Probability Notes

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Contents

1 Measure theory

Definition 1.1. *Let Ω be a set. A* <u>*topology*</u> *on Ω is a collection $\mathscr{A} \subset P(\Omega)$ that is closed under unions and finite intersections with $\Omega, \emptyset \in \mathscr{A}$. For $E \subset \Omega$, if $E \in \mathscr{A}$, we call E open, and if $E^C \in \mathscr{A}$, we call E closed.*

For example, if (X, d) is a metric space, $\mathscr{A} = \{E | E = \cup_i N_{r_i}(x_i)\}$ is a topology on X.

Gap!!! The example should be proven.

Definition 1.2. *Let Ω be a set and $\mathcal{S} \subset P(\Omega)$. Then the* <u>*topology generated by*</u> *$\mathcal{S}$ is*

$$\mathscr{A} = \{E \subset \Omega | E \text{ is a union of finite intersections of sets in } \mathcal{S}.\}$$

Gap!!! The definition should be more explicit.

Gap!!! Examples and Furstenberg's theorem should be added here

Definition 1.3. *Let Ω be a set. An* <u>*algebra*</u> *is a collection $\mathscr{A} \subset P(\Omega)$ that is closed under finite unions and compliments with $\Omega \in \mathscr{A}$. If an algebra is also closed under countable unions, we call it a* <u>*σ-algebra*</u>*.*

Theorem 1.4. *Algebras are closed under finite intersections and σ-algebras are closed under countable intersections.*

Proof. Let $\mathscr{A}$ be an algebra and let $A_1, \ldots, A_n \in \mathscr{A}$. Then $A_1^C, \ldots, A_n^C \in \mathscr{A}$, so $\left(\cap_{i=1}^n A_i\right)^C = \cup_{i=1}^n A_i^C \in \mathscr{A}$, so $\cap_{i=1}^n A_i \in \mathscr{A}$. The proof for σ-algebras is similar. $\square$

Definition 1.5. *Let $\mathscr{F}$ be a σ-algebra on a set Ω. A* <u>*measure*</u> *on $\mathscr{F}$ is a function $\mu : \mathscr{F} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and for all disjoint $(A_i)_{i \in \mathbb{N}} \in \mathscr{F}$ we have $\mu(\cup_i A_i) = \sum_i \mu(A_i)$. We call $(\Omega, \mathscr{F}, \mu)$ a* <u>*measure space*</u>*.*

Definition 1.6. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a measure space. If $\mathbb{P}(\mathscr{F}) = 1$, we call μ a* <u>*probability measure*</u>*, we call $(\Omega, \mathscr{F}, \mathbb{P})$ a* <u>*probability space*</u>*, and we call an element of $\mathscr{F}$ an* <u>*event*</u>*.*

Gap!!! Add examples of measures and probability measures.

Theorem 1.7. *An intersection of σ-algebras is also a σ-algebra.*

Proof. Let $(\mathscr{A}_i)$ be uncountably many σ-algebras. Let $A \in \cap \mathscr{A}_i$. Then $A^C \in \mathscr{A}_i$ for all $\mathscr{A}_i$, so $A^C \in \cap \mathscr{A}_i$. Similarly, let $(A_j)_{j \in \mathbb{N}} \in \cap \mathscr{A}_i$. Then $\cup_j A_j \in \mathscr{A}_i$ for all $\mathscr{A}_i$, so $\cup_j A_j \in \cap \mathscr{A}_i$. $\square$

Definition 1.8. *Let $\mathscr{A} \subset \Omega$. The* <u>*$\sigma$-algebra generated by $\mathscr{A}$*</u> *is the intersection of all σ-algebras that contain $\mathscr{A}$. We denote it by $\sigma(\mathscr{A})$.*

Due to Theorem 1.7, we can think of $\sigma(\mathscr{A})$ as the smallest σ-algebra containing $\mathscr{A}$.

Notation 1.9. *A Borel σ-algebra is a σ-algebra generated by a topology. We will use $\mathscr{B}$ to denote the Borel σ-algebra of $\mathbb{R}$ that is generated by the usual open set topology.*

Theorem 1.10. *Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. TBC*