# AN APPLICATION OF BANACH'S FIXED POINT THEOREM TO THE NONLINEAR INVERSE PROBLEMS

# ỨNG DỤNG ĐỊNH LÍ ĐIỂM BẤT ĐỘNG BANACH ĐỐI VỚI BÀI TOÁN KHẢ NGHỊCH PHI TUYẾN

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***Abstract:*** *The fixed point theory has applications in many problems, such as variational inequalities, the approximation theory, nonlinear analysis, etc. Banach's fixed point theorem is one of the famous results in mathematical analysis. The main aim of this survey is to present an application of Banach's fixed point theorem to the nonlinear inverse problems.*

***Keywords:*** *Nonlinear inverse problems, Banach's fixed point theorem, contractive mapping.*

***Tóm tắt:*** *Lý thuyết điểm bất động có nhiều ứng dụng trong các lĩnh vực như bất đẳng thức biến phân, lý thuyết xấp xỉ, giải tích phi tuyến,... Định lí điểm bất động Banach là một trong những kết quả nổi tiếng của Giải tích toán học. Mục đích chính của bài báo là trình bày ứng dụng của định lí điểm bất động Banach đối với bài toán khả nghịch phi tuyến.*

***Từ khóa:*** *Bài toán khả nghịch phi tuyến, định lí điểm bất động Banach, ánh xạ co.*

## 1. INTRODUCTION

Nonlinear inverse problems arise from many practical applications that include inverse scattering problems, biomedical imaging and so on. A nonlinear inverse problem is difficult problem and it is often represented in the form of an operator equation,

$$Fx = y, \qquad (1)$$

where $F$ is a nonlinear operator mapping between two spaces $X, Y$. Various sufficient conditions for the inevitability of nonlinear operators were studied by many authors, see [1, 4, 6-8] and references therein.

Banach's fixed point theorem states sufficient conditions for the existence and uniqueness of a fixed point for a class of operators, called contractions. It also gives an iterative process by which we can obtain approximations to the fixed point and error bounds, see [2, 3, 5] for details. This theorem has many application in several domains, such as differential equations, integral equations, economics and several others. In this paper, we consider the application of Banach's fixed point theorem to the nonlinear inverse problems.

The paper is organized as follows. Section 2 presents some notations, definition and lemma needed to establish our main results. In Section 3, we give some necessary and sufficient conditions for the invertibility of nonlinear operator, which map from a normed linear space into a Banach space. Some conclusions are drawn in Section 4.

## 2. PRELIMINARIES

Throughout this section, we assume that $X, Y$ are normed linear spaces. Let $Lip(X,Y)$ be the space of all operator $F: X \to Y$ such that

$$\|F\|^* = sup\left\{\|Fx_1 - Fx_2\|\|x_1 - x_2\|^{-1} : x_1, x_2 \in X, x_1 \neq x_2\right\} < \infty \tag{2}$$

and let $Lip_0(X,Y) = \{F \in Lip(X,Y) : F(0) = 0\}$. Clearly, $\|\cdot\|^*$ is a seminorm on $Lip(X,Y)$ and a norm on $Lip_0(X,Y)$. If $F \in Lip_0(X,Y)$ is the linear operator, then $\|F\|^*$ concides with customary norm $\|F\|$.

The following definition and lemma play an important role in our consideration.

**Definition 1.** Let $X, Y$ be two normed linear spaces. For any operator $F: X \to Y$, we introduce the numerical characteristic

$$\|F\|_* = \inf\left\{\|Fx_1 - Fx_2\|\|x_1 - x_2\|^{-1} : x_1, x_2 \in X, x_1 \neq x_2\right\} < \infty. \tag{3}$$

**Remark 1.** If $\|F\|_* > 0$ then obviously $F$ is injective operator.

**Lemma 1.** *Assume that the operator $A: X \to Y$ has an inverse and $\|A\|_* > 0$. Then $A^{-1} \in Lip(Y,X)$ and*

$$\|A^{-1}\|^* \leq \frac{1}{\|A\|_*}. \tag{4}$$

*Proof.* It follows from the assumptions that

$$\|Ax_1 - Ax_2\| \geq \|A\|_* \|x_1 - x_2\|, \forall x_1, x_2 \in X, x_1 \neq x_2.$$

Then

$$\|y_1 - y_2\| \geq \|A\|_* \|A^{-1}y_1 - A^{-1}y_2\|,$$

where $y_i = Ax_i, i = 1, 2$. By Remark 1, we have

$$\frac{\|A^{-1}y_1 - A^{-1}y_2\|}{\|y_1 - y_2\|} \leq \frac{1}{\|A\|_*},$$

which implies that $A^{-1} \in Lip(Y,X)$ *and* $\|A^{-1}\|^* \leq \frac{1}{\|A\|_*}$. The proof is complete. W

## 3. MAIN RESULTS

We first establish necessary and sufficient conditions of invertibility for nonlinear operator.

