STAT 421 Assignment #8

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5.9

a.

$$\begin{aligned} 1 &= \int_0^1 \int_0^{y_2} k(1-y_2)dy_1 dy_2 \\ &= k\int_0^1 \Bigg[y_1(-y_2+1) \Bigg]_0^{y_2} dy_2 \\ &= k\int_0^1 y_2(-y_2+1)dy_2 \\ &= k\Bigg[-\frac{1}{3}y_2^3 + \frac{1}{2}y_2^2 \Bigg]_0^1 \\ &= \frac{k}{6} \\ &\boxed{k=6} \end{aligned}$$

b.

$$\begin{aligned} P(Y_1 \leq \frac{3}{4}, Y_2 \geq \frac{1}{2}) &= \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 6(1-y_2)dy_1 dy_2 + \int_{\frac{1}{2}}^{\frac{3}{4}} \int_{y_2}^1 6(1-y_2)dy_2 dy_1 \\ &= 6\int_{\frac{1}{2}}^1 \frac{1-y_2}{2} dy_2 + 6\int_{\frac{1}{2}}^{\frac{3}{4}} \frac{3}{32} dy_1 \\ &= \frac{24}{64} + \frac{7}{64} \\ &= \boxed{\frac{31}{64}} \end{aligned}$$

5.16

a.

$$
\begin{aligned}
P(Y_1 < \frac{1}{2}, Y_2 > \frac{1}{4}) &= \int_{\frac{1}{4}}^{1} \int_{0}^{\frac{1}{2}} (y_1 + y_2) dy_1 dy_2 \\
&= \int_{\frac{1}{4}}^{1} \frac{1}{2} y_2 + \frac{1}{8} dy_2 \\
&= \int_{\frac{1}{4}}^{1} \frac{1}{2} y_2 + \frac{1}{8} \\
&= \frac{1}{2} \int_{\frac{1}{4}}^{1} y \, dy_2 + \frac{15}{64} \\
&= \frac{1}{8} - \frac{1}{8 * 4} + \frac{15}{64} \\
&= \boxed{\frac{21}{64}}
\end{aligned}
$$

b.

$$
\begin{aligned}
P(Y_1 + Y_2 \leq 1) &= P(Y_1 \leq 1 - Y_2) \\
P(Y_1 \leq 1 - Y_2) &= \int_{0}^{1} \int_{0}^{1-y_2} (y_1 + y_2) dy_1 dy_2 \\
&= \boxed{\frac{1}{3}}
\end{aligned}
$$

5.24

a.

$$f_{Y_1}(y_1) = \int_0^1 1 dy_2$$
$$= [y_2]_0^1$$
$$= 1$$
$$f_{Y_2}(y_2) = \int_0^1 1 dy_1$$
$$= [y_1]_0^1$$
$$= 1$$

So,

$$\boxed{f_{Y_1}(y_1) = 1;\ 0 \leq y_1 \leq 1}$$
$$\boxed{f_{Y_2}(y_2) = 1;\ 0 \leq y_1 \leq 1}$$

b.

$$P(0.3 < Y_1 < 0.5) = \int_{0.3}^{0.5} f_{Y_1}(y_1) dy_1$$
$$= \int_{0.3}^{0.5} 1\ dy_1$$
$$= \boxed{0.2}$$

$P(0.3 < Y_2 < 0.5)$ is the same integral so we also have

$$P(0.3 < Y_2 < 0.5) = \boxed{0.2}$$

c.

$$f_{Y_1|Y_2}(y_1|y_2) = \frac{f(y_1, y_2)}{f(y_2)}$$
$$= 1$$

Where

$0 < y_1 < 1$

$0 < y_2 < 1$

therefore,

$$\boxed{0 \leq y_2 \leq 1}$$

d.

$$\begin{aligned} f(y_1|y_2) &= \frac{f(y_1, y_2)}{f(Y_2 = y_2)} \\ &= 1 \\ f(y_1) &= 1 \\ &\text{therefore,} \\ &\boxed{0 \leq y_1 \leq 1} \end{aligned}$$

e.

$$\begin{aligned} P(0.3 < Y_1 < 0.5|Y_2 = 0.3) &= \int_{0.3}^{0.5} f(y_1|Y_2 = 0.3)dy_1 \\ &= \int_{0.3}^{0.5} 1dy_1 \\ &= 0.5 - 0.3 \\ &= \boxed{0.2} \end{aligned}$$

f.

$$\begin{aligned} P(0.3 < Y_1 < 0.5|Y_2 = 0.5) &= \int_{0.3}^{0.5} f(y_2|Y_1 = 0.5)dy_2 \\ &= \int_{0.3}^{0.5} 1dy_2 \\ &= \boxed{0.2} \end{aligned}$$

g.

