Discretization Methods HW2 – Metehan Dündar

Exercise 1

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--- Comparison ---
Compares total grid points used (N_even vs N_odd+1)
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k  | Min Points (Even Method, N_even) | Min Points (Odd Method, N_odd+1) | More Accurate?
---|----------------------------------|----------------------------------|---------------
2  | 20                               | 23                               | Even
4  | 32                               | 33                               | Even
6  | 40                               | 43                               | Even
8  | 54                               | 53                               | Odd
10 | 62                               | 61                               | Odd
12 | 70                               | N/A                              | N/A
----------------------------------------------------------------------
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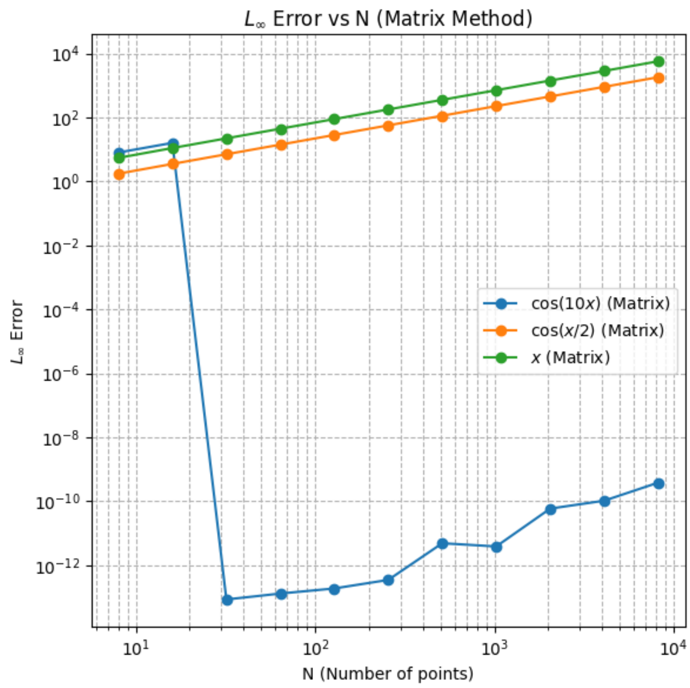
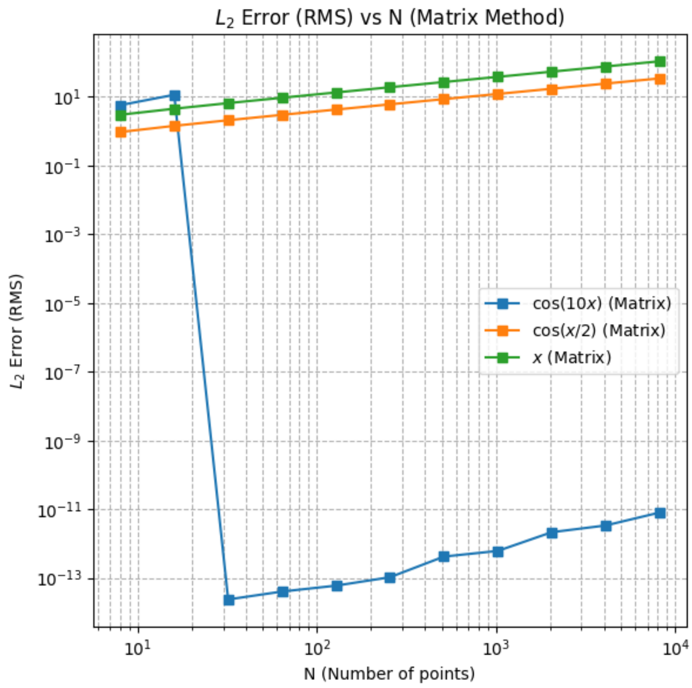
The function u(x)=exp(ksinx) becomes highly oscillatory for large k. For k=12, it varies extremely rapidly, requiring a very fine grid (large N) to differentiate accurately. It's plausible that the Odd method, for this specific function and high k, requires more than 201 points to achieve the **target accuracy**. Numerical differentiation methods can lose accuracy or require significantly more points when dealing with very high frequencies or wavenumbers relative to the grid spacing. It might also be that the specific formulation of the Odd method matrix (using 1/sin) is slightly less numerically robust than the Even method (using cot) in this high-k regime, requiring more points to suppress errors.

