

Lecture slides

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Lecture 1

- ▶ Introduction to MA42002
- ▶ Conservation equations
- ▶ Examples of spatially homogeneous models

Conservation equations

$$\left(\begin{array}{c} \text{rate of change} \\ \text{in the population density} \end{array} \right) = (\text{spatial movement}) + \left(\begin{array}{c} \text{birth, growth, death,} \\ \text{production or degradation} \\ \text{due to chemical reactions} \end{array} \right)$$

Spatially homogeneous models (MA32009 revision)

Example problem - bacteria in a dish

$$N(t + \Delta t) = N(t) + KN(t)\Delta t.$$

A model for cell growth under nutrient depletion

$$\begin{aligned}\frac{dN}{dt} &= K(c)N = \kappa cN, \\ \frac{dc}{dt} &= -\alpha \frac{dN}{dt} = -\alpha \kappa cN,\end{aligned}\tag{1}$$

Leading to the logistic growth equation

The last equation can be rewritten as

$$\frac{dN}{dt} = \rho N \left(1 - \frac{N}{B}\right) \quad N(0) = N_0, \quad (2)$$

Can also consider other biological processes

Exercise

Consider a well mixed bio reactor.

A biologist cultures an initial cell population of size N_0 in the bioreactor for 72 h.

Cells undergo division with a period of 14 h.

Each cell produces a non-degradable waste product, W , at rate k_1 .

When total waste levels exceed a threshold, W^* , cell division stops. Otherwise the cell population grows exponentially.

How many cells are there at the end of the experiment?

Model development

i Model checklist

1. Variables (dependent, independent ?)
2. Schematic diagram - what processes are being modelled?
3. Governing equations?
4. Define model parameters?
5. Initial conditions?

Exercise solution

Recap

- ▶ Is course layout clear
- ▶ Introduction to conservation equation
- ▶ Deriving spatially homogeneous models

Lecture 2

- ▶ Continue example
- ▶ Introduce SIR model
- ▶ Introduce an activator inhibitor model
- ▶ Derive a conservation equation

Exercise

Consider a well mixed bio reactor.

A biologist cultures an initial cell population of size N_0 in the bioreactor for 72 h.

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Each cell produces a non-degradable waste product, W , at rate k_1 .

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How many cells are there at the end of the experiment?

The SIR model (used in Chapter 7)

Consider the SIR model equations:

$$\begin{aligned}\frac{dS}{dt} &= -rIS, \\ \frac{dI}{dt} &= rIS - aI, \\ \frac{dR}{dt} &= aI.\end{aligned}$$

What are the variables? What are the parameters?

Identify an expression for the reproduction number, R_0 .

Hence explain why the condition $R_0 < 1$ is necessary to avoid an epidemic?

SIR model Calculations

$$\frac{dS}{dt} = -rIS,$$

$$\frac{dI}{dt} = rIS - aI,$$

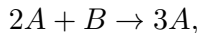
$$\frac{dR}{dt} = aI.$$

An activator inhibitor model (used in Chapter 6)

Assume that species A is produced at constant rate k_1 and degrades at rate k_2 .

Assume that B is produced at a constant rate, k_4 .

Consider the reaction schematic



with reaction rate k_3 .

Write down governing ODEs.

Activator-inhibitor model

Consider the ODEs

$$\begin{aligned}\frac{da}{dt} &= k_1 - k_2 a + k_3 a^2 b, \\ \frac{db}{dt} &= k_4 - k_3 a^2 b,\end{aligned}$$

Identify the steady state of the ODEs. How would you compute linear stability of the steady state?

