Lecture slides

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Lecture 1

- ▶ Introduction to MA42002
- ▶ Conservation equations
- ▶ Examples of spatially homogeneous models

Conservation equations

$$\begin{pmatrix} \text{rate of change} \\ \text{in the population density} \end{pmatrix} = \begin{pmatrix} \text{spatial movement} \end{pmatrix}$$

$$+ \begin{pmatrix} \text{birth, growth, death,} \\ \text{production or degradation} \\ \text{due to chemical reactions} \end{pmatrix}$$

Spatially homogeneous models (MA32009 revision)

Example problem - bacteria in a dish

$$N(t + \Delta t) = \quad N(t) + \quad KN(t)\Delta t.$$

A model for cell growth under nutrient depletion

$$\frac{dN}{dt} = K(c)N = \kappa c N,$$

$$\frac{dc}{dt} = -\alpha \frac{dN}{dt} = -\alpha \kappa c N, \tag{1}$$

Leading to the logistic growth equation

The last equation can be rewritten as

$$\frac{dN}{dt} = \rho N \left(1 - \frac{N}{B}\right) \qquad N(0) = N_0, \qquad (2)$$

Can also consider other biological processes

Exercise

Consider a well mixed bio reactor.

A biologist cultures an initial cell population of size N_0 in the bioreactor for 72 h.

Cells undergo division with a period of 14 h.

Each cell produces a non-degradable waste product, W, at rate k_1.

When total waste levels exceed a threshold, W^*, cell division stops. Otherwise the cell population grows exponentially.

How many cells are there at the end of the experiment?

Model development

> **ℹ Model checklist**
>
> 1. Variables (dependent, indepedent ?)
> 2. Schematic diagram - what processes are being modelled?
> 3. Governing equations?
> 4. Define model parameters?
> 5. Initial conditions?

Exercise solution

Recap

- Is course layout clear
- Introduction to conservation equation
- Deriving spatially homogeneous models

Lecture 2

▶ Continue example
▶ Introduce SIR model
▶ Introduce an activator inhibitor model
▶ Derive a conservation equation

Exercise

Consider a well mixed bio reactor.

A biologist cultures an initial cell population of size N_0 in the bioreactor for 72 h.

Cells undergo division with a period of 14 h.

Each cell produces a non-degradable waste product, W, at rate k_1.

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How many cells are there at the end of the experiment?

The SIR model (used in Chapter 7)

Consider the SIR model equations:

$$\frac{dS}{dt} = -rIS,$$
$$\frac{dI}{dt} = rIS - aI,$$
$$\frac{dR}{dt} = aI.$$

What are the variables? What are the parameters?

Identify an expression for the reproduction number, R_0.

Hence explain why the condition $R_0 < 1$ is necessary to avoid an epidemic?

SIR model Calculations

$$\frac{dS}{dt} = -rIS,$$
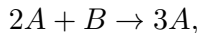$$\frac{dI}{dt} = rIS - aI,$$
$$\frac{dR}{dt} = aI.$$

An activator inhibitor model (used in Chapter 6)

Assume that species A is produced at constant rate k_1 and degrades at rate k_2.

Assume that B is produced at a constant rate, k_4.

Consider the reaction schematic

$$2A + B \rightarrow 3A,$$

with reaction rate k_3.

Write down governing ODEs.

Activator-inhibitor model

Consider the ODEs

$$\frac{da}{dt} = k_1 - k_2 a + k_3 a^2 b,$$

$$\frac{db}{dt} = k_4 - k_3 a^2 b,$$

Identify the steady state of the ODEs. How would you compute linear stability of the steady state?

Recap

- Introduced SIR and activator-inhibitor models
- Computed steady states and stability analysis

Lecture 3 Spatiotemporal models

▶ Derive conservation PDEs
▶ Consider different models of fluxes

Spatiotemporal models - derivation

Consider a spatial domain V. A conservation equation can be written either in terms of the mass or number of particles of a species as follows:

$$\begin{pmatrix} \text{rate of change of} \\ \text{number of particles} \\ \text{per unit time} \end{pmatrix} = \begin{pmatrix} \text{rate of entry of} \\ \text{particles into } V \\ \text{per unit time} \end{pmatrix} - \begin{pmatrix} \text{rate of exit of} \\ \text{particles from } V \\ \text{per unit time} \end{pmatrix}$$

$$+ \begin{pmatrix} \text{rate of degradation} \\ \text{or creation of particles} \\ \text{in } V \text{ per unit time} \end{pmatrix}$$

