INTRODUCTION TO LAPLACE TRANSFORMS

DUNCAN SUTHERLAND

1. Mock lecture

These notes accompany my mock lecture for the senior lecturer position in the School of Mathematics and Statistics at the University of Melbourne. The mock lecture is 20 minutes long and must explain what is meant by a Laplace transform, explain how Laplace transforms may be applied to solve ODEs, and introduce and illustrate this for a second-order linear ODE. The students, while of diverse backgrounds will have seen the fundamental concepts of functions of several variables and vector calculus; and standard one-variable calculus and elementary real analysis. The mock lecture fits into a broader ODEs course. Students will have been introduced to linear ODEs including initial value problems, first-order linear ODEs and second-order linear ODEs. The purpose of these notes is to provide some context to the students to help them place the new material in the course, ensure that students have all relevant reference material to hand, and to provide a more comprehensive representation of my teaching style. These notes are not meant to be a complete set of notes upon Laplace transforms as the notes really only cover the 20 minute mock lecture.

2. Outline

(1) Definition of the Laplace transform, some conditions, and the inverse Laplace transform
(2) Using the table of Laplace transforms
(3) The Laplace transform of derivatives
(4) Solving linear ordinary differential equations with initial conditions using Laplace transforms

3. Definition and notation

The Laplace transform is an example of a much broader class of transforms called *integral transforms*. Integral transforms map a function in one *original* function space to another *transformed* function space using integration. The transformed function is often easier to manipulate and characterise than the original function. For example, Laplace transforms are often used to convert linear ordinary differential equations, particularly those with discontinuous or impulsive forcing terms, to algebraic equations which are far easier to solve.
Let $f(t)$ be a function defined for $t \geq 0$ The Laplace transform of the function $f(t)$, denoted as $\mathcal{L}\{f(t)\} = F(s)$ is defined as follows:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt\,. \tag{1}$$

Notationally, we use the fancy $\mathcal{L}\{f(t)\}$ to denote the Laplace transform is applied to the function $f(t)$ and we use the capital $F(s)$ to denote the transformed $f(t)$. We will also colloquially refer to $t-$space to refer to the original function space and we will use $s-$space to refer to the new functional space after the Laplace transform has been applied. We will adopt the common convention of using lower case letters to denote the original function and using upper case letters to denote the transformed function.

3.1. **Some technical restrictions.** The Laplace transform is defined by an improper integral, and so the Laplace transform is only defined when the integral converges. That is when the following limit exists:

$$\lim_{x\to\infty}\int_0^x f(t)e^{-st}\,dt\,.$$

If $f(t)$ is a *piecewise continuous* function and if

$$|f(t)| \le Ke^{ct}$$

for all $t > M$, where K and c are constants, then the Laplace transform converges for all $s > c$ (we will not prove this). The test to see if a function satisfies this boundedness constraint is to determine if the limit

$$\lim_{t\to\infty}\frac{f(t)}{e^{ct}}$$

exists.

These are not a particularly restrictive conditions and for most functions that arise in practical applications, such as polynomials, exponential functions sine and cosine, the Laplace transforms are well defined. However, there are some notable exceptions, such as the function: $f(t) = e^{t^2}$. Note that any bounded function, that is $f(t)$ such that $|f(t)| \le M$ for all t satisfies this constraint. A function $f(t)$ is piecewise continuous if the interval on which the function is defined can be divided into a finite number of subintervals $I_n = [t_n, t_{n+1}]$ and if $f(t)$ is continuous on each open subinterval (t_n, t_{n+1}) and $f(t)$ approaches a finite limit as $t \to t_{n+1}$. Importantly, it is not necessary for $f(t)$ to be defined at the end points of I_n. An example of a piecewise continuous function is shown in Figure 1.

