Pop. dynamics worksheet

Introduction

The aim of this demonstration is to show how we can use ideas from calculus to study dynamical systems.

- It is *not* intended that you work through all the questions in the available time.
- You are encouraged to use your phone to explore the linked apps

Recap

You might have previously encountered differentiation. Suppose that y is some function of x.

Consider the differential equation

$$\frac{dy}{dx} = 1$$

Upon integration

$$y(x) = x + C$$

where C is an integration.

Suppose that

$$\frac{dy}{dx} = x$$

Can you integrate this ordinary differential equation and identify the solution $y = y(x)$?

Modelling population dynamics

Formulating a model of population dynamics

Let's consider a model for the number of people in a room at a given time. Let t represent time and $N(t)$ represent the number of people in the room at time t.

Suppose that there are initially no people in the room, but people enter at a constant rate, k.

We could formulate a model of population dynamics given by

$$\frac{dN}{dt} = k, \quad N(0) = 0.$$

- Can you integrate this ODE (Hint is mathematically equivalent to the ODE introduced in Section)?

- Can you use the model to determine the amount of time taken for the number of people in the room to reach capacity, N_C.

- Can you use the app (Figure 2) to identify what the entry rate, k, needs to be such that the room reaches capacity of 40 people after 20 minutes?

Figure 1: https://dundeemath.github.io/Admissions/posts/PopulationDynamicsIntro.html.

What if people enter the room at a constant rate but also leave the room at random?

Taking the previous model as a starting point, we now assume that people leave the room at a rate proportional to the number of people in the room

We could formulate a model of population dynamics as before, but not accounting for people entering and leaving the room. The model equation is now given by

$$\frac{dN}{dt} = k - dN, \quad N(0) = 0.$$

It is possible to integrate this ODE (hint: try using an *integrating factor*)?

Given that the solution is

$$N(t) = \frac{k}{d}(1 - e^{-dt})$$

can you use the model to determine the amount of time taken for the number of people in the room to reach capacity, N_C. Does a solution always exist?

Can you identify what the entry rate needs to be such that the room reaches capacity of 40 people after 20 minutes given $d = 0.1$?

The SIR model

The SIR model is used to study infectious disease.

A population is split into three groups:

- suspectible (S)
- infectious (I)
- recovered (R)

Unlike in the previous example, the population dynamics of each group depend on the levels of the other populations.

The governing equations are:

$$\begin{aligned}
\frac{dS}{dt} &= -rIS, \\
\frac{dI}{dt} &= rIS - aI, \\
\frac{dR}{dt} &= aI.
\end{aligned} \tag{1}$$

with initial conditions

$$\begin{aligned}
S(t=0) &= S_0, \\
I(t=0) &= I_0, \\
R(t=0) &= R_0.
\end{aligned}$$

You can explore solution behaviour using this app

Figure 2: https://dundeemath.github.io/Admissions/posts/TheSIRModel.html

Note

At Dundee, the mathematical tools needed are developed in modules:

- Maths 1A, 1B, 2A and 2B (Core maths modules)
- Computer algebra and dynamical systems
- Mathematical Biology I
- Mathematical Biology II

At Levels 2, 3 and 4 you will learn how to use computer programming to explore and communicate mathematical concepts.
You can find out more about these modules here.