

An Introduction to Latent Order LOGistic (LOLOG) Network Models

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Getting Started

This vignette is based on the vignette from the `ergm` package, and is designed to give a working introduction to LOLOG modeling.

This vignette will utilize the statnet suite of packages for data and network manipulation. To install these:

```
install.packages("statnet")
```

To install the latest version of the package from CRAN (not yet available):

```
install.packages("lolog")
```

To install the latest development version from github run the following (you'll need a sh shell and git):

```
git clone https://github.com/fellstat/lolog.git
sh lolog/mkdist
```

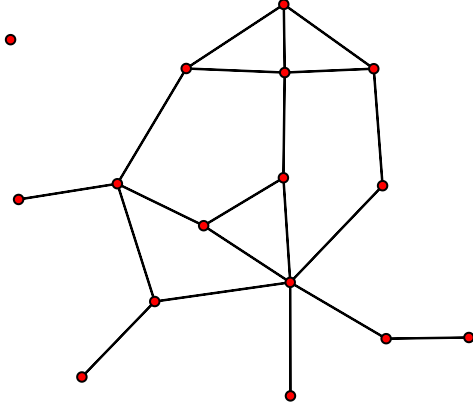
Statistical Network Modeling: The lolog Command and lolog Object

Make sure the lolog package is attached:

```
library(lolog)
set.seed(1)
```

The `ergm` package contains several network data sets that you can use for practice examples.

```
suppressPackageStartupMessages(library(ergm))
#data(package='ergm') # tells us the datasets in our packages
data(florentine) # loads flomarriage and flobusiness data
flomarriage # Let's look at the flomarriage data
#> Network attributes:
#>   vertices = 16
#>   directed = FALSE
#>   hyper = FALSE
#>   loops = FALSE
#>   multiple = FALSE
#>   bipartite = FALSE
#>   total edges= 20
#>   missing edges= 0
#>   non-missing edges= 20
#>
#> Vertex attribute names:
#>   priorates totalties vertex.names wealth
#>
#> No edge attributes
plot(flomarriage) # Let's view the flomarriage network
```



Networks tend to evolve over time. This evolution can take the form of the addition or deletion of a tie from the network, or the addition or deletion of a vertex. LOLOGs are motivated by a growth process, where each edge variable is sequentially considered for edge creation, and edges are not deleted.

Remember a LOLOG represents the probability of a tie, given the network grown up to a time-point as

$$\text{logit}(p(y_{s_t} = 1 | \eta, y^{t-1}, s_{\leq t})) = \theta \cdot c(y_{s_t} = 1 | y^{t-1}, s_{\leq t})$$

where $s_{\leq t}$ is the growth order of the network up to time t , y^{t-1} is the state of the graph at time $t - 1$. $c(y_{s_t} | y^{t-1}, s_{\leq t})$ is a vector representing the change in graph statistics from time $t - 1$ to t if an edge is present, and θ is a vector of parameters.

An Erdos-Renyi model

We begin with the simplest possible model, the Bernoulli or Erdos-Renyi model, which contains only an edge term.

```
flomodel.01 <- lollog(flomarriage~edges) # fit model
#> Initializing using variational fit
#>
#> Model is dyad independent. Replications are redundant. Setting nReplicates <- 1L.
#> Model is dyad independent. Returning maximum likelihood estimate.
flomodel.01
#> MLE Coefficients:
#> edges
#> -1.609438

summary(flomodel.01) # look in more depth
#> observed_statistics theta se pvalue
#> edges 20 -1.609438 0.2449451 0
```

How to interpret this model? The log-odds of any tie occurring is:

$$= -1.609 \times \text{change in the number of ties} = -1.609 \times 1$$

for all ties, since the addition of any tie to the network changes the number of ties by 1!

Corresponding probability is:

$$\exp(-1.609)/(1 + \exp(-1.609)) = 0.1667$$

which is what you would expect, since there are 20/120 ties.

