University of Dundee

MA32011

MA32011

Philip Murray

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Preface

Welcome to the module MA32011 Dynamical systems.

My name is Philip Murray and I am the module lead.

How to contact me?

- email: pmurray@dundee.ac.uk
- office: G11, Fulton Building
- Teams: PM me

Lecture notes

You can find lecture notes for the module on this page. If you would like a pdf this can be easily generated by clicking on the pdf link of the webpage. I will occasionally edit/update the notes as we proceed through lectures. If you spot any errors, typos or omissions please let me know.

Reading

Nonlinear dynamics and chaos, Steven Strogatz Strogatz (2001) Mathematical Biology I, Murray (2002)

Python codes

I have provided Python codes for most of the figures in the notes (you can unfold code section by clicking 'Code'). Note that the Python code does not appear in the pdf.

Many of you have taken the Introduction to Programming module at Level 2 and have therefore some experience using Python. I strongly encourage you to use the provided codes as a tool to play around with numerical solutions of the various models that we will be working on. The codes should run as standalone Python codes.

> **Note**
>
> To access Python on Uni machines:
>
> 1. Launch Anaconda from AppsAnywhere
> 2. When a folder opens, double click on *Spyder*.
> 3. Paste a code from lecture notes into the editor on the left-hand side.
> 4. Click on the green arrow to run the code.
> 5. The plots should appear in the plots tab on the right-hand side.
> 6. Experiment with the code. When you change a model parameter, does the solution change in an expected way?

I have also provided some examples of how to use Python as a symbolic calculator. This uses a Python library called *sympy* and is quite similar to Maple.

Assessment

- Final exam (80 %)
- 2 class tests (8 % each), Week 7 and 11
- 4 quizes (1 % each), Week 2,4,6 and 9

Plan

Table 1: Projected delivery

Week	Up to Section	Tutorial sheet	Assessment
1		1	
2		1	Quiz 1
3		2	
4		2	Quiz 2
5		3	
6		3	Quiz 3
7		4	Test 1
8		4	
9		5	Quiz 4 4
10		5	

Week	Up to Section	Tutorial sheet	Assessment
11		Test 2	

References

1 Introduction

The goal of this module is to provide an introduction to dynamical systems. We will introduce key mathematical concepts and explore examples from physics and biology.

To begin with we introduce some key terminology.

1.1 Discrete v continuous time

1.1.1 Differential equations

Let t be a continuous variable and $x = x(t)$. $\dot{x}$ is used to denote the time derivative, i.e.

$$\dot{x} := \frac{dx}{dt}.$$

Similarly, the second order derivative is represented by

$$\ddot{x} := \frac{d^2x}{dt^2}.$$

1.1.1.1 Population dynamics

Many of the biological examples that we will encounter later describe population dynamics. Let $N(t)$ represent the size of a population at time t.

Suppose that individuals are born at per capita constant rate b and die at a rate d.

We can write

$$N(t + \Delta t) = N(t) + bN(t)\Delta t - dN(t)\Delta t$$

Gathering terms and taking the limit $\Delta t \to 0$ yields

$$\dot{N} = (b - d)N.$$

If we allow the birth and death rates to depend on population size then we can derive a more general equation

$$\dot{N} = H(N),$$

where H is some prescribed function. An example is the logistic growth equation

$$\dot{N} = rN(1 - N). \tag{1.1}$$

1.1.1.2 Newton's second law

Many of the physics examples that we will examine later originate from Newton's Second Law, i.e.

$$m\ddot{x} = F(x, \dot{x}),$$

where $x(t)$ represents the position of a particle at time, t, m represents a constant particle mass and F a resultant force.

Consider the case in which F represents a linear restoring force, i.e.

$$F(x, \dot{x}) = -kx.$$

The equation of motion can be written as

$$\ddot{x} = -\mu x, \tag{1.2}$$

where $\mu = -k/m$.

Example 1.1. The app encodes a numerical solution of the second order ODE

$$a\ddot{x} + b\dot{x} + cx = 0.$$

Use the app to:

- identify appropriate values of the parameters a, b and c so that the solution captures the case of a particle of mass (m) equal to 3 subjected to a linear restoring force with spring constant (k) equal to 5.
- explore the dependence of oscillation period on initial conditions.

Is the behaviour of the numerical solution consistent with Equation 1.2?

1.1.2 Difference equations

Suppose that n is a discrete variable. Let y_n represent a dependent variable at iteration n. Consider the difference equation

$$y_{n+1} = r y_n (1 - y_n),$$

where $r \in \Re$.

Example 1.2. Use the app below to 1. explore how the solution behaviour (top row) changes qualitatively as the parameter r increases. 2. connect the branching structure in the bifurcation diagram (second and third rows) to the qualitative behaviour of the solution (top row). 3. demonstrate self similarity (by zooming in on the bifurcation diagram (bottom row) show that the structure appears to repeat at finer scales).

1.1.3 Key questions to ask of a dynamical system

- do solutions exist? If so are they unique?
- Is there an explicit solution?
- Can we qualitatively describe solution behaviour?
- How do the solutions depend on the model parameters?
- Are their critical values of parameters where solution behaviour changes?

1.2 Autonomous v nonautonomous ODEs

For an autonomous system, the update does not explicitly depend on the independent variable. Equation 1.1 is autonomous. But

$$\dot{x} = rx(1 - x + t)$$

is nonautonomous (because of the explicit time dependence on the right-hand side).

1.3 Linear v Nonlinear

Linear systems satisfy a linear supposition principle: a sum of solutions is itself a solution. In general, this property does not hold for nonlinear systems.

In linear dynamical systems, the dynamics are a function of linear sums of the dependent variables. Hence

$$\dot{x} = -x \tag{1.3}$$

is a linear ordinary differential equation (ODE). But

$$\dot{x} = -x^2 \tag{1.4}$$

is nonlinear.

Example 1.3. Integrate each of Equation 1.3 and Equation 1.4. Use the solutions to demonstrate that the principle of linear superpositions holds for Equation 1.3 but not for Equation 1.4.

1.4 Quantitative v qualitative solutions

You are likely used to solving problems in which an explicit solution can be found. For example, consider the ODE

$$\dot{x} = -kx, \quad x(0) = x_0$$

where $k, x_0 \in \Re^+$.

We can integrate and express the solution as

$$x(t) = x_0 e^{-kt}$$

Using the explicit solution we can then answer questions about its behaviour. For example, let's say we want to find the time, t^*, at which the solution is half it's maximum. Hence

$$x(t*) = x_0/2 \implies t^* = \frac{\ln 2}{k}.$$

However, almost all problems that we will encounter in the study of nonlinear systems will not have an explicit solution. For example, consider the nonlinear ODE

$$\dot{x} = -\frac{k\sin(x) + \sqrt{x}}{1+x}, \quad x(0) = x_0,$$

where $0 < x_0 < \pi$.

I cannot integrate this equation in order to find solutions in terms of standard functions. Hence I cannot *quantitatively* describe the solution. However, I can identify that

$$\dot{x} < 0, \ \forall \ 0 < x < \pi.$$

Hence the solution will decrease in value from the given initial condition and tend to zero as $t \to \infty$. This is an example of a *qualitative* analysis.

1.5 Representing solutions

It is useful to define some important concepts that are used to describe the solutions of a dynamical system. Consider an ODE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0,$$

where $\mathbf{f}$ is a prescribed function and $\mathbf{x}_0$ is an initial condition.

- *phase space* - a Cartesian coordinate system with dependent variables represented on Cartesian axes
- *phase point* - value of the solution at given time point
- *vector field* - the derivative of the solution, i.e. $\mathbf{f}$.
- *trajectory* - a line in phase space that traces out a solution as time evolves (must be tangential to vector field)
- *phase portrait* - collection of trajectories (i.e. solutions with different initial conditions)

1.6 Fixed points and their stability

Many of the dynamical systems that we will study will be nonlinear. Hence it will not be possible to compute exact solutions.

The behaviour of dynamical systems can often be understood by considering the fixed points, i.e. values of the dependent variables at which the dynamics are at steady state.

Stability analyses are used to investigate the dynamics of perturbations about the steady state.

1.7 Uniqueness and existence

We will restrict ourselves to problems in which the vector fields are sufficiently *well behaved* such that unique solutions exist.

Theorem 1.1.

Theorem

Suppose that

$$\dot{x} = f(x), \quad x(0) = x_0. \tag{1.5}$$

If f is continuously differentiable on an open interval D of the x axis and x_0 is a point in D, Equation 1.5 possesses a unique solution on some time interval $(-\tau, \tau)$.

However, it is noted that problems can be identified where solutions do not exist or where multiple solutions exist.

Example 1.4. Show that the solution to the ODE

$$\dot{x} = x^2 \quad x(0) = 2$$

blows up after a finite time.

Is this result consistent with Theorem 1.1?

Example 1.5. Show that the solution to the ODE

$$\dot{x} = \sqrt{x}, \quad x(0) = 0$$

is given by

$$x = \begin{cases} 0, & t \leq \delta \\ \frac{(t-\delta)^2}{4} > \delta, & t > \delta \end{cases}$$

for any $\delta > 0$. Why does this form imply that the solution is not unique? Is this result consistent with Theorem 1.1?

1.8 Nondimensionalisation

In real world problems, variables and parameters typically have units (e.g. time - seconds, Force - Newtons etc.). We can nondimensionalise problems by defining rescaled variables. This process can be used to justify simplifications to models and to reduce the number of parameters.

1.9 Numerical solutions

Numerical solutions are used to numerically compute approximate solutions to problems. The simplest example of a numerical method in a dynamical system is the forward Euler method. Suppose we want to study the ODE

$$\dot{x} = f(x), \quad x(0) = x_0, \quad 0 < t < T.$$

Discretise the independent variable t by defining $t = 0, \Delta t, 2\Delta t, \ldots, T = N\Delta T$.

Approximate the time derivative

$$\dot{x} = \frac{dx}{dt} \sim \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

Hence the solution at time $t + \Delta t$ can be approximated by

$$x(t+\Delta t) = x(t) + \Delta t f(x(t)).$$

Given an initial condition $x(0) = a$ we can compute the approximate solution at time Δt. Further iteration then allows an approximate solution to be calculated.

Numerical solutions provide a a very useful way to explore solution behaviour. However, they describe the quantitative behaviour of a solution for a particular initial condition and set of parameter values.

Example 1.6. The app below uses the Forward Euler method to compute numerical solutions of the ODE

$$\dot{x} = a + bx + cx^2$$

in the domain $[0, T]$ with initial condition

$$x(0) = x_0.$$

1. Choose model parameters so that the app solves the ODE

   $$\dot{x} = -bx, \quad x(0) = x_0.$$

   and show that the numerical solution agrees (eye ball norm) with the exact solution (dashed line).
2. Show that the numerical error increases with Δt.

Part I

1D flows

2 Flows on the line

Here we consider ODE models with a single dependent variable that are first order in time.

Let $x = x(t)$ and consider the ODE

$$\dot{x} = f(x). \tag{2.1}$$

It is assumed that f is smooth and real valued.

2.1 Geometric

For many problems an explicit solution can either not be constructed or is not of practical use.

Example 2.1. Let $x(t)$. Consider the ODE

$$\dot{x} = \sin x.$$

For the initial condition $x(0) = \pi/4$, describe solution behaviour as $t \to \infty$.

After applying separation of variables, an implicit solution is given by

$$t = -\ln|\csc x + \cot x| + C,$$

where C is an integration constant.

However, this does not help me to describe the limiting behaviour of the solution as $t \to \infty$.

Instead let's use a graphical method. In Figure 2.1 we sketch a graph of f, the right-hand side of the ODE. The arrows depict the vector field. Hence when $f > 0$, $\dot{x} > 0$ and the solution increases. In contrast, when $f < 0$, $\dot{x} < 0$ and the solution decreases.

Note that $f > 0 \; \forall \; 0 < x < \pi$. Hence for the initial condition $x_0 = \pi/4$, $\dot{x}_{t=0} > 0$. The solution will increase until it reaches π. At $x = \pi$, $f = 0$.

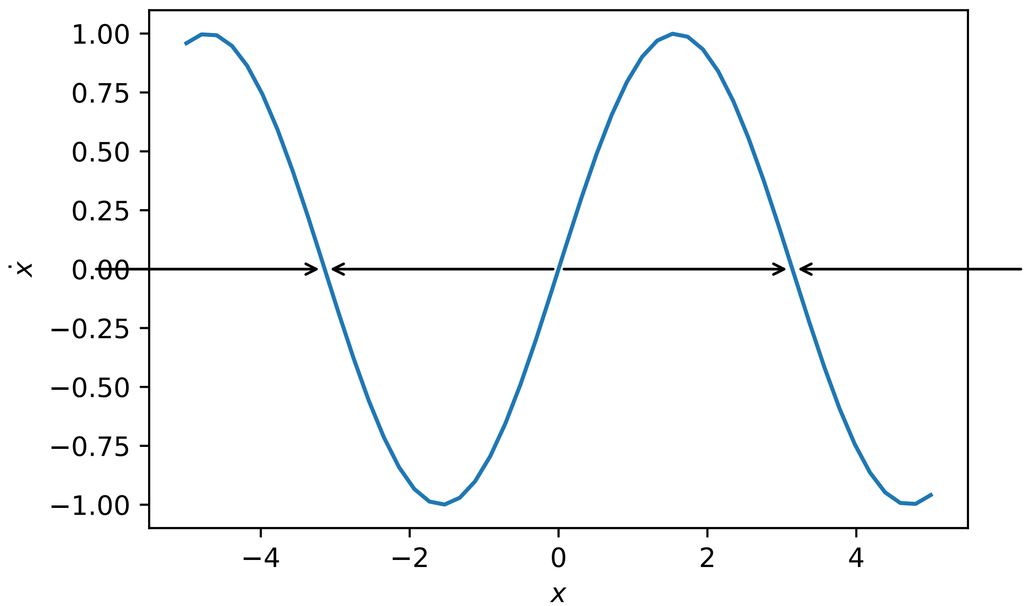

Figure 2.1

2.2 Fixed points and linear stability

2.2.1 Fixed points

Let $x = x^*$ be a fixed point of Equation 2.1. At $x = x^*$

$$\dot{x} = 0 \implies f(x^*) = 0.$$

There are a number of interpretations of x^*:

- roots of f (algebraic)
- stagnation points of the flow (topological)

Corollary 1

Any trajectory initialised at a fixed point remains there for all t.

