Use Theorem 1.7.1.)*

The ordinals thus provide us with a continuation of the natural numbers into the transfinite. Further discussion of ordinals will have to be postponed until we have developed the axiomatic foundation of our set theory.

1.8 Problems

1. (Boolean Algebras)

A boolean algebra, $\mathcal{B}$, is a structure consisting of a set B with a unary operation (*complement*) and two binary operations $\wedge$ (*meet*) and $\vee$ (*join*). The axioms to be satisfied by this structure are:

(B1) $b \vee c = c \vee b$, $b \wedge c = c \wedge b$;

(B2) $b \vee (c \vee d) = (b \vee c) \vee d$, $b \wedge (c \wedge d) = (b \wedge c) \wedge d$;

(B3) $(b \wedge c) \vee c = c$, $(b \vee c) \wedge c = c$;

(B4) $b \wedge (c \vee d) = (b \wedge c) \vee (b \wedge d)$, $b \vee (c \wedge d) = (b \vee c) \wedge (b \vee d)$;

(B5) $(b \wedge -b) \vee b = b$, $(b \vee -b) \wedge b = b$.

Prove the following:

A. The elements $b \wedge -b$ are all equal and denoted by 0 (zero).

B. The elements $b \vee -b$ are all equal and denoted by 1 (unity).

C. Any nonempty set $\mathcal{F}$ of subsets of a set X that is closed under union, intersection, and complement with respect to X is a boolean algebra under the operations meet = intersection, join = union, complement = complement in X.

Such a set $\mathcal{F}$ is called a *field of subsets* of X. For example, $\mathcal{P}(x)$ is a boolean algebra under the above boolean operations. It can be shown that every boolean algebra is isomorphic to a field of sets. (This is Stone's Theorem. See [6] for details.)

D. Let X be a topological space. Let $\mathcal{C}$ denote the set of all clopen (i.e. closed and open) subsets of X. $\mathcal{C}$ is a field of sets and, hence, is a boolean algebra.

E. Let X be a topological space. Let $\mathcal{R}$ be the set of all closed sets A such that A = closure interior A. Define $A \vee B = A \cup B$, $A \wedge B$ = closure interior $A \cap B$, $-A$ = closure $(X - A)$. Then $\mathcal{R}$ is a boolean algebra. $\mathcal{R}$ is not usually a field of sets, since, in general, $\wedge$ is not the same as $\cap$.

We may define a binary relation on the boolean algebra $\mathcal{B}$ by

$$b \leq c \quad \text{if and only if} \quad b = b \wedge c.$$

Prove the following:

F. For any b, c, $b \leq c$ if and only if $b \vee c = c$.

G. $\leq$ is a partial ordering of B; 0 is the unique minimum element under $\leq$, and 1 is the unique maximum.

H. For any b, c, $b \wedge c \leq b \leq b \vee c$.

It is possible to define a boolean algebra as a poset satisfying certain conditions. In this case, $b \vee c$ turns out to be the unique least upper bound of b and c, and $b \wedge c$ is the unique greatest lower bound.

2. (Ideals and Filters)

Let B be a boolean algebra. A nonempty subset I of B is called an *ideal* if and only if:

(a) $b, c \in I \rightarrow b \vee c \in I$;

(b) $[b \in I \text{ and } c \in B] \rightarrow b \wedge c \in I$.

Prove the following:

A. $I \subseteq B$ is an ideal if and only if (a) and (b)$'$ hold, where

(b)$'$ $[b \in I \text{ and } c \in B] \rightarrow c \in I$.

B. $0 \in I$ for every ideal I; if $1 \in I$, then $I = B$.

C. If $b \in B$, then $\{c \in B \mid c \leq b\}$ is an ideal; it is called the *principal ideal* generated by b. Any ideal not of this form is said to be *nonprincipal.*

D. Let X be an infinite set. Let I be the set of all finite subsets of X. I is a nonprincipal ideal in the field of sets $\mathcal{P}(X)$.

A *measure* on a boolean algebra B is a function $\mu\colon B \rightarrow [0,1]$ such that:

(i) $\mu(0) = 0$, $\mu(1) = 1$;

(ii) if $b \wedge c = 0$, then $\mu(b \vee c) = \mu(b) + \mu(c)$.

E. Prove that, if μ is a measure on B, then $\{b \in B \mid \mu(b) = 0\}$ is an ideal in B.

F. Let B be a boolean algebra. Show that, if $I_t, t \in T$, are ideals in B, so too is

$$\bigcap_{t \in T} I_t.$$

Deduce that if $X \subseteq B$, there is a unique smallest ideal containing X; it is called the ideal *generated* by X.

A nonempty set $F \subseteq B$ is called a *filter* if and only if:

(a) $b, c \in F \rightarrow b \wedge c \in F$;

(b) $[b \in F \text{ and } c \in B] \rightarrow b \vee c \in F$.

G. Show that in the above definition, (b) can be replaced by

(b)$'$ $[b \in F \text{ and } b \leq c] \rightarrow c \in F$.

H. Prove that a subset $F \subseteq B$ is a filter if and only if the set $\{-b \mid b \in F\}$ is an ideal. The filter $\{-b \mid b \in I\}$ is called the *dual* of the ideal I; the ideal $\{-b \mid b \in F\}$ is the *dual* of the filter F.

An ideal in the field of sets $\mathcal{P}(X)$ is sometimes said to be an ideal *on* the set X; similarly a filter *on* the set X.

3. (The Order Topology)

Let $(X, <)$ be a toset. The *order topology* on X is the topology determined by taking as open subbase all sets of the form $\{x \in X \mid x < a\}$ or $\{x \in X \mid x > a\}$ for $a \in X$.

A. Prove that the order topology on X is the smallest topology with the property that whenever $a, b \in X$ and $a < b$, there are neighborhoods U of a and V of b such that $U < V$ (i.e. such that $x < y$ whenever $x \in U$ and $y \in V$).

B. Prove that, if X is connected (under the order topology), then X is complete as a toset; i.e. every nonempty subset with an upper bound has a least upper bound.

If there are points a, b in X such that $a < b$ and for no c in X is $a < c < b$, we say X has a *gap*.

C. Prove that X is connected (with the order topology) if and only if X is complete (as a toset) and has no gaps.

D. Prove that X is complete (as a toset) if and only if every closed (in the order topology), bounded subset of X is compact.

2

The Zermelo–Fraenkel Axioms

In this chapter, I develop an axiomatic framework for set theory. For the most part, the axioms will be simple existence assertions about sets, and it may be argued that they are all self-evident 'truths' about sets. But why axiomatize set theory in the first place? Well, for one thing, it is well known that set theory provides a unified framework for the whole of pure mathematics, and surely if anything deserves to be put on a sound basis it is such a foundational subject. "But surely," you say, "the concept of a set is so simple that nothing further need be said. We simply regard any collection of objects as a single entity in its own right, and that provides us with our set theory." Alas, nothing could be further from the truth. Certainly, the idea of being able to regard any collection of objects as a single entity forms the very core of set theory. But a great deal more needs to be said about this.

First, what is to determine a 'collection'. In the case of a (small?) finite collection, one may simply list the elements of the collection in order to determine it. But what about infinite (or even large finite) collections? Well, we could allow just those collections that are describable by means of a sentence in the English language. But there are only countably many sentences of the English language, so this would not provide us with many sets. Moreover, we would be faced with many collections that are not strictly mathematical, since the expressive power of the English language greatly transcends the realm of mathematics. And we are, after all, looking for a rigorous framework for our set theory.

But it would seem that the idea of taking for our 'collections' just those collections that are somehow *describable* is quite reasonable. It is just a question of fixing a suitable 'language'. The 'language' must be sufficiently restrictive to allow only the construction of 'mathematical' collections, and sufficiently powerful to allow the construction of any set we may require in mathematics. So we commence our study of the concept of a 'set' by

describing such a language. Later on we shall see whether or not this language helps us in our task of rigorizing set theory.

2.1 The Language of Set Theory

I shall describe a language suitable for, and adequate for, describing mathematical collections. The language will have a precisely determined set of symbols (the 'words' of the language) and a rigid syntax ('grammar'). This will ensure that the concept of a 'collection describable in the language' will be rigorously defined. As such, the language is an example of a *formal language*. Being the language of set theory, let us give it the name LAST. (This stands for LAnguage of Set Theory. Admittedly this sounds like the name of a computer programming language. But this is no bad thing, since programming languages are also formal languages, having the same rigid construction as our own LAST.)

Our language must have a facility for referring to specific sets, so we want a collection of names that we can use to denote sets. Now, at no time shall we be able to refer simultaneously to infinitely many different, specific sets, by name. This is, after all, a language we are defining, and as such its sentences will just be finite sequences of words of the language. On the other hand, we could conceivably wish to refer to an arbitrarily large finite number of sets at some time, so there should be no *a priori* upper bound on the lengths of our sentences, or the number of names of sets that occur in them. So, what we require is a countably infinite collection of names. Thus, our first requirement is

(1) *Names (for sets)*: $w_0, w_1, w_2, \ldots, w_n, \ldots$.

These names will be used to denote specific sets. Of course, on one occasion the name w_0 may be used to denote one set, on another occasion quite a different set. But this does not matter. During the course of any one description of a set there are enough names to denote all of the sets involved in that description, and it is only a duplication of names occurring in the course of the same description that must be avoided. (Just as the existence of two persons named John Smith only becomes problematical when they live in the same district or work for the same company, etc.) Besides referring to specific sets by giving them (temporary) names, we also wish to refer to *arbitrary* sets. In other words, we need variables for sets. The same argument as we used for the names leads to our taking a countably infinite collection of variables:

(2) *Variables (for sets)*: $v_0, v_1, v_2, \ldots, v_n, \ldots$.

Next we need to be able to make simple identity assertions about sets. We need to be able to say that two sets are equal, or that one is an element of the other. So we need:

(3) *Membership symbol*: $\in$,

(4) *Equality symbol*: $=$.

We further need to be able to combine any finite number of assertions, or clauses, to produce one big assertion. So we need

(5) *Logical connectives*: $\wedge$ (and), $\vee$ (or)

and

(6) *Negation symbol*: $\neg$ (not) .

The intended meaning and use of these symbols is self-evident, but I shall, in any case, make this precise when I describe the syntax of LAST.

Also required are

(7) *Quantifier symbols*: $\forall$ (for all), $\exists$ (there exists).

Finally, to serve as punctuation symbols, keeping various clauses apart, we need

(8) *Brackets*: (,) .

This then is the lexicon for the formal language LAST. The reader may be surprised to discover, as she will presently, that this simple language is adequate for expressing the most complex of mathematical descriptions. But indeed it is.

As for the syntax, that too is simple. We may build *formulas* (i.e. 'clauses', or 'phrases', or 'sentences') as follows.

(a) Any expression of the forms

$$(v_n = v_m) \quad (v_n = w_m) \quad (w_m = v_n) \quad (w_n = w_m)$$
$$(v_n \in v_m) \quad (v_n \in w_m) \quad (w_m \in v_n) \quad (w_n \in w_m)$$

is a formula of LAST.

(b) If ϕ, ψ are formulas of LAST, so too are

$$(\phi \wedge \psi) \quad , \quad (\phi \vee \psi).$$

(c) If ϕ is a formula of LAST, so too is

$$(\neg\phi).$$

(d) If ϕ is a formula of LAST, then so too are

$$(\forall v_n \phi) \quad , \quad (\exists v_n \phi).$$

No other methods are allowed in the construction of formulas of LAST.

Notice that the variables are used in two distinct ways in LAST. If ϕ is a formula of LAST which does not contain a quantifier of the form $\forall v_n$ or $\exists v_n$, then any occurrence of v_n in ϕ is said to be *free* (since v_n is free to denote any set in ϕ). If we now construct the formula $(\forall v_n)\phi$ or $(\exists v_n)\phi$, then all occurrences of v_n in this new formula are said to be *bound.* In this case, v_n is no longer free to denote an arbitrary set; it is an integral part of the quantifier construction.

A formula that contains no free variables is called a *sentence.* If ϕ is a sentence of LAST, then, once we know which sets any names in ϕ refer to, ϕ can be read as an assertion about sets, and as such will either be true or false.[1] Thus, a sentence actually makes some assertion. A formula that contains one or more free variables makes no assertion, because there is no meaning available for the free variables. Of course, if we *assign* specific sets to the free variables we can say whether or not the formula is true *for those assignments*; but on its own the formula has no meaning.

We often write $\phi(v_0, \ldots, v_n)$ (etc.) to indicate that ϕ is a formula all of whose free variables, if any, are amongst the list $v_0, \ldots, v_n$. Given specific sets $a_0, \ldots, a_n$ if we subsequently write $\phi(a_0, \ldots, a_n)$, we mean ϕ with a_i interpreting v_i in ϕ for $i = 0, \ldots, n$.

We are now in a position to define the notion of a LAST-*describable collection.* Let $\phi(v_n)$ be a formula of LAST. Suppose we know which sets the various names in ϕ refer to. Then, given any set x, we can determine whether or not $\phi(x)$. Hence 'the collection of all sets x for which $\phi(x)$' is a well-defined collection. And it is clearly a mathematical collection. The question now is: can we obtain all describable mathematical collections in this manner?

[1] There is one possible cause of confusion. Suppose we have a formula $(\forall v_0 \phi)$, and we extend this to a formula such as $((v_0 = w_0) \wedge (\forall v_0 \phi))$. How do we resolve the apparent conflict in the use of v_0? The answer lies in the meaning. In the clause $(\forall v_0 \phi)$, v_0 is totally 'bound' by the quantifier $\forall v_0$, and as such we no longer have any 'access to it. When we add the conjunct $(v_0 = w_0)$, the v_0 here is, in a sense, a totally different v_0. The formula $((v_0 = w_0) \wedge (\forall v_0 \phi))$ thus has exactly one free occurrence of v_0, that occurrence being in the first conjunct. This is, of course, very clear when the meaning of the formula is considered.

We can now go on to construct the formula $\exists v_0((v_0 = w_0) \wedge (\forall v_0 \phi)))$. Again, this is unambiguous.

One could avoid this kind of complication by altering the syntax somewhat, but there seems no point in doing so when the meaning is so clear.

To put the above question a little more precisely: if a collection has any mathematical description, does it have a description in LAST? Of necessity, a formal answer is not possible. The notion of a 'mathematical description', though probably well understood, is not a precisely defined notion, whereas the notion of a LAST description is very precise. But by investigating the expressive power of LAST, it soon becomes abundantly clear that it is indeed adequate for any 'mathematical description' that one could imagine. Part of this investigation has in fact already been carried out for us. It is well known how to express all of the concepts of analysis, algebra, etc. in terms of sets. So what we must show here is that LAST is adequate for expressing any concept of set theory.

Now, since LAST is so rudimentary, it is clear that, except in the case of very simple assertions, the expression in LAST of any set-theoretical assertion will be unbelievably cumbersome, and totally unreadable. And although this is of no consequence to the fact of adequacy or otherwise, it might appear to make our task of demonstrating adequacy very difficult. But remember that we are, after all, only interested in showing that LAST is *capable* of expressing any set-theoretical assertion; we do not wish to actually construct such expressions. So, we are justified in enriching our formal language by the introduction of abbreviations.

For instance, we may introduce the implication symbol as an abbreviation, with

$$(\phi \to \psi)$$

abbreviating

$$((\neg\phi) \vee \psi)$$

for any pair ϕ, ψ of formulas of LAST.

We may then introduce the *if and only if* symbol $\leftrightarrow$ as an abbreviation, with

$$(\phi \leftrightarrow \psi)$$

abbreviating

$$((\phi \to \psi) \wedge (\psi \to \phi)).$$

Now, once an abbreviation has been introduced, it may itself be used in order to define new abbreviations. So what we are really doing is this. In set theory, we commence with the very simple notions of sets, equality of sets, and membership of sets, and proceed to develop the whole framework of ordered pairs, functions, partial orderings, etc., from this simple beginning. Our language LAST is adequate for describing the basic part of the development and, hence, the whole development. And in order to make this clearer, we may expand LAST by introducing abbreviations that correspond to each new development in the set theory. For example, in parallel with

the development of set theory carried out in Chapter 1, we introduce the following abbreviations into LAST. (I use x, y, z to denote arbitrary names or variables of LAST.)

$x \subseteq y$ abbreviates $(\forall v_n((v_n \in x) \to (v_n \in y)))$
where $v_n \neq x, y$;

$x = \bigcup y$ abbreviates $(\forall v_n((v_n \in x) \leftrightarrow \exists v_m((v_n \in v_m) \wedge (v_m \in y))))$
where $n \neq m$ and $v_n, v_m \neq x, y$;

$x = \{y\}$ abbreviates $(\forall v_n((v_n \in x) \leftrightarrow (v_n = y)))$
where $v_n \neq x, y$;

$x = \{y, z\}$ abbreviates $(\forall v_n((v_n \in x) \leftrightarrow ((v_n = y) \vee (v_n = z))))$
where $v_n \neq x, y, z$;

$x = (y, z)$ abbreviates $(\forall v_n((v_n \in x) \leftrightarrow ((v_n = \{y\}) \vee (v_n = \{y, z\}))))$
where $v_n \neq x, y, z$;

$x = y \cup z$ abbreviates $x = \bigcup\{y, z\}$.

Exercise 2.1.1. *Develop* LAST *further to allow the expression of the following concepts. Feel free to make use of any abbreviations you introduce, once they are available. (The first one is done for you.)*

(i) *x is an ordered pair.* $[\ (\exists v_n(\exists v_m(x = (v_n, v_m))))\ ]$

(ii) *x is a function.*

(iii) *$x = y \times z$.*

(iv) *x is an n-ary function from y to z.*

(v) *x is a poset.*

(vi) *x is a toset.*

(vii) *x is a woset.*

(viii) *x is an ordinal.*

(ix) *x and y are isomorphic wosets.*

(x) *x is a group.*

(xi) *x is an abelian group.*

If the reader has faithfully done all of the above exercise, he will no doubt appreciate how it is that our rudimentary language LAST is indeed capable of expressing very powerful and complex concepts. In essence, it is, of course, because set theory has itself such expressive power.

2.2 The Cumulative Hierarchy of Sets

Having developed our language of set theory to the point we have, it is very tempting to say that a set is simply a collection that is describable by a formula of LAST. According to this definition, x will be a set if and only if there is a formula $\phi(v_n)$ of LAST, having just the one free variable v_n, and sets $a_1, \ldots, a_m$ which the names in ϕ denote, such that x is the collection of all those objects a for which $\phi(a)$. This definition will certainly provide us with all the sets x that are describable in mathematics.[2] Moreover, we are clearly unable to describe any *non*-mathematical collections by formulas of LAST, so this definition will only lead to mathematical sets. So what is wrong with this simple idea?

The answer is immediate: it leads to an inconsistent theory! Indeed, the inconsistency is easily arrived at. Let ϕ be the LAST formula

$$(\neg(v_0 \in v_0)).$$

According to the above definition of sets, ϕ defines a set. Thus there is a set x such that

$$x = \{a \mid a \notin a\}.$$

Now, since x is a set, it must either be the case that $x \in x$ or $x \notin x$. If $x \in x$, then x must satisfy the condition imposed by ϕ, that is, $x \notin x$. On the other hand, if $x \notin x$, then x must fail to satisfy ϕ, which means that $x \in x$. So we have a contradiction.

The question now is: 'Why, exactly, does this simple definition of sets fail?' The answer is inherent in the very idea about a theory of sets that I expressed at the beginning of Chapter 1: fundamental to set theory, is the concept of being able to regard any collection of objects as a single entity.

Now surely, before we can form a collection of objects, those objects must first be 'available' to us! For instance, in our development of naive set

[2]The freedom to refer by 'name' to any other sets is what overcomes the 'handicap' of only having a countable language. This is why there is no bound to the number of sets that we obtain in this way.

theory, we commenced with some initial collection of objects, then considered sets of these objects, then later sets of these sets of objects, and so on. Before we can build sets of sets of objects, we must have the sets of objects out of which to build these sets. The crucial word here, of course, is 'build'. Naturally we are not thinking of actually building sets in any constructive sense. But our set theory should certainly reflect this idea. In the case of our previous definition this was not the case. If we try to form the 'set'

$$x = \{a \mid a \notin a\},$$

the 'set' x itself will not be available for consideration as an element. So how can we ever form this set? Indeed, when we form *any* set u, the set u cannot yet be 'available' to us, so it can surely never be the case that $u \in u$!

Putting these vague considerations into a more precise setting, we see that set theory is essentially hierarchical in nature. We commence with some initial collection, M_0, of objects. We then have a collection, M_1, of sets of members of M_0. Then comes a collection, M_2, of sets of members of $M_0 \cup M_1$, and so on. In order to obtain a precise theory now, we must answer three questions:

(i) What collection do we take as our initial collection, M_0?

(ii) Which 'sets' of objects from lower levels of the hierarchy do we take as elements of each new level of the hierarchy?

(iii) 'How far' does the hierarchy extend?

Well, since we require our set theory to serve as a foundation for mathematics, it should be as simple and intuitive as possible, with no unnecessary and restrictive assumptions. So, in answer to question (i), we commence with *nothing*, that is to say, the empty set. Accordingly, we set

$$V_0 = \emptyset$$

where V_0 denotes the first level of the *set-theoretic hierarchy.*

Avoiding question (ii) for the moment, let us answer question (iii). Since our set theory is to have as few restrictions as possible, there should be no point at which we cannot 'construct' new sets. Thus, for each ordinal number α, there should be a corresponding level V_α in the hierarchy, the members of V_α being sets whose elements all lie in $\bigcup_{\beta<\alpha} V_\beta$.

Finally, let us turn to question (ii). Suppose we have defined the level V_α. Which 'sets' of members of V_α are we to take as the members of $V_{\alpha+1}$? Or, to put it another way, since the intention is that $V_{\alpha+1}$ will consist of

'all' sets of elements of V_α (this being the 'purpose' of the hierarchy), what rules are we to adopt in deciding what is to constitute a 'set'?

One natural answer is to allow just those collections that are describable in LAST. And once a few initial difficulties are overcome, this leads to an extremely rich and powerful theory of sets. But there is another possibility, of a much more general nature. For, when we say that a collection can only be said to exist if there is some formula of LAST that defines it, we are giving a precise definition of the set concept: indeed we are adopting a fundamental axiom of set theory, *the Axiom of Constructibility*, to be discussed in Chapter 5. But what if we are not so specific and decide to interpret the word 'collection' in the widest possible sense?

According to this conception, given V_α, we shall say that $V_{\alpha+1}$ will consist of *all* subsets of V_α, without attempting to say what the word 'all' really entails. This is, of course, much more vague than in the former case, but is nonetheless a conceptually reasonable approach. We all have, do we not, some conception of what the collection of *all* subsets of a set means? Since the Axiom of Constructibility approach will be a sort of 'special case', where we actually make the notion of 'all subsets' more precise, it is not unreasonable to take this second, less restrictive notion of set as basic, and see how far we get with that.

Thus, we shall take as a basic, undefined (but, hopefully, understood) notion, the so-called *unrestricted* power set operation. That is, we shall simply assume that, given any set x, there just *is* a set, $\mathcal{P}(x)$, the *power set* of x, which consists of all and only the subsets of x.

Then, given the level V_α of the hierarchy, we set

$$V_{\alpha+1} = \mathcal{P}(V_\alpha).$$

Now, the above definition tells us how to go from V_α to $V_{\alpha+1}$. But what do we take for V_α when α is a limit ordinal? (Recall that there are two distinct kinds of ordinals, *successor ordinals* and *limit ordinals*.) One answer might be that we do much the same as above, taking

$$V_\alpha = \mathcal{P}(\bigcup_{\beta<\alpha} V_\beta).$$

Indeed, when we come to investigate the set-theoretic hierarchy more thoroughly we shall see that $V_{\alpha+1} = \mathcal{P}(\bigcup_{\beta\leq\alpha} V_\beta)$, so this answer is extremely tempting. But it turns out to be technically more convenient to take instead the definition

$$V_\alpha = \bigcup_{\beta<\alpha} V_\beta.$$

The reason is that this reflects more accurately just what is going on at a limit ordinal. When we 'form' a limit ordinal, we are really just collecting

together all the previous ordinals, without introducing anything new. And this is just what we do in defining V_α as above. Of course, this point does not affect the set theory as a whole; it just makes the hierarchy itself more amenable to the demands we shall be making of it. It is not entirely fatuous to say that, in set theory, it is nice to have time to pause for breath and 'collect' oneself every now and then!

I summarize, and at the same time formalize a little, the discussion so far. We take as basic the unrestricted power-set operation, $\mathcal{P}(x)$, where $\mathcal{P}(x)$ is the set of all and only the subsets of x. The *cumulative hierarchy* of sets (or the *Zermelo hierarchy*, so named after its inventor) is defined thus:

$$\begin{aligned} V_0 &= \emptyset, \\ V_{\alpha+1} &= \mathcal{P}(V_\alpha), \\ V_\alpha &= \bigcup_{\beta<\alpha} V_\beta, \text{ if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

Any set will be an element of some V_α. Because we commence with the empty set, this will mean that, although we place no restriction on the power-set operation, only genuine mathematical objects will be allowed as sets. Letting V denote the 'collection' of all sets, called the *universe of sets*, we can express the above conception of a set by the equation

$$V = \bigcup_\alpha V_\alpha.$$

Notice, however, that this is just a convenient shorthand notation. V is not a set, even though it is a well-defined collection. This is because of the 'unending' nature of the ordinal numbers.

We have now almost arrived at what is known as *Zermelo–Fraenkel set theory*, named after Ernst Zermelo and Abraham Fraenkel, who first formulated and made rigorous this theory. (The intuitive development presented here is essentially due to Zermelo. Fraenkel provided some of the analysis leading to the axiomatization of the theory, to be described shortly.) There are just two principles missing. First, there is the Axiom of Choice, which we will consider later. Second, and more fundamentally, since we have not described the power-set operation at all, how can we be sure that all the sets we require in mathematics will appear in our collection, V, of all sets? We need the following fundamental axiom:

> *Axiom of Subset Selection*: Let x be a set, and let $\phi(v_n)$ be a formula of LAST (which may, as usual, refer to some particular sets by name). Then amongst the sets in $\mathcal{P}(x)$ appears the set of all those members a of x for which $\phi(a)$.

It should be noted that the axiom of subset selection, as stated above, cannot be written as a single sentence of LAST, since LAST has no facility for handling formulas of LAST themselves. This difficulty may be overcome by regarding the axiom of subset selection as an axiom *schema*, each appropriate formula ϕ giving rise to a specific instance of this schema. Given any formula ϕ of LAST with the single free-variable v_n, the following sentence of LAST expresses the ϕ-instance of the axiom:

$$\forall v_i \exists v_m \forall v_n [(v_n \in v_m) \leftrightarrow (v_n \in v_i \wedge \phi(v_n))].$$

The Axiom of Subset Selection says that all the sentences of LAST of this kind are true.

In essence, Zermelo–Fraenkel set theory can be summarized as the theory of sets with the assumptions:

(I) $V = \bigcup_\alpha V_\alpha$;

(II) Axiom of Subset Selection;

(III) Axiom of Choice (see later).

Exercise 2.2.1. *Show that if $y \in V_\alpha$ and $x \in y$, then $x \in V_\alpha$. (A set M is said to be* transitive *if $x \in M \rightarrow x \subseteq M$. Thus we can rephrase this exercise by saying that each V_α transitive. Why is the word 'transitive' used here?)*

Exercise 2.2.2. *Show that, for any ordinal α, $V_\alpha = \bigcup_{\beta<\alpha} \mathcal{P}(V_\beta)$.*

Exercise 2.2.3. *Show that, if $\alpha < \beta$, then $V_\alpha \subset V_\beta$. (This explains the use of the phrase 'cumulative hierarchy of sets' to describe the V_α-hierarchy.)*

Exercise 2.2.4. *Check the following:*

(i) $V_0 = \emptyset$;

(ii) $V_1 = \{\emptyset\}$;

(iii) $V_2 = \{\emptyset, \{\emptyset\}\}$;

(iv) $V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

Exercise 2.2.5. *What are V_4 and V_5?*

Exercise 2.2.6. *How many elements has V_n, where n is a positive integer?*

2.3 The Zermelo–Fraenkel Axioms

The development of our theory of sets so far depends upon the construction of the cumulative hierarchy of sets, and this, in turn, depends on the ordinal number system. We are thus *assuming* a considerable amount of 'set theory' in order to *specify* our set theory. There is, of course, no real dilemma here. What we have done is to analyze what we mean by the concept of a 'set', and our set theory has been the result of this analysis. We have not yet given an axiomatic presentation of the theory. This will be our next step. By analyzing still further, we shall isolate those fundamental assumptions about sets that are implicitly required in order to obtain the set theory developed above, and then, by taking these assumptions as the axioms of set theory, we shall turn the whole process round, obtaining a well-defined set theory, based on a set of axioms.

We commence by taking the ordinal number system as given and asking what principles of set formation are used, perhaps implicitly, in constructing the V_α-hierarchy of sets.

Well, for a start we took the power-set operation as basic. So we are assuming that for any set x, there is a set that consists of all and only the subsets of x, i.e. the power set of x. Formulating this as an axiom of set theory, we have:

> *Power Set Axiom.* If x is a set, there is a set that consists of all and only the subsets of x.

Exercise 2.3.1. *Write down a sentence of* LAST *which expresses the power set axiom.*

The power set axiom allows us to pass from V_α to $V_{\alpha+1}$ in the construction of the cumulative hierarchy of sets. What about the definition of V_α when α is a limit ordinal? Well, in this case we have

$$V_\alpha = \bigcup_{\beta<\alpha} V_\beta,$$

so we must be able to form the union of any collection of sets:

> *Axiom of Union.* If x is a set, there is a set whose members are precisely the members of the members of x, i.e. the set $\bigcup x$.

Exercise 2.3.2. *Express the axiom of union in the language* LAST.

The axiom of union allows us to obtain V_α, when α is a limit ordinal, as

$$V_\alpha = \bigcup\{V_\beta \mid \beta < \alpha\}.$$

But wait a moment. How do we know that $\{V_\beta \mid \beta < \alpha\}$ is a set? Well, $\alpha = \{\beta \mid \beta < \alpha\}$ is a set.[3] And one can obtain $\{V_\beta \mid \beta < \alpha\}$ from the set $\{\beta \mid \beta < \alpha\}$ by replacing each element β of α by the set V_β. This leads us to the formulation of the *Axiom of Replacement.* It is perhaps one of the least appreciated axioms of set theory. And yet it is undoubtedly one of the most powerful axioms. The main reason why the nonexpert finds it hard to appreciate the axiom of replacement is that it is rarely required in most areas of mathematics. It is predominantly an axiom for the set theorist. There are, however, several instances where it is known for certain that it is necessary for results in everyday mathematics, so it should not be ignored.

Roughly speaking, what the axiom of replacement says is that, if we have a set x, and we *replace* each element a of x by a new set a', then the collection of all a' so obtained is a set. The immediate question is: what is to determine a 'replacement'? If x is finite, we can list the elements a of x and alongside them the new sets a', and in this manner we can say exactly what the replacement procedure is. But what in the general case? The answer should, by now, be obvious. We allow any replacement procedure that can be described by a formula of LAST.

> *Axiom of Replacement.* Let $\phi(v_n, v_m)$ be any formula of LAST (which may refer by name to any finite number of specific sets), such that for each set a there is a unique set b such that $\phi(a, b)$. Let x be a set. Then there is a set y consisting of just those b such that $\phi(a, b)$ for some a in x.

Exercise 2.3.3. *As in the case of the Axiom of Subset Selection, it is not possible to transcribe the axiom of replacement as stated above to a sentence of* LAST, *since* LAST *has no facility for handling formulas of* LAST *themselves. This difficulty may be overcome by regarding the axiom of replacement as an axiom* schema, *each appropriate formula ϕ giving rise to a specific instance of this schema. Given any such formula ϕ of* LAST *with free variables v_n and v_m only, write down the sentence of* LAST *that expresses the ϕ-instance of the axiom of replacement. The Axiom of Replacement says that all the sentences of* LAST *of the kind you have (I hope) written down are true.*

[3] Remember that for the time being we are taking the ordinal numbers as basic. Later, we shall see what assumptions are needed for the construction of the ordinals.

We now have the axioms we need in order to construct the cumulative hierarchy of sets, given the ordinal number system, and we turn to the question of what is needed in order to construct the ordinals. It turns out that only a few very simple requirements remain to be formulated as axioms. These are as follows.

> *Null Set Axiom.* There is a set which has no members. (This set is denoted by the symbol $\emptyset$.)
>
> *Axiom of Infinity.* There is a set x such that $\emptyset \in x$, and such that $\{a\} \in x$ whenever $a \in x$.

Some comment concerning this last axiom is warranted. Axioms such as the Power Set Axiom, although providing us with new sets, require the existence of sets before they can function, and as such do not in themselves guarantee that our set-theoretic universe, V, will be nontrivial. Only two of our axioms do this: the Null Set Axiom and the Axiom of Infinity. Taken with the Null Set Axiom, the other axioms of set theory (leaving aside the Axiom of Infinity for the moment) allow us to construct many finite sets. But without the Axiom of Infinity we are unable to pass into the realm of the transfinite, and this is, after all, what set theory is all about. Now, in order to obtain all the infinite sets we need, it suffices that we commence with just one infinite set. The precise nature of this set turns out to be quite irrelevant, so we have some freedom in the way we formulate the Axiom of Infinity. (But notice that the notion of 'infinite' is not itself a basic notion in our theory.) The formulation chosen has the advantage of being easy to state.

The reader should bear in mind that although in our subsequent development we shall be able to construct sets that are, in every way imaginable, immeasurably larger than the set of natural numbers (say), no further 'axioms of infinity' will be required to do this; the one leap provided by the Axiom of Infinity is sufficient. As such, the Axiom of Infinity is an extremely powerful assumption. Indeed, the knowledge that Zermelo–Fraenkel set theory is not able to resolve all the questions about sets that may be formulated in the theory has led various people to consider extensions of the theory obtained by introducing additional 'axioms of infinity', trying to mimic at a higher level the jump from the finite to the infinite provided by the Axiom of Infinity. In no case could it be said that the attempt came anywhere near to achieving its aim. (See Chapter 3 for further details.)

Of course, since the Axiom of Infinity guarantees the existence of at least one set, we can prove the Null Set Axiom by a simple application of the Axiom of Subset Selection: given some set a, we have

$$\emptyset = \{x \in a \mid x \neq x\}.$$

So we could omit the Null Set Axiom from the axioms of set theory if we wished. However, in view of its fundamental nature it is usual to include it as an axiom in its own right.

One final remark: Our formulation of the Axiom of Infinity requires the existence of the operation $a \to \{a\}$. Many texts include a 'Pairing Axiom' in their axiomatization of set theory, guaranteeing the existence of the unordered pair $\{a, b\}$ of any sets a, b. A special case of this then provides singletons, of course. However, all finite sets can easily be obtained by applying the Axiom of Replacement to the sets $\mathcal{P}(\emptyset)$, $\mathcal{PP}(\emptyset)$, $\mathcal{PPP}(\emptyset)$, etc. (*Exercise*: *Check this.*). Consequently we shall not regard the 'Pairing Axiom' as a fundamental axiom.

Exercise 2.3.4. *Express the above two axioms in* LAST.

Have we forgotten anything? Well, we have not mentioned the Axiom of Subset Selection in the above list, but the adoption of this principle has already been acknowledged. Anything else? The answer is 'Yes', but the remaining axiom is so very fundamental that it could easily be forgotten. We are, after all, considering a theory of sets, and a set is just a collection of objects, so we have an axiom that reflects this fact within the theory, namely:

> *Axiom of Extensionality.* If two sets have identical elements, then they are equal.

The converse to the above assertion is also valid, of course, but that need not be included here, since it is a theorem of logic.

Exercise 2.3.5. *Express the Axiom of Extensionality in* LAST.

Our analysis is now complete. The following collection of axioms suffices for the construction of the ordinal number system and the cumulative hierarchy of sets:

1. Axiom of Extensionality.
2. Null Set Axiom.
3. Axiom of Infinity.
4. Power Set Axiom.
5. Axiom of Union.
6. Axiom of Replacement.

7. Axiom of Subset Selection.

The axioms of Zermelo–Fraenkel set theory then consists of the above seven statements, together with the following two:

8. $V = \bigcup_\alpha V_\alpha$.

9. Axiom of Choice.

Leaving aside the formulation of the Axiom of Choice for the time being, the above description of Zermelo–Fraenkel set theory, whilst accurate, is not in its most concise form. The problem is the formulation of Axiom 8. In order to state this axiom, we have had to assume a fair development of the theory based on the other axioms, at least as far as the construction of the ordinal number system and the cumulative hierarchy of sets. It would be better if we could replace statement 8 by a more basic assertion. This turns out to be quite easy. In the presence of axioms 1–7, statement 8 is equivalent to the fact that the binary relation of set membership ($\in$) is well-founded. So we may replace Axiom 8 by the more fundamental axiom:

Axiom of Foundation. $\in$ is a well-founded relation.

A more explicit way of expressing the above axiom is: for every nonempty set x, there is a set $a \in x$ such that $a \cap x = \emptyset$.

Exercise 2.3.6. *Prove the result just claimed, that the relation $\in$ is well-founded if and only if, for every nonempty set x, there is a set $a \in x$ such that $a \cap x = \emptyset$.*

Exercise 2.3.7. *Assuming Axioms 1–7 in the above list, prove that the above statement of the Axiom of Foundation is equivalent to the equality*

$$V = \bigcup_\alpha V_\alpha.$$

We finish the section by summarizing the Zermelo–Fraenkel axioms.

(1) *Axiom of Extensionality.* If two sets have the same elements, then they are equal.

(2) *Null Set Axiom.* There is a set, $\emptyset$, which has no members.

(3) *Axiom of Infinity.* There is a set x such that $\emptyset \in x$ and such that $\{a\} \in x$ whenever $a \in x$.

(4) *Power Set Axiom.* If x is a set, there is a set, $\mathcal{P}(x)$, consisting of all and only the subsets of x.

(5) *Axiom of Union.* If x is a set, there is a set, $\bigcup x$, consisting of all elements of all elements of x.

(6) *Axiom of Replacement.* Let $\phi(v_n, v_m)$ be any formula of LAST, such that for each set a there is a unique set b such that $\phi(a, b)$. Let x be a set. Then there is a set y consisting of just those b such that $\phi(a, b)$ for some a in x.

(7) *Axiom of Subset Selection.* Let x be a set, and let $\phi(v_n)$ be a formula of LAST. Then there is a set consisting of just those a in x for which $\phi(a)$.

(8) *Axiom of Foundation.* If x is a set, there is an $a \in x$ such that $a \cap x = \emptyset$.

(9) *Axiom of Choice.* (See Section 2.7.)

The theory whose Axioms are 1–8 above is usually denoted by ZF. If we add Axiom 9, we denote the resulting theory by ZFC. This is at slight variance with the fact that 'Zermelo–Fraenkel set theory' has all nine axioms as its basic assumptions, but the nomenclature is now standard.

Exercise 2.3.8. *The nine axioms listed above are not all independent. For instance, we have already observed that the null set axiom may be deduced from the other* ZF *axioms. A more challenging exercise is to deduce the axiom of subset selection from the remaining axioms. This requires clever use of the axiom of replacement. Given a set x and a formula ϕ of* LAST, *consider the replacement rule F defined by*

$$F(\alpha) = \begin{cases} \{a\} & , \text{ if } \phi(a) \\ \emptyset & , \text{ otherwise} \end{cases}$$

Then consider the set $\bigcup\{F(a) \mid a \in x\}$. (You should formulate your solution in a way that allows you to apply the Axiom of Replacement as stated.)

Exercise 2.3.9. *Examine the development of the ordinal numbers in Section 1.7 and see how the various axioms are used, paying particular attention to the use of the Axiom of Replacement in the proof of Theorem 1.7.11.*

2.4 Classes

From the point of view of set theory, *sets* are completed entities — points in the space of all sets one might say. Our axioms tell us how to construct and handle these entities. Now, as we know, a set is a collection of objects, those objects also being sets. But does it follow that any collection of objects is a set? Well, before we can answer this, we have to ask ourselves what is meant by the words 'collection' and 'object' here.

By 'object' (i.e. a point in the space) we surely mean 'set'. But just what do we mean by 'collection'? Naturally, any *set* (i.e. any point in the space) is a 'collection'. But what about that case where a formula of LAST determines a 'collection'? Are all 'collections' determined by formulas of LAST *sets*?

The answer is 'no'. For instance, the collection, V, of all sets is not itself a set. If it were, then by the axiom of subset selection,

$$\{x \in V \mid x \notin x\}$$

would be a set, and we have already seen what happens then! And yet V is a well-defined collection. Indeed, if $\phi(v_0)$ is the formula $(v_0 = v_0)$ of LAST, then V is just the collection of all sets x for which $\phi(x)$ is true; i.e.,

$$V = \{x \mid x = x\}.$$

Another LAST-definable collection that is not a set is the collection of all ordinals.

Thus there are collections of sets, definable by formulas of LAST, that are not sets. Since these collections are not sets, the Zermelo–Fraenkel axioms do not tell us how to handle them. They are somehow 'too big' to be 'completed collections' in the space of all sets. Now, it would be a nuisance if we could not discuss such collections *qua* 'collections'. Indeed, we just have referred to the *collection* of all sets and the *collection* of all ordinals! But what does such discussion amount to, and can/should it be formalized?

In fact it is possible to formalize such discussions, by enlarging the axiom system to handle these 'large' collections. In this case we are then no longer doing *set* theory, or course, but something else. This extended theory is generally known as *class theory*. It includes set theory as a subsystem. The objects under discussion in class theory are known as *classes*. All sets are classes. Classes that are not sets are known as *proper classes*; these are collections (of sets) that are somehow 'too big' to qualify as sets. The most common axiomatic treatment of class theory that extends the Zermelo–Fraenkel system is due to Bernays and Gödel and is described in [7].

However, this is not the route I shall follow here. As I see it, the main disadvantages with developing such a system are, (1) it results in a loss of the intuitive naturalness of set theory that the Zermelo–Fraenkel axioms manage to capture, and, (2) it is not necessary. In my view, it is quite natural to base set theory on the idea of iteratively constructing new sets from old ones (so in set theory one is always climbing upward), whereas in class theory the 'universe' of sets is presented as a completed whole (so one looks downward at the universe of sets from a high vantage point). Moreover, when I say it is not necessary to develop an axiomatic class theory, I am not just expressing a vague idea. One can prove that any result about sets that is provable in Bernays-Gödel class theory is already provable in Zermelo–Fraenkel set theory.

My preferred way of dealing with 'big collections' is as follows. Simply introduce the notion of a *class* as a convenient abbreviational device. Given any formula $\phi(v_n)$ of LAST, whose names refer to specific sets, the collection

$$\{x \mid \phi(x)\}$$

of all x for which $\phi(x)$ is said to be a *class*.

Now, all sets are classes. Indeed, if a is a set, the LAST formula $(v_0 \in w_0)$ defines the class a when w_0 denotes a; i.e.,

$$a = \{x \mid x \in a\}.$$

But, as we saw above, not all classes will be sets. For instance, V is a class that is not a set. Such classes will be called *proper classes*.

Since proper classes are not sets, we are not able to handle classes as we do sets. For instance, we cannot ask ourselves if one class is a member of another. This question has no meaning in set theory. A proper class is an 'uncompleted collection' and, hence, is *never* available for being in any other collection. It is not just false to write '$V \in V$', it is set-theoretically meaningless, as is the statement '$V \notin V$'.

"So what," you may ask, "is the point of introducing (proper) classes?" Well, classes are collections and, hence, will exhibit many of the properties of sets. And providing we exercise a little care, we can handle classes quite often just as if they were sets. Indeed, the only thing we must never do is treat a proper class as a 'completed whole' or 'point in the space'.

"But surely," you say, "what we have now done is enlarged our theory to incorporate proper classes?" Well no, because we shall only use them as abbreviations. If A is some class, then there will be a LAST formula $\phi(v_0)$ such that

$$A = \{x \mid \phi(v_0)\}.$$

In discussing the 'class' A, we are simply avoiding the explicit mention of ϕ. If challenged by a 'purist', we could always stop referring to A and deal with ϕ instead. For instance, if I wrote

$$a \in A$$

and you were upset by my use of the symbol A to denote something that, by my own admission, does not really exist (in the sense of set theory), I could instead write

$$\phi(a).$$

The two statements clearly have the same meaning, but in the second no use is made of classes. Again, if

$$A = \{x \mid \phi(x)\},$$

$$B = \{x \mid \psi(x)\}$$

and if I wrote

$$A = B,$$

then I could always replace this by the totally harmless statement

$$\forall x(\phi(x) \leftrightarrow \psi(x)).$$

Likewise,

$$A \subseteq B$$

can be replaced by

$$\forall x(\phi(x) \rightarrow \psi(x)).$$

Exercise 2.4.1. *Let A, B, ϕ, ψ be as above. Let C be the class $A \cup B$. Express the assertion*

$$x \in C$$

as a sentence in set theory.