Recap

- ▶ Introduced SIR and activator-inhibitor models
- ▶ Computed steady states and stability analysis

Lecture 3 Spatiotemporal models

- ▶ Derive conservation PDEs
- ▶ Consider different models of fluxes

Spatiotemporal models - derivation

Consider a spatial domain V . A conservation equation can be written either in terms of the mass or number of particles of a species as follows:

$$\begin{aligned} \left(\begin{array}{c} \text{rate of change of} \\ \text{number of particles} \\ \text{per unit time} \end{array} \right) &= \left(\begin{array}{c} \text{rate of entry of} \\ \text{particles into } V \\ \text{per unit time} \end{array} \right) - \left(\begin{array}{c} \text{rate of exit of} \\ \text{particles from } V \\ \text{per unit time} \end{array} \right) \\ &+ \left(\begin{array}{c} \text{rate of degradation} \\ \text{or creation of particles} \\ \text{in } V \text{ per unit time} \end{array} \right) \end{aligned}$$

Deriving a conservation equation in 1D

$$\begin{aligned} \frac{\partial}{\partial t} \int_x^{x+\Delta x} c(\tilde{x}, t) A d\tilde{x} &= J(x, t) A - J(x + \Delta x, t) A \\ &+ \int_x^{x+\Delta x} f(\tilde{x}, t, c(\tilde{x}, t)) A d\tilde{x}. \end{aligned} \tag{3}$$

A conservation PDE in 1D

$$\frac{\partial}{\partial t}c(x,t) = -\frac{\partial}{\partial x}J(x,t) + f(x,t,c(x,t)). \quad (4)$$

Generalising to R^n

$$\frac{\partial}{\partial t} \int_V c(x, t) \, dx = - \int_S J(x, t) \cdot \mathbf{n} \, d\sigma + \int_V f(x, t, c) \, dx.$$

Fluxes - Fickian diffusion

$$\mathbf{J} = -D\nabla c, \quad (5)$$

Fluxes - Nonlinear diffusion

$$D = D(c), \quad \text{e.g. } D(c) = D_0 c^m, \quad D_0 > 0,$$

Hence

$$J = -D(c)\nabla c$$

Fluxes - Convection/advection

$$\mathbf{J} = \mathbf{v}c, \quad (6)$$

Fluxes - Taxis

$$\mathbf{J} = \chi(a)c\nabla a,$$

Domain of definition of the problem

Lecture 4

- ▶ Boundary and initial conditions
- ▶ Nondimensionalisation
- ▶ Model formulation

Boundary conditions

- ▶ Dirichlet
- ▶ Neumann
- ▶ Robin

Initial conditions

Formulating a model

Lecture 5

- ▶ Introduce a linear reaction diffusion model
- ▶ Diffusion

Linear reaction diffusion equation

$$\frac{\partial c}{\partial t} = D \nabla^2 c + f(c), \quad c \equiv c(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^n, \quad t > 0.$$

so in 1D Cartesian coordinates

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} + f(c), \quad x \in \mathbb{R}, \quad t > 0.$$

1D diffusion equation with delta IC

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}, \quad x \in \mathbb{R}, \quad t > 0. \quad (7)$$

$$c(x_0, 0) = \delta_0(x) \quad x \in \mathbb{R}, \quad (8)$$

where δ_0 is a *Dirac delta distribution* (Dirac measure) satisfying

$$\int_{-\infty}^{+\infty} \delta_0(x) = 1 \quad \text{and} \quad \int_{-\infty}^{+\infty} f(x) \delta_0(x) = f(0), \quad \text{for continuous } f.$$