Example 2.2. Find all the fixed points of

$$\dot{x} = x^2 - 1, \tag{2.2}$$

Solution

The fixed points are point, x^* defined such that

$$\dot{x} = 0 \implies f(x^*) = 0.$$

Hence fixed points satisfy

$${x^*}^2 - 1 = 0.$$

The solutions are

$$x^* = \pm 1.$$

See Figure 2.2 for graphical solution.

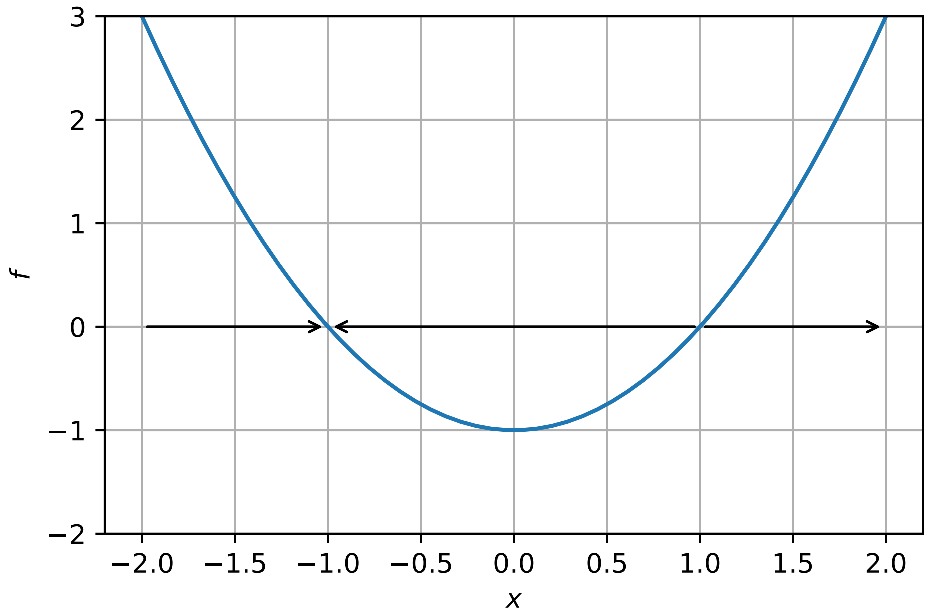

Figure 2.2: Graphical solution of Equation 2.2.

2.2.2 Linear stability analysis

Let $x = x^*$ be a fixed point of Equation 2.1.

2.2.2.1 A change of dependent variable

To perform a linear stability analysis we make the change of variables

$$x(t) = x^* + \hat{x}(t)$$

where the new dependent variable, $\hat{x}(t)$, is a perturbation about the fixed point.

The time derivative on the left-hand side of Equation 2.1 transforms to

$$\dot{x} = \frac{d}{dt}(x^*) + \frac{d}{dt}(\hat{x}(t)) = \dot{\hat{x}}.$$

Hence Equation 2.1 transforms to

$$\dot{\hat{x}} = f(x^* + \hat{x}(t)). \tag{2.3}$$

2.2.2.2 Taylor expansion and a linear system

Employing the Taylor expansion on the right-hand side of Equation 2.1 and making the assumption that perturbations are small

$$\dot{\hat{x}} = f(x^*) + f'(x^*)\hat{x}(t) + f''(x^*)\hat{x}^2(t) + h.o.t.$$

Noting that

$$f(x^*) = 0$$

and retaining linear terms yields

$$\dot{\hat{x}} = f'(x^*)\hat{x}(t)$$

with solution

$$\hat{x}(t) = \eta e^{f'(x^*)t}.$$

Here η is an initial perturbation about the steady-state that is determined by initial conditions.

2.2.2.3 A condition for linear stability

When $f'(x^*) > 0$ the perturbation grows exponentially fast and the steady-state is linearly unstable. When $f'(x^*) < 0$ the perturbation decays exponentially fast and the steady-state is linearly stable.

Example 2.3. Determine the linear stability of the fixed points of

$$\dot{x} = x^2 - 1.$$

Example 2.4. What can be said about the stability of the fixed points of the following ODEs:

1. $$\dot{x} = -x^3.$$

2. $$\dot{x} = x^3.$$

3. $$\dot{x} = 0.$$

2.3 Validity of linear classification

It is worth highlighting here that

$$f'(x*)$$

can be interpreted as an eigenvalue of the linearised problem

$$\dot{\hat{x}} = \lambda \hat{x},$$

where

$$\lambda = f'(x^*).$$

Definition 2

A fixed point is said to be *hyperbolic* when the eigenvalues of its linearisation are nonzero.

Theorem 3: Hartman-Grobman

If a system has a hyperbolic FP, the classification of the nonlinear system at the fixed point is determined by the linear classification.

If a fixed point is non-hypberbolic, it's classification requires consideration of higher order terms.

Example 2.5. Apply the Hartman Grobman theorem to the classification of the problem

$$\dot{x} = -x^3.$$

2.4 Case study: population dynamics

Let $N = N(t)$. The logistic model of population growth, due to Verhulst, takes the form

$$\dot{N} = rN(t)\left(1 - \frac{N(t)}{K}\right), \tag{2.4}$$

where r is the linear growth rate and K is carrying capacity. We consider both $r, K \in \Re^+$.

Questions to ask of such a model are: what type of biologically realistic solutions does it possess? Are there fixed points? If so, are they stable or unstable?

2.4.0.1 Numerical solutions

In Figure 2.3 we present numerical solutions of equation using different initial conditions. Note the limiting behaviour of solutions as $t \to \infty$. In Figure 2.3 it is clear that even though some solutions are initialised at $N_0 = 0.1$, much closer to $N^* = 0$ than $N^* = K$, they tend to the limit $N = K$. Why do solutions not tend to $N^* = 0$?

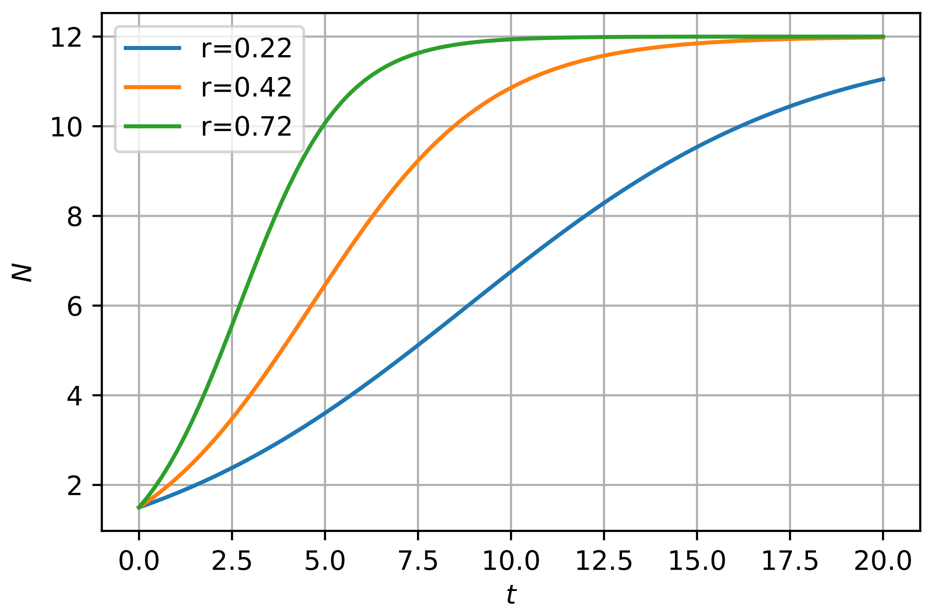

Figure 2.3: Numerical solution of the logistic growth model

2.4.0.2 Dimensional analysis and nondimensionalisation

N represents the population density and has units of one over area (say $1/m^2$) and t has units of time (say, seconds, s). Hence the left-hand side of Equation 2.4 has units of $1/(m^2 s)$. The first term on the right-hand side of Equation 2.4 is rN. N has units $1/m^2$ hence the parameter r must have units of $1/s$ for dimensional consistency. This is consistent as r represents the linear growth rate.

The second term has the form rN^2/K. Given the chosen units for r and N, the parameter K must have dimensions $1/m^2$. Again, this is consistent as K is a carrying capacity (i.e. it has units of population density).

We define the nondimensionalised variables

$$n = \frac{N}{\tilde{N}} \qquad \tau = \frac{t}{\tilde{T}}$$

where $\tilde{N}$ and $\tilde{T}$ are constants that have units of population density and time, respectively. Hence Equation 2.4 transforms, upon change of variables, to

$$\frac{\tilde{N}}{\tilde{T}}\frac{dn}{d\tau} = r\tilde{N}n(1 - \frac{n\tilde{N}}{K}).$$

In the case of the logistic equation there is only one time scale and density scale in the problem, hence we choose

$$\tilde{T} = \frac{1}{r} \quad \textit{and} \quad \tilde{N} = K$$

and the dimensionless model is

$$\frac{dn}{d\tau} = n(1 - n) \tag{2.5}$$

Note that we can retrieve the original equation by rescaling and calculating $N = \tilde{N}n$ and $t = \tilde{T}\tau$.

2.4.1 Fixed points and linear stability

Fixed points satisfy

$$n^*(1 - n^*) = 0.$$

Hence

$$n^* = 0, \qquad n^* = 1.$$

To determine linear stability we compute

$$H'(n) = (1 - 2n).$$

When $n = n^* = 0$ we obtain

$$H'(n) = 1.$$

Hence the origin is a linearly unstable fixed point.

At the steady-state $n^* = 1$

$$H'(n^*) = -1$$

hence $n^* = 1$ is linearly stable.

Note that the linear stability analysis can explain the observations regarding the numeric solutions presented in Figure 2.3.

2.4.2 Graphical analysis

In Figure 2.4 we plot the right-hand side of Equation 2.5. We can qualitatively describe model solutions by considering the arrow along the n axis. Suppose we consider an initial condition with $0 < n_0 < 1$. Using the graph of $H(n)$, $dn/d\tau$ is positive, hence n increases as a function of time until $n(\tau) \to 1$.

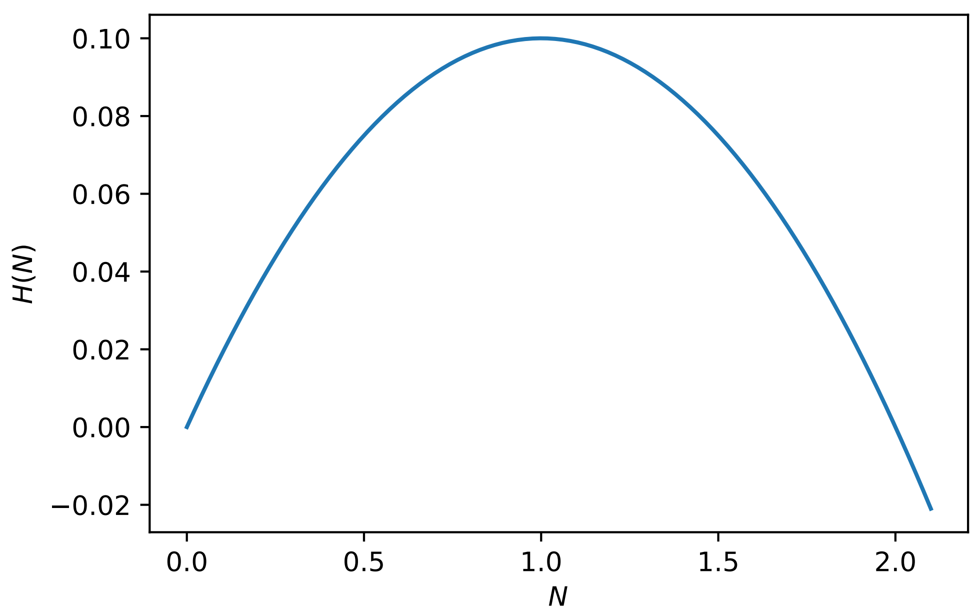

Figure 2.4: Right-hand side of the logistic ODE

Example 2.6. Use separation of variables to show that the solution can be written explicitly as

$$N(t) = \frac{N_0 K e^{rt}}{K + N_0(e^{rt} - 1)}$$

Solution

$$\int \frac{dN}{N(1-\frac{N}{K})} = r\int dt.$$

Using partial fractions

$$\int \frac{dN}{N} + \frac{1}{K}\int \frac{dN}{1-\frac{N}{K}} = r\int dt.$$

Integration yields

$$\ln N - \ln\left(1-\frac{N}{K}\right) = \ln \frac{N}{1-\frac{N}{K}} = rt + C.$$

Hence

$$N = \frac{De^{rt}}{1+\frac{D}{K}e^{rt}}$$

Given an initial condition $N(0) = N_0$, we obtain

$$N(t) = \frac{N_0 K e^{rt}}{K + N_0(e^{rt}-1)}$$

2.4.2.1 Qualitative analysis of the exact solution

As $t \to \infty$, $N \to K$. At $t = 0$, $N = N_0$ and that for small $N_0 \ll K$ the initial growth phase is exponential, i.e.

$$N(t) \sim N_0 e^{rt} \qquad N_0 \ll K, t \ll \frac{1}{r}.$$

2.5 Impossibility of oscillations

In 1D flows with well behaved f, the range of permissible qualitiative behaviours is limited by the geometry of the line. Solutions must have one of the following behaviours:

- tend towards a stable fixed point
- move away from an unstable fixed point
- stay at a fixed point for all time
- tend to $\pm\infty$

Oscillatory solutions to Equation 1.5 are impossible, i.e. first order autonomous ODEs (with one dependent variable) cannot oscillate.

This can be argued using geometrical constraints of dynamics on the line.

Example 2.7. Consider the integral

$$\int_t^{t+T} f(x(t))\frac{dx}{dt}dt,$$

where T is the oscillation period. Use proof by contradiction to show that periodic solutions are impossible.