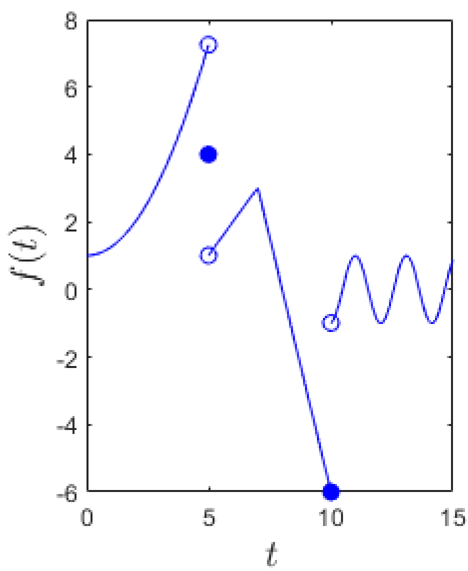

FIGURE 1. A example of a piecewise continuous function

3.2. **Inverse Laplace transforms.** We have so far defined the Laplace transform from $t-$space to $s-$space; an inverse transform is needed to transform from $s-$space back to $t-$space. The inverse transform does have a formal integral definition, however, it is defined by a complex line integral and requires knowledge beyond the scope of the course to calculate explicitly. The inverse transform is also unique; the proof of this is also beyond the scope of the course.
Notationally, we write $f(t) = \mathcal{L}^{-1}\{F(s)\}$ as the inverse Laplace transform. In practice, we will use a table of known Laplace transforms to perform our calculations.

3.3. **Linearity.** Because the Laplace transform is an integral transform, then the *linearity property* of integrals also applies to Laplace transforms. Let $g(x)$ and $h(x)$ be integrable functions, let a and b be scalar constants and recall that:

$$\int ag(x) + bh(x)\,dx = a\int g(x)\,dx + b\int h(x)\,dx\,.$$

The linearity property is vitally important for manipulating Laplace transforms. The inverse Laplace transform is also linear.

4. Tables of Laplace transforms

Table 1 shows Laplace transforms that commonly arise in many problems with practical applications. To use the table to compute a Laplace transform of a function $f(t)$, we identify the function in the left column, and then read the Laplace transform in the right column. To compute the inverse Laplace transform of a function $F(s)$ we identity the function in the right column and read the inverse Laplace transform from the left column. Some manipulation, such as the use of partial fractions, is often required to compute the inverse Laplace transform of more complicated functions; this will be illustrated later in case-by-case worked examples, tutorial problems, or the supplementary videos. Note that a more comprehensive table is on the course website and will be provided as reference material in assessments.

4.1. **The Heaviside function.** The Laplace transform is typically used to handle linear ordinary differential equation problems where there are jumps in the forcing term. One useful function to define is the Heaviside step function:

$$u(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t \le 0 \end{cases}\,.$$

A plot of the Heaviside step function is shown in Figure 2.

4.2. **Example Laplace transforms.** Laplace transforms may be explicitly computed by simply applying the integral definition equation 1. The functions in Table 1 are either polynomials or exponential. Recall Euler's result allows us to rewrite a complex exponential as a sum of a sine and cosine:

$$e^{i\theta} = cos(\theta) + i\sin(\theta)\,,$$

and recall that the hyperbolic functions are defined in terms of exponential functions:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}\,, \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}\,.$$

Therefore we need only consider the Laplace transforms of $f(t) = t^n$ and $f(t) = e^{bt}$. The other transforms in Table 1 may be deduced by manipulating the Laplace transform of the exponential and these will be covered in the tutorial problems or in the supplementary videos.

TABLE 1. Table of some useful Laplace transforms.

$f(t)$	$F(s)$
$1\ (n=0,1,2,\cdots)$	$\frac{1}{s}$ $\mathrm{Re}(s)>0$
$t^n\ (n=0,1,2,\cdots)$	$\frac{n!}{s^{n+1}}$ $\mathrm{Re}(s)>0$
e^{at}	$\frac{1}{s-a}$ $\mathrm{Re}(s)>a$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$ $\mathrm{Re}(s)>0$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$ $\mathrm{Re}(s)>0$
$\cosh(wt)$	$\frac{s}{s^2-w^2}$ $\mathrm{Re}(s)>\|w\|$
$\sinh(wt)$	$\frac{w}{s^2-w^2}$ $\mathrm{Re}(s)>\|w\|$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$tf(t)$	$F'(s)$
$e^{at}f(t)$	$F(s-a)$
$u(t-a)f(t-a)$	$e^{-as}F(s)$ $a>0$
$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
$\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$

