Discretization Methods HW1 - Metehan Dündar

Exercise 1

We will use Taylor Series expansions for each term $v_{j+m} = v(x_j + m\Delta x)$ around the point x_j. Let $v_j^{(n)}$ denote the n-th derivative of v evaluated at x_j

$$v_{j+1} = v_j + v_j'\Delta x + \frac{v_j''}{2!}(\Delta x)^2 + \frac{v_j'''}{3!}(\Delta x)^3 + \frac{v_j^{(4)}}{4!}(\Delta x)^4 + \frac{v_j^{(5)}}{5!}(\Delta x)^5 + \frac{v_j^{6}}{6!}(\Delta x)^6 + \frac{v_j^{7}}{7!}(\Delta x)^7 + O(\Delta x^7)$$

$$v_{j-1} = v_j - v_j'\Delta x + \frac{v_j''}{2!}(\Delta x)^2 - \frac{v_j'''}{3!}(\Delta x)^3 + \frac{v_j^{(4)}}{4!}(\Delta x)^4 - \frac{v_j^{(5)}}{5!}(\Delta x)^5 + \frac{v_j^{(6)}}{6!}(\Delta x)^6 - \frac{v_j^{(7)}}{7!}(\Delta x)^7 + O(\Delta x)^7$$

$$v_{j+2} = v_j + v_j'(2\Delta x) + \frac{v_j''}{2!}(2\Delta x)^2 + \frac{v_j'''}{3!}(2\Delta x)^3 + \frac{v_j^{(4)}}{4!}(2\Delta x)^4 + \frac{v_j^{(5)}}{5!}(2\Delta x)^5 + \frac{v_j^{(6)}}{6!}(2\Delta x)^6 + \frac{v_j^{(7)}}{7!}(2\Delta x)^7 + O(\Delta x)^8$$

$$v_{j-2} = v_j - v_j'(2\Delta x) + \frac{v_j''}{2!}(2\Delta x)^2 - \frac{v_j'''}{3!}(2\Delta x)^3 + \frac{v_j^{(4)}}{4!}(2\Delta x)^4 - \frac{v_j^{(5)}}{5!}(2\Delta x)^5 + \frac{v_j^{(6)}}{6!}(2\Delta x)^6 - \frac{v_j^{(7)}}{7!}(2\Delta x)^7 + O(\Delta x)^8$$

$$v_{j+3} = v_j + v_j'(3\Delta x) + \frac{v_j''}{2!}(3\Delta x)^2 + \frac{v_j'''}{3!}(3\Delta x)^3 + \frac{v_j^{(4)}}{4!}(3\Delta x)^4 + \frac{v_j^{(5)}}{5!}(3\Delta x)^5 + \frac{v_j^{(6)}}{6!}(3\Delta x)^6 + \frac{v_j^{(7)}}{7!}(3\Delta x)^7 + O(\Delta x^8)$$

$$v_{j-3} = v_j - v_j'(3\Delta x) + \frac{v_j''}{2!}(3\Delta x)^2 - \frac{v_j'''}{3!}(3\Delta x)^3 + \frac{v_j^{(4)}}{4!}(3\Delta x)^4 - \frac{v_j^{(5)}}{5!}(3\Delta x)^5 + \frac{v_j^{6}}{6!}(3\Delta x)^6 - \frac{v_j^{(7)}}{7!}(3\Delta x)^7 + O(\Delta x^8)$$

We will substitute above equations into the numerator of the formula;

$$N = -v_{j-3} + 9v_{j-2} - 45v_{j-1} + 45v_{j+1} - 9v_{j+2} + v_{j+3}$$

Then, collect the coefficient for the each derivative term

$v_j = -1 + 9 - 45 + 45 - 9 + 1 = 0$

$v_j'\Delta x = -(-3) + 9(-2) - 45(-1) + 45(+1) - 9(2) + 1(3) = 60$

$\frac{v_j''}{2!}(\Delta x)^2 = -(3^2) + 9(2^2) - 45(1^2) + 45(1^2) - 9(2^2) + 1(3^2) = 0$

