33-777

Today

- General Relativity
- Friedmann Equations

In the last lecture we saw that a Universe constrained by the cosmological principle must have a Riemannian geometry with the Friedmann-Robertson-Walker (FRW) Metric where the space-time "distance" defined by,

$$ds^2 = a^2(\tau)\left[d\tau^2 - d\chi^2 + f_K^2(\chi)\left(d\alpha^2 + \sin^2(\alpha)\, d\phi^2\right)\right]$$

In this lecture we will add general relativity to derive the evolution, $a(\tau)$.

- General Relativity

So far we have shown how the cosmological principle (the Universe is homogeneous and isotropic on large scales) and Riemannian geometry gives the FRW metric. The FRW metric has two parameters which should give us pause,

(1) $a(t) \rightarrow$ governs expansion/contraction
(2) $K \rightarrow$ "curvature" of space

In order to shed some light on these parameters we need to discuss <u>gravity</u>.

Let's first review Newtonian gravity. Recall that the gravitational force may be written as a central force vector, $\vec{F}_g$, between two massive bodies,

$$\vec{F}_g = -G\frac{M_1 M_2}{r^2}$$

The gravitation force may be written as the gradient of a scaler,

$$\vec{g} = -\nabla \Phi$$

This scaler, Φ, is the gravitational potential. The gravitational potential is related to a matter density distribution, ρ, via the Poisson equation,

$$\nabla^2 \Phi = 4\pi G \rho$$

Using this field description for gravity we can derive Newton's formulation for a point mass by using Green's function. The gravitation potential at a distance r from a point mass, M, is given by,

$$\Phi(r) = -G \frac{M}{r},$$

Before discussing some problems with Newtonian gravity, let's first review some useful terms.

- frame of reference : a standard to which motion and rest may be measured. Any set of points at rest to each other can serve as a frame of reference.

- inertial frame : a frame of reference which has a constant velocity with respect to the distant stars. I.e. a frame that is at rest or moving at a constant speed in a straight line. It is a non-accelerating frame in which the laws of physics take on their simplest forms (there are no fictitious forces).

- non-inertial frame : a reference frame which is accelerating. In this case motion is affected by fictitious forces, e.g. centrifugal + coriolis force.

- <u>an invariant</u> : a property / quantity that remains unchanged under a transformation of the frame of reference

- <u>covariance</u> : the invariance of the physical laws under some transformation of the frame of reference.

With these properties in our pocket, let's discuss the development of general relativity. In the early 20th century Einstein was thinking about some problems with Newtonian gravity. These problems can be summarized as follows:

1. while Newtonian gravity appears accurate, it offers no physical explanation for gravity (action at a distance)

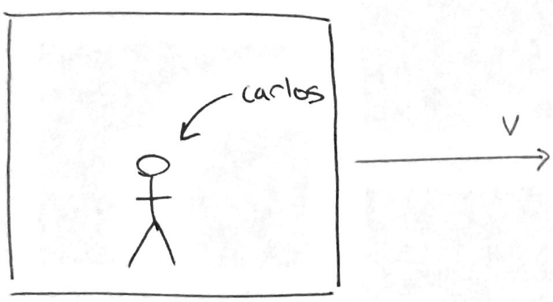
(2) Newton's law of gravity only holds in inertial systems and it is covariant under Galilean transformations. <u>But</u>, according to special relativity, inertial systems transform under Lorentz transformations

(3) Inertial systems can not exist since one can not shield oneself from gravity (no equivalent to Faraday cage).

(4) Newtonian gravity violates special relativity since moving a massive object has an immediate effect throughout all of space

These problems motivated Einstein to search for a manifest covariant, relativistic theory of gravity. In 1915 he presented the Einstein field equations to the Prussian Academy of Science.

We can trace the development of GR with a series of thought experiments. These experiments are based on a simple fact. Galileo (and Newton) realized that there is no physical experiment that can reveal the velocity of an inertial frame.



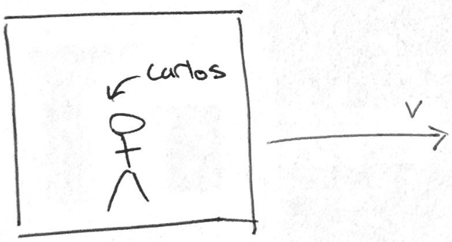
If Carlos throws a ball in the air, it looks to him the same as it would if he were at rest wrt the distant stars.