Since we know $f(y_1) = f(y_1|Y_2 = y_2)$. Since $f(y_1|Y_2 = y_2)$ is independent of y_2. Therefore the answers are the same.

5.38

a.

The joint density function for Y_1 and Y_2 has the limits $0 \leq y_2 \leq y_1 \leq 1$, therefore the marginal density function of Y_2 is given by

$$
\begin{aligned}
f(y_2) &= \int_{y_1}^{1} f(y_1, y_2) \\
&= \int_{y_1}^{1} \frac{1}{y_1} dy_1 \\
&= \Big[ln(y_1) \Big] \\
&= -ln(y_2) \\
\text{Therefore, } f(y_2) &= \begin{cases} -ln(y_2), & 0 \le y_2 \le 1 \\ 0, & \text{otherwise} \end{cases} \\
f(y_1, y_2) &= f(y_2|y_1) f(y_1) \\
&= \frac{1}{y_1} * 1 \\
&= \boxed{\begin{cases} 1/y_1, & 0 \le y_2 \le y_1, 0 \le y_1 \le 1 \\ 0, & \text{otherwise} \end{cases}}
\end{aligned}
$$

b.

$$
\begin{aligned}
P(Y_2 > \frac{1}{4} | Y_1 = \frac{1}{2}) &= \int_{\frac{1}{4}}^{\frac{1}{2}} f(y_2 | Y_1 = \frac{1}{2}) dy_2 \\
&= \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{(1/2)} dy_2 \\
&= 2 \int_{\frac{1}{4}}^{\frac{1}{2}} dy_2 \\
&= 2(\frac{1}{2} - \frac{1}{4}) \\
&= \boxed{\frac{1}{2}}
\end{aligned}
$$

There is a 1 in 2 chance that she sells more than a quarter ton given the supplier stocks a half ton of the item.

c.

$$
\begin{aligned}
P(Y_2 > \frac{1}{2}|Y_1 = \frac{1}{4}) &= \int_{\frac{1}{2}}^{1} f(y_1|Y_2 = \frac{1}{4})dy_1 \\
&= \int_{\frac{1}{2}}^{1} \frac{-1}{y_1 * ln(\frac{1}{4})}dy_1 \\
&= -\frac{1}{ln(4)} * [ln(y_1)]_{\frac{1}{2}}^{1} \\
&= \frac{1}{1.3863} * (ln(1 - ln(\frac{1}{2}))) \\
&= \boxed{\frac{1}{2}}
\end{aligned}
$$

Therefore the probability that she has stocked more than a half ton given that the supplier sold a quarter ton is 1 out of 2.

5.48

Given that Y_1 and Y_2 are independent we have

$$
\begin{aligned}
p(Y_1 = 0, Y_2 = 0) &= p(Y_1 = 0)p(Y_2 = 0) \\
0.38 &= 0.55(0.76) \\
&= 0.418 \\
0.38 &\neq 0.418 \\
\text{Therefore, } &Y_1 \text{ and } Y_2 \text{ are not independent.}
\end{aligned}
$$

5.68

a.