Exercise 2.4.2. *As above, but now take*

$$C = A \cap B.$$

Now, so far, it may not be apparent that there is a great deal to be gained by introducing classes. Indeed, in the sense of achieving a stronger theory, there is no gain at all: they are just abbreviations. But do they help us to understand things better, and do they ever help clarify various concepts? The answer is an emphatic 'yes'. For instance, consider the statement of the axiom of replacement given earlier. This runs as follows:

> Let $\phi(v_n, v_m)$ be any formula of LAST that for each set a there is a unique set b such that $\phi(a, b)$. Let x be a set. Then there is a set y consisting of just those b such that $\phi(a, b)$ for some a in x.

Quite a mouthful, and difficult to read. The difficulty can be totally eliminated by introducing classes. Such a formula ϕ clearly defines a 'class function'. That is, the class

$$F = \{(a, b) \mid \phi(a, b)\}$$

has all the appearances and properties of a function, except for the fact that it is not a set. In terms of F, what the axiom of replacement says is that for any set x, the class

$$\{F(a) \mid a \in x\}$$

is a set, which is simple, concise, intuitive, and totally unambiguous.

To summarize then:

(1) Classes are just abbreviations. Their use can always be eliminated by replacing them by the formulas of LAST that define them.

(2) Proper classes may be thought of as 'big collections'.

(3) Proper classes can be handled as sets, except that the class is not a completed whole, eligible to be a member of anything else; for example, $\mathcal{P}(A)$ has no meaning if A is a proper class.

(4) All sets are classes. Some classes, the proper classes, are not sets.

Exercise 2.4.3. *Let* On *be the class of all ordinals. Prove that* On *is a proper class. (Hint. Show that if* On *were a set, it would be an ordinal, whence we would have* On $\in$ On.)

Exercise 2.4.4. *Show that the following assertions are equivalent:*

(a) $(\forall x \in V)(\exists y \in \text{On})(\exists f \in V)[f : x \leftrightarrow y]$;

(b) *Every set can be well-ordered.*

Exercise 2.4.5. *Let*

$$A = \{x \mid (\exists y \in \text{On})(\exists f)(f : x \leftrightarrow y)\}.$$

Show that condition (b) *in Exercise 2.4.4 can be expressed as the class identity*

$$V = A.$$

2.5 Set Theory as an Axiomatic Theory

We were led to our axiomatization of set theory by an analysis of the Zermelo hierarchy of sets. Now that we have obtained these axioms, we can take them as basic and develop set theory rigorously, with these axioms as the starting point. This is Zermelo–Fraenkel set theory, ZFC. Providing the ZFC axioms are consistent, we can be sure that anything we prove in ZFC set theory is meaningful. Indeed, if we 'believe' the axioms, we can conclude that anything proved from them is 'true'. (The reader who so wishes is permitted to delete the quotation marks from the last sentence.)

Now, it is a consequence of a classical theorem of logic due to Gödel that we cannot hope to prove that ZFC is consistent. In order to prove the consistency of ZFC, one would need to carry out the proof itself in a theory even stronger than ZFC, whose own consistency would be even more in doubt, of course. With a foundational subject like set theory, one is *forced* to make an assumption of consistency somewhere along the line. In fact, we can go one step beyond assuming the system ZFC is free of contradictions. Another theorem of Gödel shows that if ZFC were an inconsistent theory, then so too would be ZF, the theory ZFC minus the Axiom of Choice. So one simply needs to assume that ZF is consistent in order to be sure that ZFC is consistent.

I shall assume throughout that ZF is consistent; for otherwise, there would be no point in my writing this book. Granted this consistency assumption, anything we prove from the axioms ZFC will thus be a meaningful assertion about sets.

One obvious question that remains to be answered now is this. When we formulated the ZFC axioms for set theory, did we miss anything fundamental? More precisely, we formulated the axioms in an attempt to make precise the basic assumptions about sets that we must implicitly make when we wish to develop a hierarchical theory of sets in the manner outlined in Section 2.2. Do the ZFC axioms in fact do this? This reduces at once to the more precise question: assuming only the ZFC axioms, can we define the Zermelo hierarchy, V_α, $\alpha \in \mathrm{On}$? Well, how do we define the Zermelo hierarchy?

To commence we set

$$V_0 = \emptyset.$$

Then, given V_α, we set

$$V_{\alpha+1} = \mathcal{P}(V_\alpha).$$

And, if λ is a limit ordinal and V_α is defined for all $\alpha < \lambda$, we set

$$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha.$$

Now, it is easily seen that this three-cases definition can be expressed more concisely by the single clause:

$$V_\alpha = \bigcup_{\beta<\alpha} \mathcal{P}(V_\beta)$$

(for all α).

Exercise 2.5.1. *Prove the equivalence of these two definitions of the Zermelo hierarchy.*

Working with this alternative definition of the hierarchy, it is clear that in order to define V_α, not only do we need to have first defined all the sets V_β, for $\beta < \alpha$, we need to have available the sequence (or function)

$$\langle V_\beta \mid \beta < \alpha \rangle ,$$

which assigns to each ordinal $\beta < \alpha$ the corresponding set V_β. For, letting f denote this function, we actually define V_α as

$$V_\alpha = \bigcup \{ \mathcal{P}(f(\beta)) \mid \beta < \alpha \}.$$

By the axioms of power set, replacement, and union, this is an admissible definition of a set.

Definitions of the above kind are sometimes referred to as definitions 'by induction'. More correctly they are definitions by *recursion.* (*Induction* is a method of proof, not of definition.) Letting $f : \mathrm{On} \rightarrow V$ (use of class notation!) be the 'function' $f(\alpha) = V_\alpha$, we define $f(\alpha)$ in terms of $f \quad \alpha$ (i.e. in terms of $\langle f(\beta) \mid \beta < \alpha \rangle$). Indeed, we have

$$f(\alpha) = \bigcup \{ \mathcal{P}((f \quad \alpha)(\beta)) \mid \beta < \alpha \}.$$

That such definitions are possible in ZF set theory is a consequence of the recursion principle, which I consider next.

2.6 The Recursion Principle

Although basically simple in concept and application, the recursion principle is often not fully appreciated. However, it plays a central role in set theory, and its importance cannot be overemphasized. Intuitively, what it says is that, in the ZF system, it is possible to define functions by recursion. It can be, and often is, applied without being fully understood, but some awkward complications arise when one tries to state and prove the recursion principle. It is these complications that sometimes prevent students

from gaining a proper understanding of the principle. Before we start, let me therefore warn the casual reader that you may well find the following discussion rather hard to follow. However, I can reassure you that if you content yourself with the knowledge that recursive definitions are always possible, you can read the rest of the book without any further loss. (For the reader interested in set theory *per se*, there is no such escape clause, of course, so if you are such a reader, you should prepare yourself for some hard work.)

The recursion principle is a result about ZF; it does not require the Axiom of Choice.

Now, starting with the ZF axioms, the ordinal number system can be developed as in Section 1.7. Assuming this development from now on, I first state a simplified recursion principle.

Theorem 2.6.1 [Recursion on an Ordinal] Let h: On$\times V \to V$ be a 'class function'. Let λ be an ordinal. Then there exists a unique function $f : \lambda \to V$ such that, for every $\alpha \in \lambda$,

$$f(\alpha) = h(\alpha, f \quad \alpha).$$

I shall prove Theorem 2.6.1 presently, but first let me demonstrate how the use of classes can be eliminated from the statement of the theorem.[4] As it stands, Theorem 2.6.1 is an assertion that implicitly involves a universal quantifier, $\forall h$, ranging over proper classes. This is not possible in the ZF system. But now let us fix our attention on a single, but arbitrary, h. Let $\phi(v_0, v_1, v_2)$ be that formula of LAST that defines h. That is, for $\alpha \in$ On and $x, y \in V$,

$$h(\alpha, x) = y \quad \text{if and only if} \quad \phi(\alpha, x, y).$$

What Theorem 2.6.1 really says is that, starting with the formula ϕ, we can prove, on the basis of the ZF axioms, that there exists a unique function $f : \lambda \to V$ such that, for every $\alpha \in \lambda$,

$$\phi(\alpha, f \quad \alpha, f(\alpha)).$$

In other words, Theorem 2.6.1 as stated is not a single theorem provable in the theory ZF but a schema of theorems of ZF. For each h, there is a

[4]This discussion concerns a rather subtle point, and you may well find it difficult to see what is going on—in which case you should perhaps postpone reading it in detail until later. Indeed, for the casual reader it can safely be ignored. Only the intending mathematical logician needs eventually to master the point discussed.

corresponding ZF theorem that asserts the existence, for every λ, of a corresponding f. (The reason why I stated the theorem the way I did should, however, be fairly clear. Reformulation of the result as a theorem-schema along the lines just indicated results in a rather complex, and certainly less intuitive, assertion.)

Before proving Theorem 2.6.1, it is perhaps worth our while seeing how this helps us to define the Zermelo hierarchy. In fact, all Theorem 2.6.1 tells us is that, for every λ, the hierarchy

$$\langle V_\alpha \mid \alpha < \lambda \rangle$$

exists (as a function with domain λ). In due course, when we have proved Theorem 2.6.1, we shall see how it can be extended to give the full hierarchy (which is, of course, a class 'function', with 'domain' On).

So, fixing λ, consider $h : \mathrm{On} \times V \to V$ defined by

$$h(\alpha, x) = \begin{cases} \bigcup_{\xi \in \mathrm{dom}(x)} \mathcal{P}(x(\xi)), & \text{if } x \text{ is a function,} \\ \emptyset, & \text{otherwise.} \end{cases}$$

By the axioms of power set, replacement, and union, h is a well-defined function. By Theorem 2.6.1, there is a function $f : \lambda \to V$ such that

$$f(\alpha) = h(\alpha, f \quad \alpha)$$

for all α. By definition of h, this means that for all $\alpha < \lambda$

$$f(\alpha) = \bigcup_{\xi < \alpha} \mathcal{P}(f(\beta)).$$

Indeed, Theorem 2.6.1 tells us that this f is unique. Clearly, f is what we want: f is the sequence $\langle V_\alpha \mid \alpha < \lambda \rangle$. In other words, $\langle V_\alpha \mid \alpha < \lambda \rangle$ is the unique function that Theorem 2.6.1 guarantees us when we define h as above.

I turn now to the proof of Theorem 2.6.1. Let $h : \mathrm{On} \times V \to V$, and let $\lambda \in \mathrm{On}$. Using only the axioms of ZF, I prove that there is a unique function $f : \lambda \to V$ such that

$$f(\alpha) = h(\alpha, f \quad \alpha)$$

for all $\alpha < \lambda$.

I first prove uniqueness.

Lemma 2.6.2 Let $\mu \leq \lambda$. Suppose $f_i : \mu \to V$, $i = 1, 2$, are such that, for all $\alpha < \mu$,

$$f_i(\alpha) = h(\alpha, f_i \quad \alpha).$$

Then $f_1 = f_2$.

Proof: By induction on μ. (Remember that, by Theorem 1.7.1, we can prove results by induction on ordinals.)

For $\mu = 0$, the result is trivial.

Now assume $\mu > 0$ and that the result holds for all $\mu' < \mu$. Thus, for $\mu' < \mu$, $f_1 \quad \mu' = f_2 \quad \mu'$. If μ is a limit ordinal, then it follows at once that $f_1 = f_2$. Otherwise, let $\mu = \nu + 1$. Then we have, by the induction hypothesis, $f_1 \quad \nu = f_2 \quad \nu$. Hence

$$f_1(\nu) = h(\nu, f_1 \quad \nu) = h(\nu, f_2 \quad \nu) = f_2(\nu).$$

Thus,

$$f_1 = (f_1 \quad \nu) \cup \{(\nu, f_1(\nu))\} = (f_2 \quad \nu) \cup \{(\nu, f_2(\nu))\} = f_2,$$

which completes the proof. □

Turning to the proof of the existence part of Theorem 2.6.1, let M be the class

$$M = \{f \mid (\exists \mu \leq \lambda)[(f : \mu \to V) \wedge (\forall \alpha \in \mu))(f(\alpha) = h(\alpha, f \quad \alpha))]\}.$$

In order to prove Theorem 2.6.1, it suffices to show that there is a function $f \in M$ such that $\mathrm{dom}(f) = \lambda$.

Lemma 2.6.3 Let $f, g \in M$. Let $\mu = \mathrm{dom}(f)$, $\nu = \mathrm{dom}(g)$, and suppose $\mu < \nu$. Then $f = g \quad \mu$.

Proof: For all $\alpha \in \mu$, we have

$$\begin{aligned} f(\alpha) &= h(\alpha, f \quad \alpha), \\ g(\alpha) &= h(\alpha, g \quad \alpha). \end{aligned}$$

So, by Lemma 2.6.2, $f = g \quad \mu$. □

Now define

$$A = \{\mu \mid (\exists f \in M)(\mathrm{dom}(f) = \mu)\}.$$

I show that $\lambda \in A$.

Suppose not. Then $\lambda \in (\lambda + 1) - A$, so $(\lambda + 1) - A \neq \emptyset$. Let μ be the least element of this set. Thus $\mu \leq \lambda$ and, for each $\nu < \mu$, there is an $f \in M$ with $\mathrm{dom}(f) = \nu$. By Lemma 2.6.3, for each $\nu < \mu$, let $F(\nu)$ be the unique $f \in M$ such that $\mathrm{dom}(f) = \nu$. By the axiom of replacement, $F[\mu]$ is a set. Let

$$f_0 = \bigcup F[\mu].$$

Using Lemma 2.6.3, it is easily seen that f_0 is a function. Moreover, for each $\nu < \mu$, $f_0 \quad \nu = F(\nu)$, so for all $\nu < \mu$, we have

$$(\forall \alpha \in \nu)(f_0(\alpha) = h(\alpha, f_0 \quad \alpha)).$$

If μ is a limit ordinal, then this implies that $f_0 \in M$ and $dom(f_0) = \mu$, contrary to the choice of μ. So μ must be a successor ordinal, say $\mu = \nu + 1$. Now set

$$f_0' = f_0 \cup \{(\nu, h(\nu, f_0))\}.$$

Then $f_0' \in M$ and $\mathrm{dom}(f_0') = \mu$, a contradiction. This completes the proof of Theorem 2.6.1.

Exercise 2.6.1. *Write out a proof of Theorem 2.6.1 in which the class F is replaced by explicit use of a* LAST *formula that defines it.*

We are now ready to state the full ordinal recursion principle. This will provide us with the complete Zermelo hierarchy $\langle V_\alpha \mid \alpha \in \mathrm{On} \rangle$.

Theorem 2.6.4 [Ordinal Recursion] Let $h\colon \mathrm{On} \times V \to V$ be a class 'function'. Then there exists a unique class 'function' $f\colon \mathrm{On} \to V$ such that, for every $\alpha \in \mathrm{On}$,

$$f(\alpha) = h(\alpha, f \quad \alpha).$$

Clearly, Theorem 2.6.4 is a sort of 'limiting' version of Theorem 2.6.1. But now when we come to examine what is really meant by the usage of the classes h, f we must be more careful than before.[5] Theorem 2.6.4 is not a theorem of set theory, provable from the ZF axioms. Nor does Theorem 2.6.4 represent a schema of existence theorems, each instance being a theorem of ZF, as was the case for Theorem 2.6.1. Rather, Theorem 2.6.4 is a metatheorem about ZF, a theorem of formal logic that guarantees that we can make recursive definitions within the ZF framework. Expressed precisely, it says the following.

Suppose $\phi(v_0, v_1, v_2)$ is a formula of LAST such that

$$(\forall \alpha \in \mathrm{On})(\forall x)(\exists y)(\forall z)[z = y \leftrightarrow \phi(\alpha, x, z)].$$

Then there is a formula $\psi(v_0, v_1)$ of LAST such that the following are provable in ZF:

[5] Readers who decided to skip this point in the context of Theorem 2.6.1 should do likewise here. For the reader only concerned with applications of the recursion principle, Theorem 2.6.4 as stated will cause no problems.

(a) $(\forall\alpha \in \mathrm{On})(\exists y)(\forall z)[z = y \leftrightarrow \psi(\alpha, z)]$;

(b) $(\forall\alpha)(\forall y)[\psi(\alpha, y) \leftrightarrow (\exists z)(z \text{ is a function} \wedge \mathrm{dom}(z) = \alpha) \wedge (\forall\xi \in \alpha)\phi(\xi, z \quad \xi, z(\xi)) \wedge \phi(\alpha, z, y)]$.

I shall not give the proof in detail. In fact, the idea is much as in Theorem 2.6.1, only now we cannot apply the replacement axiom to produce our function as we did then. Indeed, we cannot produce a *function* at all (working in ZF), since what we eventually get is a proper class. The only way to prove this is to start with the *formula* ϕ and explicitly produce an appropriate *formula* ψ as above.

We take for our ψ precisely the LAST formula that appears on the right of the double arrow in (b) above, namely,

$$(\exists z)(z \text{ is a function} \wedge \mathrm{dom}(z) = \alpha) \wedge (\forall\xi \in \alpha)\phi(\xi, z \quad \xi, z(\xi)) \wedge \phi(\alpha, z, y).$$

This makes condition (b) trivially true and leaves us only to prove (a). (Actually, we should also check uniqueness, but this is really implicit in (b).) I sketch the proof, using classes instead of formulas.

Let $h : \mathrm{On} \times V \to V$. Define a class f by

$$f = \{(\alpha, x) \mid (\alpha \in \mathrm{On}) \wedge (\exists z)[(z \text{ is a function }) \wedge \mathrm{dom}(z) = \alpha \wedge (\forall\xi \in \alpha)(z(\xi) = h(\xi, z \quad \xi)) \wedge x = h(\alpha, z)]\}.$$

It is easily seen that if (α, x), $(\alpha, x') \in f$, then $x = x'$. And if there were an α such that no x existed with $(\alpha, x) \in f$, then consideration of the least such α would lead speedily to a contradiction. Hence $f : \mathrm{On} \to V$. And clearly,

$$f(\alpha) = h(\alpha, f \quad \alpha)$$

for all α. Finally, if $g : \mathrm{On} \to V$ is such that $g(\alpha) = h(\alpha, g \quad \alpha)$, then by induction on α we get $f(\alpha) = g(\alpha)$ for all α, so $f = g$.

Exercise 2.6.2. *Fill in the details in the above sketch. Then give the proof without any use of classes.*

2.7 The Axiom of Choice

There is one axiom of set theory that we have not yet discussed: the Axiom of Choice. In its simplest form, this may be expressed as follows:

> *Axiom of Choice* (AC). Let $\mathcal{F}$ be a set of pairwise-disjoint, nonempty sets. Then there is a set M that consists of precisely one element from each member of $\mathcal{F}$. The set M is called a *choice set* for $\mathcal{F}$.

Now, in the case where $\mathcal{F}$ is finite, the existence of such a choice set M is not problematical: we may prove it from the ZF axioms. But in general, when $\mathcal{F}$ is infinite, the existence of such an M cannot be proved in ZF.

This is not to say that the existence of a choice set can never be proved. There are cases where it can be. For instance, suppose $\mathcal{F}$ is a set of pairwise-disjoint, non-empty sets of ordinals. In this case we may define

$$M = \{\alpha \in \bigcup \mathcal{F} \mid (\exists X \in \mathcal{F})(\alpha \text{ is the least member of } X)\}.$$

M is a well-defined set by virtue of the axioms of union and subset selection. And M clearly consists of exactly one element from each member of $\mathcal{F}$. The reason why we are able to construct a choice set in this case is that we have some *rule* for picking out (or 'choosing') one element of each set in $\mathcal{F}$.

Now, in general, no such rule as above will be available to us. But even then, it is intuitively clear that a choice set M should always 'exist'. We know that *all* subsets of $\bigcup \mathcal{F}$ 'exist', as *bona fide* sets, since our set theory is totally unrestrictive with regard to the power set operation. And surely 'all subsets' will include a choice set, for if it did not, we would then seem to have some implicit restriction on our ability to form sets.

On an intuitive level then, it seems that AC must be 'true', being implicit in our avowed freedom to form arbitrary new sets from old ones. And yet there is no possibility of proving $\neg$AC from the ZF axioms. Thus we adopt it as an axiom.

Though AC as formulated above is the simplest version of the Axiom of Choice to state, it is by no means the most useful as far as applications are concerned. I shall establish various alternative formulations.

Now, normally we assume the entire Zermelo–Fraenkel axiom system (i.e. ZFC) as our basic set theory. But when we are proving that various forms of the Axiom of Choice are 'equivalent', we clearly do not want to be using the Axiom of Choice itself as a fundamental axiom. When we say that statement Φ is equivalent to AC, we mean, of course, that this equivalence can be established in the system ZF alone! To emphasize this point, I shall mark all the relevant theorems as being provable in ZF.

Our first reformulation of AC concerns choice functions. Let $\mathcal{F}$ be a set of nonempty sets. A *choice function* for $\mathcal{F}$ is a function $f : \mathcal{F} \to \bigcup \mathcal{F}$ such that, for each $X \in \mathcal{F}$, $f(X) \in X$.
Consider the assertion

(AC′) Every set of nonempty sets has a choice function.

Theorem 2.7.1 (in ZF) AC $\leftrightarrow$ AC′.

Proof: ($\rightarrow$) Let $\mathcal{F}$ be a set of nonempty sets. For each $X \in \mathcal{F}$, let $X^* = X \times \{X\}$. By the axiom of replacement, let $\mathcal{F}^*$ be the set

$$\mathcal{F}^* = \{X^* \mid X \in \mathcal{F}\}.$$

Clearly, $\mathcal{F}^*$ is a set of nonempty, pairwise-disjoint sets. By AC, let $M \subseteq \bigcup \mathcal{F}^*$ be a set such that $M \cap X^*$ has exactly one element, for each $X \in \mathcal{F}$. Let $f^*(X)$ denote the unique element of $M \cap X^*$, for each $X \in \mathcal{F}$. More formally, set

$$f^*(X) = \bigcup (M \cap X^*).$$

Define $f : \mathcal{F} \rightarrow \bigcup \mathcal{F}$ by

$$f(X) = (f^*(X))_0.$$

(Recall that $(-)_0, (-)_1$ denote the inverses to the ordered-pair operation.) Clearly, f is a choice function for $\mathcal{F}$.

($\leftarrow$) Let $\mathcal{F}$ be a set of pairwise-disjoint, nonempty sets. By AC$'$, let f be a choice function for $\mathcal{F}$. Let $M = f[\mathcal{F}]$. Clearly, M contains exactly one member of each X in $\mathcal{F}$. □

Some authors regard AC$'$ as the 'basic' form of the Axiom of Choice. In fact, as the above proof shows, AC and AC$'$ are very close to each other, and after this chapter I shall not bother to distinguish between the two formulations, referring to both as AC and using the most convenient version in any instance.

Our next equivalence to AC is *Zermelo's Well-Ordering Principle*:

(WO) Every set can be well-ordered.

I shall prove that AC and WO are equivalent. First I need a lemma.

Lemma 2.7.2 (in ZF) Assume AC$'$. Let A be any set. Then there is a function $f : \mathcal{P}(A) \rightarrow A \cup \{A\}$ such that

(i) $f(A) = A$;

(ii) $f(X) \in A - X$, whenever $X \subset A$.

Proof: Let

$$B = \{A - X \mid X \subset A\}.$$

By AC$'$, let g be a choice function for B. Thus $g : B \rightarrow \bigcup B$ and $g(Y) \in Y$ for all $Y \in B$. Define

$$f : \mathcal{P}(A) \rightarrow A \cup \{A\}$$

by

$$\begin{aligned} f(A) &= A, \\ f(X) &= g(A - X), \quad \text{if } X \subset A. \end{aligned}$$

Clearly, f is as required. □

Theorem 2.7.3 (in ZF) AC $\leftrightarrow$ WO.

Proof: ($\rightarrow$) By the recursion principle, given a set A we may define a class 'function' $h : \mathrm{On} \rightarrow V$ by

$$h(\alpha) = \begin{cases} f(h[\alpha] \cap A) & , \text{ if } A \not\subseteq h[\alpha], \\ \{A\} & , \text{ otherwise} \end{cases}$$

where $f : \mathcal{P}(A) \rightarrow A$ is as in Theorem 2.7.2.

I claim that for some α, $h(\alpha) = \{A\}$. For suppose otherwise. Then $h(\alpha) \in A$ for all α. Hence by the axiom of subset selection,

$$X = h[\mathrm{On}] = \{a \in A \mid \exists \alpha (h(\alpha) = a)\}$$

is a set, and $h : \mathrm{On} \rightarrow X$ is a surjection. In fact, h is a bijection. For if $\alpha < \beta$, then $h(\alpha) \in h[\beta] \cap A$, so as $f(h[\beta] \cap A)$ cannot lie in $h[\beta] \cap A$ (by choice of f), $h(\alpha) \neq h(\beta)$.

Hence the inverse class 'function' $h^{-1} : X \rightarrow \mathrm{On}$ exists. By the axiom of replacement, therefore, On is a set, contrary to Exercise 2.4.3.

Now let α be least such that $h(\alpha) = \{A\}$. Thus

$$\gamma < \alpha \rightarrow h(\gamma) \in A.$$

But if $h[\alpha] \neq A$, then by definition of h, $h[\alpha] \neq \{A\}$. Hence

$$h : \alpha \leftrightarrow A.$$

We can thus well-order A by

$$a <_A b \leftrightarrow h^{-1}(a) \in h^{-1}(b).$$

($\leftarrow$) Let $\mathcal{F}$ be a set of pairwise-disjoint, nonempty sets. Let $X = \bigcup \mathcal{F}$. By WO, let $<_X$ be a well-ordering of X. Let

$$M = \{x \in X \mid (\exists A \in \mathcal{F})(x \text{ is the } <_X\text{-least member of } A)\}.$$

Clearly, M satisfies AC for $\mathcal{F}$. □

In conjunction with Theorem 1.7.12, the above result yields at once the following corollary.

Corollary 2.7.4 (in ZF) AC $\leftrightarrow$ For every set X there is an ordinal α and a bijection $f : \alpha \leftrightarrow X$.

Our next axiom equivalent to AC is perhaps the one most familiar to the working mathematician outside of set theory. For historical reasons it is known as a 'lemma', but it is indeed just another formulation of the Axiom of Choice.

Let $(P, \leq)$ be a poset. An element a of P is said to be *maximal* in P if and only if there is no b in P such that $a < b$. A poset can have many maximal elements. The concept of a maximal element should not be confused with that of a *maximum* element: a *maximum* element of P is an element a of P such that $b \leq a$ for all b in P, and there can clearly be at most one such element. (In the case of tosets, the two concepts do, however, coincide, as is easily seen.) A subset X of a poset P is called a *chain* if it is totally ordered by $\leq$.

The following assertion is known as *Zorn's Lemma*:

(ZL) If a poset $(P, <_P)$ has the property that every chain in P has an upper bound in P, then P has a maximal element.

Theorem 2.7.5 (in ZF) AC$\rightarrow$ ZL.

Proof: Let $(P, \leq_P)$ be a poset such that every chain in P has an upper bound in P. By Theorem 2.7.4, let λ be an ordinal and let $j : \lambda \leftrightarrow P$. For each $\xi < \lambda$, let $p_\xi = j(\xi)$. Then

$$P = \{p_\xi \mid \xi < \lambda\}.$$

By the recursion principle, define $f : \mathrm{On} \rightarrow \lambda + 1$ so that $f(0) = 0$ and, for $\eta > 0$,

$$f(\eta) = \begin{cases} \text{the least } \zeta \text{ such that } \xi < \eta \rightarrow p_{f(\xi)} <_P p_\zeta, & \text{if such a } \zeta \text{ exists,} \\ \lambda, & \text{otherwise.} \end{cases}$$

I claim that $f(\eta) = \lambda$ for some η. For suppose not. By the Axiom of Subset Selection, $X = f[\mathrm{On}]$ is a well-defined subset of λ. Since f is one–one, f has a well-defined inverse, g, on X. Then $g : X \rightarrow \mathrm{On}$ is surjective. But by the Axiom of Replacement, $g[X]$ is a set, so we have a contradiction.

Let η be least such that $f(\eta) = \lambda$. If η is a limit ordinal, then the sequence $\langle p_{f(\xi)} \mid \xi < \eta \rangle$ is a chain in P with no upper bound, which is impossible. Hence $\eta = \nu + 1$, for some ν. Clearly, $p_{f(\nu)}$ is a maximal element of P. □

The following variant of Zorn's Lemma is also common:

(ZL′) If $(P, \leq_P)$ is a poset such that every chain in P has an upper bound in P, then for every $p \in P$ there is a $q \in P$ such that $p \leq_P q$ and q is maximal in P.

Theorem 2.7.6 (in ZF) ZL $\rightarrow$ ZL′.

Proof: Let $(P, \leq_P)$ be as above, and let $p \in P$ be given. Set

$$Q = \{q \in P \mid p \leq_P q\}.$$

With the induced ordering, Q is a poset that satisfies the hypotheses of ZL. By ZL, let q be a maximal element in Q. Then $p \leq_P q$, and q is clearly maximal in P. □

Instead of proving directly that ZL′ $\rightarrow$ AC, I construct a chain of implications that will end up with AC, thereby establishing a whole collection of equivalences to AC.

The Hausdorff Maximal Principle says:

(HP) If $(P, \leq_P)$ is a poset, then every chain in P can be extended to a maximal chain.[6]

Theorem 2.7.7 (in ZF) ZL′ $\rightarrow$ HP.

Proof: Let $(P, \leq_P)$ be a given poset. Let $\mathcal{F}$ be the set of all chains in P. $\mathcal{F}$ is partially ordered by inclusion. I claim that the poset $(\mathcal{F}, \subseteq)$ has the property that every chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$. In fact, if $\mathcal{C}$ is a chain in $\mathcal{F}$, then $\bigcup \mathcal{C}$ is easily shown to be a member of $\mathcal{F}$ and, hence, is an upper bound of $\mathcal{C}$. Hence, applying ZL′ to the poset $(\mathcal{F}, \subseteq)$, we can conclude that every member of $\mathcal{F}$ extends to a maximal member of $\mathcal{F}$. This proves HP. □

A set A is said to have *finite character* if $A \neq \emptyset$, and for any set X, X is a member of A if and only if every finite subset of X is a member of A.

Exercise 2.7.1. *Let A be any set. Let $\mathcal{F}$ be the set of all subsets of $\mathcal{P}(A)$ that consist only of disjoint subsets of A (i.e. if $X \in \mathcal{F}$, then $X \subseteq \mathcal{P}(A)$ and $S, T \in X \rightarrow S \cap T = \emptyset$). Show that $\mathcal{F}$ is a set of finite character.*

Exercise 2.7.2. *Let A, B be any sets. Let $\mathcal{F}$ be the set of all functions f such that* $\mathrm{dom}(f) \subseteq A$ *and* $\mathrm{ran}(f) \subseteq B$. *Show that if we regard $\mathcal{F}$ as a*

[6] A *maximal* chain is one to which no further elements may be added so that the resulting set is still a chain.

subset of $\mathcal{P}(A \times B)$ *(which, strictly speaking, it is), then* $\mathcal{F}$ *is a set of finite character.*

Exercise 2.7.3. *Show that if we modify the example of Exercise 2.7.2 by insisting that* f *be one–one, then* $\mathcal{F}$ *is still a set of finite character.*

Tukey's Lemma says:

(TL) Every set of finite character has an element that is maximal with respect to inclusion.

The concept of finite character is, at first sight, rather strange. The proof of the following result should indicate the type of circumstance under which TL can be applied.

Theorem 2.7.8 (in ZF) TL $\rightarrow$ AC′.

Proof: Let $\mathcal{F}$ be a set of nonempty sets. We seek a function $f : \mathcal{F} \rightarrow \bigcup\mathcal{F}$ such that $f(X) \in X$ for every $X \in \mathcal{F}$. Setting $A = \bigcup\mathcal{F}$, it suffices to find a function $f : \mathcal{P}(A) \rightarrow A$ such that $f(X) \in X$ for every nonempty $X \subset A$. Set

$$G = \{f \mid f : \mathcal{P}(A) \rightarrow A\}.$$

For each $f \in G$, let

$$\mathcal{C}(f) = \{X \subseteq A \mid f(X) \in X\}.$$

Thus, for each such f, f is a choice function for the family $\mathcal{C}(f)$ of subsets of A. Set

$$K = \{f \mid (\exists g \in G)(f \subseteq g \quad \mathcal{C}(g))\}.$$

K is a set of subsets of $\mathcal{P}(A) \times A$. It is easily seen that K has finite character. So, by TL, K has a maximal element, f_0.

Suppose $\mathrm{dom}(f_0) \neq \mathcal{P}(A) - \{\emptyset\}$. Then we can find $X \subseteq A$, $X \notin \mathrm{dom}(f_0)$, $X \neq \emptyset$. Pick $x \in X$ arbitrarily, and set $f_0' = f_0 \cup \{(X, x)\}$. Then $f_0' \in K$ and $f_0 \subset f_0'$, contrary to the choice of f_0. Hence $\mathrm{dom}(f_0) = \mathcal{P}(A) - \{\emptyset\}$. Thus f will be as required, where we let a be any element of A and set

$$f = f_0 \cup \{(\emptyset, a)\}. \qquad \square$$

The next result completes our chain of implications proving that AC, ZL, HP, and TL are equivalent.

Theorem 2.7.9 (in ZF) HP $\rightarrow$ TL.

Proof: Let $\mathcal{F}$ be a set of finite character. Regarding $\mathcal{F}$ as a poset under inclusion, let $\mathcal{C}$ be a maximal chain in $\mathcal{F}$ (by HP). Now, if $\mathcal{C}$ were to have a greatest element, then the maximality of $\mathcal{C}$ would mean that such an element would be maximal in $\mathcal{F}$ and so we would be done. I show that $\mathcal{C}$ does in fact have a greatest element.

Suppose otherwise. Set $A = \bigcup \mathcal{C}$. Since $\mathcal{C}$ has no greatest element, $A \notin \mathcal{C}$. Hence we have $X \in \mathcal{C} \rightarrow X \subset A$. Now, if A were an element of $\mathcal{F}$, $\mathcal{C} \cup \{A\}$ would be a chain in $\mathcal{F}$ extending $\mathcal{C}$. Hence $A \notin \mathcal{F}$. Thus as $\mathcal{F}$ has finite character, there is a finite set $a \subseteq A$ such that $a \notin \mathcal{F}$. Since $a \subseteq A = \bigcup \mathcal{C}$ is finite and $\mathcal{C}$ has no last member, there is an $X \in \mathcal{C}$ such that $a \subseteq X$. But $X \in \mathcal{F}$ and $\mathcal{F}$ has finite character. Hence $a \in \mathcal{F}$, a contradiction. □

2.8 Problems

1. ($\in$-Induction, $\in$-Recursion)

More general than the notions of induction and recursion on ordinals are $\in$-induction and $\in$-recursion.

A. Prove that if A is a class of sets such that, for every x,

$$(\forall y \in x)(y \in A) \rightarrow (x \in A)$$

(i.e. $x \subseteq A \rightarrow x \in A$), then $A = V$. (This is the Principle of Proof by $\in$-Induction.)

B. Show that whenever $h : V \times V \rightarrow V$, there exists a unique $f : V \rightarrow V$ such that, for every set x,

$$f(x) = h(x, f \quad x).$$

(This is the Principle of $\in$-Recursion.)

C. Define the function $\rho : V \rightarrow \text{On}$ by the $\in$-recursion

$$\rho(x) = \bigcup\{\rho(y) + 1 \mid y \in x\}.$$

Show that for any set x, $\rho(x)$ is the least γ such that $x \in V_{\gamma+1}$. (ρ is called the *rank* function, and $\rho(x)$ is called the *rank* of x.)

2. (Ideals and Filters)

For basic definitions see Problems 2 in Chapter 1. Let $\mathcal{B}$ be a boolean algebra, I an ideal in $\mathcal{B}$, F a filter in $\mathcal{B}$. I (respectively, F) is said to be *prime* if and only if for each b in $\mathcal{B}$, either $b \in I$ or $-b \in I$ (respectively, F). (Prime filters are often referred to as *ultrafilters.*)

A. Show that I (respectively F) is prime if and only if it is maximal, i.e. is not equal to $\mathcal{B}$ and is not contained in any ideal (respectively filter) other than $\mathcal{B}$ itself.

B. Let $\mathcal{F}$ be a field of subsets of a set X. Let $x \in X$. Show that the set of all sets A in $\mathcal{F}$ with $x \notin A$ is a maximal ideal in $\mathcal{F}$, and that the set of all sets A in $\mathcal{F}$ with $x \in A$ is a maximal filter in $\mathcal{F}$.

C. Let X be an infinite set, and let $\mathcal{F}$ be the set of all sets $A \subseteq X$ such that either A or $X - A$ is finite. Prove that $\mathcal{F}$ is a field of sets. Show further that the set of all finite sets in $\mathcal{F}$ is a maximal ideal in $\mathcal{F}$ and that the set of all infinite sets is a maximal filter in $\mathcal{F}$.

D. Show that there is a natural, one–one correspondence between the maximal ideals in $\mathcal{B}$, the maximal filters in $\mathcal{B}$, and the boolean morphisms from $\mathcal{B}$ into the two-element algebra $\mathbf{2} = \{0, 1\}$.

E. Show that any ideal in $\mathcal{B}$, other than $\mathcal{B}$, can be extended to a maximal ideal. (This requires the use of AC.) Similarly for filters.

3. (Use of AC)

Prove each of the following results. They all make essential use of the axiom of choice. In some cases, it requires careful thought to spot the usage.

A. The union of a countable set of countable sets is countable.

B. Any vector space has a basis.

C. There is a set of real numbers that is not Lebesgue measurable.

D. A product of compact topological spaces is compact. (Tychonoff's Theorem)

E. In a Banach space $\mathcal{B}$, any bounded linear functional defined on a subspace of $\mathcal{B}$ extends to a bounded linear functional, having the same norm, defined on all of $\mathcal{B}$. (The Hahn–Banach Theorem)

F. Any subgroup of a free abelian group is free abelian. (The Nielsen–Schreier Theorem)

G. Every boolean algebra is isomorphic to a field of sets. (Stone's Theorem)

3

Ordinal and Cardinal Numbers

3.1 Ordinal Numbers

The concept of an ordinal number (or *ordinal*) was introduced in Section 1.7, where an ordinal was defined to be a woset $(X, <)$ such that

$$a = \{x \in X \mid x < a\}$$

for every $a \in X$. We saw that any two ordinals are either identical or else nonisomorphic (as ordered sets), and that, if X, Y are nonidentical ordinals, then either $X \in Y$ or $Y \in X$. We also noted that, if $(X, <)$ is an ordinal, then the ordering $<$ is just $\subset$ on X, which (in the case of ordinals) is just $\in$ on X. (This justifies my referring simply to X, Y above.)

In Theorem 1.7.12, we proved that every well-ordered set $(X, \leq_X)$ is isomorphic to a unique ordinal, which we denoted by $\mathrm{Ord}(X)$ (more precisely, $\mathrm{Ord}(X, \leq_X)$).

The first ordinal is 0, the second is $1 = \{0\}$, and the $(n+1)$'th is $n = \{0, 1, \ldots, n-1\}$. The first infinite ordinal is $\omega = \{0, 1, 2, \ldots, n, n+1, \ldots\}$, the second infinite ordinal is $\omega + 1 = \{0, 1, 2, \ldots, n, \ldots, \omega\}$, and so on. In general, the first ordinal after α is $\alpha + 1 = \alpha \cup \{\alpha\}$. Any ordinal of the form $\gamma = \alpha + 1$ (i.e. $\gamma = \alpha \cup \{\alpha\}$) is called a *successor ordinal*, and we sometimes write $\mathrm{succ}(\gamma)$. An ordinal δ that is not a successor ordinal is called a *limit ordinal*, and we sometimes write $\lim(\delta)$.

The general notational convention is that lower case Greek letters denote ordinals, with ω having the specific meaning of the first infinite ordinal.

The following set-theoretic characterization of ordinals is very useful. A set X is called *transitive* if and only if

$$[x \in X \wedge a \in x] \rightarrow a \in X.$$

Lemma 3.1.1 A set X is an ordinal if and only if it is transitive and totally ordered by $\in$.

Proof: I should first note that when I say a *set* X is an ordinal, strictly speaking I mean X together with the partial ordering $\subset$.

Suppose first that X is an ordinal. That is, $(X, \subset)$ is a woset, and for every $x \in X$, $x = \{a \in X \mid a \subset x\}$. Since $x \in X \rightarrow x \subseteq X$, X is transitive. And we know that, as an ordinal, X is totally ordered by $\in$.

Conversely, let X be a transitive set that is totally ordered by $\in$. By the axiom of foundation, X is thus well-ordered by $\in$. Now let $x \in X$. Since X is transitive, $a \in x \rightarrow a \in X$, so $x = \{a \in X \mid a \in x)$. Thus X is an ordinal, and we are done. □

Using Lemma 3.1.1, we can prove (from the ZF axioms) that there are infinitely many ordinals. By the null set axiom, the ordinal 0 exists. The existence of all the finite ordinals now follows from the following lemma.

Lemma 3.1.2 If α is an ordinal, then $\alpha \cup \{\alpha\}$ is an ordinal.

Proof: If α is transitive and totally ordered by $\in$, so too is $\alpha \cup \{\alpha\}$. Now apply Lemma 3.1.1. □

The next lemma is instrumental in proving that there are many limit ordinals.

Lemma 3.1.3 If A is a set of ordinals, then $\bigcup A$ is an ordinal.

Proof: Let $x \in a \in \bigcup A$. For some $b \in A$, $a \in b$. Since b is an ordinal, $x \in a \in b$ implies $x \in b$. Hence $x \in \bigcup A$. Thus A is transitive.

Again, let $x, y \in \bigcup A$. Pick $a, b \in A$ with $x \in a$, $y \in b$. Either $a \subseteq b$ or $b \subseteq a$. Assume, for the sake of argument, that $a \subseteq b$. Then $x, y \in b$. Hence either $x \in y$ or $y \in x$ (or $x = y$). Hence $\bigcup A$ is totally ordered by $\in$. Thus $\bigcup A$ is an ordinal. □

Having observed that the ZF axioms guarantee the existence of all the finite ordinals, the next step is to obtain ω. Now, the existence of the ordinal ω follows from the axiom of infinity (together with some other axioms), but the actual construction of the set ω presents some technical difficulties, so instead of giving the proof here, I shall leave it as an exercise for the reader. (It is not hard, but it does require some thought.)

Given ω, the existence of the ordinals $\omega + 1$, $\omega + 2$ $(= (\omega + 1) + 1))$, etc., now follows using Lemma 3.1.2 much as for the finite ordinals.

Now let $\omega + \omega$ denote the next limit ordinal, i.e. the 'set'

$$\{0, 1, 2, \ldots, \omega, \omega + 1, \ldots\}.$$

That this set really 'exists' (i.e. can be formed using the ZF axioms) may be demonstrated as follows. Let the 'function'

$$f : \omega \to V$$

be defined by

$$f(n) = \omega + n.$$

By the axiom of replacement, the collection

$$E = \{f(n) \mid n \in \omega\}$$

is a set. Let

$$A = \bigcup E.$$

By the axiom of union, A is a set. By Lemma 3.1.3, A is an ordinal. Clearly, A is the ordinal $\omega + \omega$.

Then we have the ordinals $\omega + \omega + 1$, $\omega + \omega + 2$, $\ldots, \omega + \omega + \omega$, and so on.

3.2 Addition of Ordinals

Given ordinals α, β, we define the *ordinal sum* $\alpha + \beta$. Intuitively, $\alpha + \beta$ is the ordinal that 'commences' with α and continues beyond α for β more steps. That is to say, $\alpha + \beta$ is α 'followed by' β. Formally, we set

$$A = (\alpha \times \{0\}) \cup (\beta \times \{1\}),$$

and we define a well-ordering of A by

$$(\nu, i) <_A (\tau, j) \leftrightarrow (i < j) \vee (i = j \wedge \nu < \tau).$$

(It is easily seen that this is indeed a well-ordering of A.) We then set

$$\alpha + \beta = \mathrm{Ord}(A, <_A).$$

It is immediate that the ordinal sum $\alpha + 1$ is the successor ordinal to α, so our previous notation for successor ordinals causes no problems. More generally, $\alpha + n$ is the n'th ordinal beyond α for any natural number n, and indeed $\alpha + \beta$ is the β'th ordinal beyond α for any ordinal β.