A Triangle Model

Let's add a term often thought to be a measure of clustering: the number of completed triangles.

```
flomodel.02 <- lollog(flomarriage~edges()+triangles(), verbose=FALSE)
summary(flomodel.02)
#>      observed_statistics      theta      se pvalue
#> edges                20 -1.6350235 0.2610002 0.0000
#> triangles              3  0.2007203 0.7161925 0.7793
```

```
coef1 = flomodel.02$theta[1]
coef2 = flomodel.02$theta[2]
logodds = coef1 + c(0,1,2) * coef2
expit = function(x) 1/(1+exp(-x))
ps = expit(logodds)
coef1 = round(coef1, 3)
coef2 = round(coef2, 3)
logodds = round(logodds, 3)
ps = round(ps, 3)
```

Again, how to interpret coefficients?

Conditional log-odds of two actors forming a tie is:

$$-1.635 \times \text{change in the number of ties} + 0.201 \times \text{change in number of triangles}$$

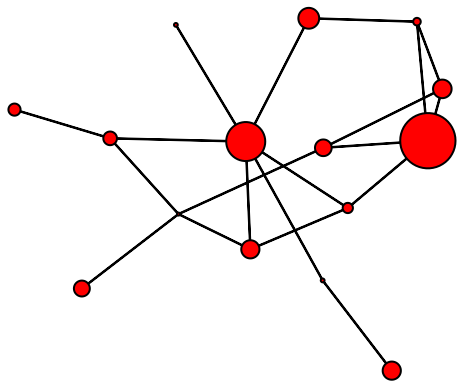
- if the tie will not add any triangles to the network, its log-odds is: -1.635 .
- if it will add one triangle to the network, its log-odds is: $-1.635 + 0.201 = -1.434$
- if it will add two triangles to the network, its log-odds is: $-1.635 + 0.201 \times 2 = -1.234$
- the corresponding probabilities are 0.163, 0.192, and 0.226.

Let's take a closer look at the `lollog` object itself:

```
class(flomodel.02) # this has the class lollog
#> [1] "lollogGmm" "lollog"      "list"

names(flomodel.02) # let's look straight at the lollog obj.
#> [1] "method"      "formula"      "auxFromula"
#> [4] "theta"       "stats"        "estats"
#> [7] "auxStats"    "obsStats"     "net"
#> [10] "grad"        "vcov"         "likelihoodModel"
```

```
flomodel.02$theta
#>      edges triangles
#> -1.6350235  0.2007203
flomodel.02$formula
#> flomarriage ~ edges() + triangles()
wealth <- flomarriage %v% 'wealth' # the %v% extracts vertex
wealth # attributes from a network
#> [1] 10 36 55 44 20 32 8 42 103 48 49 3 27 10 146 48
plot(flomarriage, vertex.cex=wealth/25) # network plot with vertex size
```



proportional to wealth

The Effect of Wealth

We can test whether edge probabilities are a function of wealth:

```
flomodel.03 <- lollog(flomarriage~edges+nodeCov('wealth'))
#> Initializing using variational fit
#>
#> Model is dyad independent. Replications are redundant. Setting nReplicates <- 1L.
#> Model is dyad independent. Returning maximum likelihood estimate.
summary(flomodel.03)
#>               observed_statistics          theta          se pvalue
#> edges                20 -2.59492903 0.536051763 0.0000
#> nodecov.wealth       2168  0.01054591 0.004674236 0.0241
```

Yes, there is a significant positive wealth effect on the probability of a tie.

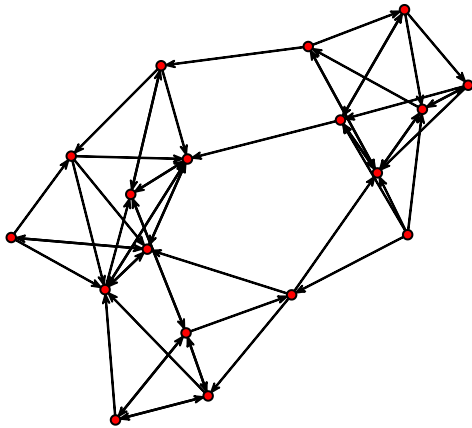
Reciprocity

Let's try a model or two on:

