A Discontinuous Galerkin Method for GR-Hydrodynamics in `thornado`

Samuel J. Dunham[1,2], Eirik Endeve[3,2], Anthony Mezzacappa[2,4], Jesse Buffaloe[2]

[1]Vanderbilt University

[2]University of Tennessee-Knoxville

[3]Oak Ridge National Laboratory

[4]Joint Institute for Computational Sciences

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Outline

Motivation/Background

Numerical Method

Results

Summary/Future Work

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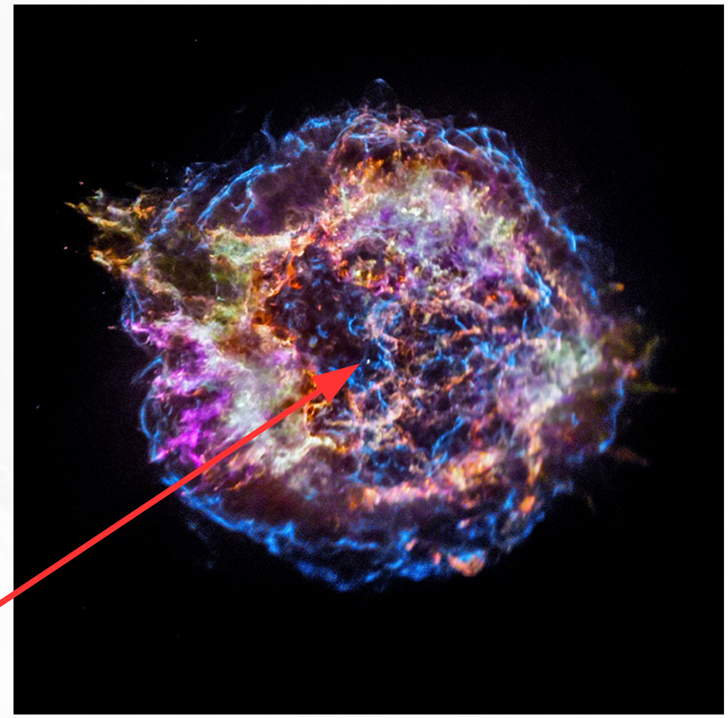
Motivation: Core-Collapse Supernova Explosion Mechanism

- Nucleosynthesis
  - Heavy element distribution
- Gravitational waves
- Neutrinos
- Nuclear matter



Neutron star!

30 light-years

Cas A, color-coded by chemical composition. Figure courtesy of APOD

What Are We Doing?

- Developing new code to study CCSNe: **t**oolkit for **h**igh-**or**der **n**eutrino-r**ad**iation hydr**o**dynamics (`thornado`)
- 3+1 (CFA)
  - Hydrodynamics
  - Neutrino Transport
  - Gravity
- Runge-Kutta Discontinuous Galerkin (RKDG)

$$\partial_t \left(\sqrt{\gamma}\, \boldsymbol{U} \right) + \partial_i \left(\alpha \sqrt{\gamma}\, \boldsymbol{F}^i \left(\boldsymbol{U} \right) \right) = \alpha \sqrt{\gamma}\, \boldsymbol{S}$$

3+1 GR-Hydro Equations

$$\partial_t \left(\sqrt{\gamma}\,\boldsymbol{U}\right) + \partial_i \left(\alpha\,\sqrt{\gamma}\,\boldsymbol{F}^i\left(\boldsymbol{U}\right)\right) = \alpha\,\sqrt{\gamma}\,\boldsymbol{S}$$

Geometry

γ : Determinant of spatial three-metric

α : Lapse function

$\boldsymbol{\beta}$: Shift vector

$\boldsymbol{K}$: Extrinsic curvature tensor

CFA: $\gamma_{ij} = \psi^4\,\bar{\gamma}_{ij} \overset{\text{sph}}{=} \text{diag}\left(\psi^4, \psi^4\,r^2, \psi^4\,r^2\,\sin^2\theta\right)$

$$K_{ij} = \frac{1}{2}\pounds_{\boldsymbol{n}}\gamma_{ij}$$

Extrinsic curvature: describes how the 3D spatial hypersurface is curved in the 4D spacetime manifold

3+1 GR-Hydro Equations

$$\partial_t \left(\sqrt{\gamma}\,\boldsymbol{U}\right) + \partial_i \left(\alpha\,\sqrt{\gamma}\,\boldsymbol{F}^i \left(\boldsymbol{U}\right)\right) = \alpha\,\sqrt{\gamma}\,\boldsymbol{S}$$

$$\boldsymbol{U} = \boldsymbol{U}\left(\vec{x}, t\right),\ \ \vec{x} \in \left[\vec{x_L}, \vec{x_U}\right],\ t \in \left[0, t_f\right]$$

$$\boldsymbol{U}\left(\vec{x}, 0\right) = \boldsymbol{U}_0\left(\vec{x}\right),\ \boldsymbol{U}\left(\vec{x_L}, t\right) = \boldsymbol{b}_L\left(t\right),\ \boldsymbol{U}\left(\vec{x_U}, t\right) = \boldsymbol{b}_U\left(t\right)$$