Lemma 3.2.1 Ordinal addition is associative; that is, for all α, β, γ,

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma.$$

Proof: An easy exercise. □

Notice that ordinal addition is not commutative. For example, as is easily verified,

$$1 + \omega = \omega$$

but

$$\omega + 1 > \omega.$$

Indeed, for any integer n, we have

$$n + \omega = \omega,$$

whereas

$$\omega < \omega + 1 < \omega + 2 < \omega + 3 < \dots .$$

Using ordinal addition, we can now obtain a fuller 'picture' of the ordinal number system, namely:

$$0,\ 1,\ 2,\ \dots,\ n,\ \dots,\ \omega,\ \omega+1,\ \omega+2,\ \dots,\ \omega+n,\ \dots,\ \omega+\omega,$$
$$\omega+\omega+1,\ \omega+\omega+2,\ \dots,\ \omega+\omega+n,\ \dots,\ \omega+\omega+\omega,$$
$$\omega+\omega+\omega+1,\ \omega+\omega+\omega+2,\ \dots .$$

3.3 Multiplication of Ordinals

Let λ be an ordinal, and let $\langle \alpha_\eta \mid \eta < \lambda \rangle$ be a λ-sequence of ordinals. The *ordinal sum*

$$\sum_{\eta<\lambda} \alpha_\eta$$

is defined as follows. Set

$$A = \bigcup_{\eta<\lambda} (\alpha_\xi \times \{\xi\}).$$

Define a well-ordering of A by

$$(\nu, \xi) <_A (\nu', \xi') \leftrightarrow (\xi < \xi') \vee (\xi = \xi' \wedge \nu < \nu').$$

Let

$$\sum_{\xi<\lambda} \alpha_\xi = \mathrm{Ord}(A, <_A).$$

Clearly, the intuitive 'picture' of $\sum_{\xi<\lambda} \alpha_\xi$ is the ordinal that commences with α_0, then has α_1 more steps, then another α_2 steps, and so on, up through all $\xi < \lambda$. For instance, we have

$$\begin{aligned}
\textstyle\sum_{\xi<2} \alpha_\xi &= \alpha_0 + \alpha_1, \\
\textstyle\sum_{\xi<3} \alpha_\xi &= \alpha_0 + \alpha_1 + \alpha_2, \\
\textstyle\sum_{\xi<n} \alpha_\xi &= \alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1}.
\end{aligned}$$

Notice that, in particular,

$$\sum_{n<\omega} n = \sum_{n<\omega} 1 = \omega.$$

We may now define *ordinal multiplication* as iterated addition. That is, we define

$$\alpha \cdot \beta = \sum_{\xi<\beta} \alpha.$$

Thus, $\alpha \cdot \beta$ denotes 'β copies of α', or to express it another way 'α followed by α followed by $\alpha \ldots$ (β times)'. In particular, for any finite ordinal n,

$$\alpha \cdot n = \underbrace{\alpha + \alpha + \ldots + \alpha}_{n \text{ times}}.$$

The first thing to notice about ordinal multiplication is that it is not commutative. For instance, we clearly have

$$2 \cdot \omega = \omega \quad \text{but} \quad \omega \cdot 2 = \omega + \omega > \omega.$$

Indeed, for any finite ordinal n, $n \cdot \omega = \omega$ but

$$\omega < \omega \cdot 2 < \omega \cdot 3 < \omega \cdot 4 < \ldots .$$

We do have a distributive law, namely:

Lemma 3.3.1 For any α, β, γ,

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma.$$

Proof: An easy exercise. □

The other distributivity property is false. For example

$$(1 + 1) \cdot \omega = 2 \cdot \omega = \omega$$

but

$$1 \cdot \omega + 1 \cdot \omega = \omega + \omega > \omega.$$

Finally, we have associativity of ordinal multiplication.

Lemma 3.3.2 For any α, β, γ,

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$$

Proof: A moderately easy exercise. □

Using ordinal multiplication, we may now describe the ordinal number system even more fully than before:

$0,\ 1,\ 2,\ \ldots\ ,\ n,\ \ldots\ ,\ \omega,\ \omega+1,\ \omega+2,\ \ldots\ ,\ \omega+n,\ \ldots\ ,\ \omega+\omega,$
$\omega+\omega+1,\ \omega+\omega+2,\ \ldots\ ,\ \omega+\omega+n,\ \ldots\ ,\ \omega\cdot 3,$
$\omega\cdot 3+1,\ \omega\cdot 3+2,\ \ldots\ ,\ \omega\cdot 3+n,\ \ldots\ ,\ \omega\cdot 4,$
$\omega\cdot 4+1,\ \omega\cdot 4+2,\ \ldots\ ,\ \omega\cdot 5,\ \ldots\ ,\ \omega\cdot n,\ \ldots$
$\omega\cdot\omega,\ \omega\cdot\omega+1,\ \omega\cdot\omega\cdot 2,\ \ldots\ ,\ \omega\cdot\omega\cdot n,$
$\ldots\ ,\ \omega\cdot\omega\cdot\omega,\ \ldots\ ,\ \omega\cdot\omega\cdot\omega\cdot\omega,\ \ldots\ .$

Notice that the limit ordinals are just those ordinals of the form $\omega\cdot\alpha$ for some ordinal α. This suggests that the ordinals consist of nothing more than an 'endless' sequence of copies of ω placed one after the other. However, although this is strictly speaking true, it provides the beginner with a picture that is almost certainly false. The deep implications that lie behind the word 'endless' here mean that there are many limit ordinals that do not resemble ω in in the least, even though they are of the form $\omega\cdot\alpha$ for some α. This will become clear when we are able to describe some ordinals that are much bigger than any mentioned above.

The remaining basic arithmetical operation on ordinals is exponentiation, but before we can introduce this notion, we need to establish some fundamental results about sequences of ordinals.

3.4 Sequences of Ordinals

Let λ be a limit ordinal, and let $\langle\alpha_\xi \mid \xi<\lambda\rangle$ be a λ-sequence of ordinals. We write

$$\alpha=\lim_{\xi<\lambda}\alpha_\xi$$

if and only if

$$(\forall\beta<\alpha)(\exists\xi<\lambda)(\forall\zeta)(\xi<\zeta<\lambda\rightarrow\beta<\alpha_\zeta\leq\alpha).$$

If such an α exists, it is clearly unique, and we call it the *limit* of the sequence $\langle\alpha_\xi \mid \xi<\lambda\rangle$. Our next lemma shows that many sequences do have limits.

Lemma 3.4.1 Let λ be a limit ordinal, and let $\langle\alpha_\xi \mid \xi<\lambda\rangle$ be an increasing sequence of ordinals. Then this sequence has a (unique) limit; and indeed,

$$\lim_{\xi<\lambda}\alpha_\xi=\bigcup_{\xi<\lambda}\alpha_\xi.$$

Proof: An easy exercise. □

Lemma 3.4.2 Let λ, μ be limit ordinals, and let $f : \mu \to \lambda$ be an order-preserving function such that $\lim_{\xi<\mu} f(\xi) = \lambda$. Let $\langle \alpha_\xi \mid \xi < \lambda \rangle$ be an increasing sequence. Then

$$\lim_{\xi<\lambda} \alpha_\xi = \lim_{\xi<\mu} \alpha_{f(\xi)}.$$

Proof: An easy exercise. □

Lemma 3.4.3 Let λ be a limit ordinal, and let $\langle \alpha_\xi \mid \xi < \lambda \rangle$, $\langle \beta_\zeta \mid \zeta < \lambda \rangle$ be increasing sequences such that

(a) $(\forall \xi < \lambda)(\exists \zeta < \lambda)(\beta_\zeta > \alpha_\xi)$,

(b) $(\forall \zeta < \lambda)(\exists \xi < \lambda)(\alpha_\xi > \beta_\zeta)$.

Then

$$\lim_{\xi<\lambda} \alpha_\xi = \lim_{\zeta<\lambda} \beta_\zeta.$$

Proof: An easy exercise. □

Lemma 3.4.4 Let λ be a limit ordinal, and let $\langle \alpha_\xi \mid \xi < \lambda \rangle$ be any λ-sequence of ordinals. For each $\mu < \lambda$, let

$$\sigma_\mu = \textstyle\sum_{\xi<\mu} \alpha_\xi.$$

Then

$$\textstyle\sum_{\xi<\lambda} \alpha_\xi = \lim_{\mu<\lambda} \sigma_\mu.$$

Proof: I leave the proof as an exercise. □

Let $f : \lambda \to \lambda$, and let $\alpha \in \lambda$ be a limit ordinal. We say f is *continuous at* α if and only if

$$f(\alpha) = \lim_{\xi<\alpha} f(\xi).$$

For example, the identity function on λ is continuous at every limit ordinal in λ. We shall see many more examples of continuity later.

Exercise 3.4.1. *Let λ be endowed with the order topology (see Problems 1.3). Show that a function $f : \lambda \to \lambda$ is continuous at α in the sense just defined if and only if it is continuous at α with respect to the order topology on λ.*

A function $f : \lambda \to \lambda$ is said to be a *normal function* if and only if it is both order preserving and continuous at every limit ordinal in λ.

Lemma 3.4.5 Let $f : \mu \to \mu$ be a normal function, and let $\lambda \in \mu$ be a limit ordinal. If $\langle \alpha_\xi \mid \xi < \lambda \rangle$ is an increasing sequence of ordinals in μ and $\lim_{\xi<\lambda} \alpha_\xi < \mu$, then

$$f(\lim_{\xi<\lambda} \alpha_\xi) = \lim_{\xi<\lambda} f(\alpha_\xi).$$

Proof: An easy exercise. □

The following lemma is often useful when normal functions are concerned.

Lemma 3.4.6 Let $f : \lambda \to \lambda$ be order preserving. Then $f(\alpha) \geq \alpha$ for all $\alpha \in \lambda$.

Proof: By induction on α. For $\alpha = 0$ there is nothing to prove. Assuming $f(\alpha) \geq \alpha$, then $f(\alpha + 1) > f(\alpha) \geq \alpha$, so $f(\alpha + 1) \geq \alpha + 1$. Finally, if α is a limit ordinal and $f(\beta) \geq \beta$ for all $\beta < \alpha$, then since $f(\alpha) > f(\beta)$ for all $\beta < \alpha$, we have $f(\alpha) > \beta$ for all $\beta < \alpha$, so $f(\alpha) \geq \bigcup_{\beta<\alpha} \beta = \alpha$. □

Let $f : \lambda \to \lambda$. We say $\alpha \in \lambda$ is a *fixed-point* of f if and only if $f(\alpha) = \alpha$.

Lemma 3.4.7 [Fixed-Point Theorem] Let $f : \mathrm{On} \to \mathrm{On}$ be a normal function (in the class sense). For every α there is a fixed-point γ of f such that $\gamma \geq \alpha$.

Proof: Let α be given. If $f(\alpha) = \alpha$, there is nothing further to prove. So assume otherwise. Then, by Lemma 3.4.6, $f(\alpha) > \alpha$. By recursion, we define a function $g : \omega \to \mathrm{On}$ so that

$$\begin{aligned} g(0) &= \alpha \\ g(n+l) &= f(g(n)). \end{aligned}$$

An easy induction proves that g is order-preserving. By Lemma 3.4.1, let $\gamma = \lim_{n<\omega} g(n)$. Notice that $\gamma > g(0) = \alpha$. I finish by proving that $f(\gamma) = \gamma$. Since f is a normal function, we have, by Lemma 3.4.5

$$f(\gamma) = f(\lim_{n<\omega} g(n)) = \lim_{n<\omega} f(g(n)) = \lim_{n<\omega} g(n+1) = \gamma.$$

as required. □

The above result does not, in general, hold if $f : \lambda \to \lambda$. For instance, the function $f : \omega \to \omega$ defined by $f(n) = n + 1$ has no fixed points. There do exist ordinals λ such that every normal function $f : \lambda \to \lambda$ has a fixed point, and indeed arbitrarily large fixed-points in λ, but we shall not be able to characterize these ordinals until later.

3.5 Ordinal Exponentiation

Let $\alpha \in \mathrm{On}$. By recursion, we define a function $f_\alpha : \mathrm{On} \to \mathrm{On}$ so that:

$$\begin{aligned} f_\alpha(0) &= 1, \\ f_\alpha(\beta+1) &= f_\alpha(\beta) \cdot \alpha, \\ f_\alpha(\beta) &= \lim_{\gamma<\beta} f_\alpha(\gamma), \text{ if } \beta \text{ is a limit ordinal.} \end{aligned}$$

We write α^β instead of $f_\alpha(\beta)$. Thus, α^β is defined by 'the recursion':

$$\begin{aligned} \alpha^0 &= 1, \\ \alpha^{\beta+1} &= \alpha^\beta \cdot \alpha, \\ \alpha^\beta &= \lim_{\gamma<\beta} \alpha^\gamma, \text{ if } \beta \text{ is a limit ordinal.} \end{aligned}$$

Thus, α^β corresponds to the product of α with itself taken β times. In particular, $\alpha^1 = \alpha$, $\alpha^2 = \alpha \cdot \alpha$, $\alpha^3 = \alpha \cdot \alpha \cdot \alpha$, ...

Lemma 3.5.1 Let α, β, γ be ordinals. Then:

(i) $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$.

(ii) $(\alpha^\beta)^\gamma = \alpha^{(\beta \cdot \gamma)}$.

Proof: In each case fix α and β and argue by induction on γ. The details are left as an exercise. □

Lemma 3.5.2 Let α be a fixed ordinal. Regarded as functions of β, the functions $\alpha + \beta$, $\alpha \cdot \beta$, α^β are normal functions.

Proof: Exercise. □

Corollary 3.5.3 For any α, there are arbitrarily large ordinals β, γ, δ such that

$$\alpha + \beta = \beta, \ \alpha \cdot \gamma = \gamma \ , \ \alpha^\delta = \delta.$$

Proof: By Lemma 3.4.7. □

Exercise 3.5.1. *Show that for any* α, $\alpha + \alpha \cdot \omega = \alpha \cdot \omega$ *and* $\alpha \cdot \alpha^\omega = \alpha^\omega$.

Exercise 3.5.2. *By the above exercise,* $\beta = \alpha.\omega$ *and* $\gamma = \alpha^\omega$ *are specific instances of ordinals guaranteed to exist by Corollary 3.5.3. Find a specific value for* δ.

Is it possible for any of β, γ, δ to be a successor ordinal?

Exercise 3.5.3. *Show that for any finite $n > 1$, $n^\omega = \omega$.*

We can use the notion of ordinal exponentiation in order to extend our picture of the ordinal number system still further than before.

$0,\ 1,\ 2, \ldots,\ \omega,\ \omega + 1,\ \ldots,\ \omega \cdot 2,\ \ldots,\ \omega \cdot 3,\ \ldots,\ \omega \cdot \omega,\ \ldots$

$\omega^3,\ \omega^3 + 1,\ \ldots,\ \omega^3 + \omega,\ \ldots,\ \omega^3 + \omega^2,\ \ldots,\ \omega^4,\ \omega^4 + 1, \ldots$

$\omega^\omega,\ \omega^\omega + 1,\ \ldots,\ \omega^{\omega \cdot 2},\ \ldots,\ \omega^{\omega \cdot 3},\ \ldots,\ \omega^{\omega \cdot \omega},\ \ldots$

$\omega^{(\omega^2)},\ \ldots,\ \omega^{(\omega^3)},\ \ldots,\ \omega^{(\omega^\omega)},\ \ldots\ .$

We are thus able to picture very many ordinal numbers. Nevertheless, as we shall see in the remaining parts of this chapter, the above picture does not even *begin* to describe the true situation. The above 'sequence' is only an 'infinitesimal' initial part of the sequence of all ordinal numbers. Even the 'giant' ordinal

$$\omega^{\omega^\omega}$$

where the iteration of the ω-exponentiation is ω steps long, is tiny in comparison with 'most' ordinal numbers.

3.6 Cardinality, Cardinal Numbers

We are now in a position to assign to every set a quantity that represents the 'size' or 'number of elements' of that set. In the case of a finite set, our notion will be the number of elements of the set in the usual, everyday sense. For infinite sets we shall obtain a generalization of this finite concept.

I commence by considering finite sets. If A is a finite set, let $n(A)$ denote the number of elements of A. In essence $n(A)$ is some sort of abstraction from A, with the property that if A and B are two finite sets, then

(Property 1) $n(A) = n(B)$ if and only if A and B can be put into one-one correspondence.

What exactly is the object $n(A)$? It is a finite ordinal. The ordinal m is a set with exactly m elements. Thus:

(Property 2) $n(m) = m$.

By (1) and (2), we have, for A finite still,

(Property 3) $n(A) = m$ if and only if A and m can be put into one-one correspondence.

We turn now to the general case. By Corollary 2.7.4 (which uses AC), if X is any set, there is an ordinal α and a bijection $f : \alpha \leftrightarrow X$. Now, this might suggest that we can extend our previous notion of 'number of elements' from the finite to the infinite realm by just using the ordinals, but this does not work. The problem is that if X is infinite, Corollary 2.7.4 does not yield a unique α, but infinitely many such. For example, the set $\omega = \{0, 1, 2, \ldots\}$ can be put into one-one correspondence with the ordinal ω by means of the identity map, and with the ordinal $\omega \cdot 2$ by means of the bijection

$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ \omega + (n-1)/2, & \text{if } n \text{ is odd.} \end{cases}$$

Thus, although the finite ordinals provide us with an excellent number system for measuring the size of finite sets, the same cannot be said of the infinite ordinals for infinite sets. At least, not if we try to do it in a naive manner. But if we make use of the fact that the ordinals are well-ordered by $\in$, we can easily obtain a suitable number system for 'measuring' arbitrary sets, as I now show.

As before, we know that for any set X *there is* an ordinal α and a bijection $f : \alpha \leftrightarrow X$. The *cardinality* of X, denoted by $|X|$, is the *least* ordinal α for which there exists a bijection $f : \alpha \leftrightarrow X$. Clearly, $|X|$ is uniquely defined, and may be taken to represent the 'number of elements' of X. It is immediate that, if X is finite, then $|X| = n(x)$ as defined earlier. Moreover, it is clear from the definition that analogues of properties 1, 2, and 3 above hold in the generalized situation.

Of course, although we are using the ordinal number system to 'measure' sets, we are not using *all* the ordinal numbers. For instance, our remark above shows that the ordinal $\omega \cdot 2$ is never the cardinality of a set. A *cardinal number* (or *cardinal*) is an ordinal α such that for no $\beta < \alpha$ does there exist a bijection $f : \beta \leftrightarrow \alpha$.

It is immediate that the cardinality of any set is in fact a cardinal number, and, conversely, any cardinal number is the cardinality of some set. (In fact, the cardinal number α is the cardinality of the set $\alpha = \{\beta \mid \beta < \alpha\}$.)

It is customary to restrict the letters κ, λ, μ to denote cardinals, although λ and μ are sometimes used to denote arbitrary limit ordinals.

Theorem 3.6.1 (i) Every finite ordinal is a cardinal.

(ii) ω is a cardinal.

(iii) Every infinite cardinal is a limit ordinal.

Proof: (i) and (ii) are immediate. I prove (iii). Let $\alpha \geq \omega$. I show that $\alpha + 1$ is not a cardinal. Define $f : \alpha \to \alpha + 1$ by

$$\begin{aligned} f(0) &= \alpha, \\ f(n+1) &= n, \text{ for } n < \omega, \\ f(\xi) &= \xi, \text{ if } \omega \leq \xi < \alpha. \end{aligned}$$

Clearly, f is a bijection. Hence $\alpha + 1$ is not a cardinal. □

Now, the notions of cardinality and of cardinal number were defined using bijections. But it is often quite tricky to construct a bijection to verify some assertion about cardinality or cardinal numbers. In such instances, the theorem proved below is often helpful. We need a simple lemma.

Lemma 3.6.2 Let X, Y be sets. Then $|X| \leq |Y|$ if and only if there is an injection $f : X \to Y$.

Proof: Let $\kappa = |X|$, $\lambda = |Y|$, and let

$$i : \kappa \leftrightarrow X, \quad j : \lambda \leftrightarrow Y.$$

Suppose first that there is an injection $f : X \to Y$. Let $h = j^{-1} \circ f \circ i$. Then $h : \kappa \to \lambda$ is an injection. Let $U = h[k]$. Since $U \subseteq \lambda$, U is well-ordered (by the ordinal relation $<$). Let $\gamma = \mathrm{Ord}(U)$ (see p.22), and let $\pi : \gamma \leftrightarrow U$.

By definition of cardinality, $|U| \leq \gamma$. But clearly, $\gamma \leq \lambda$. Hence $|U| \leq \lambda$. Since $h : \kappa \leftrightarrow U$, we have $|\kappa| = |U|$, and it follows that $|\kappa| \leq \lambda$, i.e. $\kappa \leq \lambda$.

Conversely, suppose $\kappa \leq \lambda$. Then $j \circ i^{-1} : X \to Y$ is a well-defined injection. □

Theorem 3.6.3 [Schröder-Bernstein] Let X, Y be sets. If there are injections $i : X \to Y$ and $j : Y \to X$, then there is a bijection $f : X \leftrightarrow Y$.

Proof: Let $\kappa = |X|$, $\lambda = |Y|$, and let $h : \kappa \leftrightarrow X$, $k : \lambda \leftrightarrow Y$. By Lemma 3.6.2, $\kappa \leq \lambda$ and $\lambda \leq \kappa$. Hence $\kappa = \lambda$. Let $f = k \circ h^{-1}$. □

Exercise 3.6.1. *The above theorem was proved with the aid of the Axiom of Choice, using the notion of cardinality. (Where exactly is* AC *used in the proof?) The 'classical' proof of the result, though a little more complicated, proceeds by a direct combinatorial argument which does not use the Axiom of Choice. The proof is outlined below. Your task is to fill in the details.*

(1) *The first step is to show that if* X *is any set and* $h : \mathcal{P}(X) \to \mathcal{P}(X)$ *is such that*

$$A \subseteq B \subseteq X \ \to \ h(A) \subseteq h(B)$$

then there is a set $T \subseteq X$ *such that* $h(T) = T$. *(Hint: Set*

$$T = \bigcup\{A \subseteq X \mid A \subseteq h(A)\}.)$$

(2) *Given sets* X, Y *and injections* $i : X \to Y$, $j : Y \to X$ *now, define a function* $^* : \mathcal{P}(X) \to \mathcal{P}(X)$ *by setting*

$$A^* = X - j[Y - i[a]]$$

for each $A \subseteq X$. *Show that*

$$A \subseteq B \subseteq X \ \to \ a^* \subseteq B^*.$$

(3) *Combining parts (1) and (2), show that there is a set* $T \subseteq X$ *such that* $T^* = T$, *i.e. such that*

$$T = X - j[Y - i[T]].$$

(4) *Define* $f : X \to Y$ *now, by*

$$f(x) = \begin{cases} i(x), & \text{if } x \in T, \\ j^{-1}(x), & \text{if } x \in X - T. \end{cases}$$

Prove that f *is a bijection, as required.*

Using the Schröder–Bernstein Theorem, we obtain an alternative characterization of cardinal numbers. First a simple lemma.

Lemma 3.6.4 Let X, Y be nonempty sets. The following are equivalent:

(i) There is an injection $f : X \to Y$.

(ii) There is a surjection $g : Y \to X$.

Proof: (i) $\rightarrow$ (ii). Choose $x_0 \in X$ arbitrarily. Define $g : Y \rightarrow X$ by

$$g(y) = \begin{cases} x\,, & \text{if } x \text{ is the unique member of } X \text{ such that } f(x) = y, \\ x_0\,, & \text{if there is no } x \in X \text{ such that } f(x) = y. \end{cases}$$

Clearly, g is a surjection.

(ii) $\rightarrow$ (i). Let $<_Y$ be a well-ordering of Y. Define $f : X \rightarrow Y$ by setting

$$f(x) = \text{the } <_Y\text{-least } y \in Y \text{ such that } g(y) = x.$$

Clearly, f is an injection. □

Lemma 3.6.5 An ordinal α is a cardinal if and only if for no ordinal $\beta < \alpha$ is there a surjection $f : \beta \rightarrow \alpha$.

Proof: If α is not a cardinal, there is a $\beta < \alpha$ and a bijection $f : \beta \leftrightarrow \alpha$, so we are done.

Now suppose that there is a $\beta < \alpha$ and a surjection $f : \beta \rightarrow \alpha$. By Lemma 3.6.4, there is thus an injection $g : \alpha \rightarrow \beta$. But $\beta < \alpha$, so $\mathrm{id}_\beta : \beta \rightarrow \alpha$ is an injection. By the Schröder–Bernstein Theorem, there is thus a bijection $h : \beta \leftrightarrow \alpha$. Hence α cannot be a cardinal. □

So far we have only met one infinite cardinal, ω. Our next result, due to Cantor, shows that there are at least infinitely many infinite cardinals.

Lemma 3.6.6 If κ is a cardinal, there is a cardinal greater than κ.

Proof: Let $X = \mathcal{P}(\kappa)$, $\lambda = |X|$. I show that $\lambda > \kappa$.

Since the map $j : \kappa \rightarrow X$ defined by

$$j(\alpha) = \{\alpha\}$$

is an injection, Lemma 3.6.2 tells us that $\lambda \geq \kappa$. Suppose $\lambda = \kappa$. Thus there is a bijection $f : \kappa \leftrightarrow X$. Let

$$A = \{\alpha \in \kappa \mid \alpha \notin f(\alpha)\}.$$

Clearly, A is a well-defined subset of κ. So, as f is onto, for some $\alpha_0 \in \kappa$, we must have $A = f(\alpha_0)$. Then,

$$\alpha_0 \in A \;\leftrightarrow\; \alpha_0 \notin f(\alpha_0) \;\leftrightarrow\; \alpha_0 \notin A.$$

This contradiction completes the proof. □

Of course, since the ordinals are well-ordered by $\in$, so too are the cardinals. Hence, for each cardinal κ there is a unique least cardinal greater than κ; this cardinal is denoted by κ^+, and is referred to as the *successor cardinal* to κ or, if there is no possible confusion with the successor ordinal $\kappa + 1$, simply as the *successor* to κ.

The first cardinal after ω is denoted by ω_1, the next cardinal by ω_2, and so on, providing an infinite sequence of infinite cardinals

$$\omega,\ \omega_1,\ \omega_2,\ \ldots\ ,\ \omega_n,\ \omega_{n+1},\ \ldots\ .$$

Our next result shows that the sequence does not stop after ω steps.

Lemma 3.6.7 Let δ be a limit ordinal, and let $\langle \kappa_\xi \mid \xi < \delta \rangle$ be a strictly increasing sequence of infinite cardinals. Let $\kappa = \lim_{\xi<\delta} \kappa_\xi$. Then κ is a cardinal.

Proof: By Lemma 3.4.1, $\kappa = \bigcup_{\xi<\delta} \kappa_\xi$. Suppose κ were not a cardinal. Then there would be an ordinal $\alpha < \kappa$ and a surjection

$$f : \alpha \to \kappa.$$

For some $\xi < \delta$, we have $\alpha < \kappa_\xi$. Define $g : \alpha \to \kappa_\xi$ by

$$g(\nu) = \begin{cases} f(\nu), & \text{if } f(\nu) \in \kappa_\xi, \\ 0, & \text{if } f(\nu) \notin \kappa_\xi. \end{cases}$$

Clearly, g is a surjection, contrary to κ_ξ being a cardinal. Hence κ is a cardinal. □

It follows that the class of all cardinals is in one-one correspondence with the class of all ordinal numbers, and hence there is a proper class of cardinals. Indeed, by the recursion principle the 'sequence' (in the class sense) of all infinite cardinal numbers may be defined thus

$$\begin{aligned} \omega_0 &= \omega \\ \omega_{\alpha+1} &= \omega_\alpha^+ \\ \omega_\delta &= \lim_{\alpha<\delta} \omega_\alpha, \quad \text{if } \delta \text{ is a limit ordinal.} \end{aligned}$$

Now, very shortly I shall define an arithmetic of cardinal numbers, which will not at all resemble the arithmetic we defined for ordinal numbers. However, since every cardinal number is an ordinal number, and since I shall use the same notation $\kappa + \lambda$, $\kappa \cdot \lambda$, κ^λ as before for the basic arithmetical

operations of addition, multiplication, and exponentiation, there arises a possibility of confusion. To try and eliminate this, I adopt the following convention. The notation

$$\omega_\alpha$$

is to be used whenever I am considering ω_α as an *ordinal*. If, however, I am using ω_α as a *cardinal*, I write instead

$$\aleph_\alpha.$$

[$\aleph$ is the letter 'aleph', the first letter of the Hebrew alphabet.]

Thus, for example, if I write

$$\omega_\alpha + \omega_\beta$$

it is understood that *ordinal* addition is meant, whereas

$$\aleph_\alpha + \aleph_\beta$$

will imply *cardinal* addition.

But bear in mind that the two notations are purely for our convenience. The identity

$$\aleph_\alpha = \omega_\alpha$$

is strictly valid. Indeed, experts in the field often use ω_α at all times, relying on experience to keep out of trouble.

We are now in a position to give a formal definition to the terms 'finite', 'countable', and 'uncountable':

- A set is *finite* if its cardinality is less than $\aleph_0$.
- A set is *countable* if its cardinality is at most $\aleph_0$.
- A set is *uncountable* if its cardinality is at least $\aleph_1$.

Thus, $\aleph_\alpha$ is the α'th *uncountable* cardinal.

Let us return now to our picture of the ordinal number system. Although we were able to extend this picture quite a way into the transfinite by using our arithmetical notions for ordinals, all of the ordinals considered (even the 'giant'

$$\omega^{\omega^\omega}$$

where the exponentiation is iterated ω times) were countable. (We shall presently be in a position to prove this.) Hence already ω_1 is much bigger than any of these ordinals. The following picture is much more 'complete'.

$0,\ 1,\ 2,\ \ldots\ ,\ \omega,\ \omega+1,\ \ldots\ ,\ \omega\cdot 2,\ \ldots\ ,\ \omega\cdot 3,\ \ldots\ ,\ \omega\cdot n,\ \ldots$

$\omega\cdot\omega,\ \ldots\ ,\ \omega^3,\ \ldots\ ,\ \omega^4,\ \ldots\ ,\ \omega^n,\ \ldots\ ,\ \omega^\omega,\ \ldots,\omega^{\omega^\omega},\ \ldots$

$\omega_1,\ \ldots,\ \omega_2,\ \ldots\ ,\ \omega_n,\ \ldots\ ,\ \omega_\omega,\ \ldots\ ,\ \omega_{\omega_1},\ \ldots\ ,\ \omega_{\omega_2},\ \ldots\ ,\ \omega_{\omega_\omega},\ \ldots$

In fact, now we can consider the real 'giant'

$$\omega_{\omega_\omega}$$

where the subscript ω is iterated ω times.

This particular cardinal has an interesting property which we study below. First, let me make a rather obvious observation, immediate from the definition.

Lemma 3.6.8 The function $\aleph : \mathrm{On} \to \mathrm{On}$ is a normal function.

It follows from this lemma that $\omega_\alpha \geq \alpha$ for all α. In general, $\omega_\alpha > \alpha$. The iterated subscript ordinal just considered is the smallest cardinal κ such that $\aleph_\kappa = \kappa$. By Theorem 3.4.7, there is in fact a proper class of such κ.

Notice what this means in terms of size of infinity. Since the jump from ω_α to $\omega_{\alpha+1}$ is absolutely enormous in the ordinal sense, even though it is only a step of one up in the cardinal sense, the cardinals increase in size way in advance of the ordinals. Nonetheless, as we have just observed, there are arbitrarily large cardinals κ that are simultaneously the κ'th ordinal and the κ'th uncountable cardinal. Such cardinals are truly 'enormous'.

I shall end this section with a simple point, which, for all its simplicity, rapidly leads into a rather hazardous region.

By our definitions, every set has a unique cardinality. Hence, for each set X there is a unique ordinal α such that $|X| = \aleph_\alpha$. This much is known. Calculation of the α involved for a particular set X is, however, not always easy, and, for some sets X, the α concerned simply cannot be calculated on the basis of the ZFC axioms alone. We return to this issue later.

3.7 Arithmetic of Cardinal Numbers

Let $\langle \kappa_\alpha \mid \alpha < \beta \rangle$ be a sequence of cardinal numbers. The *cardinal sum*

$$\sum_{\alpha<\beta}\kappa_\alpha$$

is defined to be

$$\left|\bigcup_{\alpha<\beta}(\kappa_\alpha \times \{\alpha\})\right|$$

where the cardinals κ_α are regarded as sets in this definition. By manipulation of bijections, it is easily seen that

$$\sum_{\alpha<\beta}\kappa_\alpha = |\bigcup_{\alpha<\beta}A_\alpha|$$

where $\{A_\alpha \mid \alpha < \beta\}$ is any set of pairwise disjoint sets with $|A_\alpha| = \kappa_\alpha$ for all $\alpha < \beta$.

We write $\kappa_0 + \kappa_1$ in place of $\sum_{\alpha<2}\kappa_\alpha$. Thus

$$\kappa + \lambda = |(\kappa \times \{0\}) \cup (\lambda \times \{1\})|.$$

The following two lemmas are immediate from these definitions.

Lemma 3.7.1 Let κ, λ, μ be cardinals. Then:

(i) $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$;

(ii) $\kappa + \lambda = \lambda + \kappa$.

Lemma 3.7.2 Let $\langle \kappa_\alpha \mid \alpha < \beta \rangle$ be any sequence of cardinals, and let $\langle \lambda_\gamma \mid \gamma < \delta \rangle$ be a rearrangement of this sequence. Then

$$\sum_{\alpha<\beta}\kappa_\alpha = \sum_{\gamma<\delta}\lambda_\gamma.$$

If $\langle A_\alpha \mid \alpha < \beta \rangle$ is a sequence of sets, the *Cartesian product* of this sequence is defined to be the set

$$\prod_{\alpha<\beta}A_\alpha = \{f \mid (f : \beta \to \bigcup_{\alpha<\beta}A_\alpha) \wedge (\forall\alpha < \beta)(f(\alpha) \in A_\alpha)\}.$$

If $\langle \kappa_\alpha \mid \alpha < \beta \rangle$ is a sequence of cardinals, the *cardinal product*

$$\prod^{\#}_{\alpha<\beta}\kappa_\alpha$$

is defined to be

$$|\prod_{\alpha<\beta}\kappa_\alpha|$$

where, as in the case of addition, we make use of the fact that the cardinal numbers κ_α are just sets. It is easily seen that

$$\prod^{\#}_{\alpha<\beta}\kappa_\alpha = |\prod_{\alpha<\beta}A_\alpha|$$

where $\langle A_\alpha \mid \alpha < \beta \rangle$ is any sequence of sets with $|A_\alpha| = \kappa_\alpha$ for all $\alpha < \beta$.

We write $\kappa_0 \cdot \kappa_1$ in place of $\prod^{\#}_{\alpha<2}\kappa_\alpha$. Since $\prod_{\alpha<2}A_\alpha$ is canonically isomorphic to the usual 'Cartesian product'

$$A_0 \times A_1 = \{(a_0, a_1) \mid a_0 \in A_0 \wedge a_1 \in A_1\}$$

we have

$$\kappa \cdot \lambda = |\kappa \times \lambda|.$$

The following three lemmas follow easily from the definitions.

Lemma 3.7.3 Let κ, λ, μ be cardinals. Then:

(i) $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$;

(ii) $\kappa \cdot \lambda = \lambda \cdot \kappa$.

Lemma 3.7.4 If $\langle \kappa_\alpha \mid \alpha < \beta \rangle$ is any sequence of cardinals, and if $\langle \lambda_\gamma \mid \gamma < \delta \rangle$ is a rearrangement of this sequence, then

$$\prod^{\#}_{\alpha<\beta} \kappa_\alpha = \prod^{\#}_{\gamma<\delta} \lambda_\gamma.$$

Lemma 3.7.5 Let κ, λ, μ be cardinals. Then

$$\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu.$$

Thus, cardinal addition and multiplication are commutative and associative, and multiplication distributes over addition. It is also easily seen that

$$\kappa + \kappa = 2 \cdot \kappa.$$

If κ, λ are cardinals, the *cardinal power*

$$\kappa^\lambda$$

is defined to be

$$\prod^{\#}_{\alpha<\lambda} \kappa.$$

It follows at once that

$$\kappa^\lambda = |\{f \mid f : \lambda \to \kappa\}|.$$

Set theorists sometimes write ${}^\lambda\kappa$ to denote the set

$$\{f \mid f : \lambda \to \kappa\}.$$

In terms of this notation, we have

$$\kappa^\lambda = |{}^\lambda\kappa|.$$

Lemma 3.7.6 Let κ, λ, μ be cardinals. Then:

(i) $\kappa^\lambda \cdot \kappa^\mu = \kappa^{(\lambda+\mu)}$;

(ii) $\kappa^\lambda \cdot \mu^\lambda = (\kappa \cdot \mu)^\lambda$;

(iii) $(\kappa^\lambda)^\mu = \kappa^{(\lambda \cdot \mu)}$.

Proof: An easy exercise. □

Thus, not only do addition and multiplication behave just as in the finite case, so too does exponentiation. Moreover, $\kappa^2 = \kappa \cdot \kappa$, as is easily verified.

Note that all of the above arithmetical notions for arbitrary cardinals reduce to the usual notions in the case where the cardinals are finite. Hence, in the finite case, cardinal and ordinal arithmetic coincide. But in general these arithmetics are quite distinct. For instance, both cardinal addition and cardinal multiplication are commutative, but neither of the ordinal analogues is commutative.

As we have just seen, the arithmetical operations defined on cardinals have all the algebraic properties of their finite counterparts. But this does not mean that the arithmetic of infinite cardinals is directly comparable to finite arithmetic. In fact, infinite cardinal arithmetic is essentially trivial, as our next results show.

Theorem 3.7.7 Let $\kappa \geq \aleph_0$ Then $\kappa \cdot \kappa = \kappa$.

Proof: Suppose not. Let κ be the least infinite cardinal such that $\kappa \cdot \kappa \neq \kappa$. Thus, for all cardinals $\lambda < \kappa$, we have $\lambda \cdot \lambda = \lambda < \kappa$.

Let $P = \kappa \times \kappa$. Thus $|P| = \kappa \cdot \kappa > \kappa$. For each $\xi < \kappa$, let

$$P_\xi = \{(\alpha, \beta) \in P \mid \alpha + \beta = \xi\}.$$

Clearly, $\xi \neq \zeta$ implies $P_\xi \cap P_\zeta = \emptyset$.

Moreover,

$$P = \bigcup_{\xi < \kappa} P_\xi.$$

To see this, suppose first that $(\alpha, \beta) \in P_\xi$, where $\xi < \kappa$. Thus $\alpha + \beta = \xi$, which implies $\alpha, \beta < \kappa$, and hence that $(\alpha, \beta) \in P$. Conversely, let $\alpha, \beta < \kappa$. Thus $|\alpha|, |\beta| < \kappa$. Let $\lambda = \max(|\alpha|, |\beta|)$. By choice of κ, we have $\lambda \cdot \lambda < \kappa$. But

$$|\alpha + \beta| = |\alpha| + |\beta| \leq \lambda + \lambda = 2 \cdot \lambda \leq \lambda \cdot \lambda = \lambda < \kappa.$$

Hence $\alpha + \beta < \kappa$. Thus, setting $\xi = \alpha + \beta$, we have $(\alpha, \beta) \in P_\xi$.

Thus P_ξ, $\xi < \kappa$, constitutes a partition of P.

For each $\xi < \kappa$, define a well-ordering $<_\xi$ of P_ξ by

$$(\alpha, \beta) <_\xi (\alpha', \beta') \quad \leftrightarrow \quad (\alpha < \alpha') \vee (\alpha = \alpha' \wedge \beta < \beta').$$

Then define a well-ordering $<_*$ of P by

$$\begin{aligned}(\alpha, \beta) <_* (\alpha', \beta') \quad \leftrightarrow \quad & [(\alpha, \beta) \in P_\xi \wedge (\alpha', \beta') \in P_\eta \wedge \xi < \eta] \vee \\ & [(\alpha, \beta), (\alpha' e, \beta') \in P_\xi \wedge (\alpha, \beta) <_\xi (\alpha', \beta')].\end{aligned}$$

Let $\theta = \mathrm{Ord}(P <_*)$. Since $|P| > \kappa$, we have $\theta > \kappa$. It follows that there is a point (α_0, β_0) in P such that $\mathrm{Ord}(Q, <_*) = \kappa$, where

$$Q = \{(\alpha, \beta) \in P \mid (\alpha, \beta) <_* (\alpha_0), \beta_0)\}.$$

Pick $\xi_0 < \kappa$ with $(\alpha_0, \beta_0) \in P_{\xi_0}$.Thus $\alpha_0 + \beta_0 = \xi_0$. Then, if $(\alpha, \beta) \in Q$, we have $(\alpha, \beta) <_* (\alpha_0, \beta_0)$, so $\alpha, \beta \leq \xi_0$. Hence

$$Q \subseteq (\xi_0 + 1) \times (\xi_0 + 1).$$

But $\xi_0 + 1 < \kappa$, so $|\xi_0 + 1| < \kappa$, and we have

$$|Q| \leq |\xi_0 + 1| \cdot |\xi_0 + 1| < \kappa,$$

contrary to $\mathrm{Ord}(Q, <_*) = \kappa$. The proof is complete. □

Corollary 3.7.8 Let κ, λ be cardinals, $\kappa \leq \lambda$, $\lambda \geq \aleph_0$. Then $\kappa + \lambda = \kappa \cdot \lambda = \lambda$.

Proof: We have

$$\lambda \leq \kappa + \lambda \leq \lambda + \lambda = 2 \cdot \lambda \leq \lambda \cdot \lambda = \lambda$$

and

$$\lambda \leq \kappa \cdot \lambda \leq \lambda \cdot \lambda = \lambda$$

and the result follows immediately. □

Corollary 3.7.9 Let $\kappa \geq \aleph_0$. Then

$$\kappa^+ = |\{\alpha \mid \kappa \leq \alpha < \kappa^+\}|.$$

In words, the set of all ordinals of cardinality κ has cardinality κ^+.

Proof: We have:

$$\begin{aligned} \kappa^+ &= |\{\alpha \mid \alpha < \kappa^+\}| \\ &= |\{\alpha \mid \alpha < \kappa\} \cup \{\alpha \mid \kappa \leq \alpha < \kappa^+\}| \\ &= |\{\alpha \mid \alpha < \kappa\}| + |\{\alpha \mid \kappa \leq \alpha < \kappa^+\}| \\ &= \kappa + |\{\alpha \mid \kappa \leq \alpha < \kappa^+\}|. \end{aligned}$$

By Corollary 3.7.8, we must have

$$\kappa^+ = |\{\alpha \mid \kappa \leq \alpha < \kappa^+\}|,$$

as required. □

Corollary 3.7.10 Let κ be an infinite cardinal. The union of at most κ sets of cardinality at most κ has cardinality at most κ . In particular, the union of countably many countable sets is countable.

Proof: If $|A_\alpha| \leq \kappa$, for each $\alpha < \lambda$, where $\lambda \leq \kappa$, then

$$|\bigcup_{\alpha<\lambda} A_\alpha| \leq \kappa \cdot \lambda \leq \kappa \cdot \kappa = \kappa,$$

as required. □

Corollary 3.7.11 For any α, β,

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \aleph_{\max(\alpha,\beta)}.$$

Proof: Immediate. □

Corollary 3.7.12 For any α,

$$\aleph_\alpha = \sum_{\beta\leq\alpha} \aleph_\beta.$$

Proof: For each $\beta < \alpha$, let

$$Z_\beta = \{\xi \mid \omega_\beta \leq \xi < \omega_{\beta+1}\}.$$

Then

$$\omega_\alpha = \omega \cup (\bigcup_{\beta<\alpha} Z_\beta).$$

Using Corollary 3.7.9, we get

$$\aleph_\alpha = |\omega_\alpha| = \aleph_0 + \sum_{\beta<\alpha} \aleph_{\beta+1} \geq \aleph_0 + \sum_{\beta<\alpha} \aleph_\beta.$$

So, by Corollary 3.7.8,

$$\aleph_\alpha = (\aleph_0 + \sum_{\beta<\alpha} \aleph_\beta) + \aleph_\alpha = \sum_{\beta\leq\alpha} \aleph_\beta,$$

and the corollary is proved. □

Corollary 3.7.13 If α is a limit ordinal, then

$$\aleph_\alpha = \sum_{\beta<\alpha} \aleph_{\beta+1}.$$

Proof: Arguing as in the proof of Corollary 3.7.12, we get

$$\aleph_\alpha = \aleph_0 + \sum_{\beta<\alpha} \aleph_{\beta+1}.$$

But since α is a limit ordinal,

$$\sum_{\beta<\alpha} \aleph_{\beta+1} = \sum_{\beta<\alpha} \aleph_\beta.$$

Hence, using Corollary 3.7.8,

$$\aleph_\alpha = \sum_{\beta<\alpha} \aleph_\beta,$$

and the proof is complete. □

Cardinal exponentiation turns out to be much more difficult to handle than addition and multiplication, so I shall postpone a discussion of this topic until later (Section 3.9) and end the present investigation of cardinal arithmetic where it stands.

