Railgun and Targeting Model

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1 Railgun Model

```
In [1]: # Configure Jupyter so figures appear in the notebook
        %matplotlib inline

        # Configure Jupyter to display the assigned value after an assignment
        %config InteractiveShell.ast_node_interactivity='last_expr_or_assign'

        # import functions from the modsim.py module
        from modsim import *

        import math
        import matplotlib
```

1.1 Introduction

Railguns are powerful tools that can be used to launch projectiles at extremely high velocity using electromagnetic forces. While connotations of railguns are typically in the realm of weaponry and military usage, they could also be used to assist in launching satellites into space, propelling plasma, or launching baseballs at very high velocities.

A typical railgun is composed of the following primary components: - Two long parallel rails made of magnetic material

- A power supply capable of discharging very quickly, connected by the positive lead to one end of the rails and by the negative lead to the same end of the other rail.
- A conductive armature (the object propelled by the Lorentz force)
- A projectile pushed by the armature (if the armature is not the projectile itself)

A diagram is included later that further details the anatomy of railguns.

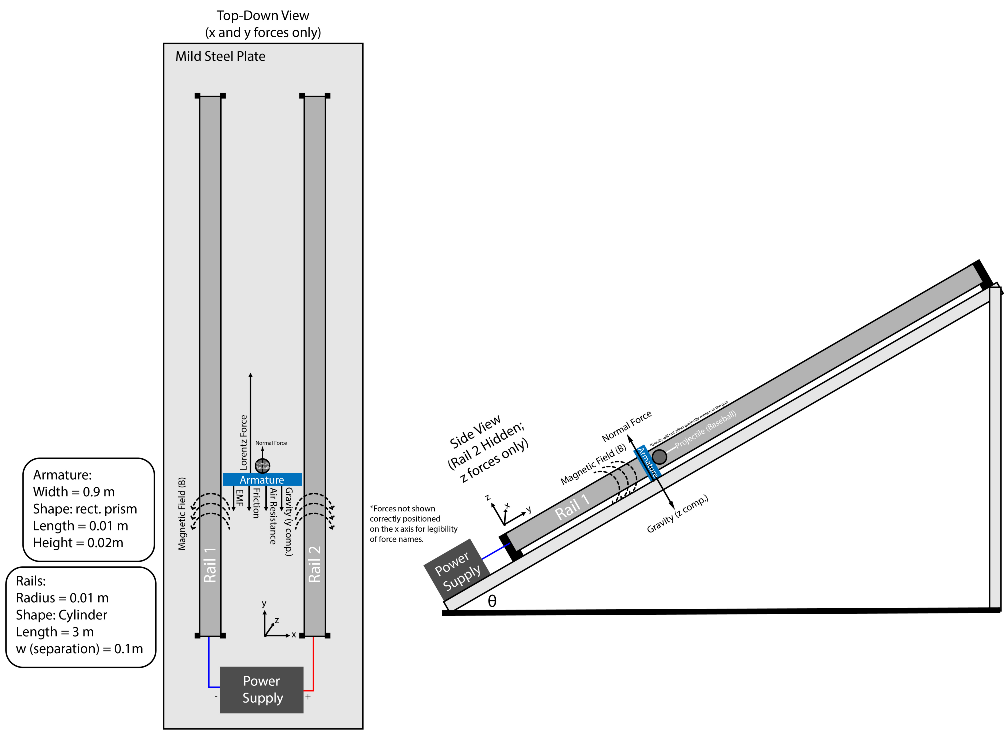
1.2 Questions

1. What is the launch velocity of a projectile (in this case, a baseball) from a railgun with a given current and given specifications of the armature, projectile, rails, etc.?

2. With a target of a given location (in a 2D plane in line with the railgun's rails) and a given launch velocity determined by the railgun model, what angle from the horizontal axis will result in the baseball striking the target?

1.2.1 Force Diagram

This cartoon doubles as our free body diagram of the railgun model. The forces acting on the armature are shown as force vectors and represented below mathematically.



1.2.2 Equations used for railgun

$$F_{Lorentz} = \frac{\mu_0 I^2}{2\pi} \ln \frac{R+w}{R}$$

$$F_{EMF} = -B\gamma w, \text{ where:}$$

$$B = \frac{\mu_0 I}{2\pi w} \ln \frac{R+w}{R}$$

$$F_{friction} = \mu_d F_{g_y} m_{arm}, \text{ where:}$$

$$F_{g_y} = F_g cos(\theta)$$

Variables are labeled in the parameter objects with their units and what they represent, except for `rail_r` which is expressed as R and `current` which is expressed as I. Derivation for equations can be found in the paper (linked at the bottom) on railgun physics.