Given that the **Even method performs better** at the challenging high end (k=12) and for lower k values, it appears to be the **more generally accurate and robust method** of the two for this problem, despite the Odd method needing slightly fewer points for k=8 and k=10 in this run.

Exercise 2





a) $f(x) = \cos(10x)$

→ for small N = 8, 16 the errors are large. This is because these grid sizes are insufficient to resolve the oscillations of $\cos(10x)$ because nyquist criterion suggests $N > 2 \cdot 10 = 20$. Aliasing effects dominate. When n increasing to 32 which is > 20, the error drops dramatically to machine precision. for the next increases in N from 64 to 65536, the error remains very small, essentially at the limit of floating point accuracy. The slight increase in error for very large N is likely due to the accumulation of floating point round-off errors in the FFT calculations.

→ Convergence rates demonstrates spectral convergence. The error decreases extremely rapidly once N is sufficient to resolve the function's highest frequency.

b) $f(x) = \cos(x/2)$

→ The errors L_{inf} and L_2 increase consistently and significantly as N increases. The method is diverging for this function. The error does not decrease; it gets worse as the resolution N increases

c) $f(x) = x$

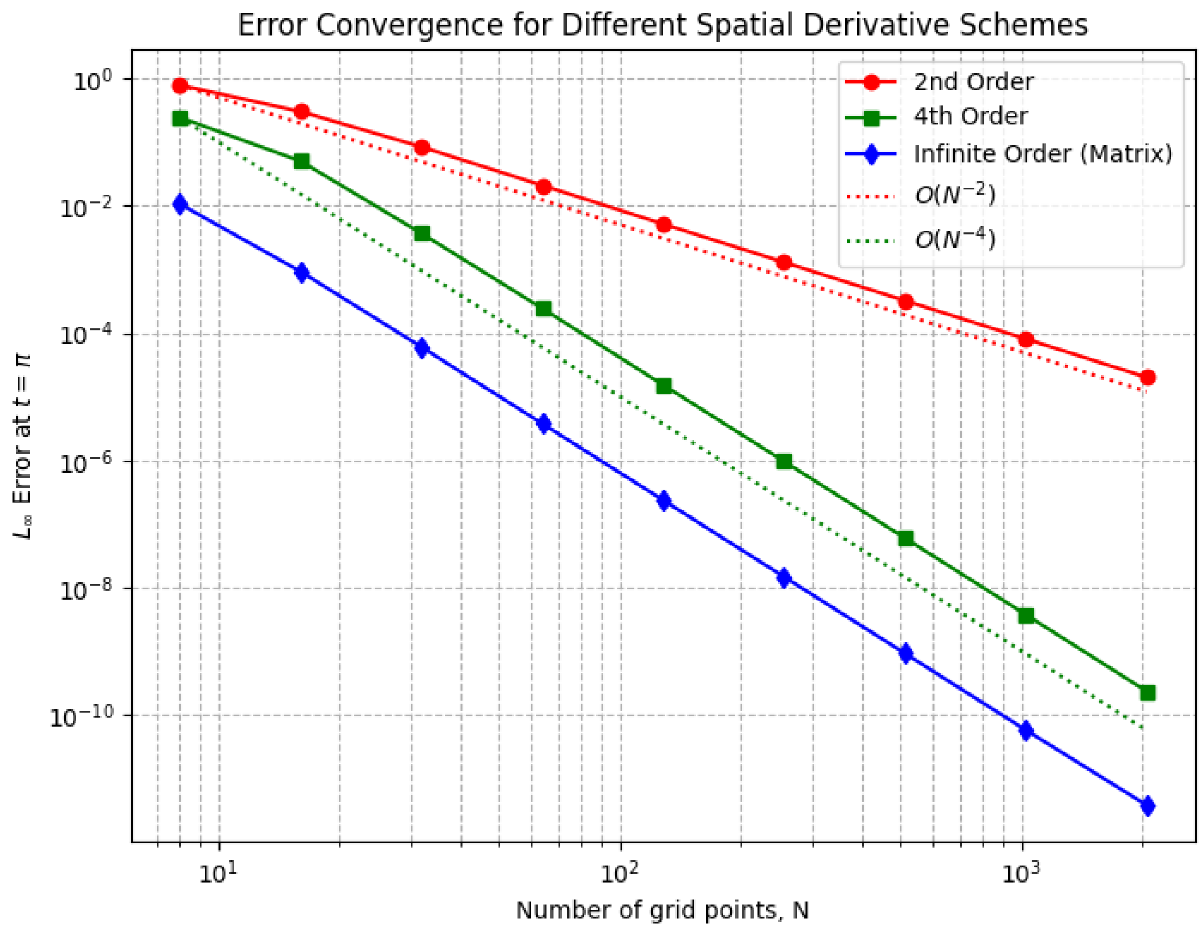
→ Similar to $\cos(x/2)$, the errors L_{inf} and L_2 increase consistently and significantly as N increases. The method is also diverging for this function.

