Tutorial 8

- Paper
- Pen
- Calculator

Q1. A 256ms portion of an analog signal is sampled at a rate of 16kHz and the resulting L samples are saved for further processing. What is L? The 256 point DFT of these samples is computed.

a. What is the frequency spacing in Hz of the computed DFT values?
b. What is the total number of required multiplications if the computations are done directly using the definition of DFT?
c. What is the total number of required multiplications if the L samples are first wrapped modulo 256 and then 256-point DFT is computed?
d. What is the total number of required multiplications if a 256-point FFT is computed of the wrapped signal?

Q2. By definition, the first and second of Fibonacci numbers are 0 and 1 (e.g. h(0)=0 and h(1)=1), and each subsequent number is the sum of the previous two.

a. Write the Fibonacci sequence h(n) for n=0, 1, …,9 described by the above definition?
b. Calculate 4-point FFT of the sequence of ten numbers in (a) using the definition in matrix form.
c. Recomputed by first reducing x modulo 4 (wrapped signal) and then computing the 4-DFT of the result.
d. Finally, compute the 4-point IDFT of the result and verify that you recover the mod-4 wrapped version of x.

Q3. Compute the 8-point FFT of the length-8 signal x, in which these samples are the first 8 samples of $x(n)=4\cos(\pi n/2)+\cos(\pi n)$, discuss whether the 8 computed FFT values accurately represent the expected spectrum of x(n). What FFT indices correspond to the two frequencies of the cosinusoids?

Q4. The 8-point DFT X of an 8-point sequence x is given by

$$X=[0,4,-4j,4,0,4,4j,4].$$

a. Using the FFT algorithm, compute the inverse IFFT: x=IFFT(X).
b. Using the given FFT X, express x as a sum of real-valued (co)sinusoidal signals.

Q5.

a. Compute the DFT of the sequence $x(n) = \{1,0,2,0\}$
b. Without explicitly computing the DFT sum, find the DFT of the sequence $y(n) = \{0,2,0,1\}$, using your answer to part (a)
c. Find the inverse DFT of the sequence $X(m) = \{1, e^{-j3\pi/4}, 0, e^{j3\pi/4}\}$
d. Without explicitly computing the inverse DFT sum, find the inverse DFT of the sequence $Y(m) = \{1, e^{-j\pi/4}, 0, e^{j\pi/4}\}$, using your answer to part(c).

Q1. A 256ms portion of an analog signal is sampled at a rate of 16kHz and the resulting L samples are saved for further processing. What is L? The 256 point DFT of these samples is computed.

a. What is the frequency spacing in Hz of the computed DFT values?
b. What is the total number of required multiplications if the computations are done directly using the definition of DFT?
c. What is the total number of required multiplications if the L samples are first wrapped modulo 256 and then 256-point DFT is computed?
d. What is the total number of required multiplications if a 256-point FFT is computed of the wrapped signal?

Q1 (answer)

Use the formula $T_R = LT = L/f_s$ to solve for $L = f_s T_R$, or in words

$$\boxed{\text{no. of samples} = (\text{no. of samples per second}) \times (\text{no. of seconds})}$$

Thus, $L = 8 \text{ kHz} \times 128 \text{ msec} = 1024$ samples. The frequency spacing is $\Delta f = f_s/N = 8000/256 = 31.25$ Hz. If the DFT is done directly, we would require $L \cdot N = 1024 \cdot 256 = 262144$ multiplications. If the signal is reduced mod-256, then the number of multiplications would be $N \cdot N = 256 \cdot 256 = 65536$. And if the FFT is used, $N \log_2 N/2 = 256 \cdot 8/2 = 1024$.
The cost of performing the mod-256 reduction is $N(M-1) = 256 \cdot 3 = 768$ additions, where $M = L/N = 1024/256 = 4$ is the number of segments. This cost may be added to the costs of FFT or reduced DFT.