This argues that the concept of absolute velocity is ill-defined. Only relative motion is measurable in physics. This concept is called <u>Newtonian Relativity</u>.

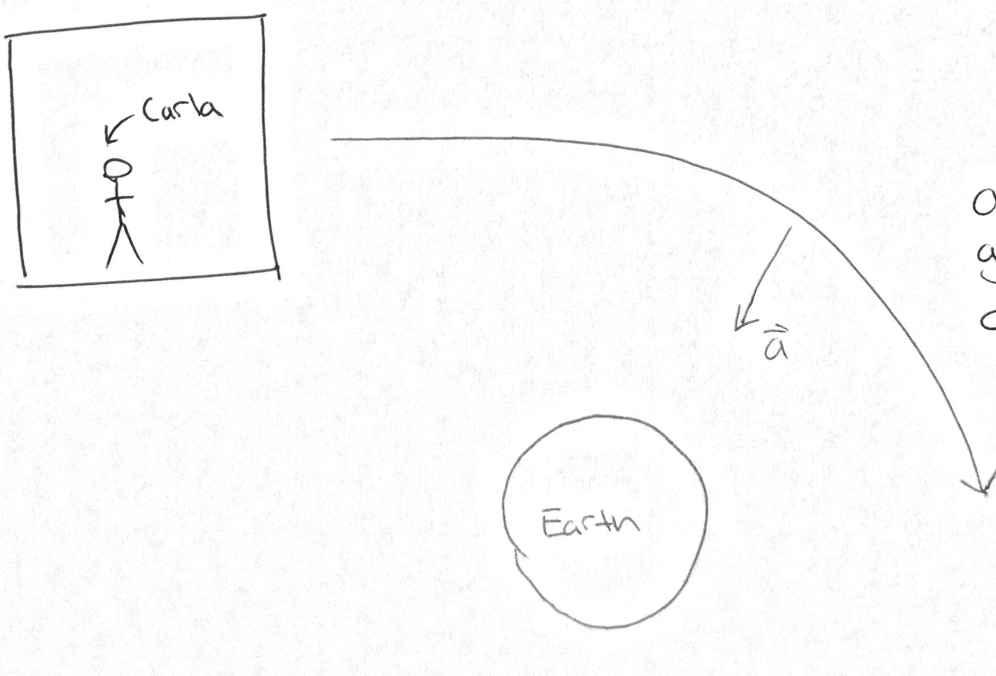
$$\boxed{\text{All uniform motion is relative}}$$

Newtonian relativity is still valid in special relativity. The primary change is that inertial frames transform via Lorentz transformations. This is required to satisfy the constancy of the speed of light.

Thought Experiment # 1



constant uniform motion.