**Theorem 1.** *Let $X$ be a normed linear space, $Y$ be a Banch space and let $F: X \to Y$. Then the inverse $F^{-1}: Y \to X$ exists if and only if there exists an invertible operator $A: X \to Y$ such that $\|A\|_* > 0$ and $\|F - A\|^* < \|A\|_*$.*

*Proof.* Let such an invertible operator $A: X \to Y$ *such that* $\|A\|_* > 0$ *and* $\|F - A\|^* < \|A\|_*$ exists.

Then equation (1) is equivalent to

$$Ax + (F - A)x = y. \tag{5}$$

We rewrite equation (5) as

$$u = \Phi u, \tag{6}$$

where $u = Ax, \Phi u = -(F - A)A^{-1}u + y$. It follows from the assumptions and Lemma 1 that

$$\begin{aligned}\|\Phi u_1 - \Phi u_2\| &= \left\|(F-A)A^{-1}u_1 - (F-A)A^{-1}u_2\right\| \\ &\le \|F-A\|^* \left\|A^{-1}u_1 - A^{-1}u_2\right\| \\ &\le \|F-A\|^* \left\|A^{-1}\right\|^* \|u_1 - u_2\| \\ &\le \|F-A\|^* \frac{1}{\|A\|_*} \|u_1 - u_2\| \\ &= q\,\|u_1 - u_2\|\end{aligned}$$

for any $u_1, u_2 \in Y,$ where $q = \dfrac{\|F-A\|^*}{\|A\|_*} < 1.$ Hence $\Phi$ is a contractive operator with contraction coefficient equal to $q<1$. By Banach's fixed point theorem, equation (6) has a unique solution for any $y \in Y$, i.e., equation (1) has a unique solution for any $y \in Y.$ Consequently, $F$ is invertible, i.e., the inverse $F^{-1}: Y \to X$ exists.

Conversely, assume that the inverse $F^{-1}: Y \to X$ exists. If we set $A = F$ then $\|A\|_* = \|F\|_* > 0$ and $\|F-A\|^* = 0 < \|A\|_*$ . This completes the proof. W

The next result show that the inverse $F^{-1}$ is Lipschitz continuous.

**Theorem 2.** *Let the assumptions of Theorem 1 be sastisfied. Then*

$$\left\|F^{-1}y_1 - F^{-1}y_2\right\| \le \frac{\left\|A^{-1}\right\|^*}{1-q}\|y_1 - y_2\|, \forall y_1, y_2 \in Y,$$

*where* $q = \dfrac{\|F-A\|^*}{\|A\|_*} < 1.$

*Proof.* As in the proof of Theorem 1, equation (1) is equivalent to (5)

$$Ax + (F-A)x = y.$$

We rewrite equation (5) as

$$x = Qx, \tag{7}$$

where $Qx = -A^{-1}(F-A)x + A^{-1}y.$ Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $Fx_1 = y_1, Fx_2 = y_2$ . From the assumptions and Lemma 1, we have

$$\begin{aligned}\|x_1 - x_2\| &= \|Qx_1 - Qx_2\| \\ &= \left\|-A^{-1}(F-A)x_1 + A^{-1}y_1 + A^{-1}(F-A)x_2 - A^{-1}y_2\right\| \\ &\le \left\|A^{-1}(F-A)x_1 - A^{-1}(F-A)x_2\right\| + \left\|A^{-1}y_1 - A^{-1}y_2\right\| \\ &\le \left\|A^{-1}\right\|^* \left\|(F-A)x_1 - (F-A)x\right\| + \left\|A^{-1}\right\|^* \|y_1 - y_2\| \\ &\le \left\|A^{-1}\right\|^* \|F-A\|^* \|x_1 - x_2\| + \left\|A^{-1}\right\|^* \|y_1 - y_2\| \\ &\le \frac{\|F-A\|^*}{\|A\|_*}\|x_1 - x_2\| + \left\|A^{-1}\right\|^* \|y_1 - y_2\| \\ &= q\|x_1 - x_2\| + \left\|A^{-1}\right\|^* \|y_1 - y_2\|,\end{aligned}$$

where $q = \dfrac{\|F-A\|^*}{\|A\|_*} < 1.$ Hence

$$\|x_1 - x_2\| \le \frac{\left\|A^{-1}\right\|^*}{1-q}\|y_1 - y_2\|,$$

i.e.,

$$\left\|F^{-1}y_1 - F^{-1}y_2\right\| \le \frac{\left\|A^{-1}\right\|^*}{1-q}\|y_1 - y_2\|, \forall y_1, y_2 \in Y.$$