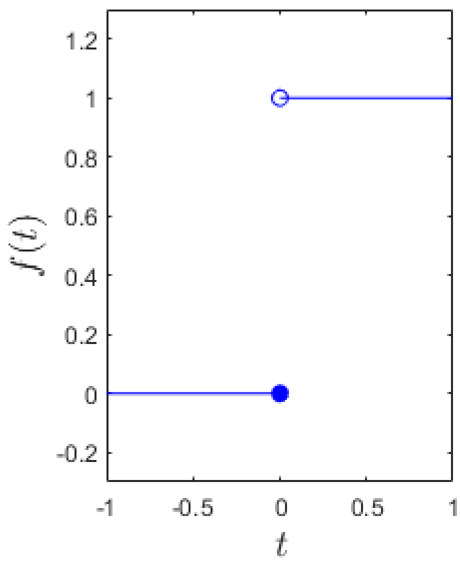

FIGURE 2. A plot of the Heaviside step function

Worked example: The Laplace transform of $f(t)=t^n$.
t^n is continuous everywhere for $t\geq 0$ and t^n grows slower than an exponential function, so the limit

$$\lim_{t\to\infty}\frac{f(t)}{e^{ct}}$$

will exist and the Laplace transform will be well defined. The first step is to substitute the form of $f(t)$ into the definition of the Laplace transform:

$$\mathcal{L}\{t^n\} = F(s) = \int_0^\infty t^n e^{-st}\,dt\,.$$

This integral may be evaluated by repeated application of integration by parts. Integrating once by parts gives

$$\int_0^\infty t^n e^{-st}\,dt \;=\; \lim_{x\to\infty} \frac{t^n e^{-st}}{-s}\Big|_0^x - \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt\,,$$

because the decaying exponential e^{-sx} will dominate the x^n term as $x \to \infty$ (this may be formally proved with L'Hôpital's rule) the upper bound of the first term gives zero. The lower bound of the first term is also obviously zero. Hence we have

$$F(s) = \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt\,,$$

which we notice is $\frac{n}{s}\mathcal{L}\{t^{n-1}\}$. We can repeat the same steps of integrating by parts to obtain:

$$F(s) = \frac{n(n-1)}{s^2}\int_0^\infty t^{n-2}e^{-st}\,dt\,.$$

If we continue to repeat the same steps of integrating by parts we will eventually arrive at

$$F(s) = \frac{n(n-1)(n-2)\cdots 2\times 1}{s^n}\int_0^\infty e^{-st}\,dt\,,$$

and a final trivial integration yields:

$$\begin{aligned}
F(s) &= \frac{n(n-1)(n-2)\cdots 2\times 1}{s^n}\lim_{x\to\infty}\left(\frac{1}{-s}\right)e^{-st}\Big|_0^x\,,\\
&= \frac{n!}{s^n}\left(\frac{1}{-s}\right)(0-1)\,,\\
&= \frac{n!}{s^{n+1}}\,.
\end{aligned}$$

Worked example: The Laplace transform of $f(t) = e^{at}$.
e^{at} is continuous everywhere for $t \geq 0$ but the limit

$$\lim_{t\to\infty}\frac{f(t)}{e^{ct}} = \lim_{t\to\infty}\frac{e^{at}}{e^{ct}} = \lim_{t\to\infty} e^{(a-c)t}$$

only exists if $c \geq a$. Therefore, we will find some restrictions upon the Laplace transform of $f(t) = e^{at}$.
The first step is to substitute the form of $f(t)$ into the definition of the Laplace transform:

$$\mathcal{L}\{e^{at}\} = F(s) = \int_0^\infty e^{at}e^{-st}\,dt\,.$$

Fortunately, this yields an integral which is straightforward. Rearranging slightly gives

$$F(s) = \int_0^\infty e^{(a-s)t}\,dt\,,$$

and upon integrating we obtain:

$$F(s) = \lim_{x\to\infty} \frac{1}{a-s} e^{(a-s)t}\Big|_0^x ,$$
$$= \lim_{x\to\infty} \frac{1}{a-s} e^{(a-s)x} - \frac{1}{a-s} ,$$

and as expected, we have encountered a problem. This limit exists only when $s > a$ (strictly $\mathrm{Re}(s) > a$ and if that is true then

$$F(s) = \frac{1}{s-a} .$$

The restriction $s > a$ is not particularly problematic as it essentially just reduces the domain of the transformed function.