Q2. By definition, the first and second of Fibonacci numbers are 0 and 1 (e.g. $h(0)=0$ and $h(1)=1$), and each subsequent number is the sum of the previous two.

a. Write the Fibonacci sequence $h(n)$ for $n=0, 1, \ldots, 9$ described by the above definition?
b. Calculate 4-point FFT of the sequence of ten numbers in (a) using the definition in matrix form.
c. Recomputed by first reducing x modulo 4 (wrapped signal) and then computing the 4-DFT of the result.
d. Finally, compute the 4-point IDFT of the result and verify that you recover the mod-4 wrapped version of x.

$$
A = \left[\begin{array}{cccc|cccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & W_4 & W_4^2 & W_4^3 & W_4^4 & W_4^5 & W_4^6 & W_4^7 \\
1 & W_4^2 & W_4^4 & W_4^6 & W_4^8 & W_4^{10} & W_4^{12} & W_4^{14} \\
1 & W_4^3 & W_4^6 & W_4^9 & W_4^{12} & W_4^{15} & W_4^{18} & W_4^{21}
\end{array}\right]
$$

$$
= \left[\begin{array}{cccc|cccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & W_4 & W_4^2 & W_4^3 & 1 & W_4 & W_4^2 & W_4^3 \\
1 & W_4^2 & W_4^4 & W_4^6 & 1 & W_4^2 & W_4^4 & W_4^6 \\
1 & W_4^3 & W_4^6 & W_4^9 & 1 & W_4^3 & W_4^6 & W_4^9
\end{array}\right]
$$

$$
= \left[\begin{array}{rrrr|rrrr}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -j & -1 & j & 1 & -j & -1 & j \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & j & -1 & -j & 1 & j & -1 & -j
\end{array}\right] = [\widetilde{A}, \widetilde{A}]
$$

Solution: The corresponding DFT is

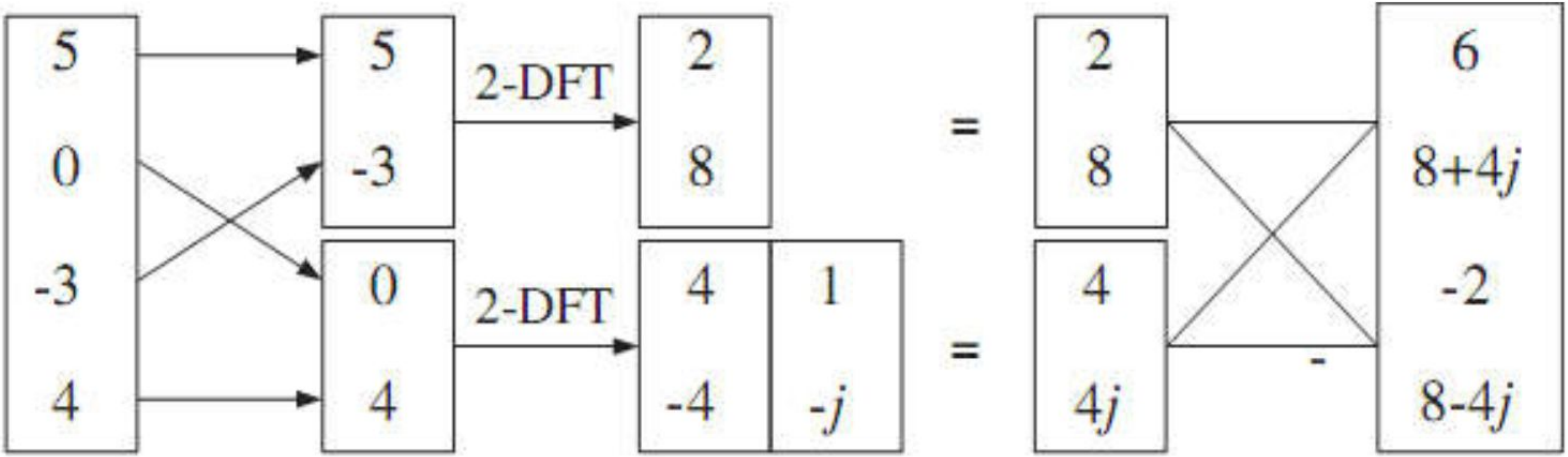
$$\mathbf{X} = A\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j & 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \\ 3 \\ 4 \\ -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8+4j \\ -2 \\ 8-4j \end{bmatrix}$$