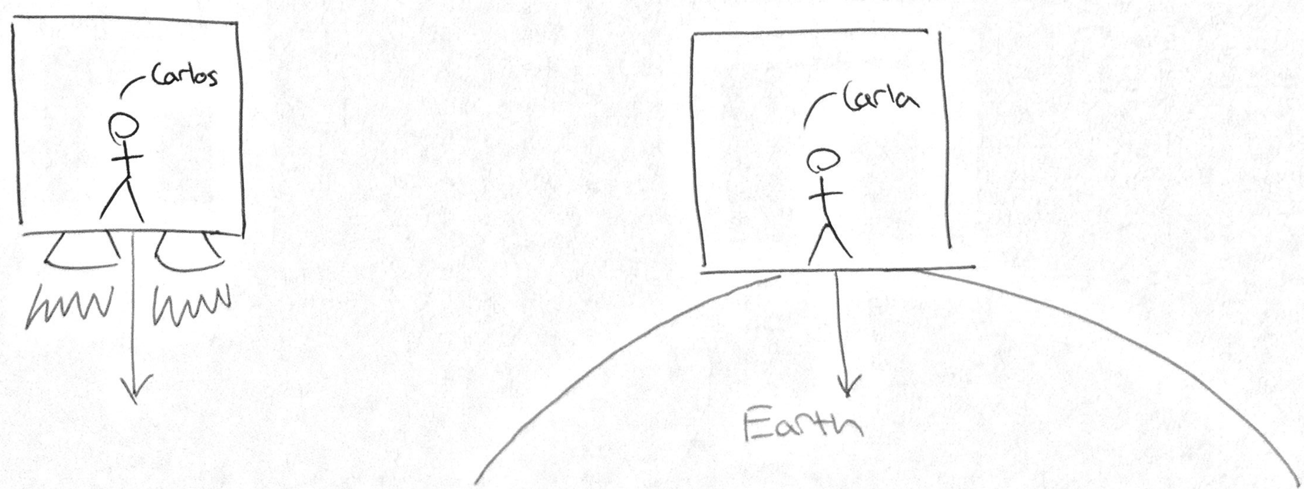


accelerated due to gravitational field of Earth.

In this experiment, neither Carlos or Carla in their windowless lab can perform any experiment to reveal their motion. Carlos can not because of Newtonian relativity. Carla can not because the gravitation force is exactly balanced by the centrifugal force

From this we can conclude that gravity can be transformed away by going to a non-inertial (free fall) frame.

Thought Experiment # 2



Carlos's frame is accelerated by some (external) rockets. As a result, he experiences an inertial force, i.e. he has a non-zero weight.

Carla's lab is inhibited from free-fall due to the normal force on Earth's surface. She experiences a gravitational force and has a non-zero weight.

There is no experiment that Carlos and Carla can preform to distinguish between acceleration and gravity.

From these two experiments we can summarize the principles of relativity,

1. it is not possible to distinguish being at rest, in constant motion, or in free-fall.

2. it is not possible to distinguish between being in an accelerated frame or in a gravitational field.

This was Einstein's "happiest thought".

A corollary is the <u>weak equivalence principle</u>.

$$\vec{F} = M_i \vec{a} \quad \text{(Newton's 2nd law)}$$

$$\vec{F}_g = M_g \vec{g} \quad \text{(Newtonian gravity)}$$

$$\Downarrow$$

$$\vec{a} = \frac{M_g}{M_i} \vec{g}$$

The weak equivalence principle states that gravitational mass and inertial mass are the same,

$$\boxed{\frac{M_g}{M_i} = 1}$$

The weak equivalence principle has been experimentally confirmed to very high precision. In 1889 Eötvös showed that

$$\left| \frac{M_i}{M_g} - 1 \right| < \frac{1}{2\times 10^7}$$

Using a torsion balance. Modern experiments have constrained

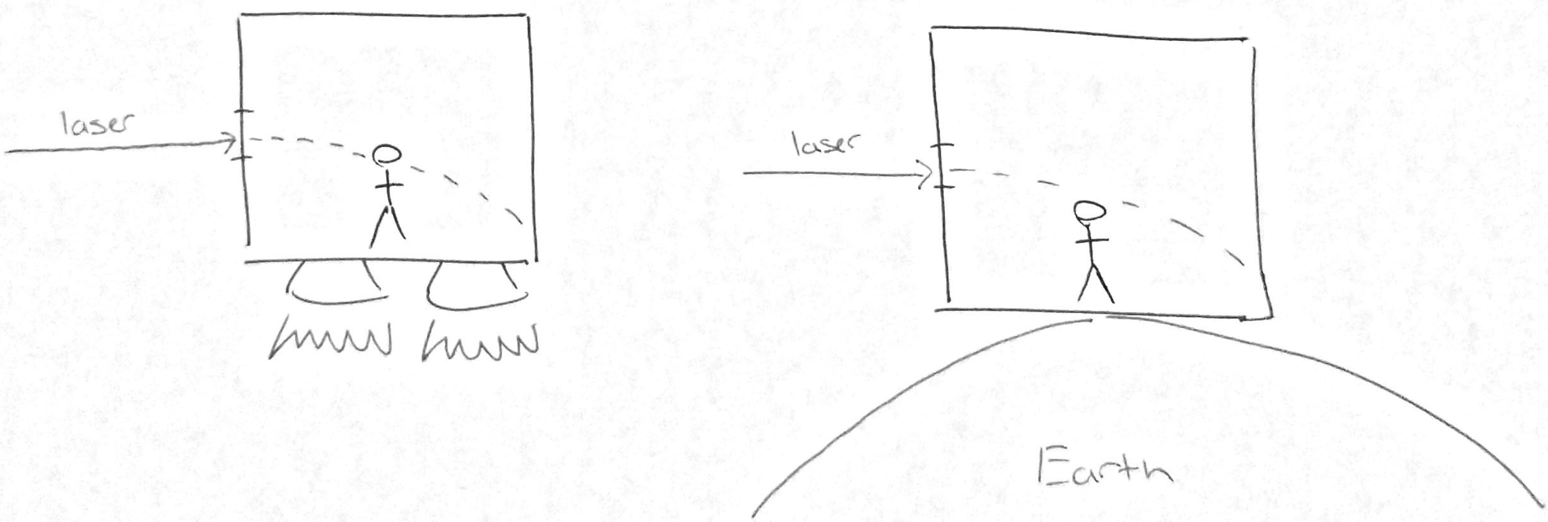
$$\left| \frac{M_i}{M_g} - 1 \right| < \frac{1}{10^{15}}$$