1.2.3 Assumptions and Constraints

To use the aforementioned equations, the following assumptions/constraints must be considered:

- Constant Lorentz force, which is dependent on:
  - Constant current
  - The rails are made of a magnetic material with a magnetic permeability of free space constant: $\mu_0 = 10\,\frac{H}{m}$
  - Rails are separated enough (by length w) relative to their diameter
  - Rails are long enough relative to their separation (w) so they can be treated by the equations as rails of infinite length.
  - Armature is made of copper (conductive)
  - Rails are cylindrical and wire-like (small diameter)
  - Uniform current density
  - Thin armature
  - Constant μ_0
  - No energy loss to magnetization
  - No energy lost to eddy currents
- Coefficient of drag for armature is constant and is 2
- Flow speed around armature equals the armatures velocity

These assumptions and constraints are incorporated into our model parameters:

In [2]:
```
num_sweeps = 15
angle_elevation = Params(angle_elevation = pi / 4);

params = Params(mass_proj = 0.145, #kg
                rail_r = 0.01, #m rail radius
                rail_l = 3, #m rail length
                arm_w = 0.9, #m armature width in x axis
                arm_l = 0.01, #m armature length in y axis
                arm_d = 0.02, #m armature depth in z axis
                w = 0.1, #m rail separation
                mu = 10, #H/m permeability constant of rails
                rho_cu = 8960, #g/m^3 density of Cu (copper)
                v_init = 0, #m/s
                y_init = 0, #m
                t_end = 1000,
                coef_fric = 0.36, #dimensionless, dynamic friction
                                  #between copper and mild steel
                                  #(the surface that the copper armature
                                  #slides against)
                g = 9.8, #m/s*s
                angle_elevation = angle_elevation.angle_elevation, #radians
```

```
                                                                      #from
                                                                      #horizontal;
                                                                      #being swept
                     cd = 2, # approximate drag coefficient of a rectangular box
                     rho_air = 1.225, # kg/m^3 density of air
                     current = 1000 # amperes
                     )
    ;
```

Out[2]: ''

In [3]:
```
def make_system(params):
    """Make a system object.

    params: parameter object

    returns: System object
    """

    unpack(params)

    mass_arm = (arm_w * arm_l * arm_d) * rho_cu

    # Magnetic force in railgun on armature
    lorentz_force = (((mu * current**2)/(2 * np.pi))
                    * (math.log((rail_r + w) / rail_r)))

    # Magnetic field strength
    B = ((mu * current)/(2 * np.pi * w)) * (math.log((rail_r + w) / rail_r))

    # Friction force on armature
    arm_friction = coef_fric * (g * mass_arm * math.cos(angle_elevation))

    # Force of gravity in y direction
    gravity_y_direc = g * mass_arm * math.sin(angle_elevation)

    # Cross-sectional area of armature
    front_area = arm_w * arm_d #m^2 frontal areaa of the armature

    init = State(v=v_init, y=y_init)

    return System(params, lorentz_force=lorentz_force, mass_arm = mass_arm,
                  B = B, arm_friction = arm_friction,
                  gravity_y_direc = gravity_y_direc,
                  front_area = front_area, init=init)
```

In [4]:
```
system = make_system(params);
```

1.2.4 Modeling Decisions

```
In [5]: def slope_func(state, t, system):
            """Define differential equations for velocity and acceleration

            state: v = velocity, y = position in railgun
            t: time
            system: System object

            returns: acceleration and velocity
            """

            unpack(system)
            v, y = state

            # Electromagnetic force
            emf = B * v * w

            # The u of the drag force equation will be set equal to velocity (v)
            air_resistance = 0.5 * rho_air * (v**2) * cd * front_area

            # Acceleration of the armature and projectile
            dvdt = ((lorentz_force - emf - arm_friction - gravity_y_direc - air_resistance)
                   / (mass_proj + mass_arm))

            # Velocity of armature and projectile
            dydt = v

            return dvdt, dydt
```

```
In [6]: slope_func(system.init, 0, system)
```

```
Out[6]: (2171096.6306084436, 0.0)
```

```
In [7]: def event_func(state, t, system):
            """Event function to stop the simulation when the
               armature reaches the end of the rails

            state: v = velocity, y = position in railgun
            t: time
            system: System object

            returns: the difference between armature position and length of rails
            """

            v, y = state
            return system.rail_l - y
```

```
In [8]: results, details = run_ode_solver(system, slope_func, events = event_func,
```

```
                                            max_step=0.00001)
        details.message
```

