Homework 1

Duncan Gates

9/28/2020

Question 1

Prove: $\sum_{k=0}^{n} \binom{n}{k}(-1)^k = 0$

$\sum_{k=0}^{n} \binom{n}{k}(-1)^k = \sum_{k=0}^{n} \binom{n}{k}(-1)^k(1)^{n-k}$

$\{1-1\}^n = 0^n = 0$

Question 2

Prove $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

Given $\binom{n}{k} = \binom{n}{n-k}$

apply this to $\sum_{k=0}^{n} \binom{n}{k}^2$ to get $\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$ by the multiplicative rule of counting

Which is $\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \ldots + \binom{n}{n}\binom{n}{0}$ when summed over k going from 0 to n.

So we can conclude that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

Question 3

In the word REDRESSER there is one D, three E's, three R's, and two S's.

The distinct number of letter combinations is therefore:

$\frac{9!}{3!3!2!} = \frac{362{,}880}{(6)(6)(2)} = \boxed{5,040}$

Question 4

We have 12 people in committees of 3, 4, and 5. Going first by committee size 3 we have

$\binom{12}{3}\binom{9}{4}\binom{5}{5}$

$\frac{12!}{3!*9!} * \frac{9!}{4!*5!}\frac{5!}{5!*0!}$ using $\binom{n}{x} = \frac{n!}{x!(n-x)!}$

We can then cancel $\frac{12!}{3!*\cancel{9!}} * \frac{\cancel{9!}}{4!*\cancel{5!}}\frac{\cancel{5!}}{5!*0!}$

To get $\frac{12!}{3!*4!*5!}$

Which is $\boxed{27,720}$, so there are 27,720 possible divisions.

Question 5

The number of positive solutions of the equation is $x_1 + x_2 + ... + x_r = n$ is $\binom{n-1}{k-1}$

Therefore the number of solutions of $x_1 + x_2 + x_3 + x_4 = 49$ is $\binom{49-1}{4-1} = \binom{48}{3}$

$\frac{48!}{3!*(48-3)!} = \frac{48*47*46}{3*2*1}$

Which gives us $\boxed{17,296}$ possible integer solutions.

Question 6

Given $x_1 + x_2 + x_3 + x_4 = 49$,

Let $y_1 = x_1,\ y_2 = x_2 - 1,\ y_3 = x_3 - 2,\ y_4 = x_4 - 3$

So $x_1 = y_1,\ x_2 = y_2 + 1,\ x_3 = y_3 + 2,\ x_4 = y_4 + 3$

Then $x_1 + x_2 + x_3 + x_4 = 49$ becomes $y_1 + y_2 + y_3 + y_4 = 49 - (1 + 2 + 3) = 43$

Where $y_i \geq 0,\ 1 \leq i \leq 4$

Thus we have $n = 43,\ r = 4$ which gives us $\binom{43-1}{4-1}$

Which is $\frac{43!}{3!(43-3)!} = \frac{42*41*40*\cancel{39!}}{3*2*1*\cancel{39!}}$

So there are $\boxed{11,480}$ possible integer solutions.

Question 7

Let the total number of steps be $n = 12$, the steps to the right be $b_1 = 7$, and the steps up be $b_2 = 5$

Using the multinomial rule we know $\binom{n}{b_1, b_2} = \frac{n!}{b_1!b_2!}$

So we have $\frac{12!}{7!*5!} = \frac{12*11*10*9*8}{5*4*3*2*1} = 792$

There $\boxed{792}$ possible paths.

Question 8

There are $\binom{7}{3}$ paths from A to C and $\binom{5}{2}$ paths from C to B.

Therefore, by the multiplicative rule of counting we have $\binom{7}{3}\binom{5}{2}$

Which is $\frac{7!}{3!(7-3)!} * \frac{5!}{2!(5-2)!} = 350$

So there are $\boxed{350}$ possible paths.

Question 9

There are 4 aces and 52 cards, and each of the 4 players will get $52/4 = 13$ cards

Further there are $4!$ ways to distribute aces so that each person receives one and

The remaining 48 cards must be distributed so that each person receives 12 of them

So we have $\frac{48!}{12!*12!*12!*12!}$ ways to be distributed.

Together we have $\boxed{4! * \frac{48!}{12!^4}}$ which is a very large number.