3.8 Regular and Singular Cardinals

A cardinal of the form κ^+ is called a *successor cardinal.* For instance, 1, 2, 3, ... are all successor cardinals; so too are $\aleph_1, \aleph_2$, and $\aleph_3$. Indeed, an infinite cardinal will be a successor cardinal if and only if it is of the form $\aleph_{\alpha+1}$ for some ordinal α; or, to rephrase this slightly, an infinite cardinal $\aleph_\gamma$ is a successor *cardinal* if and only if the index γ is a successor *ordinal.*

A cardinal that is not a successor cardnial is called a *limit cardinal.* Examples of limit cardinals are 0, $\aleph_0$, $\aleph_\omega$, $\aleph_{\omega+\omega}$, $\aleph_{\omega\cdot\omega}$, $\aleph_{\omega_1}$. Indeed, an uncountable cardinal $\aleph_\gamma$ will be a limit *cardinal* if and only if the index γ is a limit *ordinal.*

The properties of limit cardinals, including many of their arithmetical properties, is closely bound up with the notion of *cofinality*, which we examine next.

Let λ be a limit ordinal. A set $A \subseteq \lambda$ is said to be *bounded* in λ if and only if there is a $\gamma < \lambda$ such that $A \subseteq \gamma$; otherwise, we say A is *unbounded* in λ. Thus, A is unbounded in λ if and only if

$$(\forall \alpha \in \lambda)((\exists \beta \in A)(\beta > \alpha).$$

Now let θ be a limit ordinal, and let $\langle \gamma_\nu \mid \nu < \theta \rangle$ be an increasing sequence of ordinals in λ. We say the sequence $\langle \gamma_\nu \mid \nu < \theta \rangle$ is *cofinal* in λ if and only if the set $\{\gamma_\nu \mid \nu < \theta\}$ is unbounded in λ.

Lemma 3.8.1 $\langle \gamma_\nu \mid \nu < \theta \rangle$ is cofinal in λ if and only if $\bigcup_{\nu<\theta} \gamma_\nu = \lambda$.

Proof: Trivial. □

The *cofinality* of λ, denoted by $\mathrm{cf}(\lambda)$, is the least limit ordinal θ such that there is an increasing θ-sequence that is cofinal in λ.

Lemma 3.8.2 $\mathrm{cf}(\lambda)$ is a cardinal.

Proof: An easy exercise. □

Lemma 3.8.3 If $\mathrm{cf}(\lambda) = \theta$, there is an increasing θ-sequence, cofinal in λ, which is continuous at every limit ordinal in θ.

Proof: An easy exercise. □

An infinite cardinal κ is said to be *regular* if and only if $\mathrm{cf}(\kappa) = \kappa$; otherwise, κ is said to be *singular*. Thus κ is singular if and only if $\mathrm{cf}(\kappa) < \kappa$.

For instance, $\aleph_0$ is clearly regular. And consideration of the sequence $\langle \omega_n \mid n < \omega \rangle$ indicates that $\mathrm{cf}(\aleph_\omega) = \omega$, so $\aleph_\omega$ is singular.

Lemma 3.8.4 For any limit ordinal λ, $\mathrm{cf}(\lambda)$ is a regular cardinal.

Proof: An easy exercise. □

The following theorem relates the notion of cofinality to cardinal arithmetic.

Theorem 3.8.5 Let κ be an infinite cardinal. Let θ be the least ordinal such that there is a sequence $\langle \kappa_\nu \mid \nu < \theta \rangle$ of cardinals $\kappa_\nu < \kappa$ with

$$\kappa = \sum_{\nu<\theta} \kappa_\nu.$$

Then $\theta = \mathrm{cf}(\kappa)$.

Proof: Let $\lambda = \mathrm{cf}(\kappa)$. Notice that by Corollary 3.7.11, $\theta \geq \omega$, and by Lemma 3.7.2, θ is a cardinal.

Suppose first that $\theta < \lambda$. Thus, for some $\gamma < \kappa$,

$$\{\kappa_\nu \mid \nu < \theta\} \subseteq \gamma.$$

Then, $\kappa_\nu \leq |\gamma|$, for all $\nu < \theta$, so

$$\kappa = \sum_{\nu<\theta} \kappa_\nu \leq \sum_{\nu<\theta} |\gamma| = \theta \cdot |\gamma| = \max(\theta, \gamma|) < \kappa,$$

which is absurd.

Now suppose that $\lambda < \theta$. By Lemma 3.8.3, let $\langle \gamma_\nu \mid \nu < \lambda \rangle$ be a normal sequence, cofinal in κ. We may assume that $\gamma_0 = 0$. Then, by Lemma 3.8.1,

$$\kappa = \bigcup_{\nu<\lambda} \gamma_\nu = \bigcup_{\nu<\lambda} (\gamma_{\nu+1} - \gamma_\nu).$$

Setting

$$\mu_\nu = |\gamma_{\nu+1} - \gamma_\nu|,$$

we get

$$\kappa = |\kappa| = \sum_{\nu<\lambda} \mu_\nu,$$

contrary to $\lambda < \theta$. □

Using Theorem 3.8.5, we can now show that there are many regular cardinals.

Theorem 3.8.6 Every infinite successor cardinal is regular.

Proof: Let κ be any infinite cardinal. We show that κ^+ is regular. Let $\lambda = \mathrm{cf}(\kappa^+)$. By Theorem 3.8.5, we can find cardinals $\kappa_\nu < \kappa^+$ for all $\nu < \lambda$, such that

$$\kappa^+ = \sum_{\nu<\lambda} \kappa_\nu.$$

For each $\nu < \lambda$, $\kappa_\nu \leq \kappa$, so

$$\kappa^+ \leq \sum_{\nu<\lambda} \kappa = \lambda \cdot \kappa.$$

Hence by Theorem 3.7.5, $\lambda = \kappa^+$. □

Corollary 3.8.7 Every singular cardinal is a limit cardinal.

Proof: Immediate. □

In Section 3.10, I discuss the converse to Corollary 3.8.7. Meanwhile, let us consider a few examples. By Theorem 3.8.6, $\aleph_1$, $\aleph_2$, $\aleph_3$, ... are all regular. $\aleph_\omega$ is singular, of cofinality ω. $\aleph_{\omega+1}$, $\aleph_{\omega+2}$, ... are all regular. $\aleph_{\omega+\omega}$ is singular of cofinality ω. $\aleph_{\omega_1}$ is singular of cofinality ω_1; $\aleph_{\omega_2}$ is singular of cofinality ω_2, etc. As a general lemma, we have

Lemma 3.8.8 If α is a limit ordinal, then

$$\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\alpha).$$

Proof: Trivial. □

In Section 3.9, we shall meet cases where the cardinal arithmetic is affected by the cofinality of the cardinals concerned. As a first example of cofinality properties, however, I consider Theorem 3.4.7, the fixed-point theorem for normal functions. Previously we were only able to prove this result for 'class functions' $f : \mathrm{On} \to \mathrm{On}$. We may now state and prove a genuine set-theoretic version of the theorem.

Theorem 3.8.9 [Fixed-Point Theorem] Let λ be a limit ordinal such that $\mathrm{cf}(\lambda) > \omega$. If $f : \lambda \to \lambda$ is a normal function, then for every $\alpha \in \lambda$ there is a fixed-point γ of f such that $\gamma \geq \alpha$.

Proof: Let $\alpha \in \lambda$ be given. If $f(\alpha) = \alpha$, there is nothing further to prove. So assume otherwise. Then by Lemma 3.4.6, $f(\alpha) > \alpha$. By recursion, define a function $g : \omega \to \lambda$ so that $g(0) = 0$ and $g(n+l) = f(g(n))$. By induction, $g(n) < g(n+l)$ for all n and g maps into λ. Let

$$\gamma = \lim_{n<\omega}(g(n)).$$

Since $\mathrm{cf}(\lambda) > \omega$, g cannot be cofinal in λ, so $\gamma < \lambda$. But clearly, $f(\gamma) = \gamma$. The proof is complete. □

3.9 Cardinal Exponentiation

I consider now the function κ^λ. First a useful characterization of 2^κ.

Lemma 3.9.1 For any cardinal κ,

$$2^\kappa = |\mathcal{P}(\kappa)|.$$

Proof: By definition,

$$2^\kappa = |{}^\kappa 2| = |\{f \mid f : \kappa \to 2\}|.$$

But there is a well-known one-one correspondence between the sets $\{f \mid f : \kappa \to 2\}$ and $\mathcal{P}(\kappa)$, where we associate with each set $X \subseteq \kappa$ its characteristic function $\chi_X : \kappa \to 2$, defined by

$$\chi_X(\xi) = 1 \leftrightarrow \xi \in X.$$

The lemma follows at once. □

Corollary 3.9.2 For any cardinal κ,

$$2^\kappa > \kappa.$$

Proof: The proof of Lemma 3.6.6 shows that $|\mathcal{P}(\kappa)| > \kappa$. Now apply the above theorem. □

Theorem 3.9.3 Let κ, λ be cardinals, λ infinite, $\kappa \leq \lambda$. Then

$$\kappa^\lambda = 2^\lambda.$$

Proof: Clearly, $2^\lambda \leq \kappa^\lambda$. I show that $\kappa^\lambda \leq 2^\lambda$. Since λ is infinite and $\kappa \leq \lambda$, $\kappa \cdot \lambda = \lambda$. Let $j : \lambda \times \kappa \leftrightarrow \lambda$. For each function $h : \lambda \to \kappa$, we have, formally, $h \subseteq \lambda \times \kappa$, so we can define $G(h) = j[h]$. Thus $G(h) \subseteq \lambda$. Clearly, $G :^\lambda \kappa \to \mathcal{P}(\lambda)$ is an injection. Hence, using Theorem 3.9.1,

$$\kappa^\lambda = |^\lambda \kappa| \leq |\mathcal{P}(\lambda)| = 2^\lambda,$$

and the proof is complete. □

By the above result, if λ is infinite, the behaviour of κ^λ as κ varies up to λ is known. For $\kappa > \lambda$, the picture is more complex. We have, for example:

Theorem 3.9.4 Let κ be an infinite cardinal. Then

$$(\kappa^+)^\kappa = 2^\kappa.$$

Proof: Clearly,

$$\kappa^+ \cdot 2^\kappa \leq (\kappa^+)^\kappa \cdot 2^\kappa = (\kappa^+)^\kappa.$$

By Theorem 3.8.6, κ^+ is regular, so

$$^\kappa(\kappa^+) \subseteq \bigcup_{\alpha<\kappa^+}^\kappa \alpha,$$

which gives

$$\begin{aligned}(\kappa^+)^\kappa &\leq \textstyle\sum_{\alpha<\kappa^+} |\kappa_\alpha| \\ &= \textstyle\sum_{\alpha<\kappa^+} |\alpha|^\kappa \\ &= \kappa^+ \cdot \kappa^\kappa \\ &= \kappa^+ \cdot 2^\kappa = 2^\kappa.\end{aligned}$$

This completes the proof. □

In general there is little more that can be said. For instance, although we know that there is an ordinal α such that $2^{\aleph_0} = \aleph_\alpha$, the ZFC axioms do not provide us with enough information to decide which α 'solves' this equation. The 'obvious' guess is, perhaps, that $2^{\aleph_0} = \aleph_1$. This was already proposed by Cantor at the beginning of this century. By considering the representations of the real numbers in the unit interval $(0, 1)$ as non-terminating decimal expansions, one sees easily that the cardinality of this interval is $10^{\aleph_0} = 2^{\aleph_0}$. Since $(0, 1)$ is known to be homeomorphic to the whole real line, $\mathbb{R}$, it follows that $|\mathbb{R}| = 2^{\aleph_0}$. Hence the question as to which α solves $2^{\aleph_0} = \aleph_\alpha$ can be expressed thus: How many real numbers are there? Expressed in this manner, the question became known as the *continuum problem.*

Cantor's hypothesis $2^{\aleph_0} = \aleph_1$ (i.e. $|\mathbb{R}| = \aleph_1$) became known as the *continuum hypothesis*, or CH for short. Despite the relative ease with which CH can be stated, efforts over the years to resolve the continuum problem met with no success. In the 1930's, Kurt Gödel used techniques of logic to show that CH could certainly not be disproved (on the basis of the ZFC axioms), but a moment's thought will indicate that this does not in itself *prove* the CH. And, in fact, it cannot be proved on the basis of the ZFC axioms, as Paul Cohen demonstrated in 1963. The combined effect of the results of Gödel and Cohen is to show that the ZFC axioms simply do not resolve the continuum problem one way or the other and, indeed, do not resolve a great many of the questions one can raise about cardinal exponentiation. In Chapter 5, I provide some explanation as to why CH is not decidable in the ZFC system. And in Chapter 6, I outline the methods by which it can be *proved* that a particular statement, such as CH, is not decidable in the ZFC system. In the meantime, in order not to leave too many loose ends, I present the one and only positive result about the size of $2^{\aleph_0}$ that we have (and indeed can ever have). We can prove that $2^{\aleph_0}$ cannot equal $\aleph_\omega$, or $\aleph_{\omega+\omega}$, or indeed any cardinal of cofinality ω. In order to prove this, I first establish a very general result on cardinal arithmetic.

Theorem 3.9.5 Let β be any ordinal, and for each ordinal $\alpha < \beta$, let $\kappa_\alpha, \lambda_\alpha$ be cardinals such that $\kappa_\alpha < \lambda_\alpha$. Then

$$\sum_{\alpha<\beta}\kappa_\alpha < \prod^{\#}_{\alpha<\beta}\lambda_\alpha.$$

Proof: Define

$$f : \bigcup_{\alpha<\beta}(\kappa_\alpha \times \{\alpha\}) \to \prod_{\alpha<\beta}\lambda_\alpha$$

by taking $f(\xi,\gamma)$ to be that element of $\prod_{\alpha<\beta}\lambda_\alpha$ that takes the value $\xi \in \lambda_\gamma$ in the γ'th place, and the value 0 elsewhere. That is ,

$$f(\xi,\gamma)(\nu) = \begin{cases} \xi & \text{, if } \nu = \gamma, \\ 0 & \text{, otherwise.} \end{cases}$$

Clearly, f is an injection. Hence

$$\sum_{\alpha<\beta}\kappa_\alpha = |\bigcup_{\alpha<\beta}(\kappa_\alpha \times \{\alpha\})| \leq |\prod_{\alpha<\beta}\lambda_\alpha| = \prod^{\#}_{\alpha<\beta}\lambda_\alpha.$$

Suppose that $\sum_{\alpha<\beta}\kappa_\alpha = \prod^{\#}_{\alpha<\beta}\lambda_\alpha$. Let

$$f : \bigcup_{\alpha<\beta}(\kappa_\alpha \times \{\alpha\}) \xrightarrow{\text{onto}} \prod_{\alpha<\beta}\lambda_\alpha.$$

For $\alpha < \beta$, let f_α be the projection of f onto λ_α; that is,

$$f_\alpha(\xi,\gamma) = f(\xi,\gamma)(\alpha).$$

Then

$$f_\alpha \quad (\kappa_\alpha \times \{\alpha\}) : \kappa_\alpha \times \{\alpha\} \to \lambda_\alpha.$$

Since $\kappa_\alpha < \lambda_\alpha$ and $|\kappa_\alpha \times \{\alpha\}| = \kappa_\alpha$, the function $f_\alpha \quad (\kappa_\alpha \times \{\alpha\})$ cannot be surjective. Hence we can pick $\delta_\alpha \in \lambda_\alpha - f_\alpha[\kappa_\alpha \times \{\alpha\}]$. Let

$$\sigma = \langle \delta_\alpha \mid \alpha < \beta \rangle.$$

Then $\sigma \in \prod_{\alpha<\beta}\lambda_\alpha$, so for some ξ, α we must have $\sigma = f(\xi,\alpha)$. Thus, in particular,

$$\delta_\alpha = f(\xi,\alpha)(\alpha) = f_\alpha(\xi,\alpha) \in f_\alpha[\kappa_\alpha \times \{\alpha\}],$$

which is absurd. This contradiction proves the theorem. □

Corollary 3.9.6 For any infinite cardinal κ,

$$\kappa^{\mathrm{cf}(\kappa)} > \kappa.$$

Proof: Let $\lambda = \mathrm{cf}(\kappa)$. By Theorem 3.8.5, we can find cardinals $\kappa_\alpha < \kappa$, for all $\alpha < \lambda$, such that $\kappa = \sum_{\alpha<\beta}\kappa_\alpha$. Since $\kappa_\alpha < \kappa$ for all $\alpha < \beta$, Theorem 3.9.5 gives

$$\kappa = \sum_{\alpha<\beta}\kappa_\alpha < \prod^{\#}_{\alpha<\beta}\kappa = \kappa^\lambda,$$

as required. □

As a consequence of Corollary 3.9.6, we have our promised result on $2^{\aleph_0}$:

Theorem 3.9.7 For any infinite cardinal κ,

$$\mathrm{cf}(2^\kappa) > \kappa.$$

Hence, in particular,

$$\mathrm{cf}(2^{\aleph_0}) > \omega,$$

giving $2^{\aleph_0} \neq \aleph_\omega$, etc.

Proof: Suppose $\mathrm{cf}(2^\kappa) \leq \kappa$. Then, setting $\lambda = 2^\kappa$, we get

$$\lambda^{\mathrm{cf}(\lambda)} = \lambda^{\mathrm{cf}(2^\kappa)} \leq \lambda^\kappa = (2^\kappa)^\kappa = 2^{\kappa \cdot \kappa} = 2^\kappa = \lambda,$$

contrary to Corollary 3.9.6 (for λ). □

3.10 Inaccessible Cardinals

In Corollary 3.8.7, I proved that every singular cardinal is a limit cardinal. What about the converse? Is every limit cardinal singular? Well, $\aleph_0$ is a regular limit cardinal. But are there any others? If you attempt to find any, you will (for certain) fail. Any uncountable limit cardinal that one can 'construct' in the ZFC system is singular. Or, to put it another way, in ZFC, it is not possible to prove that there is an uncountable regular limit cardinal. On the other hand, most set theorists believe that it is extremely unlikely that one could prove (in ZFC) that no such cardinal can exist. Indeed, though the existence of an uncountable regular limit cardinal cannot be proved in ZFC, it is arguable that the existence of such cardinals is implicit in the motivation that leads to the ZFC axioms. Accordingly we give such cardinals a name—*weakly inaccessible cardinals*—and study them. There are at least three reasons for doing this.

First, suppose we are trying to prove some mathematical result by an induction on cardinals. (There are many instances of such proofs, in different areas of mathematics.) The induction step at a singular limit cardinal often makes use of properties peculiar to singular cardinals. For the induction step at regular limIt cardinals—i.e. weakly inaccessible cardinals—a separate argument must be used, making use of properties of weakly inaccessible cardinals. (For this purpose, a study of such cardinals is necessary, leaving aside any questions as to whether weakly inaccessible cardinals 'exist'.)

The second reason for looking at weakly inaccessible cardinals will appeal perhaps only to the set theorist. The existence of such cardinals cannot be proved, but it is (arguably) inherent in the basic ideas of set theory (see later), and hence the notion of an inaccessible cardinal is worthy of study as an aspect of pure set theory. Of course, to do so one needs to adjoin to the

ZFC system an axiom which asserts the existence of a weakly inaccessible cardinal. This is directly analogous to the inclusion of the Axiom of Infinity in the ZFC system. Without the axiom of infinity, one cannot prove (in ZFC) the existence of an infinite set. Because we want infinite sets, we introduce an axiom which gives us one. Weakly inaccessible cardinals are just uncountable analogues of $\aleph_0$ (recall that $\aleph_0$ is a regular limit cardinal).

This brings me to the argument that weakly inaccessible cardinals are in fact inherent in our intuition on set theory. The cardinal $\aleph_0$, being both regular and a limit cardinal, is very much larger than any of its predecessors. Neither the replacement axiom nor the cardinal successor function gets up to $\aleph_0$ from below. But our set-theoretic inverse should surely possess a uniform character! The cardinal $\aleph_0$ should not be so unusual; there ought to be a proper class of such cardinals. In fact, we can take this a step further.

Let us call an uncountable cardinal κ *strongly inaccessible* (or just *inaccessible*) if it is regular and

$$(\forall \lambda < \kappa)(2^\lambda < \kappa).$$

Clearly, every strongly inaccessible cardinal is weakly inaccessible. (We discuss the converse later.) Also, apart from not being uncountable, $\aleph_0$ is strongly inaccessible. An inaccessible cardinal is one that cannot be constructed using any of the ZFC axioms. In particular, its definition precludes construction using the axioms of replacement and power set, the two powerful construction axioms that have provided all of our cardinal existence results hitherto.

Since $\aleph_0$ cannot be constructed using the ZFC axioms without the Axiom of Infinity, uniformity could be said to demand the existence of a proper class of such cardinals. In fact, one rarely makes any fuss about this, because the adjunction of 'many' inaccessible cardinal axioms to the ZFC system does not seem to increase the richness of the set theory very much. At most one usually just studies one or two inaccessible cardinals.

And so to our third reason. As I mentioned, inaccessible cardinals and weakly inaccessible cardinals resemble $\aleph_0$ to some extent. And we all feel that there is some fundamental difference between finite and infinite that is not shared by any other division, such as between countable and uncountable. Finiteness is a very special property. By studying inaccessibility properties, one can hope to gain some insight into how 'finiteness' behaves, without getting at all involved in the finite itself.

But now it is high time to get down to the business of looking at inaccessible cardinals. I commence by restating the definitions.

A cardinal κ is *weakly inaccessible* if and only if it is an uncountable, regular limit cardinal.

A cardinal κ is (*strongly*) *inaccessible* if and only if it is an uncountable regular cardinal and

$$(\forall \lambda < \kappa)(2^{\lambda} < \kappa).$$

Clearly, every inaccessible cardinal is weakly inaccessible. The converse is not in general true. One circumstance in which the two notions of inaccessibility do coincide is under the assumption of the following generalization of Cantor's Continuum Hypothesis.

The *Generalized Continuum Hypothesis* (GCH) is the assertion

$$(\forall \kappa \geq \aleph_0)[2^{\kappa} = \kappa^{+}].$$

This may also be written as

$$\forall \alpha[2^{\aleph_\alpha} = \aleph_{\alpha+1}].$$

The GCH is known to be consistent with the ZFC axioms.

Clearly, if we assume GCH as an additional axiom of set theory, then the notions of weak inaccessibility and inaccessibility coincide. But it is also consistent with the ZFC axioms that these two notions are quite different. Indeed, it is not possible to prove in ZFC that $2^{\aleph_0}$ is not weakly inaccessible, though it is trivially not inaccessible.

Lemma 3.10.1 If $\aleph_\kappa$ is a weakly inaccessible cardinal, then (κ is a cardinal and) $\aleph_\kappa = \kappa$.

Proof: Suppose $\kappa \neq \aleph_\kappa$. Thus $\kappa < \aleph_\kappa$. Since κ is a limit ordinal, $\mathrm{cf}(\aleph_\kappa) = \mathrm{cf}(\kappa) < \aleph_\kappa$, contrary to $\aleph_\kappa$ being regular. □

Our next result gives an application of inaccessibility to cardinal arithmetic.

Theorem 3.10.2 If κ is an inaccessible cardinal, then

$$\sum_{\lambda<\kappa} \kappa^{\lambda} = \kappa.$$

Proof: Assume κ is inaccessible. Since κ is regular, if $\lambda < \kappa$ then

$$^{\lambda}\kappa = \bigcup_{\alpha<\kappa}^{\lambda} \alpha.$$

Hence, for $\lambda < \kappa$,

$$\kappa^{\lambda} \leq \sum_{\alpha<\kappa} |\alpha|^{\lambda}.$$

But if $\lambda, \alpha < \kappa$, then since κ is inaccessible, $|\alpha|^{\lambda} < \kappa$. Hence, for $\lambda < \kappa$,

$$\kappa^{\lambda} \leq \sum_{\alpha<\kappa} \kappa = \kappa \cdot \kappa = \kappa.$$

Thus,

$$\sum_{\lambda<\kappa}\kappa^{\lambda} \leq \sum_{\lambda<\kappa}\kappa = \kappa \cdot \kappa = \kappa.$$

The theorem is proved. □

Assuming GCH (as an additional axiom of set theory), we can extend Theorem 3.10.2 as follows.

Theorem 3.10.3 Assume GCH. Let κ be a limit cardinal. Then κ is inaccessible if and only if

$$\sum_{\lambda<\kappa}\kappa^{\lambda} = \kappa.$$

Proof: Theorem 3.10.2 gives one half of the result. So let us suppose κ is not inaccessible. By GCH, κ is not weakly inaccessible. So, being a limit cardinal, κ must be singular. Thus $\mathrm{cf}(\kappa) < \kappa$. But by Corollary 3.9.6, $\kappa^{\mathrm{cf}(\kappa)} > \kappa$. Hence $\sum_{\lambda<\kappa}\kappa^{\lambda} > \kappa$, which completes the proof. □

The most significant fact concerning inaccessible cardinals is the following, which I do not prove here.[1]

If κ is inaccessible, then the level V_κ in the cumulative hierarchy is a 'fixed point' or 'closure point' with respect to the ZFC axioms. That is, if we use the ZFC axioms in any manner to define new sets from sets in V_κ, then these new sets will themselves lie in V_κ; the axioms of ZFC will not lead out of V_κ. Consequently, V_κ is a 'model' of the ZFC axioms.

3.11 Problems

1. (Ordinal Arithmetic) The operations of ordinal addition and multiplication may be defined by recursion.

A. For each ordinal α, define a function $a_\alpha : \mathrm{On} \to \mathrm{On}$ by the recursion

$$\begin{aligned} a_\alpha(0) &= \alpha, \\ a_\alpha(\beta+1) &= a_\alpha(\beta)+1, \\ a_\alpha(\gamma) &= \bigcup_{\beta<\gamma} a_\alpha(\beta), \quad \text{if } \lim(\gamma). \end{aligned}$$

Prove that $a_\alpha(\beta) = \alpha + \beta$.

[1]This brief discussion is not very precise, and will not stand up to much detailed consideration. To make it precise would, however, lead us far from our goal, so we shall have to content ourselves with an evocative, if not totally rigorous, remark.

B. For each ordinal α, define a function $m_\alpha : \mathrm{On} \to \mathrm{On}$ by the recursion

$$\begin{aligned} m_\alpha(0) &= 0 \\ m_\alpha(\beta+1) &= m_\alpha(\beta) + \alpha \\ m_\alpha(\gamma) &= \bigcup_{\beta<\gamma} m_\alpha(\beta), \quad \text{if } \lim(\gamma). \end{aligned}$$

Prove that $m_\alpha(\beta) = \alpha \cdot \beta$.

C. Prove an ordinal recursion principle which allows for parameters, and use it to modify the above definitions so as to obtain recursive definitions of $+$ and $\cdot$ as functions from $\mathrm{On} \times \mathrm{On}$ to On.

2. (Cardinal Arithmetic)

A. Let β be an ordinal, and let κ_α, $\alpha < \beta$, be infinite cardinals with $\kappa = \sum_{\alpha<\beta} \kappa_\alpha$. Prove that, for any cardinal λ,

$$\lambda^\kappa = \prod^{\#}_{\alpha<\beta} \lambda^{\kappa_\alpha}.$$

B. Show that if κ is regular and $2^\mu \leq \kappa$ for all $\mu < \kappa$, then

$$\sum_{\mu<\kappa} \kappa^\mu = \kappa.$$

C. Assume GCH. Prove that for any infinite cardinal κ, $\kappa^\lambda = \kappa$ for all $\lambda < \mathrm{cf}(\kappa)$. Deduce that if κ is regular, then

$$\sum_{\mu<\kappa} \kappa^\mu = \kappa.$$

D. Prove that for any infinite cardinal κ,

$$\kappa^\kappa \geq \sum_{\lambda<\kappa} \kappa^\lambda \geq \sum_{\lambda<\kappa} 2^\lambda \geq \kappa.$$

E. Show that if $\kappa = \lambda^+$, then

$$\sum_{\mu<\kappa} \kappa^\mu = \sum_{\mu<\kappa} 2^\mu = 2^\lambda.$$

F. Prove that GCH is equivalent to the identity $\sum_{\mu<\kappa} 2^\mu = \kappa$ for all infinite κ.

G. Prove that GCH is equivalent to the identity $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$ for all infinite κ.

H. Show that for any infinite cardinal κ, $\sum_{\lambda<\kappa}\kappa^{\lambda} = \kappa$ if and only if κ is regular and $\sum_{\lambda<\kappa}2^{\lambda} = \kappa$.

3. (The Order Topology on ω_1) The order topology was defined in Problems 1.3. Here we consider the particular spaces given by the order topology on ω_1 and $\omega_1 + 1$.

A. Show that ω_1 is first countable but not second countable. Show further that $\omega_1 + 1$ is not even first countable.

B. Let $A \subseteq \omega_1$. Prove that A is closed if and only if, whenever γ is a limit ordinal in ω_1 such that $A \cap \gamma$ is unbounded in γ, then $\gamma \in A$. Similarly with $\omega_1 + 1$ in place of ω_1.

C. Let $\alpha \in \omega_1$. Show that $\{\alpha\}$ is open (i.e. the point α is isolated) if and only if α is a successor ordinal.

D. Prove that both ω_1 and $\omega_1 + 1$ are Hausdorff spaces.

E. Prove that ω_1 is locally compact but not compact. Show also that $\omega_1 + 1$ is not locally compact.

F. Prove that both ω_1 and $\omega_1 + 1$ are Lindelöf.

G. Prove thast both ω_1 and $\omega_1 + 1$ are normal.

H. Show that the product space $\omega_1 \times (\omega_1 + 1)$ is not normal.

4

Topics in Pure Set Theory

In this chapter, we take a look at a number of topics in pure set theory. Some of the proofs are fairly complex, and at first reading, can be glossed over, or even ignored, without affecting the ability to follow the remainder of the book. The various sections in the chapter are all largely independent of each other.

4.1 The Borel Hierarchy

Recall from Problems 1.1 that a *field of sets* is a collection of subsets of a set, X, that is closed under pairwise union, pairwise intersection, and complementation with respect to X. A field of set $\mathcal{F}$ is called a *σ-field* if, whenever $A_n \in \mathcal{F}$, $n = 0, 1, 2, \ldots$, then $\bigcup_{n<\omega} A_n \in \mathcal{F}$ and $\bigcap_{n<\omega} A_n \in \mathcal{F}$.

It is easily seen that the intersection of any family of σ-fields of subsets of a set X is again a σ-field of subsets of X. It follows that for any collection $\mathcal{H}$ of subsets of X, there is a unique smallest σ-field of subsets of X that contains all the elements of $\mathcal{H}$. We call that σ-field the σ-field of subsets of X *generated by* $\mathcal{H}$.

The σ-field, $\mathcal{B}$, of subsets of $\mathbb{R}$ generated by the collection of all open sets of reals is called the *Borel algebra*. A member of $\mathcal{B}$ is said to be a *Borel set*.

We show that there are $2^{\aleph_0}$ Borel sets and that they lie in a natural, ramified hierarchy.

Let $\mathcal{B}_0$ be the set of all open intervals (a, b), where $a, b \in \mathbb{R}$. By recursion, we define a hierarchy $\langle \mathcal{B}_\alpha \mid \alpha < \omega_1 \rangle$ as follows.

Let $\mathcal{B}_{\alpha+1}$ be the set of all subsets of $\mathbb{R}$ that are either a countable union of members of $\mathcal{B}_\alpha$ or a countable intersection of members of $\mathcal{B}_\alpha$ or the difference of two members of $\mathcal{B}_\alpha$. If λ is a countable limit ordinal, set $\mathcal{B}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{B}_\alpha$.

Clearly, $\nu < \tau \rightarrow \mathcal{B}_\nu \subseteq \mathcal{B}_\tau$.

Theorem 4.1.1 $\mathcal{B} = \bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha$.

Proof: Let $U = \bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha$. We must prove that $\mathcal{B} = U$.

By induction on α, we first of all prove that $\mathcal{B}_\alpha \subseteq \mathcal{B}$ for all $\alpha < \omega_1$. For $\alpha = 0$ this is clear. Also, the induction step at limit ordinals is trivial. Now suppose $\mathcal{B}_\alpha \subseteq \mathcal{B}$. If $X \in \mathcal{B}_{\alpha+1}$, then X is either a countable union of members of $\mathcal{B}_\alpha$ (and hence of $\mathcal{B}$), or a countable intersection of members of $\mathcal{B}_\alpha$ (and hence of $\mathcal{B}$), or the difference of two members of $\mathcal{B}_\alpha$ (and hence of $\mathcal{B}$. But $\mathcal{B}$ is a σ-field and, hence, is closed under such operations. Thus $X \in \mathcal{B}$. This completes the induction. Hence $U \subseteq \mathcal{B}$.

We now prove that $\mathcal{B} \subseteq U$. It suffices to show that U is a σ-field of subsets of $\mathbb{R}$ which contains all open sets (since $\mathcal{B}$ is the smallest such σ-field). Well, since every open set can be expressed as a countable union of open intervals, $\mathcal{B}_1$ (and hence U) contains all open sets. We show that U is a σ-field.

Suppose $X, Y \in U$. Then for some $\alpha, \beta < \omega_!$, $X \in \mathcal{B}_\alpha$, $Y \in \mathcal{B}_\beta$. Let $\gamma = \max(\alpha, \beta)$. Then $X, Y \in \mathcal{B}_\gamma$, so $X - Y \in \mathcal{B}_{\gamma+1} \subseteq U$.

Now suppose $X_n \in U$ for all $n < \omega$. For each n, pick $\alpha_n < \omega_1$ so that $X_n \in \mathcal{B}_{\alpha_n}$. Since $\mathrm{cf}(\omega_1) > \omega$, there is a $\gamma < \omega_1$ such that $\alpha_n < \gamma$ for all $n < \omega$. Then $\{X_n \mid n < \omega\} \subseteq B_\gamma$. Hence

$$\bigcup_{n<\omega} X_n,\ \bigcap_{n<\omega} X_n \in \mathcal{B}_{\gamma+1} \subseteq U.$$

The proof is complete. □

We call the sequence $\langle \mathcal{B}_\alpha \mid \alpha < \omega_1 \rangle$ the *Borel hierarchy.*

The *rank* of a Borel set, X, is defined to be the least ordinal γ such that $X \in \mathcal{B}_{\gamma+1}$. The rank of a Borel set can be thought of as a measure of its 'complexity' as a Borel set. It tells us how many steps are required in order to construct the set starting from open intervals and using the operations of countable union, countable intersection, and set-difference.

We shall make use of the Borel hierarchy in order to establish the cardinality of the Borel algebra.

Theorem 4.1.2 The set $\mathcal{B}$ of Borel sets has cardinality $2^{\aleph_0}$.

Proof: Since there are $2^{\aleph_0}$ open intervals, we certainly have $|\mathcal{B}| > 2^{\aleph_0}$. To establish the opposite inequality, it is enough to show that $|\mathcal{B}_\alpha| < 2^{\aleph_0}$ for all $\alpha < \omega_1$, since then we would have

$$|\mathcal{B}| = |\bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha| \leq \sum_{\alpha<\omega_1} |\mathcal{B}_\alpha| \leq \aleph_1 . 2^{\aleph_0} = 2^{\aleph_0}.$$

The proof is by induction. For $\alpha = 0$, we clearly have $\mathcal{B}_0 = 2^{\aleph_0}$. Now suppose $\alpha < \omega_1$ is a limit ordinal and that $|calb_\beta| \leq 2^{\aleph_0}$ for all $\beta < \alpha$.

Then

$$|\mathcal{B}_\alpha| = |\bigcup_{\beta<\alpha}\mathcal{B}_\beta| \leq \sum_{\beta<\alpha}|\mathcal{B}_\beta| \leq |\alpha|.2^{\aleph_0} = \aleph_0.2^{\aleph_0} = 2^{\aleph_0}.$$

Finally suppose that $\alpha < \omega_1$ and that $\mathcal{B}_\alpha \leq 2^{\aleph_0}$. Now, expressing the definition of $\mathcal{B}_{\alpha+1}$ in a formal manner, we have

$$\begin{aligned}\mathcal{B}_{\alpha+1} &= \{X - Y \mid (X,Y) \in \mathcal{B} \times \mathcal{B}\} \cup \\ &\quad \{\bigcup f[\omega] \mid f : \omega \to \mathcal{B}_\alpha\} \cup \\ &\quad \{\bigcap f[\omega] \mid f : \omega \to \mathcal{B}_\alpha\}.\end{aligned}$$

Thus

$$\begin{aligned}|\mathcal{B}_{\alpha+1}| &\leq |\mathcal{B}_\alpha \times \mathcal{B}_\alpha| + |{}^\omega\mathcal{B}_\alpha| + |{}^\omega\mathcal{B}_\alpha| \\ &= |\mathcal{B}_\alpha|.|\mathcal{B}_\alpha| + |\mathcal{B}_\alpha|^{\aleph_0} + |\mathcal{B}_\alpha|^{\aleph_0} \\ &\leq 2^{\aleph_0}.2^{\aleph_0} + (2^{\aleph_0})^{\aleph_0} + (2^{\aleph_0})^{\aleph_0} \\ &= 2^{\aleph_0}\end{aligned}$$

as required. □

4.2 Closed Unbounded Sets

Let λ be a limit ordinal. A set $C \subseteq \lambda$ is said to be *closed* in λ if and only if $\bigcup(C \cap \alpha) \in C$ for every limit ordinal $\alpha < \lambda$. Equivalently, C is closed in λ if and only if, whenever s is an increasing sequence of ordinals in C, which is bounded in C, and whose domain is a limit ordinal, then $\lim(s) \in C$. A further characterization, using the order topology (see Problems 1.3 and 3.3) is that C is closed in the above sense if and only if it is (topologically) closed in the order topology on λ.

The following lemma is immediate, regardless of the definition of 'closed' that is assumed:

Lemma 4.2.1 If A, B are closed subsets of λ, so too is $A \cap B$.

A subset C of λ, which is at the same time closed and unbounded in λ, is said to be *club* in λ. Now, if $\mathrm{cf}(\lambda) = \omega$, any ω-sequence cofinal in λ determines a club set, namely, its range. But if $\mathrm{cf}(\lambda) > \omega$, any club set in λ is 'large', in a sense made precise by the following result.

Lemma 4.2.2 Suppose $\mathrm{cf}(\lambda) > \omega$. If A, B are club in λ, so too is $A \cap B$.

Proof: By virtue of Lemma 4.1.2, we need only prove that $A \cap B$ is unbounded in λ. Let $\alpha \in \lambda$ be given. We seek a $\gamma \in A \cap B$, $\gamma > \alpha$.

Choose $\alpha_0 \in A$, $\alpha_0 > \alpha$. Since A is unbounded in λ, this is always possible. Now choose $\alpha_1 \in B$, $\alpha_1 > \alpha_0$. Since B is unbounded, this is always possible, By recursion now, define a sequence $\langle \alpha_n \mid n < \omega \rangle$ so that $\alpha_{n+1} > \alpha_n$ and $\alpha_{2n} \in A$, $\alpha_{2n+1} \in B$. Let $\gamma = \lim_{n<\omega} \alpha_n$. Since $\mathrm{cf}(\lambda) > \omega$, $\gamma \in \lambda$. Clearly, $\gamma = \lim_{n<\omega} \alpha_{2n}$, so as $\alpha_{2n} \in A$ for all n and A is closed, $\gamma \in A$. Similarly, $\gamma = \lim_{n<\omega} \alpha_{2n+1}$, so $\gamma \in B$. Hence $\gamma \in A \cap B$, and we are done. □

The next result relates club sets to the normal function, at least in the case of ω_1.

Theorem 4.2.3 *A set $C \subseteq \omega_1$ is club in ω_1 if and only if it is the range of a normal function $f : \omega_1 \to \omega_1$.*

Proof: Let $C \subseteq \omega_1$ be club in ω_1. Define a function $f : \omega_1 \to \omega_1$ by the recursion:

$$f(0) = \text{the smallest member of } C,$$

$$f(\alpha) = \text{the smallest member of } C - f[\alpha].$$

Since C is unbounded in ω_1 and ω_1 is regular, $|C| = \aleph_1$. Hence f is well-defined. And clearly, f is order-preserving. Let $\alpha < \omega_1$ be a limit ordinal. Since $f[\alpha] \subseteq C$ and C is closed in ω_1, $\bigcup f[\alpha] \in C$. Thus by the definition of f, $f(\alpha) = \bigcup f[\alpha]$. So by Theorem 3.4.1, $f(\alpha) = \lim_{\beta<\alpha} f(\beta)$. Hence f is a normal function. Since we clearly have $C = \mathrm{ran}(f)$, this proves one half of the result.

Conversely, if $f : \omega_1 \to \omega_1$ is a normal function, then $C = \mathrm{ran}(f)$ is clearly a club in ω_1. The proof is complete. □

In fact, the above theorem generalizes immediately from ω_1 to any uncountable, regular cardinal. The same is true for all the results we present in this section, but for definiteness we shall concentrate only on ω_1 from now on. (In addition to providing a specific example, the proofs tend to be marginally simpler for the case ω_1, though in all cases the general proof is essentially the same.)

Theorem 4.2.4 Let $f : \omega_1 \to \omega_1$ be a normal function. Then the set

$$C = \{\alpha \in \omega_1 \mid f(\alpha) = \alpha\}$$

of all fixed-points of f is club in ω_1.

Proof: It is immediate that C is closed. And by Theorem 3.8.9, C is unbounded in ω_1. □

Theorem 4.2.5 Let $f : \omega_1 \to \omega_1$, and set

$$C = \{\alpha \in \omega_1 \mid f[\alpha] \subseteq \alpha\}.$$

Then C is club in ω_1.

Proof: Since $f[\alpha] = \bigcup_{\beta<\alpha} f[\beta]$, for any limit ordinal α, it is easily seen that C is closed in ω_1. We prove unboundedness.

Let $\alpha_0 \in \omega_1$ be given. By recursion, we define an increasing sequence $\langle \alpha_n \mid n < \omega \rangle$ of countable ordinals by setting α_{n+1} to be the least ordinal such that $f[\alpha_n] \subseteq \alpha_{n+1}$. Let

$$\alpha = \lim_{n<\omega} \alpha_n.$$

Then

$$f[\alpha] = \bigcup_{n<\omega} f[\alpha_n] \subseteq \bigcup_{n<\omega} \alpha_{n+1} = \alpha,$$

so $\alpha \in C$, and we are done. □

Strengthening Theorem 4.2.2 we have:

Theorem 4.2.6 Let A_n, $n = 0, 1, 2, \ldots$, be club subsets of ω_1. Then the set

$$A = \bigcap_{n<\omega} A_n$$

is club in ω_1.

Proof: That A is closed is immediate. To prove unboundedness, for each n, set

$$B_n = A_0 \cap \ldots \cap A_n.$$

By applying Lemma 4.2.2 iteratively, each set B_n is club in ω_1. Moreover, we clearly have

$$B_0 \supseteq B_1 \supseteq B_2 \supseteq \ldots$$

and $A = \bigcap_{n<\omega} B_n$.

Let $\alpha_0 \in \omega_1$ be given now. Since each B_n is unbounded, we can use the recursion principle to define a sequence $\langle \alpha_n \mid n < \omega \rangle$ such that for every n,

$$\alpha_{n+1} \text{ is the least ordinal greater than } \alpha_n \text{ in } B_{n+1}.$$

Let $\alpha = \lim_{n<\omega} \alpha_n$. For each n, we have $\alpha = \lim_{m<\omega} \alpha_{n+m}$, so $\alpha \in B_n$. Hence $\alpha \in A$, as required. □.

4.3 Stationary Sets and Regressive Functions

This section depends on Section 4.2. As we did there, we shall concentrate on ω_1, but all our results will hold for any uncountable, regular cardinal, with proofs differing only slightly from those we give for ω_1.

A set $E \subseteq \omega_1$ is said to be *stationary* in ω_1 if and only if $E \cap C \neq \emptyset$ for every club set $C \subseteq \omega_1$.

By Lemma 4.2.2, every club set is stationary. The converse fails trivially. For example, the set $E = \omega_1 - \{\omega\}$ is obviously not closed in ω_1, since the limit of the sequence $\langle n \mid n < \omega \rangle$ is not in E, but since E intersects every unbounded subset of ω_1, E is certainly stationary. Nevertheless, stationary sets are 'large'. They are certainly unbounded. To see this, observe that each set $C_\alpha = \omega + 1 - \alpha$ is club, for $\alpha < \omega_1$, so a stationary set must intersect each C_α.

The following observation, though trivial, is sufficiently important to be worth stating as a lemma.

Lemma 4.3.1 A set $E \subseteq \omega_1$ is stationary if and only if its complement $\omega_1 - E$ does not contain a club set.

Among the subsets of ω_1, it is not easy to construct examples of stationary sets that are not in fact club or else simple modifications of clubs. In the case of ω_2, however, the following sets provide an example of a pair of disjoint stationary sets:

$$\{\alpha \in \omega_2 \mid \alpha \text{ is a limit ordinal and } \mathrm{cf}(\alpha) = \omega\},$$

$$\{\alpha \in \omega_2 \mid \alpha \text{ is a limit ordinal and } \mathrm{cf}(\alpha) = \omega_1\}.$$

However, the difficulty of actually finding nontrivial examples of stationary subsets of ω_1 does not mean they do not exist. Indeed, the following classical result, which I shall not prove here, tells us that there are stationary subsets of ω_1 that do not at all resemble club sets.