Is there a statistically significant tendency for ties to be reciprocated ('mutuality')?

```
data(samplk)
ls() # directed data: Sampson's Monks
#> [1] "coef1"      "coef2"      "expit"      "fit"        "flobusiness"
#> [6] "flomarriage" "flomodel.01" "flomodel.02" "flomodel.03" "lazega"
#> [11] "logodds"    "ps"         "samplk1"    "samplk2"    "samplk3"
#> [16] "seniority"  "senority"   "wealth"
samplk3
#> Network attributes:
#> vertices = 18
#> directed = TRUE
#> hyper = FALSE
#> loops = FALSE
#> multiple = FALSE
#> bipartite = FALSE
```

```
#> total edges= 56
#> missing edges= 0
#> non-missing edges= 56
#>
#> Vertex attribute names:
#> cloisterville group vertex.names
#>
#> No edge attributes
plot(samplk3)
```

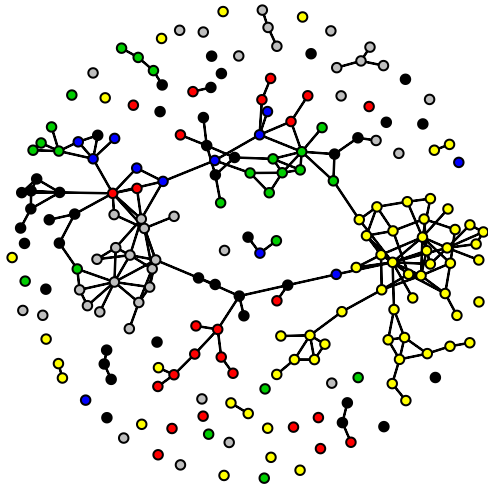


```
sampmodel.01 <- lolog(samplk3~edges+mutual, verbose=FALSE)
summary(sampmodel.01)
#> observed_statistics theta se pvalue
#> edges 56 -1.780017 0.1710626 0e+00
#> mutual 15 2.487496 0.6743379 2e-04
```

Modeling Larger Networks

Let's try a larger network. First we will fit a dyad independent model.

```
data(faux.mesa.high)
mesa <- faux.mesa.high
mesa
#> Network attributes:
#> vertices = 205
#> directed = FALSE
#> hyper = FALSE
#> loops = FALSE
#> multiple = FALSE
#> bipartite = FALSE
#> total edges= 203
#> missing edges= 0
#> non-missing edges= 203
#>
#> Vertex attribute names:
#> Grade Race Sex
#>
#> No edge attributes
plot(mesa, vertex.col='Grade')
```



```
#legend('bottomleft',fill=7:12,legend=paste('Grade',7:12),cex=0.75)
mesa %v% "GradeCat" <- as.character(mesa %v% "Grade")
fauxmodel.01 <- lolog(mesa ~edges + nodeMatch('GradeCat') + nodeMatch('Race'))
#> Initializing using variational fit
#>
#> Model is dyad independent. Replications are redundant. Setting nReplicates <- 1L.
#> Model is dyad independent. Returning maximum likelihood estimate.
summary(fauxmodel.01)
#>
#>      observed_statistics      theta      se pvalue
#> edges                203 -6.2254111 0.1737005 0.0000
#> nodematch.GradeCat    163  2.8278899 0.1773356 0.0000
#> nodematch.Race        103  0.4266748 0.1427599 0.0028
```

Now lets try adding in transitivity and 2-stars (a measure of degree spread)

```
# This may take a minute or two
fauxmodel.02 <- lolog(mesa ~edges + nodeMatch('GradeCat') + nodeMatch('Race') +
                      triangles + star(2), verbose=FALSE)
summary(fauxmodel.02)
#>
#>      observed_statistics      theta      se pvalue
#> edges                203 -6.32253753 0.23009490 0.0000
#> nodematch.GradeCat    163  2.59264481 0.19642150 0.0000
#> nodematch.Race        103  0.41817975 0.16839142 0.0130
#> triangles              62  3.07545054 0.75245219 0.0000
#> star.2                659  0.02291813 0.05504247 0.6771
```

