# MTH 463 Homework #2

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## Problem 1

There are $\binom{9}{2}$ possible combinations that can be ordered in 2! ways.

So we have $\binom{9}{2} * 2! = 9 * 8 = 72$

Each five digit combination will have a permutation consisting of a pair as $\binom{5}{2}$.

So in total we have $72 * \binom{5}{2} = 72 * 10 = 720$

Therefore the total number of ways a 5 digit number can be formed is $9 + 720 = \boxed{729}$

## Problem 2

We can calculate $P(A \cap B)$ as $\frac{2*4!}{6!!}$. With 2 representing the probability of card 1 being either second or third following the one way to place card 5 at second from the top. There are then $4!$ ways to organize the rest of the cards. Finally there are $6!$ total configurations yielding $P(A \cap B) = \frac{1}{15}$.

We know $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A) = \frac{3}{6} = \frac{1}{2}$ and $P(B) = \frac{1}{6}$

Therefore $P(A \cup B) = \frac{1}{2} + \frac{1}{6} - \frac{1}{15}$

## Problem 3

Simplifying

$$\begin{aligned}
&= (E \cup F) \cap (E \cup \overline{F}) \\
&= (E \cup F) \cap \overline{F} && \text{by the distributive law} \\
&= (F \cup \overline{F}) \cap E && \text{by the associative law} \\
&= E
\end{aligned}$$

We also have

$$\begin{aligned}
&= (E \cup F) \cap (\overline{E} \cup F) \cap (E \cup \overline{F}) \\
&= F \cap (E \cup \overline{F}) && \text{by the distributive law} \\
&= (F \cap E) \cup (F \cap \overline{F}) \\
&= E \cap F
\end{aligned}$$

## Problem 4

Given events E, F, and G

**a. At least one of the three events occurs**

At least one is the complement of none of them i.e.

$$\overline{(\overline{E} \cap \overline{F} \cap \overline{G})}$$

Using De Morgan's Law becomes

$$\boxed{E \cup F \cup G}$$

So for the given events E, F, and G at least one of the three events occurs

**b. At most one of the three events occurs.**

One of the three events occurring can be written as

$$[(E \cap F) \cup (F \cap G) \cup (G \cap E)]$$

So at most it will be

$$\boxed{(E \cap \overline{F} \cap \overline{G}) \cup (\overline{E} \cap F \cap \overline{G}) \cup (\overline{E} \cap \overline{F} \cap G) \cap (\overline{E} \cap \overline{F} \cap \overline{G})}$$

**c. Exactly two of them occur.**

$$\boxed{(E \cap F \cap \overline{G}) \cup (E \cap \overline{F} \cap G) \cup (\overline{E} \cap F \cap G)}$$

**d. At most two of the three events occur.**

At most two of the three events occur is the complement of all three events occur

All three events occur can be written as $E \cap F \cap G$

The complement of which is

$$\boxed{\overline{(E \cap F \cap G)} = (\overline{E} \cap \overline{F} \cap \overline{G})}$$

**e. All three events occur.**

$$\boxed{(E \cap F \cap G)}$$

**f. None of the three events occurs.**

None of the three events is the complement of each event

$$\boxed{(\overline{E} \cap \overline{F} \cap \overline{G})}$$

**g. At most three of the events occur.**

This includes zero, one, two, and three events occurring, so

$$\boxed{\text{This includes the entire sample space S}}$$

**h. E or F, but not G occur.**

$\boxed{(E \cup F) \cap \overline{G}}$

**i. Both E and F, but not G occur.**

$\boxed{E \cap F \cap \overline{G}}$

**j. Exactly one of the three events occurs.**

This can be thought of as each event occurring against its complement

$\boxed{[E \cap (\overline{F} \cap \overline{G})] \cup [F \cap (\overline{E} \cap \overline{G})] \cup [G \cap (\overline{E} \cap \overline{F})]}$

## Problem 5

We have

$$\begin{aligned} P(E \cap F \cap G) &= P(E) + P(F) + P(G) - P(\overline{E} \cap F \cap G) \\ &\quad - P(E \cap \overline{F} \cap G) - P(E \cap F \cap \overline{G}) - 2P(E \cap F \cap G) \\ &\text{which gives} \\ P(E \cap F \cap G) &= P(E) + P(F) + P(G) - P(\overline{E} \cap F) \\ &\quad - P(E \cap G) - P(F \cap G) + P(E \cap F \cap G) && \text{by the inclusion-exclusion rule} \end{aligned}$$

after plugging in

$$\begin{aligned} P(E \cap F) &= P(E \cap F \cap G) + P(E \cap F \cap \overline{G}) \\ P(E \cap G) &= P(E \cap F \cap G) + P(E \cap \overline{F} \cap G) \\ P(F \cap G) &= P(E \cap F \cap G) + P(\overline{E} \cap F \cap G) \end{aligned}$$

We have the equation

$$\boxed{\begin{aligned} P(E \cap F \cap G) &= P(E) + P(F) + P(G) - P(\overline{E} \cap F \cap G) - P(E \cap \overline{F} \cap G) - P(E \cap F \cap \overline{G}) \\ &\quad - 2P(E \cap F \cap G) \end{aligned}}$$

