WALDSCHMIDT CONSTANT OF CERTAIN SETS OF POINTS IN PROJECTIVE PLANE WITH TWO SUPPORTING LINES

HẰNG SỐ WALDSCHMIDT CỦA TẬP ĐIỂM NẰM TRONG MẶT PHẲNG XÃ ẢNH VỚI HAI ĐƯỜNG THẲNG ĐI QUA

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ABSTRACT: *We computed the Waldschmidt constant for some special cases of points in projective plane with two supporting lines, each line contains at least 3 points.*

Keywords: *Waldschmidt constant, initial degree, zero-scheme, fat points.*

TÓM TẮT: *Chúng tôi đã tính toán hằng số Waldschmidt cho một số trường hợp đặc biệt của các tập điểm trong mặt phẳng xạ ảnh với 2 đường thẳng đi qua, mỗi đường thẳng chứa ít nhất 3 điểm.*

Từ khóa: *Hằng số Waldschmidt, bậc dẫn đầu, chiều không, điểm béo.*

1. INTRODUCTION TOWALDSCHMIDT CONSTANT OF A SET OF POINTS

Definition. Let $P_1,...,P_r$ be a set of r distinct points in $\mathrm{P}_k^{\,n}$, where k is an algebraically closed field. Let ρ_i be the ideal corresponding to the point P_i for $1 \le i \le r$.

Let $Z = m_1P_1 + ... + m_rP_r = \sum_{i=1}^{r} m_iP_i$ where $m_1,...,\mathrm{m}_r$ are positive integers, be the fat point in P^n corresponding to the ideal $J = \bigcap_{i=1}^{r} \rho^{m_i}$. Denote $\alpha(\mathrm{J}) = \min\{\mathrm{t} \mid J_t \ne 0\}$ the initial degree of J. The constant $\alpha(\mathrm{J})$ is the least degree of the hypersurfaces containing P_i with multiplicity at least m_i, for $1 \le i \le r$, $\lim_{m\to\infty} \frac{\alpha(\mathrm{I}^{(m)})}{m}$ is called the Waldschmidt constant of I or of the set X and denoted by $\gamma(I)$ or $\gamma(\mathrm{X})$.

Lemma 1.1.

(1) $\gamma(I)$ is well defined and $1 \le \gamma(I) \le \frac{\alpha(\mathrm{I}^{(m)})}{m} \le \alpha(\mathrm{I})$, $\forall m \ge 1$.

(2) $\gamma(I) \le \sqrt[n]{r}$.

Proof. See [4].

The constant is firstly introduced by Waldschmidt [18]. Since then, some results about lower bounds of this constant was achieved as in [3, 7, 10, 14, 15, 19]. Those papers also gave many interesting conjectures for lower bounds of the constants, most of them recently had been proved as in [1, 5, 6,8]. Computation of $\alpha(I^{(m)})$ and $\gamma(I)$ is very hard in general, even for cases of small numbers of points in the projective plane. Some results of computation of the constants could be mentioned as for free square monomial ideals as in [2], for points in star configuration as in [1], for some certain sets as in [16] or for set of r + 1 almost collinear points in P^2, see [11]. For a set of small number of points in general position in P^2, the Waldschmidt constant was known only for $1 \le r \le 9$ or r is a perfect square, as Nagata's results, see [12,13]. Recently, the constanst were computer for certain sets of $r+s$ points s points where there are r collinear points and 1 ≤ s ≤ 7, see [17]. Note that, if s = 1, it is the case of almost collinear as in [11]. In this paper, we will continue to consider some special cases with two supporting lines. For proofs in next section, we need to use following results of Bezout.

Theorem 1.2 ([5], I.7.7). Let Y be a variety of dimension at least 1 in P^n, and let H be a hypersurface not containing Y . Let $Z_1,\dots,Z_r$ be the irreducible components of Y ∩ H. Then

$$\sum_{j=1}^{s} i(H,Y,Z_j)\deg Z_j = (\deg Y)(\deg H).$$

Note that $i(H,Y,Z_j)$ is the intersection multiplicity if Y and H along Z_j.

Corollary 1.3 (Bezout's Theorem). Let Y, Z be distinct curves in P^2 having degrees d, e. Let Y ∩ Z = {P1, ..., Ps}. Then

$$\sum_{j=1}^{s} i(H,\mathrm{Z},\mathrm{P}_j) = de.$$

Note that $i(H,\mathrm{Z},\mathrm{P}) \ge \mathrm{mult}_P(H)\cdot \mathrm{mult}_P(Z)$ and the equality holds if and only if H and Z have no tangent in common at P, see [4].

2. MAIN RESULTS

We will compute the Waldschmidt constant for following cases:

• Sets of at least 5 points lying on 2 lines, each line contains at least 3 points; or all the points lying on an irreducible conic. This computation implies the initial degree of some relevant sets with equal multiplicity.

