MTH 463 Homework #4

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Problem 1

I check that f(x) is a probability density function by finding if the cumulative distribution function is 1

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \int_{-2}^{\infty} \frac{8dx}{x^3} \\ &= \left[\frac{8x^{-2}}{-2}\right]_2^{\infty} \\ &= \boxed{1}\end{aligned}$$

So we know that f(x) is indeed a probability density function.

To find $P(X > 5)$

$$
\begin{aligned}
P(X > 5) &= \int_5^\infty \frac{8dx}{x^3} \\
&= \left[\frac{8x^{-2}}{-2}\right]_5^\infty \\
&= \boxed{\frac{4}{25}}
\end{aligned}
$$

To find $E[X]$ we use

$$
\begin{aligned}
E[X] &= \int_{-\infty}^\infty xf(x)dx \\
&= \int_2^\infty \frac{8dx}{x^2} \\
&= \left[\frac{8x^{-1}}{-1}\right]_2^\infty \\
&= \boxed{4}
\end{aligned}
$$

Problem 2

I find c by integrating knowing the probability density function is equal to 1

$$
\begin{aligned}
1 &= \int_{-\infty}^\infty f(x)dx \\
&= \int_1^2 c(x-1)^4 dx \\
&= \left[\int_1^2 \frac{c(x-1)^5}{5}\right]_1^2 \\
&= \frac{c}{5}
\end{aligned}
$$

$$
\text{therefore, } \boxed{c = 5}
$$

To find $E[X]$ we have

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} xf(x)dx \\
&= \int_{1}^{2} 5x(x-1)^4 dx \\
&= \int_{1}^{2} 5((x-1)+1)(x-1)^4 dx \\
&= \int_{1}^{2} 5(x-1)^5 dx + \int_{1}^{2} 5(x-1)^4 dx \\
&= \left[\frac{5(x-1)^6}{6}\right]_1^2 + 1 \\
&= \frac{5}{6} + 1 \\
&= \boxed{\frac{11}{6}}
\end{aligned}$$

Problem 3

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} f(x)dx \\
&= \int_{0}^{1} (ax^2 + bx)dx \\
&= \left[ax^3\frac{1}{3} + bx^2\frac{1}{2}\right] \\
&= \frac{1}{3}a + \frac{1}{2}b
\end{aligned}$$

$$
\begin{aligned}
0.75 &= E(X) \\
&= \int_{-\infty}^{\infty} x f(x) dx \\
&= \int_0^1 (ax^3 + bx^2) dx \\
&= \left[a\frac{x^4}{4} + b\frac{x^3}{3} \right]_0^1 \\
&= \frac{1}{4}a + \frac{1}{3}b
\end{aligned}
$$

So we have $1 = \frac{1}{3}a + \frac{1}{2}b$ and $0.75 = \frac{1}{4}a + \frac{1}{3}b$ which reduces to $b = 0, a = 3$.

Therefore,

$$
\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx \\
&= \int_1^0 3x^4 dx \\
&= \left. \frac{3x^5}{5} \right|_0^1 \\
&= \boxed{\frac{3}{5}}
\end{aligned}
$$

and,

$$
\begin{aligned}
Var(X) &= E(X^2) - E(X)^2 \\
&= \frac{3}{5} - 0.75^2 \\
&= \boxed{0.0375}
\end{aligned}
$$

Problem 4

$$
\begin{aligned}
P(1 < X < 3) &= \int_1^3 F(x)dx \\
&= \int_{-\infty}^3 F(x)dx - \int_{-\infty}^1 F(x)dx \\
&= F(3) - F(1) \\
&= \frac{15}{6} - \frac{3}{4} \\
&= \boxed{\frac{3}{16}}
\end{aligned}
$$

Problem 5

Given $4x^2 + 4xY - Y + 6 = 0$ with $\lambda = 3$, the roots of Y ($\frac{-4Y \pm \sqrt{16Y^2 + 16(Y-6)}}{8}$) will only be positive iff $16Y^2 + 16(Y-6) \geq 0$. So we need to find

$$
\begin{aligned}
P(16Y^2 + 16(Y-6) \geq 0) &= P(Y^2 + Y - 6 \geq 0) \\
&= P((Y+3) * (Y-2) \geq 0) \\
&= P(Y \leq -3) + P(Y \geq 2) \\
&= 0 + e^{-2\lambda} \\
&= \boxed{e^{-6}}
\end{aligned}
$$

Problem 6

$$
\begin{aligned}
\tau(\alpha+1) &= \int_0^\infty e^{-y}y^\alpha dy \\
&= \int_0^\infty (-e^{-y})'y^\alpha dy \\
&= (-e^{-y}y^\alpha)\Big|_0^\infty - \int_0^\infty e^{-y}(y^\alpha)'dy \\
&= 0 + \int_0^\infty e^{-y}y^{\alpha-1}dy \\
&= \alpha\tau(\alpha) \\
\tau(1) &= \int_0^\infty e^{-y}dy \\
&= 1 = 0!
\end{aligned}
$$

Therefore $\tau(2) = 1 * \tau(1) = 1!$

Problem 7

$$
\begin{aligned}
E[X^k] &= \int_0^\infty (\frac{y}{\lambda})^k e^{-y}dy \\
&= \frac{1}{\lambda^k}\int_0^\infty e^{-y}y^{(k+1)-1}dy \\
&= \frac{\tau(k+1)}{\lambda^k} \\
&= \frac{k!}{\lambda^k}
\end{aligned}
$$

Problem 8

$$
\begin{aligned}
E[e^{-x}] &= \frac{1}{\tau(\alpha)}\int_0^\infty \lambda e^{-x}e^{-\lambda x}(\lambda x)^{\alpha-1}dx \\
&= \frac{\lambda}{\tau(\alpha)}\int_0^\infty e^{-(\lambda+1)x}(x)^{\alpha-1}dx && \text{Let } y=(\lambda+1)x \\
&= \frac{\lambda}{\tau(\alpha)}\int_0^\infty e^{-y}\frac{y^{\alpha-1}}{(\lambda+1)^{\alpha-1}}\frac{1}{\lambda+1}dy \\
&= \left(\frac{\lambda}{\lambda+1}\right)^\alpha \frac{1}{\tau(\alpha)}\int_0^\infty e^{-y}y^{\alpha-1}dy \\
&= \left(\frac{\lambda}{\lambda+1}\right)^\alpha
\end{aligned}
$$

Problem 9

We have $t > 0$, $f(t) = \lambda e^{-\lambda t}$, $F(t) = 1 - e^{-\lambda t}$.

$$
\begin{aligned}
h(t) &= \frac{f(t)}{1-F(t)} \\
&= \frac{\lambda e^{-\lambda t}}{1-(1-e^{-\lambda t})} \\
&= \lambda
\end{aligned}
$$

The memorylessness property therefore makes it such that the “hazard rate” is constant at all times