Theorem 4.3.2 If $E \subseteq \omega_1$ is stationary, then there are stationary sets A_α, $\alpha < \omega_1$, such that:

(i) $\alpha \neq \beta$ implies $A_\alpha \cap A_\beta = \emptyset$;

(ii) $\bigcup_{\alpha<\omega_1} A_\alpha = E$.

We shall obtain a very useful characterization of stationary sets in terms of a certain kind of function on ω_1.

A function $f : \omega_1 \to \omega_1$ is said to be *regressive* if and only if $f(\alpha) < \alpha$ for every nonzero α in ω_1. More generally, if $E \subseteq \omega_1$, we say a function $f : E \to \omega_1$ is *regressive* if and only if $f(\alpha) < \alpha$ for every non-zero α in E.

In order to obtain the promised characterization of stationary sets, we need a generalization of Theorem 4.2.6.

Let $\langle C_\alpha \mid \alpha < \omega_1 \rangle$ be an ω_1-sequence of club sets. Now, it is not necessarily the case that $\bigcap_{\alpha,\omega_1} C_\alpha$ is club; indeed, it may be empty, as occurs when $C_\alpha = \omega_1 - \alpha$. But consider the following superset of the complete intersection:

$$\Delta_{\alpha<\omega_1} C_\alpha = \{\gamma \in \omega_1 \mid (\forall \alpha < \gamma)(\gamma \in C_\alpha)\}.$$

The set $\Delta_{\alpha<\omega_1} C_\alpha$ is known as the *diagonal intersection* of the sequence $\langle C_\alpha \mid \alpha < \omega_1 \rangle$. Clearly,

$$\gamma \in \Delta_{\alpha<\omega_1} C_\alpha \text{ if and only if } \gamma \in \bigcap_{\alpha<\gamma} C_\alpha.$$

Theorem 4.3.3 If $\langle C_\alpha \mid \alpha < \omega_1 \rangle$ is an ω_1-sequence of club sets in ω_1, then the diagonal intersection $\Delta_{\alpha<\omega_1} C_\alpha$ is club in ω_1.

Proof: Let $C = \Delta_{\alpha<\omega_1} C_\alpha$. We start by proving that C is closed in ω_1. Let γ be a countable limit ordinal. We show that $\bigcup(C \cap \gamma) \in C$. If $C \cap \gamma$ is bounded in γ, then

$$\bigcup(C \cap \gamma) = \max(C \cap \gamma) \in C$$

and we are done. So assume $C \cap \gamma$ is unbounded in γ. Then $\bigcup(C \cap \gamma) = \gamma$, so we have to prove that $\gamma \in C$. Let $\langle \gamma_n \mid n < \omega \rangle$ be a strictly increasing sequence of elements of C cofinal in γ. For each n,

$$\{\gamma_n, \gamma_{n+1}, \gamma_{n+2}, \ldots\} \subseteq \bigcap_{\alpha<\gamma_n} C_\alpha.$$

But $\bigcap_{\alpha<\gamma_n} C_\alpha$ is club in ω_1 by Theorem 4.2.6. Thus

$$\gamma = \lim_{m<\omega} \gamma_{n+m} \in \bigcap_{\alpha<\gamma_n} C_\alpha.$$

So, as n here is arbitrary, we have, using Theorem 4.2.6 again,

$$\gamma \in \bigcap_{n<\omega} \bigcap_{\alpha<\gamma_n} C_\alpha = \bigcap_{\alpha<\gamma} C_\alpha.$$

Thus $\gamma \in C$, as required.

We turn now to the proof that C is unbounded in ω_1. Let $\alpha_0 \in \omega_1$ be given. By Theorem 4.2.6, the intersection $\bigcap_{\alpha<\alpha_0} C_\alpha$ is club, so we can find an $\alpha_1 \in \bigcap_{\alpha<\alpha_0} C_\alpha$ such that $\alpha_1 > \alpha_0$. Thus, using the recursion principle now, we can define a sequence $\langle \alpha_n \mid n < \omega \rangle$ such that $\alpha_{n+1} > \alpha_n$ and $\alpha_{n+1} \in \bigcap_{\alpha<\alpha_n} C_\alpha$. Let

$$\gamma = \lim_{n<\omega} \alpha_n.$$

Since

$$\{\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \ldots\} \subseteq \bigcap_{\alpha<\alpha_n} C_\alpha$$

for each n and $\gamma = \lim_{m<\omega} \alpha_{n+m}$, we have $\gamma \in \bigcap_{\alpha<\alpha_n} C_\alpha$ for each n, so by Theorem 4.2.6 again,

$$\gamma \in \bigcap_{n<\omega} \bigcap_{\alpha<\alpha_n} C_\alpha = \bigcap_{\alpha<\gamma} C_\alpha,$$

which means that $\gamma \in C$. Since $\gamma > \alpha_0$, we are done. □

Theorem 4.3.4 Let $E \subseteq \omega_1$. The following are equivalent:

(i) E is stationary.

(ii) If $f : E \to \omega_1$ is regressive, then for some $\gamma \in \omega_1$, $f^{-1}[\gamma]$ is stationary in ω_1.

(iii) If $f : E \to \omega_1$ is regressive, then for some $\gamma \in \omega_1$, $f^{-1}[\gamma]$ is unbounded in ω_1.

Proof: (i) → (ii). Suppose that (i) holds but, contrary to (ii), there is a regressive function $f : E \to \omega_1$ such that for no $\gamma \in \omega_1$ is $f^{-1}[\gamma]$ stationary. Thus for each $\gamma \in \omega_1$ we can find a club set $C_\gamma \subseteq \omega_1$ such that $f^{-1}[\gamma] \cap C_\gamma = \emptyset$. Let $C = \Delta_{\gamma<\omega_1} C_\gamma$. We prove that $C \cap E = \emptyset$, contradicting (i).

Suppose otherwise. Let $\alpha \in C \cap E$. Since $\alpha \in C$, we have $\alpha \in \bigcap_{\gamma<\alpha} C_\gamma$. Since $\alpha \in E$, we know that $\gamma = f(\alpha) < \alpha$ is defined. Thus $\alpha \in f^{-1}[\gamma]$. This implies that $\alpha \notin C_\gamma$. Hence $\alpha \notin \bigcap_{\gamma,\alpha} C_\gamma$, a contradiction. This proves that $C \cap E = \emptyset$ and completes the proof that (i) implies (ii).

(ii) → (iii). Trivial.

(iii) → (i). Assume ¬(i). Let $C \subseteq \omega_1$ be club with $C \cap E = \emptyset$. Define $f : E \to \omega_1$ be setting

$$f(\alpha) = \bigcup (C \cap \alpha).$$

Since C is club and $C \cap e = \emptyset$,

$$f(\alpha) = \max (C \cap \alpha) < \alpha$$

for all nonzero $\alpha \in E$. That is, f is regressive. Let $\gamma \in \omega_1$. Since C is unbounded in ω_1, we can pick $\alpha \in C$ such that $\alpha > \gamma$. Now, if $\delta \in E$ is greater than α, we have $f(\delta) \geq \alpha$, so $f(\delta) \neq \gamma$. Hence $f^{-1}[\gamma] \subseteq \alpha + 1$. Since γ was arbitrary, this proves ¬(iii) for this f. □

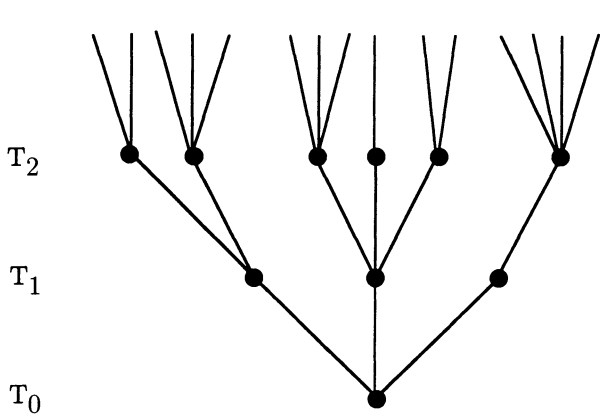

Figure 4.1: A tree

4.4 Trees

A *tree* is a poset $\mathcal{T} = \langle T, <_T \rangle$ such that for every $x \in T$, the set

$$\widehat{x} = \{y \in T \mid y <_T x\}$$

of all predecessors of x is well-ordered by $<_T$.

The ordinal number $\mathrm{Ord}(\widehat{x}, <_T)$ is called the *height* of x in $\mathcal{T}$, denoted by $\mathrm{ht}_T(x)$ (or simply $\mathrm{ht}(x)$).

If we set

$$T_\alpha = \{x \in T \mid \mathrm{ht}(x) = \alpha\}$$

for each α, we obtain a stratification of T into *levels*; T_α is the α'th *level* of $\mathcal{T}$. An element x of T_α will have exactly α predecessors in $\mathcal{T}$ (ordered by $<_T$).

Notice that no two elements of the same level T_α will be comparable under $<_T$.

We may picture a tree as in Figure 4.1. The elements (or *nodes*) of the tree are denoted by points, and a vertical (or near vertical) line drawn between two points indicates that the higher point immediately succeeds the lower in the tree ordering.

Notice that if we follow a 'path' through the tree, moving upward, each choice of direction is irrevocable; no two paths ever coincide once they have separated. Or, expressing the same fact another way, starting from any point in the tree there is one and only one path down to level T_0.

Clearly, if $\mathcal{T}$ is a tree, and if $T_\alpha \neq \emptyset$, then $T_\beta \neq \emptyset$ for all β, α. In particular, if we pick $x \in T_\alpha$, then x has a unique predecessor on each level T_β for $\beta < \alpha$.

Since T is a set, there is a unique least ordinal λ such that $T_\lambda = \emptyset$. By our previous observation, it will be the case that $T_\alpha = \emptyset$ for all $\alpha \geq \lambda$. We call this ordinal λ the *height* of $\mathcal{T}$, denoted by $\mathrm{ht}(\mathcal{T})$.

A *chain* in $\mathcal{T}$ is a linearly ordered subset of T. A *branch* in $\mathcal{T}$ is a chain that is closed under predecessors. For example, for any $x \in T$, the set $\widehat{x}$ is a branch in $\mathcal{T}$. However, branches do not have to be of this form; they may have order-type ω, having no last element.

In this book we shall be concerned almost exclusively with infinite trees. One very basic, and useful, question that can be asked about an infinite tree is, does it have an infinite branch? If $T_\omega \neq \emptyset$, the answer is trivially 'Yes' of course. The following theorem provides conditions under which the answer is always 'Yes' for trees having no level ω.

Theorem 4.4.1 [König Tree Lemma] Let $\mathcal{T} = \langle T, \leq_T \rangle$ be a tree of height ω such that every level of $\mathcal{T}$ is finite. The $\mathcal{T}$ has an infinite branch.

Proof: For each $x \in T$, let $[x]$ denote the set of all successors of x in $\mathcal{T}$:

$$[x] = \{y \in T \mid x <_T y\}.$$

Clearly, the sets $[x]$, for $x \in T_0$, constitute a disjoint partition of $T - T_0$ into finitely many subsets. Since T is infinite and T_0 is finite, $T - T_0$ is infinite. Hence, as

$$T - T_0 = \bigcup_{x \in T_0} [x],$$

$[x]$ must be infinite for at least one $[x]$ in T_0. Let x_0 be such an x.

Again, $[x_0] - T_1$ is infinite, and the sets $[x]$ for $x \in T_1 \cap [x_0]$ partition $[x_0] - T_1$ into finitely many disjoint subsets, so we can pick $x_1 \in T_1 \cap [x_0]$ so that $[x_1]$ is infinite.

Proceeding in this manner (more formally, be appealing to the recursion principle), we can define a sequence $\langle x_n \mid n < \omega \rangle$ such that, for each n, $x_{n+1} \in T_{n+1} \cap [x_n]$ and $[x_{n+1}]$ is infinite. Clearly, $\{x_n \mid n < \omega\}$ is an infinite branch of $\mathcal{T}$. □

Corollary 4.4.2 There are uncountably many reals.

Proof: What we actually prove is that there are uncountably many members of the set ${}^\omega 2$. By considering binary representations of reals in the unit interval $(0, 1)$, this is easily seen to yield the desired result.

Suppose otherwise. That is, suppose that $2^{\aleph_0} = \aleph_0$. Let $\langle \epsilon_n \mid n < \omega \rangle$ enumerate ${}^\omega 2$. Let

$$T_n = \{\epsilon \in {}^n 2 \mid (\forall m < n)[\epsilon \quad (m+1) \neq \epsilon_m \quad (m+1)]\}$$

and set

$$T = \bigcup_{n<\omega} T_n.$$

Ordered by inclusion (i.e. functional extension), T is a tree, as is easily seen. Also easy to verify is that T is infinite and has height at most ω. Moreover, since

$$|{}^n 2| = 2^n < \aleph_0$$

for any n, each level T_n of T is finite. In particular, this implies that T has height exactly ω. By König's Lemma, let b be an infinite branch. Set

$$f = \bigcup b.$$

Clearly, $f \in^{\omega} 2$. But for all n, $f \quad (n+1) \in T$, so $f \quad (n+1) \neq \epsilon_n \quad (n+1)$. Thus $f \notin \{\epsilon_n \mid n < \omega\}$, a contradiction. □

Of course, in the above example, one gains nothing by using König's Lemma, since the proof that T is infinite amounts to a thinly disguised version of the classical diagonalization argument Cantor used to prove the uncountability of the reals. What the corollary does do is illustrate how one can view recursive procedures of the Cantor type as applications of the König Lemma, and this can be an advantage in more complex situations.

With the König Tree Lemma now behind us, what do you think is the answer to the following question?

> Let $\mathcal{T}$ be a tree of height ω_1, all of whose levels are countable. Does $\mathcal{T}$ necessarily have an uncountable branch?

At first glance, one might think that the proof of Theorem 4.4.1 will generalize easily to give a positive answer to this question, and rare is the beginner who sees at once that this is not the case. In a moment I shall present a construction of a tree of height ω_1, all of whose levels are countable, having no uncountable branch. But first let us try to see what goes wrong when we simply try to generalize the proof of the König Tree Lemma from ω to ω_1.

Suppose $\mathcal{T}$ is a tree of height ω_1, all of whose levels are countable. Pick $x_0 \in T_0$ to have uncountably many extensions. Then pick $x_1 \in T_1$ to extend x_0 so that x_1 has uncountably many extensions. And so on. This procedure works fine for the first ω steps, defining a branch $\langle x_n \mid n < \omega \rangle$. The next step is to pick $x_\omega \in T_\omega$ so that $x_n <_T x_\omega$ for all $n < \omega$ and x_ω has uncountably many extensions. But how do we know that there is *any* element of T_ω that extends the branch $\langle x_n \mid n < \omega \rangle$? Indeed, since there are uncountably many such branches (this is easily seen), many of them will not have an extension on T_ω. Of course, your initial reaction is that

provided we are careful when we choose x_0, x_1, x_2, etc., we can ensure that we do end up with a branch that extends onto T_ω. True enough. But it is a long way to go to ω_1, and we cannot allow *in advance* for all future limit levels. Sooner or later we will reach a limit stage that we have not been able to allow for, and then the same problem arises.

Of course, you are still not completely convinced, are you? So let me now present you with the incontravertible evidence of a proof.

Theorem 4.4.3 [N. Aronszajn] There is a tree $\mathcal{T} = \langle T, \leq_T \rangle$ such that:

(i) $\mathcal{T}$ has height ω_1;

(ii) $|T_\alpha| \leq \aleph_0$ for all $\alpha < \omega_1$;

(iii) if $x \in T_\alpha$ and $\alpha < \beta < \omega_1$, there is a $y \in T_\beta$ such that $x <_T y$;

(iv) $\mathcal{T}$ has no uncountable branch.

Proof: The elements of T_α will be strictly increasing α-sequences of rational numbers that are bounded above. The ordering of $\mathcal{T}$ will be inclusion (i.e. sequence extension). Notice at once that this will yield condition (iv) of the theorem, since an uncountable branch of such a tree would present us with a strictly increasing ω_1-sequence of rationals, which is impossible. Notice also that condition (i) follows from condition (iii). Thus our task is to construct the tree to satisfy both (ii) and (iii). This requires some care.

Since the ordering is inclusion, we are only concerned with which sequences each T_α will contain. The definition is by recursion on the levels. That is, we define T_α from $\bigcup_{\beta<\alpha} T_\beta$. We use $T \quad \alpha$ to denote both the set $\bigcup_{\beta<\alpha} T_\beta$ and the tree on this set determined by the inclusion order. The recursion is carried out to preserve the following condition:

(*) If $s \in T_\alpha$ and $\alpha < \beta < \omega_1$, then, for each rational number $q > \sup(s)$, there is a $t \in T_\beta$ such that $s \subset t$ and $\sup(t) < q$.

To commence, we set $T_0 = \{\emptyset\}$. If $T \quad (\alpha+1)$ is defined, we define $T_{\alpha+1}$ as

$$T_{\alpha+1} = \{s \in {}^{\alpha+1}\mathbb{Q} \mid s \quad \alpha \in T_\alpha\}$$

where $\mathbb{Q}$ is the set of rationals. If $|T_\alpha| \leq \aleph_0$, then since $|\mathbb{Q}| = \aleph_0$, we have $|T_{\alpha+1}| = \aleph_0$. Moreover, if (*) is valid for $T \quad (\alpha+1)$, it will clearly be valid for $T \quad (\alpha+2)$.

There remains the case where $T \quad \alpha$ is defined for α a limit ordinal. Let us call a branch b of $T \quad \alpha$ *cofinal* if it intersects each level of $T \quad \alpha$ (i.e. if its order-type under the tree-ordering is α). In order to define T_α, we must extend some cofinal branches of $T \quad \alpha$. Indeed, any element of T_α will necessarily be of the form $\bigcup b$, where b is a cofinal branch of $T \quad \alpha$.

Now, if $\bigcup b \in T_\alpha$, $\bigcup b$ must be bounded above in $\mathbb{Q}$, so it must be the case that the set $\{\sup(s) \mid s \in b\}$ is bounded above in $\mathbb{Q}$. (As will become clear in a moment, it was in order to ensure that such branches b can always be found that we introduced the requirement $(*)$.) Now, we cannot simply extend all such branches, since there are uncountably many of them, which would make T_α uncountable. On the other hand, we must ensure that $(*)$ holds for $T \quad (\alpha+1)$. So we proceed as follows.

Notice first that $(*)$ will hold for $T \quad \alpha$ providing it holds for each $T \quad \beta$ for $\beta < \alpha$.

Let $\langle \alpha_n \mid n < \omega \rangle$ be a strictly increasing sequence of ordinals cofinal in α. For each $s \in T \quad \alpha$, and each rational number $q > \sup(s)$, we define an element $b(s,q)$ of ${}^{\alpha}\mathbb{Q}$ as follows.

Let $n(s)$ be least such that $s \in T \quad \alpha_{n(s)}$. By $(*)$, pick $s_{\alpha(n)} \in T_{\alpha_{n(s)}}$ so that $s \subset s_{n(s)}$ and $\sup(s_{n(s)} < q$.

We define $s(n)$ for $n(s) < n < \omega$ now by recursion. Let $s_{n+1} \in T_{\alpha_{n+1}}$ be such that $s_n \subset s_{n+1}$ and $\sup(s_{n+1}) < q$. If $\sup(s_n) < q$, then by $(*)$, such an s_{n+1} can always be found.

Now set

$$b(s,q) = \bigcup_{n(s)<n<\omega} s_n.$$

Clearly, $b(s,q) \in {}^{\alpha}\mathbb{Q}$ and $s \subset b(s,q)$. Moreover, $\sup(b(s,q)) \leq q$. We define

$$t_\alpha = \{b(s,q) \mid s \in T \quad \alpha \wedge q \in \mathbb{Q} \wedge q > \sup(s)\}.$$

If $|T \quad \alpha| \leq \aleph_0$, then $|t_\alpha| \leq \aleph_0$. Moreover, $t \quad (\alpha+1)$ satisfies $(*)$ by virtue of the construction.

That completes the definition of T. An easy induction on the levels shows that condition (ii) holds. (The induction steps have already been noted.) And (iii) follows directly from $(*)$. The proof is complete. □

4.5 Extensions of Lebesgue Measure

We commence by recalling some standard definitions from measure theory.

Let $\mathcal{F}$ be a σ-field of subsets of a set X. (See Problems 1.1 and Section 4.1 of this chapter for the relevant definitions.) A *measure* on $\mathcal{F}$ is a function μ from $\mathcal{F}$ to the unit interval $[0,1]$ such that:

(i) $\mu(\emptyset) = 0$, $\mu(X) = 1$;

(ii) if $\{E_n\}$ is a finite or infinite sequence of disjoint elements of $\mathcal{F}$, then

$$\mu(\bigcup_n E_n) = \sum_n \mu(E_n).$$

The classic example of such is where $X = [0,1]$, $\mathcal{L}$ is the σ-field of all Lebesgue measurable subsets of $[0,1]$, and μ is the Lebesgue measure on $\mathcal{L}$.

Now, it is known that $\mathcal{L} \neq \mathcal{P}([0,1])$. (The proof uses the Axiom of Choice; in the absence of such an assumption, the result is not necessarily valid.) A natural question to ask is whether it is possible to extend the Lebesgue measure on $\mathcal{L}$ to a measure defined on all subsets of $[0,1]$. The usual proof that $\mathcal{L} \neq \mathcal{P}([0,1])$ is easily modified to show that any such extension would fail to be translation invariant, but does not preclude the existence of such an extension. It turns out that this simple-sounding question has a rather surprising consequence, namely, the following theorem, the proof of which occupies the remainder of this section.

Theorem 4.5.1 Assume there is an extension of Lebesgue measure (or, more generally, any measure) defined on all subsets of $[0,1]$. Then there is a weakly inaccessible cardinal $\kappa \leq 2^{\aleph_0}$. □

As an immediate consequence of this theorem, we have:

Corollary 4.5.2 Assume CH. Then there is no extension of Lebesgue measure to $\mathcal{P}([0,1])$, nor indeed any measure defined on $\mathcal{P}([0,1])$.

Proof: Clearly, $2^{\aleph_0}$ cannot at the same time equal $\aleph_1$ and dominate a weakly inaccessible cardinal. □

For the remainder of this section, we shall assume that there is a measure defined on $\mathcal{P}([0,1])$.

Since $|[0,1]| = 2^{\aleph_0}$, the measure on $\mathcal{P}([0,1])$ induces a measure on $\mathcal{P}(2^{\aleph_0})$ in a trivial fashion.

The idea is to show that, starting from any measure, μ, on $\mathcal{P}(2^{\aleph_0})$, it is possible to find an uncountable cardinal $\kappa \leq 2^{\aleph_0}$, such that there is what is called a *κ-additive* measure, σ, on $\mathcal{P}(\kappa)$, and then make use of the measure σ to prove that κ must be weakly inaccessible.

We say that a measure σ on the power set of an uncountable cardinal κ is *θ-additive*, for $\theta \leq \kappa$ an uncountable cardinal, if and only if, whenever $\gamma < \theta$ and E_ν, $\nu < \gamma$, are sets of measure zero, then $\bigcup_{\nu<\gamma} E_\nu$ has measure zero.

In particular, by definition, any measure is $\aleph_1$-additive.

Now, there is clearly a largest cardinal κ such that μ is κ-additive. Obviously, $\kappa \geq \aleph_1$. Moreover, since $2^{\aleph_0} = \bigcup_{\alpha<2^{\aleph_0}} \{\alpha\}$ and $\mu(2^{\aleph_0}) = 1$, we have $\kappa \leq 2^{\aleph_0}$.

By definition of κ, there is a set A, of positive measure, for which there are disjoint sets A_ν, $\nu < \kappa$, of measure zero, with

$$A = \bigcup_{\nu<\kappa} A_\nu.$$

Define a map $f : A \to \kappa$ by

$$f(a) = \nu \quad \leftrightarrow \quad a \in A_\nu.$$

For $B \subseteq \kappa$, set

$$\sigma(B) = \frac{\mu(f^{-1}[B])}{\mu(A)}.$$

It is easily seen that σ is a κ-additive measure on $\mathcal{P}(\kappa)$. We work with the measure σ on $\mathcal{P}(\kappa)$ from now on.

We complete the proof of Theorem 4.5.1 by showing that κ is weakly inaccessible.

Lemma 4.5.3 If $\xi < \kappa$, then $\xi = \{\alpha \mid \alpha < \xi\}$ has measure zero.

Proof: Because σ is κ-additive. □

Lemma 4.5.4 κ is regular.

Proof: Suppose not. Then there is a $\theta < \kappa$ and ordinals $\kappa_\nu < \kappa$ for $\nu < \theta$, such that

$$\kappa = \bigcup_{\nu<\theta} \kappa_\nu.$$

By Lemma 4.5.3, $\sigma(\kappa_\nu) = 0$ for all $\nu < \theta$. So, as σ is κ-additive, $\sigma(\kappa) = 0$, which is impossible. □

Lemma 4.5.5 κ is a limit cardinal.

Proof: Suppose not. Let $\kappa = \lambda^+$. We define a $\kappa \times \lambda$ matrix of subsets of κ,

$$\{A_{\alpha\nu} \mid \alpha < \kappa, \nu < \lambda\},$$

such that each column consists of pairwise disjoint sets and each row contains all but λ-many elements of κ. (Such a matrix is sometimes called an *Ulam matrix*.)

For each $\xi < \kappa$, let f_ξ be a function defined on λ such that $\xi \subseteq \mathrm{ran}\,(f_\xi)$. For $\alpha < \kappa, \nu < \lambda$, define $A_{\alpha\nu}$ by

$$\xi \in A_{\alpha\nu} \quad \leftrightarrow \quad f_\xi(\nu) = \alpha.$$

If $\nu < \lambda$, then for each $\xi \in \kappa$, there is only one α such that $\xi \in A_{\alpha\nu}$, namely, $\alpha = f_\xi(\nu)$. Hence:

(*) if $\alpha, \beta < \kappa, \alpha \neq \beta$, then $A_{\alpha\nu} \cap A_{\beta\nu} = \emptyset$ for all $\nu < \lambda$.

Moreover, if $\alpha < \kappa$, then for each $\xi > \alpha$, there is a $\nu < \lambda$ such that $f_\xi(\nu) = \alpha$, so

$$\kappa - \bigcup_{\nu<\lambda} A_{\alpha\nu} \subseteq \alpha + 1,$$

so

(**) for each $\alpha < \kappa$, the set $\kappa - \bigcup_{\nu<\lambda} A_{\alpha\nu}$ has cardinality at most λ.

Now, since σ is κ-additive, by (**) we see that for each $\alpha < \kappa$ there is a $\nu_\alpha < \lambda$ such that $\sigma(A_{\alpha\nu_\alpha}) > 0$. For some set $W \subseteq \kappa$ of cardinality κ, we must have $\nu_\alpha = \nu$ for all $\alpha \in W$, for some fixed ν. Then, using (*), $\{A_{\alpha\nu} \mid \alpha \in W\}$ is a family of pairwise disjoint sets of positive measure, which is absurd, since $\sigma(\kappa) = 1$.

The lemma is proved. □

That completes the proof of Theorem 4.5.1.

4.6 A Result About the GCH

I have already indicated (p.97) that the GCH cannot be proved in Zermelo–Fraenkel set theory. In fact, using techniques of the kind outlined in Chapter 6, it may be shown that, for any uncountable regular cardinal κ, it is consistent with the ZFC axioms that the GCH holds below κ (i.e. $(\forall\lambda < \kappa)[2^\lambda = \lambda^+]$) but fails at κ itself (i.e. $2^\kappa > \kappa^+$). For instance, it is consistent with the ZFC axioms that $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$, $2^{\aleph_2} = \aleph_3$, and $2^{\aleph_3} = \aleph_{17}$

The regularity of κ is essential here; or almost: if κ is singular of cofinality ω it may be possible for the GCH to hold below κ and still fail at κ, but the situation is rather complex. However, if κ is singular of uncountable cofinality, the validity of the GCH below κ implies its validity at κ. The proof of this result, which I present here, is nontrivial and provides a good illustration of an argument in combinatorial set theory and cardinal arithmetic. The proof requires a knowledge of Sections 4.2 and 4.3.

We fix from now on a singular cardinal κ of uncountable cofinality. We assume that $2^\lambda = \lambda^+$ for all $\lambda < \kappa$. We prove that $2^\kappa = \kappa^+$.

Let $\theta = \mathrm{cf}\,(\kappa)$. Thus $\omega_1 \leq \theta < \kappa$. Let $\langle \kappa_\nu \mid \nu < \theta \rangle$ be a normal sequence of cardinals that is cofinal in κ.

Lemma 4.6.1 Let $E \subseteq \theta$ be stationary. Let $f : E \to \kappa$ be such that $f(\alpha) < \kappa_\alpha$ for all $\alpha \in E$. Then there is a $\gamma < \theta$ and a stationary set $E' \subseteq E$ such that

$$\alpha \in E' \text{ implies } f(a) < \kappa_\gamma.$$

Proof: Let

$$C = \{\alpha \in \theta \mid \alpha \text{ is a limit ordinal}\}.$$

For $\alpha \in C$, we have $\kappa_\alpha = \lim_{\nu<\alpha}\kappa_\nu$, so, for each $\alpha \in E \cap C$ there is a $\nu < \alpha$ such that $f(\alpha) < \kappa_\nu$. Let $g(\alpha)$ be the least such ν. Clearly, $g : E \cap C \to \theta$ is regressive. But $E \cap C$ is stationary in θ and C is club in θ and θ is an uncountable regular cardinal. So, $E \cap C$ is stationary and, by Theorem 4.3.4, there is a stationary set $E' \subseteq E \cap C$ and a $\gamma < \theta$ such that

$$\alpha \in E' \text{ implies } g(\alpha) = \gamma.$$

Thus

$$\alpha \in E' \text{ implies } f(\alpha) < \kappa_\gamma,$$

as required. □

Now, by assumption, $2^{\kappa_\alpha} = \kappa_\alpha^+$ for each $\alpha < \theta$. Let $\langle A_\xi^\alpha \mid \xi < \kappa_\alpha^+\rangle$ be an enumeration of $\mathcal{P}(\kappa_\alpha)$.

For $A \subseteq \kappa$, define $f_A : \theta \to \kappa$ by

$$f_A(\alpha) = \xi \leftrightarrow A \cap \kappa_\alpha = A_\xi^\alpha.$$

Notice that if $A, B \subseteq \kappa$ and $A \neq B$, then for some $\alpha < \theta$, $A \cap \kappa_\alpha \neq B \cap \kappa_\alpha$, whence $f_A(\beta) \neq f_B(\beta)$, whenever $\beta \geq \alpha$, which means that the set

$$\{\alpha \in \theta \mid f_A(\alpha) = f_B(\alpha)\}$$

is bounded in θ.

We define a relation R on $\mathcal{P}(\kappa)$ by

$$R(A, B) \text{ if and only if } \{\alpha \in \theta \mid f_A(\alpha) < f_B(\alpha)\} \text{ is stationary.}$$

Lemma 4.6.2 Let $A, B \subseteq \kappa$, $A \neq B$. Then $R(A, B)$ or $R(B, A)$ (or both).

Proof: Clearly,

$$\begin{aligned}\theta \;=\; & \{\alpha \in \theta \mid f_A(\alpha) < f_B(\alpha)\} \cup \{\alpha \in \theta \mid f_B(\alpha) < f_A(\alpha)\} \\ & \cup \{\alpha \in \theta \mid f_A(\alpha) = f_B(\alpha)\}.\end{aligned}$$

By our earlier discussion, there is a $\gamma < \theta$ such that

$$\{\alpha \in \theta \mid f_A(\alpha) = f_B(\alpha)\} \subseteq \gamma.$$

Suppose that neither $R(A,B)$ nor $R(B,A)$ held. Then we could find club sets $C_1, C_2 \subseteq \theta$ such that:

$$\alpha \in C_1 \rightarrow f_A(\alpha) \text{ is not less than } f_B(\alpha),$$

$$\alpha \in C_2 \rightarrow f_B(\alpha) \text{ is not less than } f_A(\alpha).$$

By Theorem 4.2.2, the set $C = C_1 \cap C_2 - (\gamma + 1)$ is club in θ. But, by the choice of γ, we must have $C = \emptyset$, which is a contradiction. This proves that at least one of $R(A,B)$ and $R(B,A)$ must be valid. (They could both be valid.) □

Our aim is to prove $2^\kappa = \kappa^+$. We assume, on the contrary, that $2^\kappa > \kappa^+$ and work toward a contradiction.

Lemma 4.6.3 There is a $B \subseteq \kappa$ such that $|\{A \subseteq \kappa \mid R(A,B)\}| \geq \kappa^+$.

Proof: Let $X \subseteq \mathcal{P}(\kappa)$, $|X| = \kappa^+$. If there is a $B \in X$ with the required property we are done, so assume otherwise. For each $B \in X$, let $R^{-1}(B)$ denote the set $\{A \subseteq \kappa \mid R(A,B)\}$. Let

$$Y = \bigcup\{(R^{-1}(B) \mid B \in X\}.$$

Now, $|X| = \kappa^+$ and, by our assumption, $|R^{-1}(B)| \leq \kappa$ for all $B \in X$, so $|Y| \leq \kappa^+$. So, as $|\mathcal{P}(\kappa) > \kappa^+$, there is a $B \subseteq \kappa$ such that $B \notin Y$.

Now, if $A \in X$, then $B \notin R^{-1}(A)$, so $R(B,A)$ fails. Hence, by Lemma 4.6.2, $A \in X$ implies $R(A,B)$. Thus as $|X| = \kappa^+$, B is as required. □

We fix B as in Lemma 4.6.3 from now on.

Now, for each $\alpha < \theta$, $f_B(d\alpha) < \kappa^+$, so we can fix some one-one mapping

$$g_\alpha : f_B(\alpha) \rightarrow \kappa_\alpha.$$

Suppose now that $A \subseteq \kappa$ is such that $R(A,B)$. Let

$$S_A = \{\alpha \in \theta \mid f_A(\alpha) < f_B(\alpha)\}.$$

Then S_A stationary, and for each $\alpha \in S_A$,

$$g_\alpha \circ f_A(\alpha) < \kappa_\alpha.$$

So by Lemma 4.6.1 there is a stationary set $T_A \subseteq S_A$ and an ordinal $\gamma_A < \theta$ such that

$$\alpha \in T_A \rightarrow g_\alpha \circ f_A(\alpha) < \kappa_{\gamma_A}.$$

Now, using the fact that $\theta < \kappa$,

$$|\{(T_A, \gamma_A) \mid A \subseteq \kappa \wedge R(A, B)\}| \leq |\mathcal{P}(\theta) \times \theta| = 2^\theta . \theta = \theta^+.$$

So, as there are at least κ^+ many sets $A \subseteq \kappa$ with $R(A, B)$ and κ^+ is regular, there is a pair (T, γ) such that

$$|\{A \subseteq \kappa \mid R(A, B) \wedge T_A = T \wedge \gamma_A = \gamma\}| \geq \kappa^+.$$

But

$$|{}^T\kappa_\gamma| = {\kappa_\gamma}^\theta \leq \max(\kappa_\gamma^{\kappa_\gamma}, \theta^\theta) = \max(2^{\kappa_\gamma}, 2^\theta) = \max(\kappa_\gamma^+, \theta^+) \leq \kappa.$$

Hence there must be sets $A_1, A_2 \subseteq \kappa$, $A_1 \neq A_2$, such that $R(A_1, B), R(A_2, B)$, $T_{A_1} = T_{A_2} = T$, $\gamma_{A_1} = \gamma_{A_2} = \gamma$, and

$$g_\alpha \circ f_{A_1} \quad T = g_\alpha \circ f_{A_2} \quad T.$$

Since g_α is one-one, this implies that

$$f_{A_1} \quad T = f_{A_2} \quad T.$$

But $\{\alpha \in \theta \mid f_{A_1}(\alpha) = f_{A_2}(\alpha)\}$ is known to be bounded in θ, so we have a contradiction. We have thus proved the following theorem:

Theorem 4.6.4 Let κ be a singular cardinal of uncountable cardinality. If $2^\lambda = \lambda^+$ for all $\lambda < \kappa$, then $2^\kappa = \kappa^+$. □

5

The Axiom of Constructibility

5.1 Constructible Sets

Before reading this chapter, the reader should go back and reread Sections 2.2 and 2.3 of Chapter 2, where we developed the concept of the set-theoretic hierarchy, $\langle V_\alpha \mid \alpha \in On \rangle$.

Now, in defining the set-theoretic hierarchy, we took as a basic notion the unrestricted power set operation $\mathcal{P}(x)$. Given the level V_α of the hierarchy, we took

$$V_{\alpha+1} = \mathcal{P}(V_\alpha).$$

That is, $V_{\alpha+1}$ is the set of all subsets of V_α. But we did not say just what does constitute a subset of V_α, in that we never really defined the notion of what a set is! (Of course, as I said in Chapter 2, a set is a collection of sets, but this does not tell us what a set is unless we know what a collection is.)

Now, for a large part of mathematics, indeed the greatest part, this lack of specificity is not important. Usually in mathematics, when one needs to refer to a particular set, one has a description of that set (i.e. a definition of the set), and thus the Axiom of Subset selection suffices to provide that set. The only exception is (usually) when an Axiom of Choice or Zorn's Lemma argument is involved, when one simply appeals to the axiom to provide a raw existence assertion. (This of course explains why some people still feel uneasy about the use of Axiom of Choice arguments: one obtains a set that one cannot 'imagine'.)

But when we come to a question such as whether $2^{\aleph_0} = \aleph_1$ or not, the situation is quite different. Here we want to know how many elements the set $\mathcal{P}(\omega)$ has. But since we have at no point determined what is to constitute an arbitrary subset of ω, how could we expect to answer this question? It turns out that indeed we cannot. The Zermelo–Fraenkel axioms do not decide the Continuum Problem one way or the other. We sketch a proof of this in Chapter 6.

Of course, one could assume that this type of question is the only type that results in an undecidable statement and ignore it, but this is not reasonable. There are many simple statements of analysis, for instance, which have an easy proof if $2^{\aleph_0} = \aleph_1$, but apparently no proof otherwise, and these questions demand an answer.

To which problem one obvious solution might be to take CH (or even GCH) as an additional axiom of set theory. But why? What possible intuition could lead to our taking GCH as a 'reasonable' assertion about sets? There is indeed none. And anyway, even if we were to take GCH as an axiom, our problems would not be over. There are several fundamental questions of pure mathematics that cannot be resolved even if we assume GCH; I shall state two.

1. (Whitehead Problem) Suppose $\mathcal{G}$ is an abelian group with the property that whenever $\mathcal{H}$ is an abelian group extending the group, $\mathbb{Z}$, of integers, such that $\mathcal{H}/\mathbb{Z} \simeq \mathcal{G}$, then $\mathcal{H} \simeq \mathbb{Z} \oplus \mathcal{G}$ (direct sum). Is $\mathcal{G}$ necessarily free? ($\mathcal{G}$ being free is a sufficient condition for this to hold.)

2. (Souslin Problem) Let $(X, <)$ be a Dedekind complete toset with no end points, such that between each pair of elements of X lies a third element of X. Suppose that there is no uncountable collection of pairwise disjoint open intervals of X. Is it necessarily the case that $(X, <) \cong \mathbb{R}$? (If the last condition is strengthened to X having a countable dense subset, the answer is 'Yes'.)

Assuming that we feel that our foundational set theory should be able to provide the means of resolving questions such as these, we had better re-examine our set theory.

I shall describe one natural and highly successful solution to the dilemma, a solution that certainly resolves the two questions above, as well as a good many more.

The idea is to provide a precise definition of the notion of a 'set' (or 'collection').

Suppose we take as our basic idea of a set, the notion of a *describable* collection (of sets). We can make this a bit more precise by restricting our 'descriptions' to those expressible in our formal language LAST. This will allow us to refer to existing sets in order to describe new sets, because LAST includes a facility for such references. And it will clearly provide us with all the sets we need in mathematics, except perhaps for the 'undescribed' sets which we obtain by using the Axiom of Choice—but let us leave the problem about the Axiom of Choice for the time being. (It will turn out that this is a wise decision. This is one of the rare occasions when a problem disappears as a result of its being ignored!)

Since the original motivation for sets forming a hierarchy would still appear to be in order, let us now try to redefine the set-theoretic hierarchy, replacing the unrestricted (and undescribed) power set operation by the more precise notion of the 'describable power set' operation.

Thus, we shall start with the empty set, and at limit levels we shall collect everything together just as before. But in proceeding from stage α to stage $\alpha+1$, we shall introduce just those subsets (of what we now have) which we can describe using LAST.

To indicate that the hierarchy is being defined differently, I denote the α'th level not by V_α now, but by L_α. Thus, we have the (tentative, and as yet informal) definition

$$\begin{aligned} L_0 &= \emptyset, \\ L_\lambda &= \bigcup_{\alpha<\lambda} L_\alpha, \text{ if } \lambda \text{ is a limit ordinal,} \\ L_{\alpha+1} &= \text{all collections of elements of } L_\alpha \text{ that are describable} \\ &\quad \text{by means of a formula of LAST.} \end{aligned}$$

All we need to do now is to make the last clause in this definition precise. We will not run into any problems providing we keep in mind the fundamental intuition that, when we are trying to define $L_{\alpha+1}$, *those sets in L_α and only those sets are at our disposal.*

Assume then that we have constructed the set L_α. If $\phi(v_n)$ is a formula of LAST having the single free variable v_n, and if $a_1, \ldots, a_m$ are sets in L_α which the names (i.e. the w_i's) in ϕ denote, then the collection of all those sets x in L_α for which $\phi(x)$ is true is well-defined. $L_{\alpha+1}$ will consist of all such collections.

Thus, $X \in L_{\alpha+1}$ if and only if there is a formula $\phi(v_n)$ of LAST, with the single free variable v_n, and sets $a_1, \ldots, a_m$ in L_α, which interpret the names involved in ϕ, such that X is the collection of all x in L_α for which $\phi(x)$ is true.

One point needs a little clarification here. Suppose the formula ϕ involves the quantifier $\forall v_i$. What do we mean by saying '$\phi(x)$ is true'. Well, at stage α, the only sets available are those in L_α. So we are only in a position to 'check' whether all interpretations of v_i *in* L_α are as required. In other words, the only possible meaning that the quantifier $\forall v_i$ can have at stage α is 'for all $v_i \in L_\alpha$'. Similarly for an existential quantifier: at stage α, $\exists v_j$ can only mean 'there exists a v_j *in* L_α'. (Strictly, at stage α there is no need for the qualification 'in L_α' here, since L_α really is all there is!) Thus the truth or falsity of $\phi(x)$ at stage α need not be related to its eventual 'truth' or 'falsity'. Rather, since L_α is a well-defined set, the notion of '$\phi(x)$ being true with respect to the partial universe L_α' is

certainly well-defined, and this notion of 'truth' is what gives us $L_{\alpha+1}$ as a precisely defined collection of sets.

That then defines our hierarchy. We set

$$L = \bigcup_{\alpha} L_{\alpha},$$

the union being over all ordinals.

We call the class L the *constructible universe.* Sets corresponding to this notion of set (i.e. members of the class L) are called *constructible sets.* The hierarchy

$$\langle L_{\alpha} \mid \alpha \in \mathrm{On} \rangle$$

is known as the *constructible hierarchy.*

It should be pointed out that our notion of 'describable collection' is very strong, so one should not read too much into the present use of the word 'constructible'. For instance, in constructible set theory the real line turns out to have a constructible well-ordering!

5.2 The Constructible Hierarchy

A brief examination of the definitions in the previous section shows that:

- $L_{\alpha} \subseteq L_{\beta}$ for $\alpha \leq \beta$;
- each L_{α} is transitive;
- $L_{\alpha} \cap \mathrm{On} = \{\beta \mid \beta < \alpha\} = \alpha$.

These properties are shared by the Zermelo (V_{α}) hierarchy, of course. But there the similarity ends. For example, because the language LAST is countable, we have

$$|L_{\alpha}| = |\alpha|$$

for every infinite ordinal α. Hence, in particular,

$$|L_{\omega+1}| = \aleph_0.$$

But since $\mathcal{P}(\omega) \subseteq V_{\omega+1}$,

$$|V_{\omega+1}| > \aleph_0.$$

Thus the constructible hierarchy grows much more slowly than the Zermelo hierarchy.

And now, before we go on, let me explain a point that may just have begun to worry the reader. Does not the fact that $L_{\omega+1}$ is countable contradict the fact that $\mathcal{P}(\omega)$ is uncountable?