Deriving a conservation equation in 1D

$$\frac{\partial}{\partial t} \int_x^{x+\Delta x} c(\tilde{x}, t) A d\tilde{x} = J(x, t) \, A - J(x + \Delta x, t) \, A$$
$$+ \int_x^{x+\Delta x} f(\tilde{x}, t, c(\tilde{x}, t)) \, A \, d\tilde{x}. \tag{3}$$

A conservation PDE in 1D

$$\frac{\partial}{\partial t} c(x,t) = -\frac{\partial}{\partial x} J(x,t) + f(x,t,c(x,t)). \qquad (4)$$

Generalising to R^n

$$\frac{\partial}{\partial t} \int_V c(x,t)\, dx = -\int_S J(x,t) \cdot \mathbf{n}\, d\sigma + \int_V f(x,t,c)dx.$$

Fluxes - Fickian diffusion

$$\mathbf{J} = -D\nabla c, \qquad (5)$$

Fluxes - Nonlinear diffusion

$$D = D(c), \qquad \text{e.g.} \ \ D(c) = D_0 c^m, \quad D_0 > 0,$$

Hence

$$J = -D(c)\nabla c$$

Fluxes - Convection/advection

$$\mathbf{J} = \mathbf{v}c, \tag{6}$$

Fluxes - Taxis

$$\mathbf{J} = \chi(a)c\nabla a,$$

Domain of definition of the problem

Lecture 4

- ▶ Boundary and initial conditions
- ▶ Nondimensionalisation
- ▶ Model formulation

Boundary conditions

- Dirichlet
- Neumann
- Robin

Initial conditions

Formulating a model

Lecture 5

▶ Introduce a linear reaction diffusion model
▶ Diffusion

Linear reaction diffusion equation

$$\frac{\partial c}{\partial t} = D\nabla^2 c + f(c), \quad c \equiv c(\mathbf{x}, t), \;\; \mathbf{x} \in \mathbb{R}^n, \; t > 0.$$

so in 1D Cartesian coordinates

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} + f(c), \quad x \in \mathbb{R}, \; t > 0.$$

1D diffusion equation with delta IC

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}, \quad x \in \mathbb{R}, \ t > 0. \tag{7}$$

$$c(x_0, 0) = \delta_0(x) \qquad x \in \mathbb{R}, \tag{8}$$

where δ_0 is a *Dirac delta distribution* (Dirac measure) satisfying

$$\int_{-\infty}^{+\infty} \delta_0(x) = 1 \quad \text{and} \quad \int_{-\infty}^{+\infty} f(x)\delta_0(x) = f(0), \text{ for continuous } f.$$
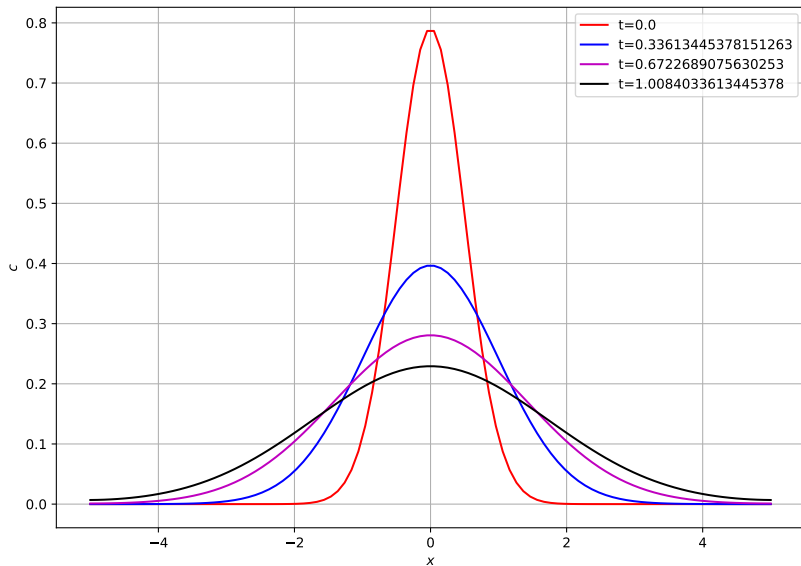
Numerical solution

Figure 1: Numerical solution of diffusion equation.