• Sets of 6 points, except exactly one, lying on 2 lines, each line contains 3 points. This computation implies the initial degree of sets of 6 relevant points with equal multiplicity.

We will use the following lemma as a result in [17].

Lemma 2.1. Let $X = (P_1,\dots,P_r,Q_1,Q_2) \subset \mathrm{P}^2$ where $P_1,\dots,P_r$, for $r \ge 2$, are collinear and Q_1,Q_2 are not on the line containing $P_1,\dots,P_r$. Then $\gamma(I)=2.$

Proof. Let $P_i \subset k[x,y,z]$ be the ideal corresponding to P_i for $1 \le i \le r$, let $Q_1,Q_2 \subset k[x,y,z]$ be the ideals corresponding to Q_1,Q_2. Let $L_0 = V(z)$ be the line containing $P_1,\dots,P_r$ and $V(l)$ be the line connecting Q_1,Q_2. It is easy to see that $\alpha(\mathrm{I}^{(\mathrm{m})}) \le 2m$ since $z^m l^m \in I^{(m)}_{2m}$, where $I = \bigcap_{i=1..r} \rho_i \cap Q_1 \cap Q_2$.

Suppose that there exits a nonzero $f \in I^m_{2m-1}$. We see that $f(z=0)$ has degree at most 2m − 1 but vanishes at $P_1,\dots,P_r$ with multiplicity at least m, therefore by Bezout's Theorem, we have L_0 is a component of $V(\mathrm{f})$. Similarly, we see that $V(\mathrm{l})$ is another component of $V(\mathrm{f})$. Then $f = z^a l^b g$, where zz Œc g, lŒ g, $\deg(g) = \mathrm{c}$ and

$a+b+c=2m-1.$ We see that $ra+c \geq rm,\ 2b+c \geq 2m.$ Then $ra+2b+2c \geq rm+2m.$ It implies that $(r-2)a+2(a+b+c)$ or $= (r-2)a+2(2m-1) \geq rm+2m$ $(r-2)a \geq (r-2)m+2$ then $a > m.$ Since $a+b+c = 2m-1$ we have $b+c < m-1.$

But $2b+c = b+(b+c) \geq 2m,$ it implies that $b > m+1.$ But then it contradicts to $a+b+c=2m-1.$ Therefore we have $\alpha(I^{(m)}) \geq 2m.$ It means that $\alpha(I^{(m)}) = 2m$ and $\gamma(\mathrm{X}) = 2.$

Theorem 2.2. *Let* $X=\{P_1,...,P_r\} \subset \mathrm{P}^2$ *for* $r \geq 5,$ *let* $I = \bigcap_{i=1}^{r} \rho_i^{m_i}$ *such that they lie either on 2 distinct lines, each line contains at least 3 points or on an irreducible conic. Then* $\alpha(I^{(m)}) = 2m$ *and* $\gamma(\mathrm{X}) = 2.$.

Proof. Consider the case they lie on two lines, each line contains at least 3 points. Let $\ell_1 = V(x)$, $\ell_2 = V(z)$, be the two lines, see Figure 1. It is easy to see that $x^m z^m \in I_{2m}^{(m)}$, thus $\alpha(I^{(m)}) \leq 2m$ for all $m \geq 1$. Then $\gamma(X) \leq 2$.

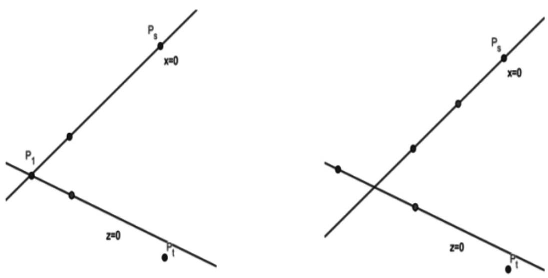

Figure 1. *r* points on 2 lines

Let $f \in I_{2m-1}^{(m)}$ for $m \geq 1$. When $m=1$, it is easy to see $I_1 = 0$. We see that $x \mid f, z \mid f$. Therefore, we can write $f = xzf_1.$ If there is no point in the corresponding to $Y=\{P_1,...,P_r\} \subset \mathrm{P}^2.$ If each line of l_1, l_2 contains at least 3 points of Y then by the above $I_{2(m-1)-1}^{(m-1)} = 0.$ If l_1 consists of 2 points of Y then by Lemma 2.1, we have $I_{2(m-1)-1}^{(m-1)} = 0.$ Thus $f_1 = 0$ then $f = 0$ and $\alpha(\mathrm{I}^{(m)}) \geq 2m, \gamma(X) \geq 2.$

Now consider the case all point of X lie on an irreducible conic $C = Y(g_0)$, see Figure 2. Since $g_0^m \in I_{2m}^{(m)}$, for $m \geq 1,$ we have $\alpha(I^{(m)}) \leq 2m.$ Thus $\gamma(X) \leq 2$.

