33-777

Today

- FRW Cosmologies
- Age of the Universe

In this lecture we will apply the Friedmann equation to study "cosmological dynamics" for various possible Universes.

- Friedmann-Robertson-Walker Cosmologies

In the last lecture, we saw that by combining:

1. FRW metric
2. Einstein field equation
3. perfect fluid energy-momentum tensor

we can derive the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

Along with an equation of state for the material contents of the Universe,

$$\frac{d\rho}{da} + 3\left(\frac{\rho + P/c^2}{a}\right) = 0$$

Recall we derived this from the 1^{st} + 2^{nd} laws of thermodynamics

and some initial conditions (K, Λ), the Friedmann equation allows us to solve for the size of the Universe (a) and the density (ρ).

By considering the cosmological constant as an energy component of the Universe with "density" given by,

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}$$

and an equation of state,

$$P_\Lambda = \omega \rho_\Lambda c^2$$

where $\omega = -1$, we can rewrite the Friedmann equation as,

$$\underbrace{\left(\frac{\dot{a}}{a}\right)^2}_{\text{Hubble parameter}} = H^2(t) = \frac{8\pi G}{3}\overbrace{\left[\underbrace{\rho_{m,0}\left(\frac{a_0}{a}\right)^3}_{\text{Matter contribution}} + \underbrace{\rho_{r,0}\left(\frac{a_0}{a}\right)^4}_{\text{radiation contribution}} + \underbrace{\rho_{\Lambda,0}}_{\text{Vacuum energy}}\right]}^{\rho(a)} - \underbrace{\frac{Kc^2}{a^2}}_{\text{curvature}}$$

Note that $\underbrace{a_0 = a(t_0)}_{\text{usually taken to be } t_{now}}$.

Recall that the curvature signiture, K, controlls the geometry of the Universe

* $K = -1$

  negative curvature (open universe)

* $K = 0$

  zero curvature (flat universe)

* $K = +1$

  positive curvature (closed universe)

We use the $K=0$ case to define the critical density. Filling $K=0$ into the Friedmann equation allows one to solve for,

$$\rho_{crit}(t) = \frac{3H^2(t)}{8\pi G}$$

Now, by examination of the Friedmann equation, One can see that if,

* $\rho < \rho_{crit} \Rightarrow K < 0$

  negative curvature "open" Universe

* $\rho = \rho_{crit} \Rightarrow K = 0$

  flat Universe

* $\rho > \rho_{crit} \Rightarrow K > 0$

  positive curvature "closed" universe

It is common to parameterize the density components of the cosmological fluid as,

$$\Omega_x(t) \equiv \frac{\rho_x(t)}{\rho_{crit}(t)} \quad \text{(note } \Omega_x \text{ is unitless)}$$

The total density is then found by summation,

$$\Omega(t) = \sum_x \Omega_x(t) = \Omega_m(t) + \Omega_r(t) + \Omega_\Lambda(t)$$

If you recall that the scale factor is related to the redshift by,

$$a = \frac{1}{1+z} \quad (\text{we have taken } a_0 = 1)$$

then we can use this new notation to rewrite the Friedmann equation as,

$$H^2(t) = H_0^2\left[\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_\Lambda\right] - \frac{Kc^2}{a^2}$$

At the present $(t = t_0, z = 0)$ this equation becomes,

$$H_0^2 = H_0^2\,\Omega_0 - \frac{Kc^2}{a_0^2}$$

Solving for the curvature term,

$$-\frac{Kc^2}{a_0^2} = (1 - \Omega_0)H_0^2$$

Dividing both sides by a^2,

$$-\frac{Kc^2}{a_0^2}\frac{1}{a^2} = (1-\Omega_0) H_0^2 \frac{1}{a^2}$$

Using $a_0 = 1$ and $a = \frac{1}{1+z}$

$$\Rightarrow -\frac{Kc^2}{a^2} = (1-\Omega_0) H_0^2 (1+z)^2$$

This allows us to write curvature as another component where,

$$\Omega_{K,0} \equiv -\frac{Kc^2}{H_0^2 a_0^2} = 1 - \Omega_0$$

And the Friedmann equation becomes,

$$H^2(t) = H_0^2 \left[\Omega_{\Lambda,0} + (1-\Omega_0)(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 \right]$$

Equivalently this is often written as,

$$H^2(z) = H_0^2 E^2(z)$$

$$E(z) = \left[\Omega_{\Lambda,0} + (1-\Omega_0)(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 \right]^{1/2}$$