Solution

Suppose that a periodic solution exists such that

$$x(t+T) = x(t), \quad x(t+s) \neq x(t+T) \forall 0 < s < T.$$

Consider

$$\int_t^{t+T} f(x(t))\frac{dx}{dt}dt.$$

This integral can be written as

$$\int_t^{t+T} f(x(t))^2 dt > 0,$$

as f is not identically zero.
Changing variables yields

$$\int_{x(t)}^{x(t+T)} f(x)dx = 0.$$

Hence there is a contradiction and no periodic solutions exist.

2.6 Potential flows

Consider the ODE

$$\dot{x} = f(x).$$

Suppose that

$$f(x) = -\frac{dV(x)}{dx}.$$

Now consider

$$\dot{V}.$$

Applying the chain rule

$$\dot{V} = \frac{dV}{dx}\dot{x} = -(\frac{dV}{dx})^2 \leq 0.$$

Hence for a potential flow V is never increasing. Hence particle move to points of lower potential until they reach equilibrium given by

$$f(x) = -\frac{dV(x)}{dx} = 0.$$

Example 2.8. Graph the potential for the system

$$\dot{x} = -x$$

and identify equilibrium points.

3 Bifurcations

3.1 Introduction

The qualitative behaviour of solutions of

$$\dot{x} = f(x), \quad x(0) = x_0$$

are limited. Solutions either *flow* to a fixed point, remain at a fixed point or tend to $\pm\infty$. So what is interesting about the study of such problems?

Now we introduce the idea of bifurcations. These arise when the structure of the solutions (i.e. the number and/or stability of fixed points) changes at particular parameter values.

At a bifurcation point of a 1D system the eigenvalue is 0. Hence the fixed point is *not* hyperbolic.

Consider the 1D system

$$\dot{x} = f(x, r)$$

where r is a parameter.

Suppose that

- there is a fixed point $x = x^*$.
- at some $r = r_c$ the eigenvalue of the linearised system vanishes.

Upon Taylor expansion of f about (x^*, r_c), we obtain

$$\dot{x} = f(x^*, r_c) + \frac{\partial f}{\partial x}_{(x^*, r_c)}(x - x^*) + \frac{\partial f}{\partial r}_{(x^*, r_c)}(r - r_c) + \frac{1}{2}(x - x_c)^2 \frac{\partial^2 f}{\partial x^2}_{(x^*, r_c)} + ...$$

As x^* is a fixed point

$$\dot{x} = \frac{\partial f}{\partial r}_{(x^*, r_c)}(r - r_c) + \frac{1}{2}(x - x^*)^2 \frac{\partial^2 f}{\partial x^2}_{(x^*, r_c)} + \frac{1}{2}(r - r_c)^2 \frac{\partial^2 f}{\partial r^2}_{(x^*, r_c)} + (r - r_c)(x - x^*) \frac{\partial^2 f}{\partial r \partial x}_{(x^*, r_c)} ...$$

Close to a bifurcation point, the stability classification will be determined by the higher order derivatives of f w.r.t. x and r.

It can be shown that generic bifurcations of the system

$$\dot{x} = f(x, r), \quad x(0) = x_0$$

can be reduced to one of three *normal forms*:

- saddle node bifurcations
- transcritical bifurcations
- pitchfork bifurcations

3.2 Saddle node

At a saddle node bifurcation, two fixed points move towards one another and mutually annihilate. The canonical form is given by

$$\dot{x} = f(x, r) = r + x^2, \tag{3.1}$$

where $r \in \Re$.

Example 3.1. Identify the fixed points of Equation 3.1 and determine their linear stability.

Solution

The fixed points are

$$x^*_\pm = \pm\sqrt{-r}.$$

In the case $r < 0$

$$f'(x^*) = \pm 2\sqrt{-r}.$$

Hence x^*_- is linearly stable and x^*_+ is linearly unstable.
For $r = 0$ the fixed point is *half-stable* and it vanishes for $r > 0$. Upon plotting x^* against r we obtain a bifurcation diagram.

In Figure 3.1 we plot f at three different values of r. Note that when $r > 0$, $f > 0$ and the solution is an increasing function of time.

At $r = 0$ there is a double root of f. Here the fixed point is *half stable*. For the initial condition $x_0 < 0$ the solution will increase until $x(t = 0)$. It is stable to perturbations along the negative x axis. However, for $x_0 > 0$ $f > 0$ and the solution is an increasing function of time. Hence

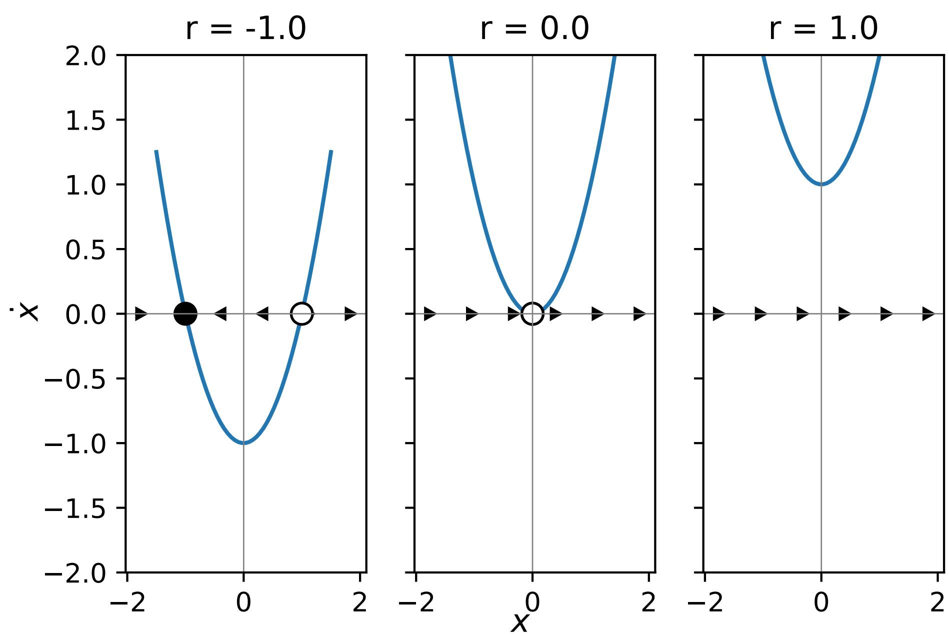

Figure 3.1

the solution is unstable to perturbations with $x_0 > 0$. Hence the fixed point $x^* = 0$ is defined to be half-stable when $r = 0$.

In Figure 3.2 we plot the fixed points against the parameter r. For $r < 0$ there are two fixed points ($\sqrt{r}$ is linearly unstable whilst $-\sqrt{r}$ is linearly stable). Usually some annotation is used to denote the stability. For $r > 0$ there are no fixed points.

Example 3.2. Consider the following ODE:

$$\dot{x} = 1 + rx + x^2, \quad x(0) = x_0.$$

in the case $r \in \mathfrak{R}^+$.

1. Sketch the bifurcation diagram.
2. Show that upon introducing rescaled variables, the ODE can be written in the normal form

$$\dot{y} = r' + y^2.$$

   Hint: try completing the square.

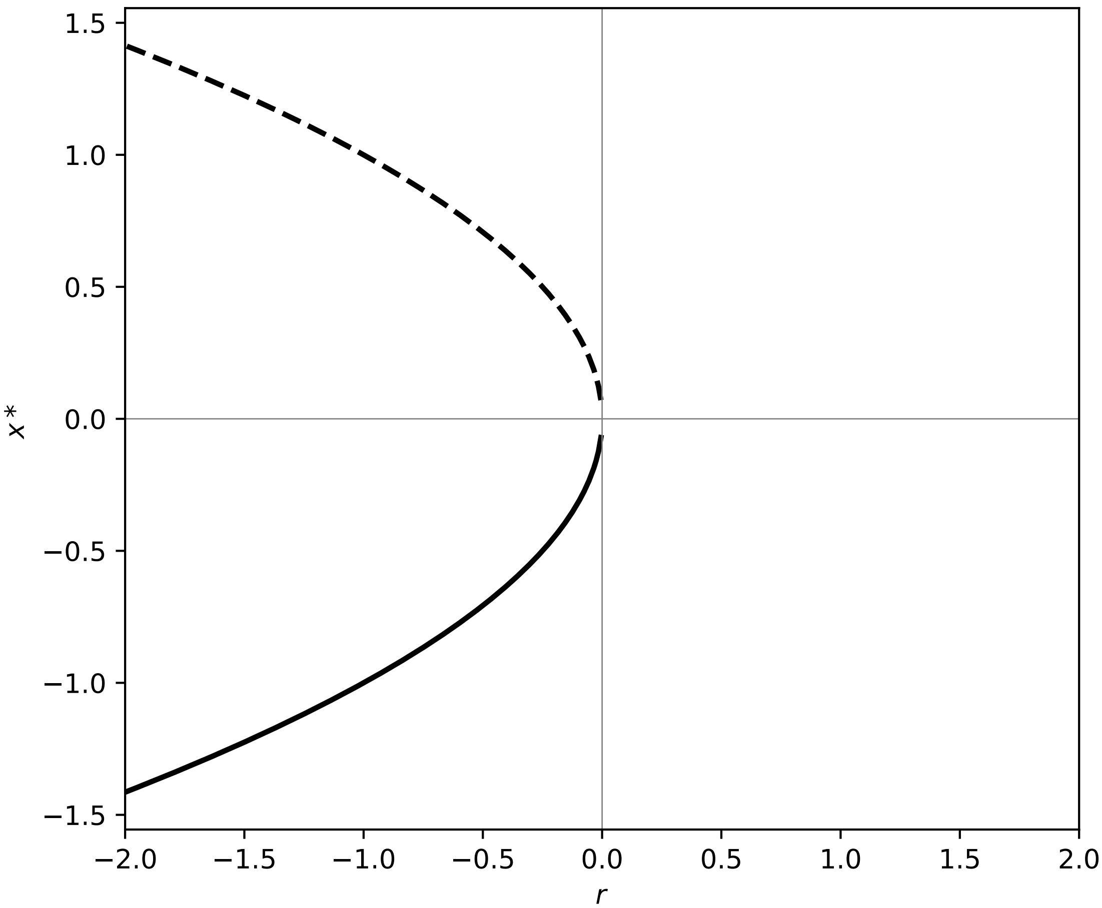

Figure 3.2

3.3 Transcritical

At a transcritical bifurcation the stability of a fixed point changes as a parameter is varied. However, fixed points do not *disappear*, as was the case with the saddle node bifurcation.

The *normal form* for a transcritical bifurcation is

$$\dot{x} = rx - x^2,$$

Example 3.3. Identify the fixed points and their linear stability

Solution

The fixed points are $x^* = 0$ and $x^* = r$. For $r < 0$, $x^* = r$ is linearly unstable and $x^* = 0$ is linearly stable. At $r = 0$ the fixed points coalesce and the fixed point is half stable. For $r > 0$ $x^* = 0$ is linearly unstable and $x^* = r$ is linearly stable.

In Figure 3.3 the function f is plotted for different values of the parameter r. The origin is always a fixed point. For $r < 0$ there is a fixed point on the negative real axis whilst for $r > 0$ there is a fixed point on the positive real axis. Hence the number of fixed points is two for $|r| > 0$. This bifurcation is fundamentally different to the saddle node bifurcation (where there are no fixed points on one side of the bifurcation).

In Figure 3.4 we plot a bifurcation diagram for the transcritical bifurcation. Note that $x^* = 0$ is always a fixed points but it's stability changes at $r = 0$.

Example 3.4. Sketch the bifurcation diagram for

$$\dot{x} = rx + x^2.$$

Example 3.5. Consider the ODE

$$\dot{x} = r \ln x + x - 1.$$

1. Show that $x^* = 1$ is a fixed point.
2. Make a change of variables $u = x - 1$ in order to analyse perturbations about the fixed point.
3. Use a Taylor expansion to show that this system can be approximated by the transcritical normal form. Hence deduce that the bifurcation vaue is $r = -1$.
4. Show that the system can be reduced to the normal form

   $$\dot{X} = RX - X^2$$

   by making an appropriate change of variables.

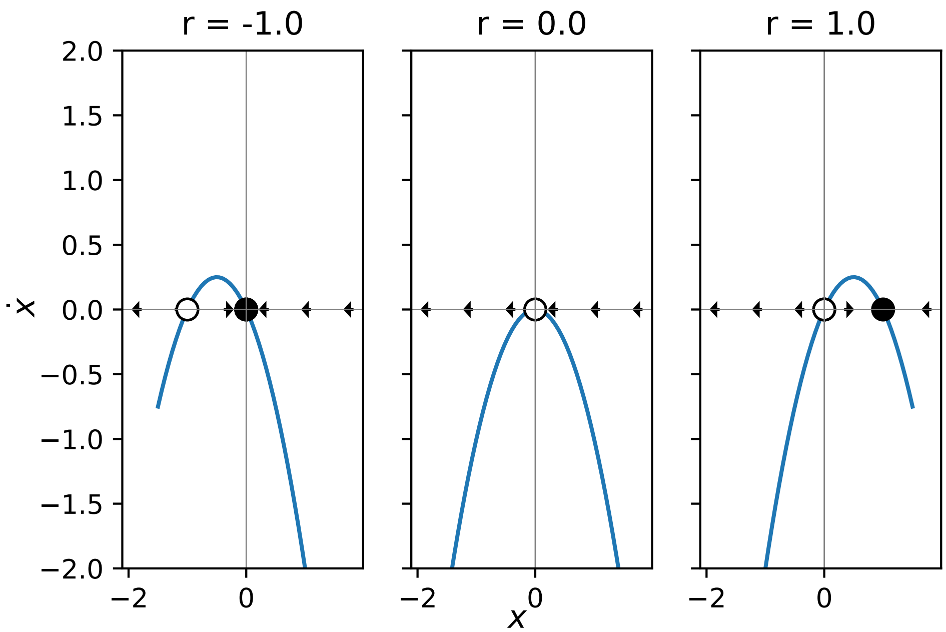

Figure 3.3

3.4 Pitchfork

Pitchfork bifurcations often arise in situations with symmetry. Typically, two or more fixed points appear/disappear together.