Evolved Quantities

D : Conserved rest-mass density

$\boldsymbol{S}$: Conserved momentum-density

τ : Conserved energy-density

$$D = \rho W$$
$$S_j = \rho\, h\, W^2\, v_j$$
$$\tau = \rho\, h\, W^2 - p - \rho\, W$$

$$W = \left(1 - \vec{v} \cdot \vec{v}\right)^{-1/2}$$
$$h = 1 + \left(e + p\right)/\rho$$

$$p = p\left(e\right) = \left(\Gamma - 1\right) e,\ \ \Gamma \in (1, 2]$$

3+1 GR-Hydro Equations

$$\partial_t \left(\sqrt{\gamma}\,\boldsymbol{U}\right) + \partial_i \left(\alpha \sqrt{\gamma}\,\boldsymbol{F}^i\left(\boldsymbol{U}\right)\right) = \alpha \sqrt{\gamma}\,\boldsymbol{S}$$

Fluxes

$$F^i_D = \left(v^i - \alpha^{-1}\beta^i\right) D$$

$$F^i_{S_j} = P^i{}_j - \alpha^{-1}\beta^i S_j$$

$$F^i_\tau = S^i - D\,v^i - \alpha^{-1}\beta^i\,\tau$$

$$P^{ij} = \rho\, h\, W^2\, v^i\, v^j + \gamma^{ij}$$

3+1 GR-Hydro Equations

$$\partial_t \left(\sqrt{\gamma}\,\boldsymbol{U}\right) + \partial_i \left(\alpha\,\sqrt{\gamma}\,\boldsymbol{F}^i\left(\boldsymbol{U}\right)\right) = \alpha\,\sqrt{\gamma}\,\boldsymbol{S}$$

Sources

$$S_D = 0$$

$$S_{S_j} = \frac{1}{2} P^{ik}\,\partial_j \gamma_{ik} + \alpha^{-1}\,S_i\,\partial_j\,\beta^i - \alpha^{-1}\,(\tau + D)\,\partial_j\,\alpha$$

$$S_\tau = P^{ij}\,K_{ij} - \alpha^{-1}\,S^j\,\partial_j\,\alpha$$

3+1 GR-Hydro Equations

$$\partial_t \left(\sqrt{\gamma}\,\boldsymbol{U}\right) + \partial_i \left(\alpha\,\sqrt{\gamma}\,\boldsymbol{F}^i\left(\boldsymbol{U}\right)\right) = \alpha\,\sqrt{\gamma}\,\boldsymbol{S}$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\sqrt{\gamma}\,D\\ \sqrt{\gamma}\,S_j\\ \sqrt{\gamma}\,\tau\end{pmatrix} + \frac{\partial}{\partial x^i}\begin{pmatrix}\alpha\,\sqrt{\gamma}\left(v^i - \alpha^{-1}\,\beta^i\right)D\\ \alpha\,\sqrt{\gamma}\left({P^i}_j - \alpha^{-1}\,\beta^i\,S_j\right)\\ \alpha\,\sqrt{\gamma}\left(S^i - D\,v^i - \alpha^{-1}\,\beta^i\,\tau\right)\end{pmatrix}$$

$$= \alpha\,\sqrt{\gamma}\begin{pmatrix}0\\ \frac{1}{2}\,P^{ik}\,\partial_j\gamma_{ik} + \alpha^{-1}\,S_i\,\partial_j\,\beta^i - \alpha^{-1}\,\left(\tau + D\right)\,\partial_j\,\alpha\\ P^{ij}\,K_{ij} - \alpha^{-1}\,S^j\,\partial_j\,\alpha\end{pmatrix}$$

Outline

Motivation/Background - CCSNe

Numerical Method

Results

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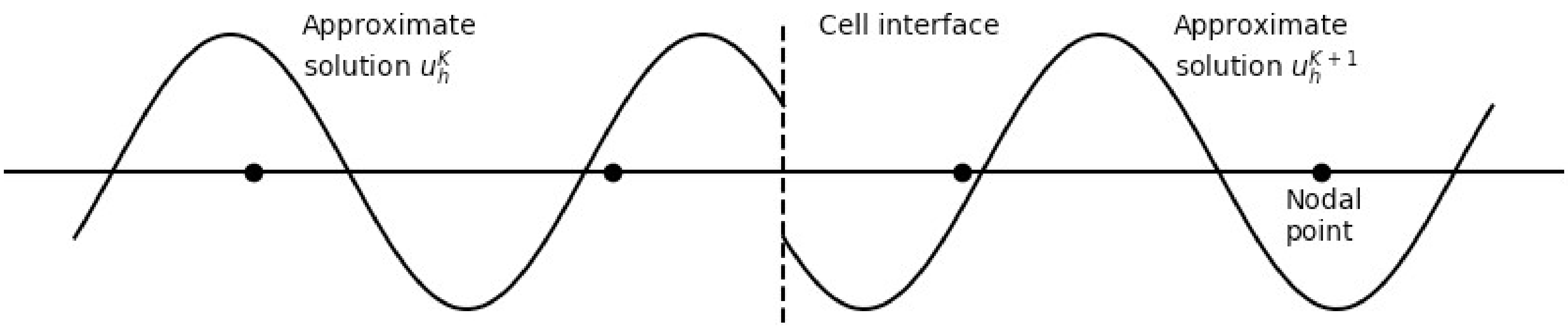
Numerical Method