For $(Y_1, Y_2) = (0, 0)$ the joint probability is:

$$
\begin{aligned}
p(Y_1, Y_2) &= p_1(y_1)(y_2) \\
&= \binom{2}{y_1}(.2)^{y_1}(1 - 0.2)^{2-y_1} * \binom{1}{y_2}(0.3)^{y_2}(1 - 0.3)1 - y_2 \\
&= \frac{2!}{0!2!} * \frac{1!}{0!1!} * 0.2^0 * 0.3^0 * 0.8^2 * 0.7^1 \\
&= \boxed{0.448}
\end{aligned}
$$

For $(Y_1, Y_2) = (1, 0)$ the joint probability is:

$$\begin{aligned} &= \frac{2!}{1!1!} * \frac{1!}{0!1!} * 0.2^1 * 0.3^0 * 0.8^1 * 0.7^1 \\ &= \boxed{0.224} \end{aligned}$$

For $(Y_1, Y_2) = (2, 0)$ the joint probability is:

$$\begin{aligned} &= \frac{2!}{0!2!} * \frac{1!}{0!1!} * 0.2^2 * 0.3^0 * 0.8^0 * 0.7^1 \\ &= \boxed{0.028} \end{aligned}$$

For $(Y_1, Y_2) = (0, 1)$ the joint probability is:

$$\begin{aligned} &= \frac{2!}{0!2!} * \frac{1!}{0!1!} * 0.2^0 * 0.3^1 * 0.8^2 * 0.7^0 \\ &= \boxed{0.192} \end{aligned}$$

For $(Y_1, Y_2) = (1, 1)$ the joint probability is:

$$\begin{aligned} &= \frac{2!}{1!1!} * \frac{1!}{0!1!} * 0.2^1 * 0.3^1 * 0.8^1 * 0.7^0 \\ &= \boxed{0.096} \end{aligned}$$

For $(Y_1, Y_2) = (2, 1)$ the joint probability is:

$$\begin{aligned} &= \frac{2!}{2!0!} * \frac{1!}{0!1!} * 0.2^2 * 0.3^1 * 0.8^0 * 0.7^0 \\ &= \boxed{0.012} \end{aligned}$$

b.

The probability of interest is

$$\begin{aligned} P(Y_1 + Y_2 \leq 1) &= p(0,0) + p(1,0) + p(0,1) \\ &= \boxed{0.864} \end{aligned}$$

5.72

a.

$$
\begin{aligned}
E(Y_1) &= \sum y_1 p(y_1) \\
&= 0 * \frac{4}{9} + 1 * \frac{4}{9} + 2 * \frac{1}{9} \\
&= \boxed{\frac{2}{3}}
\end{aligned}
$$

b.

$$
\begin{aligned}
E(Y_1)^2 &= \sum y_1^2 p(y_1) \\
&= 0^2 * \frac{4}{9} + 1^2 * \frac{4}{9} + 2^2 * \frac{1}{9} \\
&= \boxed{\frac{8}{9}}
\end{aligned}
$$

c.

$$
\begin{aligned}
V(Y_1) &= E(Y_1)^2 - E(Y_1^2) \\
&= \frac{8}{9} - (\frac{2}{3})^2 \\
&= \boxed{\frac{4}{9}}
\end{aligned}
$$

5.84

a.

$$
\begin{aligned}
E(Y_1) &= \frac{1}{p} \\
E(Y_2) &= \frac{1}{p} \\
E(Y_1 - Y_2) &= E(Y_1) - E(Y_2) \\
&= \frac{1}{p} - \frac{1}{p} \\
&= \boxed{0}
\end{aligned}
$$

b.

$$E(Y_1^2) = \boxed{\frac{2-p}{p^2}}$$

$$E(Y_2^2) = \boxed{\frac{2-p}{p^2}}$$

c.

$$\begin{aligned} E(Y_1 - Y_2)^2 &= E(Y_1^2) + E(Y_2^2) - 2E(Y_1Y_2) \\ &= \frac{2-p}{p^2} + \frac{2-p}{p^2} - \frac{2}{p^2} \\ &= \frac{2(1-p)}{p^2} \\ V(Y_1 - Y_2) &= V(Y_1^2) + V(Y_2^2) - 0 \\ &= (E(Y_1^2) - E(Y_1)^2) + (E(Y_2^2) - E(Y_2)^2) \\ &= \left(\frac{2-p}{p^2} - \frac{1}{p^2}\right) + \left(\frac{2-p}{p^2} - \frac{1}{p^2}\right) \\ &= \boxed{\frac{2(1-p)}{p^2}} \end{aligned}$$

d.

