MA32011

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Preface

Welcome to the module MA32011 Dynamical systems.

My name is Philip Murray and I am the module lead.

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Lecture notes

You can find lecture notes for the module on this page. If you would like a pdf this can be easily generated by clicking on the pdf link of the webpage. I will occasionally edit/update the notes as we proceed through lectures. If you spot any errors, typos or omissions please let me know.

Reading

Nonlinear dynamics and chaos, Steven Strogatz Strogatz (2001) Mathematical Biology I, Murray (2002)

Python codes

I have provided Python codes for most of the figures in the notes (you can unfold code section by clicking 'Code'). Note that the Python code does not appear in the pdf.

Many of you have taken the Introduction to Programming module at Level 2 and have therefore some experience using Python. I strongly encourage you to use the provided codes as a tool to play around with numerical solutions of the various models that we will be working on. The codes should run as standalone Python codes.

> **Note**
>
> To access Python on Uni machines:
>
> 1. Launch Anaconda from AppsAnywhere
> 2. When a folder opens, double click on *Spyder*.
> 3. Paste a code from lecture notes into the editor on the left-hand side.
> 4. Click on the green arrow to run the code.
> 5. The plots should appear in the plots tab on the right-hand side.
> 6. Experiment with the code. When you change a model parameter, does the solution change in an expected way?

I have also provided some examples of how to use Python as a symbolic calculator. This uses a Python library called *sympy* and is quite similar to Maple.

Assessment

- Final exam (80 %)
- 2 class tests (8 % each), Week 7 and 11
- 4 quizes (1 % each), Week 2,4,6 and 9

Plan

Table 1: Projected delivery

Week	Up to Section	Tutorial sheet	Assessment
1		1	
2		1	Quiz 1
3		2	
4		2	Quiz 2
5		3	
6		3	Quiz 3
7		4	Test 1
8		4	
9		5	Quiz 4 4
10		5	

Week	Up to Section	Tutorial sheet	Assessment
11		Test 2	

References

1 Introduction

The goal of this module is to provide an introduction to dynamical systems. We will introduce key mathematical concepts and explore examples from physics and biology.

To begin with we introduce some key terminology.

1.1 Discrete v continuous time

1.1.1 Differential equations

Let t be a continuous variable and $x = x(t)$. Consider the ordinary differential equation (ODE)

$$\dot{x} = rx(1-x). \tag{1.1}$$

$\dot{x}$ is used to denote the time derivative, i.e.

$$\dot{x} := \frac{dx}{dt}.$$

Similarly, the second order derivative is represented by

$$\ddot{x} := \frac{d^2x}{dt^2}.$$

Many of the second order ODE problems that we will examine originate from Newton's Second Law, i.e.

$$m\ddot{x} = F(x)$$

where $x(t)$ represents the position of a particle at time, t, m represents a constant particle mass and F a resultant force.

Consider the case in which F represents a linear restoring force, i.e.

$$F(x) = -kx.$$

The equation of motion can be written as

$$\ddot{x} = -\mu x, \tag{1.2}$$

where $\mu = -k/m$.

Example 1.1. The app in Figure **??** encodes a numerical solution of the second order ODE

$$a\ddot{x} + b\dot{x} + cx = 0.$$

Choose an appropriate value of the parameters a, b and c so that the solution captures the case of a particle of mass (m) equal to 3 subjected to a linear restoring force with spring constant (k) equal to 5.

Is the behaviour of the numerical solution consistent with Equation **??**.

1.1.2 Difference equations

Suppose that n is a discrete variable. Let y_n represent a dependent variable at iteration n. Consider the difference equation

$$y_{n+1} = ry_n(1 - y_n),$$

where $r \in \Re$.

See this link for exploration of model solutions.

1.1.3 Key questions to ask of a dynamical system

- do solutions exist? If so are they unique?
- Is there an explicit solution?
- Can we qualitatively describe solution behaviour?
- How do the solutions depend on the model parameters?
- Are their critical values of parameters where solution behaviour changes?

1.2 Autonomous v nonautonomous ODEs

For an autonomous system, the update does not explicitly depend on the independent variable. Equation **??** is autonomous. But

$$\dot{x} = rx(1 - x + t)$$

is nonautonomous (because of the explicit time dependence on the right-hand side).

1.3 Linear v Nonlinear

Linear systems satisfy a linear supposition principle: a sum of solutions is itself a solution. In general, this property does not hold for nonlinear systems.

In linear dynamical systems, the dynamics are a function of linear sums of the dependent variables. Hence

$$\dot{x} = -x$$

is a linear ordinary differential equation (ODE). But

$$\dot{x} = -x^2$$

is nonlinear.

1.4 Quantitative v qualitative solutions

You are likely used to solving problems in which an explicit solution can be found. For example, consider the ODE

$$\dot{x} = -kx, \quad x(0) = x_0$$

where $k, x_0 \in \Re^+$.

We can integrate and express the solution as

$$x(t) = x_0 e^{-kt}$$

Using the explicit solution we can then answer questions about its behaviour. For example, let's say we want to find the time, t^*, at which the solution is half it's maximum. Hence

$$x(t*) = x_0/2 \implies t^* = \frac{\ln 2}{k}.$$

However, in the study of nonlinear systems, most problems will not have an explicit solution. For example, consider the nonlinear ODE

$$\dot{x} = -\frac{k\sin(x) + \sqrt{x}}{1+x}, \quad x(0) = x_0$$

where $0 < x_0 < \pi$.

I cannot integrate this equation in order to find solutions in terms of standard functions. Hence I cannot *quantitatively* describe the solution. However, I can identify that

$$\dot{x} < 0, \ \forall \ 0 < x < \pi.$$

Hence the solution will decrease in value from the given initial condition and tend to zero as $t \to \infty$. This is an example of a *qualitative* analysis.

1.5 Representing solutions

It is useful to introduce some nomenclature to describe the solutions of a dynamical system. Consider an ODE

$$\dot{x} = f(x), \quad x(0) = x_0,$$

where f is a prescribed function and x_0 is an initial condition.

- phase space - put dependent variable on Cartesian axes
- *phase point* - value of the solution at given time point
- *vector field* - the derivative of the solution, i.e. f.
- *trajectory* - a line in phase space that traces out a solution as time evolves
- *phase portrait* - collection of trajectories (i.e. solutions with different initial conditions)

1.6 Fixed points and their stability

Many of the dynamical systems that we will study will be nonlinear. Hence it will not be possible to compute exact solutions.

The behaviour of dynamical systems can often be understood by considering the fixed points, i.e. values of the dependent variables at which the dynamics are at steady state.

Stability analyses are used to investigate the dynamics of perturbations about the steady state.

1.7 Uniqueness and existence

We will restrict ourselves to problems in which the vector fields are sufficiently well behaved such that unique solutions exist. However, problems can be identified where solutions do not exists or where multiple solutions exist.

Example 1.2. Show that the solution to the ODE

$$\dot{x} = x^2 \quad x(0) = 2$$

is not defined for all $t > 0$.