33-777

Today

- Geometry of Space-Time
- Thermodynamics in an Expanding Universe

In this lecture we will begin our discussion of cosmology, the study of the structure and evolution of the Universe as a whole. We will see that modern cosmology is based on General Relativity (GR) which dictates that the structure of space-time is governed by its matter and energy density. Note that this is a significant departure from a classical picture where space and time are eternal and absolute, not relying on the existence/absence of matter.

Cosmology (without perturbations) turns out to be a very simple application of GR. Because of this, we will only cover the very basics of GR, those aspects needed for a general understanding of modern cosmology. In this lecture we will focus on the geometry of space-time. This will include a description of:

- Riemannian Geometry
- metrics
- geometry of space-time
- the cosmological principle
- fundamental observers
- distance measures in cosmology
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Finally we will conclude by discussing thermodynamics in an expanding space-time.

- Geometry of Space-Time

To specify where and when an event occurs in any space or space-time we require a coordinate system. We will specify a coordinate system by:

$$x^{\mu} \quad \text{where } \mu \text{ is an index}$$

It is important to note that a coordinate system has no physical importance. That is, any valid physical law should be independent of the choice of coordinate system. In general, we can make physical laws manifest invariant (valid for any coordinate system) by writing them in a <u>tensor</u> form.

Recall that tensors are simply geometric objects that can be represented as mult-dimensional arrays of numerical values. The rank of a tensor is then the number of dimensions of this array. For example,

- tensor of rank 0 (scaler)

  Newtonian potential

  $\Phi(x^\mu)$

- tensor of rank 1 (vector)

  Electric field

  $\vec{E}(x^\mu)$

- ~~the~~ tensor of rank 2

  Metric

  $g_{\alpha\beta}(x^\nu)$

The defining properties are their transformation rules, i.e. how the values of a tensor changes under a coordinate transformation, $x^i \to x'^i$.

These transformations define different types of tensors:

* Scalars

$$\Phi(x'^j) = \Phi(x^i)$$

Note that the value of a scalar field at a particular location is independent of the coordinate system

* Contra-variant vector

$$A'^k = \frac{\partial x'^k}{\partial x^i} A^i$$

* Covariant vector

$$A'_k = \frac{\partial x^i}{\partial x'^k} A_i$$

* Covariant (rank 2) tensor

$$T'_{ik} = \frac{\partial x^m}{\partial x'^i} \frac{\partial x^n}{\partial x'^k} T_{mn}$$

etc ...

Here we should make two quick notes regarding notation. First is the Einstein summation convention. Here when an index appears twice in a term, it implies a summation of that term over all the values of that index.

$$c_i x^i \equiv \sum_{i=1}^{3} c_i x^i = c_1 x^1 + c_2 x^2 + c_3 x^3$$

note that the superscript is not an exponent, but simply an index

Second, the index for contra-variant tensors is written as a superscript while for a covariant tensor they are subscripts. Also note that you can have "mixed" tensors, e.g.

$$T'^{i}_{k} = \frac{\partial x'^{i}}{\partial x^{m}} \frac{\partial x^{n}}{\partial x'^{k}} T^{m}_{n} \qquad \text{(mixed rank 2)}$$

In order to describe space-time in a coordinate independent we will focus on a physical invariant, the distances between events/locations. The spatial distance, dl, may be written,

$$dl^2 = g_{ij}\, dx^i dx^j \qquad \text{(space)}$$

And the space-time distance, ds, may be written,

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \qquad \text{(space-time)}$$

This implies that for a given coordinate system, x^α, the geometry of space or space-time is described by the metric, Note that in general this metric may depend on location,

$$g_{\mu\nu} = g_{\mu\nu}(x^\alpha)$$

and in general, the numerical values of this Metric tensor depend on the choice of coordinate system.

To see how this works, let's consider a few example geometries.