Numerical solution

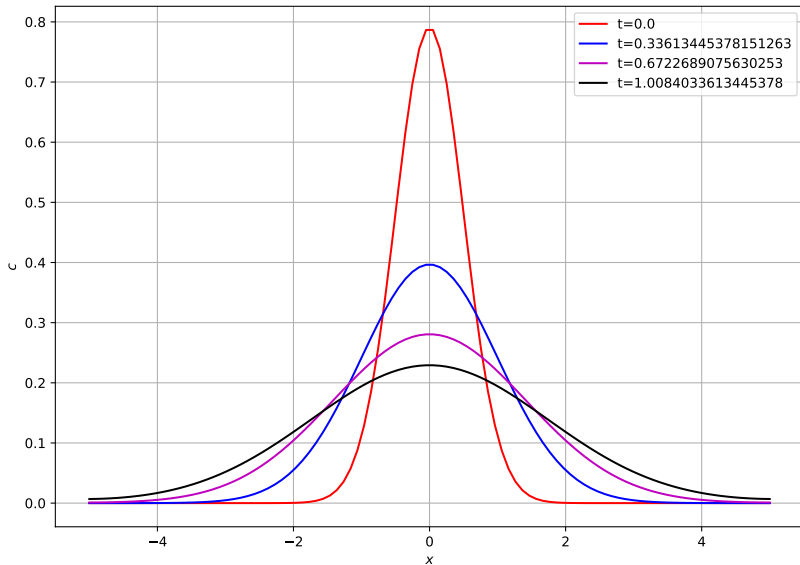


Figure 1: Numerical solution of diffusion equation.

An exact solution computed using a *similarity* variable

Consider the diffusion Equation 7 with initial condition Equation 8.

Introduce the similarity variable

$$\eta = \frac{x}{\sqrt{Dt}}$$

and look for solution of the form

$$c(x, t) = \frac{1}{\sqrt{Dt}} F(\eta).$$

Hence it can be shown that the explicit (analytic) solution is given by

$$c(x, t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right). \quad (9)$$

The 1D diffusion equation for arbitrary initial condition

For a general initial condition $c(x, 0) = c_0(x)$ for $x \in \mathbb{R}$:

$$c(x, t) = \int_{-\infty}^{+\infty} \frac{c_0(y)}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x-y)^2}{4Dt}\right) dy.$$

Key properties of the (linear) diffusion equation (heat equation)

- ▶ The solution is infinitely smooth.
- ▶ The solution $c(x, t)$ stays positive for all $t > 0$ and $x \in \mathbb{R}$ if $c(x, 0) > 0$ for $x \in \mathbb{R}$.
- ▶ The solution “propagates” with infinite speed i.e. for any $t > 0$, the solution is everywhere in \mathbb{R} .
- ▶ If we change the initial data $c(x, 0)$ (continuously) then the solution also changes (continuously).

Diffusive transit time

$$D \frac{d^2 c}{dx^2} = 0 \quad \text{in } (0, L), \quad c(0) = C_0, \quad c(L) = 0.$$

Diffusion as a description of random walk

Suppose that the probability of a particle hopping distance Δx to the right in time Δt is

$$\lambda_R \Delta t.$$

Similarly, the probability of hopping a distance Δx to the left is

$$\lambda_L \Delta t.$$

Numerical simulation

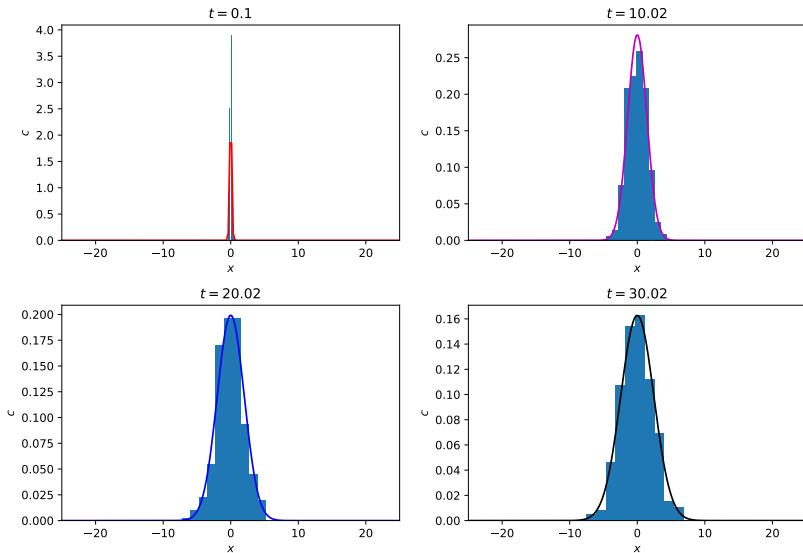


Figure 3: Numerical implementation of random walk

Derivation

Let $c(x, t)$ represent the particle density at spatial location x and time t .