Well, since we have not yet examined the consequences of our new notion of a set, it may be that in our new set theory $\mathcal{P}(\omega)$ is in fact countable. But before my readers throw up their hands in horror, let me hasten to say that this is not in fact the case: $\mathcal{P}(\omega)$ is indeed uncountable in constructible set theory. The confusion, if there is any, lies in the fact that $\mathcal{P}(\omega)$ will not be contained in $L_{\omega+1}$. Certainly, some subsets of ω will lie in $L_{\omega+1}$. For instance, the set of all even numbers is there, as too is the set of all multiples of 3. Indeed, $L_{\omega+1}$ will contain infinitely many subsets of ω. But, to be in $L_{\omega+1}$, a subset of ω will have to be describable in terms of sets in L_ω. This only allows the formation of relatively simple sets of numbers. L_ω does not contain enough 'information' to enable us to define 'complex' sets of integers. Now, when we come to define $L_{\omega+2}$, our expressive power has increased enormously. In describing sets, we may now refer to all the new sets that went into $L_{\omega+1}$. Thus $L_{\omega+2}$ will contain many new sets of integers, not previously 'constructible'. And so on.

Thus, not only does the constructible hierarchy grow more slowly than the Zermelo hierarchy, it in fact grows in quite a different manner.

5.3 The Axiom of Constructibility

We are now in a position analogous to the one we were in at the end of Section 2.2. By means of an analysis of the notion of 'set', we have arrived at a picture of the way the set-theoretic universe should look. Instead of the picture represented by the two 'axioms'

(Z1) $V = \bigcup_\alpha V_\alpha$

(Z2) Axiom of subset selection,

we now have two principles

(Ll) $V = \bigcup_\alpha L_\alpha$

(L2) Axiom of subset selection.

(So far in the discussion, I have not mentioned (L2), but of course we shall need this if our set theory is to be of any use to us as mathematicians. The remarks I made after defining the constructible hierarchy should indicate why it may be necessary to include this principle as an axiom, even though the constructible hierarchy is built up by defining sets; namely, defining subsets of L_α at stage α is not at all the same as defining subsets of L_α *over the entire universe*, which is what the Axiom of Subset Selection concerns.)

The next step is to do what we did in Section 2.3 for the Zermelo–Fraenkel set theory: analyze the two principles (Ll) and (L2), and thereby isolate all those assumptions about sets which the construction makes implicit use of.

Well, we certainly need the ordinal number system. We also need the recursion principle. (The formal definition of the constructible hierarchy will be as a recursion on ordinals, of course) In fact, the only difference between the constructible hierarchy and the Zermelo hierarchy lies in what we do at successor stages. With the Zermelo hierarchy, since the power set operation is guaranteed by the ZF axioms, theZF system suffices for the entire construction. But the definition of $L_{\alpha+1}$ from L_α is a little more complex. Here we use logical formulas, assignments of sets to names, interpretation of variables, and the truth of formulas within a certain partial universe L_α.

Now, admittedly mathematical logic (or rather the parts of it that concern us here) deals with some of the fundamental concepts that *lie behind* the notion of a set. Nevertheless, mathematical logic, in common with all other areas of pure mathematics, can be developed rigorously *within set theory.* In particular, all of the concepts required for the passage from L_α to $L_{\alpha+1}$ are capable of definition and analysis within set theory. Indeed, ZF suffices!

In other words, the construction of the constructible hierarchy is possible on the basis of the ZF axioms, just as is the construction of the Zermelo hierarchy.[1] Hence, constructible set theory can be axiomatized as follows.

(i) The ZF axioms;

(ii) $V = \bigcup_\alpha L_\alpha$. (The Axiom of Constructibility)

The ZF axioms enable us to define the hierarchy $\langle L_\alpha \mid \alpha \in \mathrm{On}\rangle$. The Axiom of Constructibility tells us that the universe of sets is the limit of this hierarchy.

Consequently, we see that constructible set theory is an *extension* of ZF, obtained by adjoining the Axiom of Constructibility.

Since we have introduced the symbol L to denote the class $\bigcup_\alpha L_\alpha$, the axiom of constructibility may be abbreviated as

$$V = L.$$

And constructible set theory may be denoted as

[1]The development of mathematical logic within set theory is not particularly difficult, but it would constitute too great a digression to go into details here. All that we need to know for our discussion is that within ZF one can define a function

$$\mathrm{Def} : V \to V$$

such that $\mathrm{Def}(L_\alpha) = L_{\alpha+1}$ for all α. $\mathrm{Def}(X)$ is the set of all 'definable' subsets of X, for any set X, where 'definable" here means 'definable over the partial universe X by means of a formula of LAST, with one free variable, whose names refer only to sets in X'.

$$\text{ZF} + (V = L).$$

Now, on the basis of the ZF axioms, we can still define the Zermelo hierarchy, regardless of whether $V = L$ or not. Hence, $V = L$ does not affect the validity of the equation

$$V = \bigcup_\alpha V_\alpha.$$

But $V = L$ certainly does affect the meaning of this equation. In the context of ZF alone, the power set operation is left totally undescribed, which means that there is a great degree of 'freedom' built in to the Zermelo hierarchy. But if we assume $V = L$ (in addition to the ZF axioms), then the notion of what constitutes a set is made very precise, which means that the power set operation is a rigidly determined operator.

And now to the Axiom of Choice. The one obvious advantage of leaving the power set unrestricted is that it allows one to postulate the existence of choice sets, and thereby to introduce the Axiom of Choice. But if we adopt as our system of set theory the theory $ZF + (V = L)$, we no longer have this freedom. Either AC will be true, or it will be false. Fortunately for us it turns out to be true:

Theorem 5.3.1 [In the system ZF + $(V = L)$] Every set can be well-ordered. □

I shall not prove this theorem, but I can give a brief indication of how the proof goes. The idea is to prove, by induction, that each set L_α can be well-ordered. (Since, in constructible set theory, each set is a subset of some L_α , this clearly suffices.) We do this as follows. Clearly, L_0 can be well-ordered. And if α is a limit ordinal, and each L_β can be well-ordered for $\beta < \alpha$, then $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$ can be well-ordered by combining the well-orderings of the L_β, $\beta < \alpha$. (This is only a sketch, remember.)

Now suppose L_α can be well-ordered. It is easy to define a well-ordering of the formulas of LAST that have one free variable. Thus, using the well-ordering of L_α, we can define a well-ordering of all these formulas of LAST, coupled with the interpretations of the names in the formulas as elements of L_α. But this, in effect, provides us with a well-ordering of $L_{\alpha+1}$.

In view of Theorem 5.3.1, we can in fact regard constructible set theory as an extension not just of ZF but of ZFC, i.e. full Zermelo–Fraenkel set theory. And we now have the added bonus that there is no 'question' about whether the assumption of AC is justifiable: it is provable from the other axioms.

At present, it is as a possible extension of ZFC that constructible set theory is usually regarded. For most applications of set theory, it is not

necessary to define precisely the concept of a set, and the Zermelo–Fraenkel picture of the universe suffices. So why assume more? Hence we take ZFC as the basic set theory for mathematics.

But the ZFC axioms leave some questions in mathematics unresolved. To answer these questions we need to be more precise as to what a set really is. Whether or not we regard constructible set theory as 'more natural' than the Zermelo–Fraenkel system (and some mathematicians do), if we are subsequently able to solve the problem in constructible set theory, then the effect is that by assuming an additional axiom the problem can be solved. In this case, what the Axiom of Constructibility amounts to is fixing a precise definition of the set concept.

If you regard constructible set theory as a reasonable theory of sets, any result proved in it will be simply a 'theorem'. If, on the other hand, you do not regard constructible set theory in this way, its results will just be 'theorems based on an additional assumption'. (Some people continue to regard AC in this manner as well.)

Now, regardless of the manner in which you view constructible set theory, it is worth noting whenever the notion of constructibility is needed for a result. Since ZFC is taken as basic, we never mention the use of the ZFC axioms. (Except that we sometimes mention that AC is necessary for a result.) Consequently, when we prove a result in the system ZFC + $(V = L)$, it suffices to prefix the result with the statement 'Assume V = L.' What this tells the reader is that the theorem concerned is to be proved within the framework of constructible set theory, and not just using the Zermelo–Fraenkel axioms.

5.4 The Consistency of $V = L$

I have indicated earlier (see Section 2.5) that, in a theory of sets, one cannot ever hope to prove, within that system, the consistency of the theory. Thus, just as we cannot prove within the system ZFC that ZFC is a consistent theory, so too are we unable to prove within the system ZFC + $(V = L)$ that this system is consistent. Thus, if we take constructible set theory as our basic set theory, we must simply assume that, as a formalization of our intuitions concerning sets, it is a consistent system. But as we have just noted, we can regard constructible set theory as an extension of Zermelo–Fraenkel set theory, obtained by adding an extra axiom, the axiom of constructibility. Viewed in this light, constructible set theory could be said to be somewhat more 'suspect' than ZFC with regards to consistency, on the grounds that the more axioms we have, the more chance there is that there will be an internal inconsistency. Were it true, such an allegation

could be used as an argument against constructible set theory. But in fact there is no such danger, by virtue of the following theorem of Gödel.

Theorem 5.4.1 If ZF is a consistent theory, so too is ZF + $(V = L)$. □

A rigorous proof of this theorem is beyond the scope of this book. Intuitively the idea is as follows. In order to prove a system of axioms is consistent, what one usually does is exhibit a 'model' for that system. Starting with a model of ZF (which exists as a consequence of the assumption that ZF is consistent), one can carry out the construction of the 'constructible universe' within that model. This miniature 'constructible universe' turns out to constitute a model of ZFC + $(V = L)$. (Incidentally, the proof of Theorem 5.4.1 itself takes place in a very simple fragment of ZF.)

Since AC is a theorem of the theory ZFC + $(V = L)$, Theorem 5.4.1 at once implies the corollary:

Corollary 5.4.2 If ZF is a consistent theory, so too is ZFC. □

5.5 Use of the Axiom of Constructibility

One of the simplest consequences of the axiom of constructibility is the solution to the Continuum Problem.

Theorem 5.5.1 Assume $V = L$. Then GCH holds. □

Unfortunately, even a sketch of the proof is beyond the scope of this book. This is not because the proof is particularly complex. The difficulty lies in the fact that it requires a reasonable knowledge of techniques of mathematical logic. This is to be expected with proofs that make an essential use of the Axiom of Constructibility. The fact that a result is not provable in ZFC alone already means that a detailed analysis of the notion of sets is required for its solution. And such an investigation is, of course, a matter of mathematical logic. Now, since mathematical logic is a well-defined mathematical discipline, the proofs within this field resemble proofs in any area of mathematics; they do not stand out as unusual in any way. But to follow such a proof naturally requires a degree of familiarity with the field.

For instance, to prove that, under the assumption of $V = L$, $2^{\aleph_0} = \aleph_1$, one demonstrates that, although new subsets of ω keep appearing as

we proceed up the constructible hierarchy, through $L_{\omega+1}, L_{\omega+2}, \ldots$, this process terminates by stage ω_1, so that

$$\mathcal{P}(\omega) \subseteq L_{\omega_1}.$$

Since $|L_{\omega_1}| = \aleph_1$, this implies at once that $2^{\aleph_0} = \aleph_1$. However, to prove that the process of new sets of integers appearing stops at stage ω_1 requires a fairly deep analysis of the constructible hierarchy and the way it grows.

The fact that proofs involving $V = L$ involve a good knowledge of mathematical logic means, of course, that most working mathematicians are in general unable to work in constructible set theory. But this is not always the case. Set theorists have obtained various principles of combinatorial set theory within the system ZFC + $(V = L)$, and for many applications these consequences of $V = L$ are all that is required.

For instance, one of the most common combinatorial consequences of $V = L$ is the following, known as $\diamondsuit$.

> There is a sequence $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ such that for each $\alpha < \omega_1$, $S_\alpha \subseteq \alpha$, and whenever $X \subseteq \omega_1$, then for some infinite ordinal $\alpha \in \omega_1$, $X \cap \alpha = S_\alpha$.

As I said, $\diamondsuit$ is a consequence of $V = L$. And $\diamondsuit$ clearly implies CH. (CH does not, however, imply $\diamondsuit$.) Many results in set theory and topology can be proved by a fairly straightforward argument that makes use of the principle $\diamondsuit$. The mathematical logic involved in constructibility lies in the proof of $\diamondsuit$, not its application. To apply $\diamondsuit$ one needs to know nothing of mathematical logic.

For example, assuming $\diamondsuit$, it is quite a straightforward matter to obtain a negative answer to the Souslin Problem, stated in Section 5.1. (The Whitehead Problem requires another, rather similar, set-theoretic principle, but again the argument *from* this principle needs no logic.)

And of course, any result proved using GCH is automatically a theorem of constructible set theory, though such proofs usually do not involve any logic. For any further details on the usage of $V = L$, I refer the reader to my monograph [4].

6

Independence Proofs in Set Theory

6.1 Some Undecidable Statements

The following statements are known to be undecidable in the system ZFC. (Though they are all decidable in constructible set theory, by the way.)

(1) The Whitehead Problem. (See Chapter 5 for a statement of this problem.)

(2) The Souslin Problem. (Ditto.)

(3) Borel's Conjecture. Let $X \subseteq \mathbb{R}$, and suppose that, whenever $\{\epsilon_n\}$ is a sequence of positive reals, there is a sequence $\{I_n\}$ of open intervals such that $\text{length}(I_n) < \epsilon_n$ for each n and $X \subseteq \bigcup_{n=1}^{\infty} I_n$. Then X is countable.

(4) The union of fewer than $2_0^{\aleph}$ many sets of reals of (Lebesgue) measure zero has measure zero.

(5) The Continuum Hypothesis. If $X \subseteq \mathbb{R}$ is not equinumerous with $\mathbb{R}$, then X is countable.

(6) There is a well-ordering of $\mathbb{R}$ that is definable in analysis.

6.2 The Idea of a Boolean-Valued Universe

I shall attempt to motivate a method by which one can prove, in ZFC, that statements such as those listed above are undecidable in ZFC. To do this, I commence with a re-examination of the Zermelo hierarchy. Recall the basic definition:

$$\begin{aligned} V_0 &= \emptyset, \\ V_{\alpha+1} &= \mathcal{P}(V_\alpha), \\ V_\lambda &= \bigcup_{\alpha<\lambda} V_\alpha, \quad \text{if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

Suppose now that we decide to develop our set theory using not sets themselves but rather characteristic functions of sets. Consider the following definition

$$\begin{aligned} V_0^F &= \emptyset, \\ V_{\alpha+1}^F &= {}^{(V_\alpha^F)}2, \\ V_\lambda^F &= \bigcup_{\alpha<\lambda} V_\alpha^F, \quad \text{if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

In passing from V_α^F to $V_{\alpha+1}^F$, we take not $\mathcal{P}(V_\alpha^F)$ but the set of all the characteristic functions of the members of $\mathcal{P}(V_\alpha^F)$. In essence, this will give us a functional equivalent of the Zermelo hierarchy. The correspondence is not quite trivial, of course, because there will be many different functions in the V_α^F-hierarchy that correspond to each set in the V_α-hierarchy: for instance, if $x \subseteq V_\alpha^F$ and $f : (V_\alpha^F) \to 2$ is the characteristic function of x, then $f' \in (V_{\alpha+2}^F)$ also corresponds to x, where

$$f' = f \cup \{(a, 0) \mid a \in V_{\alpha+1}^F - (V_\alpha^F)\}.$$

The point is that, when functions are involved, different domains mean different functions, even though the different functions may be 'essentially' the same. But, discounting this minor technical problem, the two hierarchies $\langle V_\alpha \mid \alpha \in \mathrm{On}\rangle$ and $\langle V_\alpha^F \mid \alpha \in \mathrm{On}\rangle$ are essentially equivalent.

Setting

$$V^F = \bigcup_\alpha V_\alpha^F$$

we obtain a universe of characteristic functions of 'sets'. (I write 'sets' in quotation marks here because, of course, each function in V^F is itself defined on functions and not on sets. So in V^F there is really only one kind of entity: a characteristic function.)

It is intuitively clear that anything we can do with V we could do with V^F. In other words, we could carry out our entire development of set theory using the members of V^F instead of the pure sets of V. (Few people would regard this as a worthy exercise, of course, and I am not for one moment suggesting that it should be done. But it is certainly possible.)

Now let us ask ourselves what the significance is of the fact that I have only allowed functions mapping into the set 2 in the above? Well, the elements 1 and 0 of the set 2 correspond to the two truth values, T and F ('true' and 'false', respectively). If $f \in V^F$ and $f(x) = 1$, then the statement '$x \in f$' (interpreted in V^F) is true, i.e. has the truth value T; and if $f(y) = 0$, the statement '$y \in f$' (interpreted in V^F) has the truth value F. Hence, the restriction to functions mapping into 2 corresponds to the fact that our logic admits only two possibilities, true or false.

But why not allow more possibilities? Certainly we are all aware that in real life there are more than just two truth values, as the following anecdote of P. Vopenka illustrates. According to Charles Darwin, there is a finite toset, S, whose first element is a monkey and whose last element is you, dear reader. Let $M(x)$ denote the statement 'x is a monkey'. Let x_0 be the first member of S, x_1 the second, and so on, with you being x_n. By assumption, $M(x_0)$. In two valued logic, we clearly have

$$M(x_m) \to M(x_{m+1})$$

for any m. (The offspring of a monkey is a monkey.) Hence, by a simple induction we conclude that $M(x_n)$. Assuming you agree that we have now arrived at a contradiction, let us see what has gone wrong. Well, nothing really, except that it is not valid to use two valid logic here. Although $M(x_0)$ holds and $M(x_n)$ fails, in between there is a gradual change in truth values, with $M(x_m)$ becoming 'less true' as m increases.

Of course, the above anecdote does not in itself constitute a sufficient reason for adopting a many-valued logic in mathematics. But it does illustrate that such a concept is not entirely devoid of meaning. And it turns out that this is the idea that we can utilize to obtain undecidability results.

So what sort of sets can we replace 2 by and still obtain a 'universe of sets' that has some useful properties? What is so special about the set $\{0, 1\}$? The answer is that the only critical feature is that this set does correspond to truth values. For instance, in V^F, if $f : V_\alpha^F \to 2$, and if $g : V_\alpha^F \to 2$ is defined by $g = 1 - f$, then we have, for any $x \in V_\alpha^F$, $f(x) = 1$ if and only if $g(x) = O$. And if we set $h = \min(f, g)$, then $h(x) = 1$ if and only if $f(x) = 1$ and $g(x) = 1$. And so on.

In summary, if our functional hierarchy is to provide us with a type of 'set theory', then the values of the functions must behave like truth values. Well, what kinds of sets do behave like truth values? The answer is well known: boolean algebras! (See Problem 1 in Chapter 1 for relevant definitions.) Providing $\mathcal{B}$ is a boolean algebra, we obtain a reasonable 'universe of sets' by means of the following definition:

$$V_0^{\mathcal{B}} = \emptyset,$$

$$\begin{aligned} V_{\alpha+1}^{\mathcal{B}} &= \{f \mid f : V_\alpha^{\mathcal{B}} \to \mathcal{B}, \\ V_\lambda^{\mathcal{B}} &= \bigcup_{\alpha<\lambda} V_\alpha^{\mathcal{B}}, \text{ if } \lambda \text{ is a limit ordinal}, \\ V^{\mathcal{B}} &= \bigcup_\alpha V_\alpha^{\mathcal{B}}. \end{aligned}$$

An element of $V^{\mathcal{B}}$ is called a *boolean-valued set*, or, more precisely, a $\mathcal{B}$-valued set. $V^{\mathcal{B}}$ is a *boolean-valued universe*, or more precisely the B-valued universe. If $x \in V^{\mathcal{B}}$ and $f \in V_{\alpha+1}^{\mathcal{B}}$, $f(x)$ (which is an element of $\mathcal{B}$) is a measure of the truth of the statement '$x \in f$' in terms of $V^{\mathcal{B}}$. If $f(x) = 0$, then x is certainly not a member of f; if $f(x) = 1$, then x is a member of f; and if $0 < f(x) < 1$, then x is partly not in f and partly in f, with (x) telling us 'to what extent' x is a member of f.

6.3 The Boolean-Valued Universe

I shall now formalize the discussions of Section 6.2. For technical reasons I shall set things up in a slightly different manner. For a start, I shall not use an arbitrary boolean algebra $\mathcal{B}$ but rather a complete boolean algebra. (A boolean algebra is *complete* if and only if every subset, X, of $\mathcal{B}$ has a least upper bound, denoted by $\bigvee X$, and a greatest lower bound, denoted by $\bigwedge X$.) Second, I shall not demand that the $\mathcal{B}$-valued characteristic functions are defined on some $V_\alpha^{\mathcal{B}}$; they can have arbitrary domains. (Since there will in any case be a great deal of duplication, with many members of $V^{\mathcal{B}}$ denoting the same boolean 'set', owing to differing domains, this causes no extra hardship and simplifies matters a little.)

So fix now some complete boolean algebra $\mathcal{B}$. By recursion on ordinals, we define the hierarchy of *$\mathcal{B}$-valued sets* as follows:

$$V_\alpha^{\mathcal{B}} = \{u \mid u \text{ is a function } \wedge \operatorname{ran}(u) \subseteq \mathcal{B} \wedge (\exists \beta < \alpha)\operatorname{dom}(u) \subseteq V_\beta^{\mathcal{B}})\}.$$

This formulation allows for the cases $\alpha = 0, \alpha$ is a successor ordinal, and α is a limit ordinal, all in one go. It is easily seen to be equivalent to taking

$$\begin{aligned} V_0^{\mathcal{B}} &= \emptyset, \\ V_{\alpha+1}^{\mathcal{B}} &= \{u \mid \operatorname{dom}(u) \subseteq V_\alpha^{\mathcal{B}} \wedge \operatorname{ran}(u) \subseteq \mathcal{B}\}, \\ V_\lambda^{\mathcal{B}} &= \bigcup_{\alpha<\lambda} V_\alpha^{\mathcal{B}}, \text{ if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

Clearly,

$$\alpha < \beta \to V_\alpha^{\mathcal{B}} \subseteq V_\beta^{\mathcal{B}}.$$

We set

$$V^{\mathcal{B}} = \bigcup_{\alpha} V_{\alpha}^{\mathcal{B}}.$$

$V^{\mathcal{B}}$ is the *$\mathcal{B}$-valued universe.* The elements of $V^{\mathcal{B}}$ are called *$\mathcal{B}$-valued sets.* Thus a $\mathcal{B}$-valued set is a $\mathcal{B}$-valued function defined on $\mathcal{B}$-valued sets.

Having defined the $\mathcal{B}$-valued universe, the next task is to assign $\mathcal{B}$-truth values to the various set-theoretical assertions we can make about the members of $V^{\mathcal{B}}$.

For each sentence ϕ of LAST, providing we know which 'sets' in $V^{\mathcal{B}}$ the names in ϕ refer to, we should be able to assign to ϕ a unique 'truth value', which measures the degree to which ϕ is true. We shall denote this truth value by

$$||\phi||.$$

$||\phi||$ is a member of $\mathcal{B}$. If $||\phi|| = 0$, ϕ will be false in $V^{\mathcal{B}}$. If $||\phi|| = 1$, ϕ will be true in $V^{\mathcal{B}}$. In all other cases, ϕ will be partly false and partly true in $V^{\mathcal{B}}$.

The definition of $||\phi||$ is obtained by unravelling the construction of ϕ. We consider first the case where ϕ is an 'atomic' sentence of the form $w_i \in w_j$ or $w_i = w_j$. To avoid talking of 'names' and their 'meanings', I shall henceforth just use x, y, z, u, v, w, etc., to denote both names and their meanings. This accords with common usage both in and out of logic.

If $u, v \in V^{\mathcal{B}}$, how should we define $||u \in v||$ and $||u = v||$? Well, intuitively, $v(u)$ measures the degree to which u is an element of v, so why not take as our definition

$$||u \in v|| = v(u)?$$

Well, because this only works when $u \in \mathrm{dom}(v)$, whereas we want $||u \in v||$ to have a meaning for all $u, v \in V^{\mathcal{B}}$. A similar difficulty arises with $||u = v||$, which needs to be defined even if $\mathrm{dom}(u) \neq \mathrm{dom}(v)$. To overcome this difficulty, we recall the following extensionality principles:

$$u \in v \quad \leftrightarrow \quad (\exists y \in v)(u = y),$$

$$u = v \quad \leftrightarrow \quad (\forall x \in u)(x \in v) \wedge (\forall y \in v)(y \in u).$$

Accordingly, we make the definitions:

$$||u \in v|| \;=\; \bigvee_{y \in \mathrm{dom}(u)} [v(y) \wedge ||u = y||],$$

$$||u = v|| \;=\; \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow ||x \in v||] \wedge \bigwedge_{y \in \mathrm{dom}(v)} [v(y) \Rightarrow ||y \in u||],$$

where, for $b, c \in \mathcal{B}$, the element $b \Rightarrow c$ of $\mathcal{B}$ is defined by

$$b \Rightarrow c = -b \vee c.$$

Thus the definition is by a 'double recursion'. We define $||u \in v||$ and $||u = v||$ simultaneously. In order to calculate $||u \in v||$, we need to know all the values of $||u = y||$ for $y \in \mathrm{dom}(v)$. And in order to calculate $||u = v||$, we need all the values of $||x \in v||$ for $x \in \mathrm{dom}(u)$ and all the values of $||y \in u||$ for $y \in \mathrm{dom}(v)$. It can be shown that this does provide us with a sound recursive definition.

By recalling the connection between the boolean operations $\wedge, \vee, -$ and their logical counterparts $\wedge, \vee, \neg$ (respectively), the various parts of the definition make sense. And it is easy to see why '$\bigvee_{y \in \mathrm{dom}(v)}$' corresponds to '$(\exists y \in \mathrm{dom}(v))$'and '$\bigwedge_{y \in \mathrm{dom}(u)}$' to '$(\forall y \in \mathrm{dom}(u))$'.

However, the definition will no doubt still seem a little odd. Unfortunately, to try to clarify matters further would involve so great a digression that I shall leave the matter with the remark that this is the best definition that does all we want of it.

The assignments of $\mathcal{B}$-truth values to compound sentences is now quite straightforward. The conditions for the recursion are

$$\begin{aligned}
||\phi \vee \psi|| &= ||\phi|| \vee ||\psi||; \\
||\phi \wedge \psi|| &= ||\phi|| \wedge ||\psi||; \\
||\neg\phi|| &= -||\phi||; \\
||\exists u \phi(u)|| &= \bigvee_{u \in V^{\mathcal{B}}} ||\phi(u)||; \\
||\forall u \phi(u)|| &= \bigwedge_{u \in V^{\mathcal{B}}} ||\phi(u)||.
\end{aligned}$$

An immediate consequence of the above definitions is

$$||\phi \rightarrow \psi|| = ||\phi|| \Rightarrow ||\psi||.$$

Notice the duplication in notation in the above definition, with the symbols $\vee$ and $\wedge$ being used in two different ways, to denote both logical connectives and boolean operations. By these very definitions, there is no harm in this clash, and indeed it helps to highlight the reason why we need to have a boolean algebra for our set of 'truth values'.

The definitions of $||u \in v||$ and $||u = v||$, and the two clauses dealing with quantification, indicate why the boolean algebra should be complete.

6.4 $V^{\mathcal{B}}$ and V

Now, all of our development so far has taken place within the framework of ZFC. (After all, $V^{\mathcal{B}}$ is just the result of a simple set-theoretic construction by recursion.) Hence $V^{\mathcal{B}}$ a well-defined class within V:

$$V^{\mathcal{B}} \subseteq V.$$

But, in a sense, $V^{\mathcal{B}}$ is an 'extension' of V. For consider the particular boolean-valued universe V^2 where 2 is the two-element algebra $\{0,1\}$. Clearly V^2 should be 'isomorphic' to V in some sense. In fact, if we define a 'relation' $\sim$ on V^2 by

$$u \sim v \text{ if and only if } ||u = v|| = 1$$

then $\sim$ is an 'equivalence relation', and if we 'factor out' V^2 by this 'relation' we do obtain an isomorph to V. (The quotation marks are necessary because we are dealing with proper classes here, in a manner which, strictly, is not permitted within ZFC. An equivalent argument can be formulated within the ZFC framework, but it is a little more complicated.)

Now, since 2 is a complete subalgebra of $\mathcal{B}$, it is easily seen that:

(i) $V^2 \subseteq V^{\mathcal{B}}$;

(ii) if $u, v \in V^2$, then

$$\begin{aligned} ||u \in v||^2 &= ||u \in v||^{\mathcal{B}} \\ ||u = v||^2 &= ||u = v||^{\mathcal{B}}. \end{aligned}$$

Hence V^2 is an isomorphic copy of V sitting inside $V^{\mathcal{B}}$. This is the sense in which $V^{\mathcal{B}}$ 'extends' V.

In fact, there is a canonical embedding of V into $V^{\mathcal{B}}$ that is often useful. By recursion, we define $\widehat{\ } : V \to V^{\mathcal{B}}$ by

$$\begin{aligned} \mathrm{dom}(\widehat{x}) &= \{\widehat{y} \mid y \in x\} \\ \widehat{x}(a) &= 1, \text{ for all } a \in \mathrm{dom}(\widehat{x}). \end{aligned}$$

Thus, $\widehat{x} = \{(\widehat{y}, 1) \mid y \in x\}$.

Then, for $x, y \in V$,

$$\begin{aligned} x = y \quad &\text{if and only if} \quad ||\widehat{x} = \widehat{y}||^{\mathcal{B}} = 1, \\ x \in y \quad &\text{if and only if} \quad ||\widehat{x} = \widehat{y}||^{\mathcal{B}} = 1. \end{aligned}$$

There is one great source of difficulty for the beginner concerning the use of the symbols $=, \in$, etc. On the one hand, we ourselves carry out all our arguments in regular ZFC set theory, where a set is a set! On the other hand, some of our arguments involve the internal properties of the universe in $V^{\mathcal{B}}$, where all 'sets' are $\mathcal{B}$-valued sets. Let me stress that, as mathematicians, we continue to use regular set theory and logic. Within this framework, we discuss boolean-valued sets and boolean-valued logic. Unfortunately, only experience can really overcome the problems that arise from this situation.

6.5 Boolean-Valued Sets and Independence Proofs

We shall wish to consider $\mathcal{B}$-valued arguments within the universe $V^{\mathcal{B}}$. Accordingly, we need to know that the usual rules of logic are valid in the $\mathcal{B}$-valued case. That they are is quite easily proved, but we content ourselves here with a simple statement of the result.

Lemma 6.5.1 (i) All the rules and axioms of propositional logic are $\mathcal{B}$-valid.

(ii) All the rules and axioms of first-order predicate logic are $\mathcal{B}$-valid.

(iii) All the axioms of equality are $\mathcal{B}$-valid. □

Let me remark that (i) was known to Boole, and is an immediate consequence of the definition of a boolean algebra; (ii) was proved by Sikorski; neither (i) nor (ii) has anything particular to do with $V^{\mathcal{B}}$; (iii) depends upon the definition of $||u = v||$ with respect to $V^{\mathcal{B}}$.

The following theorem, which is proved within ZFC set theory (as are all our theorems about $V^{\mathcal{B}}$), is nontrivial, and is the key to our method for obtaining independence results.

Theorem 6.5.2 If ϕ is an axiom of ZFC, then $||\phi|| = 1$. □

As a corollary to Lemma 6.5.1 and Theorem 6.5.2, we have at once:

Theorem 6.5.3 If ϕ is a theorem of ZFC, then $||\phi|| = 1$. □

Suppose now that we wish to prove that a certain statement ϕ is undecidable in the theory ZFC. Here is one way we might try to do this. The algebraic structure of a complete boolean algebra $\mathcal{B}$ has a considerable effect upon the structure of the universe $V^{\mathcal{B}}$. (This is a fact that I both know and appreciate. Now you know it; unfortunately, space does not permit me to help you appreciate it.) Suppose that by examination of the statement ϕ we are able to find (or construct) an algebra $\mathcal{B}$ such that, when interpreted in $V^{\mathcal{B}}$ we get

$$0 < ||\phi|| < 1.$$

By Theorem 6.5.3, it will follow that ϕ is not a theorem of ZFC. But since $||\phi|| > 0$,

$$||\neg\phi|| = -||\phi|| < 1,$$

so $\neg\phi$ is also not a theorem of ZFC. Hence ϕ is shown to be undecidable in ZFC.

This, briefly, outlines the most common method for proving undecidability results for ZFC. Since $V^{\mathcal{B}}$ is a sort of boolean-valued 'model' of the system ZFC, we often refer to the method as the method of 'boolean-valued models of set theory'. The method has a model-theoretic analogue where there is no explicit use of boolean-valued logic, and in this form it is then referred to as the method of 'forcing'. Once the basic theory is known, any specific independence proof thus takes the following form:

(i) Examine the statement ϕ whose independence is suspected.

(ii) Find or construct an algebra that might do the trick.

(iii) Calculate $||\phi||$ in $V^{\mathcal{B}}$ and see that it is neither 0 nor 1.

Each of steps (i) and (ii) can involve an enormous amount of effort. Very often, one is forced to adopt a different procedure:

(i) Examine the statement ϕ whose independence is suspected.

(ii) Find two algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ 'related' to ϕ.

(iii) Show that $||\phi||^{\mathcal{B}_1} < 1$ and $||\neg\phi||^{\mathcal{B}_2} < 1$.

It is celar that this also suffices to establish the undecidability of ϕ.

Thus, although one is ultimately proving that some statement is unprovable, what is actually involved in an independence proof is just a regular *proof* in classical set theory.

6.6 The Nonprovability of the CH

I should warn the reader that this section assumes a considerable acquaintance with boolean algebras, together with a little measure theory. Moreover, even with the necessary prerequisites, you should not expect to gain more than a general impression of the proof. I do not strive for completeness in my account, and several tricky points are glossed over without mention. For a rigorous account you should consult, for example, Bell's book [3].

As an example of how boolean-valued techniques are applied, I sketch a proof of the fact that the CH is not provable in ZFC. As well as being the first independence proof, it is perhaps also the easiest of all.

I commence by defining the boolean algebra. Now, for most independence proofs there is no 'standard' algebra that suffices. One has to use one's 'appreciation' of the statement whose undecidability is to be shown, in order to construct a very special algebra that will work. (Such constructions can be very delicate and occasionally stretch into fifty pages or so.) But for CH a 'standard' algebra suffices.

Let $X = 2^{\omega \times \omega_2}$, a generalized Cantor space. That is, let 2 have the discrete topology and give X the product topology induced from 2. Let $\mathbb{B}$ be the field of all Borel subsets of X. $\mathbb{B}$ is a σ-field, of course. Now make X into a measure space by taking the usual measure on 2 and forming the product measure on X. Let Δ be the σ-ideal of all Borel sets of measure zero. Let

$$\mathcal{B} = \mathbb{B}/\Delta,$$

the quotient algebra. It can be shown that $\mathcal{B}$ is complete. (In fact, the measure on X induces a measure on $\mathcal{B}$, so completeness is almost a triviality.) The nonprovability of CH follows from the fact that

$$||\, 2^{\aleph_0} > \aleph_1 \,||^{\mathcal{B}} > 0.$$

(Hence $||\mathrm{CH}||^{\mathcal{B}} < 1$.) I sketch a proof of this fact.

By definition, ω_1 is the first uncountable ordinal. Hence, by isomorphism,

$$||\, \widehat{\omega_1} \text{ is the first uncountable ordinal} \,||^2 = 1.$$

But $V^{\mathcal{B}}$ contains many more 'sets' than does V^2. And perhaps among these extra 'sets' is one that is (in $V^{\mathcal{B}}$ terms) a map of $\widehat{\omega}$ onto $\widehat{\omega_1}$. Thus, it is possible that

$$||\, \widehat{\omega_1} \text{ is countable} \,||^{\mathcal{B}} = 1.$$

Or to put it another way, we may have

$$||\, \widehat{\omega_1} < \omega_1 \,||^{\mathcal{B}} = 1$$

where ω_1 without a hat means the ω_1 of $V^{\mathcal{B}}$, the first uncountable ordinal in the universe $V^{\mathcal{B}}$. This is not the same as $\widehat{\omega_1}$, this just being the image of the first uncountable ordinal in V under the embedding $\widehat{\ } : V \to V^{\mathcal{B}}$.

Indeed, for many boolean algebras, the above situation does arise. But in the present situation it does not. This follows from the fact that, being a measure algebra, $\mathcal{B}$ satisfies the countable chain condition.

Lemma 6.6.1 $||\widehat{\omega_1} = \omega_1||^{\mathcal{B}} = 1$.

Proof: Suppose not. Since $V^{\mathcal{B}}$ is 'bigger' than V^2, it cannot happen that $||\omega_1 < \widehat{\omega_1}||^{\mathcal{B}} > 0$. Thus $||\widehat{\omega_1} < \omega_1||^{\mathcal{B}} > 0$. Hence

$$(*) \qquad ||(\exists f)(f : \omega \xrightarrow{\text{onto}} \widehat{\omega_1})|| > 0.$$

Now, by induction on $n \in \omega$,

$$||\,\widehat{n} \text{ is the } n\text{'th natural number}\,|| = 1,$$

so we clearly have

$$||\,\widehat{\omega} = \omega|| = 1.$$

Hence $(*)$ can be written

$$||(\exists f)(f : \widehat{\omega} \xrightarrow{\text{onto}} \widehat{\omega_1})|| > 0.$$

So, for some $f \in V^{\mathcal{B}}$,

$$b = ||f : \widehat{\omega} \xrightarrow{\text{onto}} \widehat{\omega_1}|| > 0.$$

Then,

$$b \leq ||(\forall \alpha \in \widehat{\omega_1})(\exists n \in \widehat{\omega})(f(n) = \alpha)||,$$

so

$$b \leq \bigwedge_{\alpha \in \omega_1} \bigvee_{n \in \omega} ||f(\widehat{n}) = \widehat{\alpha}||.$$

So, for each $\alpha \in \omega_1$, we can pick an $n(\alpha) \in \omega$ such that

$$b \wedge ||f(\widehat{n}) = \widehat{\alpha}|| > 0.$$

Since ω_1 is uncountable, we can find an uncountable set $X \subseteq \omega_1$ such that $n(\alpha) = n$ (say) for all $\alpha \in X$. For each $\alpha \in X$, set

$$b_\alpha = b \wedge ||f(\widehat{n}) = \widehat{\alpha}||.$$

Thus, $b_\alpha > 0$. But, if $\alpha \neq \beta$ are elements of X, then

$$b_\alpha \wedge b_\beta = b \wedge ||f(\widehat{n}) = \widehat{\alpha}|| \wedge ||f(\widehat{n}) = \widehat{\beta}|| \leq ||\widehat{\alpha} = \widehat{\beta}|| = 0.$$

Hence $\{b_\alpha \mid \alpha \in X\}$ is pairwise disjoint, contrary to the countable chain condition for $\mathcal{B}$. □

The above proof should indicate how it can be that the set theory of $V^{\mathcal{B}}$ is effected by the algebraic properties of $\mathcal{B}$. A similar proof now yields (using Lemma 6.6.1):

Lemma 6.6.2 $||\widehat{\omega_2} = \omega_2||^{\mathcal{B}} = 1.$ □

For $\alpha < \omega_2$ now, define functions $u_\alpha : \mathrm{dom}(\widehat{\omega}) \to \mathcal{B}$ by

$$u_\alpha(\widehat{n}) = \{p \in X \mid p(n, \alpha) = 1\}/\Delta.$$

Clearly,

$$||u_\alpha \subseteq \widehat{\omega}|| = 1.$$

Moreover, a straightforward calculation shows that

$$\begin{aligned} ||u_\alpha = u_\beta|| &= \bigwedge_{n\in\omega} [(u_\alpha(\widehat{n}) \Rightarrow u_\beta(\widehat{n})) \wedge (u_\beta(\widehat{n}) \Rightarrow u_\alpha(\widehat{n}))] \\ &= \{p \in X \mid (\forall n \in \omega)(p(n, \alpha) = p(n, \beta))\}/\Delta. \end{aligned}$$

This calculation, though it is indeed quite straightforward as such arguments go, does require a considerable facility with the definitions of boolean-valued truth, so you are not urged to try to reconstruct it, unless you really feel you need to.

Suppose now that $\alpha < \beta < \omega_2$. Set

$$S = \{p \in X \mid (\forall n \in \omega)(p(n, \alpha) = p(n, \beta))\}.$$

Let $n_1, \ldots, n_k \in \omega$, and set

$$\{p_1, \ldots p_{2^k}\} = {}^{\{n_1,\ldots,n_k\}}2.$$

For $l = 1, \ldots, 2^k$, let

$$\begin{aligned} U_l = \{p \in X \mid p(n_1, \alpha) = (p_1, \beta) = p_l(n_1) \wedge \ \ldots \\ \ldots \wedge p(n_k, \alpha) = p(n_k, \beta) = p_l(n_k)\}. \end{aligned}$$

Clearly,

$$S \subseteq U_1 \cup \ldots \cup U_{2^k}.$$

But, if μ denotes the measure on X, we have

$$\mu(U_l) = (1/2)^{2k}.$$

Hence,

$$\mu(S) \leq 2^k.(1/2)^{2k} = (1/2)^k.$$

So, as k is arbitrary,

$$\mu(S) = 0.$$

Thus,

$$||u_\alpha = u_\beta|| = S/\Delta = 0.$$

Hence, in $V^{\mathcal{B}}$, the sets u_α, $\alpha < \omega_2$, are distinct subsets of ω. But $\widehat{\omega_2}$ is the ω_2 of $V^{\mathcal{B}}$ (by Lemma 6.6.2). It follows that

$$|| \, |\mathcal{P}(\omega)| \geq \aleph_2||^{\mathcal{B}} = 1.$$

This completes the proof that CH is not provable in ZFC. (At least, it completes my sketch of the proof. To fill in all the details entails a considerable amount of work. The interested reader should consult Bell's book [3] for more details.)

7

Non-Well-Founded Set Theory

The approach to set theory that has motivated and dominated the study presented so far in this book has essentially been one of *synthesis*: from an initial set of axioms, we build a framework of sets that can be used to provide a foundation for all of mathematics. By starting with pure sets provided by the Zermelo–Fraenkel axioms, and progressively adding more and more structure, we may obtain all of the usual structures of mathematics. And then, of course, we may make use of those mathematical structures to model various aspects of the world we live in. In this way, set theory may be used to provide ways to model 'mathematical' aspects of our world.

But there is an alternative way to approach set theory, namely in an *analytic* fashion, where we start with all of the various 'mathematical' structures we observe in the world and progressively *strip away* structure until all that is left are pure sets.

As you might expect, there is no *a priori* reason that these two approaches will lead to the same theory of sets. Indeed, some very familiar real-world structures give rise to a dramatically different conception of set from the now-familiar Zermelo–Fraenkel notion.

For example, suppose I try to model set-theoretically the items of information in some information-storage device, say this very book. Let $\mathcal{B}$ be the set of all sets explicitly referred to in this book. Clearly, since $\mathcal{B}$ is referred to in this book (I am just now referring to it), we have

$$\mathcal{B} \in \mathcal{B}.$$

More generally, it is not hard to think up examples of 'real world' sets having closed loops of membership:

$$a_1 \in a_2 \in \ \ldots \ \in a_n \in a_1.$$

Such sets are said to be *circular*. With the growing tendency to apply set-theoretic methods in computer and information science, it is getting

steadily harder to avoid having to deal with such sets in a formal and rigorous manner.

Now, in Zermelo–Fraenkel set theory, the Axiom of Foundation explicitly rules out the formation of circular sets or sets having themselves as members. So at the very least, if we are to approach set theory in an *analytic* fashion, in a manner that will, for instance, allow us to capture some of the self-referential structure that arises in information systems, we will have to dispense with this particular axiom. But just how significant a step will this be? Will it, for instance, mean that we shall be working within a framework quite unlike that used in other parts of mathematics?

The answer turns out to be 'no'. Simply dropping the Axiom of Foundation from the axioms of set theory results in practically no change in almost all of present day mathematics (or its applications). The reason is that this axiom is totally irrelevant as far as most applications of set theory are concerned. The kinds of sets that arise in, say, Analysis or Algebra, simply *are*, as a matter of fact, noncircular. No axiom is required to guarantee this. It is really only within set theory itself that the Axiom of Foundation is important.

Thus, in contemplating the introduction of a set theory that violates the Axiom of Foundation, which is what this chapter is all about, we are not starting out along a path that will bring us into conflict with the bulk of current mathematical practice. We shall simply find ourselves using sets of a different nature than those used elsewhere (for different purposes).

Of course, in developing a set theory as a conceptual abstraction from, say, information structures in the world, there may turn out to be other features that *do* conflict with the set theory used elsewhere in mathematics. But as far as is known, this is not the case. Indeed, it is possible to regard the universe of sets described below as an *extension* of the Zermelo–Fraenkel universe, one that enlarges the domain of study to include all those circular sets that the Axiom of Foundation normally excludes from consideration.

In this respect, what we are doing is analogous to the extension procedure that takes you from the real numbers to the complex numbers. New 'numbers' are introduced to enlarge the real number system to a richer structure in which more equations have solutions, etc. No properties of the real numbers are violated by this extension. More things become possible at no cost in terms of existing theory.