Example: 2D Euclidean Space

Here, the cartesian coordinate system is
$x^i = (x, y)$

We know the distance between points is given by

$$dl^2 = dx^2 + dy^2$$

From here, we can read off the metric

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In fact we can generalize this for any Cartesian coordinate system in n-dimensional Euclidean space, E_n, the metric is given by

$$g_{ij} = \delta_{ij} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

Kronecker delta function

Euclidean spaces are a subset of the more general Riemannian manifolds. Euclidean manifolds are characterized by having zero curvature everywhere.

Now, let's see what happens when we choose a new coordinate system. For example, consider the polar coordinate system,

$$x^i = (r, \theta)$$

We can derive the metric for this coordinate system by utilizing the fact that dl^2 is invariant. The coordinates are related via,

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta = \cos(\theta)\, dr - r\sin(\theta)\, d\theta$$

$$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta = \sin(\theta)\, dr + r\cos(\theta)\, d\theta$$

$$\Rightarrow dl^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$$

$$\Rightarrow \quad g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Note that $g_{ij} \neq \delta_{ij}$ even though this space is still Euclidean. This just demonstrates that the metric depends on the choice of the coordinate system. However, it is still the case that,

$$dl^2 = g_{ij}\, dx^i dx^j$$

As a side note, a space is called Euclidean if there exists a coordinate system for which $g_{ij} = \delta_{ij}$.

Example: 2D surface of a sphere

Let's now consider a slightly more complicated space, the surface of a sphere. Here it is useful (although not necessary) to embed this 2D surface in 3D Euclidean space. This allows us to specify a point in this space using the Cartesian coordinates, (x, y, z).

In order to be on the surface,

$$x^2 + y^2 + z^2 = a^2$$

Alternatively, we can define a spherical polar coordinate system,

$$x^i = (\chi, \alpha)$$

The parameter is just a "scale factor" which appears in the transformation relations,

$$x = a \sin(\chi) \sin(\alpha)$$
$$y = a \sin(\chi) \cos(\alpha)$$
$$z = a \cos(\chi)$$

In this case,

$$d\ell^2 = dx^2 + dy^2 + dz^2$$
$$= a^2 (d\chi^2 + \sin^2(\chi) d\alpha^2)$$

$$\Rightarrow g_{ij} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sin^2(\chi) \end{pmatrix}$$

With a slight change of notation,

$$r \equiv \sin(\chi)$$

$$\Rightarrow \quad x = ar\sin(\alpha)$$
$$y = ar\cos(\alpha)$$
$$z = a(1-r^2)^{1/2}$$

Now, the distance between two points becomes,

$$d\ell^2 = dx^2 + dy^2 + dz^2$$
$$= a^2\left[\frac{dr^2}{1-r^2} + r^2 d\alpha^2\right]$$

$$\Rightarrow \quad g_{ij} = \begin{pmatrix} \frac{a^2}{1-r^2} & 0 \\ 0 & a^2 r^2 \end{pmatrix}$$

This turns out to be a Riemann space with <u>constant</u> <u>positive</u> curvature.

Example: 2D "saddle" surface

Let's consider a 2D geometry with negative curvature. Again, we can embed this in a 3D space, although this one is pseudo-Euclidean. Here points are constrained to be on a 2D surface such that,

$$x^2 + y^2 - z^2 = -a^2$$

Here the distance between points is given by,

$$dl^2 = dx^2 + dy^2 - dz^2$$

Again, let's define an intrinsic coordinate system

$$x^i = (\chi, \alpha)$$

where,

$$x = a \sinh(\chi) \sin(\alpha)$$
$$y = a \sinh(\chi) \cos(\alpha)$$
$$z = a \cosh(\chi)$$

Similarly, if we define $r \equiv \sinh(\chi)$ the distance equation becomes,

$$dl^2 = dx^2 + dy^2 - dz^2$$
$$= a^2 \left[\frac{dr^2}{1+r^2} + r^2 d\alpha^2 \right]$$

$$\Rightarrow \quad g_{ij} = \begin{pmatrix} \frac{a^2}{1+r^2} & 0 \\ 0 & a^2 r^2 \end{pmatrix}$$

Note that for each of these surfaces, the metric is independent of position. That is, all points are equivalent.