A conservation equation for c is given by

$$c(x, t + \Delta t) = c(x, t) + \lambda_R \Delta t c(x - \Delta x, t) - \lambda_R \Delta t c(x, t) + \lambda_L \Delta t c(x + \Delta x, t) - \lambda_L \Delta t c(x, t).$$

Lecture 6

- ▶ Random walk as a model for the diffusion equation
- ▶ Linear reaction diffusion

Recap from last week

A conservation equation for c is given by

$$c(x, t + \Delta t) = c(x, t) + \lambda_R \Delta t c(x - \Delta x, t) - \lambda_R \Delta t c(x, t) + \lambda_L \Delta t c(x + \Delta x, t) - \lambda_L \Delta t c(x, t).$$

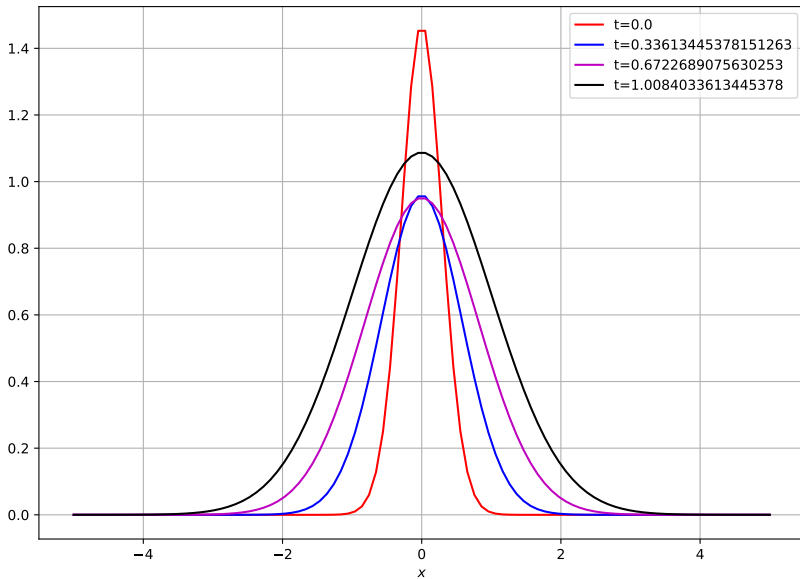
Linear reaction term

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} + \rho c, \quad x \in \mathbb{R}, \quad t > 0, \quad (10)$$

where $\rho \in \mathbb{R}$ is a constant. with initial condition

$$u(x, 0) = M\delta_0(x), \quad x \in \mathbb{R}. \quad (11)$$

Numerical solution



Muskrat invasion dynamics

$$\frac{\partial u}{\partial t} = D \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right) + \rho u, \quad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, \quad t > 0,$$

with initial condition

$$u(\mathbf{x}, 0) = M \delta_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2. \quad (12)$$

$$u_1(\mathbf{x}, t) = \frac{M}{4\pi Dt} \exp \left(\rho t - \frac{r_1^2}{4Dt} \right).$$

Lecture 7

Travelling waves

i Travelling wave

A travelling wave is a solution of a PDE that has a constant profile (shape) and a constant propagation speed.

Fisher's equation

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + \rho u \left(1 - \frac{u}{K}\right), \quad x \in \mathbb{R}, \quad t > 0$$

with initial condition

$$u(x, 0) = u_0(x). \tag{13}$$

Nondimensional form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1 - u), \quad x \in \mathbb{R}, \ t > 0$$