Applying Chebyshev's theorem with k = 3 we have

$$\begin{aligned} \text{Limits} &= \mu \pm 3\sigma \\ &= \frac{2(1-p)}{p^2} \pm 3\left(\frac{\sqrt{2(1-p)}}{p}\right) \\ &= \boxed{\left(\frac{2(1-p)}{p^2} - 3\left(\frac{\sqrt{2(1-p)}}{p}\right), \frac{2(1-p)}{p^2} + 3\left(\frac{\sqrt{2(1-p)}}{p}\right)\right)} \end{aligned}$$

5.94

a.

Given,

$$
\begin{aligned}
Cov(Y_1, Y_2) &= E[(Y1 - \mu_1)(Y2 - \mu_2)] \\
&= E(Y1Y2) - E(Y1)E(Y2) \\
&\text{therefore,} \\
Cov(U_1, U_2) &= E[(U_1 - \mu_1)(U_2 - \mu_2)] \\
&= E(U_1U_2) - E(U_1)E(U_2)
\end{aligned}
$$

So, substituting U_1, U_2 we have

$$
\begin{aligned}
Cov(U_1, U_2) &= E((Y_1 + Y_2)(Y_1 - Y_2)) - E(Y_1 + Y_2)E(Y_1 - Y_2) \\
&= E(Y_1^2 - Y_2^2) - (E(Y_1)^2 - E(Y_2)^2)^2
\end{aligned}
$$

and we know that $E(Y_1) = \mu_1$, $E(Y_2) = \mu_2$, $\sigma = E((X - \mu)^2)$, so we have

$$
\begin{aligned}
E(Y_1^2 - Y_2^2) - (E(Y_1)^2 - E(Y_2)^2)^2 &= (\sigma_1^2 + \mu_1^2) - (\sigma_2^2 + \mu_2^2) - (\mu_1^2 - \mu_2^2)^2 \\
&= \boxed{\sigma_1^2 - \sigma_2^2}
\end{aligned}
$$

b.

Given that $p(U_1, U_2) = \frac{Cov(U_1,U_2)}{\sqrt{Var(U_1)}\sqrt{Var(U_2)}}$ we have

$$
\frac{\sigma_1^2 - \sigma_2^2}{\sqrt{\sigma_1^2 + \sigma_2^2}\sqrt{\sigma_1^2 - \sigma_2^2}} = \boxed{\frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 - \sigma_2^2}}
$$

c.

It is possible for $Cov(U_1, U_2) = 0$, this occurs when there is no correlation between U_1 and U_2, which is to say mathematically, when $\sigma_1^2 = \sigma_2^2$.

5.100

a.

As Z is a standard normal random variable is has $\mu = 0, \sigma = 1$. Given that $Y_1 = Z$, and $Y_2 = Z^2$

$$
\begin{aligned}
E(Y_1) &= E(Z) \\
&= \boxed{0} \\
E(Y_2) &= E(Z^2) \\
&= 1 + 0 \\
&= \boxed{1}
\end{aligned}
$$

b.

$$
\begin{aligned}
E(Y_1 Y_2) &= E(Z^3) \\
&= \int_{-\infty}^{\infty} z^3 f(z) dz \quad \text{all infinitely bounded odd functions integrate to 0} \\
&= \boxed{0}
\end{aligned}
$$

c.

$$
\begin{aligned}
Cov(Y_1, Y_2) &= E(Z^3) - E(Z)E(Z^2) \\
&= \boxed{0}
\end{aligned}
$$

d.

Given that $P(Y2 > 1 | Y1 > 1) = 1$, we have that

$$
\begin{aligned}
P(Y_2 > 1 | Y_1 > 1) &= 1 \\
P(Z^2 > 1 | Z > 1) &= 1
\end{aligned}
$$

and by definition for 2 events to be independent $P(A|B) = P(A)$, but $P(Z^2 > 1 \neq 1)$ since

$$
\begin{aligned}
P(Y_2 > 1 | Y_1 > 1) &= 1 \\
\Rightarrow \frac{P(Y_2 > 1 | Y_1 > 1)}{P(Y_1 > 1)} &= 1 && \text{by } P(A|B) = P(A) \\
P(Y_2 > 1 | Y_1 > 1) &= P(Y_1 > 1) \\
&\neq P(Y_2 > 1) * P(Y_1 > 1)
\end{aligned}
$$

and therefore Y_1 and Y_2 are dependent.