We can summarize these results with the following metric,

$$g_{ij} = \begin{pmatrix} \frac{a^2}{1-Kr^2} & 0 \\ 0 & a^2 r^2 \end{pmatrix}$$

where $K = (1, 0, -1)$ is a curvature parameter.

* $K = 1$, 2D sphere
* $K = 0$, 2D Euclidean space
* $K = -1$, 2D saddle

Now, we should consider how to expand this to space-time. Let's consider one more example.

<u>Example</u>: 4D Minkowski space

Here we have one time-like coordinate and three spatial coordinates.

$$x^\mu = (ct, x, y, z)$$

Events in Minkowski space are seperated by,

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

The metric for this space is then

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

This is another example of a pseudo-Euclidean space. Note that the signature of the metric is arbitrary. It is perfectly acceptable to write $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$, one only needs to be consistent.

Minkowski space metric is of course the metric for special relativity. In this case ds is the invariant that replaces dl.

Note that according to special relativity, photons travel at the speed of light. This implies that after a time dt photons travel a spatial distance $c\,dt = (dx^2 + dy^2 + dz^2)^{1/2}$.

Thus, for photons $ds = 0$.

- $ds^2 > 0$ (time-like seperated)

  an observer may witness both events.

- $ds^2 < 0$ (space-like seprated)

  no observer may witness both events.

This brings us to an important question: what is the appropiate geometry of space-time for the Universe governed by GR?

Any metric which governs the geometry of the Universe must obey the cosmological principle, the Universe is homogeneous and isotropic on large scales. This principle is in effect a grand extension of the Copernican principle, i.e. our location in the Universe is in no way special.

* Homogeneity $\Rightarrow$ metric must be independent of location, $g_{\mu\nu}(x^\alpha) = g_{\mu\nu}$
* Isotropy $\Rightarrow$ the only dynamics of space-time possible are global expansion | contraction, $a = a(t)$.

According to the cosmological principle, the only possible geometry is a 3D Riemann space embedded in a (3+1)D space-time.

The metric for such a space is:

$$d\ell^2 = a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2\left(d\alpha^2 + \sin^2(\alpha)\,d\phi^2\right)\right]$$

note that in this form (r, K, α, ϕ) are all unitless. Only $a(t)$ has unit length.

If we embed this 3D Riemann space in a (3+1)D space time we arrive at the Friedmann-Robertson-Walker (FRW) metric.

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2\left(d\alpha^2 + \sin^2(\alpha)\,d\phi^2\right)\right]$$

Here it becomes useful to define a <u>fundamental observer</u> as an observer in a FRW metric who observes the Universe to be isotropic. The set of all such observers defines a cosmological rest-frame.

FRW Metric Notes

* (r, α, ϕ) are coordinates which label the positions of fundamental observers. These are called "comoving coordinates". Comoving coordinates don't change as a result of global expansion/contraction.

* t is referred to as the "proper time". This is the time on a standard clock of a fundamental observer.

* a is called the "scale factor". The scale factor relates (r, α, ϕ) to true physical distances.

* K is the "curvature parameter". It indicates the global curvature of space-time. $K = (+1, 0, -1)$.

It is useful to be precise by what we mean by the seperation of objects/events in the Universe. As a result we define:

- <u>proper distance</u> - distance between to fundamental observers at same proper time t.

$$l = \int dl = a(t) \int_0^r \frac{dr'}{\sqrt{1-Kr'^2}} = a(t)\chi(r)$$

- <u>comoving distance</u> - in the above expression $\chi(r)$ defines the comoving distance between the fundamental observers.

$$\chi(r) = \begin{cases} \sin^{-1}(r) & \text{if } K=+1 \\ r & \text{if } K=0 \\ \sinh^{-1}(r) & \text{if } K=-1 \end{cases}$$

The scale factor converts the comoving distance into the proper distance.

