# Lecture slides

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# Lecture 1

- Introduction to MA42002
- Conservation equations
- Examples of spatially homogeneous models

# Conservation equations

$$\begin{pmatrix} \text{rate of change} \\ \text{in the population density} \end{pmatrix} = (\text{spatial movement}) + \begin{pmatrix} \text{birth, growth, death,} \\ \text{production or degradation} \\ \text{due to chemical reactions} \end{pmatrix}$$

# Spatially homogeneous models (MA32009 revision)

# Example problem - bacteria in a dish

$$N(t+\Delta t) = \quad N(t) + \quad KN(t)\Delta t.$$

# A model for cell growth under nutrient depletion

$$
\begin{aligned}
\frac{dN}{dt} &= K(c)N = \kappa c N, \\
\frac{dc}{dt} &= -\alpha \frac{dN}{dt} = -\alpha \kappa c N,
\end{aligned}
\tag{1}
$$

# Leading to the logistic growth equation

The last equation can be rewritten as

$$\frac{dN}{dt} = \rho N \, (1 - \frac{N}{B}) \qquad\qquad N(0) = N_0, \tag{2}$$

# Can also consider other biological processes

# Exercise

Consider a well mixed bio reactor.

A biologist cultures an initial cell population of size $N_0$ in the bioreactor for 72 h.

Cells undergo division with a period of 14 h.

Each cell produces a non-degradable waste product, $W$, at rate $k_1$.

When total waste levels exceed a threshold, $W^*$, cell division stops. Otherwise the cell population grows exponentially.

How many cells are there at the end of the experiment?

# Model development

**Model checklist**

1. Variables (dependent, indepedent ?)
2. Schematic diagram - what processes are being modelled?
3. Governing equations?
4. Define model parameters?
5. Initial conditions?

# Exercise solution

# Recap

- Is course layout clear
- Introduction to conservation equation
- Deriving spatially homogeneous models

# Lecture 2

- Continue example
- Introduce SIR model
- Introduce an activator inhibitor model
- Derive a conservation equation

# Exercise

Consider a well mixed bio reactor.

A biologist cultures an initial cell population of size $N_0$ in the bioreactor for 72 h.

Cells undergo division with a period of 14 h.

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When total waste levels exceed a threshold, $W^*$, cell division stops. Otherwise the cell population grows exponentially.

How many cells are there at the end of the experiment?

# The SIR model (used in Chapter 7)

Consider the SIR model equations:

$$\begin{aligned} \frac{dS}{dt} &= -rIS, \\ \frac{dI}{dt} &= rIS - aI, \\ \frac{dR}{dt} &= aI. \end{aligned}$$

What are the variables? What are the parameters?

Identify an expression for the reproduction number, $R_0$.

Hence explain why the condition $R_0 < 1$ is necessary to avoid an epidemic?

# SIR model Calculations

$$\frac{dS}{dt} = -rIS,$$
$$\frac{dI}{dt} = rIS - aI,$$
$$\frac{dR}{dt} = aI.$$

# An activator inhibitor model ((used in Chapter 6))

Consider the reaction schematic

$$2A + B \to A.$$

Assume that species A is produced at constant rate $k_1$ and degrades at rate $k_2$.

Assume that A is linearly degraded.

Write down governing ODEs.

# Activator-inhibitor model

Consider the ODEs

$$\frac{dA}{dt} = k_1 - k_2 A + k_3 A^2 B,$$
$$\frac{dB}{dt} = k_4 - k_3 A^2 B,$$

Identify the steady state of the ODEs.

# Spatiotemporal models

# Spatiotemporal models - derivation

Consider a spatial domain $V$. A conservation equation can be written either in terms of the mass or number of particles of a species as follows:

$$\begin{aligned}\begin{pmatrix}\text{rate of change of}\\ \text{number of particles}\\ \text{per unit time}\end{pmatrix} &= \begin{pmatrix}\text{rate of entry of}\\ \text{particles into } V\\ \text{per unit time}\end{pmatrix} - \begin{pmatrix}\text{rate of exit of}\\ \text{particles from } V\\ \text{per unit time}\end{pmatrix}\\ &+ \begin{pmatrix}\text{rate of degradation}\\ \text{or creation of particles}\\ \text{in } V \text{ per unit time}\end{pmatrix}\end{aligned}$$

# Deriving a conservation equation in 1D

$$\frac{\partial}{\partial t}\int_x^{x+\Delta x} c(\tilde{x},t)A d\tilde{x} = J(x,t)\,A - J(x+\Delta x,t)\,A + \int_x^{x+\Delta x} f(\tilde{x},t,c(\tilde{x},t))\,A\,d\tilde{x}. \tag{3}$$

# A conservation PDE in 1D

$$\frac{\partial}{\partial t}c(x,t) = -\frac{\partial}{\partial x}J(x,t) + f(x,t,c(x,t)). \tag{4}$$

# Generalising to $R^n$

$$\frac{\partial}{\partial t}\int_V c(x,t)\,dx = -\int_S J(x,t)\cdot\mathbf{n}\,d\sigma + \int_V f(x,t,c)dx.$$

# Spatiotemporal models - fluxes