From this form of the Friedmann equation it should be clear that the redshift evolution of the various components can be written as,

$$\Omega_\Lambda(z) = \frac{\Omega_{\Lambda,0}}{E^2(z)}$$

$$\Omega_m(z) = \frac{\Omega_{m,0}(1+z)^3}{E^2(z)}$$

$$\Omega_r(z) = \frac{\Omega_{r,0}(1+z)^4}{E^2(z)}$$

And the total density,

$$\Omega(z) - 1 = \frac{(\Omega_0 - 1)(1+z)^2}{E^2(z)}$$

Given the matter and radiation terms in $E(z)$, it should be clear that as $z \to \infty$, $\Omega(z) \to 1$ as long as $\Omega_{m,0}$ and/or $\Omega_{r,0}$ are greater than zero.

This means that at a sufficiently high redshift (far enough in the past), the density approaches the critical density ($\Omega_0 = 1$) and the Universe was close to flat. This is sometimes called the "flatness problem" for reasons that will become clear

Because Ω_x scales with redshift differently for different components, the Universe goes through different "epochs" where one of the components may dominate the others.

- Radiation dominated epoch

In the absence of an earlier contracting phase ("a bounce") at sufficiently early times (high redshift, z) the Universe must be dominated by radiation.

This is relatively easy to see if we notice that

$$\Omega_r \propto (1+z)^\alpha \quad ; \quad \alpha = 4$$

while for all other components

$$\Omega_x \propto (1+z)^\beta \quad ; \quad \beta < 4$$

Thus at sufficiently high redshift,

$$\Omega_r >> \Omega_m, \Omega_\Lambda, \Omega_K$$

and $\Omega \simeq \Omega_r$, $\rho \simeq \rho_r$. In this case, the Friedmann equation simplifies to,

$$(a\dot{a})^2 = \frac{8\pi G \rho_{r,0}}{3} a_0^4$$

The solution to this equation is,

$$\left(\frac{a}{a_0}\right) = \left(\frac{32\pi G \rho_{r,0}}{3}\right)^{1/4} t^{1/2}$$

As the universe expands, other components become important and start to influence the evolution of the universe. We have already noted that for radiation,

$$\rho_r = \rho_{r,0} (1+z)^4$$

Cold matter (dust) has the next steepest dependence on redshift,

$$\rho_M = \rho_{M,0} (1+z)^3$$

The redshift when these two components become equal is known as "matter-radiation equality" and defines

$$\rho_M = \rho_r$$

$$\Rightarrow z_{eq} = \frac{\rho_{M,0}}{\rho_{r,0}} - 1 \simeq 1000$$

for our universe

- Matter dominated Universe

If we assume that $\Omega_{\Lambda,0} = 0$, then at late enough times, $z \ll z_{eq}$, the radiation content of the Universe has almost no effect on the evolution of the scale factor.

Taking $\rho = \rho_M$ and $\Lambda = 0$, the Friedmann equation takes on the form,

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_{M,0}\left(\frac{a_0}{a}\right)^3 - \frac{Kc^2}{H_0^2 a_0^2}\left(\frac{a_0}{a}\right)^2\right]$$

We can solve this for $a(t)$ for the three cases of curvature.

① $K = 0$ (Einstein - de Sitter universe)

$$\left(\frac{a}{a_0}\right) = \left(\frac{3}{2}H_0 t\right)^{2/3}$$

② negative curvature ($K=-1$)

This solution is slightly more complicated. It turns out to be useful to cast the Friedmann equation as a function of conformal time, τ. Recall that

$$c\,dt = a\,d\tau$$

← note that some authors instead use η instead of τ.

In this case, the Friedmann equation takes the form,

$$\frac{c^2}{a^4}\left(\frac{da}{d\tau}\right)^2 + \frac{Kc^2}{a^2} = \frac{8\pi G}{3}\rho$$

$$\Rightarrow \left(\frac{da}{d\tau}\right) = \pm\left(\frac{8\pi G}{3c^2}\rho a^4 - Ka^2\right)^{1/2}$$

$$\Rightarrow \tau = \pm\int\left(\frac{8\pi G}{3c^2}\rho a^4 - Ka^2\right)^{-1/2} da$$

For negative curvature the solution takes on a parametric form,

$$\left(\frac{a}{a_0}\right) = \frac{1}{2}\frac{\Omega_{M,0}}{(1-\Omega_{M,0})}\left(\cosh(\tau)-1\right)$$

note that τ goes from $0 \to \infty$

and

$$H_0 t = \frac{1}{2}\frac{\Omega_{M,0}}{(1-\Omega_{M,0})^{3/2}}\left(\sinh(\tau)-\tau\right)$$