Let $f \in I_{2m-1}^{(m)}$, for $m \geq 1$. When $m=1$, it is clearmthat $I_1^{(1)} = 0.$ For $m \geq 2,$ since $2(2m-1) < \mathrm{rm},$ we have $g_0 \mid f$ by Bezout's. Then we can write $f = g_0 f_1.$ Since g_0 has multiplicity exacly 1 at each point, we see that $f_1 \in I_{2(m-1)-1}^{(m-1)}.$ By induction, we have $I_{2(m-1)-1}^{(m-1)} = 0$ then $f = 0.$ Thus $\alpha(I^{(m)}) \geq 2m$ and $\gamma(X) \geq 2.$

For all the cases, we have $\alpha(I^{(m)}) = 2m$ and $\gamma(X) = 2.$

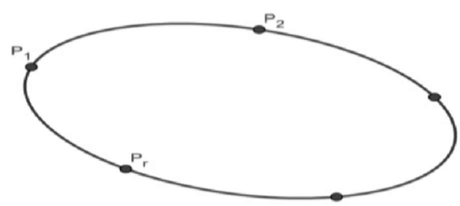

Figure 2. *r* points on an irreducible conic

Theorem 2.3. Let $\{\mathrm{P}_1,...,\mathrm{P}_6\}$ such that $P_1, P_2, P_3 \in l_1,$ $P_1, P_4, P_5 \in l_2,$

$P_6 \notin l_1 \cup l_2$. Then $\gamma(\mathrm{X}) = \frac{7}{3}$.

Proof. Let $I = \cap_{i=1}^{6} \rho_i \subset k[\mathrm{x},\mathrm{y},\mathrm{z}]$. Let $l_1 = V(\mathrm{z}), \mathrm{l}_2 = V(x)$ and l be the linear form vanishing at P_1 and P_6, see Figure 3. intersection of two lines, then $f_1 \in I_{2(m-1)-1}^{(m-1)}$, and by induction, we have $f_1 = 0$, and then $f = 0$. If $P_1 \in l_1 \cap l_2$ and $m \geq 2$. Then, $f_1 \in I_{2(m-1)-1}^{(m-1)}$, where J be the ideal Let $C = V(f_0)$ be a conic containing 5 points $\mathrm{P}_2, P_3, ..., \mathrm{P}_6$.

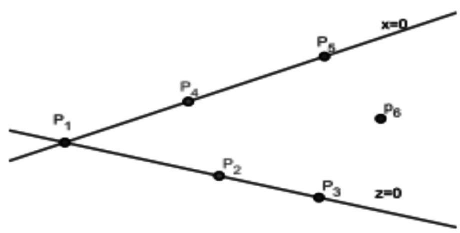

Figure 3. 6 points with two supporting lines

Then, $xzlf_0^2 \in I_7^{(3)}$. Thus, $\gamma(X) \leq 7/3$.

For a given $m \geq 1$, suppose that there exists $f \in I_{7m-1}^{(3m)}$. Since $3.3m = 9m > 7m-1$, we see that $x \mid f, z \mid f$. Let $f = z^a x^b g$ where $\deg(g) = c$ and no one of z, x is a divisor of g. Then $a+b+c = 7m-1$. We see that $2c < 2(3m-a)+2(3m-b)+3m = 15m-2(a+b)$ since $2(a+b+c) = 2(7m-1) < 15m$. Thus, $f_0 \mid g$. Let $f = z^a x^b f_0^d h$, where $\deg(h) = e$ and no of x, z, f_0 is a divisor of h. We have $a+b+2\mathrm{d}+e = 7m-1$. Since f_0 is not a divisor of h, we have

$$2\mathrm{e} \geq 2(3m-a-d)+2(3m-b-d)+(3m-d)$$
$$= 15m-2(a+b)-5\mathrm{d}.$$

This implies that $d \geq m+2 \geq 3$. Consider the line $V(l)$ containing P_1 and P_6. We see that

$$e < (3m-a-b)+(3m-d) = 6m-(a+b+2\mathrm{d})+d$$
$$= 6m-(7m-e)+d = e+d-m.$$

Since $d \geq m+2$. It implies that $l \mid h$. Now we can write $f = z^a x^b f_0^d f_1$ where $f_1 \in I_{7(m-1)-1}^{(3m-3)}$. For $m = 1$ There is a contradiction since $d \geq 3$ but $a+b+2\mathrm{d}+e = 6$. Therefore, $I_{6m}^{(3m)} = 0$. By induction, we have $I_{7m-1}^{(3m)} = 0$. Thus, $\gamma(X) \geq \frac{7}{3}$.

Corollary 2.4. *Let* $X = \{\mathrm{P}_1, ..., \mathrm{P}_6\}$ *as in the theorem. Let* $I = \cap_{i=1}^{6} \rho_i$. Then $\alpha(I^{(3m)}) = 7m$, for any $m \geq 1$.

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