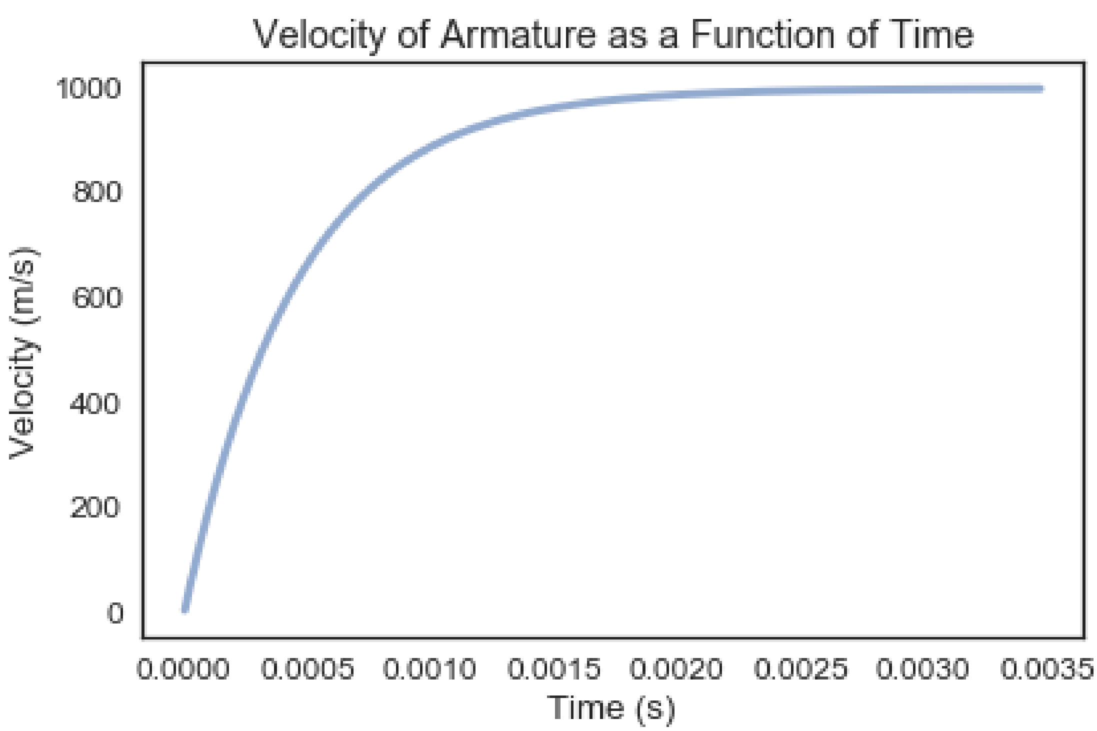
```
Out[8]: 'A termination event occurred.'
```

1.2.5 Results

```
In [9]: print('The final velocity of the armature (or the exit velocity of '
              'the projectile) is {} m/s'.format(get_last_value(round(results.v, 3))))
```

```
The final velocity of the armature (or the exit velocity of the projectile) is 993.798 m/s
```

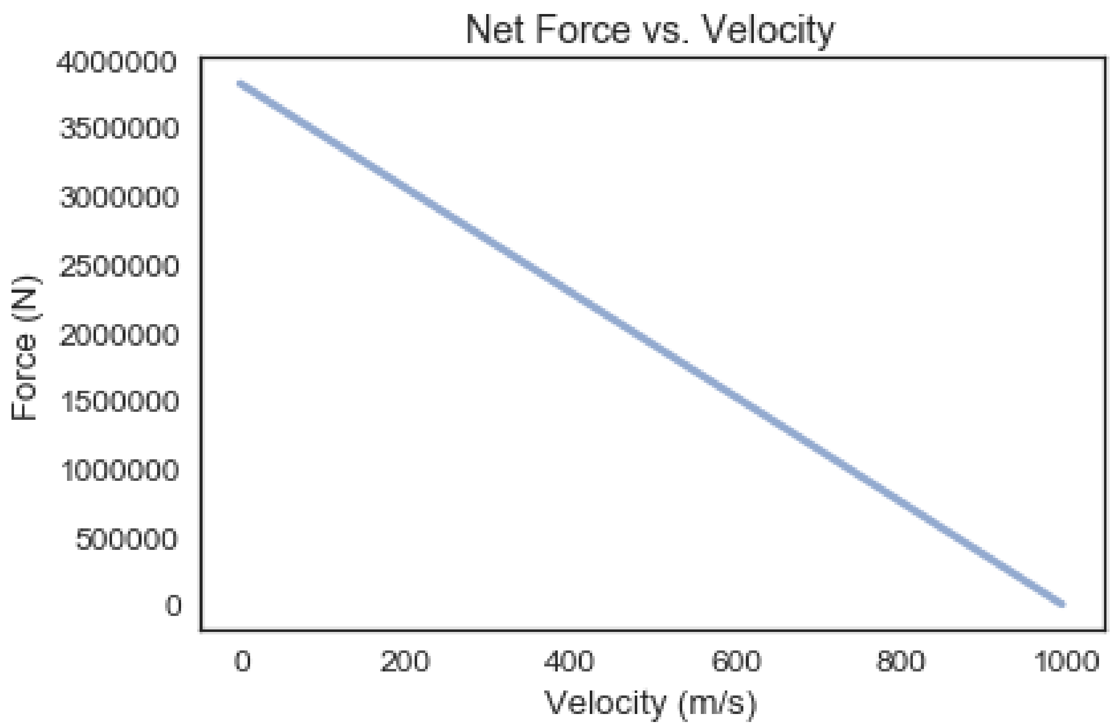
```
In [10]: plot(results.v)

         decorate(xlabel='Time (s)',
                  ylabel='Velocity (m/s)',
                  title='Velocity of Armature as a Function of Time',
                  legend=False)
```