Together these equations give a solution to $a(t)$. We can look at some limits though. At early times (small τ),

$$a \propto t^{2/3}$$

This solution looks just like the Einstein-de Sitter solution precisely because at early times curvature can be ignored. At later times $\tau \gg 1$ $(\sinh(\tau) \simeq \cosh(\tau))$ and

$$a \propto t \quad \text{(free expansion)}$$

(3) positive curvature ($K=+1$)

Here, the method to find a solution is similar to the $K=-1$ case. The parametric solution is given by,

$$\left(\frac{a}{a_0}\right) = \frac{1}{2}\frac{\Omega_{m,0}}{(\Omega_{m,0}-1)}(1-\cos(\tau))$$

and

$$H_0 t = \frac{1}{2}\frac{\Omega_{m,0}}{(\Omega_{m,0}-1)^{3/2}}(\tau - \sin(\tau))$$

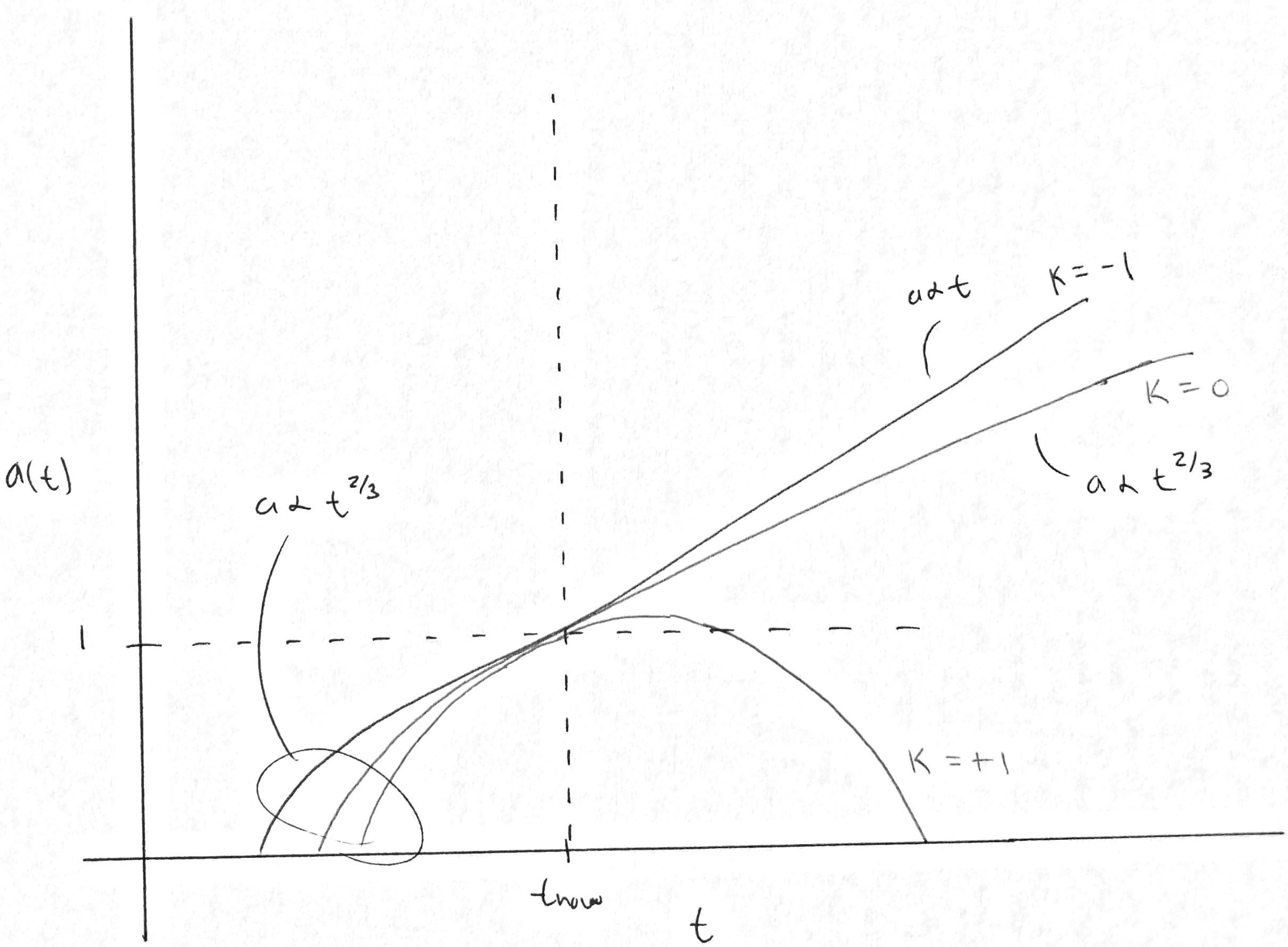
In this case τ is limited to be between $[0, 2\pi]$. For this solution, the scale factor reaches a maximum size before the Universe begins to collapse.