with initial condition

$$u(x, 0) = u_0(x). \tag{14}$$

Numerical solution

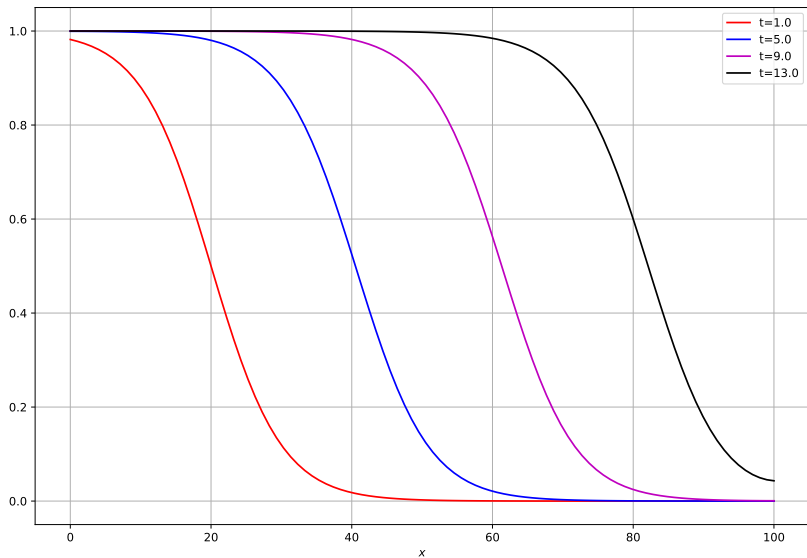


Figure 4: Numerical solution of Fisher's equation.

Spatially homogeneous solutions

Travelling wave solutions

In travelling wave coordinates

$$\frac{d^2W}{dz^2} + v \frac{dW}{dz} + W(1 - W) = 0.$$

A pair of first order ODEs

$$\frac{dW}{dz} = P = F(W, P),$$

$$\frac{dP}{dz} = -vP - W(1 - W) = G(W, P).$$

Numerical solution

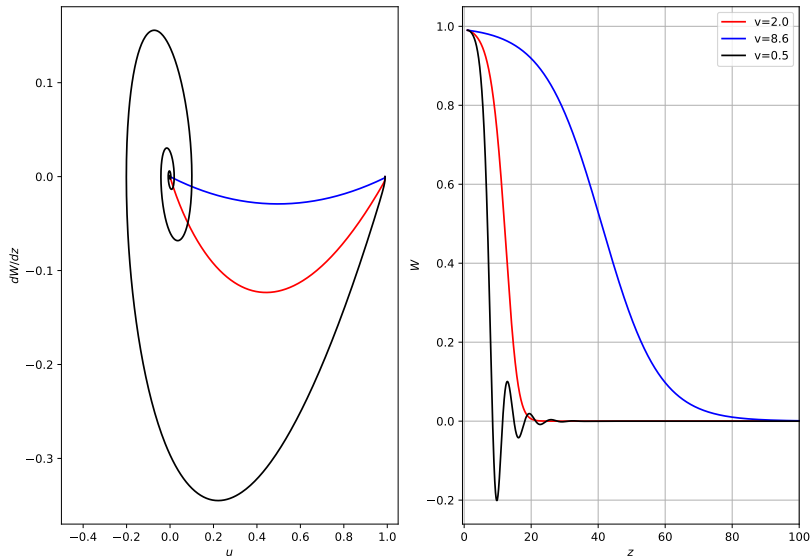


Figure 5: Numerical solution of the travelling wave problem in the phase plane

Lecture 8 Recap

- ▶ Two steady states (saddle plus stable node)
- ▶ Confined set
- ▶ no oscillations

i PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1 - u), \quad x \in \mathbb{R}, \quad t > 0$$

i Travelling wave solution

$$\frac{d^2 W}{dz^2} + v \frac{dW}{dz} + W(1 - W) = 0.$$

Recap

i Pair of first order ODEs

$$\frac{dW}{dz} = P = F(W, P),$$

$$\frac{dP}{dz} = -vP - W(1 - W) = G(W, P).$$

Steady state: $(0,0)$, $(1,0)$

Linear stability analysis

$(0,0)$ is either a stable node or a stable spiral

$(1,0)$ is a saddle

A heteroclinic trajectory

A minimal wave speed

Existence of a travelling wave solution

Strategy:

- ▶ identify a confined set in \mathfrak{R}^2
- ▶ show no other steady states in confined set
- ▶ show no oscillatory solutions

Hence: trajectory that leaves $(1,0)$ via unstable manifold must connect to stable manifold at $(0,0)$

A confined set

Consider

$$T = \{(W, P) : 0 \leq W \leq 1, P \leq 0, P \geq \mu W\}$$

for some $\mu < 0$.