3.4.1 Supercritical pitchfork

The normal form of the supercritical pitchfork bifurcation is

$$\dot{x} = rx - x^3,$$

Example 3.6. Identify the fixed points and determine their linear stability

Solution

The fixed points satisfy

$$rx^* - {x^*}^3 = 0$$

Hence x^* is a fixed point. Other fixed points satisfy

$$r - {x^*}^2 = 0$$

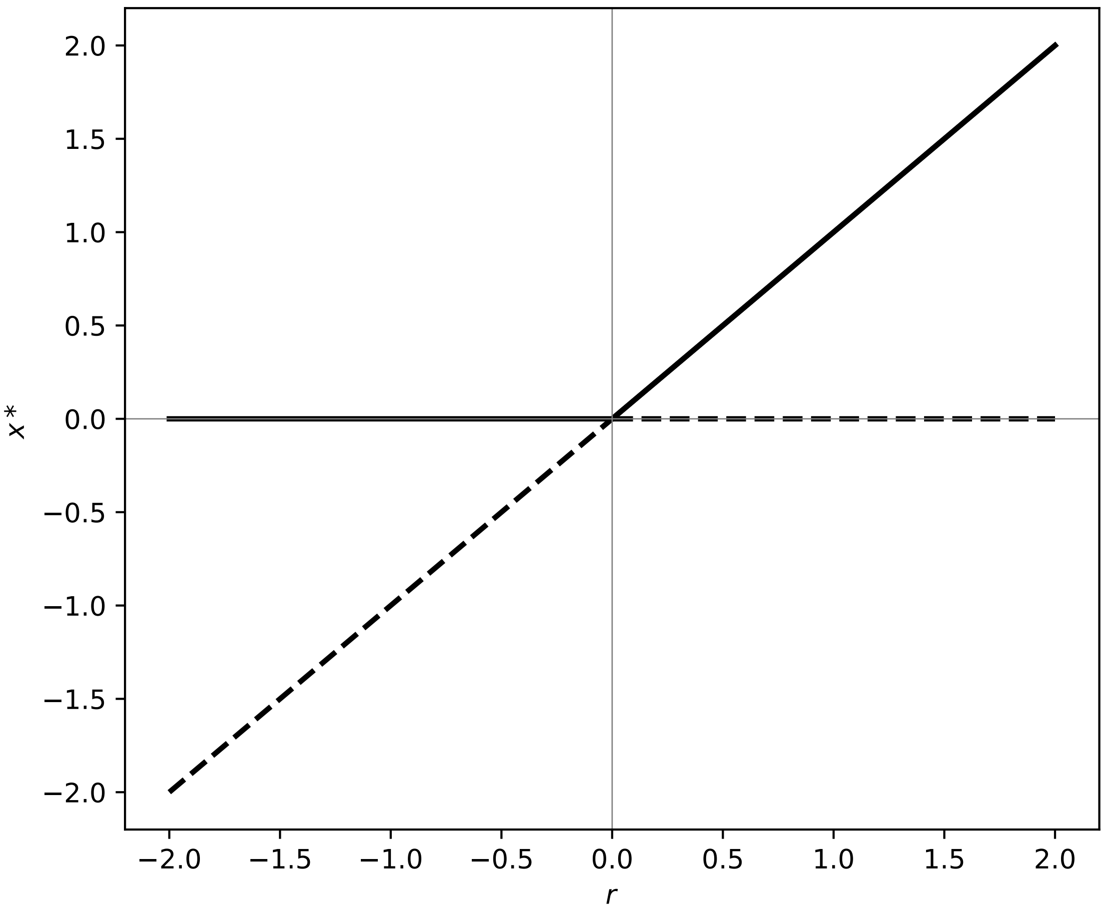

Figure 3.4

Hence there is a pair of fixed points given by

$$x^* = \pm\sqrt{r},$$

defined for $r > 0$.
For

$$f(x,r) = rx - x^3$$

differentiation yields

$$f'(x) = r - 3x^2.$$

At $x^* = 0$

$$f'(0) = r.$$

For $r < 0$, $x^* = 0$ is linearly stable. For $r > 0$, $x^* = 0$ is linearly unstable. There is a bifurcation at $r = 0$.
At $x^* = \pm\sqrt{r}$

$$f'(\pm\sqrt{r}) = r - 3r = -2r.$$

Hence when real, non-trivial fixed points exist $r > 0$ they are linearly stable.
At $r = 0$ the fixed points coalesce and the fixed point is half stable.

In Figure 3.5 we plot f for three different values of r. For $r < 0$ it is clear that $x^* = 0$ is a uniquye fixed point and that it is linearly stable. For $r > 0$ f is a cubic with three real roots. Note that the non-zero roots are linearly stable and symmetrically distributed about the origin. We refer to a system with two linearly stable fixd points as being *bistable*.

In Figure 3.6 we plot a bifurcation diagram for the supercritical pitchfork. Note that when $r < 0$ there is a single fixed point that is linearly stable. For $r > 0$ there are three fixed points.

3.4.2 Subcritical pitchfork

In the case of a subcritical pitchform the governing equation is

$$\dot{x} = rx + x^3,$$

with $r \in R$. As this case is very similar to the supercritical pitchfork working out the algebraic details is left as an exercise. In Figure 3.8 note that for $r < 0$ there are three fixed points. This time the nontrivial points are linearly unstable whilst $x^* = 0$ is linearly stable. For $r > 0$ there is a single fixed point that is linearly unstable. At $r = 0$ there is a single non-hyperblic fixed point at which $f'(0) = 0$. This is unstable. See (**subcritbfc?**) for bifurcation diagram.

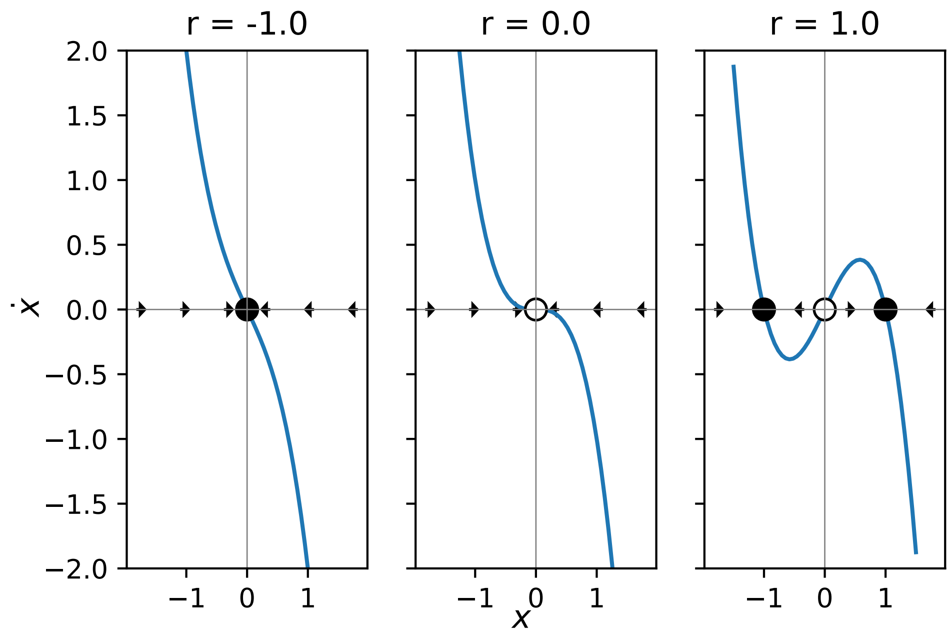

Figure 3.5

3.5 Application: the spruce budworm model

The spruce budworm is a destructive and widely distributed forest defoliator in North America. Massive outbreaks occur periodically and can destroy large quantities of valuable spruce and fir. To understand the outbreak behaviour and develop and management strategies, a series of mathematical models have been developed, beginning with Ludwig et al. (1978). The goal of the models is to explain the qualitative pattern of sudden outbreaks and then a sudden collapse.

3.5.1 Model development

Letting $N(t)$ represent the population size at time t, it is assumed that budworm exhibits logistic growth and is subject to predation at rate $p(N)$. A governing ordinary differential equation is given by

$$\frac{dN}{dt} = r_B N\left(1 - \frac{N}{K_B}\right) - p(N), \tag{3.2}$$

where

$$p(N) = \frac{BN^2}{A^2 + N^2},$$

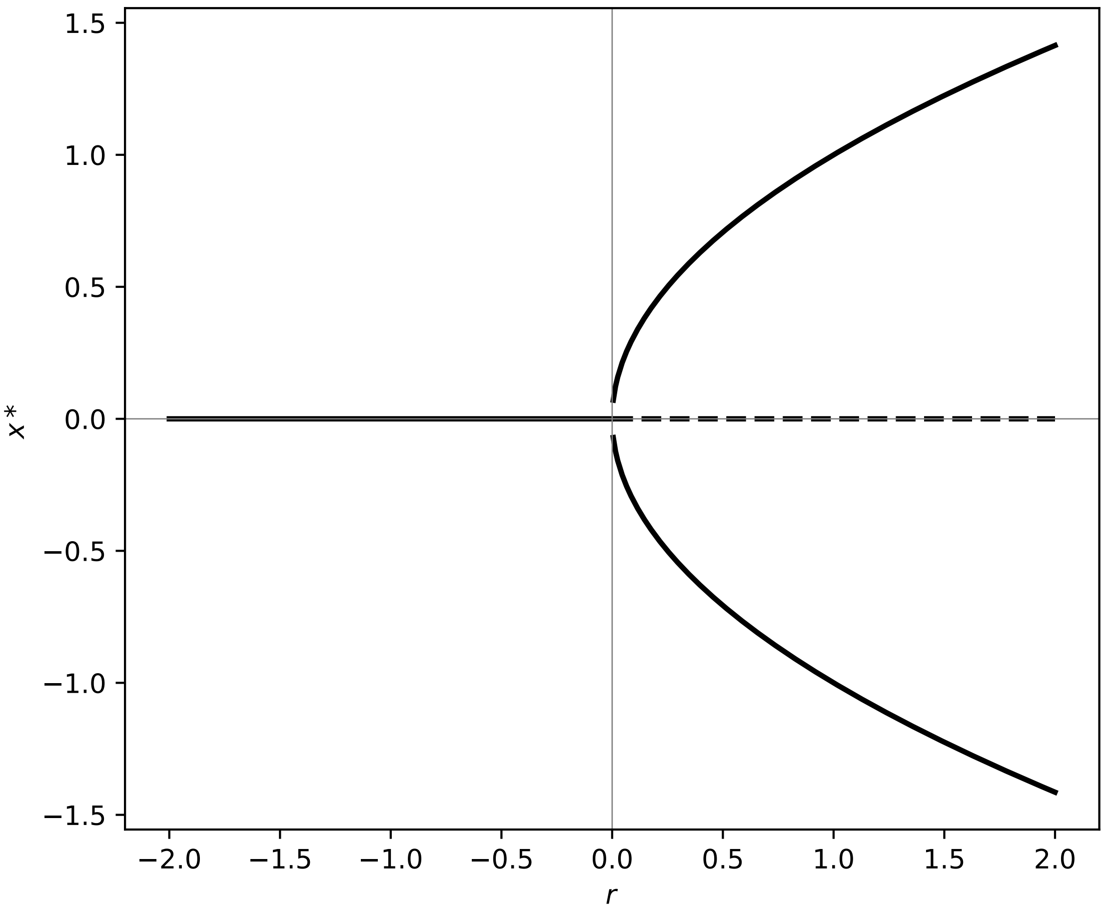

Figure 3.6

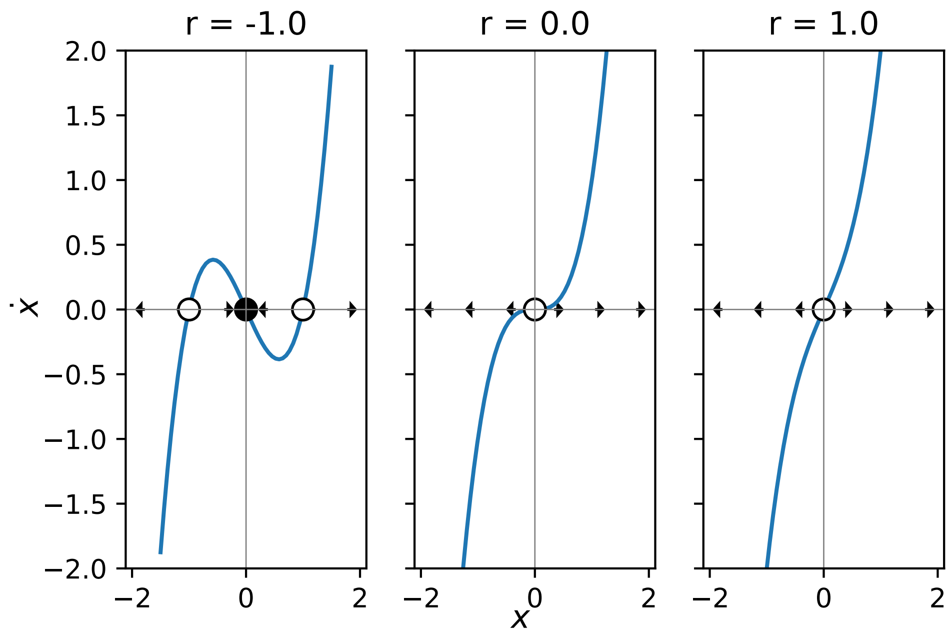

Figure 3.7

r_B is the linear growth rate, K_B is the carrying capacity, B is the maximum rate of predation and A is a measure of budworm population where predation switches on (specifically, A represents the budworm density at which predation is half its maximum value).

It is informative to graph the predation term

$$p(N) = \frac{BN^2}{A^2 + N^2},$$

and annotate the parameters A and B.