The same DFT can be computed by the DFT matrix x~ acted on the wrapped signal x~

$$\tilde{\mathbf{X}} = \tilde{A}\tilde{\mathbf{x}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8+4j \\ -2 \\ 8-4j \end{bmatrix}$$

The two methods are the same

4-FFT



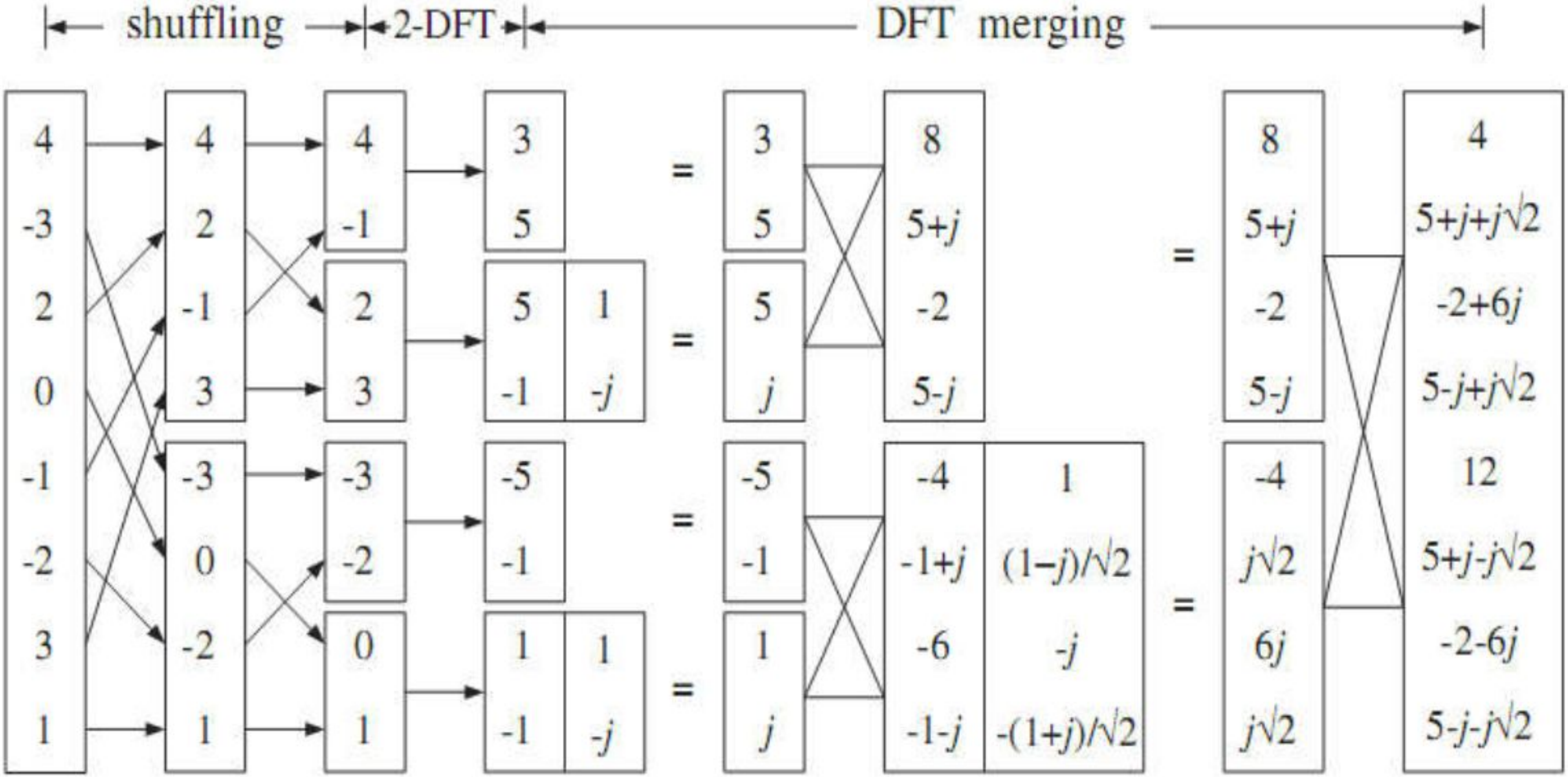
4-IDFT

$$\boxed{\tilde{\mathbf{x}} = \mathrm{IDFT}(\mathbf{X}) = \frac{1}{N}\tilde{A}^*\mathbf{X}}$$

$$\tilde{\mathbf{x}} = \mathrm{IDFT}(\mathbf{X}) = \frac{1}{N}\tilde{A}^*\mathbf{X} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 6 \\ 8+4j \\ -2 \\ 8-4j \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 4 \end{bmatrix}$$

Q3. Compute the 8-point FFT of the length-8 signal x, in which these samples are the first 8 samples of $x(n)=4\cos(\pi n/2)+\cos(\pi n)$, discuss