1.2.6 Interpretation

```
In [11]: net_force = []
         v_list = []
```

```
for i in results.index:
    v_list.append(results.v[i])
    net_force.append(-(system.B * results.v[i] * system.w)
                 - (0.5 * system.rho_air * (results.v[i]**2)
                   * system.cd * system.front_area)
                 - system.arm_friction
                 - system.gravity_y_direc
                 + system.lorentz_force)
plot(v_list, net_force, label = 'Net Force')

decorate(xlabel = 'Velocity (m/s)',
         ylabel = 'Force (N)',
         legend = False,
         Title = 'Net Force vs. Velocity')
```



The velocity of the armature and projectile as a function of time reach is asymptotic, which is explained through the balancing of forces. The plot above shows the net force as a function of velocity. As the velocity increases, the net force decreases proportionally until the net force, and therefore acceleration, become zero.

1.3 Projectile Motion

The projectile in the railgun is a baseball and will be modeled with a given initial launch angle, height, and velocity. The baseball's drag will be velocity dependent, as the coefficient of drag (c_d)

will vary according to interpolated data from baseball_drag.csv.

Projectile Question Restated With a target of a given location (in a 2D plane in line with the railgun's rails) and a given launch velocity determined by the railgun model, what angle from the horizontal axis will result in the baseball striking the target?

The parameters for our target are the target's x and y position in a 2D plane. The coordinate system between the railgun and the projectile are reset, where x represents horizontal distance between the railgun and the target, and y is the vertical height from the ground.

1.3.1 Assumptions

- The projectile is a baseball and follows the drag coefficient curve given by data/baseball_drag.csv
- No wind in any direction
- No angular velocity of the baseball

```
In [12]: params_proj = Params(x = 0, #m
                              g = 9.8, #m/s^2
                              mass = 145e-3, #kg mass of projectile
                              diameter = 73e-3, #m diameter of projectile
                              rho = 1.2, #kg/m^3 density of air
                              C_d = 0.33, #cooefficient of drag
                              angle = params.angle_elevation, #radians
                              velocity = get_last_value(results.v), #m / s
                              t_end = 200, #s
                              target_x = 320, #m
                              target_y = 0, #m
                              target_radius = 0.5 #m
                              )
         ;

Out[12]: ''

In [13]: def make_system_proj(params, params1):
             """Make a system object.

             params: Params object with angle, velocity, x, y,
                          diameter, duration, g, mass, rho, and C_d
             params1: Params object from railgun simulation

             returns: System object
             """
             unpack(params)

             # compute x and y components of velocity
             vx, vy = pol2cart(params.angle, velocity)

             # compute area from diameter
```

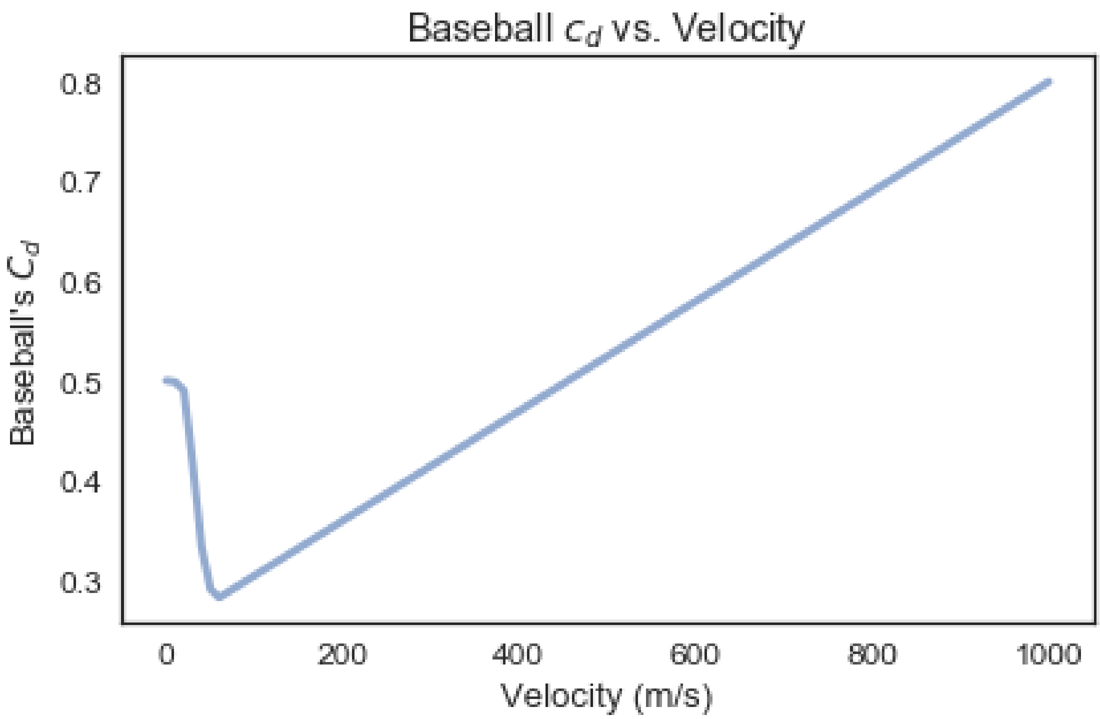
```
    area = np.pi * (diameter/2)**2

    # define y
    y = math.sin(params.angle) * system.rail_l #m

    # make the initial state
    init = State(x=x, y=y, vx=vx, vy=vy)

    return System(params, init=init, area=area, y=y)
```