The weak equivalence principle implies that in a free-falling system there is no gravity locally. In the absence of gravity space-time is described by Minkowski space.

$\Rightarrow$ space-time of a freely falling observer is Minkowski space (M_4).

This idea can be extended to the <u>Strong Equivalence Principle</u>: in free-fall in an <u>arbitrary</u> gravitational field, <u>all</u> physical processes take place as they would in the absence of a gravitational field.

One implication of the strong equivalence principle is gravitational lensing



In both the accelerated frame and in the presence of a gravitational field, a laser beam will follow a curved path.

There is one important thing to note regarding the weak/strong equivalence principle. While gravity can be transformed away by moving to a non-inertial free-falling reference frame, this is only true <u>locally</u>. Because of the tidal field in gravitational potentials,

$$\frac{\partial^2 \Phi}{\partial x \partial y}$$

we can only transform away the effects of gravity on scales that are small compared to variations in the gravitational field.

Einstein realized that the permanance of gravity (it is only possible to transform it away locally) that it must manifest itself as an intrinsic property of space-time itself.

$\Rightarrow$ Gravity manifest via the space-time metric itself, $g_{\mu\nu}$

From the equivalence principles, at any point in space, we must be able to find a coordinate system such that locally

$g_{\mu\nu} = \eta_{\mu\nu}$.

Minkowski metric

This is just the condition that defines a <u>Riemannian</u> space. This also implies that if $g_{\mu\nu} = \eta_{\mu\nu}$ everywhere, then space-time is flat everywhere and there can be no gravity.

$\Rightarrow$ gravity originates from curvature in space-time itself.

Now we know that gravity is a manifestation of a curved space-time. It is still not clear how to connect this concept to matter. For this, the goal is to find a manifest covariant version of the Poisson equation,

$$\nabla^2 \Phi = 4\pi G \rho$$

Breaking this into two steps,

1. If the gravitational potential sources the metric,

$$\Phi \leftrightarrow g_{\mu\nu}$$

it is reasonable to assume that the laplacian of Φ sources a different tensor made out of second derivatives of $g_{\mu\nu}$

$$\nabla^2 \Phi \leftrightarrow B_{\mu\nu}$$

(2) Given mass-energy equivalence, perhaps it is reasonable to replace density, ρ, in Poisson's equation with the energy-momentum (stress-energy) tensor of a fluid,

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u^\mu u^\nu - P g^{\mu\nu}$$

energy-momentum tensor for a fluid

$\rho \equiv$ mass density

$P \equiv$ pressure

$u \equiv$ 4-velocity vector

$g^{\mu\nu} \equiv$ metric

With these two assumptions, the manifest covariant form of the Poisson equation becomes,

$$B^{\mu\nu} = K T^{\mu\nu}$$

some constant

Now the goal is to determine $B_{\mu\nu}$. Recall that the geometry of any manifold is fully described by the metric tensor, $g_{\mu\nu}(\vec{x})$.

remember the metric can be a function of position.

The metric tensor has a variety of mathematical uses. For example, it can be used to raise or lower indices,

* $A^{\mu} = g^{\mu\nu} A_{\nu}$
* $A_{\mu} = g_{\mu\nu} A^{\nu}$ $\Rightarrow$ $g^{\mu\alpha} g_{\nu\alpha} = \delta^{\mu}_{\nu}$

It can also be used to construct a variety of quantities that are useful to describe a geometry. The <u>Christoffel Symbols are</u> one such example.