So too in our introduction of a 'non-well-founded set theory', as I shall refer to any theory of sets that violates the Axiom of Foundation. Indeed, the analogy with the complex numbers is an even better one. Just as the complex numbers may be *defined* in terms of the real numbers, so too the non-well-founded (or circular) sets of our new theory may be defined in terms of the more familiar, *well-founded* (i.e. noncircular) sets of the

Figure 7.1: Graphical representation of two simple sets.

Zermelo–Fraenkel theory. And just as the 'new' complex number system shares many of the fundamental properties of the 'old' real numbers—for instance, both systems are *fields*—so too the universe of non-well-founded sets will satisfy many of the axioms of the well-founded Zermelo–Fraenkel universe of sets. Indeed, it satisfies *all* axioms except for Foundation.[1]

It should perhaps be pointed out that in the case of an *analytic* approach to set theory, it is quite natural to allow for atomic (i.e. non-set) elements, or *urelements*, entities that may be used in order to construct sets, but which are not themselves analyzed in a set-theoretic fashion. Traditionally, Zermelo–Fraenkel set theory does not allow for the existence of atoms, though it is easy to amend the axioms to do so. I shall denote by ZFCA the theory ZFC amended to allow for atoms.

An excellent illustration of the application of non-well-founded set theory is provided by Barwise and Etchemendy in their book *The Liar* [2], in which they provide a set-theoretic account of the classical Liar Paradox and some other logical paradoxes.

7.1 Set-Membership Diagrams

Consider then, some very simple, circular sets of the kind that might easily arise in a discussion of information storage, say

$$a = \{3, 5\} \quad \text{and} \quad b = \{\text{Zermelo, Fraenkel}\}.$$

We may picture these sets by means of simple diagrams as in Figure 7.1.

The idea in the case of such diagrams is to represent set membership by means of directed line segments. Thus, referring to Figure 7.1, the

[1]Though, as we shall see, the Axiom of Extensionality does not always serve to distinguish non-well-founded sets as it does for well-founded sets, and another axiom will be required in order to overcome this problem.

arrows pointing from the set a to each of the two numbers 3 and 5 indicate that the set a has precisely the two elements 3 and 5, and likewise the arrows pointing from b to the two objects (atoms) 'Zermelo' and 'Fraenkel' represent the fact that the set b consists of precisely these two objects (and is thus a set consisting of two particular people). Thus Figure 7.1 provides an alternative means of indicating the set-theoretic structure of the sets a and b, other than the more familiar notation used above to introduce these sets.

Both notations show what it is that the two sets a and b have in common, as well as the way in which they differ. Any set is, of course, a purely abstract construct. In the case of set a, the elements of this set are themselves also abstract entities. Set b, on the other hand, is an abstract construct built out of two real objects in the world (or rather two objects that at one time did exist in the world). But in both cases, the set-theoretic structure itself is the same: each consists of two objects that are (conceptually) collected together to form a single (abstract) entity. With traditional set notation, this common structure is reflected in the fact that in each case precisely two objects occur between the braces { and }; in Figure 7.1, the obvious isomorphism between the two diagrams indicates the same common structure.

Now, in the case of simple sets like the two above, there seems to be little to choose between the two notations, the traditional and the diagramatic, but when it comes to indicating the hereditary (membership) structure of more complex sets, the diagramatic form can be much easier to understand, allowing as it does for the various membership paths to be traced along the connecting arrows. This is illustrated by Figure 7.2, which gives diagramatic representations of the first four ordinal numbers (under the familiar von Neumann definition used in this book, that takes any ordinal number to be just the set of its predecessors).

Both Figures 7.1 and 7.2 are examples of what are known as *graphs*. The points that occur in a graph, such as the points labeled $a, 3, 5$ in the first graph in Figure 7.1, are generally referred to as *nodes* of the graph, the lines (or arrows) connecting them as *edges*.[2]

In Figure 7.2, the ordinal 0, being the empty set, is depicted by a diagram consisting of a single node with no edges emanating from it. The graph for the ordinal 1, being the singleton set $\{\emptyset\}$, consists of two nodes, the top node depicting the ordinal (set) 1 itself, the node beneath it the single element, $\emptyset$, of that top node. And in the remaining two cases, the top node depicts the ordinal number concerned while the remainder of

[2]Strictly speaking, what we have here are *directed graphs* or *digraphs*, the adjective 'directed' indicating that the edges, being arrows, have a specified direction.

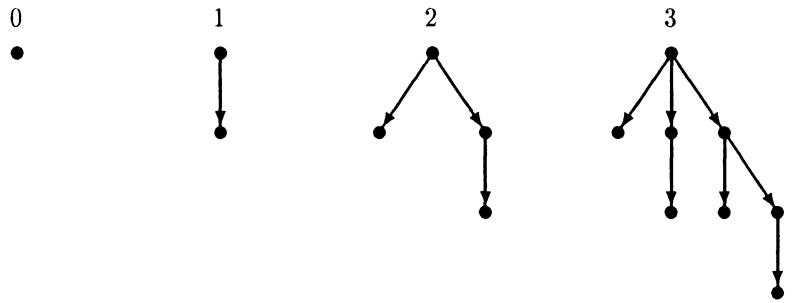

Figure 7.2: Graphical representation of the first four ordinal numbers.

the graph shows the set-theoretic structure of that ordinal number. An instructive exercise is to label each of the nodes in Figure 7.2 with the appropriate von Neumann ordinal.

One thing to notice concerning Figure 7.2 is that there was really no need to label the top nodes in each of the four cases. Since the only set depicted by a node from which no edges (arrows) emanate is the empty set, each of the bottom nodes in the four graphs must represent the empty set, so in each case we may work our way up the various paths through the graph in order to determine the exact nature of the set depicted.

This is quite unlike the situation in Figure 7.1. Here the bottom nodes all denote particular entities, as indicated by the labels attached to those nodes. In the case of the set a, if we regard the elements 3 and 5 as being sets under the von Neumann definition of an ordinal, then of course we may extend this particular graph to one without labels in the obvious way. But for the set b, such a procedure is clearly not possible, and the bottom nodes must be regarded as *atoms* or *atomic nodes* of the graph, depicting entities that either have no set-theoretic structure or whose set-theoretic structure is not pertinent.

In order to avoid confusion, I shall use hollow circles, rather than dots, to indicate atoms in graphs. Thus, the set {Zermelo, 1} will be represented graphically as in Figure 7.3.

If we allow infinite graphs in the case of infinite sets, then it is clear that any set may be represented by a membership graph in this fashion, providing a diagramatic representation of the entire hereditary structure of the set. Indeed, there is an obvious method for producing a graph that

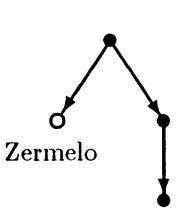

Figure 7.3: The set {Zermelo, 1}, where $1 = \{0\}$.

Figure 7.4: Alternative graphical representations of the ordinals 2 and 3.

depicts a given set.[3] Namely, start with the set concerned as top node, and then enumerate all its elements beneath it, joining the top node to each of these by means of a downward pointing arrow. Then, for each of these nodes in turn enumerate all their members beneath them, and make the appropriate edge-connections. And so on.

Now, a particular set may be represented by more than one graph. For instance, referring back to Figure 7.2, in the graph depicting the ordinal number 2 there are two nodes denoting the ordinal number 0. If we identify these two nodes then we obtain the alternative graphical representation of the ordinal 2 shown on the left of Figure 7.4. Likewise, the graph depicting the ordinal 3 in Figure 7.2 has four nodes that correspond to 0 and two corresponding to the ordinal 1, and identification of the nodes in these two groupings leads to the graph shown on the right in Figure 7.4.

Again, it is an instructive exercise to label each of the nodes in Figure 7.4 with the appropriate ordinal number and to relate these two graphs with the corresponding graphs in Figure 7.2.

By allowing the appearance of loops within graphs it is possible to depict (some) non-well-founded sets by means of finite graphs. Indeed, this

[3]This procedure can only be actually carried out in the case of reasonably small finite graphs, but it is easy to see that it will work 'in principal' for any set.

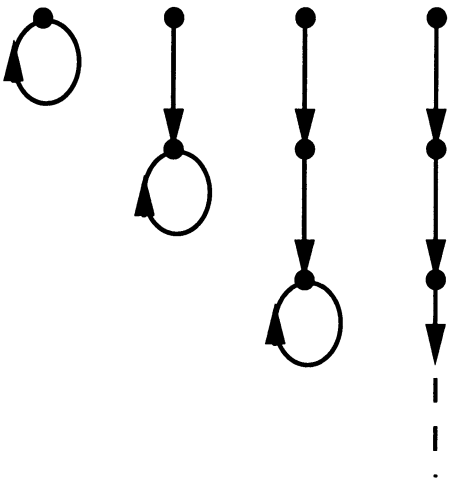

Figure 7.5: Different graphs depicting the set Ω.

is arguably the most appropriate means of depicting a circular set, since circularity is a 'looping' concept. Figure 7.5 illustrates this quite clearly, by giving a number of different graphs each of which represents the circular set

$$\Omega = \{\Omega\}.$$

Finally, consider the sets a, b, c defined as follows:

$$\begin{aligned} a &= \{b, c\}, \\ b &= \{\text{Zermelo, Fraenkel}, c\}, \\ c &= \{\text{Hilbert, Fraenkel}, b\}. \end{aligned}$$

Here we have both circularity and atoms. Figure 7.6 provides a graph depicting the set a.

Now, as things stand at the moment, all I appear to have done is exhibit a rather handy, though perhaps obvious, means of depicting sets—or rather the hereditary membership relation of sets—by means of graphs. Except, of course, that I have extended the discussion into what from the standpoint of classical (well-founded) set theory is the decidedly fanciful domain of 'sets' involving circularity. But, in fact, I have prepared the way for a significant payoff. All that needs to be done in order to collect that payoff is to recall the basic strategy of developing our theory of sets by an *analysis* of the constituency structure of the kinds of objects that arise in the real world.

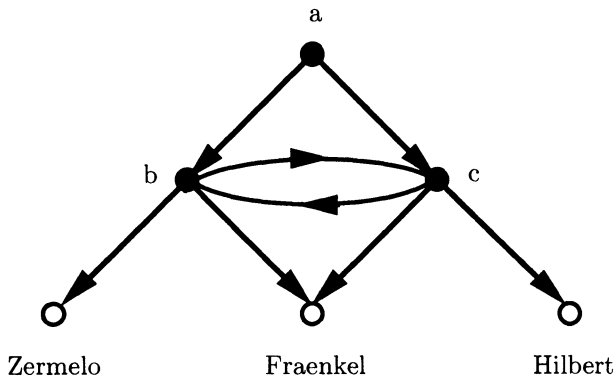

Figure 7.6: A circular set containing atoms.

According to that strategy, *graphs* (of the general forms of those discussed above) are, in a sense, *prior* to the sets they depict. Given some structured object a in the world, we may (in theory, at least) represent its hereditary constituency relation by means of a graph and thereby obtain a 'set-theoretic' model of a by moving from the graph to the set it depicts—namely, the set that corresponds to the top node of the graph.

In order for this process to work, what we need to know—and all that we need to know—is that to every graph $\mathcal{G}$ of the appropriate form (see momentarily) there is a set that $\mathcal{G}$ depicts (as its hereditary membership relation). And it is this concept of 'set from a graph' that I intend to work with.

Under this conception of 'set', all the 'usual' well-founded sets are available, since each is depicted by the graph of its hereditary membership relation, obtained as outlined above. In addition, any graph that has an infinite descending path or else contains a circuit (loop), as in Figures 7.5 and 7.6, will give rise to a non-well-founded (or circular) set. Thus non-well-founded sets arise quite naturally alongside the more familiar well-founded sets.

At this stage, I need to be precise as to just what kinds of graphs give rise to 'sets' in the above fashion.

First of all, we are restricting our attention to *directed graphs*, that is to say, graphs for which every edge has a single, designated direction. Within classical set theory, such a graph, $\mathcal{G}$, is usually defined as consisting of a nonempty set G of *nodes* (or *vertices*) and a set E of (*directed*) *edges*, where each *edge* in E is an ordered pair (x, y) of nodes. If $(x, y) \in E$, we say x and y are *joined by the edge* (x, y).

When I draw a particular graph, I represent an edge by means of an arrowed line connecting the two nodes concerned (in the appropriate direction). Thus if $(x, y) \in E$, I write $x \longrightarrow y$. In such a case, I say x is a *parent* of y or that y is a *child* of x.

It does not matter what elements of the set-theoretic universe are taken to act as the nodes of any given graph. A canonical choice—and the one I shall officially adopt—is to use the ordinal numbers for this purpose. The important issue is the graph-theoretic structure exhibited by that graph.

A *path* in a graph is a finite or infinite sequence

$$n_0 \longrightarrow n_1 \longrightarrow n_2 \longrightarrow \ldots$$

of nodes, each of which (except the first) is a child of its predecessor.

If there is a path

$$n_1 \longrightarrow n_2 \longrightarrow \ldots \longrightarrow n_k$$

from a node n_1 to a node n_k, I say that n_1 is an *ancestor* of n_k or that n_k is a *descendant* of n_1.

A graph is said to be *pointed* if there is a unique, distinguished node n_0 (called the *point* or *top node*, or sometimes the *root*, of the graph) such that all other nodes are descendants of n_0. Diagrams of pointed graphs generally show the 'top node' at the top of the picture. In this book, I shall assume all graphs are pointed. Thus, from now on, the word 'graph' should be taken to mean 'pointed, directed graph'.

It is of course the top node of a graph that corresponds to the 'set' depicted by that graph.

7.2 The Anti-Foundation Axiom

Broadly speaking, the intuitions that lead to the axioms of Zermelo–Fraenkel set theory hold true in the present situation, except for the Axiom of Foundation. So, providing we can be assured that the resulting system is consistent (i.e. consistent relative to the Zermelo–Fraenkel system itself), it is sensible to combine our new conception of a 'set determined by an arbitrary graph' with the remaining axioms. But there is a problem. To see what it is, consider the two non-well-founded sets

$$a = \{\text{Zermelo}, a\}\ ,\quad b = \{\text{Zermelo}, b\}.$$

Are the sets a and b equal or not? In the case of well-founded set theory, the answer to a question of this nature is readily obtained by applying the Axiom of Extensionality: two sets are equal if and only if they have the

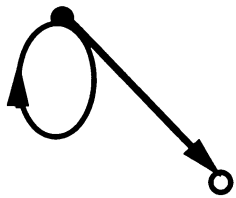

Figure 7.7: Graph depicting the unique set a such that $a = \{a, \text{Zermelo}\}$.

same elements. But in the present case, this axiom simply leads to the conclusion

$$a = b \text{ if and only if } a = b.$$

So in order to resolve identity conditions where non-well-founded sets are concerned, we will have to look for some alternative principle. Given the motivation that lies behind out present theory of sets, it seems fairly clear where we should look—and indeed what the solution to our problem should be: any given graph should (presumably) depict only one set, or, to give an alternative formulation, two sets that are depicted by the same graph should be identical.

In the case of the above example, both sets give rise to the same hereditary membership graph, namely, the one shown in Figure 7.7. Consequently, these two sets are (i.e. should be) one and the same.

This consideration leads fairly rapidly to the formulation of the following additional axiom that ought to be assumed in order to obtain an intuitive and workable theory of sets that allows for the existence of circular sets.

Every graph depicts exactly one set.

Because this principle explicitly gives rise to the existence of non-well-founded sets, I shall follow Aczel[4] and refer to this principle as the *Anti-Foundation Axiom* (AFA).

Our task now is to develop our theory of sets in a rigorous manner to incorporate this extra principle.

Obviously, since our present conception of a set requires the notion of an arbitrary graph, we need to establish some form of basic set-theoretic framework before we can even *state* the axiom AFA introduced above. This means that we need to write down some initial collection of set-theoretic

[4]The present development of a non-well-founded set theory follows closely that of Peter Aczel [1].

Figure 7.8: Decorations of the graphs shown in Figure 7.4.

principles, principles that will not effect the issues addressed by AFA one way or the other.[5] Since the present aim is to remain as close to traditional set theory as possible, while remaining true to the modeling process we have in mind, I take for this initial framework the theory ZFCA (i.e. the Zermelo–Fraenkel axioms modified to allow for atoms), modified by dropping the Axiom of Foundation. I denote this theory by the acronym ZFCA$^-$. I denote the set of atoms by $\mathcal{A}$.

Let $\mathcal{G}$ be a graph with top node n_0. A *tagging* of $\mathcal{G}$ is an assignment to every childless node of $\mathcal{G}$ of either an atom (of the underlying set theory) or else the empty set, $\emptyset$. That is, a tagging is a function from the set of childless nodes of $\mathcal{G}$ into the collection $\mathcal{A} \cup \{\emptyset\}$.

Suppose now that $\mathcal{G}$ is *tagged*, that is, there is some tagging function, t, for $\mathcal{G}$. By a *decoration* of $\mathcal{G}$ (relative to t), I mean a function, d, defined on $\mathcal{G}$ such that:

(i) if n is a childless node, then $d(n) = t(n)$;

(ii) if n is not childless, then $d(n) = \{d(n') \mid n \longrightarrow n'\}$.

For example, the two graphs shown in Figure 7.4 have the decorations shown in Figure 7.8 (assuming the one childless node is tagged with the empty set in each case).[6]

A graph is said to be *well-founded* if it has no infinite path. The following fact concerning well-founded graphs is a slight reformulation of a standard result of classical set theory.

Theorem 7.2.1 [The Collapsing Lemma] Every well-founded tagged graph has a unique decoration.

[5] Recall that I took a similar course with Zermelo–Fraenkel set theory. Some initial axiomatic development of set theory is necessary in order to properly define the cumulative hierarchy that provides the underlying conception for the entire theory.

[6] A glance at this figure should indicate why I use the word 'decoration' for this concept.

Proof: A straightforward application of definition by recursion on the well-founded graph relation, giving d as the unique function satisfying the requirements (i) and (ii) above, for each node n of the graph. [Exercise: Fill in the details.] □

Given a set x, any tagged graph that has a decoration which assigns x to its top node is called a *picture* of x.

Thus, for example, Figure 7.2 gives pictures of the first four ordinal numbers, Figure 7.4 gives alternative pictures of the ordinals 2 and 3, Figure 7.5 gives a number of different pictures of the set Ω, and Figure 7.7 gives a picture of the unique set a such that

$$a = \{a, \text{Zermelo}\}.$$

[Exercise: Give two other pictures of this particular set, one a finite graph, the other infinite.]

As an immediate consequence of Theorem 7.2.1, we see that every well-founded graph is a picture of a unique set.

By simply regarding the hereditary membership relation of a given set as a graph (i.e. $n \longrightarrow n'$ if and only if $n' \in n$), we see that every set has at least one picture. In fact, we can say more. In graph-theoretic terminology, a *tree* (see Section 4.4) is a graph such that for any node n there is a *unique* path starting from the top node and terminating at n. Then we have

Lemma 7.2.2 Every set can be pictured by a tree.

Proof: Let $\mathcal{G}$ be a graph with top node n_0 that pictures the set x. Define a new graph $\mathcal{G}'$ as follows. The nodes of $\mathcal{G}'$ are the finite paths

$$n_0 \longrightarrow n_1 \longrightarrow \ \ldots \ \longrightarrow n_k$$

starting from n_0, and the edges are the pairs

$$(n_0 \rightarrow \ldots \rightarrow n_k \ , \ n_0 \rightarrow \ldots \rightarrow n_k \rightarrow n_{k+1}).$$

It is easily seen that if d is a decoration of the graph $\mathcal{G}$, then d' is a decoration of $\mathcal{G}'$, where we define

$$d'(n_0 \rightarrow \ldots \rightarrow n_k) = d(n_k).$$

(Taggings are likewise intimately related.)

Thus $\mathcal{G}'$ also pictures the set x. I refer to $\mathcal{G}'$ as the *unfolding* of $\mathcal{G}$. □

It should be noted that even when we restrict our attention to trees, pictures of sets will not be unique. For instance, the graphs shown in

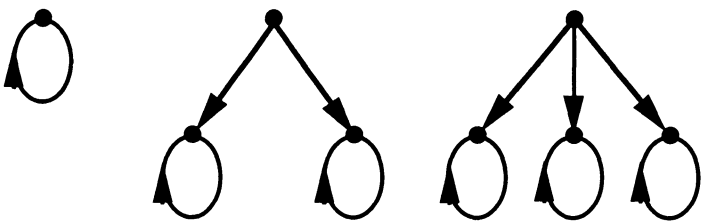

Figure 7.9: Nonisomorphic graphs for the set Ω.

Figure 7.9 all picture the set Ω, but they unfold to different (nonisomorphic) trees.

Using the newly introduced terminology, I may now state the axiom AFA:

> **The Anti-Foundation Axiom** (AFA): *Every tagged graph has a unique decoration.*

The existence part of AFA alone clearly violates the Axiom of Foundation. For instance, none of the graphs depicted in Figure 7.5 can be decorated using sets from the well-founded Zermelo–Fraenkel universe of sets.[7] On the other hand, each of these particular graphs can be decorated by assigning the non-well-founded set $\Omega = \{\Omega\}$ to each node.

By a *universe* for a theory T of sets we mean a collection V of sets that is a model of T. The following result is proved in Section 7.8.

Analogously to ZFCA, I denote by ZFC$^-$ the theory ZFC minus the Axiom of Foundation.

Theorem 7.2.3 If V is a universe for ZFC set theory (respectively, a universe for ZFCA set theory, where the atoms form a collection $\mathcal{A}$), then there is a universe V^* for ZFC$^-$ + AFA (respectively, ZFCA$^-$ + AFA with atoms from $\mathcal{A}$) such that $V \subset V^*$. □

Besides showing that the theory ZFCA$^-$ + AFA is consistent relative to ZF, the proof of this result shows how a given model of ZFC may be extended to a model of ZFC$^-$ + AFA (respectively, how a given model of ZFCA may be extended to a model of ZFCA$^-$ + AFA having the same collection of atoms).

[7] In fact the statement that no non-well-founded graph can be decorated is just a reformulation of the Axiom of Foundation.

7.3 The Solution Lemma

One of the most important consequences of AFA, as far as applications are concerned, is the way that it guarantees the existence of 'solutions' to systems of 'equations'.

The general problem is perhaps best introduced by way of a simple example.

Suppose $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are set-indeterminates, and consider the system of equations

$$\begin{aligned} \mathbf{x} &= \{\text{Zermelo}, \mathbf{y}\} \\ \mathbf{y} &= \{\text{Fraenkel}, \mathbf{z}\} \\ \mathbf{z} &= \{3, 5\} \end{aligned}$$

(where 3 and 5 are the usual von Neumann ordinal numbers).

Then it is easy to 'solve' this system of equations for the unknowns $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. The three sets concerned are

$$\begin{aligned} \mathbf{x} &= \{\text{Zermelo}, \{\text{Fraenkel}, \{3, 5\}\}\} \\ \mathbf{y} &= \{\text{Fraenkel}, \{3, 5\}\} \\ \mathbf{z} &= \{3, 5\} \end{aligned}$$

(where '3' and '5' here denote the corresponding von Neumann *sets*).

To obtain this solution, you simply observe that the last equation already gives a solution for $\mathbf{z}$, then substitute for $\mathbf{z}$ in the second equation to obtain the solution for $\mathbf{y}$, and finally substitute for $\mathbf{y}$ in the first equation to obtain the set corresponding to $\mathbf{x}$.

Now consider the amended system

$$\begin{aligned} \mathbf{x} &= \{\text{Zermelo}, \mathbf{y}\} \\ \mathbf{y} &= \{\text{Fraenkel}, \mathbf{z}\} \\ \mathbf{z} &= \{\mathbf{x}, \mathbf{y}\} \end{aligned}$$

where the sets 3 and 5 in the first system have been replaced by the indeterminates $\mathbf{x}$ and $\mathbf{y}$. Here the circularity in the system makes it impossible to derive a solution as for the first system. But, given the previous discussions, a natural approach is to investigate the graph that any solution would have to satisfy.

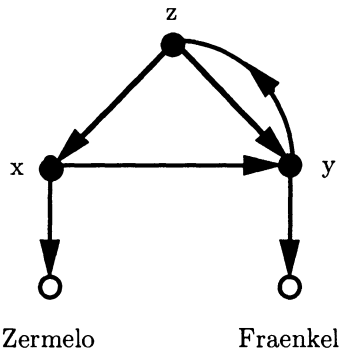

Figure 7.10: Solution of a system of equations using a graph.

A few moments analysis reveals that a graph as in Figure 7.10 provides a representation of the membership structure any solution must have. (Here I use the letters x, y, z to provide 'labels' for the nodes corresponding to the indeterminates **x**, **y**, **z**, respectively. For the sake of this informal, intuitive discussion, these labels should be regarded as nothing other than diagrammatic markers that serve to distinguish the nodes until the application of AFA yields sets to which these nodes correspond.)

By AFA, the tagged graph in Figure 7.10 has a unique decoration, d. Then, if $d(x) = X, d(y) = Y, d(z) = Z$, the sets X, Y, Z clearly solve the system of equations (for **x**, **y**, **z**, respectively). That is to say, these three sets satisfy the identities

$$\begin{aligned} X &= \{\text{Zermelo}, Y\} \\ Y &= \{\text{Fraenkel}, Z\} \\ Z &= \{X, Y\}. \end{aligned}$$

Now, intuitively, it seems clear that this approach using graphs and AFA should work for any such system of equations, involving any number of unknowns, with the set-theoretic constructions on the right-hand sides of the equations being arbitrarily complex, having as many nestings of sets as required. As long as each indeterminate appears, on its own, on the left-hand side of precisely one equation in the system, it should be possible to draw a graph depicting the membership structure that any solution will have to have, and thus, by AFA, to obtain a (presumably unique) solution to the system.

The Solution Lemma, proved using AFA, says that this is indeed the

case. In order to state the lemma properly, I need to first set up the appropriate machinery.

I denote by $V_{\mathcal{A}}$ the 'universe' of all sets (of the theory ZFCA$^-$ + AFA) built on the collection $\mathcal{A}$ of atoms. Let $\mathcal{X}$ be a collection of set-indeterminates. I denote by $V_{\mathcal{A}}[\mathcal{X}]$ the collection of all set terms that can be built up using elements of $V_{\mathcal{A}}$ and the indeterminates in $\mathcal{X}$. That is, $V_{\mathcal{A}}[\mathcal{X}]$ will be an extension of $V_{\mathcal{A}}$ that contains objects such as

$$\{a, b, \mathbf{x}, \{\mathbf{y}, c\}\}$$

$$\{a, \{\mathbf{x}, \{b, \{\mathbf{z}\}\}\}\}$$

$$\{1, 2, \{\Omega, \mathbf{x}\}\}$$

where $a, b, c \in V_{\mathcal{A}}$ and $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in \mathcal{X}$.

Formally, I regard the indeterminates in $\mathcal{X}$ as extra atoms and take

$$V_{\mathcal{A}}[\mathcal{X}] = V_{\mathcal{A} \cup \mathcal{X}}.$$

This construction is clearly analogous to the formation of the ring $\mathcal{F}[X]$ of polynomials in indeterminates from X over a field $\mathcal{F}$. And just as the members of $\mathcal{F}[X]$ give rise to systems of polynomial equations to be solved in $\mathcal{F}$, so too the members of $V_{\mathcal{A}}[\mathcal{X}]$ provide systems of set equations to be solved in $V_{\mathcal{A}}$.

By an *equation in* $\mathcal{X}$, I mean an expression of the form

$$\mathbf{x} = t$$

where $t \in V_{\mathcal{A}}[\mathcal{X}]$.

By a *system of equations in* $\mathcal{X}$, I mean a family of equations

$$\{\mathbf{x} = t_{\mathbf{x}} \mid \mathbf{x} \in \mathcal{X}\},$$

where there is exactly one equation for each indeterminate $\mathbf{x} \in \mathcal{X}$.

By a *solution* to an equation

$$\mathbf{x} = t$$

I mean an *assignment*

$$f : \mathcal{X} \to V_{\mathcal{A}}$$

of sets or atoms to indeterminates such that the equation yields a valid set-theoretic identity when each occurrence of each indeterminate in the equation is replaced by its image under f.

Thus, to use a suggestive notation familiar from formal logic, if t is an element of $V_{\mathcal{A}}[\mathcal{X}]$ that involves the indeterminates $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ..., and I write $t = t(\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots)$ to indicate this fact, then the assignment

$$f(\mathbf{x}) = a \ , \ f(\mathbf{y}) = b \ , \ f(\mathbf{z}) = c, \ldots$$

will be a solution to the above equation if and only if

$$a = t(a, b, c, \ldots).$$

More generally, I say that an assignment f of sets to the indeterminates in $\mathcal{X}$ is a *solution* to a system of equations

$$\mathbf{x} = t_{\mathbf{x}} \quad (\mathbf{x} \in \mathcal{X})$$

if and only if f is a solution for every equation in the system.

To formalize the above notions within our theory of sets, the idea is to proceed as follows. First prove that any assignment $f : \mathcal{X} \to V_{\mathcal{A}}$ extends in a natural and unique fashion to a function

$$\hat{f} : V_{\mathcal{A}}[\mathcal{X}] \to V_{\mathcal{A}}.$$

Then say that the assignment $f : \mathcal{X} \to V_{\mathcal{A}}$ is a *solution* to the equation

$$\mathbf{x} = t$$

if and only if

$$f(\mathbf{x}) = \hat{f}(t).$$

This formal development is carried out in detail in Section 7.6, where I also prove the following key result:

Theorem 7.3.1 [The Solution Lemma] Every system of equations in a collection $\mathcal{X}$ of indeterminates, over the universe $V_{\mathcal{A}}$, has a unique solution in $V_{\mathcal{A}}$. □

The general idea for the proof of this result is to develop a formal, and more general, analogue of the method used above in order to solve our sample system of three equations (where we proceeded via the graph in Figure 7.10 and then applied AFA to obtain the required sets).

It is worth remarking that the Solution Lemma is logically equivalent to AFA (over the theory ZFCA$^-$).

7.4 Inductive Definitions Under AFA

Inductive definitions pervade set theory and logic. For instance, the class of ordinals can be defined inductively as the smallest class *Ord* such that:

(i) $\emptyset \in Ord$;

(ii) if $\alpha \in Ord$, then $\alpha \cup \{\alpha\} \in Ord$;

(iii) if $x \subseteq Ord$ and x is a set, then $\bigcup x \in Ord$.

In the absence of the Axiom of Foundation, this definition serves to define the class of *well-founded* ordinals.

To see why this definition is described as *inductive*, imagine trying to construct the ordinals one by one in the following 'inductive' fashion. Start out with $0 = \emptyset$. The successor ordinal to an ordinal α is defined as the set $\alpha \cup \{\alpha\}$. In the case of limit ordinals, take unions, so that a limit ordinal α is given as

$$\alpha = \bigcup\{\beta \mid \beta < \alpha\} = \bigcup\alpha.$$

Of course, this procedure to define the ordinals cannot be carried out as described, since it assumes that the ordinals are *already available* to index the definition (i.e. to provide the domain of the sequence of ordinals being defined). But the original definition of the class *Ord* given above serves to capture the class of ordinals, by taking minimal closure under the two constructive principles (successor and union) used in this attempted iterative construction.

As a first step toward obtaining a general framework that encompasses such minimal-closure, inductive definitions, consider the function γ from sets to sets defined by

$$\gamma(x) = \{\emptyset\} \cup \{\bigcup x\} \cup \{y \cup \{y\} \mid y \in x\}.$$

For any class X now, define

$$\Gamma(X) = \bigcup\{\gamma(x) \mid x \subseteq X \wedge x \text{ is a set}\}.$$

Then clearly, Γ is an operator taking classes to classes, that is *monotone*, in the sense that

$$X \subseteq Y \quad \text{implies} \quad \Gamma(X) \subseteq \Gamma(Y).$$

Moreover, Γ is *set-based*, which means that, for any set z,

$$\text{if } z \in \Gamma(X), \text{ then } z \in \Gamma(x) \text{ for some set } x \subseteq X.$$

Clearly, a straightforward translation of our definition of the class of ordinals now is that *Ord* is the smallest class X such that $\Gamma(X) = X$. (Since $X \subseteq \Gamma(X)$ for any class X, this is equivalent to *Ord* being the smallest class X such that $\Gamma(X) \subseteq X$.)

In general now, if Γ is any class operator that is *monotone* and *set-based*, as defined above, then, as I shall prove in Section 7.7, there will be a *least fixed-point* X for Γ, that is, a smallest class X such that $\Gamma(X) = X$. I then say that the operator Γ thereby provides an *inductive definition* of the class X.

I shall also prove that every monotone, set-based operator has a *greatest fixed-point*. If Y is the greatest fixed-point of Γ, I shall say that Γ provides a *co-inductive definition* of the class Y.

In the case of the particular operator Γ defined above, the greatest fixed-point is the class, V, the entire universe of sets (this is easily seen), so the co-inductive definition gives us nothing new. But for other examples the greatest fixed-point can be both nontrivial (i.e. not just V) and distinct from the least fixed-point. And in cases where the underlying set theory is ZFCA$^-$ + AFA rather than ZFCA, it is often the greatest fixed-point that is of more use than the least fixed-point. The example below is a case in point.

Assume for simplicity that the collection $\mathcal{A}$ of atoms is finite. Consider the operator Γ that assigns to any class X the class of all finite subsets of $X \cup \mathcal{A}$. In ZFCA, this operation has a unique fixed-point, the set HF of all hereditarily finite sets. But in ZFCA$^-$ + AFA, there are many distinct fixed-points. The smallest fixed-point, HF_0, can be characterized as the smallest set satisfying the condition

$$\text{if } a \subseteq HF_0 \cup \mathcal{A} \text{ and } a \text{ is finite, then } a \in HF_0$$

(i.e. $\Gamma(HF_0) \subseteq HF_0$.)

The greatest fixed-point, HF_1, can be characterized as the largest set satisfying

$$\text{if } a \in HF_1, \text{ then } a \subseteq HF_1 \cup \mathcal{A} \text{ and } a \text{ is finite}$$

(i.e. $HF_1 \subseteq \Gamma(HF_1)$.)

It is clear that $HF_0 \subseteq HF_1$, and in ZFCA these two sets coincide. But under AFA, the inclusion is proper. In particular, it is easily demonstrated that every member of HF_0 is well-founded, but HF_1 contains non-well-founded sets. For example, Ω is a member of HF_1. Indeed, HF_1 consists of all and only those sets that can be pictured by at least one finitely branching graph. Since this latter is obviously the correct notion of hereditarily finite set under our present conception of sets as determined by graphs, in this case the co-inductive definition provides the most appropriate definition.

The above example is typical of the situation in non-well-founded set theory. A pair of inductive and co-inductive definitions that characterize the same set or class in classical set theory often yield distinct classes under

AFA. The least fixed-point, specified by the inductive definition, usually consists of the well-founded members of the largest fixed-point, given by the co-inductive definition. For reasons outlined below, it is usually the latter that is required for applications (under AFA). (Though in the case of the class *Ord* considered above, it is the inductive definition that is by far the more important of the two. But this is for the special reason that the *well-foundedness* of the ordinals that is one of their most significant properties.)

It is largely because of the way the Solution Lemma operates that, when AFA is assumed, co-inductive definitions are often more useful than inductive definitions. The situation is best explained by starting with a simple example, namely, the co-inductively defined set HF_1 of all hereditarily finite sets in the AFA universe (with a finite set of atoms).

Suppose we have some finite system of equations of the form

$$\mathbf{x} = a_{\mathbf{x}}(\mathbf{x},\mathbf{y},\dots)$$

where each $a_{\mathbf{x}}$ is in the collection $HF_1^{\mathcal{X}}$ of all hereditarily finite sets in the expanded universe $V_{\mathcal{A}}[\mathcal{X}]$ (which, you may recall, is formally the same as $V_{\mathcal{A}\cup\mathcal{X}}$). And suppose that we apply the Solution Lemma to obtain a solution f to this system of equations. Intuitively, the set-theoretic structure of each $V_{\mathcal{A}}[\mathcal{X}]$-set $a_{\mathbf{x}}$ is that of a hereditarily finite set, and consequently one might expect that the solution sets $f(\mathbf{x})$ are also hereditarily finite, that is, in the collection HF_1 as defined in the universe $V_{\mathcal{A}}$. That this is indeed the case is a special case of what is known as the Co-Inductive Closure Theorem, proved in Section 7.7. A nonrigorous argument for the present example is given below.

Recall that in my original motivation for the Solution Lemma, I showed how, in the case of a simple example at least, a system of equations may be 'unraveled' to produce a graph that any solution will have to satisfy, whence by AFA we can conclude that there is in fact a solution. As I mentioned at the time, the proof of the Solution Lemma consists of a formal analogue of this heuristic argument. The idea behind the proof of the Co-Inductive Closure Theorem is to trace through the proof of the Solution Lemma and check that closure is indeed achieved. (This requires that the class operator Γ concerned satisfies some fairly general additional requirements that will be made precise when I give the formal proof.) In the case of the present example, the following argument gives the desired result.

First of all, by introducing more indeterminates, we may assume that each equation is of one of the following simple forms:

- $\mathbf{x} = \emptyset$;
- $\mathbf{x} = a$, for some atom $a \in \mathcal{A}$;

- $\mathbf{x} = \{\mathbf{y}_1, \ldots, \mathbf{y}_n\}$,

where $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are other indeterminates with their own equations in the system.

Let f be the solution to this modified system. It is clear that the collection $HF_1 \cup \mathrm{ran}(f)$ satisfies the defining condition for HF_1. So, by the maximality of HF_1, $\mathrm{ran}(f) \subseteq HF_1$, as required.

The general statement of the Co-Inductive Closure Theorem runs roughly like this. Suppose Γ is some monotone, set-based class operator. Using Γ, we can co-inductively define a collection of objects from the universe $V_{\mathcal{A}}$ as the largest fixed point of Γ in $V_{\mathcal{A}}$. Call the objects in this collection Γ-*objects*. Likewise, we may use the same operator Γ in order to define an analogous collection in the universe $V_{\mathcal{A}}[\mathcal{X}]$. Call the objects in this collection *parametric* Γ-*objects*. What the Closure Theorem says is that, providing Γ satisfies some fairly general requirements, any system of equations involving only parametric Γ-objects will have only Γ-objects as solutions.

The combination of the Solution Lemma and the Co-Inductive Closure Theorem provides a powerful tool for handling non-well-founded sets under AFA and, in this respect, takes on the role played by the recursion principle in Zermelo–Fraenkel set theory.

7.5 Graphs and Systems

The notion of a *graph* has been precisely defined already. In order to obtain, in particular, a proof of the consistency of AFA, I require the following generalization to allow for a proper class of nodes.

By a *system* I mean a class M of *nodes* together with a class of (*directed*) *edges*, each edge being an ordered pair (n, n') of nodes. I write $n \longrightarrow n'$ if (n, n') is an edge of M. Any system is required to satisfy the requirement that, for each node n, the collection

$$\mathrm{ch}_M(n) = \{n' \in M \mid n \longrightarrow n'\}$$

of all *children* of n is a *set*.

Clearly, any graph is a system. For an example of a system that is not a graph (because the collection of nodes forms a proper class), take the collection of nodes to be the universe V of all pure (i.e. atomless) sets, with the edges given by $x \longrightarrow y$ if and only if $y \in x$.

Note that whereas graphs are assumed to have a unique *top node*, no such requirement is placed on systems.

Because of the different roles played by the two collections of atoms in our theory, *taggings* are defined as *partial* functions. Thus, a tagging of the

system M is an assignment, t, to some or all of the childless nodes, a, of M, of an atom, $t(a)$ (i.e. a member of $\mathcal{A} \cup \mathcal{X}$). I denote such a tagged system by (M, t). (Note that t may be a 'function' only in the proper class sense.)

Notice that if t is the nowhere-defined tagging on M, then the tagged system (M, t) is essentially the same as the untagged system M. Accordingly, I shall henceforth use the terms 'system' and 'graph' to mean 'tagged system' and 'tagged graph', respectively.

In order to establish the Solution Lemma, I shall need to associate atoms ('indeterminates') with nodes, as well as be able to handle the assignment to each indeterminate of a set in $V_{\mathcal{A}}$ when the equational system is solved. The following definition supplies the appropriate machinery. Since it may be necessary to associate more than one indeterminate to a given node, the 'labeling' function defined below assigns not a single set/atom but a set of sets/atoms, to each node.

A *labeling* of a (tagged) system (M, t) is a function l (possibly a 'function' in the proper class sense) defined on $M - \mathrm{dom}\,(t)$ that assigns to each node n not in $\mathrm{dom}\,(t)$, a (possibly empty) set $l(n)$ of sets/atoms.

The elements of the set $l(n)$, for any node n, are the *labels* assigned to the node n by the labeling function.

A *labeled system* then is just a system, (M, t), together with a labeling function, l. I denote such a system by (M, t, l).

A *decoration* of a labeled system (M, t, l) is an assignment d of a set $d(n)$ to each node n such that:

(i) if $n \in \mathrm{dom}(t)$, then $d(n) = t(n)$;

(ii) if $n \notin \mathrm{dom}(t)$, then

$$d(n) = \{d(n') \mid n \longrightarrow n'\} \cup l(n).$$

By virtue of the above remark, this definition includes the special case of a decoration of an unlabeled system (M, t): if $l(n) = \emptyset$ for each parent node n of M, then $d(n) = t(n)$ for all tagged nodes and $d(n) = \{d(n') \mid n \longrightarrow n'\}$ for all untagged nodes. This simply extends to (tagged) *systems*, the definition of a decoration of a (tagged) *graph* given in Section 7.2.

Our starting point is the axiom AFA:

> **The Anti-Foundation Axiom (AFA)**: *Every (tagged) graph has a unique decoration.*

I shall prove that this formulation is already enough to prove the apparently stronger result that every labeled system has a unique decoration. The following theorem provides the first of two steps toward this goal, by

showing that it is possible to go from decorations of unlabeled *graphs* to decorations of unlabeled *systems*.

Theorem 7.5.1 (Assuming AFA.) Every (tagged) system has a unique decoration.

Proof: Let (M, t) be a system. For each $n \in M$, we may define a graph M_n by taking the nodes of M_n to be all nodes of M that lie on some path of M starting from node n, and taking as edges all edges of M that connect two members of M_n. Since the collection of all children of any given node in M forms a set, it is easily seen that M_n is itself a set. Indeed, if we take $X_0 = \{n\}$ and, for each natural number i, define

$$X_{i+1} = \bigcup\{\mathrm{ch}_M(m) \mid m \in X_i\},$$

then each X_i is a set, and we have $M_n = \bigcup_{i=0}^{\infty} X_i$.

The restriction t_n of the tagging function t to M_n is obviously a tagging of the graph M_n for each n. By AFA, each (M_n, t_n) has a unique decoration d_n. Define d on M by

$$d(n) = d_n(n) \quad (\forall n \in M).$$

I show that d is the unique decoration of (M, t).

First note that if $n \in \mathrm{dom}(t)$, then n is the only node of M_n and

$$d(n) = d_n(n) = t_n(n) = t(n).$$

To handle the remaining nodes of M, we observe that if $n \longrightarrow m$ in M, then every node of M_m will be a node of M_n and the restriction of d_n to M_m will be a decoration of M_m and, hence, equal to d_m, the unique decoration of (M_m, t_m). Thus whenever $n \longrightarrow m$ in M, we have $d_n(m) = d_m(m) = d(m)$. Consequently, for each untagged node $n \in M$, we have

$$d(n) = d_n(n) = \{d_n(m) \mid n \longrightarrow m \text{ in } M_n\} = \{d(m) \mid n \longrightarrow m \text{ in } M\}.$$

Thus d is a decoration of (M, t).

To see that d is unique, simply notice that any decoration of (M, t) will restrict to a decoration of (M_n, t_n) for any node n, hence, must extend d_n, and, therefore, has to be equal to d. □

The following theorem completes our extension of AFA to cover *labeled* systems.

Theorem 7.5.2 (Assuming AFA.) Every labeled (tagged) system has a unique decoration.

Proof: Let (M, t, l) be a labeled system. Define a new, unlabeled, system (M', t') as follows. Let the nodes of M' be the members of the set

$$\{(1, n) \mid n \in M\} \cup \{(2, a) \mid a \in V_{\mathcal{A}}[\mathcal{X}]\}.$$

The edges of M' are:

- $(1, n) \longrightarrow (1, n')$, whenever $n \longrightarrow n'$ in M;
- $(1, n) \longrightarrow (2, a)$, whenever $n \in M$, $n \notin \mathrm{dom}\,(t)$, and $a \in l(n)$;
- $(2, a) \longrightarrow (2, b)$, whenever $b \in a$.

Define the tagging t' on M' by:

- $t'(1, n) = t(n)$, if $n \in \mathrm{dom}\,(t)$;
- $t'(2, a) = a$, if $a \in \mathcal{A} \cup \mathcal{X}$.