The stark difference in behavior happens because of the periodicity of the functions relative to the interval $[0, 2\pi]$

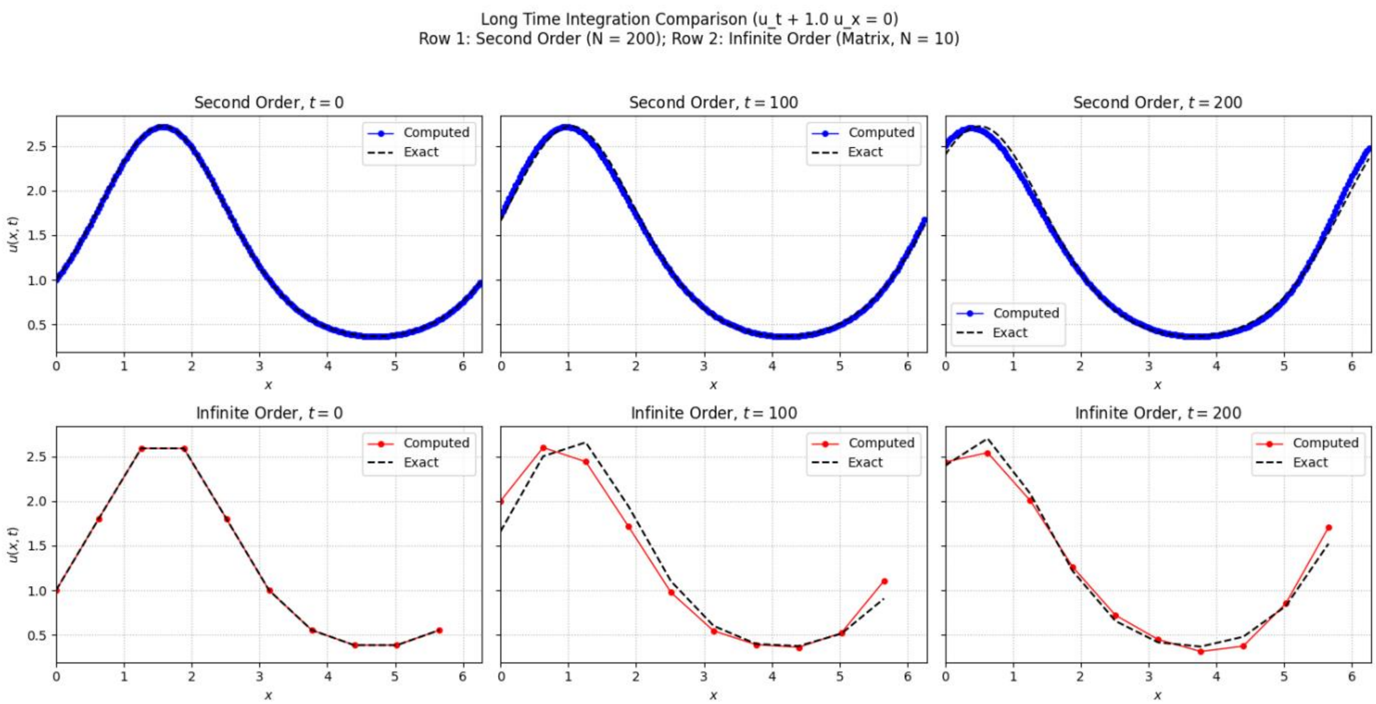
→ $\cos(10x)$ is smooth which means infinitely differentiable, and periodic on $[0, 2\pi]$. fourier spectral methods are designed for such functions. Because the functions and all its derivatives are periodic and smooth, its fourier coefficients decay very rapidly. Once N is large enough to capture the essential frequencies the DFT provides an excellent approximation, and differentiation in the frequency domain is highly accurate leading to spectral convergence.