An exact solution computed using a *similarity* variable

Consider the diffusion Equation 7 with initial condition Equation 8.

Introduce the similarity variable

$$\eta = \frac{x}{\sqrt{Dt}}$$

and look for solution of the form

$$c(x, t) = \frac{1}{\sqrt{Dt}} F(\eta).$$

Hence it can be shown that the explicit (analytic) solution is given by

$$c(x, t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right). \tag{9}$$

The 1D diffusion equation for arbitrary initial condition

For a general initial condition $c(x, 0) = c_0(x)$ for $x \in \mathbb{R}$:

$$c(x, t) = \int_{-\infty}^{+\infty} \frac{c_0(y)}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x-y)^2}{4Dt}\right) dy.$$

Key properties of the (linear) diffusion equation (heat equation)

▶ The solution is infinitely smooth.

▶ The solution $c(x,t)$ stays positive for all $t > 0$ and $x \in \mathbb{R}$ if $c(x,0) > 0$ for $x \in \mathbb{R}$.

▶ The solution "propagates'' with infinite speed i.e. for any $t > 0$, the solution is everywhere in $\mathbb{R}$.

▶ If we change the initial data $c(x,0)$ (continuously) then the solution also changes (continuously).

Diffusive transit time

$$D\frac{d^2c}{dx^2} = 0 \quad \text{in } (0, L), \quad c(0) = C_0, \, c(L) = 0.$$

Diffusion as a description of random walk

Suppose that the probability of a particle hopping distance Δx to the right in time Δt is

$$\lambda_R \Delta t.$$

Similarly, the probability of hopping a distance Δx to the left is
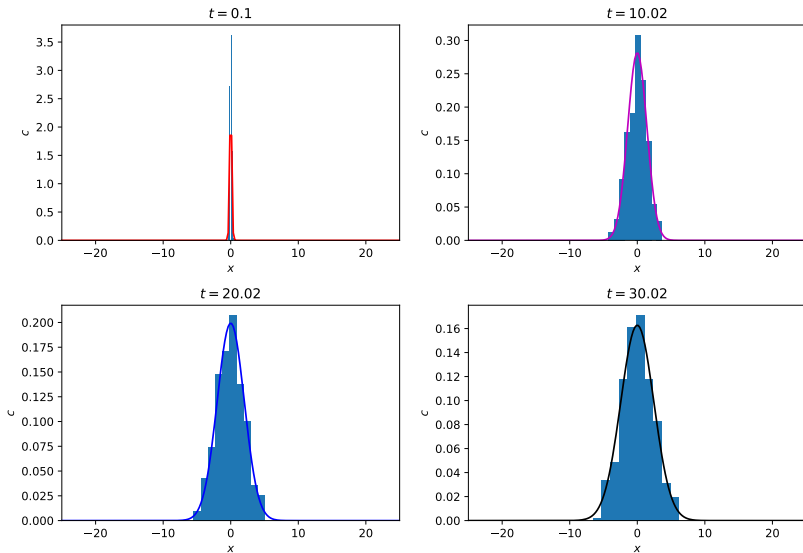
$$\lambda_L \Delta t.$$

Numerical simulation

Figure 3: Numerical implementation of random walk

Derivation

Let $c(x, t)$ represent the particle density at spatial location x and time t.

A conservation equation for c is given by

$$c(x, t + \Delta t) = c(x, t) + \lambda_R \Delta t c(x - \Delta x, t) - \lambda_R \Delta t c(x, t) + \lambda_L \Delta t c(x + \Delta x, t) - \lambda_L \Delta t c(x, t).$$