In [14]:
```
system_proj = make_system_proj(params_proj, params);
```

As the velocity of the baseball increases, the coefficient of drag (c_d) varies. This relationship is defined by `baseball_drag.csv` and is interpolated.

In [15]:
```
m = UNITS.meter
s = UNITS.second

baseball_drag = pd.read_csv('data/baseball_drag.csv')
baseball_drag.index = Quantity(baseball_drag['Velocity in mph'].values,
                               UNITS.mph).to(m/s)
baseball_drag;
```

In [16]:
```
drag_interp = interpolate(baseball_drag['Drag coefficient'])
vs = linspace(0, 1000, 101)
cds = drag_interp(vs)
plot(vs, cds)
decorate(xlabel = 'Velocity (m/s)',
         ylabel = "Baseball's $C_d$",
         title = 'Baseball $c_d$ vs. Velocity',
         legend = False)
```




```
In [17]: def drag_force(v, system):
             """finds drag force in the opposing direction to velocity

             v: velocity
             system: System object with rho, C_d, area

             returns: Vector drag force
             """

             unpack(system)
             C_d = drag_interp(v.mag)
             mag = -rho * v.mag ** 2 * C_d * area / 2
             direction = v.hat()
             f_drag = direction * mag
             return f_drag
```

```
In [18]: def slope_func_proj(state, t, system):
             """Computes derivatives of the state variables.

             state: State (x, y, x velocity, y velocity)
             t: time
             system: System object with g, rho, C_d, area, mass

             returns: sequence (vx, vy, ax, ay)
```

```
    """
    x, y, vx, vy = state
    unpack(system)

    v = Vector(vx, vy)
    a_drag = drag_force(v, system) / mass
    a_grav = Vector(0, -g)

    a = a_grav + a_drag

    return vx, vy, a.x, a.y
```

In [19]:
```
slope_func_proj(system_proj.init, 0, system_proj)
```

Out[19]:
```
(702.721098455984,
 702.721098455984,
 <Quantity(-9628.802796352897, 'dimensionless')>,
 <Quantity(-9638.602796352896, 'dimensionless')>)
```

In [20]:
```
def event_func_proj(state, t, system):
    """Stop when the y coordinate is 0.

    state: State object
    t: time
    system: System object

    returns: y coordinate
    """
    x, y, vx, vy = state
    return y
```