- conformal time τ is the total comoving distance light could have travelled.

$$\tau(t) \equiv \int_0^t \frac{c\,dt'}{a(t')}$$

A common form of the FRW metric is written in terms of χ and τ:

$$ds^2 = a^2(\tau)\left[d\tau^2 - d\chi^2 - f_K(\chi)\left(d\theta^2 + \sin^2(\theta)\,d\phi^2\right)\right]$$

where
$$f_K(\chi) = \begin{cases} \sin(\chi) & \text{if } K=+1 \\ \chi & \text{if } K=0 \\ \sinh(\chi) & \text{if } K=-1 \end{cases}$$

For what we have done so far $(r, \chi, \theta, \phi, \tau)$ are all unitless. t and $a(t)$ have units of time and length. It is much more common to redefine

$$a(t) = \frac{a(t)}{a_0}$$

where $a_0 = a(t_{now})$

In this case $a(t)$ is unitless and equal to 1 now. Now (r, χ, τ) carry units of length.

Now we can go back and recast some of our earlier results. First, let's start with the Hubble parameter. The relative velocity between two fundamental observers is simply the time derivative of the proper distance.

$$\frac{dl}{dt} \equiv H(t)\, l$$

To derive an expression for $H(t)$ recall that the proper distance is given by

$$l = a(t)\chi$$

$$\Rightarrow \frac{dl}{dt} = \dot{a}\chi = \frac{\dot{a}}{a} a\chi = \frac{\dot{a}}{a} l$$

$$\Rightarrow \boxed{H(t) = \frac{\dot{a}}{a}}$$

Next, let's revisit the concept of redshift. Photons are defined to move along geodesics such that $ds = 0$. Substituting this into the FRW metric gives,

$$ds^2 = a^2(\tau)\left[d\tau^2 - d\chi^2 - f_K^2(\chi)\left(d\theta^2 + \sin^2(\theta)\,d\phi^2\right)\right]$$

$$0 = a^2(\tau)\left[d\tau^2 - d\chi^2\right]$$

note we can always set up a coordinate system such that $d\theta = d\phi = 0$.

$$\Rightarrow d\tau = d\chi$$

$$\Rightarrow d\tau = \frac{c\,dt}{a(t)} = d\chi$$

Because the comoving distance remains fixed between two fundamental observers, proper time intervals, δt, must scale with the scale factor, $a(t)$.

As a result, the proper time interval an observer at the origin measures is related to the proper time interval for another fundamental observer by

$$\frac{\delta t_{obs}}{a(t_{obs})} = \frac{\delta t_{em}}{a(t_{em})}$$

observer frame

emitter frame

A photon's wave length is proportional to the period of the EM wave, which is a proper time interval. As a result, the observed wavelength of a photon is

$$\frac{\lambda_{obs}}{\lambda_{em}} = \frac{\delta t_{obs}}{\delta t_{em}} = \frac{a(t_{obs})}{a(t_{em})}$$

Finally, recall that redshift is defined as,

$$z \equiv \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}$$

$$\Rightarrow \quad 1 + z \equiv \frac{\lambda_{obs}}{\lambda_{em}} = \frac{a(t_{obs})}{a(t_{em})}$$

If we set $t_{obs} = t_{now}$ then $a = 1$

$$\Rightarrow \quad a = \frac{1}{1+z}$$

This tells us that a proton that has been redshifted to $z = 1$ was emitted when $a = \frac{1}{2}$, i.e. when the Universe was half its current size.

