MTH 463 Homework #5

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01 December, 2020

Problem 1

I check that f(x) is a probability density function by finding if the cumulative distribution function is 1

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)dx \\ &= \int_{1}^{\infty} \frac{cdx}{x^4} \\ &= \left[\frac{cx^{-3}}{-3}\right]_1^{\infty} \\ &= \frac{c}{3} \\ c &= \boxed{3} \end{aligned}$$

To find $E[X]$ we use

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} xf(x)dx \\
&= \int_{1}^{\infty} \frac{3xdx}{x^4} \\
&= 3\int_{1}^{\infty} \frac{dx}{x^3} \\
&= \left[\frac{3x^{-2}}{-2}\right]_1^{\infty} \\
&= \boxed{\frac{3}{2}}
\end{aligned}
$$

To find $Var[X]$ we use $Var[X] = E[X^2] - E[X]^2$

$$
\begin{aligned}
E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x)dx \\
&= \int_{1}^{\infty} \frac{3x^2dx}{x^4} \\
&= 3\int_{1}^{\infty} \frac{dx}{x^2} \\
&= 3\left[\frac{3x^{-1}}{-1}\right]_1^{\infty} \\
&= 3 \\
Var[X] &= E[X^2] - E[X]^2 \\
&= 3 - \frac{3}{2}^2 \qquad\qquad = \boxed{\frac{3}{4}}
\end{aligned}
$$

Problem 2

$$
\begin{aligned}
E[e^z] &= \int_{-\infty}^{\infty} f_Z(x)dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^x e^{\frac{-x^2}{2}}\,dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{x-\frac{-x^2}{2}}\,dx \\
&= e^{\frac{1}{2}}\frac{1}{2\pi}\left[e^{-\frac{u^2}{2}}\right]_{-\infty}^{\infty} \qquad \text{Let } u = x-1 \\
&= e^{\frac{1}{2}} \\
&= \boxed{\sqrt{e}}
\end{aligned}
$$

Problem 3

$$
\begin{aligned}
V(X+Y) &= E[(X+Y)^2] - (E[X+Y])^2 \\
&= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2 \\
&= E[X^2] - (E[X])^2 + E[Y^2] - 2E[XY] + 2E[X]E[Y] \\
&= E[X^2] - (E[X])^2 - (E[Y])^2 \\
&\text{We know } E[X*Y] = E[X]*E[Y]\text{, so} \\
&= \boxed{Var(X) + Var(Y)}
\end{aligned}
$$

Problem 4

$$
\begin{aligned}
f_{x_1+x_2+x_3}(a) &= \int_{-\infty}^{\infty} f_{x_1+x_2}(x) f_{x_3}(a-x) \\
&= \int_0^1 x f_{x)3}(a-x)dx + \int_1^2 (2-x) f_{x_3}(a-x)dx \\
&= \begin{cases} \int_0^a x dx & \text{if } 0 \le a \le 1 \\ \int_{a-1}^1 x dx + \int_1^a (2-x)dx & \text{if } 1 \le a \le 2 \\ \int_{a-1}^2 (2-x)dx & \text{if } 2 \le a \le 3 \\ 0 & \text{otherwise} \end{cases} \\
&= \boxed{\begin{cases} \frac{a^2}{2} & \text{if } 0 \le a \le 1 \\ -a^2 + 3a - \frac{3}{2} & \text{if } 1 \le a \le 2 \\ \frac{a^2}{2} - 3a + \frac{9}{2} & \text{if } 2 \le a \le 3 \\ 0 & \text{otherwise} \end{cases}}
\end{aligned}
$$

Problem 5

Let S be the number of successes. S is therefore binomial with the parameters $n = 1210, p = \frac{1}{11}$.

Thus we have $E[S] = np = 1210 * \frac{1}{11} = 110$,and $\sigma(S) = \sqrt{1210 * \frac{1}{11}(1 - \frac{1}{11})} = 10$.

By the DeMoivre-Laplace limit theorem,

$$
\begin{aligned}
P(97.5 \le S \le 116.5) &= P(-12.5 \le S - np \le -6.5) \\
&= P(-1.25 \le \frac{S - np}{\sqrt{np(1-p)}} \le 0.65) \\
&= P(-1.25 \le Z \le 0.65) \\
&= \boxed{0.6366} \text{ according to R}
\end{aligned}
$$

Problem 6

Let H be the number of heads, H is binomial and has parameters $n = 90,000$ and $p = \frac{1}{2}$.

Therefore $E[H] = 90,000 * \frac{1}{2} = 45,000$ and $\sigma(H) = \sqrt{90,000 * \frac{1}{2}(1 - \frac{1}{2})} = 150.$

Thus, by the DeMoivre-Laplace limit theorem,

$$\begin{aligned} P(45,031.5 \leq H \leq 45,169.5) &= P(31.5 \leq H - np \leq 169.5) \\ &= P(\frac{31.5}{150} \leq \frac{H - np}{\sqrt{np(1-p)}} \leq \frac{169.5}{150}) \\ &= P(0.21 \leq Z \leq 1.13) \\ &= \boxed{0.2876} \text{ according to R} \end{aligned}$$

Problem 7

Just want to solve this one the normal way first, since it says estimate I also solve with DeMoivre-Laplace.

We have $n = 18,000$ and $p = \frac{1}{6}$, so $E[X] = 3,000$ and $Var[X] = np(1-p) = 18,000 * \frac{1}{6} * \frac{5}{6} = 2500$

By the Central Limit Theorem,

$$\begin{aligned} P(Z = \frac{X - E[X]}{\sqrt{Var[X]}} \geq 3,060) &= P(\frac{X - 3,000}{50} \geq \frac{3,060 - 3,000}{50}) \\ &= P(Z \geq 1.2) \\ &= 1 - P(Z < 1.2) \\ &\approx \boxed{0.1151} \text{ in R 1-pnorm(1.2, lower.tail = T)} \end{aligned}$$

This time estimating by the DeMoivre-Laplace limit theorem where S is binomial with the above parameters

$$
\begin{aligned}
P(3,059.5 \leq S < \infty) &= P(59.5 \leq S - np < \infty) \\
&= P(\frac{59.5}{50} \leq \frac{S - np}{\sqrt{np(1-p)}} < \frac{\infty}{50}) \\
&= P(0.21 \leq Z < \infty) \\
&= \boxed{0.1170} \text{ which is pretty close to the CLT approximation}
\end{aligned}
$$