$\frac{v_j'''}{3!}(\Delta x)^3 = -(-3^3) + 9(-2^3) - 45(-1^3) + 45(1^3) - 9(2^3) + 1(3^3) = 0$

$\frac{v_j^4}{4!}(\Delta x)^4 = -(3^4) + 9(2^4) - 45(1^4) + 45(1^4) - 9(2^4) + 1(3^4) = 0$

$\frac{v_j^5}{5!}(\Delta x)^5 = -(-3^5) + 9(-2^5) - 45(-1^5) + 45(1^5) - 9(2^5) + 1(3^5) = 0$

$\frac{v_j^6}{6!}(\Delta x)^6 = -(3^6) + 9(2^6) - 45(1^6) + 45(1^6) - 9(2^6) + 1(3^6) = 0$

$\frac{v_j^7}{7!}(\Delta x)^7 = -(-3^7) + 9(-2^7) - 45(-1^7) + 45(1^7) - 9(2^7) + 1(3^7) = 2160$

So the numerator becomes

$N = 60 u_j' \Delta x + 2160 \dfrac{u_j^{(7)}}{7!} (\Delta x)^7 + O(\Delta x^9) \rightarrow$

$N = 60 u_j' \Delta x + \dfrac{2160}{5040} u_j^{(7)} (\Delta x)^7 + O(\Delta x^9) \rightarrow$

$N = 60 u_j' \Delta x + \dfrac{3}{7} u_j^{(7)} (\Delta x)^7 + O(\Delta x^9) \rightarrow$ divide by $60\Delta x$

$\dfrac{N}{60\Delta x} = u_j' + \dfrac{3}{7 \cdot 60} u_j^{(7)} (\Delta x)^6 + O(\Delta x^8) \rightarrow$

$\dfrac{N}{60\Delta x} = u_j' + \dfrac{1}{140} u_j^{(7)} (\Delta x)^6 + O(\Delta x)^7$

This formula approximates u_j' with a leading error term proportional to $(\Delta x)^6$. Therefore the formula is 6th order accurate.

Exercise 2

1) We need to show the numerical wave speed $c_j(k)$ for the 6th order approximation. Let's substitute the plane wave solution $u_j(t) = \hat{u}(t) e^{ikx_j} = \hat{u}(t) e^{ikj\Delta x}$ into the semi-discretized equation:

$$\frac{du_j}{dt} = -c \left[\frac{-u_{j-3} + 9u_{j-2} - 45u_{j-1} + 45u_{j+1} - 9u_{j+2} + u_{j+3}}{60\Delta x} \right]$$

The left hand side is $\dfrac{d\hat{u}}{dt} e^{ikj\Delta x}$

The right side's spatial derivative term becomes:

$$\frac{-e^{ik(j-3)\Delta x} + 9e^{ik(j-2)\Delta x} - 45e^{ik(j-1)\Delta x} + 45e^{ik(j+1)\Delta x} - 9e^{ik(j+2)\Delta x} + e^{ik(j+3)\Delta x}}{60\Delta x} \hat{u}(t)$$

When we factor out $e^{ikj\Delta x}$

$$\frac{e^{ikj\Delta x} \hat{u}(t)}{60\Delta x} \left[-e^{-i3k\Delta x} + 9e^{-i2k\Delta x} - 45e^{-ik\Delta x} + 45e^{ik\Delta x} - 9e^{i2k\Delta x} + e^{i3k\Delta x} \right]$$

