Discretization Methods HW 3 - Meltem Dindar

- Exercise 1

We have $\frac{\partial v}{\partial t} + \sin(x)\frac{\partial v}{\partial x} = 0$, $x \in [0, 2\pi]$ with periodic boundary conditions. We'll use the Fourier-Galerkin method to derive an approximation.

Let's approximate $v(x,t)$ as a truncated Fourier series $v_N(x,t) = \sum_{n=-N}^{N} \hat{v}_n(t) e^{inx}$

We'll take the test functions also from the same basis: $\phi_m(x) = e^{imx}$ $m \in [-N, N]$

Then, we'll substitute into the PDE:

$$\frac{\partial v_N}{\partial t} + \sin(x)\frac{\partial v_N}{\partial x} = 0$$

We project onto the test function ϕ_m, by enforcing orthogonality

$$\left\langle \frac{\partial v_N}{\partial t} + \sin(x)\frac{\partial v_N}{\partial x}, e^{imx} \right\rangle = 0$$

Then, let's compute each term:

1- Time derivative term: $\left\langle \frac{\partial v_N}{\partial t}, e^{imx} \right\rangle = \frac{d\hat{v}_m}{dt}(2\pi)$

2- Advection term: $\frac{\partial v_N}{\partial x} = \sum_{n=-N}^{N} in\hat{v}_n(t)e^{inx}$

$$\boxed{\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}}$$

$$\sin(x)\frac{\partial v_N}{\partial x} = \sum_{n=-N}^{N} in\hat{v}_n(t).\sin(x)e^{inx} = \sum_{n=-N}^{N} in\hat{v}_n(t)\frac{1}{2i}(e^{ix} - e^{-ix})e^{inx} =$$

$$= \sum_{n=-N}^{N} \frac{n\hat{v}_n(t)}{2}\left(e^{i(n+1)x} - e^{i(n-1)x}\right)$$

Now, we will compute the projection

$$\left\langle \sin(x)\frac{\partial v_N}{\partial x}, e^{imx} \right\rangle = \sum_{n=-N}^{N} \frac{n\hat{v}_n(t)}{2}\left(\delta_{m,n+1} - \delta_{m,n-1}\right)(2\pi) =$$

$$= \pi\left[(m-1)\hat{v}_{m-1}(t) - (m+1)\hat{v}_{m+1}(t)\right]$$

Therefore, the Galerkin system becomes

$$\frac{d\hat{v}_m}{dt} + \frac{1}{2}\left[(m-1)\hat{v}_{m-1} - (m+1)\hat{v}_{m+1}\right] = 0$$

$\boxed{P_N v \neq v_N}$ v_N evolves in time by solving a projected PDE, while $P_N v$ is the projection of the true solution, which solves the full PDE.

Exercise 2

We have the same PDE, but with Dirichlet boundary conditions $u(0,t) = u(\pi, t) = 0$. We now use sine basis functions due to Dirichlet boundary conditions.

$u_N(x,t) = \sum_{n=1}^{N} \hat{v}_n(t) \sin(nx)$, Then we apply Galerkin projection

$\left\langle \frac{\partial u_N}{\partial t} + \sin(x) \frac{\partial u_N}{\partial x}, \sin(mx) \right\rangle = 0$, Then we compute derivatives

$$\frac{\partial u_N}{\partial x} = \sum_{n=1}^{N} n\hat{v}_n(t) \cos(nx)$$

$$\sin(x) \frac{\partial u_N}{\partial x} = \sum_{n=1}^{N} n\hat{v}_n(t) \sin(x) \cos(nx)$$

We will use the below formula

$$\boxed{\sin(x)\cos(nx) = \frac{1}{2}\left(\sin((n+1)x) + \sin((n-1)x)\right)}$$

Then project onto $\sin(mx)$

$$\left\langle \sin(x) \frac{\partial u_N}{\partial x}, \sin(mx) \right\rangle = \sum_{n=1}^{N} \frac{n\hat{v}_n(t)}{2} \left(\delta_{m,n+1} + \delta_{m,n-1}\right) \frac{\pi}{2}$$

Then putting together

$$\frac{d\hat{v}_m}{dt} + \frac{1}{2}\left((m+1)\hat{v}_{m+1} + (m-1)\hat{v}_{m-1}\right) = 0$$

Exercise 3

In this problem cosine series does not satisfy homogeneous Dirichlet boundary conditions. So, tau method relaxes the boundary conditions and enforces them by adding constraints. Let,

$\tilde{v}_N(x,t) = \sum_{n=0}^{N} \hat{v}_n(t) \cos(nx)$ and define test functions as the first N cosine modes, which excludes highest modes. Then, we enforce PDE residual orthogonality:

$$\left\langle \frac{\partial \tilde{v}_N}{\partial t} + \sin(x) \frac{\partial \tilde{v}_N}{\partial x}, \cos(mx) \right\rangle = 0 \quad \text{for } m = 0, \dots, N-2$$

The remaining equations in order to match number of unknowns, enforce the boundary conditions. $\tilde{v}_N(0,t) = 0$, $\tilde{v}_N(\pi, t) = 0$

This leads to a differential algebraic equation system of size $N+2$ ($N_b = 2$)

Exercise 4

Let $x_j = \frac{2\pi j}{2N+1}$, $j = 0, 1, \dots, 2N$

Then represent,

$$u(x,t) \approx u_N(x,t) = \sum_{n=-N}^{N} \hat{u}_n(t) e^{inx}$$

Then, $\frac{\partial u_N}{\partial t}(x_j,t) + \frac{1}{2}\frac{\partial(u_N^2)}{\partial x}(x_j,t) = \epsilon \frac{\partial^2 u_N}{\partial x^2}(x_j,t)$ for $j = 0, \dots, M-1$

Let $u_j(t) = u_N(x_j,t)$, The equations are

$$\frac{du_j}{dt} + \frac{1}{2}\left.\frac{\partial(u_N^2)}{\partial x}\right|_{x=x_j} = \epsilon \left.\frac{\partial^2 u_N}{\partial x^2}\right|_{x=x_j}$$

$\Rightarrow$ u_j = Inverse fast fourier Transform $(\{\hat{u}_k\})$

Coefficients of $\frac{\partial u_N}{\partial x}$ are $\{ik\hat{u}_k\}$

Coefficients of $\frac{\partial^2 u_N}{\partial x^2}$ are $\{-k^2\hat{u}_k\}$

Therefore in physical space, the equation becomes:

$$\left(\frac{du}{dt}\right)_j + \frac{1}{2}\left(\frac{d}{dx}u^2\right)_j = \epsilon\left(\frac{d^2u}{dx^2}\right)_j$$

We can use FFT or IFFT to move between physical and spectral space.