5. The Laplace transform of derivatives

Laplace transforms were developed in the context of solving differential equations. The application of the Laplace transform to a differential equation in $t-$space yields an algebraic problem in $s-$space. Consider an ordinary differential equation for some independent variable $t \geq 0$ and a dependent variable $y(t)$. This ordinary differential equation will feature derivatives, e.g. $y'(t)$, however, the exact form of $y(t)$ will not be known until the equation is solved. It is therefore useful to compute the Laplace transform of $y'(t)$, to discover how it relates to the Laplace transform of $y(t)$. Similar to the functions we have previously considered, the Laplace transform of $y'(t)$ may be computed by starting with the definition.

Worked example: The Laplace transform of $y'(t)$ in terms of $\mathcal{L}\{y(t)\}$.
We will assume that $y(t)$ and $y'(t)$ satisfy the conditions for the Laplace transforms to exist. Applying the definition gives

$$\mathcal{L}\{y'(t)\} = Y'(s) = \int_0^\infty y'(t)e^{-st}\,dt .$$

Integrating by parts gives

$$Y'(s) = \lim_{x\to\infty} y(t)e^{-st}\Big|_0^x + s\int_0^\infty y(t)e^{-st}\,dt .$$

Because we assumed that $y(t)$ satisfies the conditions for the Laplace transform to exist, it must grow slower than an exponential function. Therefore, in the limit as $x \to \infty$, the first term will vanish. After applying the $t = 0$ lower bound the first term becomes $y(0)$, the value of the function at $t = 0$ often specified as an initial condition to supplement an ordinary differential equation. The second term is just the Laplace transform of $y(t)$ multiplied by s and we obtain the important result that

$$Y'(s) = sY(s) - y(0) .$$

This result may be applied repeatedly to find expressions relating the Laplace transform of higher-order derivatives to the Laplace transform of the function. Consider the second derivative and write $y''(t) = q'(t)$ so that $q(t) = y'$. Then using the previous result

$$Q'(s) = sQ(s) - q(0) ,$$

and it follows that

$$Y''(s) = sY'(s) - y'(0) .$$

Substituting for $Y'(s)$ gives

$$Y''(s) = s^2 Y(s) - sy(0) - y'(0)\,.$$

6. Solving linear ordinary differential equations with initial conditions using Laplace transforms

The diagram 3 shows the work flow for solving a linear ordinary differential equation using Laplace transforms.

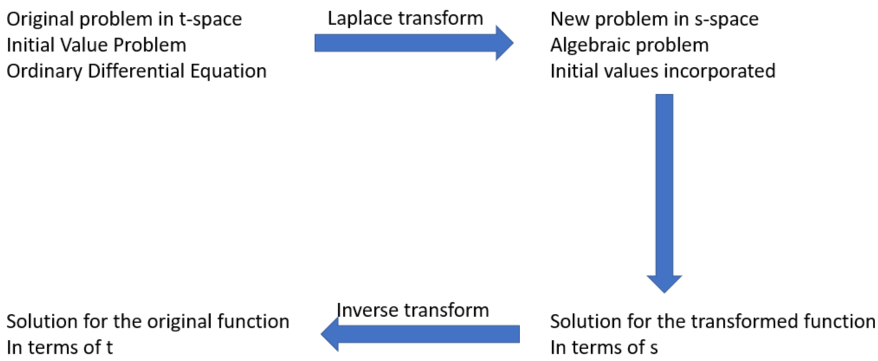

Figure 3. Diagram showing how a linear ordinary differential equation with initial conditions can be solved using Laplace transforms.