$$\frac{a_{max}}{a_0} = \frac{\Omega_{m0}}{\Omega_{m,0}-1} \; ; \; H_0 t_{max} = \frac{\Omega_{m,0}}{(\Omega_{m0}-1)^{3/2}}$$

Again, at early times,

$$a \propto t^{2/3}$$

but eventually the Universe collapses back to a singularity.



- General late time evolution

As discussed, at early times, the Universe is radiation dominated and well approximated as flat. Let's now consider the late-time evolution ($z \ll z_{eq}$) for a Universe with $\Omega_{\Lambda,0} \neq 0$.

The Friedmann equation can be written as

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_{m,0}\left(\frac{a_0}{a}\right)^3 - \frac{Kc^2}{H_0^2 a_0^2}\left(\frac{a_0}{a}\right)^2 + \Omega_{\Lambda,0}\right]$$

By defining some new variables,

$$x = \frac{a}{a_0}, \quad \eta = \left(\frac{\Omega_{m,0}}{2}\right)^{1/2} H_0 t$$

$$\lambda = \frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}, \quad \mathcal{K} = \frac{Kc^2}{H_0^2 a_0^2 \Omega_{m,0}}$$

we can write the friedmann equation as

$$\frac{1}{2}\left(\frac{dx}{d\eta}\right)^2 = \frac{1}{x} - \mathcal{K} + \lambda x^2$$

This is just the classical equation of motion of a particle in a potential

$$\Phi(x) = -\frac{1}{x} - \lambda x^2$$

with a total energy $E = -K$. Here λ depends on the contents of the Universe.

If $\lambda < 0$, Φ monotonically increases from $0 \to \infty$. All solutions end in collapse of the Universe.

If $\lambda > 0$ (the more relevant case) there are multiple possibilities.

1. $K = -1$ with $\Omega_{\Lambda,0} > 0$
   expansion forever
2. $K = +1$ with $\Omega_{\Lambda,0} > 0$
   $\Phi(x)$ contains a maximum.
   In this case the Universe may expand forever or recollapse depending on whether $E > \Phi_{max}$.

Although we will not go into great detail, there are robust constraints on the various parameters which govern the Friedmann equation. Some of the best constraints come from measurements of temperature fluctuations in the CMB (more on this later).

$$\text{CMB} \begin{cases} \Omega_0 \simeq 1.0 \Rightarrow K \simeq 0 \\ \Omega_{m,0} \sim 0.26 \\ \Omega_{\Lambda,0} \sim 0.74 \; \} \text{ type Ia Supernovae} \\ \Omega_{r,0} \sim 8 \times 10^{-5} \end{cases}$$

$$\text{Cepheid Stars} \longrightarrow H_0 \sim 72 \pm 5 \ \frac{\text{Km/s}}{\text{Mpc}}$$