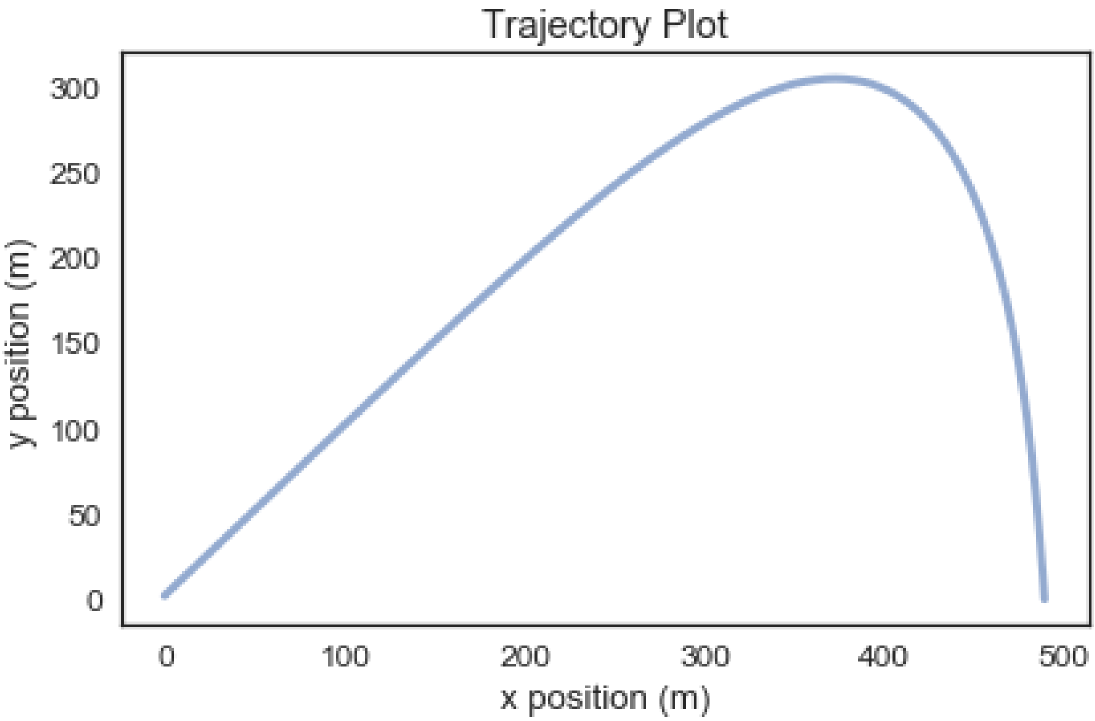
In [21]:
```
event_func_proj(system_proj.init, 0, system_proj)
```

Out[21]:
```
2.121320343559643
```

In [22]:
```
system_proj = System(system_proj, drag_interp = drag_interp)
results_proj, details_proj = run_ode_solver(system_proj, slope_func_proj,
                                             events = event_func_proj,
                                             max_step=0.05)
details_proj.message
```

Out[22]:
```
'A termination event occurred.'
```

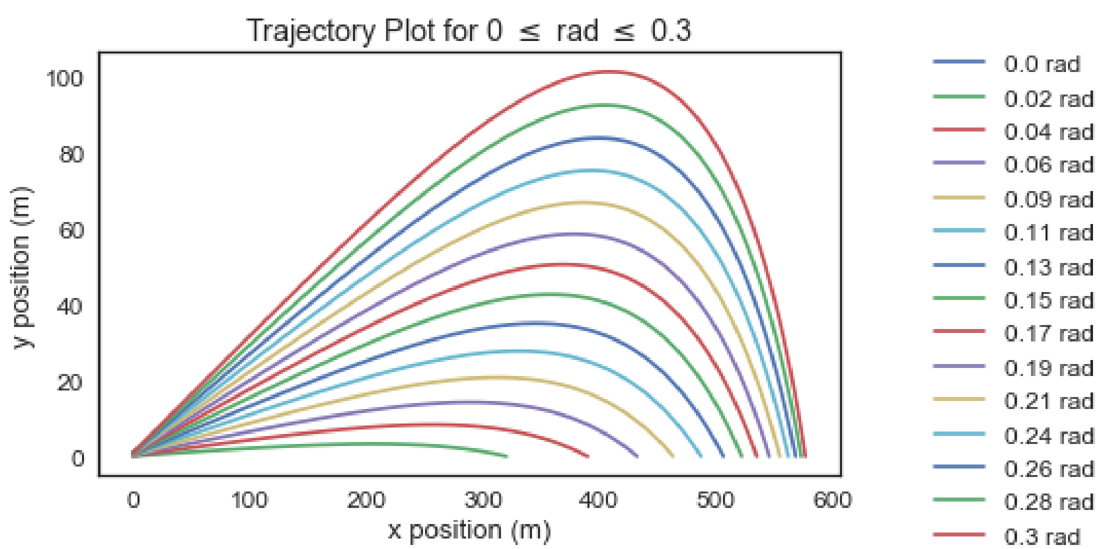
In [23]:
```
plot(results_proj.x, results_proj.y)

decorate(title='Trajectory Plot',
         xlabel='x position (m)',
         ylabel='y position (m)',
         legend = False)
```



In [24]:
```
def simulation(angle_elevation):
    """Run through all functions in simualtion

    angle_elevation: angle between the railgun and ground

    returns: Position vector of projectile
    """

    params_new = Params(params, angle_elevation = angle_elevation)
    system = make_system(params_new)
    results_sim, details_sim = run_ode_solver(system, slope_func,
                                              events = event_func,
                                              max_step = 0.00001)

    params_proj_new = Params(params_proj, angle = params_new.angle_elevation,
                             velocity = get_last_value(results_sim.v))
    system_proj = make_system_proj(params_proj_new, params_new)
    results_proj, details_proj = run_ode_solver(system_proj, slope_func_proj,
                                                events = event_func_proj,
                                                max_step=0.1)

    x = results_proj.x
    y = results_proj.y
    vx = results_proj.vx
```