A confined set - ctd

Lecture 8 Recap

- ▶ Two steady states (saddle + stable node)
- ▶ Confined set
- ▶ No oscillations

Finishing off the confined set

No oscillations

Bendixson's Negative Criterion, Dulac's Negative Criterion

If there exists a function $\varphi(W, P)$, with $\varphi \in C^1(\mathbb{R}^2)$, such that

$$\frac{\partial(\varphi F)}{\partial W} + \frac{\partial(\varphi G)}{\partial P},$$

has the same sign ($\neq 0$) almost everywhere in a simply connected region (region without holes), then the system

$$\begin{aligned}\frac{dW}{dz} &= F(W, P) \\ \frac{dP}{dz} &= G(W, P),\end{aligned}$$

has no periodic solutions in this region.

Choosing ϕ

For any $v > 2$, there exists a travelling wave solution to Fisher's equation.

Sign of the wave speed

Consider the travelling wave ODE

$$\frac{d^2W}{dz^2} + vW + W(1 - W) = 0$$

The bistable equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u), \quad x \in \mathbb{R}, \quad t > 0, \quad (15)$$

with initial condition

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}.$$

Let

$$f(0) = f(a) = f(1) = 0, \quad \text{with } 0 < a < 1.$$

There are therefore three spatially uniform steady states $u_1 = 0$, $u_2 = a$, $u_3 = 1$.

$$f'(0) < 0, \quad f'(a) > 0 \quad \text{and} \quad f'(1) < 0$$

$$f = u(u - a)(1 - u),$$

which arises in the study of nerve action potentials along nerve fibres and other problems in *excitable media*

(Gilkey, 1992, ch. 10, §10.1)

Numerical solution

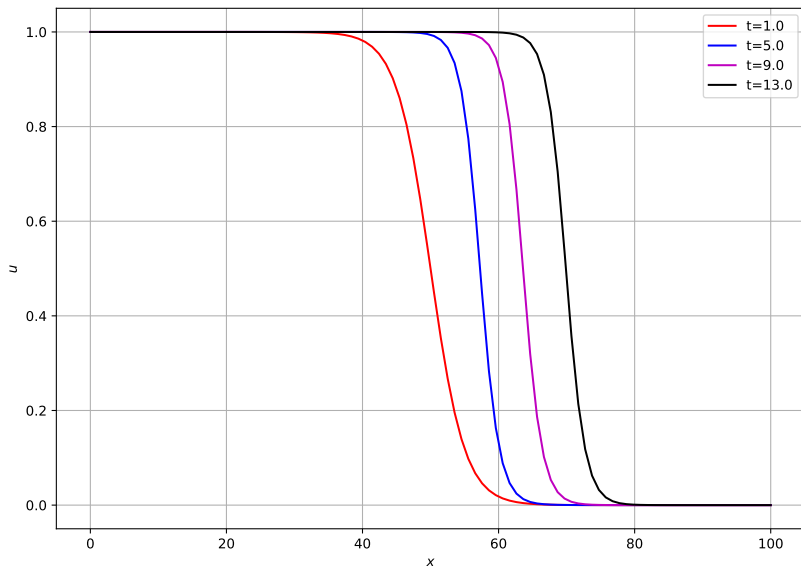


Figure 6: Travelling waves in a numerical solution of bistable PDE.

