STAT 421 Assignment #5

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3.148

We have the movement-generating function for Y as

$$m(t) = \frac{pe^t}{1-qe^t}, \text{ where } q = 1-p$$

Differentiating we have,

$$\begin{aligned}
E(Y) &= \frac{\delta}{\delta t} m(t)\Big|_{t=0} \\
&= \frac{\delta}{\delta t}\left(\frac{pe^t}{1-qe^t}\right) \\
&= \frac{pe^t(1-qe^t) - pe^t(-qe^t)}{(1-qe^t)^2}\Big|_{t=0} \\
&= \frac{pe^t}{1-qe^t} + \frac{pqe^{2t}}{(1-qe^t)^2} \\
&= \frac{p}{1-q} + \frac{pq}{(1-q)^2} \\
&= \frac{p}{p} + \frac{p(1-p)}{p^2} && \text{as } 1-p = q, p = 1-q \\
&= 1 + \frac{1-p}{p} \\
&= \frac{p+1-p}{p} \\
&= \boxed{\frac{1}{p}}
\end{aligned}$$

for $E(Y^2)$ we have

$$
\begin{aligned}
E(Y^2) &= \frac{\delta^2}{\delta t^2} m(t)\Big|_{t=0} \\
&= \frac{\delta}{\delta t}\left(\frac{\delta}{\delta t}\frac{pe^t}{1-qe^t}\right) \\
&= \frac{\delta}{\delta t}\left(\frac{pe^t(1-qe^t) - pe^t(-qe^t)}{(1-qe^t)^2}\right)\Big|_{t=0} \\
&= \frac{pe^t}{1-qe^t} + \frac{3pqe^{2t}}{(1-qe^t)^2} + \frac{2pqe^{3t}}{(1-qe^t)^3}\Big|_{t=0} \\
&= \frac{p}{1-q} + \frac{3pq}{(1-q)^2} + \frac{2pq^2e^{3t}}{(1-q)^3} \\
&= \frac{p}{p} + \frac{3pq}{p^2} + \frac{2pq^2}{p^3} \\
&= 1 + \frac{3q}{p} + \frac{2pq^2}{p^3} \\
&= \frac{p^2 + 3pq + 2q^2}{p^2} \\
&= \frac{p^2 + 3p(1-p) + 2(1-p)^2}{p^2} \\
&= \boxed{\frac{2-p}{p^2}}
\end{aligned}
$$

The variance is calculated as $V(Y) = E(Y^2) - E(Y)^2$ so we have

$$
\begin{aligned}
V(Y) &= \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 \\
&= \frac{2-p-1}{p^2} \\
&= \boxed{\frac{1-p}{p^2}}
\end{aligned}
$$

3.154

a.

We have $m(t) = ((\frac{1}{3})e^t + \frac{2}{3})^5$, so using the movement generating function of binomial random varialbe’s the mean of a random variable is

$$
\begin{aligned}
E(Y) &= \frac{\delta}{\delta t} m(t)\Big|_{t=0} \\
&= \frac{\delta}{\delta t}(\frac{1}{3}e^t + \frac{2}{3})^5 \\
&= 5(\frac{1}{3}e^t + \frac{2}{3})^{5-1} * \frac{1}{3}e^t\Big|_{t=0} \\
&= \boxed{\frac{5}{3}}
\end{aligned}
$$

For a random variable squared, and subsequently the variance we have

$$
\begin{aligned}
E(Y^2) &= \frac{\delta^2}{\delta t^2} m(t)\Big|_{t=0} \\
&= \frac{5}{3}\frac{\delta}{\delta t}(\frac{1}{3}e^t + \frac{2}{3})^{5-1})\Big|_{t=0} \\
&= \frac{20}{9}(\frac{1}{3}e^t + \frac{2}{3})^3 e^{2t} + \frac{5}{3}(\frac{1}{3}e^t + \frac{2}{3})^4 e^t\Big|_{t=0} \\
&= \frac{35}{9} \\
V(Y) &= E(Y^2) - E(Y)^2 \\
&= \frac{35}{9} - \frac{5}{3}^2 \\
&= \boxed{\frac{10}{9}}
\end{aligned}
$$

b.