This implies that the redshift of matter-radiation equality is given by,

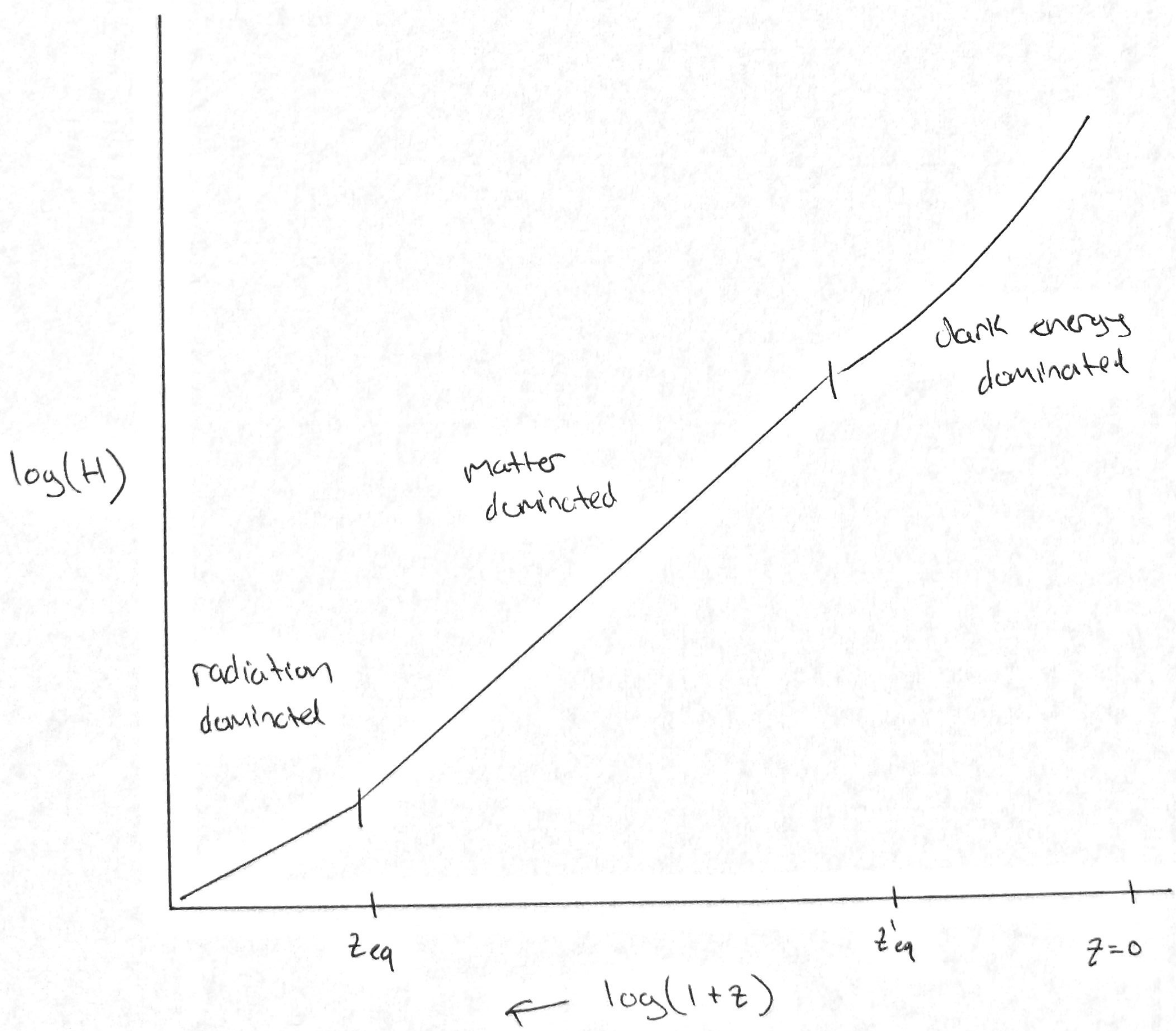
$$1 + z_{eq} \simeq 2.4 \times 10^4 \, \Omega_{m,0} h^2$$

$$\simeq 3{,}000$$

Similarly, the redshift of matter - dark energy equality can be calculated.

$$1 + z'_{eq} \simeq \left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}} \right)^{1/3} = 1.37$$

This tells us that dark energy has only relatively recently come to dominate the universe.



- Age of the Universe

If the Universe had a constant expansion rate, the age of the Universe would simply be given by the Hubble time,

$$t_H \equiv \frac{1}{H_0}$$

In such a Universe, a time t_H ago would correspond to a moment when all points were cospatial, i.e. there was a singularity. This is the concept of the <u>Big Bang</u>.

However, realistic models of the Universe show that the expansion rate evolves depending on the contents of the Universe. Given that for our purposes, it is safe to assume that $\dot{a} > 0$ for all time, it is relatively easy to calculate the age of the Universe as a function of redshift.

$$t_{age}(z) \equiv \int_0^{a(z)} \frac{da}{\dot{a}} = \frac{1}{H_0} \int_z^{\infty} \frac{dz}{(1+z)E(z)}$$

This step from application of the Friedmann equation

It is common to talk about the "look-back" time, simply defined as,

$$t_L = t_0 - t_{age}(z)$$

For a radiation dominated universe $(z \gg z_{eq})$

$$t(z) \simeq \left(\frac{1+z}{10^{10}}\right)^{-2} s$$

For an Einstein-de Sitter universe

$$t(z) = \frac{1}{H_0} \frac{2}{3} (1+z)^{-3/2} \simeq \frac{2}{3} (1+z)^{-3/2} \, 10^{10} h^{-1} \, yr$$

The solution is a little more complicated when $\Omega_M \neq 1$. However, the general trend is the same; larger $\Omega_{M,0}$ corresponds to a younger universe at all redshifts.