Pop. dynamics worksheet

1 Introduction

The aim of this demonstration is to show how we can use ideas from calculus to study dynamical systems.

- It is *not* intended that you work through all the questions in the available time.
- You are encouraged to use your phone to explore the linked apps

2 Recap

You might have previously encountered differentiation. Suppose that y is some function of x.

Consider the differential equation

$$\frac{dy}{dx} = 1.$$

Upon integration

$$y(x) = x + C$$

where C is an integration constant.

Now suppose that

$$\frac{dy}{dx} = x.$$

Question

Can you integrate this ordinary differential equation and identify the solution $y = y(x)$?

3 Modelling population dynamics

3.1 Formulating a model of population dynamics

Let's consider a model for the number of people in a room at a given time. Let t represent time and $N(t)$ represent the number of people in the room at time t.

Suppose that there are initially no people in the room, but people enter at a constant rate, k.

We could formulate a model of population dynamics given by

$$\frac{dN}{dt} = k, \quad N(0) = 0. \tag{1}$$

Question

- Can you integrate Equation 1 (Hint: it is mathematically equivalent to the ODE introduced in Section 2)?

- Can you use the solution of the model to determine the amount of time taken for the number of people in the room to reach some capacity, N_C.

- Can you use the app (see Figure 1) to identify what the entry rate, k, needs to be such that the room reaches capacity of 40 people after 20 minutes?

Figure 1: https://dundeemath.github.io/Admissions/posts/PopulationDynamicsIntro.html.

3.2 What if people enter the room at a constant rate but also leave the room at random?

Taking the previous model as a starting point, we now assume that people can also leave the room at a rate proportional to the number of people in the room

The model equation is now given by

$$\frac{dN}{dt} = k - dN, \quad N(0) = 0. \tag{2}$$

Question

It is possible to integrate Equation 2 and show that the solution is

$$N(t) = \frac{k}{d}(1 - e^{-dt}) \tag{3}$$

Can you do this? (hint: try using an *integrating factor*)?

Question

Can you use the model solution (Equation 3) to determine the amount of time taken for the number of people in the room to reach capacity, N_C. Does a solution always exist?

Question

Can you use the app or the solution (Equation 3) to identify the entry rate needs to be such that the room reaches capacity of 40 people after 20 minutes given $d = 0.1$?

4 The SIR model

The SIR model is used to study the spread of infectious disease.

In the SIR model a population is split into three groups:

- suspectible (S)
- infectious (I)
- recovered (R)

Unlike in the previous example, the population dynamics of each group depend on the levels of the other populations.

The governing equations are:

$$\begin{aligned} \frac{dS}{dt} &= -rIS, \\ \frac{dI}{dt} &= rIS - aI, \\ \frac{dR}{dt} &= aI. \end{aligned} \tag{4}$$

with initial conditions

$$\begin{aligned} S(t=0) &= S_0, \\ I(t=0) &= I_0, \\ R(t=0) &= R_0. \end{aligned}$$

You can explore solution behaviour using this app in Figure 2.

Figure 2: https://dundeemath.github.io/Admissions/posts/TheSIRModel.html

> **Note**
>
> At Dundee, the mathematical tools needed are developed in modules:
>
> - Maths 1A, 1B, 2A and 2B (Core maths modules)

- Computer algebra and dynamical systems
- Mathematical Biology I
- Mathematical Biology II

At Levels 2, 3 and 4 you will learn how to use computer programming to explore and communicate mathematical concepts.
You can find out more about these modules here.