We see strong evidence of additional transitivity, but little evidence that the degrees have higher spread than expected given the other terms. With LOLOG models, we avoid some of the degeneracy problems present in ERGM models. Let's try the same model in `ergm`

```
fauxmodel.01.ergm <- ergm(mesa ~edges + nodematch('GradeCat') + nodematch('Race') +
                          triangles + kstar(2))
#> Starting maximum likelihood estimation via MCMLE:
#> Iteration 1 of at most 20:
#> Error in ergm.MCMLE(init, nw, model, initialfit = (initialfit <- NULL), : Number of edges in a simul
```

Of course it is highly recommended that you take great care in selecting terms for use with `ergm` for precisely the reason that many can lead to model degeneracy. A much better choice here for the `ergm` model would have been to use `gwesep` and `gwdegree` in place of `triangles` and `2-stars`.

Specifying Vertex Inclusion Order

LOLOG models are motivated by a growth process where edge variables are “added” to the network one at a time. The package implements a sequential vertex ordering process. Specifically, vertices are “added” to the network in random order, and then edge variables connecting the added vertex to each other vertex already added are included in random order.

If you either observe or partially observe the inclusion order of vertices, this information can be included by specifying it in the model formula after a ‘|’ character. In this example we will consider the network relations of a group of partners at a law firm. We observe the seniority of the partner, the type of practice and the office where they work. (cSeniority and cPractice are just categorical versions of the numerically coded practice and seniority variables). It is plausible to posit that partners with more seniority entered the law firm and hence the network prior to less senior individuals.

```
library(network)
data(lazega)
seniority <- as.numeric(lazega %v% "seniority") # Lower values are more senior
fit <- lollog(lazega ~ edges() + triangles() + nodeCov("cSeniority") +
             nodeCov("cPractice") + nodeMatch("gender") + nodeMatch("practice") +
             nodeMatch("office") | seniority, verbose=FALSE)
summary(fit)
```

	observed_statistics		theta	se	pvalue
#> edges	115	-7.60173477	1.11835307	0.0000	
#> triangles	120	1.04646066	0.40911589	0.0105	
#> nodecov.cSeniority	4687	0.02756058	0.01034596	0.0077	
#> nodecov.cPractice	359	0.71222182	0.18390064	0.0001	
#> nodematch.gender	99	1.12996532	0.43492149	0.0094	
#> nodematch.practice	72	0.94769362	0.27011651	0.0005	
#> nodematch.office	85	1.59698632	0.29649304	0.0000	

In some cases only a partial ordering is observed. For example we might observed the month someone joined a group, with multiple individuals joining per month. When ties like this exist in the ordering variable, network simulations respect the partial ordering, with ties broken at random for each simulated network.

Model Terms Available for LOLOG Estimation and Simulation

Model terms are the expressions (e.g., `triangle`) used to represent predictors on the right-hand side of equations used in:

- calls to `lollog` (to estimate an LOLOG model)
- calls to `simulate` (to simulate networks from an LOLOG model fit)
- calls to `calculateStatistics` (to obtain measurements of network statistics on a dataset)

Terms Provided with `lollog`

For a list of available terms that can be used to specify a LOLOG model, type:

```
help('lollog-terms')
```

This currently produces:

- `edges` (dyad-independent) (order-independent) (directed) (undirected)
- `star(k, direction=1L)` (order-independent) (directed) (undirected)
- `triangles()` (order-independent) (directed) (undirected)
- `clustering()` (order-independent) (undirected)