There is a root at $N = 0$. In the limit $N \to \infty$, $p \to B$. Note that $N = A$, $p = A/2$. The derivative is

$$p'(N) = \frac{2BN}{A^2 + N^2} - \frac{2BN^3}{(A^2 + N^2)^2} = \frac{2BA^2N}{(A^2 + N^2)^2}$$

Thus there is a turning point at $N = 0$ and $p' > 0 \forall N > 0$.

3.5.2 Nondimensionalisation

Introducing the (as yet unspecified) dimensional scalings $\tilde{N}$ and $\tilde{T}$, the model is nondimensionalised as follows

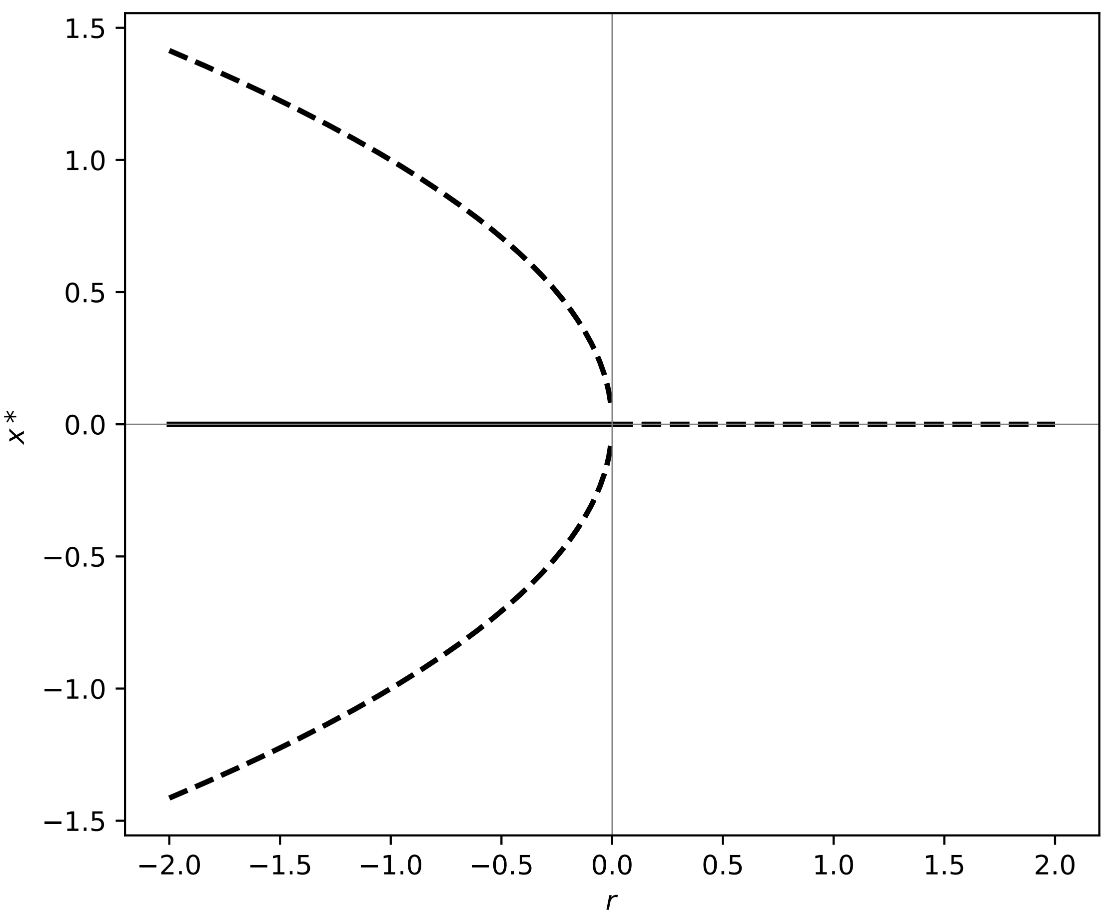

Figure 3.8

$$n = \frac{N}{\tilde{N}} \qquad \tau = \frac{t}{\tilde{T}}.$$

Changing variables in Equation 3.2 yields

$$\frac{\tilde{N}}{\tilde{T}}\frac{dn}{d\tau} = r_B \tilde{N} n \left(1 - \frac{\tilde{N} n}{K_B}\right) - \frac{B\tilde{N}^2 n^2}{A^2 + \tilde{N}^2 n^2}.$$

After some tidying

$$\frac{dn}{d\tau} = r_B \tilde{T} n \left(1 - \frac{\tilde{N} n}{K_B}\right) - \frac{B\tilde{N}\tilde{T}}{A^2}\frac{n^2}{1 + \frac{\tilde{N}^2}{A^2} n^2}.$$

A natural scale for cell density in the model is given by the parameter A, as it determines the density of budworm at which predation is half its maximal value. Hence we choose the scaling on the budworm density

$$\tilde{N} = A.$$

Similarly, a natural time scale for the model is given by

$$\tilde{T} = A/B.$$

Substituting for $\tilde{N}$ and $\tilde{T}$ yields

$$\frac{dn}{d\tau} = rn\left(1 - \frac{n}{q}\right) - \frac{n^2}{1+n^2} = H(n), \tag{3.3}$$

where we define the nondimensional parameters

$$r = \frac{r_B A}{B} \quad \text{and} \quad q = \frac{K_B}{A}.$$

Note that Equation 3.3 has two nondimensional parameters and all variables are dimensionless. See Figure 3.9 for a plot of right-hand side of equation Equation 3.3. What kind of behaviours do you expect to see from the model?

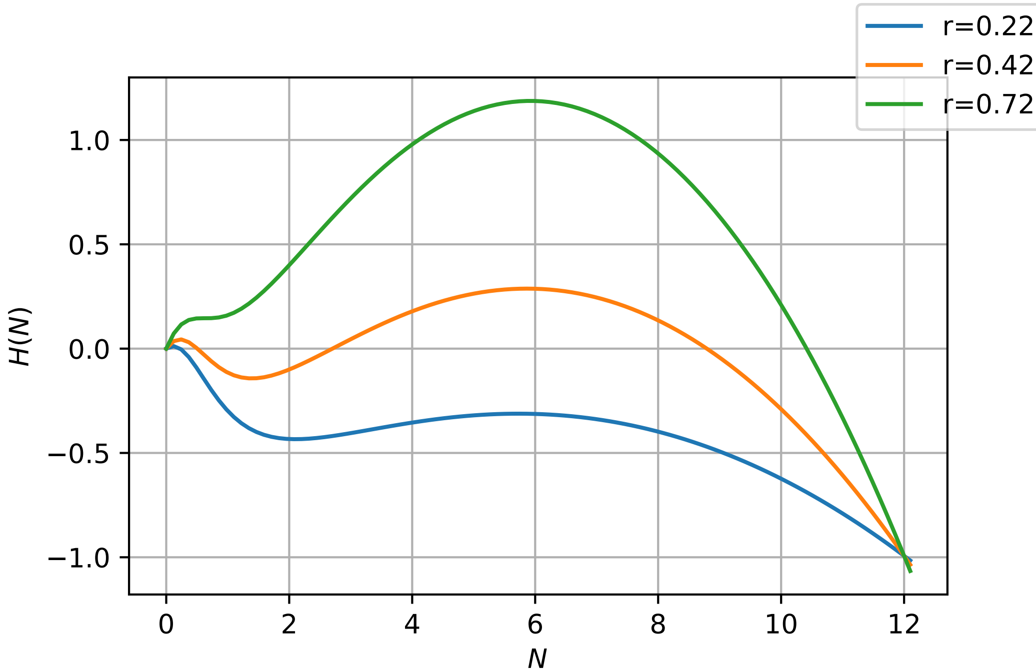

Figure 3.9: Rhs of the spruce budworm model.

3.5.3 Numerical solutions

In Figure 3.10 we plot some numerical solutions of Equation 3.3 at different values of the parameter r.

Numerical solutions of Equation 3.3 indicate that there is a single stable fixed point when r is both small and large but for intermediate values of r there are two stable fixed points.

Our goal is to analyse the model and understand why different parameter values yield these strikingly different model behaviours.

3.5.4 Fixed point analysis

Letting n^* represent fixed points of Equation 3.3 yields

$$rn^*(1-\frac{n^*}{q})-\frac{n^{*2}}{1+n^{*2}}=0.$$

Hence either

$$n^*=0,$$

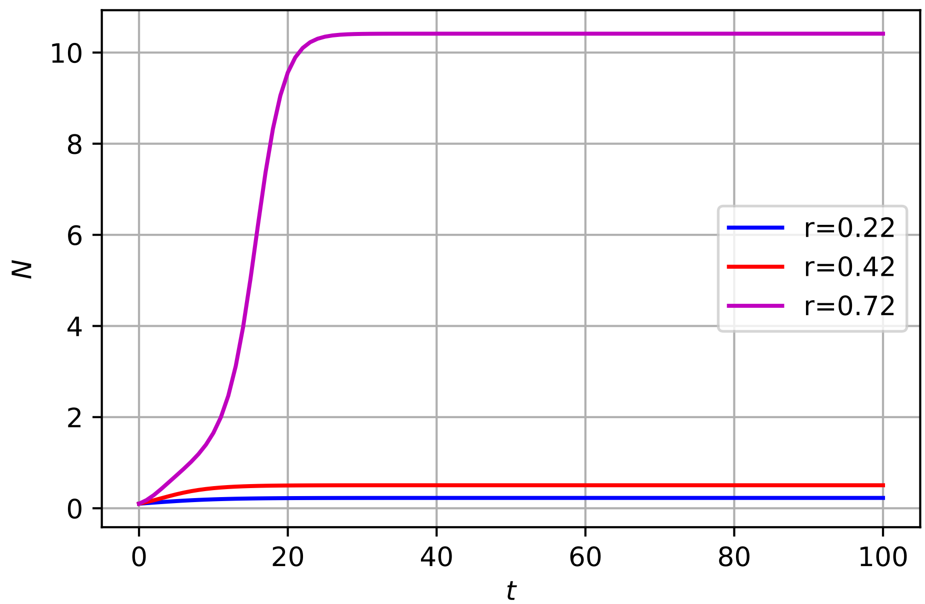

Figure 3.10: Numerical solution of spr. budworm model.

or n^* satisfies the cubic equation

$$r\left(1-\frac{n^*}{q}\right)-\frac{n^*}{1+n^{*2}}=0.$$

Explicit solutions to such a cubic can be immediately written down but they are cumbersome to work with. We proceed using a graphical/qualitative approach.

Define

$$f(n^*)=r\left(1-\frac{n^*}{q}\right) \quad \text{and} \quad g(n^*)=\frac{n^*}{1+n^{*2}}. \tag{3.4}$$

Roots occur for values of n^* that satisfy $f=g$.

In Figure 3.11 (a) we fix the parameter $q=10$ and consider model behaviour as a function of the parameter r. When $r\gg 1$ there is a nonzero steady-state corresponding to $n^*\gg 1$. When $r\ll 1$ there is a nonzero steady-state corresponding to $n^*\ll 1$. In the intermediate case there can be three intersection points.

We can use the curve sketching techniques form Tutorial Sheet 1 to sketch f and g.

f is linear. There is a root at $n^*=q$. The derivative is $-r/q$. $f(0)$=r. g has a unique root at $n^*=0$. The derivative is

$$g'=\frac{1-{n^*}^2}{(1+n^*)^2}.$$

There is a turning point at $n^* = 1$. Here $g = 1/2$. As $n^* \to \infty$, $g \to 0$. $f'(0) = 1$.

3.5.5 Linear stability analysis

The linear stability of the model is determined by the quantity

$$H'(n) = r(1 - \frac{2n}{q}) - \frac{2n}{1+n^2} + \frac{2n^3}{(1+n^2)^2}.$$

Hence at the fixed point $n^* = 0$

$$H'(0) = r$$

and the fixed point is linearly unstable.

Given the nonzero fixed points have not been calculated explicitly, we proceed using graphical analysis of stability. In Figure 3.11 (b) we plot the right-hand side of Equation 3.3 against n and examine the cases of large, small and intermediate r for a given value of q.

When r is both large and small the nonzero fixed point is stable (the derivative at the roots is negative). In the case where three biologically relevant roots exist, the intermediate root is unstable.

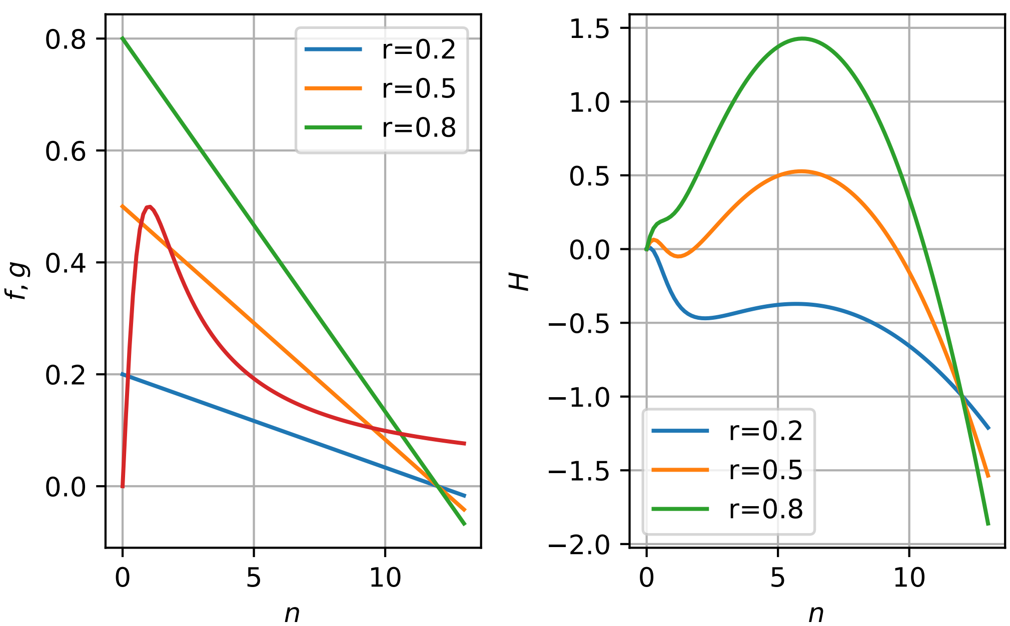

Figure 3.11: RHS of spr. budworm model.