We have $m(t) = \frac{pe^t}{(1-qe^t)}$ and the given mgf $m(t) = \frac{e^t}{2-e^t}$ which can be rewritten as $\frac{\frac{1}{2}e^t}{1-\frac{1}{2}e^t}$.

Once again for a random variable we have

$$
\begin{aligned}
E(Y) &= \frac{\delta}{\delta t} m(t)\Big|_{t=0} \\
&= \frac{\delta}{\delta t} \frac{\frac{1}{2}e^t}{1-\frac{1}{2}e^t}\Big|_{t=0} \\
&= \frac{1}{2}\frac{e^t}{1-\frac{1}{2}e^t} + \frac{1}{4}\frac{(e^t)^2}{(1-\frac{1}{2}e^t)^2}\Big|_{t=0} \\
&= \boxed{2}
\end{aligned}
$$

For a random variable squared, and subsequently the variance we have

$$
\begin{aligned}
E(Y^2) &= \frac{\delta^2}{\delta t^2} m(t)\Big|_{t=0} \\
&= \frac{\delta^2}{\delta t^2} \frac{\frac{1}{2}e^t}{1-\frac{1}{2}e^t}\Big|_{t=0} \\
&= \frac{1}{2}(\frac{\frac{1}{2}e^t}{1-\frac{1}{2}e^t}) + \frac{3}{4}(\frac{\frac{1}{2}e^t}{1-\frac{1}{2}e^t})^2 + \frac{1}{4}(\frac{\frac{1}{2}e^t}{1-\frac{1}{2}e^t})^3\Big|_{t=0} \\
V(Y) &= E(Y^2) - E(Y)^2 \\
&= 6 - (2)^2 \\
&= \boxed{2}
\end{aligned}
$$

c.

We are given the movement generating function $m(t) = e^{2(e^t-1)}$, thus the mean of a random variable is

$$
\begin{aligned}
E(Y) &= \frac{\delta}{\delta t} m(t)\Big|_{t=0} \\
&= \frac{\delta}{\delta t}(e^{2(e^t-1)})\Big|_{t=0} \\
&= 2e^t e^{2e^t-2}\Big|_{t=0} \\
&= \boxed{2}
\end{aligned}
$$

For a random variable squared, and subsequently the variance we have

$$
\begin{aligned}
E(Y^2) &= \frac{\delta^2}{\delta t^2} m(t)\Bigg|_{t=0} \\
&= \frac{\delta}{\delta t}(e^{2(e^t-1)})\Bigg|_{t=0} \\
&= 2e^t e^{2e^t-2} + 4(e^t)^2 e^{2e^t-2}\Bigg|_{t=0} \\
&= 6 \\
V(Y) &= E(Y^2) - E(Y)^2 \\
&= 6 - (2)^2 \\
&= \boxed{2}
\end{aligned}
$$

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Given $W = aY + b$ the movement generating function $m_w(t)$ is defined as

$$
\begin{aligned}
m_w(t) &= E(e^{tw}) \\
&= E(e^{t(aY+b)}) \\
&= E(e^{bt}e^{(at)Y}) \\
&= \boxed{e^{bt}m(at)}
\end{aligned}
$$

3.160

a.

$$
\begin{aligned}
E(Y*) &= E(n - Y) \\
&= n - E(Y) \\
&= n - np \\
&= n(1-p) \\
&= \boxed{nq}
\end{aligned}
$$

$$
\begin{aligned}
V(Y*) &= V(n - y) \\
&= V(y) \\
&= \boxed{npq}
\end{aligned}
$$

b.

The movement generating function of Y, given that it is binomial and has parameters p and n is given by

$$\begin{aligned} m_{Y^*}(t) &= E(e^{tY^*}) \\ &= E(e^{t(n-Y)}) \\ &= E(e^{nt}e^{(-t)Y}) \\ &= e^{nt}m(-t) \\ &= \boxed{(pe^t+q)^n} \end{aligned}$$

c.

Since we know $m_{Y^*}(t) = (pe^t + q)^n$, we can say that Y^* has a binomial distribution

d.

Since Y is the number of successes in a sample size n, then $Y^* = (n - y)$ must represent the number of failures in sample size n.

e.

The answers to parts a,b, and c are obvious based on the answer in d because the classification of success and failure in the Bernoulli trial is arbitrary.