- `transitivity()` (order-independent) (undirected)
- `mutual()` (order-independent) (directed)
- `nodeCov(name)` (dyad-independent) (order-independent) (directed) (undirected)
- `nodeMatch(name)` (dyad-independent) (order-independent) (directed) (undirected)
- `nodeMix(name)` (dyad-independent) (order-independent) (directed) (undirected)
- `degree(d, direction=0L, lessThanOrEqual=FALSE)` (order-independent) (directed) (undirected)
- `degreeCrossProd()` (order-independent) (undirected)
- `gwdsp(alpha)` (order-independent) (directed) (undirected)
- `gwest(alpha)` (order-independent) (directed) (undirected)
- `gwdegree(alpha, direction=0L)` (order-independent) (directed) (undirected)
- `esp(d)` (order-independent) (directed) (undirected)
- `geoDist(long, lat, distCuts=Inf)` (dyad-independent) (order-independent) (undirected)
- `dist(names)` (dyad-independent) (order-independent) (undirected)
- `preferentialAttachment(k=1)` (undirected)
- `sharedNbrs(k=1)` (undirected)
- `nodeLogMaxCov(name)` (order-independent) (undirected)
- `boundedDegree(lower, upper)` (order-independent) (undirected) Adds a constraint that the degrees for the network must be between lower and upper.

Key terms still to be added are:

- `ctriple`
- `ttriple`
- `triadcensus`
- ?

Coding New Terms

Full details on coding new terms is beyond the scope of this document. However, `lolog` provides facilities for extending the package to provide new terms. An example of this can be created via

```
lologPackageSkeleton()
```

and then at the command prompt, run

```
R CMD build LologExtension
R CMD INSTALL LologExtension_1.0.tar.gz
```

A new package will be installed implementing the `minDegree` (minimum degree) statistic.

Alternatively, C++ extensions can be added directly from R without the need of any package infrastructure via the inlining features of the `Rcpp` package. `lolog` provides an inline plugin for this feature. For details, see

```
help("inlineLologPlugin")
```

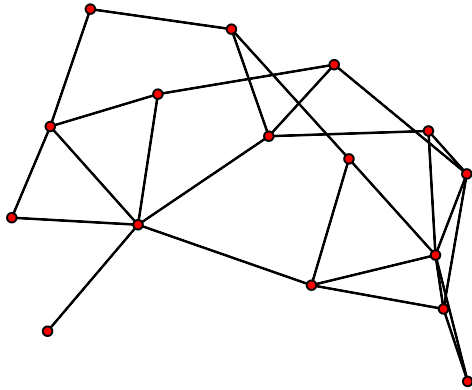
Network Simulation and Computation

Network Statistics can be calculated using the `calculateStatistics` function

```
calculateStatistics(mesa ~ edges + triangles + degree(0:15))
#>      edges triangles degree.0 degree.1 degree.2 degree.3 degree.4
#>      203       62      57      51      30      28      18
#> degree.5 degree.6 degree.7 degree.8 degree.9 degree.10 degree.11
#>      10       2       4       1       2       1       0
#> degree.12 degree.13 degree.14 degree.15
#>      0       1       0       0
```


Simulating from a fitted `lolog` object with

```
nets <- simulate(flomodel.03,nsim=10) #Generates a list of BinaryNet objects
plot(nets[[1]])
```



Voila. Of course, networks that you generate will look somewhat different. Note that the objects created are `BinaryNet` objects unless the `convert` parameter is set to `TRUE`, in which case they are `network` objects.

Working with `BinaryNet` objects

`BinaryNets` are network data structures native to the `lolog` package. They are special in a couple ways. First, they have a sparse representation of missingness, such that a directed network where half of the nodes have been egocentrically sampled takes the same amount of space as a simple fully observed network. Secondly, they may be passed up and down easily from R to C++ and vis a versa. Finally, they are extensible on the C++ level, so that their implementation may be replaced. For example, it might be useful to put in a file backed storage system for large problems.