Christoffel Symbol

$$\Gamma_{\alpha\beta\gamma} \equiv \frac{1}{2}\left(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}\right) \qquad 1^{st} \text{ Kind}$$

$$\Gamma^{\alpha}_{\beta\gamma} \equiv \frac{1}{2} g^{\alpha\mu}\left(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}\right) \qquad 2^{nd} \text{ Kind}$$

The Christoffel symbol of the 2^{nd} Kind is called the affine connection. Note we have also used a new notation for derivatives!

$$(\ldots)_{,\mu} \equiv \partial_\mu(\ldots) \equiv \frac{\partial(\ldots)}{\partial x^\mu}$$

An important property of the affine connection is in defining covariant derivatives. Consider our standard derivative,

$$A_{\mu,\nu} = \frac{\partial A_\mu}{\partial x^\nu}$$

What happens to this when we transform to a new coordinate system?

Consider a new coordinate system,

$$\bar{x}^\alpha = \bar{x}^\alpha(x)$$

Now we have that,

$$\bar{A}_{\mu,\nu} = \frac{\partial \bar{A}_\mu}{\partial \bar{x}^\nu} = \frac{\partial}{\partial \bar{x}^\nu}\left[\frac{\partial x^\alpha}{\partial \bar{x}^\mu} A_\alpha\right]$$

$$= \frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial A_\alpha}{\partial \bar{x}^\nu} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^\nu \partial \bar{x}^\mu} A_\alpha$$

$$= \frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x_\beta}{\partial \bar{x}^\nu}\frac{\partial A_\alpha}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^\nu \partial \bar{x}^\mu} A_\alpha$$

Without this term $\bar{A}_{\mu,\nu}$ would transform like a covariant tensor. Alas, it does not. This tells us that the operator ∂_μ is not covariant and therefore it can not be used in physical laws.

what we require is a <u>covariant derivative</u>,

* $$A_{\mu;\nu} \equiv D_\nu A_\mu \equiv \frac{DA_\mu}{Dx^\nu} = A_{\mu,\nu} - \Gamma^{\alpha}_{\mu\nu} A_\alpha$$

* $$A^{\mu}_{;\nu} \equiv D_\nu A^\mu \equiv \frac{DA^\mu}{Dx^\nu} = A^{\mu}_{,\nu} - \Gamma^{\mu}_{\alpha\nu} A^\alpha$$

Here the definition of the covariant derivative is given without proof. Notice that it contains the affine connection!

Some important notes about the covariant derivative.

① For scalar fields

$$\phi_{;\alpha} = \phi_{,\alpha}$$

② For the metric tensor

$$g_{\mu\nu;\alpha} = g^{\mu\nu}{}_{;\alpha} = 0$$

③ In (pseudo-) Euclidean space (like Minkowski) with Cartesian coordinates

$$\Gamma^{\alpha}_{\mu\nu} = 0$$

④ Christoffel symbols are <u>not</u> tensors.

From derivatives of the affine connection we can construct the <u>Riemann tensor</u>,

$$R^{\alpha}{}_{\beta\gamma\delta} = -\Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\alpha}_{\beta\delta,\gamma} + \Gamma^{\sigma}_{\beta\delta}\Gamma^{\alpha}_{\sigma\gamma} - \Gamma^{\sigma}_{\beta\gamma}\Gamma^{\alpha}_{\sigma\delta}$$

Note that because the Riemann tensor contains derivatives of the affine connection it is related to the <u>second</u> derivative of the metric tensor, $g_{\mu\nu}$.

From the Riemann tensor one can construct the <u>Ricci tensor</u> via contraction

$$R_{\beta\delta} \equiv R^{\gamma}{}_{\beta\gamma\delta} = g^{\alpha\gamma} R_{\alpha\beta\gamma\delta}$$

It turns out this is the only rank 2 tensor that can be constructed from the Riemann tensor by contraction.

