Math 463/563
Homework #2 - Due Wednesday, October 14

1. How many 5-digit numbers can be formed with digits $1, 2, \ldots, 9$ if no digit can appear exactly once? (For instance, 43443 and 88888 are OK, while 43413 is not counted.)

2. We shuffle a deck of six cards $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{4}$, $\boxed{5}$, and $\boxed{6}$ so that each of the 6! possible configurations (orderings) has equal probability of $\frac{1}{6!}$. Let A be the event that card $\boxed{1}$ is among the top three cards in the deck, and let B be the event that card $\boxed{5}$ ends up being second from the top. Find the probability of the event $A \cup B$.

3. Simplify $(E \cup F) \cap (E \cup \overline{F})$, and $(E \cup F) \cap (\overline{E} \cup F) \cap (E \cup \overline{F})$

4. Consider events E, F,and G. Find the expressions in E, F and G for the following events.

   (a) At least one of the three events occurs.
   (b) At most one of the three events occurs.
   (c) Exactly two of them occur.
   (d) At most two of the three events occur.
   (e) All three events occur.
   (f) None of the three events occurs.
   (g) At most three of the events occur.
   (h) E or F, but not G occur.
   (i) Both E and F, but not G occur.
   (j) Exactly one of the three events occurs.

   Example: The event that "only G occurs" is expressed as $\overline{E} \cap \overline{F} \cap G$.

5. Prove that

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(\overline{E} \cap F \cap G) - P(E \cap \overline{F} \cap G) - P(E \cap F \cap \overline{G}) - 2P(E \cap F \cap G)$$

6. Prove that

$$P(\overline{E} \cap \overline{F} \cap \overline{G}) = 1 - P(E) - P(F) - P(G) + P(E \cap F) + P(E \cap G) + P(F \cap G) - P(E \cap F \cap G)$$

7. Given events E, F and G, such that $P(F) > P(F \cap G) > 0$. Prove directly that

$$P(E|F) = P(E|F \cap G) \cdot P(G|F) + P(E|F \cap \overline{G}) \cdot P(\overline{G}|F)$$

8. Use induction to generalize Bonferroni's inequality to n events:

$$P(E_1 \cap E_2 \cap \ldots \cap E_n) \geq P(E_1) + P(E_2) + \cdots + P(E_n) - (n-1)$$

9. Prove that if $E_1, E_2, \ldots, E_n$ are independent events, then

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = 1 - \prod_{i=1}^{n} \big(1 - P(E_i)\big)$$

10. In a video game, you have a choice between two roads, "Road 1" and "Road 2". Suppose you don't know where the roads will take you, and select a road at random with probability $1/2$ for each choice. Road 1 takes you to a castle, where your character defeats the dragon and wins the game with probability $2/3$ or looses to the dragon with probability $1/3$. Road 2 takes you to a cave, where your character defeats the goblin and wins the game with probability $2/5$ or looses to the goblin with probability $3/5$. Find the probability of winning the game. Conditioned on the event that the game was won, what is the probability that you took Road 1?