Lecture 6

▶ Random walk as a model for the diffusion equation
▶ Linear reaction diffusion

Recap from last week

A conservation equation for c is given by

$$c(x, t + \Delta t) = c(x, t) + \lambda_R \Delta t c(x - \Delta x, t) - \lambda_R \Delta t c(x, t) +$$
$$\lambda_L \Delta t c(x + \Delta x, t) - \lambda_L \Delta t c(x, t).$$
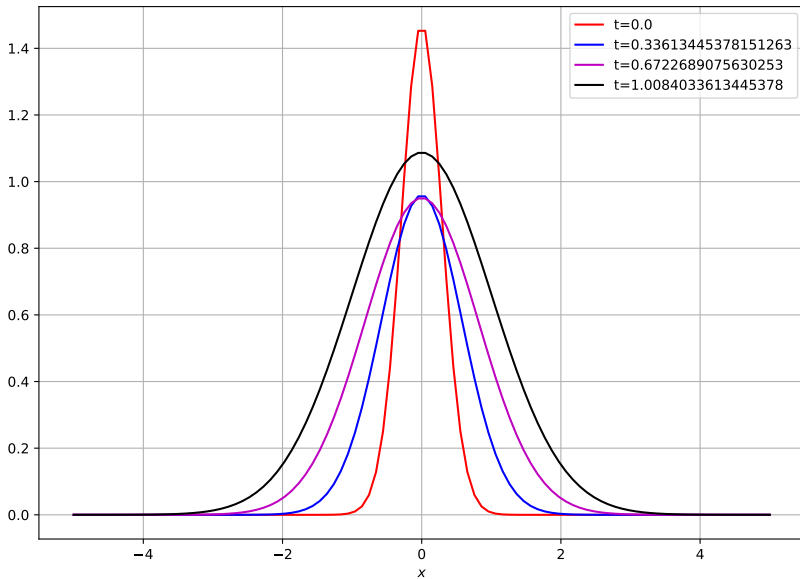
Linear reaction term

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} + \rho\, c, \quad x \in \mathbb{R},\ t > 0, \tag{10}$$

where $\rho \in \mathbb{R}$ is a constant. with initial condition

$$u(x, 0) = M\delta_0(x), \quad x \in \mathbb{R}. \tag{11}$$

Numerical solution

Muskrat invasion dynamics

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) + \rho\, u, \quad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2,\ t > 0,$$

with initial condition

$$u(\mathbf{x}, 0) = M\delta_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2. \tag{12}$$

$$u_1(\mathbf{x}, t) = \frac{M}{4\pi D t} \exp\left(\rho t - \frac{r_1^2}{4Dt}\right).$$

Lecture 7

Travelling waves

> **ℹ Travelling wave**
>
> A travelling wave is a solution of a PDE that has a constant
> profile (shape) and a constant propagation speed.

Fisher's equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \rho u(1 - \frac{u}{K}), \qquad x \in \mathbb{R}, \ t > 0$$

with initial condition

$$u(x,0) = u_0(x). \tag{13}$$

Nondimensional form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \qquad x \in \mathbb{R}, \ t > 0$$

with initial condition

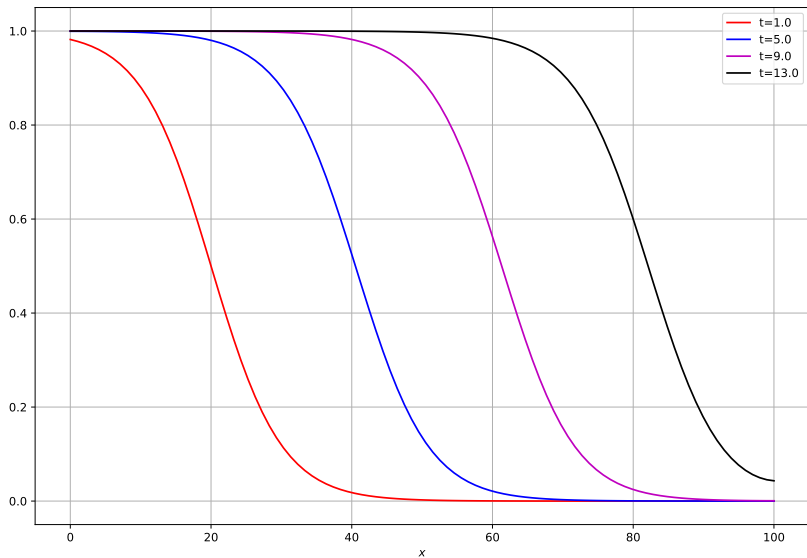$$u(x,0) = u_0(x). \tag{14}$$

Numerical solution

Figure 4: Numerical solution of Fisher's equation.