→ $\cos(x/2)$ and x are smooth within the interval $[0, 2\pi]$ but they are not periodic on $[0, 2\pi]$. The fourier method implicitly assumes the function is periodic. These two functions, the periodic extension has a jump discontinuity at the boundaries. That demonstrates that naively applying fourier differentiation to non-periodic functions leads to inaccurate results and divergence.

Exercise 3



Error Convergence for Different Spatial Derivative Schemes
2nd Order
4th Order
Infinite Order (Matrix)
$O(N^{-2})$
$O(N^{-4})$
L_∞ Error at $t = \pi$
Number of grid points, N



Long Time Integration Comparison (u_t + 1.0 u_x = 0)
Row 1: Second Order (N = 200); Row 2: Infinite Order (Matrix, N = 10)
Second Order, $t = 0$
Second Order, $t = 100$
Second Order, $t = 200$
Infinite Order, $t = 0$
Infinite Order, $t = 100$
Infinite Order, $t = 200$
Computed
Exact
$u(x, t)$
x

a) for the observed L_∞ error at $t = \pi$

→ 2nd order: errors decrease approximately at a rate of 2 once the grid is sufficiently fine. The observed convergence rates quickly settle near 2.00 which matches the theoretical expectation for a second-order scheme.

→ 4th order: after a transitional regime at coarse grids, errors begin to decrease at a rate of about 4.09 as expected. For $N = 2048$, the error is much smaller than the second-order method at the same N.

→ Infinite order: for smooth solutions spectral methods exhibit exponential convergence. Numerically, the data appear to converge at around 4th order or better until floating point round off effects dominate. At $N = 2048$, error is extremely small.

a) expectation

→ 2nd order: the scheme settles into a rate of 2.00 for sufficiently fine N, which agrees with the theory.

→ 4th order: the scheme settles into a rate of 4.00. Again, this aligns with theory.

→ Infinite order: converges faster than any fixed polynomial order, effectively limited only by round-off for smooth PDE,

a) Matching the 2nd order error at 2048

→ The target error from 2nd order scheme at N = 2048 is 2.0×10^{-5}. In order to achieve the same error

using 4th order scheme N = 128 error: 1.53×10^{-5}

using infinite order scheme N = 64 error: $3.75 \cdot 10^{-6}$

→ higher-order schemes can reach the same accuracy with a far fewer grid points

b) Long time integration comparison

1) The second order scheme with 200 grid points shows an excellent match with the exact solution at all the observed times namely $t=0$, $t=100$, and $t=200$

This indicates that even over long time integrations, the second order spatial discretization with this resolution maintains sufficient accuracy for this advection problem. The phase and amplitude errors are minimal, and cumulative error remains under control.

2) The infinite order scheme is using a very coarse grid. At $t=0$ the computed solution exactly matches the exact solution as expected, because the initial condition is smooth and can be well represented with only 10 grid points in the spectral framework.

At $t=100$ and 200, although the computed solution still remains close to the exact solution, there are noticeable slight discrepancies, this may appear as subtle differences in phase or amplitude.

While spectral methods are known for their excellent accuracy, the use of an extremely coarse grid which is $N=10$, leaves very little room for representing all the evolving scales over long times. Even though the error is very low at early times, small aliasing or time integration errors can accumulate over long iterations. Therefore, for $t=100$ and $t=200$ some mismatches appear, highlighting that in long time integrations even spectral methods may need a slightly finer grid than the minimal one required for short-time accuracy.