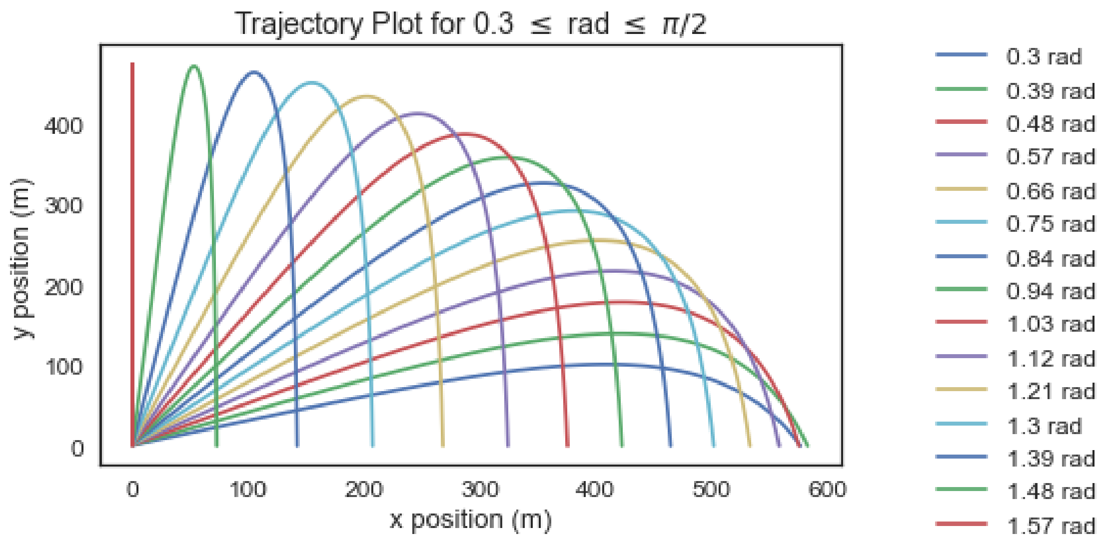
```
        vy = results_proj.vy

        xy = Vector(x, y)

        return xy, vx, vy
```

```
In [25]: # Sweep angle_elevation
         fig = plt.figure()
         ax = plt.subplot(111);

         for i in range(0, num_sweeps):
             angle_linspace = linspace(0, 0.3, num_sweeps)
             params.angle_elevation = angle_linspace[i]
             xy, vx, vy = simulation(params.angle_elevation)
             # Plot
             ax.plot(xy.x, xy.y, label = '{} rad'.format(round(angle_linspace[i], 2)))
             decorate(title='Trajectory Plot for 0 $\leq$ rad $\leq$ 0.3',
                      xlabel='x position (m)',
                      ylabel='y position (m)',
                      )
             ax.legend(bbox_to_anchor=(1.1, 1.05))
```




```
In [26]: # Sweep angle_elevation for a second time
         fig = plt.figure()
         ax = plt.subplot(111);

         for i in range(0, num_sweeps):
             angle_linspace = linspace(0.3, pi/2, num_sweeps)
             params.angle_elevation = angle_linspace[i]
```

```
xy, vx, vy = simulation(params.angle_elevation)
# Plot
ax.plot(xy.x, xy.y, label = '{} rad'.format(round(angle_linspace[i], 2)))
decorate(title='Trajectory Plot for 0.3 $\leq$ rad $\leq$ $\pi/2$',
         xlabel='x position (m)',
         ylabel='y position (m)',
         )
ax.legend(bbox_to_anchor=(1.1, 1.05))
```



As evidenced by these graphs, there will be two solutions for angles that will result in a particular impact point. Thus, we will have to indicate to `fsolve` a prediction that is ≤ ~0.3 rad.

```
In [27]: # Assuming that the projectile is on the ground (y = 0):
         def simulation_hitting_target(angle_elevation):
             """Run through all functions in simualtion

             angle_elevation: angle between the railgun and ground

             returns: difference between distance of projectile and target distance
             """

             params_new = Params(params, angle_elevation = angle_elevation)
             system = make_system(params_new)
             results_sim, details_sim = run_ode_solver(system, slope_func,
                                                       events = event_func,
                                                       max_step = 0.00001)

             params_proj_new = Params(params_proj, angle = params_new.angle_elevation,
                                      velocity = get_last_value(results_sim.v))
```

```
            system_proj = make_system_proj(params_proj_new, params_new)
            results_proj, details_proj = run_ode_solver(system_proj, slope_func_proj,
                                                        events = event_func_proj,
                                                        max_step=0.1)

            return get_last_value(results_proj.x) - params_proj.target_x
```

```
In [28]: solution = fsolve(simulation_hitting_target, x0 = 0.02);
```