Spatially homogeneous solutions

Travelling wave solutions

In travelling wave coordinates

$$\frac{d^2W}{dz^2} + v\frac{dW}{dz} + W(1 - W) = 0.$$

A pair of first order ODEs

$$\frac{dW}{dz} = P = F(W, P),$$
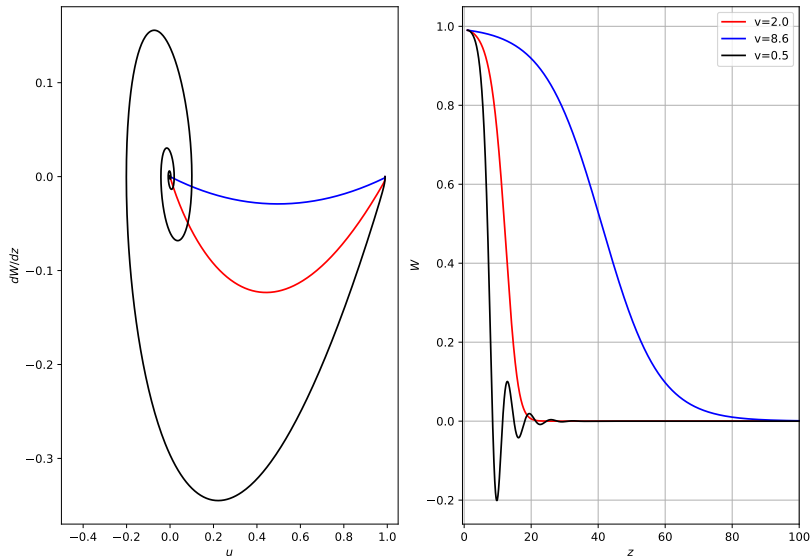$$\frac{dP}{dz} = -vP - W(1 - W) = G(W, P).$$

Numerical solution

Figure 5: Numerical solution of the travelling wave problem in the phase plane

Lecture 8 Recap

▶ Two steady states (saddle plus stable node)
▶ Confined set
▶ no oscillations

i PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \qquad x \in \mathbb{R}, \; t > 0$$

i Travelling wave solution

$$\frac{d^2 W}{dz^2} + v\frac{dW}{dz} + W(1-W) = 0.$$

Recap

> **i** Pair of first order ODEs
>
> $$\frac{dW}{dz} = P = F(W, P),$$
> $$\frac{dP}{dz} = -vP - W(1 - W) = G(W, P).$$

Steady state: (0,0), (1,0)

Linear stability analysis

(0,0) is either a stable node or a stable spiral

(1,0) is a saddle

A heteroclinic trajectory

A minimal wave speed

Existence of a travelling wave solution

Strategy:

- identify a confined set in $\mathfrak{R}^2$
- show no other steady states in confined set
- show no oscillatory solutions

Hence: trajectory that leaves (1,0) via unstable manifold must connect to stable manifold at (0,0)

A confined set

Consider

$$T = \{(W, P) : 0 \leq W \leq 1, \ P \leq 0, \ P \geq \mu W\}$$

for some $\mu < 0$.

A confined set - ctd

Lecture 8 Recap

▶ Two steady states (saddle + stable node)
▶ Confined set
▶ No oscillations

Finishing off the confined set

No oscillations

Bendixson's Negative Criterion, Dulac's Negative Criterion

If there exists a function $\varphi(W, P)$, with $\varphi \in C^1(\mathbb{R}^2)$, such that

$$\frac{\partial(\varphi F)}{\partial W} + \frac{\partial(\varphi G)}{\partial P},$$

has the same sign $(\neq 0)$ almost everywhere in a simply connected region (region without holes), then the system

$$\frac{dW}{dz} = F(W, P)$$
$$\frac{dP}{dz} = G(W, P),$$

has no periodic solutions in this region.

Choosing ϕ

For any $v > 2$, there exists a travelling wave solution to Fisher's equation.