whether the 8 computed FFT values accurately represent the expected spectrum of $x(n)$. What FFT indices correspond to the two frequencies of the cosinusoids?

8-FFT



shuffling
2-DFT
DFT merging
4
-3
2
0
-1
-2
3
1
4
2
-1
3
-3
0
-2
1
4
-1
2
3
-3
-2
0
1
3
5
5
-1
1
-j
-5
-1
1
-1
1
-j
=
=
=
=
3
5
5
j
-5
-1
1
j
8
5+j
-2
5-j
-4
-1+j
-6
-1-j
1
(1−j)/√2
-j
-(1+j)/√2
=
=
8
5+j
-2
5-j
-4
j√2
6j
j√2
4
5+j+j√2
-2+6j
5-j+j√2
12
5+j-j√2
-2-6j
5-j-j√2

Q3 (answer)

$x(n)=4\cos(\pi n/2)+\cos(\pi n)$,

The 8-point FFT is:

$$\mathbf{x} = \begin{bmatrix} 5 \\ -1 \\ -3 \\ -1 \\ 5 \\ -1 \\ -3 \\ -1 \end{bmatrix} \xrightarrow{8-FFT} \mathbf{X} = \begin{bmatrix} 0 \\ 0 \\ 16 \\ 0 \\ 8 \\ 0 \\ 16 \\ 0 \end{bmatrix}$$

It follows that the FFT spectrum will show peaks at the frequencies:

$$X(\omega_2) = X(\omega_6)^* = X(-\omega_2)^* = 16, \qquad X(\omega_4) = 8$$

Q4. The 8-point DFT X of an 8-point sequence x is given by

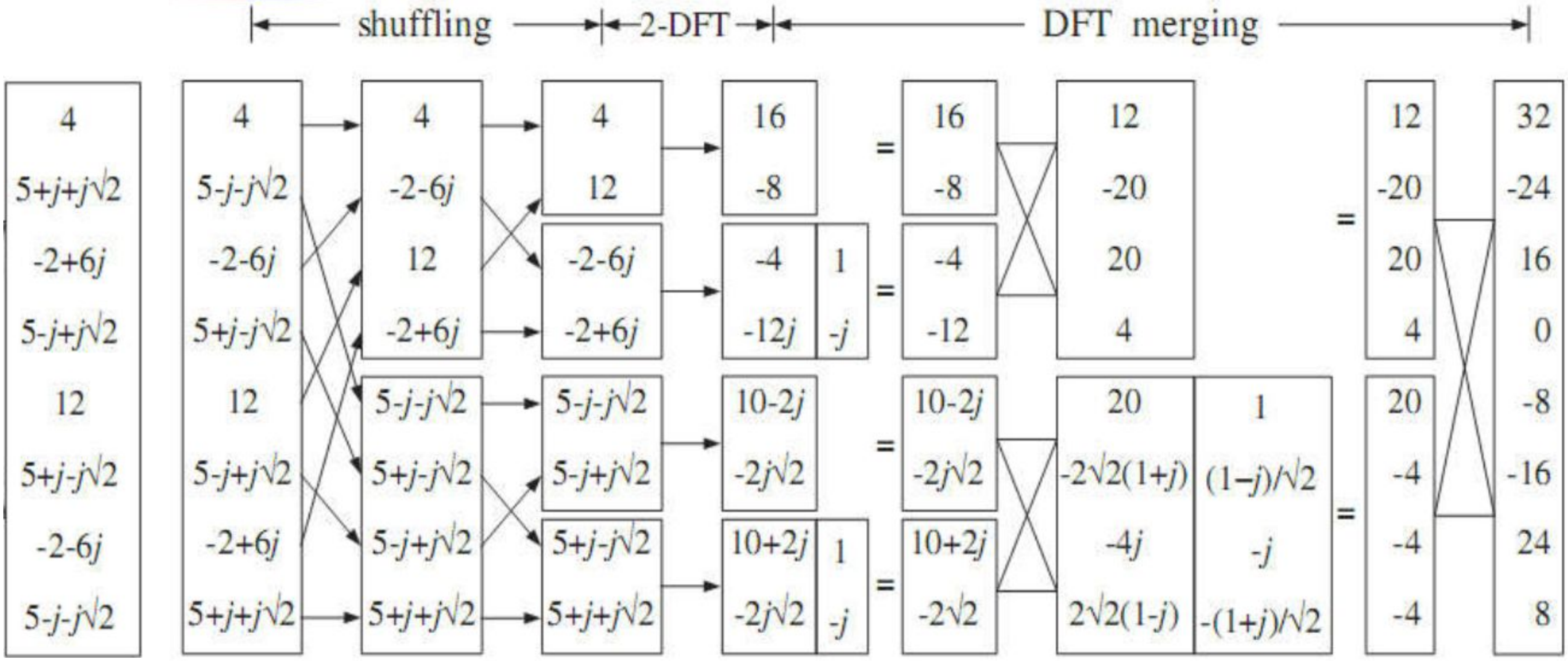
$$X=[0,4,-4j,4,0,4,4j,4].$$

a. Using the FFT algorithm, compute the inverse IFFT: x=IFFT(X).

b. Using the given FFT X, express x as a sum of real-valued (co)sinusoidal signals.

8-IFFT

$$\mathrm{IFFT}(\mathbf{X}) = \frac{1}{N}\left[\mathrm{FFT}(\mathbf{X}^*)\right]^*$$



Q4 (answer)

The IFFT is obtained by conjugating $\mathbf{X}$, computing its FFT, and dividing by 8. The required FFT of $\mathbf{X}^*$ is:

$$\mathbf{X}^* = \begin{bmatrix} 0 \\ 4 \\ 4j \\ 4 \\ 0 \\ 4 \\ -4j \\ 4 \end{bmatrix} \xrightarrow{8-FFT} \begin{bmatrix} 16 \\ 8 \\ 0 \\ -8 \\ -16 \\ 8 \\ 0 \\ -8 \end{bmatrix} \Rightarrow \quad \mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

Using the inverse DFT formula, we express $x(n)$ as a sum of sinusoids:

Q4 (answer)