Cosmological redshift is <u>not</u> a doppler shift!

what about peculiar velocity? Again, recall that the proper velocity between two particles is given by,

$$v = \frac{dl}{dt}$$

Now if we are dealing with non-fundamental observers $\chi = \chi(t)$. From the definition of proper distance,

$$l(t) = a(t)\chi(t)$$

$$\Rightarrow v = \frac{dl}{dt} = \dot{a}\chi + a\dot{\chi} \equiv v_{expansion} + v_{peculiar}$$

Here $v_{exp} = \frac{\dot{a}}{a} l = H l$ is the velocity due to expansion of the Universe.

v_{pec} is the peculiar velocity with respect to a co-spatial fundamental observer.

The observed redshift may be written as,

$$1 + z_{obs} = \underbrace{(1 + z_{cos})}_{\text{cosmological}}\underbrace{(1 + z_{pec})}_{\text{doppler shift}}$$

$$\Rightarrow \quad z_{cos} = \frac{1}{a(t_{em})} - 1$$

$$\Rightarrow \quad z_{pec} = \left(\frac{1 + v_{pec}/c}{1 - v_{pec}/c}\right)^{1/2} - 1$$

In the non-relativistic limit ($v_{pec} \ll c$) this simplifies to,

$$z_{obs} = z_{cos} + \frac{v_{pec}}{c}(1 + z_{cos})$$

So far we have mentioned two different distances, the comoving distance, χ, and the proper distance, l. It is worth mentioning two more. In a static Euclidean space, an object with an isotropic luminosity L and physical size D has observed properties,

$$\theta = \frac{D}{d_A} \qquad f = \frac{L}{4\pi d_L^2}$$

angular size   flux

In a static Euclidean space these two distances $d_A = d_L$. In an expanding space this is no longer the case. We can define the <u>angular diameter distance</u> as,

$$d_A(z) = \frac{a_0 r}{1+z}$$

and the luminosity distance as,

$$d_L(z) = a_0 r (1+z)$$

where a_0 is the scale factor for the observer (usually $a_0 = 1$ for us) and r is the FRW metric coordinate,

$$r = r(z) = f_K(\chi)$$

Note that up to this point we have shown how these distance measures depend on the scale factor | redshift. We have not yet shown how the scale factor changes as a function of time.

- Thermodynamics in an Expanding Universe

The constituents of the Universe can be modelled as a cosmological fluid consisting of multiple components. The energy density of such a fluid can be written as,

$$\rho c^2 = \underbrace{\rho_m c^2}_{\substack{\text{rest mass} \\ \text{of matter}}} + \underbrace{\rho_m \varepsilon}_{\substack{\text{internal} \\ \text{energy of} \\ \text{matter}}} + \underbrace{\rho_r c^2}_{\text{radiation}} + \underbrace{\rho_\Lambda c^2}_{\substack{\text{energy of} \\ \text{vacuum}}}$$

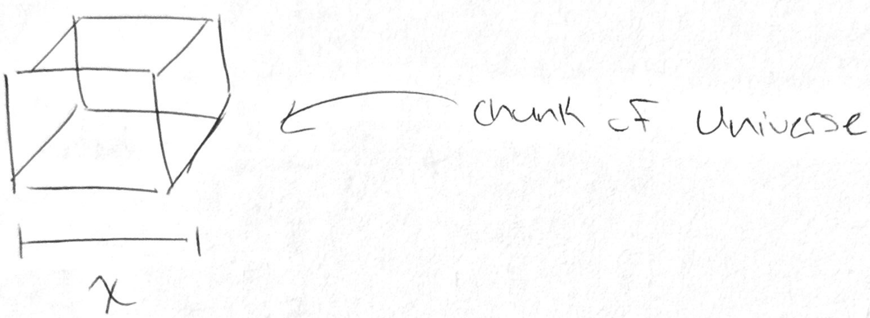
Here $\varepsilon \equiv$ internal energy per unit mass

$$\rho_r c^2 = \frac{4\sigma_{SB}}{c} T^4$$

Here we have combined baryonic matter and dark matter into one component since we suspect they evolve in the same manner.

Energy density in radiation includes all relativistic particles

Let's consider how we expect the laws of thermodynamics to apply in an expanding Universe. Consider a small chunk of the Universe with a comoving volume, V.