Sign of the wave speed

Consider the travelling wave ODE

$$\frac{d^2W}{dz^2} + vW + W(1 - W) = 0$$

The bistable equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u), \qquad x \in \mathbb{R}, \ t > 0, \qquad (15)$$

with initial condition

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}.$$

Let

$$f(0) = f(a) = f(1) = 0, \quad \text{with} \quad 0 < a < 1.$$

There are therefore three spatially uniform steady states $u_1 = 0$, $u_2 = a$, $u_3 = 1$.

$$f'(0) < 0, \quad f'(a) > 0 \quad \text{and} f'(1) < 0$$

$$f = u(u-a)(1-u),$$

which arises in the study of nerve action potentials along nerve fibres and other problems in *excitable media*
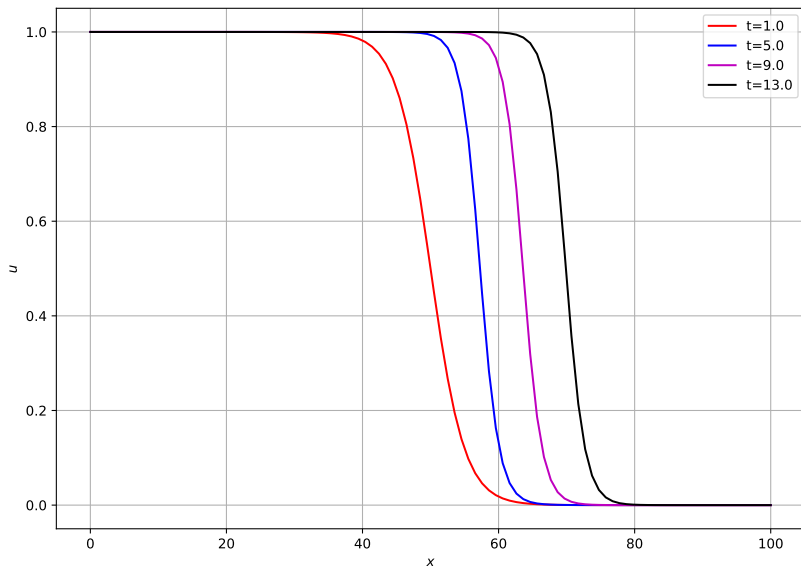(@h ..... 2000 ..... theoretical)

Numerical solution

Figure 6: Travelling waves in a numerical solution of bistable PDE.

Lecture 9

- ▶ Bistable equation

Travelling wave ansatz

$$\frac{d^2W}{dz^2} + v\frac{dW}{dz} + f(W) = 0,$$

In the phase plane

$$\frac{dW}{dz} = P = F(W, P),$$

$$\frac{dP}{dz} = -vP - f(W) = G(W, P),$$

Steady states and their linear stability

The sign of v

Recap: travelling wave analysis of bistable equation

$$\frac{dW}{dz} = P = F(W, P),$$
$$\frac{dP}{dz} = -vP - f(W) = G(W, P),$$
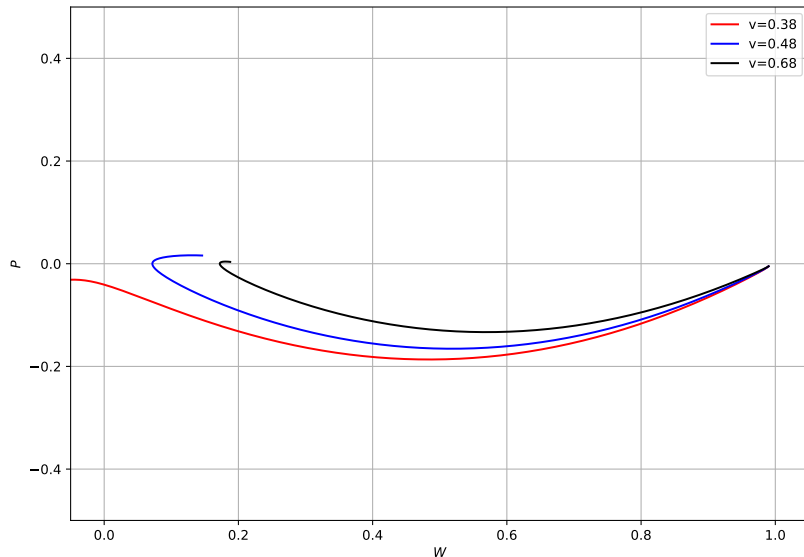
Numerical shooting

Figure 7: Using a shooting method to investigate travelling wave solutions. Continuity arguments suggest that there exists a travelling