3.5.6 Bifurcation analysis

The goal is to identify boundaries of rq parameter space where the stability changes occur and/or the number of fixed points changes.

At the transitions between different qualitative regimes pairs of fixed points can appear/disappear at critical values of r. In these cases one of the fixed points is linearly stable and the other linerly unstable. Hence these are saddle-node bifurcations.

We can define points in rq parameter space where bifurcations arise by seeking values of n^* that satisfy

$$f(n^*) = g(n^*) \qquad f'(n^*) = g'(n^*).$$

The first of these equations yields

$$r(1 - \frac{n^*}{q}) = \frac{n^*}{1 + {n^*}^2}, \tag{3.5}$$

and the latter yields

$$-\frac{r}{q} = \frac{1}{1 + {n^*}^2} - \frac{2{n^*}^2}{(1 + n^*)^2} = \frac{1 - {n^*}^2}{(1 + {n^*}^2)^2}. \tag{3.6}$$

Hence

$$\frac{r}{q} = \frac{{n^*}^2 - 1}{(1 + {n^*}^2)^2}.$$

Substituting for r/q in the first equation yields

$$r - \frac{{n^*}^2 - 1}{(1 + {n^*}^2)^2} n^* = \frac{n^*}{1 + {n^*}^2},$$

which can be written in the form

$$r = \frac{2{n^*}^3}{(1 + {n^*}^2)^2}.$$

Substituting for r in Equation 3.5 yields

$$\frac{2{n^*}^3}{(1 + {n^*}^2)^2} = \frac{2{n^*}^3}{(1 + {n^*}^2)^2} \frac{n^*}{q} + \frac{n^*}{1 + {n^*}^2}$$

which, after some algebra, yields

$$q = \frac{2{n^*}^3}{{n^*}^2 - 1}. \tag{3.7}$$

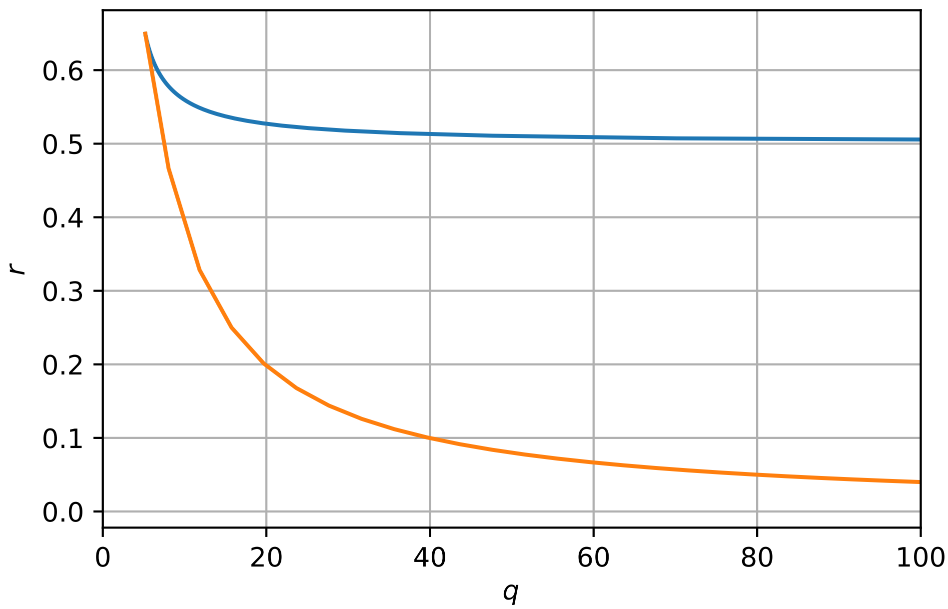

Figure 3.12: Bifurcations in the rq plane.

Hence a set of points that define bifurcations where three fixed points transform to a single fixed point are given in parametric form in qr parameter space by

$$\left(\frac{2{n^*}^3}{{n^*}^2-1}, \frac{2{n^*}^3}{(1+{n^*}^2)^2}\right) \qquad n^* > 1.$$

By varying values of n^* in Figure 3.12 we plot a region of instability. Note for example that $q \to \infty$ as $n^* \to 1$ and as $n^* \to \infty$. Note also that in these limits r must take the value $1/2$ and 0, respectively.

We can show that the cusp in Figure 3.12 is given by

$$(q, r) = \left(3^{\frac{3}{2}}, \left(\frac{\sqrt{3}}{2}\right)^3\right).$$

Note that r is a decreasing function of q for $1 < n^* < \sqrt{3}$ but that r is an increasing function of n^* for $\sqrt{3} < n^* < \infty$. This can be shown by finding the turning points of r w.r.t n^*, i.e. Given that

$$r = \frac{2{n^*}^3}{(1+{n^*}^2)^2},$$

differentiation with respect to n^* yields

$$\frac{dr}{dn^*} = \frac{6{n^*}^2}{(1+n^*)^2} - \frac{8{n^*}^4}{(1+n^*)^3}.$$

The turning point satisfies

$$\frac{6{n^*}^2}{(1+n^*)^2} - \frac{8{n^*}^4}{(1+n^*)^3} = 0$$

Solving for n^* yields

$$6(1+{n^*}^2) - 8{n^*}^2 = 0$$

Hence

$$n^* = \sqrt{3}.$$

Thus there is a turning point that minimises r at $n^* = \sqrt{3}$.

Substitution for this value of n^* yields

$$(q, r) = \left(3^{\frac{3}{2}}, \left(\frac{\sqrt{3}}{2}\right)^3\right).$$

See Figure 3.12.

3.5.7 Hysteresis

Finally, rearranging the steady-state equation

$$r(1 - \frac{n^*}{q}) = \frac{n^*}{1+{n^*}^2},$$

we obtain

$$r = \frac{n^*}{(1+{n^*}^2)(1-\frac{n^*}{q})}. \tag{3.8}$$

Considering $n^* < q$ we compute r; plotting n^* against r yields the bifurcation curve presented in Figure 3.13.

A system exhibiting hysteresis shows a response to an increases in a control parameter that is not exactly reversed when the parameter is decreased. We can show that the spruce budworm model exhibits hysteresis by considering the argument below.

Suppose r is initially small ($r < r_1$). There is only one steady-state and any initial condition will converge towards it.

Suppose we increase the value of the parameter r. There will be a critical value of r ($r = r_1$)where a second stable steady-state arises and the model enters the bistable regime, where

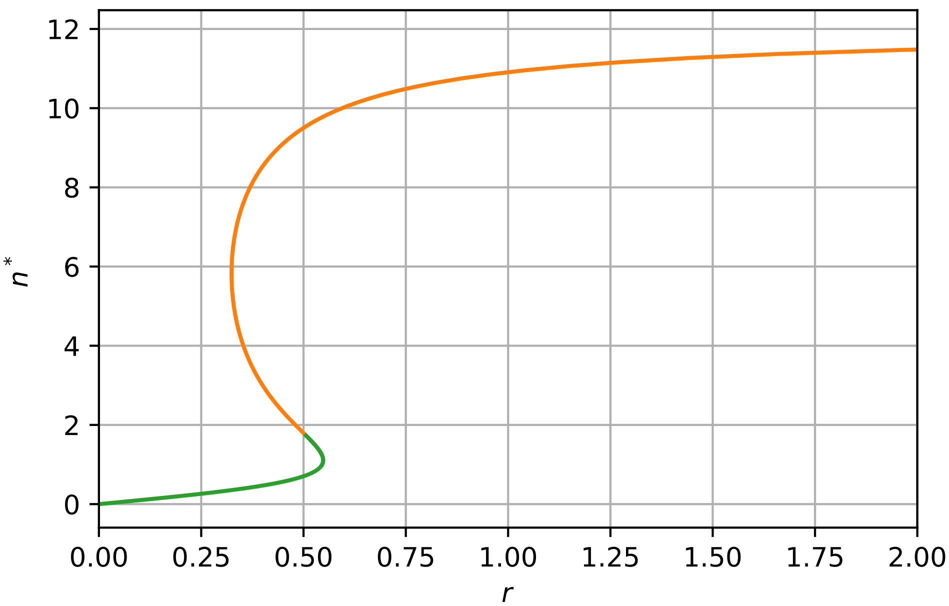

Figure 3.13: Bifurcations in the rq plane.

there are two possible stable steady-states. Given the system system was originally in the first stable fixed point, it will remain there.

Suppose we continue to increase r. Eventually we reach critical value of r ($r = r_2$) where the first stable fixed point is lost and there is again only one stable steady-state.

Suppose we now decrease the parameter r below the threshold $r = r_2$. The system again enters the bistable regime but as the second solution is stable, it remains the solution.

Suppose we continue to decrease r until eventually $r < r_1$. We return to the case where the system has only a single stable fixed point.

3.6 References

Part II

2D flows

4 Linear systems

4.1 Introduction

A 2D linear system has two dependent variables with coupled dynamics. You will have met this system before in the Differential Equations module.

Consider

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

In matrix form

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{4.1}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

4.2 A solution for linear problems

4.2.1 Linear superposition

It is straightforward to show that if $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions, the linear sum

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t),$$

where c_1 and c_2 are arbitrary constants, is itself a solution.

4.2.2 A general solution

Seeking a solution of Equation 5.2 of the form

$$\mathbf{w} = \mathbf{v}e^{\lambda t}$$

one obtains the characteristic equation

$$\lambda^2 - \lambda \mathrm{tr}(A) + \det(A) = 0,$$

which has solutions

$$\lambda = \frac{\mathrm{tr}A \pm \sqrt{\mathrm{tr}A^2 - 4\det A}}{2}.$$

Given distinct eigenvalues, the general solution is

$$\mathbf{x}(t) = C_1\mathbf{v}_1e^{\lambda_1 t} + C_2\mathbf{v}_2e^{\lambda_2 t}. \tag{4.2}$$

The behaviour of the solutions of a 2D linear system of ODEs can be classified given the knowledge of the eigenvectors/eigenvalues.

4.3 Fixed points and their stability

4.3.1 Fixed points

Fixed points of Equation 5.2 satisfy

$$A\mathbf{x} = \mathbf{0}. \tag{4.3}$$

The zero vector

$$\mathbf{x}^* = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is always a fixed point of Equation 4.3.

If A is invertible then the zero vector is a unique fixed point of Equation 5.2.

If A is not invertible there exists an infinite family of fixed points (line or plane).

4.3.2 Stability

4.3.3 Asymptotic stability

Given the general solution

$$\mathbf{x}(t) = C_1 \mathbf{v}_1 e^{\lambda_1 t} + C_2 \mathbf{v}_2 e^{\lambda_2 t},$$

the fixed point (0,0) is *asymptotically* stable if

$$\Re\{\lambda_i\} < 0 \ \forall \ i.$$

4.3.3.1 Classification via the trace-determinant plane

The stability of a linear 2D system can be determined by calculating determinant and trace of A and referring to Figure 4.1.

The different cases can be categorised as follows:

- $\det(A) < 0$ There is one positive and one negative real eigenvalue. Hence the fixed point is a saddle which is unstable.
- $\det(A) > 0$ The fixed point can be either stable or unstable, depending on $\mathrm{tr}(A)$ (and the real part of the eigenvalues).
  - If $\mathrm{tr}(A) > 0$, the fixed point is unstable.
    * If $\mathrm{tr}(A)^2 > 4\det(A)$, it is an unstable node.
    * If $\mathrm{tr}(A)^2 < 4\det(A)$ it is an unstable spiral.
  - If $\mathrm{tr}(A) < 0$, the fixed point is stable.
    * If $\mathrm{tr}(A)^2 > 4\det(A)$, it is an stable node.
    * If $\mathrm{tr}(A)^2 < 4\det(A)$, it is an stable spiral.
  - If $\mathrm{tr}(A) = 0$ the fixed point is a centre.

These different cases can be distinguished in the trace-determinant plane plotted in Figure 4.1.

Note the following:

- The condition $\det(A) > 0$ excludes the case that the eigenvalues are real but have opposite signs (i.e. it cannot be a saddle point).
- The condition $\mathrm{tr}(A) > 0$ implies that the real part of both eigenvalues are positive (i.e. the fixed point is unstable).