## Problem 6

We have

$P(\overline{E} \cap \overline{F} \cap \overline{G}) = 1 - P(E) - P(F) - P(G) + P(E \cap F) + P(E \cap G) + P(F \cap G) - P(E \cap F \cap G)$ and de Morgan's Law gives

$\overline{E} \cap \overline{F} \cap \overline{G} = \overline{E \cup F \cup G}$

So,

$P(\overline{E} \cap \overline{F} \cap \overline{G}) = 1 - P(E \cup F \cup G)$

and,

$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(E \cap G) - P(F \cap G) + P(E \cap F \cap G) \qquad \text{by the inclusion-exclusion formula}$

Thus we obtain the equation,

$$\boxed{P(E \cup F \cup G) = P(E) - P(F) - P(G) + P(E \cap F) + P(E \cap G) + P(F \cap G) - P(E \cap F \cap G)}$$

## Problem 7

Given events E, F, and G, such that $P(F) > P(F \cap G) > 0$

$$
\begin{aligned}
P(E|F) &= \frac{P(E \cap F)}{P(F)} \\
\frac{P(E \cap F)}{P(F)} &= \frac{P(E \cap F \cap G)}{P(F)} + \frac{P(E \cap F \cap \overline{G})}{P(F)} \\
&= \frac{P(E \cap F \cap G)}{P(F \cap G)} * \frac{P \cap G}{P(F)} + \frac{P(E \cap F \cap \overline{G})}{P(F \cap \overline{G})} * \frac{P(F \cap \overline{G})}{P(F)}
\end{aligned}
$$

## Problem 8

We have $P(E_1 \cap E_2 \cap ... \cap E_n) \geq P(E_1) + P(E_2) + ... + P(E_n) - (n-1)$

We know that for n = 2: $P(E_1 \cap E_2) \geq P(E_1) + P(E_2) - 1$

For $n \geq 2$ suppose we established the Bonferroni's inequality for any n events:

$P(E_1 \cap E_2 \cap ... \cap E_n) \geq P(E_1) + P(E_2) + ... + P(E_n) - (n-1)$

We need to prove the inequality for any $n+1$ events:

$P(E_1 \cap E_2 \cap ... \cap E_n \cap E_{n+1}) \geq P(E_1) + P(E_2) + ... + P(E_n) + P(E_{n+1}) - n$

We use the Bonferron's inequality for any n events $E_1, E_2, ..., E_{n-1}$ and $E_n \cap E_{n+1}$ to get

$$
\begin{gathered}
P(E_1 \cap E_2 \cap ... \cap E_{n-1}(E_n \cap E_n + 1)) \geq P(E_1) + P(E_2) + ... + P(E_{n-1}) + P(E_n \cap E_{n+1}) - (n-1) \\
\text{where } P(E_n \cap E_{n+1}) \geq P(E_n) + P(E_{n+1}) - 1, \text{ and therefore}
\end{gathered}
$$

$$\boxed{P(E_1 \cap E_2 \cap ... \cap E_n \cap E_{n+1} \geq P(E_1) + P(E_2) + ... + P(E_{n-1}) + P(E_n) + P(E_{n+1}) - 1 - (n-1)}$$

## Problem 9

Given $E_1, E_2, ..., E_n$ are independent events

$$
\begin{aligned}
&= P(E_1 \cup E_2 \cup ... \cup E_n) \\
&= 1 - P((E_1 \cup E_2 \cup ... \cup E_n)^c) \\
&= 1 - P(\overline{E_1} \cap \overline{E_2} \cap ... \cap \overline{E_n}) \qquad \text{by De Morgan's Law}
\end{aligned}
$$

Therefore by independence we have

$$
\begin{aligned}
&= P(E_1 \cup E_2 \cup ... \cup E_n) \\
&= 1 - P(\overline{E_1} \cap \overline{E_2} \cap ... \cap \overline{E_n}) \\
&= 1 - \prod_{i=1}^{n} P(\overline{E_i}) \\
&= \boxed{1 - \prod_{i=1}^{n} 1 - (P(\overline{E_i}))}
\end{aligned}
$$

## Problem 10

### Part 1

The probability of winning the game $P(W)$ is dependent on the road taken, we have

$P(W|R_1) = \frac{2}{3}$ and $P(W|R_2) = \frac{2}{5}$

Since $P(R_1) = \frac{1}{2}$ and $P(R_2) = \frac{1}{2}$,

we add $\frac{2}{3} * \frac{1}{2} + \frac{2}{5} * \frac{1}{2}$,

to get $\boxed{P(W) = \frac{8}{15}}$

### Part 2

Conditioned on the event that the game was won the probability that you took Road 1 is described as

$P(R_1|W) = \frac{P(W \cap R_1)}{P(W)}$

$\frac{P(W|R_1)P(R_1)}{P(W|R_1)P(R_1)+P(W|R_2)P(R_2)} = \frac{\frac{2}{3}*\frac{1}{2}}{\frac{2}{3}*\frac{1}{2}+\frac{2}{5}*\frac{1}{2}}$

$\boxed{P(R_1|W) = \frac{5}{8}}$, so the probability is 62.5%.