MTH 463 Homework #3

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Problem 1

$$
\begin{aligned}
E[X] &= 0 * \frac{1}{3} + 2 * \frac{1}{2} + 3 * \frac{1}{6} \\
&= \boxed{1.5} \\
E[X^2] &= 0^2 * \frac{1}{3} + 2^2 * \frac{1}{2} + 3^2 * \frac{1}{6} \\
&= \boxed{\frac{7}{2}} \\
Var(X) &= E[X^2] - E[X]^2 \\
&= \frac{7}{2} - \frac{9}{4} \\
&= \boxed{\frac{5}{4}} \\
E[X - E[X]] &= E[X - 1.5] \\
&= 1.5\frac{1}{3} + 0.5\frac{1}{2} + 1.5\frac{1}{6} \\
&= \boxed{1} \\
E[2^X] &= 2^0\frac{1}{3} + 2^2\frac{1}{2} + 2^3\frac{1}{6} \\
&= \boxed{\frac{11}{3}}
\end{aligned}
$$

Problem 2

$$
\begin{aligned}
E[\frac{1}{X+1}] &= \sum_{k=0}^{\infty} \frac{1}{k+1} e^{-\lambda} \frac{\lambda^k}{k!} \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{(k+1)!} \\
&= e^{-\lambda} \sum_{m=1}^{\infty} \frac{\lambda^{m-1}}{m} \\
&\text{where m = k + 1} \\
&= \frac{e^{-\lambda}}{\lambda} \sum_{m=1}^{\infty} \frac{\lambda^m}{m!} \\
&= \frac{e^{-\lambda}}{\lambda} (e^{\lambda} - 1) \\
&= \boxed{\frac{1 - e^{\lambda}}{\lambda}}
\end{aligned}
$$

Problem 3

$$
\begin{aligned}
E[\frac{1}{X+1}] &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} * \frac{1}{k+1} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} \frac{n!}{(k+1)!(n-k)!} p^k (1-p)^{n-k} \\
&= \frac{1}{n+1} \sum_{k=0}^{n} \frac{(n+1)!}{(k+1)!(n-k)!} p^k (1-p)^{n-k} \\
&= \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k+1} p^k (1-p)^{p-k} \\
&= \frac{1}{n+1} \sum_{m=1}^{n} \binom{n+1}{m} p^{m-1} (1-p)^{(n+1)-m} \\
&\text{where m} = \text{k} + 1 \\
&= \frac{1}{(n+1)p} \sum_{m=1}^{n} \binom{n+1}{m} p^{m} (1-p)^{(n+1)-m} \\
&= \frac{1}{(n+1)p} \sum_{m=0}^{n} \binom{n+1}{m} p^{m} (1-p)^{(n+1)-m-(1-p)^{n+1}} \\
&= \frac{1}{(n+1)p} ((p+(1-p))^n + 1 - (1-p)^{n+1}) \\
&= \boxed{\frac{1-(1-p)^{n+1}}{(n+1)^p}}
\end{aligned}
$$

Problem 4

$$
\begin{aligned}
\sum_{j=1}^{\infty} P(X \geq j) &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} P(X=k) \\
&= \sum_{k=1}^{\infty} kP(X=k) \\
&\text{since the double sum is the sum of all different integer pairs (j,k) such that} \\
&= E[X]
\end{aligned}
$$

Problem 5

$$
\begin{aligned}
P(W > j) &= \sum_{i=j+1}^{\infty} p(1-p)^{i-1} \\
&= \sum_{k=0}^{\infty} (1-p)^k \\
&= p(1-p)^j * \frac{1}{1-(1-p)} \\
&= (1-p)^j \\
&\text{from the previous problem we have} \\
E[W] &= \sum_{i=1}^{\infty} P(W \geq i) \\
&= \sum_{i=1}^{+\infty} P(W > j) \\
&= \sum_{i=1}^{+\infty} (1-p)^j \\
&= \boxed{\frac{1}{p}}
\end{aligned}
$$

Problem 6

$$
\begin{aligned}
E[a^X] &= a^1 * p + a^{-1} * (1-p) \\
&= 1 \\
&\text{therefore,} \\
&pa^2 - a + (1-p) = 0, \\
&\text{so, } a = \frac{1 \pm \sqrt{1-4p(1-p)}}{2p} = \frac{1 \pm (2p-1)}{2p} \\
&\text{which yields either 1 or } \frac{1}{1-p} \\
&\text{so, } \boxed{a = \frac{1}{1-p}}
\end{aligned}
$$

Problem 7

$$
\begin{gathered}
Var(X) = E[X - \mu^2] \geq 0 \\
\text{therefore,} \\
0 \leq Var(X) = E[X^2] - E[X]^2 \\
\text{thus,} \\
E[X^2] \geq E[X]^2
\end{gathered}
$$

Problem 8

We know that $E[aX + b] = aE[X] + b$, and

$$
\begin{aligned}
E[Y] &= \frac{1}{\sigma} E[X - \frac{\mu}{\sigma}] \\
&= \boxed{0}
\end{aligned}
$$

Therefore using the definition of variance we have

$$
\begin{aligned}
Var(Y) &= E[Y^2] - E[Y]^2 \\
&= E[Y^2] \\
&= E[\frac{(X-\mu)^2}{\sigma^2}] \\
&= \frac{1}{\sigma^2} E[(X-\mu)^2] \\
&= \frac{1}{\sigma^2} Var(X) \\
&= \boxed{1}
\end{aligned}
$$

Problem 9

We consider two separate binomial independent random variables, X as (3,p), and Y as (5,p). X obviously represents the 3-engine rocket while Y represents the 5-engine rocket. Thus we must find $P(X \geq 2) > P(Y \geq 3)$. The probability mass function of this is

$$\binom{3}{2}p^2(1-p)+\binom{3}{3}p^3 > \binom{5}{3}p^3(1-p)^2+\binom{5}{4}p^4(1-p)+\binom{5}{5}p^5$$

which reduces by dividing by p^2

$$3(1-p)+p > 10p(1-p)^2+5p^2(1-p)+p^3$$

$$3-3p > 10p-10p^3+5p^2-5p^3+p^3$$

$$0 > (-6p^2+9p-3)(1-p)$$

$$0 > 6(p-\frac{1}{2})(1-p)$$

thus by factoring p $< \frac{1}{2}$

For values of p where, $\boxed{p < \frac{1}{2}}$, a 3-engine rocket would be more reliable than a 5-engine rocket.