1.3.2 Results

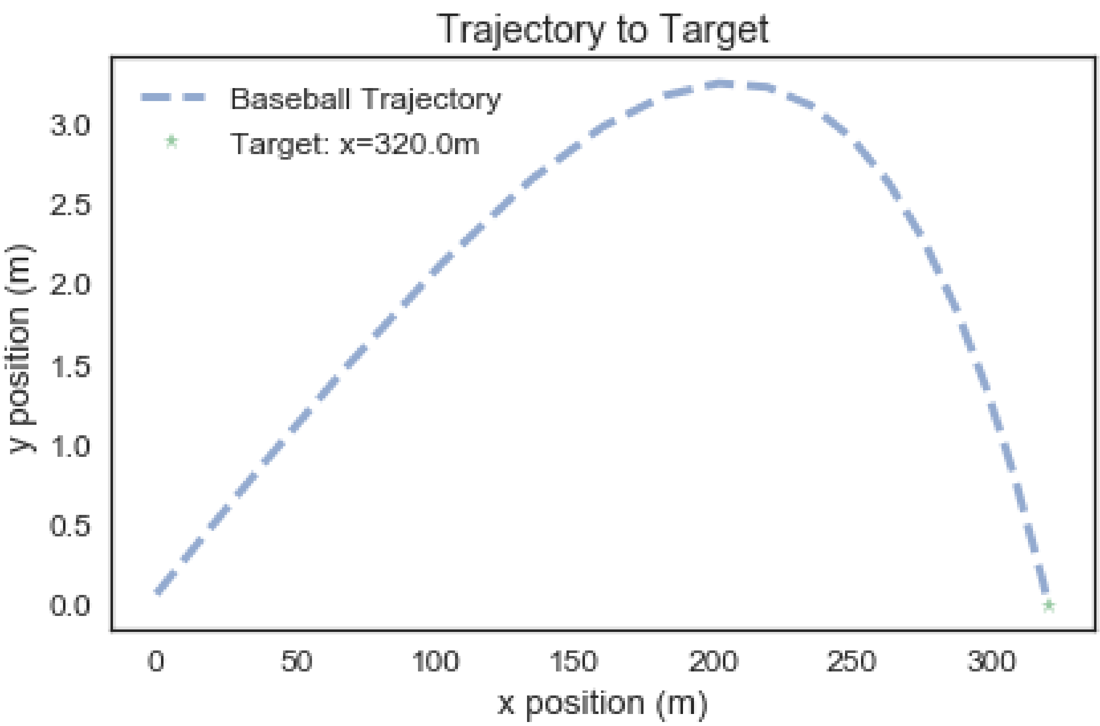
```
In [29]: print('The angle from horizontal that will cause the projectile to hit the '
               'target is {} radians.'.format(round(solution[0], 5)))
```

```
The angle from horizontal that will cause the projectile to hit the target is 0.02137 radians.
```

1.3.3 Interpretation

```
In [30]: target_sim_xy, target_sim_vx, target_sim_vy = simulation(solution[0])
         plot(target_sim_xy.x, target_sim_xy.y, '--',
              label = 'Baseball Trajectory')
         plot(params_proj.target_x, params_proj.target_y, '*',
              label = 'Target: x={}m'.format(params_proj.target_x))
         decorate(title='Trajectory to Target',
                  xlabel='x position (m)',
                  ylabel='y position (m)',
                  )
         speed = math.sqrt((get_last_value(target_sim_vx))**2
                           + (get_last_value(target_sim_vy))**2)
         kinetic_energy = 0.5 * params_proj.mass * speed **2
         print('The baseball hits the target with a final speed of {} m/s and with a '
               'kinetic energy of \n{} joules.'.format(round(speed, 2),
                                                       round(kinetic_energy, 2)))
```

```
The baseball hits the target with a final speed of 94.8 m/s and with a kinetic energy of
651.6 joules.
```



As we swept the launch angle, we used `fsolve` to find the angle at which the projectile must be launched to hit a target. From this, we found the angle was ~0.02137 radians. Solving for the final speed and kinetic energy, we found the baseball was traveling at 94.8 meters per second as it hits the target and it has a kinetic energy of 651.6 J.

Because we avoided hard-coding in the parameters, our model will work for any launch velocity and target position on the x axis, provided it is within the maximum range of the railgun.

1.3.4 Sources

- Source for coeff fric: http://www.engineershandbook.com/Tables/frictioncoefficients.htm
- Source for drag coeff: https://www.engineeringtoolbox.com/drag-coefficient-d_627.html
- Source for railgun physics: http://citeseerx.ist.psu.edu/viewdoc/download?rep=rep1&type=pdf&doi=10.
- Source for air density: https://www.engineersedge.com/calculators/air-density.htm
- Some code was adapted from: https://github.com/AllenDowney/ModSimPy/blob/master/code/soln/cl