The Laplace transform is applied to both sides of the ordinary differential equation. Because the Laplace transform of a derivative is related to the Laplace transform of the function, the derivatives can be removed from the resulting equation in $s-$space, however, this does require knowledge of $y(0)$ ($y'(0)$ and higher derivatives) which must be supplied as initial conditions. Eliminating the derivatives gives an algebraic problem, which is often solved easily. Unfortunately, finding the inverse Laplace transform of the solution is often challenging. Often, the solution in $s-$space is a rational function. Rational functions may be decomposed into a sum of more elementary terms which can then be inverted by consulting table 1. This process is best illustrated by numerous examples, and will only be mastered by practice.
Worked example: Solving $y'(t) = e^t$, $y(0) = 1$ using Laplace transforms.
We start by applying the Laplace transform to both sides of the equation. This gives

$$\mathcal{L}\{y'(t)\} = \mathcal{L}\{e^t\}\,.$$

Using the table of transforms and the first derivative result we obtain

$$sY(s) - y(0) = \frac{1}{s-1}\,,$$

for $s > 1$. Substituting for $y(0)$ and rearranging for $Y(s)$ gives

$$Y(s) = \frac{1}{s(s-1)} + \frac{1}{s}\,.$$

While $\frac{1}{s}$ can be found in the table, $\frac{1}{s(s-1)}$ is not tabulated. Partial fraction decomposition can be used to rewrite this term. Let A and B be constants and let

$$\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1}\,,$$

cross multiplying by $s(s-1)$ gives

$$1 = A(s-1) + Bs = (A+B)s - A\,,$$

we can now equation like powers of s to find the values of A and B. This gives $A = -1$ and $B = 1$. Hence

$$\begin{aligned} Y(s) &= -\frac{1}{s} + \frac{1}{(s-1)} + \frac{1}{s}\,,\\ &= \frac{1}{(s-1)}\,. \end{aligned}$$

The inverse Laplace transform may now be deduced from the table. The inverse transform (unsurprisingly) is

$$y(t) = e^t\,.$$

Worked example: Solving $y''(t) + 4y(t) = 0$, $y(0) = 0$, $y'(0) = 1$ using Laplace transforms. We apply the Laplace transform to both sides of the equation to obtain

$$\mathcal{L}\{y''(t) + 4y(t)\} = \mathcal{L}\{0\}\,.$$

Using the derivative result, and noting the Laplace transform of zero is zero gives

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = 0\,.$$

Substituting the values provided in the initial conditions and rearranging for $Y(s)$ gives

$$Y(s) = \frac{1}{s^2+4}\,.$$

This is similar to one of the forms in Table 1. If we write

$$Y(s) = \frac{2}{2(s^2+4)} = \frac{1}{2}\frac{2}{s^2+4}\,.$$

the inverse Laplace transform may now be deduced from the table without further work. The solution is

$$y(t) = \frac{1}{2}\sin(2t)\,.$$

7. Example tutorial questions

1. Let $f(t)$ be a function which has Laplace transform

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt\,.$$

(i) Differentiate both sides with respect to s.
(ii) Swap the order of differentiation and integration using the Leibniz rule.
(iii) Obtain the result:

$$\frac{dF(s)}{ds} = \int_0^\infty -tf(t)e^{-st}\,dt\,.$$

(iv) Hence conclude that $\mathcal{L}\{tf(t)\} = -\frac{dF(s)}{ds}$.

2. Consider the inhomogeneous ordinary differential equation:

$$y''(t) - 3y'(t) + 2y(t) = e^t\,,$$

with $y(0) = 0$, $y'(0) = 0$.

(i) Explain why this equation is challenging using the method of undetermined coefficients.

(ii) Apply Laplace transforms to both sides of the equation and simplify the resulting expression for $Y(s)$ using partial fractions. You should get

$$Y(s) = \frac{1}{(s-2)(s-1)^2} = -\frac{1}{s-1} - \frac{1}{(s-1)^2} + \frac{1}{s-2}\,.$$

(iii) Compute

$$\frac{d}{ds}\left(\frac{1}{s-1}\right)$$

and use the result from Q1 to solve the original problem. Compare your solution to the solution you find using the method of undetermined coefficients.

3. Prove the relation $\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)\ a > 0$.

(i) Apply the definition of the Laplace transform to $u(t-a)f(t-a)$.

(ii) Notice that the Heaviside function can be applied by making the bottom limit of the integral equal to a.

(iii) By multiplying the integrand by $e^{as}e^{-as}$ (equivalent to multiplying by one) the integral can be rewritten using the substitution $\tau = t - a$.

(iv) Conclude that, since τ is a dummy variable of integration, $\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$ $a > 0$.