`BinaryNets` can be coerced to and from `network` objects.

```
data(sampson)

#coersion
net <- as.BinaryNet(samplike)
nw2 <- as.network(net)
print(nw2)
#> Network attributes:
#> vertices = 18
#> directed = TRUE
#> hyper = FALSE
#> loops = FALSE
#> multiple = FALSE
#> bipartite = FALSE
#> total edges= 88
#> missing edges= 0
#> non-missing edges= 88
#>
#> Vertex attribute names:
#> cloisterville group vertex.names
#>
#> No edge attributes

#dyad Extraction
```

```

net[1:2,1:5]
#>      [,1] [,2] [,3] [,4] [,5]
#> [1,] FALSE TRUE  TRUE FALSE  TRUE
#> [2,]  TRUE FALSE FALSE FALSE FALSE
net$outNeighbors(c(1,2,3))
#> [[1]]
#> [1]  2  3  5  8 12 14
#>
#> [[2]]
#> [1]  1  7 12 14 15
#>
#> [[3]]
#> [1]  1  2 13 17 18

#dyad assignment
net[1,1:5] <- rep(NA,5)
net[1:2,1:5]
#>      [,1] [,2] [,3] [,4] [,5]
#> [1,] FALSE  NA  NA  NA  NA
#> [2,]  TRUE FALSE FALSE FALSE FALSE
net[1:2,1:5,maskMissing=FALSE] #remove the mask over missing values and see
#>      [,1] [,2] [,3] [,4] [,5]
#> [1,] FALSE TRUE  TRUE FALSE  TRUE
#> [2,]  TRUE FALSE FALSE FALSE FALSE
#nothing was really changed

#node variables
net$variableNames()
#> $discrete
#> [1] "cloisterville" "group"          "na"          "vertex.names"
#>
#> $continuous
#> character(0)
net[["group"]]
#> [1] Turks      Outcasts Loyal      Loyal      Loyal      Turks
#> [8] Loyal      Loyal      Loyal      Loyal      Turks      Outcasts Turks
#> [15] Turks      Turks      Outcasts Outcasts
#> Levels: Loyal Outcasts Turks
net[["rnorm"]] <- rnorm(18)
net[["rnorm"]]
#> [1]  0.350761138 -1.767448060 -0.744665531  0.778878471 -0.004706815
#> [6] -0.404851582 -0.390180373  0.996085981  0.928690498 -0.694666962
#> [11] -0.690140857  0.224105272 -0.015726322  0.555083584  1.236164093
#> [16]  0.170120622  1.136213038 -1.919781814

#See available methods
#print(DirectedNet)
#print(UndirectedNet)

```

All user facing functions in the `lolog` package accept `BinaryNets` as arguments, and will convert `network` objects to `BinaryNets` automatically.

Order Dependent Terms

LOLOG allows for network statistics that depend not just on the network, but also the (unobserved) order in which dyads were ‘added’ to the network. One model of this class is Barabasi-Albert preferential attachment model, which is closely approximated by a LOLOG model with an `edges` and `preferentialAttachment` term.

For each order dependent statistic, one or more order independent statistics must be specified as moment matching targets. In this case, we will use a two-star term:

```
flomodel.04 <- lolog(flomarriage ~ edges() + preferentialAttachment(),
                    flomarriage ~ star(2), verbose=FALSE)
summary(flomodel.04)
```

#>	observed_statistics	theta	se	pvalue
#> edges	20	-1.43951470	2.3139965	0.5339
#> preferentialAttachment	NA	0.06848808	0.9513365	0.9426