By Theorem 7.5.1, (M', t') has a unique decoration, d. Thus, for each node $n \in \mathrm{dom}\,(t)$,

$$d(1, n) = t'(1, n) = t(n),$$

and, for each $a \in \mathcal{A} \cup \mathcal{X}$,

$$d(2, a) = t'(2, a) = a.$$

Moreover, for each untagged (by t) node $n \in M$,

$$d(1, n) = \{d(1, n') \mid n \longrightarrow n' \text{ in } M\} \cup \{d(2, a) \mid a \in l(n)\},$$

and, for each nonatomic $a \in V_{\mathcal{A}}[\mathcal{X}]$,

$$d(2, a) = \{d(2, b) \mid b \in a\}.$$

Now, the assignment of the set $d(2, a)$ to each $a \in V_{\mathcal{A}}[\mathcal{X}]$ is a decoration of the system $V_{\mathcal{A}}[\mathcal{X}]$, tagged with the identity function on $\mathcal{A} \cup \mathcal{X}$. But the identity function on $V_{\mathcal{A}}[\mathcal{X}]$ is also a decoration of the same tagged system. So by Theorem 7.5.1, we must have $d(2, a) = a$ for all $a \in V_{\mathcal{A}}[\mathcal{X}]$.

Define e on M now by

$$e(n) = d(1, n).$$

Then if n is a tagged node of M,

$$e(n) = t(n),$$

and if n is an untagged node of M, then

$$\begin{aligned} e(n) &= \{e(n') \mid n \longrightarrow n' \text{ in } M\} \cup \{a \mid a \in l(n)\} \\ &= \{e(n') \mid n \longrightarrow n' \text{ in } M\} \cup l(n). \end{aligned}$$

So e is a decoration of (M, t, l).

To check uniqueness, suppose e' is also a decoration of (M, t, l). Then d' is a decoration of (M', t'), where we define

- $d'(1, n) = e'(n)$, for $n \in M$;
- $d'(2, a) = a$, for $a \in V_{\mathcal{A}}[\mathcal{X}]$.

By Theorem 7.5.1, we have $d' = d$. Hence for all $n \in M$, we have

$$e'(n) = d'(1, n) = d(1, n) = e(n),$$

so $e' = e$. □

In the future, I shall often simply refer to Theorem 7.5.2 above as AFA.

The following general result establishes the key facts I shall use in the proof of the Solution Lemma.

Theorem 7.5.3 (Assuming AFA.) Let (M, t, l) be a labeled system (in $V_{\mathcal{A}}[\mathcal{X}]$) such that $t(n) \in \mathcal{A}$ for all tagged nodes $n \in M$, and $l(n) \subseteq \mathcal{X}$ for all untagged nodes $n \in M$.

(i) Let $\pi : \mathcal{X} \to V_{\mathcal{A}}$. Then there is a unique map $\widehat{\pi} : M \to V_{\mathcal{A}}$ such that for each $n \in M$:

- if n is a tagged node of M, then $\widehat{\pi}(n) = t(n)$;
- if n is an untagged node of M, then

$$\widehat{\pi}(n) = \{\widehat{\pi}(n') \mid n \longrightarrow n' \text{ in } M\} \cup \{\pi(x) \mid x \in l(n)\}.$$

(ii) Suppose that to each $x \in \mathcal{X}$ there is assigned a node a_x of M. Then there is a unique map $\pi : \mathcal{X} \to V_{\mathcal{A}}$, such that for all $x \in \mathcal{X}$,

$$\pi(x) = \widehat{\pi}(a_x).$$

Proof: (i) Let $\pi : \mathcal{X} \to V_{\mathcal{A}}$ be given. Let l_π be a new labeling of (M, t), defined by setting

$$l_\pi(n) = \{\pi(x) \mid x \in l(n)\}$$

for all untagged nodes n of M.

Clearly, the unique decoration of the labeled system (M, t, l_π) is the desired map $\widehat{\pi}$.

(ii) Let M' be the system having the same nodes as M, and all the edges of M, together with the edges $n \longrightarrow a_x$ whenever $n \in M$ and $x \in l(n)$. By Theorem 7.5.1, the unlabeled system (M', t) has a unique decoration, d. Thus, for each tagged node $n \in M'$,

$$d(n) = t(n),$$

and, for each untagged node $n \in M'$,

$$d(n) = \{d(n') \mid n \longrightarrow n' \text{ in } M\} \cup \{d(a_x) \mid x \in l(n)\}.$$

Let $\pi(x) = d(a_x)$ for each $x \in \mathcal{X}$. Thus $\pi : \mathcal{X} \to V_{\mathcal{A}}$. Moreover, for each untagged node $n \in M$,

$$d(n) = \{d(n') \mid n \longrightarrow n' \text{ in } M\} \cup \{\pi(x) \mid x \in l(n)\}.$$

So by part (i) of the theorem, $d = \widehat{\pi}$. So, in particular, for all $x \in \mathcal{X}$, we have

$$\pi(x) = \widehat{\pi}(a_x).$$

To show that π is unique with this property, suppose that $\pi' : \mathcal{X} \to V_{\mathcal{A}}$ is such that $\pi'(x) = \widehat{\pi}'(a_x)$ for all $x \in \mathcal{X}$. Then clearly, $\widehat{\pi}'$ will be a decoration of (M', t). Thus by Theorem 7.5.1, $\widehat{\pi}' = d$. Hence for any $x \in \mathcal{X}$,

$$\pi'(x) = \widehat{\pi}'(a_x) = d(a_x) = \pi(x).$$

Thus $\pi' = \pi$. □

7.6 Proof of the Solution Lemma

I shall present the proof of the Solution Lemma in two parts. The first, which I shall call the Substitution Lemma, says that if you start with a collection, $\mathcal{C}$, of members of $V_{\mathcal{A}}[\mathcal{X}]$, and if you replace each indeterminate x that occurs (in the transitive closure of) some member of $\mathcal{C}$ by some member b_x of $V_{\mathcal{A}}$, then the result will be a family $\mathcal{C}'$ of well-defined members of $V_{\mathcal{A}}$.

Theorem 7.6.1 [Substitution Lemma] (Assuming AFA.) Let $\pi : \mathcal{X} \to V_{\mathcal{A}}$. Then there is a unique map $\widehat{\pi} : V_{\mathcal{A}}[\mathcal{X}] \to V_{\mathcal{A}}$ such that:

(i) $\widehat{\pi}(a) = a$, for all $a \in \mathcal{A}$;

(ii) $\widehat{\pi}(a) = \{\widehat{\pi}(b) \mid b \in V_{\mathcal{A}}[\mathcal{X}] \ \& \ b \in a\} \cup \{\pi(x) \mid x \in \mathcal{X} \ \& \ x \in a\}$, for all other a.

Proof: Let M be the system whose nodes are the members of $V_{\mathcal{A}}[\mathcal{X}]$ and whose edges are given by

$$a \longrightarrow b \quad \text{if and only if} \quad b \in a.$$

Let t be the identity function on $\mathcal{A}$. (So t is a tagging for M.) Define a labeling l of (M, t) by setting

$$l(a) = a \cap \mathcal{X}$$

for all $a \in V_{\mathcal{A}}[\mathcal{X}] - \mathcal{A}$. (Thus $l(a) \subseteq \mathcal{X}$ for all $a \in \mathrm{dom}(l)$.)

Let $\widehat{\pi}$ be related to (M, t, l) and π as in Theorem 7.5.3(i). Clearly, $\widehat{\pi}$ is as required. □

Theorem 7.6.2 [Solution Lemma] (Assuming AFA.) Let a_x be a member of $V_{\mathcal{A}}[\mathcal{X}]$ for each indeterminate x. Then the system of equations

$$x = a_x \quad (x \in \mathcal{X})$$

has a unique solution. That is, there is an assignment $\pi : \mathcal{X} \to V_{\mathcal{A}}$ such that

$$\pi(x) = \widehat{\pi}(a_x)$$

for all $x \in \mathcal{X}$.

Proof: Let (M, t, l) be as in the proof of Theorem 7.5.3 and apply Theorem 7.6.1(ii). □

7.7 Co-Inductive Definitions

I indicated earlier that the Solution Lemma can often be combined with co-inductive definitions in order to obtain solution sets with particular properties. In this section I develop this idea formally.

I start off by recalling that a class operator Γ is said to be *monotone* if

$$X \subseteq Y \ \Rightarrow \ \Gamma(X) \subseteq \Gamma(Y),$$

and is *set-based* if

$$a \in \Gamma(X) \ \Rightarrow \ a \in \Gamma(x), \text{ for some set } x \subseteq X.$$

Taken together, these two conditions are equivalent to the following: for any class X,

$$\Gamma(X) = \bigcup\{\Gamma(x) \mid x \subseteq X \ \wedge x \text{ is a set}\}.$$

Operators that satisfy this requirement are usually said to be *set-continuous* (or, simply, *continuous*).

It is a standard fact of ZFC^- set theory that every continuous operator, Γ, has both a least fixed-point and a greatest fixed-point. The least fixed-point of Γ is the unique smallest class I such that $\Gamma(I) \subseteq I$. The largest fixed-point is the unique largest class J such that $J \subseteq \Gamma(J)$. Our present interest is in the largest fixed-point, and accordingly I commence with a proof that such a largest class J exists.

Note that as an operator on *classes*, a class operator Γ should be thought of in terms of some defining formula, not as some form of extensional object. (The use of the word 'operator', as opposed to 'function', is intended to emphasize this point.)

Given Γ, define J by

$$J = \bigcup\{x \mid x \text{ is a set } \wedge x \subseteq \Gamma(x)\}.$$

Lemma 7.7.1 $J \subseteq \Gamma(J)$.

Proof: Let $a \in J$. Then by definition, $a \in x$ for some set x such that $x \subseteq \Gamma(x)$. Since $x \subseteq J$ and Γ is monotone, $\Gamma(x) \subseteq \Gamma(J)$. Thus $x \subseteq \Gamma(J)$. Hence $a \in \Gamma(J)$. □

Lemma 7.7.2 If $X \subseteq \Gamma(X)$, then $X \subseteq J$.

Proof: Assume $X \subseteq \Gamma(X)$, and let $a \in X$. I prove that $a \in J$.

I first show that for each set $x \subseteq X$, there is a set $x' \subseteq X$ such that $x \subseteq \Gamma(x')$. Let $x \subseteq X$. Then, by the assumption on X, $x \subseteq \Gamma(X)$. Hence as Γ is set-based,

$$(\forall y \in x)(\exists u)(y \in \Gamma(u) \wedge u \subseteq X).$$

By the Axiom of Replacement, there is a set A such that

$$(\forall y \in x)(\exists u \in A)(y \in \Gamma(u) \wedge u \subseteq X).$$

Set

$$x' = \bigcup\{u \in A \mid u \subseteq X\}.$$

Then x' is a subset of X. Moreover, as Γ is monotone, $\Gamma(u) \subseteq \Gamma(x')$ for all $u \in A$, so $x \subseteq \Gamma(x')$.

Using the above result, we can choose (using the Axiom of Choice) an infinite sequence $x_0, x_1, \ldots$ of subsets of X such that $x_0 = \{a\}$ and $x_n \subseteq \Gamma(x_{n+1})$ for all n. Set

$$x = \bigcup_{n=0}^{\infty} x_n.$$

Then x is a set. Moreover, if $y \in x$, then $y \in x_n$ for some n, so $y \in \Gamma(x_{n+1}) \subseteq \Gamma(x)$. Thus $x \subseteq \Gamma(x)$. Hence $x \subseteq J$. Since $a \in x_0 \subseteq x$, it follows that $a \in J$. □

Lemma 7.7.3 J is the unique largest fixed-point of Γ.

Proof: By Lemma 7.7.1 and the monotonicity of Γ,

$$\Gamma(J) \subseteq \Gamma(\Gamma(J)).$$

So by Lemma 7.7.2, $\Gamma(J) \subseteq J$. Thus by Lemma 7.7.1 again, $\Gamma(J) = J$, and so J is a fixed-point of Γ. By Lemma 7.7.2 again, J is the largest fixed-point of Γ. □

The task now is to establish a general result that will enable us to show that under certain conditions, the solution sets to a system of equations all satisfy a given co-inductive definition (where, you may recall, a co-inductive definition of a class is one that determines the class as the largest fixed-point of some continuous operator). The development should (continue to) be thought of as taking place in the set-theoretic universe $V_{\mathcal{A}}[\mathcal{X}]$.

Let Γ be a continuous operator. Assume Γ has the following 'absoluteness' property: for any set x, $\Gamma(x \cap V_{\mathcal{A}}) = \Gamma(x) \cap V_{\mathcal{A}}$. Let $J^{\mathcal{X}}$ be the largest fixed-point of Γ as defined in $V_{\mathcal{A}}[\mathcal{X}]$, and let J be the largest fixed-point as defined in $V_{\mathcal{A}}$. Notice that by virtue of the above absoluteness assumption on Γ, $J = J^{\mathcal{X}} \cap V_{\mathcal{A}}$. (This is easily proved.)

Let

$$x = a_x \quad (x \in \mathcal{X})$$

be a system of equations such that $a_x \in J^{\mathcal{X}}$, for all $x \in \mathcal{X}$.

The basic question to ask now is this. Given a solution

$$\pi(x) = b_x \quad (x \in \mathcal{X})$$

to this system, by sets b_x in $V_{\mathcal{A}}$, under what conditions may we conclude that each set b_x is in fact a member of J, the largest fixed-point of Γ as defined in $V_{\mathcal{A}}$? The answer, though not particularly pretty, is generally quite easy to apply in specific cases. It depends on the following definition.

Call a map $\tau : V_{\mathcal{A}}[\mathcal{X}] \to V_{\mathcal{A}}$ *faithful* (for the given system of equations) if $\tau(a) = a$ for all $a \in \mathcal{A}$, and for all other $a \in V_{\mathcal{A}}[\mathcal{X}]$,

$$\tau(a) = \{\tau(b) \mid b \in a\} \cup \{\tau(a_x) \mid x \in a \cap \mathcal{X}\}.$$

Theorem 7.7.4 [Co-Inductive Closure Theorem] (Assuming AFA.) Let Γ, $J^{\mathcal{X}}$, J, a_x $(x \in \mathcal{X})$ be as above. Suppose that for any faithful map $\tau : V_{\mathcal{A}}[\mathcal{X}] \to V_{\mathcal{A}}$, it is the case that

$$(*) \qquad a \in J^{\mathcal{X}} \Rightarrow \tau(a) \in \Gamma(K),$$

where K is the range of τ on $J^{\mathcal{X}}$.

Then the unique solution to the system of equations consists entirely of sets in J.

Proof: The Solution Lemma (Theorem 7.6.2) tells us that there is a unique map $\pi : \mathcal{X} \to V_{\mathcal{A}}$ such that

$$\pi(x) = \widehat{\pi}(a_x)$$

for all $x \in \mathcal{X}$, where $\widehat{\pi} : V_{\mathcal{A}}[\mathcal{X}] \to V_{\mathcal{A}}$ is such that $\widehat{\pi}(a) = a$ if $a \in \mathcal{A}$, and

$$\widehat{\pi}(a) = \{\widehat{\pi}(b) \mid b \in a\} \cup \{\pi(x) \mid x \in a \cap \mathcal{X}\}$$

if $a \notin \mathcal{A}$.

Since $\pi(x) = \widehat{\pi}(a_x)$ for all x, $\widehat{\pi}$ is faithful. Thus, by assumption, $\widehat{\pi}$ must satisfy condition $(*)$. So, if K is the range of $\widehat{\pi}$ on $J^{\mathcal{X}}$, we have

$$(**) \qquad a \in J^{\mathcal{X}} \Rightarrow \widehat{\pi}(a) \in \Gamma(K).$$

Now, if $b \in K$, then $b = \widehat{\pi}(a)$ for some $a \in J^{\mathcal{X}}$, so by $(**)$, $b \in \Gamma(K)$. Hence $K \subseteq \Gamma(K)$. So by the maximality of $J^{\mathcal{X}}$, $K \subseteq J^{\mathcal{X}}$. But $K \subseteq V_{\mathcal{A}}$. Hence, as $J = J^{\mathcal{X}} \cap V_{\mathcal{A}}$, $K \subseteq J$, and it follows that $\widehat{\pi}(a) \in J$. In particular, $\pi(x) = \widehat{\pi}(a_x) \in J$ for all $x \in \mathcal{X}$, as required. □

As an illustration of the use of the above result, take the example of the hereditarily finite sets discussed informally at the end of the previous chapter. The co-inductively defined collection HF of all hereditarily finite sets is the largest fixed point of the continuous operator

$$\Gamma(X) = \{a \mid a \subseteq X \cup \mathcal{A} \ \&\ a \text{ is finite}\}.$$

(As before, I assume that the collection $\mathcal{A}$ of atoms of $V_{\mathcal{A}}$ is finite here.) Notice that Γ satisfies the absoluteness requirement stipulated above for operators to which the Co-Inductive Closure Theorem may be applied. Suppose

$$x = a_x \quad (x \in \mathcal{X})$$

is a system of equations such that $a_x \in HF^{\mathcal{X}}$ for all $x \in \mathcal{X}$. Let $\tau : V_{\mathcal{A}}[\mathcal{X}] \to V_{\mathcal{A}}$ be a faithful map. I show that $(*)$ is satisfied.

Let $a \in HF^{\mathcal{X}}$. We must prove that $\tau(a) \in \Gamma(K)$, where K is the range of τ on $HF^{\mathcal{X}}$. If $a \in \mathcal{A}$ this is trivial. For the remaining cases,

$$\tau(a) = \{\tau(b) \mid b \in a\} \cup \{\tau(a_x) \mid x \in a \cap \mathcal{X}\}.$$

So, $\tau(a) \subseteq K$, and since a is finite, so too is $\tau(a)$. Thus $\tau(a) \in \Gamma(K)$, as required.

Hence, by the Co-Inductive Closure Theorem, the unique solution to the system consists of hereditarily finite sets in the sense of $V_{\mathcal{A}}$.

7.8 A Model of ZF$^-$ +AFA

This final section is fairly technical and assumes a sound knowledge of basic model theory. It is included for completeness only, since the material presented is not, at the present time, widely available.

The relative consistency result for AFA, Theorem 7.2.3, depends on an investigation of the dual questions:

- When are two sets pictured by the same graph?
- When do two graphs picture the same set?

This is the task I turn to in this section. Unless otherwise indicated, the assumed underlying set theory is ZFC$^-$; that is, Zermelo–Fraenkel set theory without the Axiom of Foundation. (I shall therefore ignore the possibility of atoms from now on. They would play no role in our development and would only be an unnecessary encumberance.)

The fundamental graph-theoretic notion that underlies our answer to the first of the above two questions is that of a *bisimulation.*[8]

Let M be a system. A binary relation R on M is called a *bisimulation* on M if, whenever aRb, then

$$(\forall x \in \mathrm{ch}_M(a))(\exists y \in \mathrm{ch}_M(b))(xRy) \wedge (\forall y \in \mathrm{ch}_M(b))(\exists x \in \mathrm{ch}_M(a))(xRy).$$

In words, if a and b are related via R, then for every child, x, of a there is a child, y, of b that is related to x, and vice versa.

The following example of this notion is basic. For two sets a, b, write $a \equiv b$ if and only if there is a graph M that is a picture of both a and b. Then $\equiv$ is a binary relation on the system V (i.e. the class of all sets, with the edge relation $x \longrightarrow y$ if and only if $y \in x$).

[8]The name comes from earlier uses of this notion in Computer Science, where it is related to a pair of processes each of which could 'simulate' the behavior of the other.

Lemma 7.8.1 The relation $\equiv$ is a bisimulation on V.

Proof: Suppose $a \equiv b$. Then there is a graph M, with top node m, and decorations d_1, d_2 of M, such that $d_1(m) = a$ and $d_2(m) = b$. Let $x \in a$. Then, as d_1 is a decoration

$$x \in \{d_1(n) \mid m \longrightarrow n\},$$

so $x = d_1(n)$ for some $n \in \mathrm{ch}_M(m)$. Let $y = d_2(n)$. Thus $y \in b$. I claim that $x \equiv y$. (By symmetry, this will be enough to establish the lemma.) In fact, the graph that pictures both x and y is just M_n, the restriction of M to all nodes that lie on some path starting from n. (The decorations that produce both x and y from this graph are simply the restrictions of d_1 and d_2 to M_n, respectively.) □

In general, a system will have many bisimulations. But, as I show below, there is always a unique maximal bisimulation. (The relation $\equiv$ of the above lemma is the maximal bisimulation on the system V.) The definition of the maximal bisimulation on a given system is straightforward.

Call a relation R on a system M *small* if it is a set. Then define a relation $\equiv_M$ on M by

$$a \equiv_M b \quad \text{if and only if} \quad aRb \text{ for some small bisimulation } R \text{ on } M.$$

As I show below, the relation $\equiv_M$ is the maximal bisimulation on M.

The following auxiliary notion will be helpful in our proof. If R is a binary relation on a system M, define the binary relation R^+ on M by aR^+b if and only if

$$(\forall x \in \mathrm{ch}_M(a))(\exists y \in \mathrm{ch}_M(b))(xRy) \wedge (\forall y \in \mathrm{ch}_M(b))(\exists x \in \mathrm{ch}_M(a))(xRy).$$

Then a relation R will be a bisimulation on M if and only if $R \subseteq R^+$, i.e. if and only if

$$aRb \;\Rightarrow\; aR^+b.$$

Note that the operator $(\,)^+$ is monotone; that is, if $R_1 \subseteq R_2$, then $R_1^+ \subseteq R_2^+$.

Lemma 7.8.2 Let M be any system. Then the relation $\equiv_M$ is the unique maximal bisimulation on M. That is:

(i) $\equiv_M$ is a bisimulation on M; and

(ii) if R is any bisimulation on M, then for any $a, b \in M$,

$$aRb \;\Rightarrow\; a \equiv_M b.$$

Proof: (i) Let $a \equiv_M b$. Thus aRb for some small bisimulation R on M. By definition of $\equiv_M$,

$$xRy \;\Rightarrow\; x \equiv_M y \quad (\forall x, y \in M).$$

So as $(\ )^+$ is monotone

$$xR^+y \Rightarrow x \equiv_M^+ y \quad (\forall x, y \in M).$$

But R is a bisimulation, so $R \subseteq R^+$. So, in particular, aR^+b, and hence $a \equiv_M^+ b$. This shows that $\equiv_M \subseteq \equiv_M^+$, which proves (i).

(ii) Let R be a given bisimulation on M, and let aRb. I show that $a \equiv_M b$. Let

$$R_0 = R \cap (M_a \times M_b).$$

It is routine to check that R_0 is a bisimulation on M such that aR_0b. But R_0 is small. Hence by definition of $\equiv_M$, $a \equiv_M b$. □

I am now in a position to show that the relation $\equiv$ on V is the maximal bisimulation on V.

Theorem 7.8.3 For all sets a, b

$$a \equiv b \;\Leftrightarrow\; a \equiv_V b.$$

Proof: By the maximality of $\equiv_V$, we know that

$$a \equiv b \;\Rightarrow\; a \equiv_V b.$$

Conversely, assume $a \equiv_V b$. Thus for some small bisimulation R on V, aRb. Define a new system M as follows. The nodes of M are the elements of R, that is, the ordered pairs (x, y) such that xRy. The edges of M are

$$(x, y) \longrightarrow (u, v) \quad \text{if and only if} \quad u \in x \;\&\; v \in y.$$

Now, if we define d_1 and d_2 on M by

$$d_1(x, y) = x,\; d_2(x, y) = y,$$

then it is easily seen that d_1 and d_2 are both decorations of M. But $(a, b) \in M$, so $M_{(a,b)}$ is a picture of both a and b. Thus by definition, $a \equiv b$. □

In general, bisimulation relations are not equivalence relations. But as the notation suggests, maximal bisimulations *are* equivalence relations.

Lemma 7.8.4 For any system M, the relation $\equiv_M$ is an equivalence relation on M.

Proof: *Reflexivity.* Since the identity relation on M is clearly a bisimulation relation, $\equiv_M$ is reflexive.

Symmetry. Suppose $a \equiv_M b$. Thus for some small bisimulation R, aRb. Let S be the reversal of R, i.e.

$$ySx \Leftrightarrow xRy.$$

It is easily seen that S is a bisimulation. Since bSa, it follows that $b \equiv_M a$.

Transitivity. Suppose $a \equiv_M b$ and $b \equiv_M c$. Let R, S be small bisimulations such that aRb and bRc. Define a relation T on M by

$$xTz \Leftrightarrow \exists y(xRy \wedge ySz).$$

It is routine to verify that T is a bisimulation on M. Since aTc, it follows that $a \equiv_M c$. □

The following simple lemma provides two conditions that imply $a \equiv_M b$.

Lemma 7.8.5 Let M be any system. Then for all $a, b \in M$:

(i) $\mathrm{ch}_M(a) = \mathrm{ch}_M(b) \Rightarrow a \equiv_M b$;
(ii) $M_a \cong M_b \Rightarrow a \equiv_M b$.

Proof: (i) Define R on M by

$$R = \{(a, b)\} \cup \{(x, x) \mid x \in M_a\}.$$

It is easily seen that R is a bisimulation on M such that aRb. Hence $a \equiv_M b$.

(ii) Let $\theta : M_a \cong M_b$, and define R on M by

$$xRy \Leftrightarrow x \in M_a \wedge y \in M_b \wedge \theta(x) = y.$$

Again it is routine to check that R is a bisimulation on M, so as aRb we again conclude that $a \equiv_M b$. □

A system M is said to be *extensional*[9] if, for all $a, b \in M$,

$$a \equiv_M b \Rightarrow a = b.$$

Theorem 7.8.6 The following are equivalent.

[9]In [1], Aczel uses the phrase 'strongly extensional' for this notion. In my development, I have no need for the weaker notion that Aczel refers to as 'extensional'.

(i) Every graph has at most one decoration.

(ii) V is extensional.

Proof: Assume (i). Let $a \equiv_V b$. Then by Theorem 7.8.3, $a \equiv b$, so there is a graph G with top node n, and decorations d_1 and d_2 of G, such that $d_1(n) = a$ and $d_2(n) = b$. By (i), $d_1 = d_2$. Hence $a = b$. This proves (ii).

Assume (ii). Let d_1 and d_2 be decorations of a graph G. If $x \in G$, then G_x is a picture of both $d_1(x)$ and $d_2(x)$, so $d_1(x) \equiv d_2(x)$. Hence by Theorem 7.8.3, $d_1(x) \equiv_V d_2(x)$. So by (ii), $d_1(x) = d_2(x)$. Hence $d_1 = d_2$. This proves (i). □

A *system map* from a system M to a system M' is a map $\pi : M \to M'$ such that for all $a \in M$, π maps the children of a in M onto the children of $\pi(a)$ in M'; i.e. for all $a \in M$,

$$\mathrm{ch}_{M'}(\pi(a)) = \{\pi(b) \mid b \in \mathrm{ch}_M(a)\}.$$

For example, any system map from a graph G into V is just a decoration of G.

The following result, which indicates how system maps preserve bisimulations, will be of use later.

Lemma 7.8.7 Let $\pi_1, \pi_2 : M \to M'$ be system maps.

(i) If R is a bisimulation on M, then $R' = (\pi_1 \times \pi_2)R$ is a bisimulation on M', where we define

$$(\pi_1 \times \pi_2)R = \{(\pi_1(a_1), \pi_2(a_2)) \mid a_1 R a_2\}.$$

(ii) If S' is a bisimulation on M', then $S = (\pi_1 \times \pi_2)^{-1}S'$ is a bisimulation on M, where we define

$$(\pi_1 \times \pi_2)^{-1}S' = \{(a_1, a_2) \in M \times M \mid (\pi_1(a_1))S'(\pi_2(a_2))\}.$$

Proof: (i) Let $b_1 R' b_2$ and suppose $b_1' \in \mathrm{ch}_{M'}(b_1)$. I show that there is a $b_2' \in \mathrm{ch}_{M'}(b_2)$ such that $b_1' R' b_2'$. Let a_1, a_2 be such that $b_1 = \pi(a_1), b_2 = \pi(a_2), a_1 R a_2$. Since $b_1' \in \mathrm{ch}_{M'}(b_1)$, there is an $a_1' \in \mathrm{ch}_M(a_1)$ such that $b_1' = \pi(a_1')$. Since R is a bisimulation, there is an $a_2' \in \mathrm{ch}_M(a_2)$ such that $a_1' R a_2'$. Let $b_2' = \pi(a_2')$. Then b_2' is as required.

Likewise, if $b_1 R' b_2$ and $b_2' \in \mathrm{ch}_{M'}(b_2)$, then there is a $b_1' \in \mathrm{ch}_{M'}(b_1)$ such that $b_1' R' b_2'$. Thus R' is a bisimulation on M'.

(ii) This is entirely analogous to the proof of part (i). □

Suppose now we have a system M and a bisimulation R on M that is also an equivalence relation on M. A system M' is said to be a *quotient* of M by R if and only if there is a surjective map $\pi : M \to M'$ such that for all $a, b \in M$,

$$aRb \Leftrightarrow \pi(a) = \pi(b).$$

Our main interest in quotients here concerns the extensional ones. The following lemma supplies some information about this.

Lemma 7.8.8 Let R be a bisimulation equivalence relation on a system M, and let $\pi : M \to M'$ be the corresponding quotient of M. Then M' is extensional if and only if R is the relation $\equiv_M$.

Proof: Suppose R is the relation $\equiv_M$. Let $\pi(a) \equiv_M \pi(b)$. I show that $\pi(a) = \pi(b)$. By Lemma 7.8.7(ii), $R' = (\pi \times \pi)^{-1}R$ is a bisimulation on M such that $aR'b$. Thus $a \equiv_M b$. But $\pi : M \to M'$ is the quotient of M by $\equiv_M$ (since this is R), so this implies that $\pi(a) = \pi(b)$.

Conversely, suppose that M' is extensional. I show that if S is any small bisimulation on M, and if aSb, then aRb, which at once implies that R is $\equiv_M$. By Lemma 7.8.7(i), $S' = (\pi \times \pi)S$ is a bisimulation on M' such that $\pi(a)S'\pi(b)$. Thus $\pi(a) \equiv_M \pi(b)$. Hence as M' is extensional, $\pi(a) = \pi(b)$. Thus aRb, as required. □

Using the above lemma, I can prove that every system, M, has an extensional quotient. The overall approach is as follows: take the bisimulation equivalence relation $\equiv_M$ on M, and construct a map π with domain M such that for all $a, b \in M$,

$$(*) \qquad \pi(a) = \pi(b) \Leftrightarrow a \equiv_M b.$$

In the case where M is a set, there is no difficulty in carrying out such a construction—it is all quite standard. The elements of the new system M' are taken to be the equivalence classes of M under the equivalence relation $\equiv_M$, and π maps each element of M to its equivalence class.

But in the case where M is a proper class, problems arise if any of the equivalence classes is a proper class. To circumvent this difficulty, the usual trick when working in well-founded Zermelo–Fraenkel set theory is to define 'equivalence classes' as being subsets of the least level of the cumulative hierarchy (of sets) at which they are nonempty. That is, given any $a \in M$, take the 'equivalence class' of a modulo $\equiv_M$ to be the *set*

$$\{b \in V_\alpha \mid b \in M \ \& \ a \equiv_M b\},$$

where α is minimal such that this collection is nonempty.

But in the absence of Foundation, this approach will not work. Instead, we adopt the following alternative.

For each $a \in M$, the set M_a is (by the Axiom of Choice) in one-one correspondence with some ordinal number, and this induces an isomorphism between the graph M_a and a corresponding graph whose domain is an ordinal. Let T_a be the class of all graphs with domain an ordinal, that are isomorphic to M_b for some $b \in M$ such that $a \equiv_M b$. Let

$$\pi(a) = \{G \in V_\alpha \mid G \in T_a\}$$

where α is the least ordinal such that this set is nonempty. I show that this definition satisfies $(*)$, as required.

If $a_1 \equiv_M a_2$, then $T_{a_1} = T_{a_2}$, so $\pi(a_1) = \pi(a_2)$. Conversely, if $a_1, a_2 \in M$ are such that $\pi(a_1) = \pi(a_2)$, then there is a graph G such that $G \in T_{a_1}$ and $G \in T_{a_2}$. Since $G \in T_{a_1}$, there is an $a_1' \in M$ such that $a_1 \equiv_M a_1'$ and $G \cong M_{a_1'}$. Likewise, as $G \in T_{a_2}$, there is an $a_2' \in M$ such that $a_2 \equiv_M a_2'$ and $G \cong M_{a_2'}$. Then $M_{a_1'} \cong M_{a_2'}$, so by Lemma 7.8.5(ii), $a_1' \equiv_M a_2'$. Thus $a_1 \equiv_M a_2$.

Theorem 7.8.9 Let M be any system. The following are equivalent:

(i) M is extensional;

(ii) for each (small) system M_0 there is at most one system map

$$\pi : M_0 \to M;$$

(iii) for each system M', every system map $\pi : M \to M'$ is one-one.

Proof: (i) $\Rightarrow$ (ii). Let $\pi_1, \pi_2 : M_0 \to M$ be system maps. By Lemma 7.8.7(i), $R = (\pi_1 \times \pi_2)(=_{M_0})$ is a bisimulation on M, where $=_{M_0}$ is the identity relation on M_0. Now, if $m \in M_0$, then $(\pi_1(m))R(\pi_2(m))$, so $\pi_1(m) \equiv_M \pi_2(m)$, and hence by (i), $\pi_1(m) = \pi_2(m)$. Thus $\pi_1 = \pi_2$, proving (ii).

(ii) $\Rightarrow$ (i). (For arbitrary systems M_0.) Let M_0 be the system whose nodes are the pairs (a, b) such that $a \equiv_M b$, and whose edges are all $(a, b) \longrightarrow (a', b')$ where $a \longrightarrow a'$ and $b \longrightarrow b'$ in M. Define $\pi_1, \pi_2 : M_0 \to M$ by $\pi_1(a, b) = a$, $\pi_2(a, b) = b$. It is routine to verify that π_1 and π_2 are system maps. Thus by (ii), $\pi_1 = \pi_2$, and hence $a = b$ whenever $a \equiv_M b$, proving (i).

(For small systems M_0.) It suffices to show that (ii) for small systems implies the unrestricted form of (ii). Let M_0 be a system, and let $\pi_1, \pi_2 : M_0 \to M$ be system maps. Let $a \in M_0$. Then $(M_0)_a$ is a small system, and $\pi_1 \quad (M_0)_a = \pi_2 \quad (M_0)_a$. In particular, $\pi_1(a) = \pi_2(a)$. But $a \in M_0$ was

arbitrary. Hence $\pi_1 = \pi_2$.

(i) $\Rightarrow$ (iii). Let $\pi : M \to M'$ be a system map. By Lemma 7.8.7(ii), $R = (\pi \times \pi)^{-1}(=_{M'})$ is a bisimulation on M (where $=_{M'}$ is the identity relation on M'). So, if $\pi(a) = \pi(b)$, then aRb, so $a \equiv_M b$, whence by (i), $a = b$. Thus π is one-one, as required.

(iii) $\Rightarrow$ (i). Let $\pi : M \to M'$ be an extensional quotient of M. By (iii), π is one-one. Hence $\pi : M \cong M'$. So, as M' is extensional, so too is M. □

I am now ready to give the construction of a model of the theory ZFC$^-$ + AFA.

Given a system M, an *M-decoration* of a graph G is just a system map $\pi : G \to M$.

Thus, in particular, a V-decoration of G is simply a decoration of G.

I call a system M *complete* if every graph has a unique M-decoration. (AFA says that V is a complete system.)

By Theorem 7.8.9, every complete system is extensional.

Let V_0 be the class of all graphs. Notice that every member of V_0 is of the form G_a, where G is a graph and a is a node of G. Using this observation, we make V_0 into a system by introducing the edges $G_a \longrightarrow G_b$ whenever G is a graph and $a \longrightarrow b$ in G.

Let $\pi_c : V_0 \to V_c$ be the extensional quotient of V_0.

Lemma 7.8.10 For each system M, there is a unique system map

$$\pi : M \to V_c.$$

Proof: If $a \in M$, then $M_a \in V_0$. Define $\pi : M \to V_0$ by $\pi(a) = M_a$. Clearly, π is a system map. Then $\pi_c \circ \pi : M \to V_c$ is a system map, which is unique by virtue of Theorem 7.8.9. □

Corollary 7.8.11 V_c is complete.

Proof: Immediate. □

Given any system M, we may obtain an interpretation of the language of set theory by letting the variables range over the nodes of M, and interpreting the predicate symbol '$\in$' by the relation $\in_M$ defined on M by

$$a \in_M b \text{ if and only if } b \longrightarrow a \text{ in } M$$

for all $a, b \in M$.

By virtue of the above corollary, the following result, which will be proved in just a moment, establishes the consistency (relative to that of the theory ZF$^-$) of the theory ZFC$^-$ + AFA.

Theorem 7.8.12 Every complete system is, under the interpretation described above, a model of ZFC$^-$ + AFA. □

Combining this theorem with Corollary 7.8.11, we see that V_c is a model of ZFC$^-$ + AFA. In fact, by virtue of Lemma 7.8.10, there is a unique system map $\pi : V \to V_c$, so V_c is a model of ZFC$^-$ + AFA that canonically embeds V. Thus we may regard our construction of the model V_c as providing an *extension* of the universe V. This gives the result stated as Theorem 7.2.3.

Call a system M *full* if for every set $u \subseteq M$, there is a unique element $a \in M$ such that $u = \mathrm{ch}_M(a)$.

For example, V is a full system, as is W, the class of all well-founded sets.

Lemma 7.8.13 Every complete system is full.

Proof: Let M be a complete system. Let $u \subseteq M$ be a set. Let G_0 be the graph consisting of all nodes and edges of M that lie on paths starting from a node in u. Obtain G from G_0 by adding one more node, t, together with edges $t \longrightarrow x$ for all $x \in u$.

Since M is complete, G has a unique M-decoration, d. Let $d_0 = d \quad G_0$. Then d_0 is an M-decoration of G_0. But the identity map is clearly the unique M-decoration of G_0. Hence $d_0(x) = x$ for all $x \in G_0$. So if we set $a = d(t)$, then $a \in M$ and

$$\begin{aligned} \mathrm{ch}_M(a) &= \{d(x) \mid t \longrightarrow x \text{ in } G\} \\ &= \{x \mid t \longrightarrow x \text{ in } G\} \\ &= u \end{aligned}$$

For uniqueness, suppose $a' \in M$ is also such that $\mathrm{ch}_M(a') = u$. Then we may define an M-decoration d' of G by setting $d'(t) = a'$, and $d'(x) = x$ for all $x \in G_0$. So by the uniqueness of d, $d' = d$. Hence, in particular,

$$a' = d'(t) = d(t) = a.$$

The proof is complete. □

Theorem 7.8.14 Every full system is a model of ZFC$^-$.

Proof: Let M be a full system. Fullness tells us that for each set $u \subseteq M$ there is a unique $a \in M$ such that $u = \mathrm{ch}_M(a)$. We shall denote this unique a by u^M. Using this notation, we check each of the axioms of ZFC$^-$ in turn.

Extensionality. Let $a, b \in M$ be such that

$$M \models (\forall x)(x \in a \leftrightarrow x \in b).$$

Then $\mathrm{ch}_M(a) = \mathrm{ch}_M(b)$. But $a = (\mathrm{ch}_M(a))^M$ and $b = (\mathrm{ch}_M(b))^M$. Hence $M \models a = b$.

Pairing. Let $a, b \in M$. Then $\{a, b\} \subseteq M$, so let $c = \{a, b\}^M$. Clearly,

$$M \models [a \in c \wedge b \in c].$$

Union. Let $a \in M$. Then $x = \bigcup\{\mathrm{ch}_M(y) \mid y \in \mathrm{ch}_M(a)\}$ is a subset of M, so let $c = x^M$. Then

$$M \models (\forall y \in a)(\forall z \in y)(z \in c).$$

Power set. Let $a \in M$. Then $x = \{y^M \mid y \subseteq \mathrm{ch}_M(a)\}$ is a subset of M, so let $c = x^M$. Then

$$M \models \forall x[(\forall z \in x)(z \in x) \rightarrow (x \in c)].$$

Infinity. Let

$$\begin{aligned} \theta_0 &= \emptyset^M, \\ \theta_{n+1} &= (\mathrm{ch}_M(\theta_n) \cup \{\theta_n\})^M, \quad \text{for } n = 0, 1, 2, \ldots . \end{aligned}$$

Then $\theta_n \in M$ for all n, so

$$\theta = \{\theta_n \mid n = 0, 1, 2, \ldots\}^M \in M.$$

Clearly,

$$M \models [\theta_0 \in \theta \wedge (\forall x \in \theta)(\exists y \in \theta)(x \in y)].$$

Separation. Let $a \in M$, and let $\phi(x)$ be a formula, possibly containing constants for elements of M, with at most the variable x free, and set

$$c = \{b \in \mathrm{ch}_M(a) \mid M \models \phi(b)\}^M.$$

Then

$$M \models \forall x(x \in c \leftrightarrow x \in a \wedge \phi(x)).$$

Collection. Let $a \in M$, and let $\phi(x, y)$ be a formula, possibly containing constants for elements of M, with at most the variables x and y free, and suppose that

$$M \models (\forall x \in a)(\exists y)\phi(x, y).$$

Then

$$(\forall x \in \mathrm{ch}_M(a))(\exists y)[y \in M \ \&\ M \models \phi(x, y)].$$

By the Collection Schema, there is a set b such that

$$(\forall x \in \mathrm{ch}_M(a))(\exists y \in b)[y \in M \ \&\ M \models \phi(x, y)].$$

Let $c = (b \cap M)^M$. Then

$$M \models (\forall x \in a)(\exists y \in c)\phi(x, y).$$

Choice. Let $a \in M$ be such that

$$M \models (\forall x \in a)(\exists y)(y \in x)$$

and

$$M \models (\forall x_1, x_2 \in a)[\exists y(y \in x_1 \wedge y \in x_2) \rightarrow (x_1 = x_2)].$$

Then

$$(\forall x \in \mathrm{ch}_M(a))(\mathrm{ch}_M(x) \neq \emptyset),$$

and, for all $x_1, x_2 \in \mathrm{ch}_M(a)$,

$$\mathrm{ch}_M(x_1) \cap \mathrm{ch}_M(x_2) \neq \emptyset \ \Rightarrow \ x_1 = x_2.$$

Thus $\{\mathrm{ch}_M(x) \mid x \in \mathrm{ch}_M(a)\}$ is a set of nonempty, pairwise-disjoint sets. So by the Axiom of Choice there is a set b such that for each $x \in \mathrm{ch}_M(a)$, the set $b \cap \mathrm{ch}_M(x)$ has a unique element $c_x \in M$. Then $c = \{c_x \mid x \in \mathrm{ch}_M(a)\}^M$ is such that

$$M \models (\forall x \in a)(\exists y \in x)(\forall u \in x)[u \in c \leftrightarrow u = y].$$

The proof is complete. □

By virtue of Lemma 7.8.13, the above result tells us that every complete system M is a model of ZFC$^-$. Thus the following completes our proof of Theorem 7.8.12.

Theorem 7.8.15 Every complete system is a model of AFA.

Proof: Let M be a complete system. For $a, b \in M$, define the "M-ordered pair" $(a,b)_M$ of a, b by

$$(a,b)_M = \{\{a\}^M, \{a,b\}^M\}^M.$$

(Thus, within M, $(a,b)_M$ has the standard set-theoretic structure of the usual ordered pair of a, b.)

Now, a *graph* is, officially, an ordered pair consisting of a set and a binary relation on that set. Thus for $c \in M$,

$$M \models \text{"}c \text{ is a graph"}$$

if and only if there are $a, b \in M$ such that $c = (a,b)_M$ and

$$M \models \text{"}b \text{ is a binary relation on } a\text{"}.$$

This last requirement reduces to

$$\mathrm{ch}_M(b) \subseteq \{(x,y)_M \mid x, y \in \mathrm{ch}_M(a)\}.$$

Hence, if $c \in M$ is such that $M \models$ "c is a graph", we may define a genuine graph G by taking a, b as above and letting the elements of $\mathrm{ch}_M(a)$ be the nodes of G and the pairs (x,y) such that $(x,y)_M \in \mathrm{ch}_M(b)$ the edges. Since M is complete, G has a unique M-decoration, d. Then $d : \mathrm{ch}_M(a) \to M$, and for all $x \in \mathrm{ch}_M(a)$,

$$d(x) = \{d(y) \mid (x,y)_M \in \mathrm{ch}_M(b)\}.$$

Set

$$f = \{(x, d(x))_M \mid x \in \mathrm{ch}_M(a)\}^M.$$

Then $f \in M$, and it is routine to verify that

$$M \models \text{"}f \text{ is the unique decoration of the graph } G\text{"}.$$

The proof is complete. □

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Glossary of Symbols

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The author is a well-known mathematician and the editor of the "Computers in Mathematics" column in the *AMS Notices* and of *FOCUS*, the magazine published by the MAA.

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