To group terms, we need to use $e^{im\theta} - e^{-im\theta} = 2i \sin(m\theta)$

$$\frac{e^{ikj\Delta x} \hat{u}(t)}{60\Delta x} \left[45(e^{ik\Delta x} - e^{-ik\Delta x}) - 9(e^{i2k\Delta x} - e^{-i2k\Delta x}) + (e^{i3k\Delta x} - e^{-i3k\Delta x}) \right]$$

$$\frac{e^{ikj\Delta x} \hat{u}(t)}{60\Delta x} \left[45(2i\sin(k\Delta x)) - 9(2i\sin(2k\Delta x)) + (2i\sin(3k\Delta x)) \right]$$

$$\frac{i e^{ikj\Delta x} \hat{u}(t)}{30\Delta x} \left[45\sin(k\Delta x) - 9\sin(2k\Delta x) + \sin(3k\Delta x) \right],$$

Now the equation becomes

$$\frac{d\hat{U}}{dt}e^{ikj\Delta x} = -c\left(i\frac{e^{ikj\Delta x}\hat{U}(t)}{30\Delta x}\left[45\sin(k\Delta x)-9\sin(2k\Delta x)+\sin(3k\Delta x)\right]\right)$$

$$\frac{d\hat{U}}{dt} = -i\left(c\,\frac{45\sin(k\Delta x)-9\sin(2k\Delta x)+\sin(3k\Delta x)}{30\Delta x}\right)\hat{U}(t)$$

This is the form of $\frac{d\hat{U}}{dt} = -ik_{eff}\,c\,\hat{U}(t)$ or $\frac{d\hat{U}}{dt} = -ik\,c_3(k)\hat{U}(t)$

where $c_3(k)$ is the numerical phase speed.

$$k\,c_3(k) = c\,\frac{\left(45\sin(k\Delta x)-9\sin(2k\Delta x)+\sin(3k\Delta x)\right)}{30\Delta x}$$

$$c_3(k) = c\,\frac{45\sin(k\Delta x)-9\sin(2k\Delta x)+\sin(3k\Delta x)}{30k\Delta x}$$ This matched the formula.

2) We need to find the leading phase error. The relative phase speed is $\frac{c_3(k)}{c}$. The phase error is related to how much this deviates. Let $p = k\Delta x$

$$\frac{c_3(k)}{c} = \frac{45\sin(p)-9\sin(2p)+\sin(3p)}{30p}$$

If we use Taylor series for $\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + O(x^9)$

$$\sin(p) = p - \frac{p^3}{6} + \frac{p^5}{120} - \frac{p^7}{5040} + \dots$$

$$\sin(2p) = 2p - \frac{(2p)^3}{6} + \frac{(2p)^5}{120} - \frac{(2p)^7}{5040} + \dots = 2p - \frac{8p^3}{6} + \frac{32p^5}{120} - \frac{128p^7}{5040} + \dots$$

$$\sin(3p) = 3p - \frac{(3p)^3}{6} + \frac{(3p)^5}{120} - \frac{(3p)^7}{5040} + \dots = 3p - \frac{27p^3}{6} + \frac{243p^5}{120} - \frac{2187p^7}{5040} + \dots$$

Numerator $N_p = 45\sin(p) - 9\sin(2p) + \sin(3p)$

Coeff of p: $45(1) - 9(2) + 1(3) = 45 - 18 + 3 = 30$

Coeff of p^3: $45\left(-\frac{1}{6}\right) - 9\left(-\frac{8}{6}\right) + 1\left(-\frac{27}{6}\right) = \frac{(-45+72-27)}{6} = 0$

Coeff of p^5: $45\left(\frac{1}{120}\right) - 9\left(\frac{32}{120}\right) + 1\left(\frac{243}{120}\right) = \frac{(45-288+243)}{120} = 0$

Coeff of p^7: $45\left(-\frac{1}{5040}\right) - 9\left(\frac{-128}{5040}\right) + 1\left(\frac{-2187}{5040}\right) = \frac{(-45+1152-2187)}{5040} = \frac{-1080}{5040} = -\frac{3}{14}$

Then $N_p = 30p - \frac{3}{14}p^7 + O(p^9)$

So $\dfrac{c_3(k)}{c} = \dfrac{30p - \frac{3}{14}p^7 + O(p^9)}{30p} = 1 - \dfrac{3}{14 \cdot 30} p^6 + O(p^8) = 1 - \dfrac{1}{140} p^6 + O(p^8)$