Examining the Quality of Model Fit - GOF

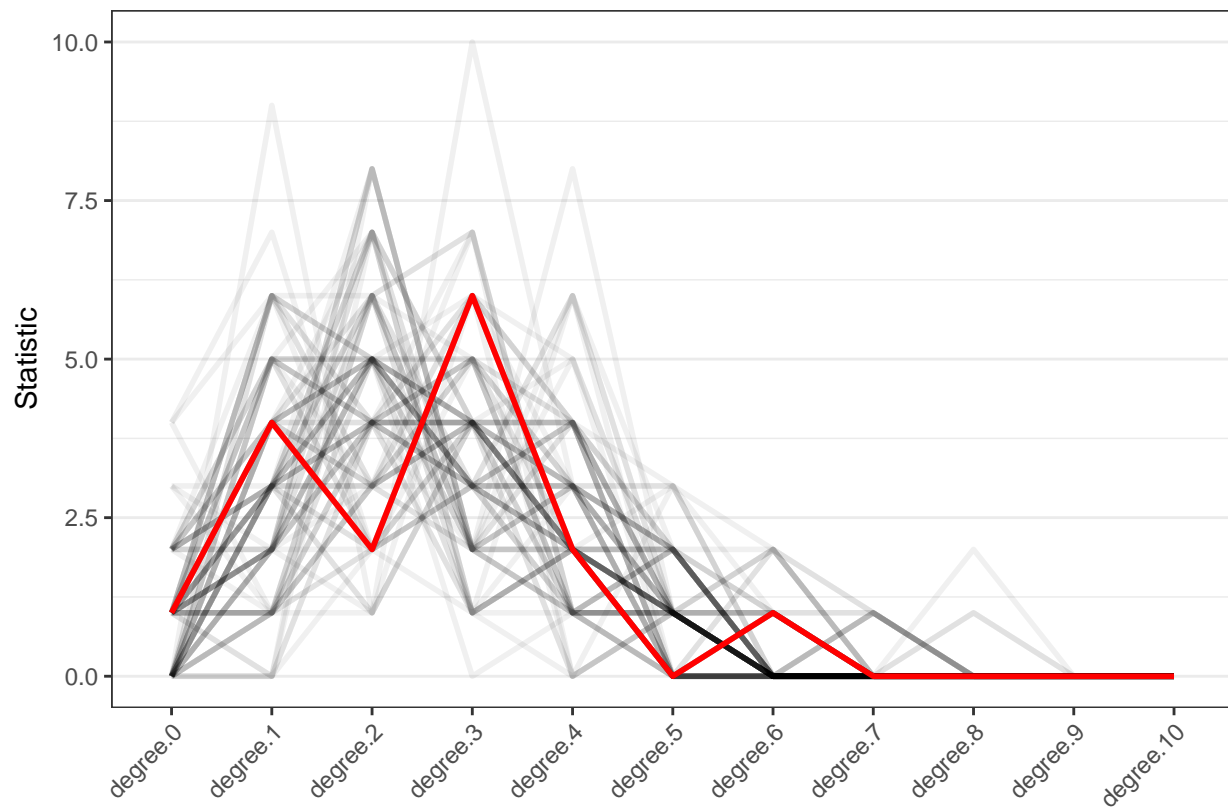
LOLOGs are generative models - that is, they represent the process that governs tie formation at a local level. These local processes in turn aggregate up to produce characteristic global network properties, even though these global properties are not explicit terms in the model. One test of whether a model “fits the data” is therefore how well it reproduces these global properties. We do this by choosing a network statistic that is not in the model, and comparing the value of this statistic observed in the original network to the distribution of values we get in simulated networks from our model.

We begin by comparing the degree structure of simulated networks compared to the observed (red)

```
gdeg <- gofit(flomodel.03, flomarriage ~ degree(0:10))
gdeg
```

#>	obs	min	mean	max	pvalue
#> degree.0	1	0	0.97	4	0.9752954
#> degree.1	4	0	3.10	9	0.6138744
#> degree.2	2	1	4.36	8	0.1844493
#> degree.3	6	0	3.59	10	0.1635244
#> degree.4	2	0	2.54	8	0.7176858
#> degree.5	0	0	1.00	3	0.2343462
#> degree.6	1	0	0.31	2	0.2484777
#> degree.7	0	0	0.09	1	0.7543497
#> degree.8	0	0	0.04	2	0.8691868
#> degree.9	0	0	0.00	0	NA
#> degree.10	0	0	0.00	0	NA

```
plot(gdeg)
```



Next we can look at the edgewise shared partner distribution in simulated networks compared with the observed distribution (red)

```
gesp <- gofit(flomodel.03, flomarriage ~ esp(0:5))
gesp
#>      obs min  mean max  pvalue
#> esp.0  12   7 13.44  20 0.6224265
#> esp.1   7   0  5.45  16 0.6408290
#> esp.2   1   0  1.07   7 0.9632507
#> esp.3   0   0  0.11   2 0.7499143
#> esp.4   0   0  0.00   0      NA
#> esp.5   0   0  0.00   0      NA
plot(gesp)
```