A shooting method to prove the existence of a traveling wave

Outline

Assume that

$$\int\limits_0^1 f(u)\, du > 0.$$

Trajectories with large v intersect the W axis with $W > 0$

Continuity there exists a trajectory with intermediate v that passes through the origin

Systems of coupled reaction diffusion equations

Lotka Volterra with diffusion

$$\frac{\partial u}{\partial t} = \rho\, u \left(1 - \frac{u}{K}\right) - \alpha\, u\, n + D_u \Delta u,$$
$$\frac{\partial n}{\partial t} = \beta\, u\, n - \gamma\, n + D_n \Delta n,$$

(16)

Nondimensional form

$$\frac{\partial u}{\partial t} = u(1 - u - n) + D\frac{\partial^2 u}{\partial x^2} = f(u,n) + D\frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, t > 0,$$

$$\frac{\partial n}{\partial t} = a\,n(u - b) + \frac{\partial^2 n}{\partial x^2} = g(u,n) + \frac{\partial^2 n}{\partial x^2}, \quad x \in \mathbb{R}, t > 0,$$

$$(17)$$

Spatially homogeneous steady states

Lecture 11

Recap

$$\frac{\partial u}{\partial t} = u(1 - u - n) + D\frac{\partial^2 u}{\partial x^2} \ = \ f(u, n) + D\frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, t > 0\,,$$

$$\frac{\partial n}{\partial t} = a\,n(u - b) + \frac{\partial^2 n}{\partial x^2} \ = \ g(u, n) + \frac{\partial^2 n}{\partial x^2}, \qquad x \in \mathbb{R}, t > 0,$$

Spatially hom. steady states:

▶ $(0, 0)$ - extinction - lin. unstable
▶ $(1, 0)$ - no predator, lin unstable
▶ $(b, 1 - b)$ - coexistence - lin. stable

Question: do travelling wave solutions exist that connect the spatially homogeneous stable steady state to either of the unstable steady states?

A travelling wave that connects $(1, 0)$ and $(b, 1 - b)$

$$u(x, t) = W(x + vt) = W(z), \quad v > 0,$$
$$n(x, t) = N(x + vt) = N(z), \quad v > 0.$$

The limit of fast diffusing predator

Three first order ODEs

$$\frac{dW}{dz} = \frac{1}{v}W(1 - W - N) = F(W, N, P),$$
$$\frac{dN}{dz} = P = G(W, N, P), \tag{18}$$
$$\frac{dP}{dz} = vP - aN(W - b) = R(W, N, P).$$

Steady states and their linear stability

Steady states and their linear stability

Lecture 12

▶ steady states: $(0, 0, 0)$, $(1, 0, 0)$, $(b, 1 - b, 0)$
▶ Heteroclinic trajectory from $(1,0,0)$ to $(b, 1 - b, 0)$
▶ (1,0,0) has a 2 dim unstable manifold
▶ Eigenvalues at $(b, 1 - b, 0)$ satisfy

$$\lambda^3 - \lambda^2(v - \frac{b}{v}) - \lambda b - \frac{1}{v}ab(1 - b) = p(\lambda) = 0.$$

TPs are independent of a

p( ) has a real positive root and two roots with negative real part
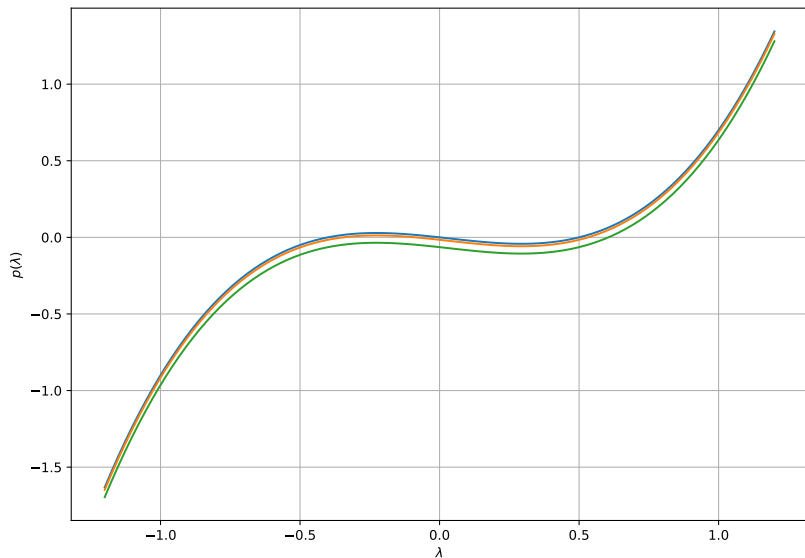
Plotting the cubic

Figure 8: Plot of cubic.