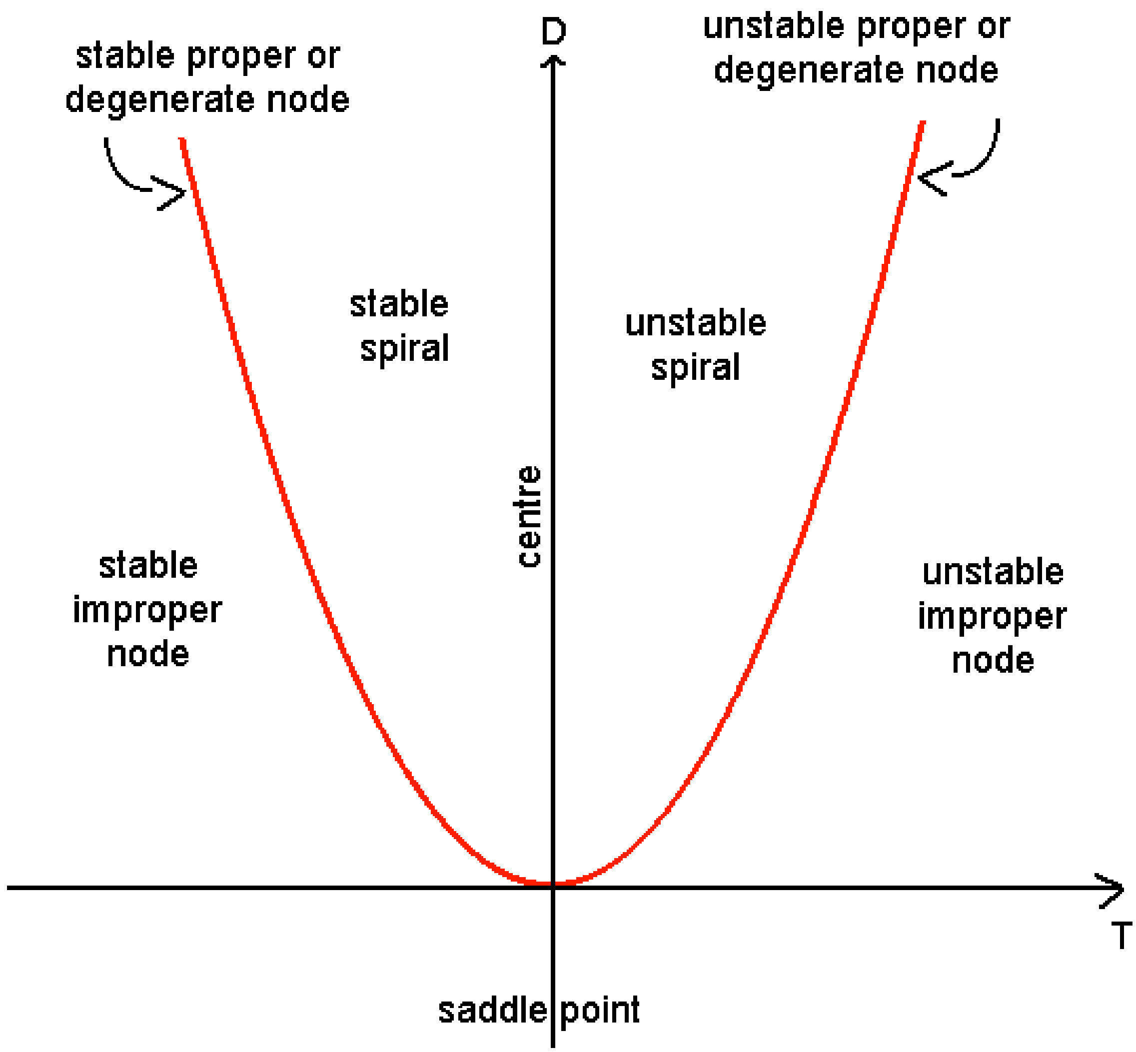

Figure 4.1: Stability in the trace determinant plane.

- The condition $\mathrm{tr}(A)^2 < 4\det(A)$ implies that the eigenvalues are complex. Hence

  $$\lambda_{\pm} = \mu \pm i\omega$$

  and the solution of the system can be written

  $$e^{\lambda t} = e^{(\mu + i\omega)t} = e^{\mu t} e^{i\omega t}.$$

  For $\mu > 0$ the magnitude of the perturbation grows but it oscillates about the fixed point. Hence the fixed point is an unstable spiral.

4.3.3.2 Introducing different types of stability

In 2D system we need to make a more nuanced description of stability:

- an *attracting* fixed point - all trajectories that start close to a fixed point approach it as $t \to \infty$.
- *globally attracting* - all trajectories tend to the fixed point as $t \to \infty$,
- *Liapunov stable* - all trajectories that get close to a fixed point stay close to it for all time.
- *neutrally stable* - a fixed point that is Liapunov stable but not attracting
- *asymptotically stable* - a fixed point that is Liapunov stable and attracting.

4.3.3.3 Stable and unstable manifolds

A *stable manifold* of a fixed point, $\mathbf{x}^*$, is a set of points such that

$$\{\mathbf{s} : \mathbf{x}(t) \to \ \mathbf{x}^* \text{ as } t \to \infty \text{ for } \mathbf{x}(0) = \mathbf{s}\}$$

An *unstable manifold* of the fixed point is the set of points such that

$$\{\mathbf{u} : \mathbf{x}(t) \to \ \mathbf{x}^* \text{ as } t \to -\infty \text{ for } \mathbf{x}(0) = \mathbf{u}\}$$

4.4 An example problem introducing phase portraits and stability

Let's consider a decoupled 2D problem given by

$$\begin{aligned} \dot{x} &= ax \\ \dot{y} &= -y, \end{aligned}$$

where $a \in \mathfrak{R}$. We will use this system to introduce some important concepts regarding phase portraits and stability.

The solution is

$$x(t) = x_0 e^{at}, \quad y(t) = y_0 e^{-t}.$$

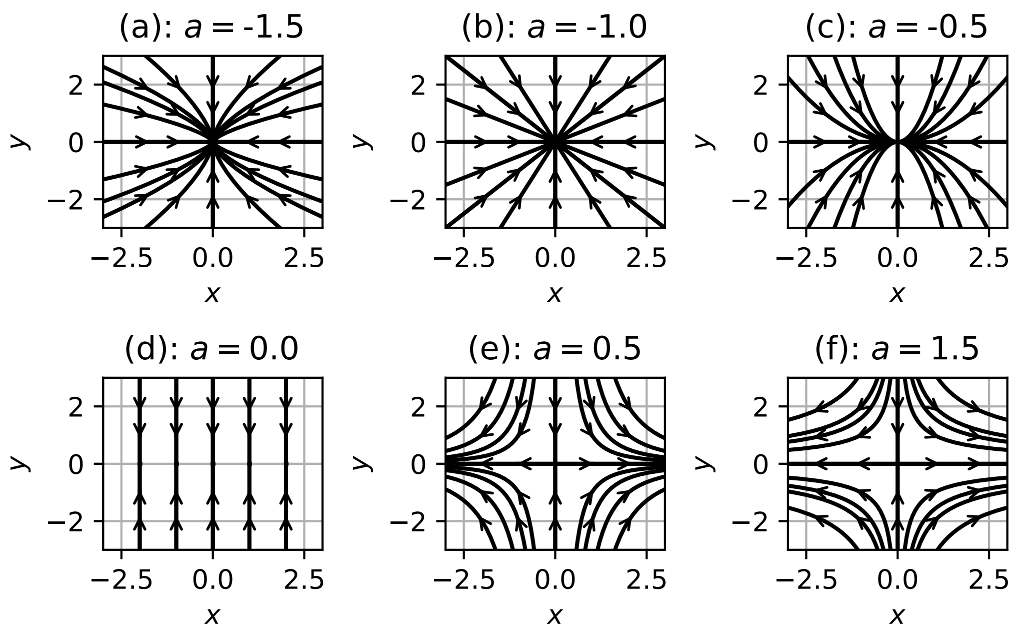

Figure 4.2

In Figure 4.2 we cna make the following classifications:

(a) stable node. As $a < 1$ the solutions decay faster along the x axis.
(b) star node - the eigenvalues are equal.
(c) stable node - not $-1 < a < 0$ so the solution decays faster along the y axis.
(d) infinite family of fixed points.
(e) saddle - one stable eigenvector and one unstable eigenvector.

(a-c) The fixed point is globally attracting. The stable manifold is

$$\{s : s \in \mathfrak{R}^2\}.$$

(d) fixed points are neutrally stable (e,f)

- unstable fixed point
- stable manifold

$$\{(x, y) : x = 0, \ \forall y\}.$$

- unstable manifold

$$\{(x, y) : y = 0 \ \forall x\}.$$

4.5 Jordan form

Consider the system of linear ODEs given by

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0,$$

where A is a $n \times n$ matrix.

Consider a change of coordinates such that

$$\mathbf{x} = P\mathbf{y}$$

where P is an $n \times n$ invertible matrix ($\det P \neq 0$). Then

$$\dot{\mathbf{x}} = A\mathbf{x}$$

implies

$$\dot{y} = P^{-1}\dot{x} = P^{-1}A\mathbf{x} = P^{-1}AP\mathbf{y} = \Gamma y$$

where

$$\Gamma = P^{-1}AP,$$

and the initial condition transforms to

$$\mathbf{y}_0 = P^{-1}\mathbf{x}_0.$$

In these new coordinates the solution is

$$\mathbf{y}(t) = e^{t\Gamma}\mathbf{y}_0.$$

Transforming back to original coordinates

$$\mathbf{x}(t) = Py(t) = Pe^{t\Gamma}\mathbf{y}_0 = Pe^{t\Gamma}P^{-1}\mathbf{x}_0.$$

Suppose that A has n distinct eigenvalues (λ_1,λ_2,...,λ_n) with associated eigenvectors ($\mathbf{e}_1$,$\mathbf{e}_2$,...,$\mathbf{e}_n$). Let P be the matrix with eigenvectors as columns, i.e.

$$P = [\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n]$$

Hence $\Gamma = P^{-1}AP$ is a diagonal matrix with eigenvalues along the diagonal axis. The component solution has the form

$$y_i(t) = y_{0i}e^{\lambda_i t}.$$

5 2D nonlinear systems

5.1 Introduction

Consider the system of nonlinear ODEs given by

$$\begin{aligned}\dot{x} &= f(x,y),\\ \dot{y} &= g(x,y),\end{aligned} \tag{5.1}$$

where f and g are smooth functions of x and y.

In vector form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

where $\mathbf{x} = (x,y)^T$ and $\mathbf{f} = (f,g)^T$.

Here $\mathbf{x}$ represents a point in the phase plane, $\dot{\mathbf{x}}$ represents the velocity. By *flowing* along the vector field, a *phase point* traces out a *trajectory* $\mathbf{x}(t)$. As each point in the plane can be an initial condition, the plane is filled with trajectories.

5.2 Existence, uniqueness and topological consequences

Theorem 1

Consider the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0,$$

Suppose that $\mathbf{f}$ is continuous and that all partial derivatives exist and are continuous in some open connected set $D \subset \mathbf{R}^n$. For $\mathbf{x}_0 \in D$, the IVP has a unique solution in some time interval $t \in (\tau, \tau)$.

Corollary 2

Different trajectories never intersect.

Corollary 3

In 2D, trajectories initialised inside a closed orbit remain trapped inside it for all time.

5.3 Fixed points and linearisation

5.3.1 Fixed points

(x^*, y^*) is defined to be a fixed point of Equation 5.1 if

$$f(x^*, y^*) = g(x^*, y^*) = 0.$$

Hence, by definition, the time derivatives of $x(t)$ and $y(t)$ are both zero at (x^*, y^*). As was the case for single species models, fixed points are obtained by solving algebraic equations.

5.3.2 Linear stability analysis

Suppose that (x^*, y^*) is a fixed point of equations Equation 5.1.

We consider a change of dependent variables such that

$$x(t) = x^* + \hat{x}(t) \quad \text{and} \quad y(t) = y^* + \hat{y}(t),$$

where $\hat{x}(t)$ and $\hat{y}(t)$ are perturbations about the fixed point.

Rewriting equation Equation 5.1 in the transformed variables yields

$$\begin{aligned}
\frac{d\hat{x}}{dt} &= f(x^* + \hat{x}, y^* + \hat{y}), \\
\frac{d\hat{y}}{dt} &= g(x^* + \hat{x}, y^* + \hat{y}).
\end{aligned}$$

Making Taylor expansions about (x^*, y^*) yields the linearised equations

$$\begin{aligned}
\frac{d\hat{x}}{dt} &= \frac{\partial f}{\partial x}_{|(x^*,y^*)} \hat{x} + \frac{\partial f}{\partial y}_{|(x^*,y^*)} \hat{y} + h.o.t, \\
\frac{d\hat{y}}{dt} &= \frac{\partial g}{\partial x}_{|(x^*,y^*)} \hat{x} + \frac{\partial g}{\partial y}_{|(x^*,y^*)} \hat{y} + h.o.t.
\end{aligned}$$

Defining

$$\mathbf{w} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix},$$

yields the matrix form

$$\frac{d\mathbf{w}}{dt} = A\mathbf{w}, +h.o.t \tag{5.2}$$

where the matrix A, known as the Jacobian matrix, takes the form

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*,y^*)}.$$

In the neighbourhood of fixed points, we expect the higher order terms to become vanishingly small. If we are willing to neglect the higher order terms then close to the fixed points we can describe the dynamics the dynamics using the linearised system

$$\frac{d\mathbf{w}}{dt} = A\mathbf{w}, \tag{5.3}$$

5.4 Phase portraits

For nonlinear problems it is usually impossible to find trajectories analytically. Instead, our goal is to describe the qualitative behaviour of trajectories. Some important features of phase portraits are:

- fixed points
- close orbits
- behaviour of trajectories close to fixed points
- the stability of any fixed points and closed orbits
- the vector field
- nullclines

We can often sketch solutions in the phase plane using the above information.

Example 5.1. Plot a phase portrait for the system:

$$\dot{x} = x + e^{-y}$$
$$\dot{y} = -y$$

Strategy: - fixed points and their linear stability - nullclines - derivative/vector field - sample trajectories

5.5 Neglecting nonlinear terms?

Definition 4

A fixed point of an n^{th} order system is hyperbolic if all the eigenvalues of the linear system lie off the imaginary axis (i.e. $\Re(\lambda_i) \neq 0$
$for all i$).

Theorem 5

Hartman-Grobman
The local phase portrait near a hyperbolic fixed point is topologically equivalent (i.e. homeomorphic) to the phase portrait of the linearisation.

Definition 6

Structural stability
A phase portrait is structurally stable if its topology cannot be changed by an arbitrarily small change to the vector field.

Example 5.2. Consider the system

$$\dot{x} = -y + ax(x^2 + y^2)$$
$$\dot{y} = x + ay(x^2 + y^2)$$

where $a \in \Re$. Show that the linearised system incorrectly predicts that the origin is a centre. By transforming to polar coordinates show that the system is actually either a stable/unspiral for $a \neq 0$.

5.6 Competition

In models of competition, two or more species compete for the same resource or in some way inhibit each other's growth. Letting $N_1(t)$ and $N_2(t)$ represent the population density of two species, we consider the ODEs

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_1}\right),$$
$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2}{K_2} - b_{21}\frac{N_1}{K_2}\right),$$

where r_1, r_2, K_1 and K_2 are positive constants. As before, the $r's$ are linear growth rates and the K's are carrying capacities. The parameters b_{12} and b_{21} measure the competitive effect of N_2 on N_1 and N_1 on N_2, respectively.