Lecture 9

- ▶ Bistable equation

Travelling wave ansatz

$$\frac{d^2W}{dz^2} + v\frac{dW}{dz} + f(W) = 0,$$

In the phase plane

$$\begin{aligned}\frac{dW}{dz} &= P = F(W, P), \\ \frac{dP}{dz} &= -vP - f(W) = G(W, P),\end{aligned}$$

Steady states and their linear stability

The sign of v

Lecture 10

Recap: travelling wave analysis of bistable equation

$$\begin{aligned}\frac{dW}{dz} &= P = F(W, P), \\ \frac{dP}{dz} &= -vP - f(W) = G(W, P),\end{aligned}$$

Numerical shooting

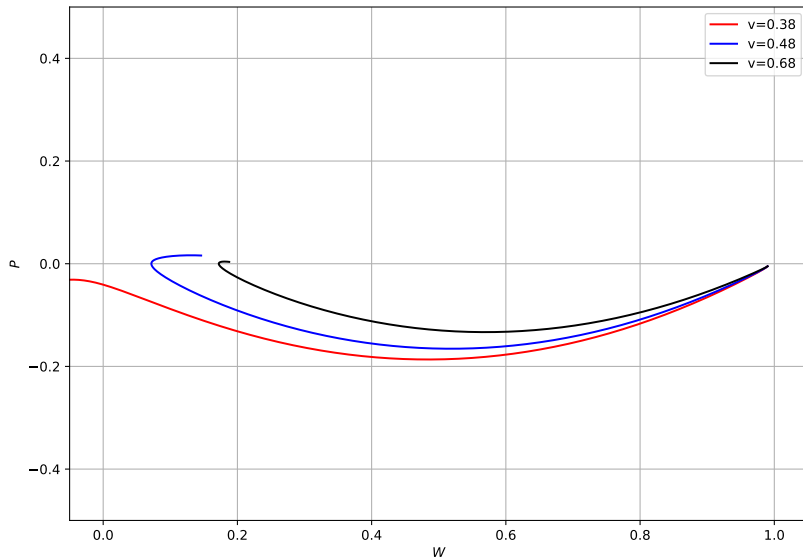


Figure 7: Using a shooting method to investigate travelling wave solutions. Continuity arguments suggest that there exists a travelling

A shooting method to prove the existence of a traveling wave

Outline

Trajectories with small v intersect the P axis with $P < 0$

Assume that

$$\int_0^1 f(u) du > 0.$$

Trajectories with large v intersect the W axis with $W > 0$

Continuity there exists a trajectory with intermediate v that passes through the origin

Systems of coupled reaction diffusion equations

Lotka Volterra with diffusion

$$\begin{aligned}\frac{\partial u}{\partial t} &= \rho u \left(1 - \frac{u}{K}\right) - \alpha u n + D_u \Delta u, \\ \frac{\partial n}{\partial t} &= \beta u n - \gamma n + D_n \Delta n,\end{aligned}\tag{16}$$

Nondimensional form

$$\begin{aligned}\frac{\partial u}{\partial t} &= u(1 - u - n) + D \frac{\partial^2 u}{\partial x^2} = f(u, n) + D \frac{\partial^2 u}{\partial x^2}, & x \in \mathbb{R}, t > 0, \\ \frac{\partial n}{\partial t} &= a n(u - b) + \frac{\partial^2 n}{\partial x^2} = g(u, n) + \frac{\partial^2 n}{\partial x^2}, & x \in \mathbb{R}, t > 0, \\ & & (17)\end{aligned}$$

Spatially homogeneous steady states

Lecture 11

Recap

$$\begin{aligned}\frac{\partial u}{\partial t} &= u(1 - u - n) + D \frac{\partial^2 u}{\partial x^2} = f(u, n) + D \frac{\partial^2 u}{\partial x^2}, & x \in \mathbb{R}, t > 0, \\ \frac{\partial n}{\partial t} &= a n(u - b) + \frac{\partial^2 n}{\partial x^2} = g(u, n) + \frac{\partial^2 n}{\partial x^2}, & x \in \mathbb{R}, t > 0,\end{aligned}$$

Spatially hom. steady states:

- ▶ $(0, 0)$ - extinction - lin. unstable
- ▶ $(1, 0)$ - no predator, lin unstable
- ▶ $(b, 1 - b)$ - coexistence - lin. stable

Question: do travelling wave solutions exist that connect the spatially homogeneous stable steady state to either of the unstable steady states?

