

# Lecture slides

Philip Murray

# Lecture 1

- ▶ Introduction to MA42002
- ▶ Conservation equations
- ▶ Examples of spatially homogeneous models

## Conservation equations

$$\left( \begin{array}{c} \text{rate of change} \\ \text{in the population density} \end{array} \right) = (\text{spatial movement}) + \left( \begin{array}{c} \text{birth, growth, death,} \\ \text{production or degradation} \\ \text{due to chemical reactions} \end{array} \right)$$

# Spatially homogeneous models (MA32009 revision)

## Example problem - bacteria in a dish

$$N(t + \Delta t) = N(t) + KN(t)\Delta t.$$

## A model for cell growth under nutrient depletion

$$\begin{aligned}\frac{dN}{dt} &= K(c)N = \kappa cN, \\ \frac{dc}{dt} &= -\alpha \frac{dN}{dt} = -\alpha \kappa cN,\end{aligned}\tag{1}$$

## Leading to the logistic growth equation

The last equation can be rewritten as

$$\frac{dN}{dt} = \rho N \left(1 - \frac{N}{B}\right) \quad N(0) = N_0, \quad (2)$$

Can also consider other biological processes



## Exercise

Consider a well mixed bio reactor.

A biologist cultures an initial cell population of size  $N_0$  in the bioreactor for 72 h.

Cells undergo division with a period of 14 h.

Each cell produces a non-degradable waste product,  $W$ , at rate  $k_1$ .

When total waste levels exceed a threshold,  $W^*$ , cell division stops. Otherwise the cell population grows exponentially.

How many cells are there at the end of the experiment?

# Model development

## **i** Model checklist

1. Variables (dependent, independent ?)
2. Schematic diagram - what processes are being modelled?
3. Governing equations?
4. Define model parameters?
5. Initial conditions?

## Exercise solution

# Recap

- ▶ Is course layout clear
- ▶ Introduction to conservation equation
- ▶ Deriving spatially homogeneous models

## Lecture 2

- ▶ Continue example
- ▶ Introduce SIR model
- ▶ Introduce an activator inhibitor model
- ▶ Derive a conservation equation

## Exercise

Consider a well mixed bio reactor.

A biologist cultures an initial cell population of size  $N_0$  in the bioreactor for 72 h.

Cells undergo division with a period of 14 h.

Each cell produces a non-degradable waste product,  $W$ , at rate  $k_1$ .

When total waste levels exceed a threshold,  $W^*$ , cell division stops. Otherwise the cell population grows exponentially.

How many cells are there at the end of the experiment?

## The SIR model (used in Chapter 7)

Consider the SIR model equations:

$$\begin{aligned}\frac{dS}{dt} &= -rIS, \\ \frac{dI}{dt} &= rIS - aI, \\ \frac{dR}{dt} &= aI.\end{aligned}$$

What are the variables? What are the parameters?

Identify an expression for the reproduction number,  $R_0$ .

Hence explain why the condition  $R_0 < 1$  is necessary to avoid an epidemic?

## SIR model Calculations

$$\frac{dS}{dt} = -rIS,$$

$$\frac{dI}{dt} = rIS - aI,$$

$$\frac{dR}{dt} = aI.$$

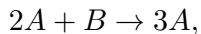


## An activator inhibitor model (used in Chapter 6)

Assume that species A is produced at constant rate  $k_1$  and degrades at rate  $k_2$ .

Assume that B is produced at a constant rate,  $k_4$ .

Consider the reaction schematic



with reaction rate  $k_3$ .

Write down governing ODEs.

## Activator-inhibitor model

Consider the ODEs

$$\frac{da}{dt} = k_1 - k_2 a + k_3 a^2 b,$$

$$\frac{db}{dt} = k_4 - k_3 a^2 b,$$

Identify the steady state of the ODEs. How would you compute linear stability of the steady state?

## Spatiotemporal models

## Spatiotemporal models - derivation

Consider a spatial domain  $V$ . A conservation equation can be written either in terms of the mass or number of particles of a species as follows:

$$\begin{aligned} \left( \begin{array}{c} \text{rate of change of} \\ \text{number of particles} \\ \text{per unit time} \end{array} \right) &= \left( \begin{array}{c} \text{rate of entry of} \\ \text{particles into } V \\ \text{per unit time} \end{array} \right) - \left( \begin{array}{c} \text{rate of exit of} \\ \text{particles from } V \\ \text{per unit time} \end{array} \right) \\ &\quad + \left( \begin{array}{c} \text{rate of degradation} \\ \text{or creation of particles} \\ \text{in } V \text{ per unit time} \end{array} \right) \end{aligned}$$

## Deriving a conservation equation in 1D

$$\begin{aligned} \frac{\partial}{\partial t} \int_x^{x+\Delta x} c(\tilde{x}, t) A d\tilde{x} &= J(x, t) A - J(x + \Delta x, t) A \\ &+ \int_x^{x+\Delta x} f(\tilde{x}, t, c(\tilde{x}, t)) A d\tilde{x}. \end{aligned} \tag{3}$$

## A conservation PDE in 1D

$$\frac{\partial}{\partial t}c(x,t) = -\frac{\partial}{\partial x}J(x,t) + f(x,t,c(x,t)). \quad (4)$$

## Generalising to $R^n$

$$\frac{\partial}{\partial t} \int_V c(x, t) \, dx = - \int_S J(x, t) \cdot \mathbf{n} \, d\sigma + \int_V f(x, t, c) \, dx.$$

# Recap

- ▶ Introduced SIR and activator-inhibitor models
- ▶ Derived conservation PDE
- ▶ Introduced concept of flux,  $J$



## Spatiotemporal models - fluxes