5.6.1 Nondimensionalisation

After nondimensionalising using the change of variables

$$n_1 = \frac{N_1}{K_1} \quad n_2 = \frac{N_2}{K_2} \quad \tau = \frac{t}{\frac{1}{r_1}},$$

we obtain the equations

$$\begin{aligned} \frac{dn_1}{d\tau} &= n_1 \left(1 - n_1 - a_{12} n_2\right) = f(n_1, n_2), \\ \frac{dn_2}{d\tau} &= \rho n_2 \left(1 - n_2 - a_{21} n_1\right) = g(n_1, n_2), \end{aligned}$$

where

$$\rho = \frac{r_2}{r_1}, \quad a_{12} = b_{12}\frac{K_2}{K_1}, \quad a_{21} = b_{21}\frac{K_1}{K_2}.$$

5.6.2 Fixed points

The fixed points of Equation 5.4 are identified in the usual manner, i.e. by seeking $({n_1}^*, {n_2}^*)$ such that

$$f({n_1}^*, {n_2}^*) = g({n_1}^*, {n_2}^*) = 0.$$

The fixed point equations are

$${n_1}^* \left(1 - {n_1}^* - a_{12}{n_2}^*\right) = 0 \qquad {n_2}^* \left(1 - {n_2}^* - a_{21}{n_1}^*\right) = 0.$$

The first equation has solution

$$n_1^* = 0$$

or

$$\left(1 - {n_1}^* - a_{12}{n_2}^*\right) \implies n_2 = \frac{1}{a_{12}}(1 - n_1^*).$$

Consider $n_1^* = 0$. Substitution in the second equation yields

$${n_2}^* \left(1 - {n_2}^*\right) = 0.$$

Hence either $n_2^* = 0$ or $n_2^* = 1$. Hence two steady states are $(0, 0)$ and $(0, 1)$.

Now consider $n_2^* = \frac{1}{a_{12}}(1 - n_1^*)$ with $n_1^* \neq 0$.

Substitution in the second fixed point equation yields

$$\frac{1}{a_{12}}(1-n_1^*)\left(1-\frac{1}{a_{12}}(1-n_1^*)-a_{21}{n_1}^*\right)$$

Hence either $n_1^* = 1$ or

$$\left(1-\frac{1}{a_{12}}(1-n_1^*)-a_{21}{n_1}^*\right)=0 \implies n_1^*=\frac{1-a_{12}}{1-a_{12}a_{21}}.$$

In the case where $n_1^* = 1$, we find that $n_2^* = 0$. Hence the fixed point is (1,0).

In the case where

$$n_1^*=\frac{1-a_{12}}{1-a_{12}a_{21}}$$

we find that

$$n_2^*=\frac{1-a_{21}}{1-a_{12}a_{21}}$$

Hence the fixed point is

$$\left(\frac{1-a_{12}}{1-a_{12}a_{21}},\frac{1-a_{21}}{1-a_{12}a_{21}}\right).$$

5.6.3 Nullclines

The nullclines for Equation 5.4 are straight lines given by

$$n_1 = 0 \qquad n_2=\frac{1-n_1}{a_{12}},$$

and

$$n_2 = 0 \qquad n_2 = 1 - a_{21}n_1.$$

Note that the fixed points $(0,0)$, $(1,0)$ and $(0,1)$ are always biologically relevant (i.e. independently of the parameter values for a_{12} and a_{21}).

However, the coexistence fixed point is only biologically relevant if the nullclines intersect in the positive quadrant and this occurs only in certain regions of the model's parameter space.

In the cases where $a_{12}, a_{21} < 1$ and $a_{12}, a_{21} > 1$ there is a coexistence fixed point (i.e. the nullclines intersect in the positive quadrant).

However, if $a_{21} < 1$ and $a_{12} > 1$ or $a_{12} < 1$ and $a_{21} > 1$ there is not a biologically relevant, coexistence fixed point (i.e. the nullclines do not intersect in the positive quadrant).

Hence there are are four qualitatively different types of solution to consider.

5.6.4 Linear stability

The linear stability of the different fixed points is determined by calculating the Jacobian matrix

$$A = \begin{pmatrix} \frac{\partial f}{\partial n_1} & \frac{\partial f}{\partial n_2} \\ \frac{\partial g}{\partial n_1} & \frac{\partial g}{\partial n_2} \end{pmatrix}_{({n_1}^*,{n_2}^*)} = \begin{pmatrix} 1-2n_1-a_{12}n_2 & -a_{12}n_1 \\ -\rho a_{21}n_2 & \rho(1-2n_2-a_{21}n_1) \end{pmatrix}_{({n_1}^*,{n_2}^*)}.$$

At (0,0)

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}.$$

Hence the eigenvalues of the Jacobian are 1 and ρ. As $\rho > 0$, the origin is therefore an unstable node (there are two real positive eigenvalues).

At (1,0)

$$A = \begin{pmatrix} -1 & -a_{12} \\ 0 & \rho(1-a_{21}) \end{pmatrix}.$$

The trace and determinant are given by

$$\det A = \rho(a_{21}-1) \quad \text{and} \quad \mathrm{tr}A = -1+\rho(1-a_{21}).$$

Hence if $a_{21} < 1$, $\det A < 0$ and (1,0) is a saddle point and thus unstable (see Figure 4.1).

If $a_{21} > 1$, $\det A > 0$ and $\mathrm{tr}A < 0$. Hence (1,0) is a stable node.

Hence the parameter a_{21}, which describes how strongly Population 1 inhibits the growth rate of Population 2, determines whether or not the fixed point representing extinction of Population 2 but not Population 1 is stable or not.

At (0,1)

$$A = \begin{pmatrix} 1-a_{12} & 0 \\ -\rho a_{21} & -\rho \end{pmatrix}.$$

In this case if $a_{12} < 1$, $\det A < 0$ and (0,1) is a saddle point. If $a_{12} > 1$, $\det A > 0$ and $\mathrm{tr}A < 0$ and (0,1) is a stable node. Hence the parameter a_{12}, which describes how strongly Population 2 inhibits the growth rate of Population 1, determines whether or not the fixed point representing extinction of Population 1 but not Population 2 is stable or not.

At the coexistence fixed point, recall the fixed point is

$$\left(\frac{1-a_{12}}{1-a_{12}a_{21}}, \frac{1-a_{21}}{1-a_{12}a_{21}}\right).$$

Note that this fixed point is only biologically relevant in the cases $a_{21} < 1, a_{12} < 1$ or $a_{21} > 1, a_{12} > 1$. Evaluating the Jacobian yields

$$A = \frac{1}{1-a_{12}a_{21}} \begin{pmatrix} a_{12}-1 & -a_{12}(1-a_{12}) \\ -\rho a_{21}(1-a_{21}) & \rho(a_{21}-1) \end{pmatrix}.$$

The determinant and trace of the Jacobian are given by

$$\begin{aligned} \det A &= \rho\left((a_{12}-1)(a_{21}-1) - a_{12}a_{21}(1-a_{12})(1-a_{21})\right)\frac{1}{(1-a_{12}a_{21})^2}, \\ &= \rho\frac{(a_{12}-1)(a_{21}-1)}{(1-a_{12}a_{21})}, \end{aligned}$$

and

$$\mathrm{tr}A = (a_{12}-1+\rho(a_{21}-1))\frac{1}{1-a_{12}a_{21}},$$

respectively.

Let's firstly consider the case where $a_{21} < 1$ and $a_{21} < 1$. This implies that $a_{21}-1 < 0$ and $a_{12}-1 < 0$, hence evaluating the signs of the different products yields

$$\det A = \rho(-)(-)(+) > 0$$

and

$$\mathrm{tr}A = \rho(-) + (-) < 0.$$

Therefore the coexistence fixed point is a stable node or spiral.

In the case where $a_{21} > 1$ and $a_{12} > 1$

$$\det A = \rho(+)(+)(-) < 0.$$

Hence the coexistence fixed point is a saddle.

5.6.5 Phase portrait

See Figure 5.2 for phase portraits of three of the four cases that we have considered. It is expected that you can sketch phase portraits. Key details to consider are the fixed points and their linear stability. You should also sketch the nullclines and depict the sign of the derivatives in the phase plane on either side of the nullclines. You should also sketch one or more sample trajectories.

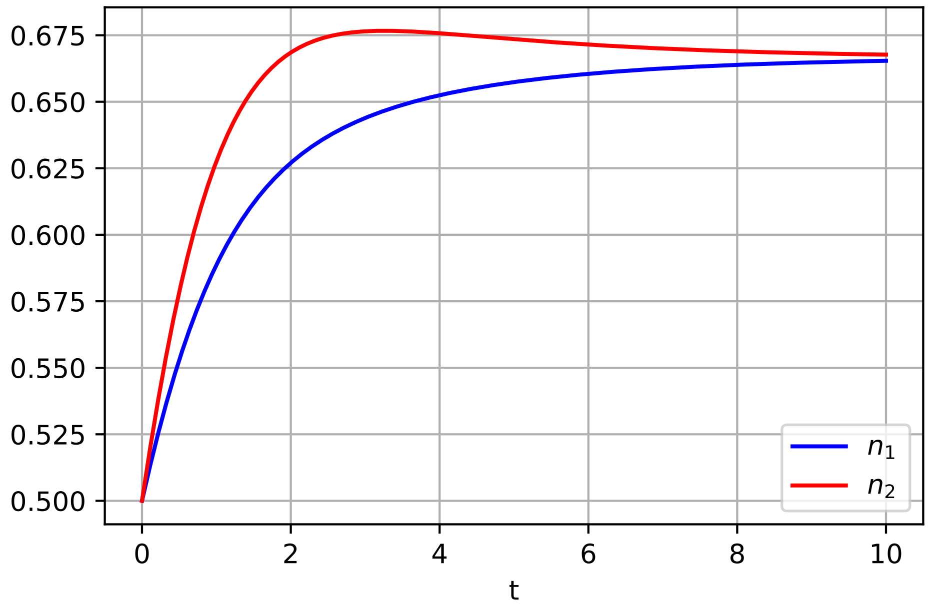

Figure 5.1: Numerical solutions of competition model

5.6.6 Insight

The model therefore has four qualitatively different behaviours that are described as follows: Consider the case where $a_{21} > 1$. This represents the case of Population 1 strongly competing with Population 2.

- If $a_{12} > 1$, Population 2 also strongly competes with Population 1. In this case, there are four biologically relevant fixed points, two of which are stable (1,0) and (0,1). The coexistence fixed point is a saddle and thus unstable. The model is bistable and the initial conditions determine whether solutions end up at (1,0) or (0,1) (see Figure 5.2). The biological interpretation of this solution is that one species will always win and completely outcompete the other. Even if the two populations are equal ($K_1 = K_2$ and $a_{21} = a_{12} > 1$), one species will always win and the other will become extinct.

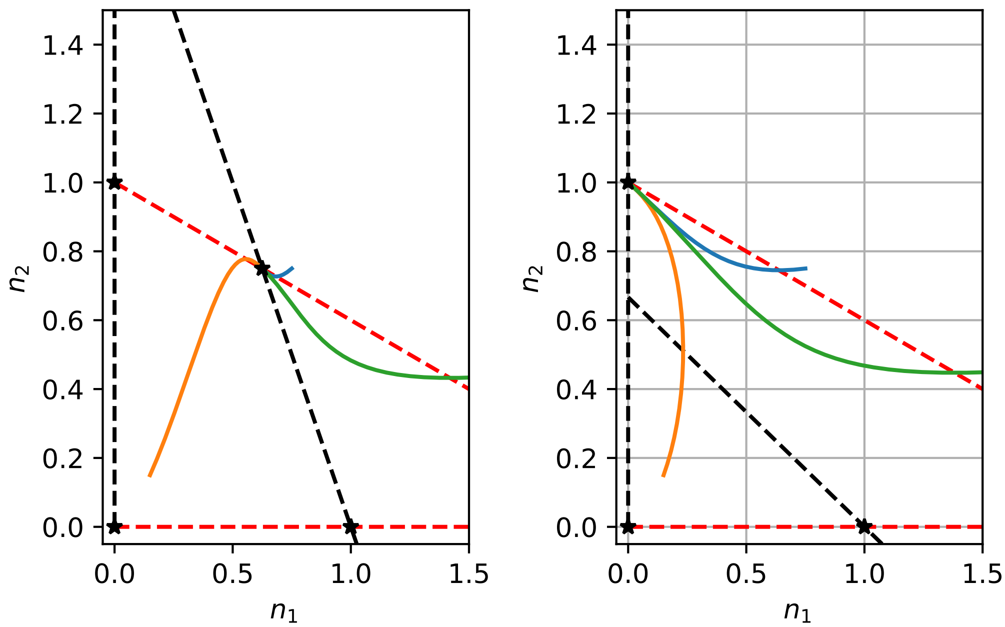

Figure 5.2: Numerical solution of the competition model

- If $a_{12} < 1$, Population 2 weakly competes with Population 1. There is no coexistence fixed point and the only stable fixed point is (1,0). Hence Population 1 always wins and Population 2 always becomes extinct.

Now consider the case where $a_{21} < 1$. This represents the case of Population 1 weakly competing with Population 2.

- If $a_{12} > 1$, Population 2 strongly competes with Population 1. There is nonexistence fixed point and the only stable fixed point is (0,1). Hence Population 2 always wins and Population 1 always becomes extinct.
- If $a_{12} < 1$, Population 2 also weakly competes with Population 1. The coexistence fixed point is stable and the fixed points (1,0) and (0,1) are unstable.

Example 5.3. Consider a particle of mass 1 moving in double well potential

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4.$$

The equation of motion is

$$m\ddot{x} = -\frac{dV}{dx}.$$

Find and classify all the fixed points for the system. Plot the phase portrait and provide an interpretation for the results.

Part III

Appendices

6 Python

6.1 Symbolic calculations

Symbolic calculations ahve been performed using the Python library Sympy.

This library comes with tutorials.

You are encouraged to familiarise yourself with the syntax by working through some of the tutorial examples provided at the links above.

Many of the calculations that we do throughout the course involve solving systems of algebraic equations

6.2 Numerical solution of difference equations

Difference equations have been solved using a for loop. Routinse have been written to solve either single or coupled system of difference equaitons.

6.3 Numerical integration of ODEs

Throughout the notes systems of ODEs have been integrated using the Scipy function odeint.

6.4 Plotting

Line graphs are plotted using the Python library Matplotlib.

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