Recall the first two laws of thermodynamics:

① $dU = dQ + dW$

② $dS = \frac{dQ}{T}$

$U \equiv$ internal energy

$Q \equiv$ heat

$W \equiv$ work

$S \equiv$ entropy

For an isolated and adiabatically expanding chunk $dQ = 0$

$$\Rightarrow \quad dU = dW = -P\,dV$$

and

$$dS = 0$$

In our fluid picture, the energy density in this chunk is ρc^2. The internal energy is then,

$$U = \rho c^2 V$$

$$\Rightarrow dU = \frac{\partial U}{\partial \rho} d\rho + \frac{\partial U}{\partial V} dV$$

$$= c^2 V d\rho + \rho c^2 dV$$

From the 1st law,

$$dU + P dV = 0$$

$$c^2 V d\rho + \rho c^2 dV + P dV = 0$$

$$V d\rho + \left(\rho + \frac{P}{c^2}\right) dV = 0$$

If we note that $V \propto a^3$

$$\Rightarrow \frac{d\rho}{da} + 3\left(\frac{\rho + P/c^2}{a}\right) = 0$$

For fluids, the equation of state (EoS) often takes the form,

$$P = P(\rho, T)$$

In cosmology it is common to parameterize the EoS as,

$$P = w \rho c^2$$

where w in general is a function of T. Substituting the EoS form into our extension of the 1st law of thermodynamics yields,

$$\frac{d\rho}{da} + 3(1+w)\frac{\rho}{a} = 0$$

$$\Rightarrow \rho \propto a^{-3(1+w)}$$

Thus w controls how the energy density of a cosmological fluid evolves.

Let's now consider the components of our cosmological fluid.

① Non-relativistic Matter

We can model this as an ideal gas.

$$P = \frac{K_B T}{\mu M_p} \rho_m$$

If we cast this in the form

$$P = w \rho c^2$$

$$w = w(T) = \frac{K_B T}{\mu M_p c^2} \left(1 + \frac{1}{\underset{\substack{\uparrow \\ \text{adiabatic index}}}{\gamma} - 1} \frac{K_B T}{\mu M_p c^2} \right)^{-1}$$

For a non-relativistic gas

$$K_B T \ll \mu M_p c^2$$

$$\boxed{\Rightarrow \quad w \simeq 0}$$

This zero pressure fluid is sometimes called a pressureless "dust".

For a pressureless dust,

$$\rho \propto a^{-3}$$

$$T \propto a^{-2}$$

$$P \propto a^{-5}$$

② Relativistic particles (radiation)

$$P = \frac{1}{3}\rho c^2$$

$$\boxed{\Rightarrow \omega = \frac{1}{3}}$$

Therefore,

$$\rho \propto a^{-4}$$

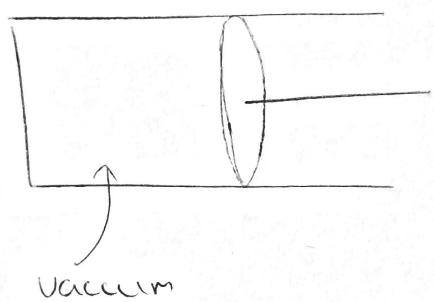
$$T \propto a^{-1}$$

$$P \propto a^{-4}$$

③ Vacuum Energy

Quantum physics allows for a non-zero vacuum energy density. This is the current best candidate for dark energy.

Consider a piston with vacuum inside



$$dU = \rho_\Lambda c^2 dV$$

An increase in vacuum volume increases the total vacuum energy

From the 1st law,

$$dU + P dV = 0$$

$$\Rightarrow P = -\rho_\Lambda c^2$$

$$\boxed{\Rightarrow \omega = -1}$$

As a result,

$$P_r \propto a^0 \quad \text{(constant)}$$

T is not defined

$$P \propto a^0 \quad \text{(constant)}$$