Aggregation via chemotaxis

▶ *Dictyostelium discoideum* (Dicty) is a slime-mold that is widely studied experimentally as a model organism.
▶ under nutrient starvation, it exhibits complex collective behaviour
▶ individual amoebae that constitute a slime-mold exhibit a range of phenomena also observed in mammalian cells e.g. differentiation, proliferation, migration.

Figure 9: Spiral wave patterns underly Dictystelium aggregation.

How do simple rules give rise to complex behaviours?

A chemotactic model

A 1D spatial domain with no-flux boundary conditions

Lecture 13

Domain:

$$x \in [0, L], t > 0$$

PDE:

$$\frac{\partial n}{\partial t} = D_n \frac{\partial^2 n}{\partial x^2} - \chi_0 \frac{\partial}{\partial x}\left(n \frac{\partial a}{\partial x}\right),$$

$$\frac{\partial a}{\partial t} = D_a \frac{\partial^2 a}{\partial x^2} + \mu n - \delta a,$$

(19)

Boundary conditions:

$$\frac{\partial a}{\partial x} = \frac{\partial n}{\partial x} = 0, \quad x = 0, L.$$
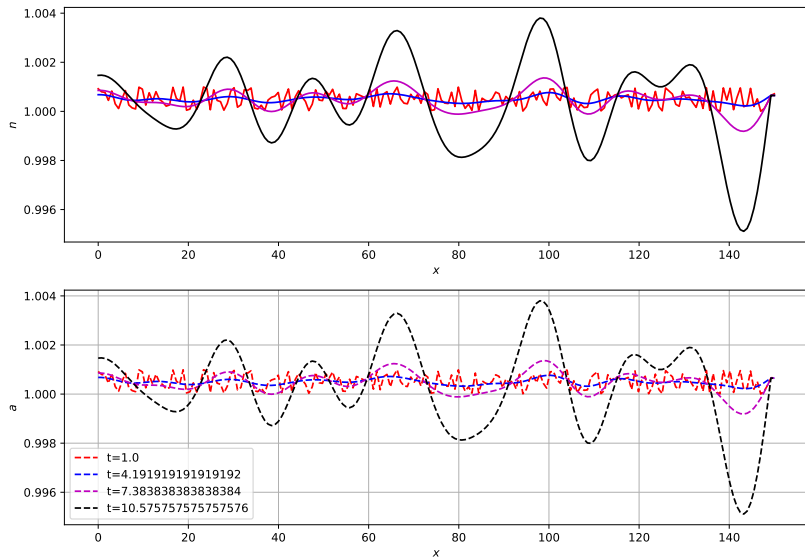
ICs:

numerical solution

Figure 10: Numerical solution of bacterial chemtaxis model.

Conservation of cell number

Spatially homogeneous solutions

Linearisation about the spatially homogeneous steady state

$$n(x,t) = n^* + \tilde{n}(x,t), \quad a(x,t) = a^* + \tilde{a}(x,t)$$

Separable solution

$$\tilde{n}(t, x) = u(t)\phi_1(x), \quad \tilde{a}(t, x) = v(t)\phi_2(x)$$

The elliptic problem

$$\frac{d^2\phi}{dx^2} = -k^2\phi \quad \text{in} \ (0, L),$$

$$\frac{d\phi}{dx} = 0 \qquad \text{for} \ x = 0, \ x = L.$$

$\phi_1 = \phi 2 = \phi$

Linear system solution

$$u(t) = C_1 e^{\lambda t} \quad \text{and} \quad v(t) = C_2 e^{\lambda t}$$

Eigenvalue equation

$$\lambda^2 + \left(D_n k^2 + D_a k^2 + \delta\right)\lambda + D_n k^2 \left(D_a k^2 + \delta\right) - \mu \chi_0 n^* k^2 = 0.$$

Conditions for instability of spatially homogeneous pattern