Finally, with one more contraction we can obtain the curvature scaler,

$$R \equiv g^{\beta\delta} R_{\beta\delta}$$

Returning to our covariant form of the Poisson equation,

$$B^{\mu\nu} = K T^{\mu\nu}$$

The stress-energy tensor has the property that,

$$T^{\mu\nu}{}_{;\nu} = 0$$

as a result of conservation of energy and momentum. This implies that

$$B^{\mu\nu}{}_{;\nu} = 0$$

Using this insight, Einstein constructed a covariantly conserved tensor using quantities related to the second derivative of the metric, namely the Ricci tensor, metric, and curvature.

This is known as the Einstein tensor,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

It can be shown that this obeys,

$$G^{\mu\nu}{}_{;\nu} = 0$$

This gives us a tensor equation called the Einstein field equation,

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

Here the constant "K" comes from this needing to reduce to Poissons equation in the Newtonian limit.

Finally, since,

$$g^{\mu\nu}{}_{;\nu} = 0$$

rather than $G^{\mu\nu}$ on the left-hand side of the Einstein field equation, we could use any combination of the form,

$$G^{\mu\nu} + \underset{\text{constant}}{K} g^{\mu\nu}$$

and still obey that $B^{\mu\nu}{}_{;\nu} = 0$. This tells us that there are other forms of the field equation. In order to obtain a static Universe, Einstein changed the form to,

$$G_{\mu\nu} - \underset{\text{"cosmological constant"}}{\Lambda} g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- The Friedmann Equation

Now we know the form of the metric from the cosmological principle,

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2(\theta)\, d\phi^2\right)\right]$$

"FRW metric"

And from general relativity we know how matter affects the curvature of space via Einstein's field equation,

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

For an isotropic and homogeneous Universe filled with a "perfect fluid",

$$T^{\mu\nu} = \mathrm{diag}(\underset{\text{energy density}}{\rho c^2}, \underbrace{-P, -P, -P}_{\text{pressure}})$$

Using this and substituting in the FRW metric into Einsteins field equation results in two equations,

① a time-time component

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3}$$

② a space-space component

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2Kc^2}{a^2} = 4\pi G\left(\rho - \frac{P}{c^2}\right) + \Lambda c^2$$

Combing these two equations yields the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

If we interpret the cosmological constant as a component of the cosmological fluid with an equation of state,

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}$$

Then we can absorb its contribution into "ρ" to write a more compact form of the Friedmann equation,

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2}}$$

We can then expand the Friedmann equation into matter, ~~ener~~ radiation, and dark energy density contributions,

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) = \frac{8\pi G}{3}\underbrace{\left[\rho_{m,0}\left(\frac{a}{a_0}\right)^{-3} + \rho_{r,0}\left(\frac{a}{a_0}\right)^{-4} + \rho_{\Lambda,0}\right]}_{\rho(t)} - \frac{Kc^2}{a^2}$$

Note that for a flat universe $K=0$ and

$$\rho(t) = \frac{3H^2(t)}{8\pi G} \equiv \rho_{crit}(t)$$

This defines the "critical density".

At $z=0$, the critical density is,

$$\rho_{crit}(z=0) = 2.78 \times 10^{11}\ h^2\ \frac{M_\odot}{Mpc^3}$$

where current measurements put $h \simeq 0.7$.

Note that cosmologist often write various density components in a unitless form,

$$\Omega_x(t) \equiv \frac{\rho_x(t)}{\rho_{crit}(t)}$$

The total density is then

$$\Omega(t) = \sum_x \Omega_x(t)$$

Using this rotation, we can rewrite the Friedmann equation as,

$$H^2(t) = H_0^2\left[\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0}\right] - \frac{Kc^2}{a^2}$$

Note that at present $(t=t_0)$ this equation becomes,

$$H_0^2 = H_0^2\Omega_0 - \frac{Kc^2}{a_0^2} \Rightarrow -\frac{Kc^2}{a_0^2} = (1-\Omega_0)H_0^2$$

$$\Rightarrow \frac{-Kc^2}{a^2} = -\frac{Kc^2}{H_0^2 a_0^2}\left(\frac{a}{a_0}\right)^{-2} H_0^2$$

$$= (1-\Omega_0)H_0^2(1+z)^2$$

If we define $E^2(z) \equiv \frac{H^2(z)}{H_0^2}$, then

we get one final form of the Friedmann equation,

$$\boxed{E(z) = \left[\Omega_{\Lambda,0} + (1-\Omega_0)(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4\right]^{1/2}}$$