A travelling wave that connects $(1, 0)$ and $(b, 1 - b)$

$$u(x, t) = W(x + vt) = W(z), \quad v > 0,$$

$$n(x, t) = N(x + vt) = N(z), \quad v > 0.$$

The limit of fast diffusing predator

Three first order ODEs

$$\begin{aligned}\frac{dW}{dz} &= \frac{1}{v}W(1 - W - N) = F(W, N, P), \\ \frac{dN}{dz} &= P = G(W, N, P), \\ \frac{dP}{dz} &= vP - aN(W - b) = R(W, N, P).\end{aligned}\tag{18}$$

Steady states and their linear stability

Steady states and their linear stability

Lecture 12

- ▶ steady states: $(0, 0, 0)$, $(1, 0, 0)$, $(b, 1 - b, 0)$
- ▶ Heteroclinic trajectory from $(1, 0, 0)$ to $(b, 1 - b, 0)$
- ▶ $(1, 0, 0)$ has a 2 dim unstable manifold
- ▶ Eigenvalues at $(b, 1 - b, 0)$ satisfy

$$\lambda^3 - \lambda^2\left(v - \frac{b}{v}\right) - \lambda b - \frac{1}{v}ab(1 - b) = p(\lambda) = 0.$$

TPs are independent of a

$p(\cdot)$ has a real positive root and two roots with negative real part

Plotting the cubic

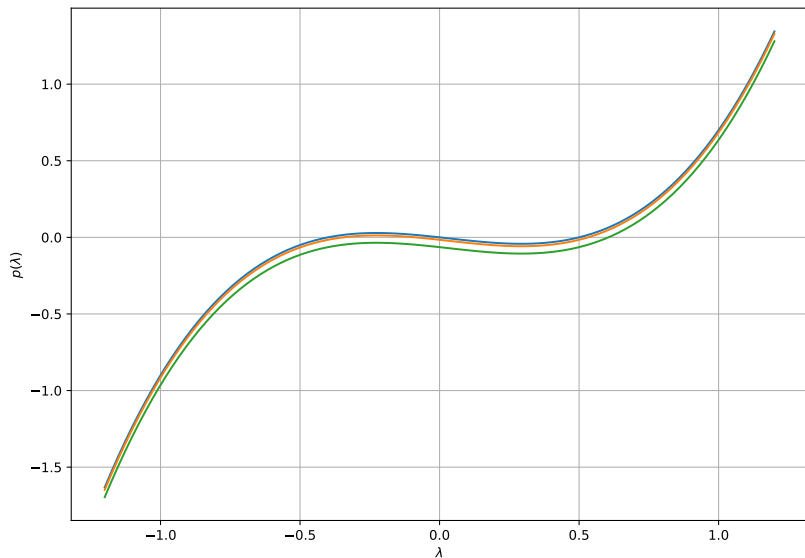


Figure 8: Plot of cubic.

Aggregation via chemotaxis

- ▶ *Dictyostelium discoideum* (Dicty) is a slime-mold that is widely studied experimentally as a model organism.
- ▶ under nutrient starvation, it exhibits complex collective behaviour
- ▶ individual amoebae that constitute a slime-mold exhibit a range of phenomena also observed in mammalian cells e.g. differentiation, proliferation, migration.



Figure 9: Spiral wave patterns underlying Dictyostelium aggregation.

How do simple rules give rise to complex behaviours?

A chemotactic model

A 1D spatial domain with no-flux boundary conditions

Spatially homogeneous solutions

Linearisation about the spatially homogeneous steady state