$\dfrac{c_3(k)}{c} = 1 - \dfrac{1}{140}(k\Delta x)^6 + O((k\Delta x)^8) \Rightarrow \dfrac{c - c_3(k)}{c} \approx \dfrac{(k\Delta x)^6}{140}$

The leading order error term is $-\frac{1}{140}(k\Delta x)^6$. This matches the error term found from the Taylor expansion analysis in previous question as expected

$e_3(p, v) = \dfrac{\pi \cdot v}{70}\left(\dfrac{2\pi}{p}\right)^6$ as required

3) We need to show that $p_3(\varepsilon_p, v) \geq 2\pi \sqrt[6]{\dfrac{\pi v}{70 \varepsilon_p}}$ and compute the given points. $\varepsilon_3(p, v) = \dfrac{\pi v}{70}\left(\dfrac{2\pi}{6}\right)^6$, Let ε_p be the maximum tolerable phase error. We need to show $\varepsilon_3(p, v) \leq \varepsilon_p$

$$\frac{\pi v}{70}\left(\frac{2\pi}{p}\right)^6 \leq \varepsilon_p \Rightarrow \left(\frac{2\pi}{p}\right)^6 \leq \frac{70\,\varepsilon_p}{\pi \cdot v} \Rightarrow \frac{2\pi}{p} \leq \sqrt[6]{\frac{70\,\varepsilon_p}{\pi \cdot v}} \Rightarrow$$

$$\frac{p}{2\pi} \geq \sqrt[6]{\frac{\pi v}{70\,\varepsilon_p}} \Rightarrow p \geq 2\pi \sqrt[6]{\frac{\pi \cdot v}{70\,\varepsilon_p}}$$

Let's assume $v = 1$ which is common choice for simplicity near the stability limit

$\varepsilon_p = 0{,}1$; $p_3 \geq 2\pi \left(\dfrac{\pi \cdot 1}{70 \cdot 0{,}1}\right)^{\frac{1}{6}} \approx 5{,}42 \Rightarrow p_3 = 6$

$\varepsilon_p = 0{,}01$: $p_3 \geq 2\pi \left(\dfrac{\pi \cdot 1}{70 \cdot 0{,}01}\right)^{\frac{1}{6}} \approx 8{,}07 \Rightarrow p_3 = 9$

α 2nd order central difference

$c_1(k) = \dfrac{c \sin(k\Delta x)}{k\Delta x} \Rightarrow \dfrac{c_1(k)}{c} = 1 - \dfrac{(k\Delta x)^2}{6} + \dots \Rightarrow e_2(p, v) = \pi v \dfrac{(2\pi/p)^2}{6} = \dfrac{2\pi^3 v}{3p^2}$

α 4th order central difference

$\dfrac{c_2(k)}{c} = 1 - \dfrac{(k\Delta x)^4}{30} + \dots \Rightarrow e_4(p, v) = \dfrac{\pi v (2\pi/p)^4}{30} = \dfrac{4\pi^5 v}{15 p^4}$

2nd order $p \geq \sqrt{\dfrac{2\pi^3 v}{3\varepsilon_p}}$ 4th order $p \geq \left(\dfrac{4\pi^5 v}{15\varepsilon_p}\right)^{\frac{1}{4}}$

We will use $v = 1$ again

$\varepsilon_p = 0{,}1$	2nd order	8,08	$\varepsilon_p = 0{,}01$	2nd order	25,55
	4th order	6,28		4th order	11,18
	6th order	5,42		6th order	8,07

The 6th order method requires fewer points per wavelength to achieve the same accuracy. This becomes more significant as higher accuracy is required. Using higher-order schemes is more efficient in problems where; accuracy of wave propagation is critical, long time simulations amplify phase errors, high-resolution results are needed without increasing grid size drastically.

Exercise 3
Please check the HW2 Exercise1