Probability Notes

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1 Some measure theory

Definition 1.1. Let Ω be a set. A topology on Ω is a collection $\mathscr{A} \subset P(\Omega)$ that is closed under unions and finite intersections with $\Omega, \emptyset \in \mathscr{A}$. For $E \subset \Omega$, if $E \in \mathscr{A}$, we call E open, and if $E^C \in \mathscr{A}$, we call E closed.

For example, if (X, d) is a metric space, $\mathscr{A} = \{E | E = \cup_i N_{r_i}(x_i)\}$ is a topology on X.

Gap!!! The example should be proven.

Definition 1.2. Let Ω be a set and $\mathscr{S} \subset P(\Omega)$. Then the topology generated by $\mathscr{S}$ is

$$\mathscr{A} = \{E \subset \Omega | E \text{ is a union of finite intersections of sets in } \mathscr{S}.\}$$

Gap!!! The definition should be more explicit.

Gap!!! Examples and Furstenberg's theorem should be added here

Definition 1.3. Let Ω be a set. An algebra is a collection $\mathscr{A} \subset P(\Omega)$ that is closed under finite unions and compliments with $\Omega \in \mathscr{A}$. If an algebra is also closed under countable unions, we call it a σ-algebra.

Theorem 1.4. *Algebras are closed under finite intersections and σ-algebras are closed under countable intersections.*

Proof. Let $\mathscr{A}$ be an algebra and let $A_1, \ldots, A_n \in \mathscr{A}$. Then $A_1^C, \ldots, A_n^C \in \mathscr{A}$, so $(\cap_{i=1}^n A_i)^C = \cup_{i=1}^n A_i^C \in \mathscr{A}$, so $\cap_{i=1}^n A_i \in \mathscr{A}$. The proof for σ-algebras is similar. $\square$

Definition 1.5. Let $\mathscr{F}$ be a σ-algebra on a set Ω. A measure on $\mathscr{F}$ is a function $\mu : \mathscr{F} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and for all disjoint $(A_i)_{i \in \mathbb{N}} \in \mathscr{F}$ we have $\mu(\cup_i A_i) = \sum_i \mu(A_i)$. We call $(\Omega, \mathscr{F}, \mu)$ a measure space.

Gap!!! Add examples of measures and probability measures.

Theorem 1.6. *An intersection of σ-algebras is also a σ-algebra.*

Proof. Let $(\mathscr{A}_i)$ be uncountably many σ-algebras. Let $A \in \cap \mathscr{A}_i$. Then $A^C \in \mathscr{A}_i$ for all $\mathscr{A}_i$, so $A^C \in \cap \mathscr{A}_i$. Similarly, let $(A_j)_{j \in \mathbb{N}} \in \cap \mathscr{A}_i$. Then $\cup_j A_j \in \mathscr{A}_i$ for all $\mathscr{A}_i$, so $\cup_j A_j \in \cap \mathscr{A}_i$. $\square$

Theorem 1.6 motivates the following definition.

Definition 1.7. Let $\mathscr{A} \subset \Omega$. The σ-algebra generated by $\mathscr{A}$ is the intersection of all σ-algebras that contain $\mathscr{A}$. We denote it by $\sigma(\mathscr{A})$.

We can think of $\sigma(\mathscr{A})$ as the smallest σ-algebra containing $\mathscr{A}$.

Notation 1.8. A Borel σ-algebra is a σ-algebra generated by a topology. We will use $\mathscr{B}$ to denote the Borel σ-algebra of $\mathbb{R}$ that is generated by the usual open set topology.

Gap!!! This section needs to be continued with product spaces, the upper/lower continuity of measures, the construction of the Lebesgue measure, the Caratheodory theorem, and the Dynkin uniqueness theorem.

2 Probability spaces

Definition 2.1. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a measure space. If $\mathbb{P}(\mathscr{F}) = 1$, we call $\mathbb{P}$ a probability measure, we call $(\Omega, \mathscr{F}, \mathbb{P})$ a probability space, and we call an element of $\mathscr{F}$ an event.

Notation 2.2. It is common to use a shorthand notation for events. We often write $\{\text{condition}\}$ to mean $\{\omega \in \Omega \mid \text{ condition}\}$ and we often write $\mathbb{P}(\text{condition})$ to mean $\mathbb{P}\left(\{\omega \in \Omega \mid \text{condition}\}\right)$.

Definition 2.3. If $(A_i)_{i \in \mathbb{N}}$ are a sequence of events, "A_i occurring infinitely often" is the event

$$\{A_i \text{ i.o.}\} = \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} A_n$$

and "A_i occurring eventually" is the event

$$\{A_i \text{ eventually}\} = \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} A_n.$$

Remark. $\{A_i \text{ i.o.}\}$ is the set of elements $\omega \in \Omega$ that belong in infinitely many of the events (A_i) and $\{A_i \text{ eventually}\}$ is the set of elements $\omega \in \Omega$ that belong in all (A_i) after a certain point. We note that $\{A_i \text{ eventually}\} \subset \{A_i \text{ i.o.}\}$.

Gap!!! Add the Borel-Cantelli Lemma here.

Definition 2.4. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. Events $(A_i)_{i \in I} \in \mathscr{F}$ are called independent if for any finite $J \subset I$ we have $\mathbb{P}\left(\cap_{j \in J} A_j\right) = \prod_{j \in J} \mathbb{P}\left(A_j\right)$.

Theorem 2.5. *If (A_n) are independent and for all n we define either $B_n = A_n$ or $B_n = A_n^C$ then all (B_n) are independent.*

Gap!!! Prove the above theorem.

Gap!!! Add the Goldberg conjecture discussion here, as an example.

Definition 2.6. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. Collections $(\mathscr{A}_i) \subset \mathscr{F}$ are called independent if for any sequence $(A_i) \in \mathscr{F}$ with $A_i \in \mathscr{A}_i$ the (A_i) are independent.

3 Random variables

Definition 3.1. Let $(\Omega_1, \mathscr{F}_1, \mu_1)$ and $(\Omega_2, \mathscr{F}_2, \mu_2)$ be measure spaces. A function $g : \Omega_1 \to \Omega_2$ is called measurable if for all $A \in \mathscr{F}_2$ we have $g^{-1}(A) \in \mathscr{F}_1$.

Definition 3.2. A random variable on a probability space $(\Omega, f, \mathbb{P})$ is a measurable function $\mathsf{X} : \Omega \to \mathbb{R}$.

Gap!!! Add some examples of random variables.