Using the inverse DFT formula, we express $x(n)$ as a sum of sinusoids:

$$x(n) = \frac{1}{8}\sum_{k=0}^{7} X(k)e^{j\omega_k n} = \frac{1}{8}\left[4e^{j\omega_1 n} - 4je^{j\omega_2 n} + 4e^{j\omega_3 n} + 4e^{j\omega_5 n} + 4je^{j\omega_6 n} + 4e^{j\omega_7 n}\right]$$

$$= \frac{1}{2}e^{j\omega_1 n} + \frac{1}{2j}e^{j\omega_2 n} + \frac{1}{2}e^{j\omega_3 n} + \frac{1}{2}e^{-j\omega_3 n} - \frac{1}{2j}e^{-j\omega_2 n} + \frac{1}{2}e^{-j\omega_1 n}$$

$$= \cos(\omega_1 n) + \sin(\omega_2 n) + \cos(\omega_3 n) = \cos(\pi n/4) + \sin(\pi n/2) + \cos(3\pi n/4)$$

which agrees with the above IFFT values at $n = 0, 1, \dots, 7$. We used the values of the DFT frequencies:

$$\omega_1 = \frac{2\pi}{8} = \frac{\pi}{4}, \quad \omega_2 = \frac{2\pi 2}{8} = \frac{\pi}{2}, \quad \omega_3 = \frac{2\pi 3}{8} = \frac{3\pi}{4}$$

The DFT frequencies ω_7, ω_6, ω_5 are the negatives (modulo-2π) of ω_1, ω_2, ω_3, respectively.

Q5.

a. Compute the DFT of the sequence $x(n) = \{1,0,2,0\}$
b. Without explicitly computing the DFT sum, find the DFT of the sequence $y(n) = \{0,2,0,1\}$, using your answer to part (a)
c. Find the inverse DFT of the sequence $X(m) = \{1, e^{-j3\pi/4}, 0, e^{j3\pi/4}\}$
d. Without explicitly computing the inverse DFT sum, find the inverse DFT of the sequence $Y(m) = \{1, e^{-j\pi/4}, 0, e^{j\pi/4}\}$, using your answer to part(c).

Q5 (answer)

(a) Given a sequence $x[n] = \{1, 0, 2, 0\}$, the DFT is

$$\begin{aligned} X[m] &= x[0] + x[1]e^{-j\frac{2\pi}{4}m} + x[2]e^{-j\frac{2\pi}{4}2m} + x[3]e^{-j\frac{2\pi}{4}3m} \\ X[0] &= 3, X[1] = -1, X[2] = 3, X[3] = -1 \end{aligned}$$

(b) Notice that $y[n] = x_{\langle n+1 \rangle_4}$,

$$\begin{aligned} Y[m] &= X[m]e^{j\frac{2\pi}{4}m} \\ &= X[m]e^{j\frac{\pi}{2}m} \end{aligned}$$

(c) The inverse DFT of the sequence is

$$\begin{aligned} x[n] &= \frac{1}{4}(X[0] + X[1]e^{j\frac{\pi}{2}n} + X[2]e^{j\pi n} + X[3]e^{j\frac{3\pi}{2}n}), n = 0, 1, 2, 3 \\ x[0] &= \frac{1}{4}(1 - \sqrt{2}), x[1] = \frac{1}{4}(1 + \sqrt{2}) \\ x[2] &= \frac{1}{4}(1 + \sqrt{2}), x[3] = \frac{1}{4}(1 - \sqrt{2}) \end{aligned}$$

(d)

$$\begin{aligned} Y[m] &= X[m]e^{j\frac{\pi}{2}m} \\ \rightarrow y[n] &= \frac{1}{4}\sum_{n=0}^{3} Y[m]e^{j\frac{2\pi}{4}nm} \\ &= \frac{1}{4}\sum_{n=0}^{3} X[m]e^{j\frac{\pi}{2}m}e^{j\frac{2\pi}{4}nm} \\ &= \frac{1}{4}\sum_{n=0}^{3} X[m]e^{j\frac{2\pi}{4}(n+1)m} \\ &= x_{\langle n+1 \rangle_4} \end{aligned}$$