This proves the Theorem. W

**Corollary 1.** *Let the assumptions of Theorem 2 be satisfied and* $A, F \in Lip_0(X,Y).$ *Then* $F^{-1} \in Lip_0(Y,X).$

*Proof.* The proof follows immediately from Theorem 2 and the assumptions $A, F \in Lip_0(X,Y).$ W

**Example 1.** Let the function $F: X \to Y$ be defined by $Fx = x - \sin\frac{x}{2}, \forall x \in X$ We define the mapping $A: X \to Y$ by $Ax = x, \forall x \in X$ Obviously, $A$ is an invertible mapping and $\|A\|_* = 1$.

On the other hand, for any $x_1, x_2 \in X, x_1 \neq x_2$ we have

$$|(F-A)x_1 - (F-A)x_2| = \left|\sin\frac{x_1}{2} - \sin\frac{x_2}{2}\right| \leq \left|\frac{x_1}{2} - \frac{x_2}{2}\right| = \frac{1}{2}|x_1 - x_2|,$$

so that $\|F-A\|^* = \frac{1}{2} < \|A\|_* = 1$. Therefore the conditions of Theorem 1 are satisfied. Then $F$ is invertible.

**Example 2.** We consider the following equation in Hilbert space $\mathrm{H}$

$$Fx = y, \tag{8}$$

where the operator $F: \mathrm{H} \to \mathrm{H}$ is strongly monotone and Lipschitz continuous.

By virtue of strong monotonicity of the operator $F$, there exists a constant $C > 0$ such that

$$\langle Fx_1 - Fx_2, x_1 - x_2 \rangle \geq C\|x_1 - x_2\|^2, \forall x_1, x_2 \in \mathrm{H}.$$

Using Schwarz's inequality, we have

$$\|Fx_1 - Fx_2\| \geq C\|x_1 - x_2\|, \forall x_1, x_2 \in \mathrm{H},$$

so that $L = \|F\|^* \geq C$, where $L$ is Lipschitz constant of the operator $F$.

We define the operator $A: \mathrm{H} \to \mathrm{H}$ by

$$Ax = rx, \forall x \in \mathrm{H},$$

where $r = \frac{L^2}{C}$. Obviously, $A$ is an invertible operator and $\|A\|_* = r > 0$.

On the other hand, for any $x_1, x_2 \in \mathrm{H}$, $x_1 \neq x_2$ we have

$$\begin{aligned}
\|(F-A)x_1 - (F-A)x_2\|^2 &= \|Fx_1 - Fx_2 - r(x_1 - x_2)\|^2 \\
&= \|Fx_1 - Fx_2\|^2 + r^2\|x_1 - x_2\|^2 - 2r\langle Fx_1 - Fx_2, x_1 - x_2 \rangle \\
&\leq (L^2 + r^2 - 2rC)\|x_1 - x_2\|^2 \\
&= \frac{L^2 + r^2 - 2rC}{r^2} r^2 \|x_1 - x_2\|^2 \\
&= \left(\frac{C^2}{L^2} + 1 - 2\frac{C^2}{L^2}\right) r^2 \|x_1 - x_2\|^2 \\
&= \left(1 - \frac{C^2}{L^2}\right) r^2 \|x_1 - x_2\|^2.
\end{aligned}$$

Hence

$$\|(F-A)x_1 - (F-A)x_2\| \leq \sqrt{1 - \frac{C^2}{L^2}}\ r\|x_1 - x_2\|,$$

which implies that

$$\|F-A\|^* = r\sqrt{1 - \frac{C^2}{L^2}} < \|A\|_* = r.$$

Consequently, the conditions of Theorem 1 are satisfied. Then $F$ is invertible, i.e., equation (8) has a unique solution.

## 4. CONCLUSION

In this paper, Banach's fixed point theorem has been successfully applied to nonlinear inverse problems. We have established necessary and sufficient conditions for the invertibility of nonlinear operator, which maps from a normed linear space into a Banach space. Several examples are provided to demonstrate the proposed results.

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