Discontinuous Galerkin (DG)

- Discretize computational domain
- Locally approximate solution as polynomial

$$u(x,t) \approx u_h(x,t) \equiv \sum_{i=1}^{N} u_h(x_i,t)\, \ell_i(x)$$

- Use weak form to evolve in time with SSP-RK methods
- Apply slope- and bound-preserving-limiters at each stage of SSP-RK algorithm



Weak Form (1D)

$$\partial_t \left(\sqrt{\gamma}\,\boldsymbol{U}\right) + \partial_x \left(\alpha \sqrt{\gamma}\,\boldsymbol{F}\left(\boldsymbol{U}\right)\right) = \alpha \sqrt{\gamma}\,\boldsymbol{S} \qquad (1)$$

- Multiply (1) by test-function (Lagrange polynomial)
  - $\ell_i\left(x\right), \quad i = 1, \ldots, N$

*Assume three-metric explicitly independent of time

- Integrate over element*
  - $dV = \sqrt{\gamma}\,dx$

- Integration-by-parts on flux term

Numerical flux obtained with approximate Riemann solver

$$\int_K \frac{\partial \boldsymbol{U}}{\partial t}\,\ell_i\,dV = -\left\{\left[\alpha \sqrt{\gamma}\,\ell_i\,\hat{\boldsymbol{F}}\right]_{x_L}^{x_U} - \int_K \alpha\,\boldsymbol{F}\,\frac{d\ell_i}{dx}\,dV - \int_K \alpha\,\boldsymbol{S}\,\ell_i\,dV\right\}$$

$$\int_K \frac{\partial u}{\partial t}\,\ell_i\,dV = -\left\{\left[\alpha\,\sqrt{\gamma}\,\ell_i\,\hat{F}\right]_{x_L}^{x_U} - \int_K \alpha\,F\,\frac{d\ell_i}{dx}\,dV - \int_K \alpha\,S\,\ell_i\,dV\right\}$$

$$\begin{aligned}\int_K \frac{\partial u}{\partial t}\,\ell_i(x)\;dV &= \int_K \frac{\partial}{\partial t}\left(\sum_{j=1}^{N} u(x_j,t)\;\ell_j(x)\right)\ell_i(x)\;dV \\ &= \sum_{j=1}^{N}\left[\frac{du(x_j,t)}{dt}\int_K \ell_i(x)\;\ell_j(x)\;dV\right] \\ &= \sum_{j=1}^{N} M_{ij}\,\frac{du_j}{dt} = \boldsymbol{M}\,\frac{d\boldsymbol{u}}{dt}, \quad M_{ij} \equiv \int_K \ell_i(x)\;\ell_j(x)\;dV\end{aligned}$$

$$\frac{d\boldsymbol{u}}{dt} = -\boldsymbol{M}^{-1}\left\{\left[\alpha\,\sqrt{\gamma}\,\boldsymbol{\ell}\,\hat{F}\right]_{x_L}^{x_U} - \int_K \alpha\,F\,\frac{d\boldsymbol{\ell}}{dx}\,dV - \int_K \alpha\,S\,\boldsymbol{\ell}\,dV\right\}$$

Time-Stepping Algorithm

Strong-Stability-Preserving Runge-Kutta (SSP-RK)

Cockburn & Shu, (2001) J. Sci. Comp., Vol. 16, No. 3

- Convex-combinations of forward-Euler time-steps

$$\frac{d}{dt}u_h = L(u_h)$$

1. $u_h^{(0)} = u_h^n;$
2. For $i = 1, \ldots, N_s$ compute the intermediate functions:

$$u_h^{(i)} = \Lambda\Pi_h \left(\sum_{j=0}^{i-1} \alpha_{ij}\, w_h^{ij} \right), \quad w_h^{ij} = u_h^{(j)} + \frac{\beta_{ij}}{\alpha_{ij}} \Delta t^n L_h\left(u_h^{(j)}\right)$$

3. $u_h^{n+1} = u_h^{N_s}$

$$\alpha_{ij},\ \beta_{ij} > 0, \quad \sum_{j=0}^{i-1} \alpha_{ij} = 1$$

Slope-limiter

Slope-Limiter

MinMod Limiter

- Higher-order methods can develop unphysical oscillations
- Map solution from nodal to modal representation
- Compare slope in target cell to approximation to derivatives in neighbors via cell-averages

Legendre Polynomials

$$u_h(x,t) = \sum_{i=1}^{N} u_h(x_i,t)\,\ell_i(x) = \sum_{n=0}^{N-1} c_n(t)\,P_n(x)$$

Parameter with value between 1 and 2

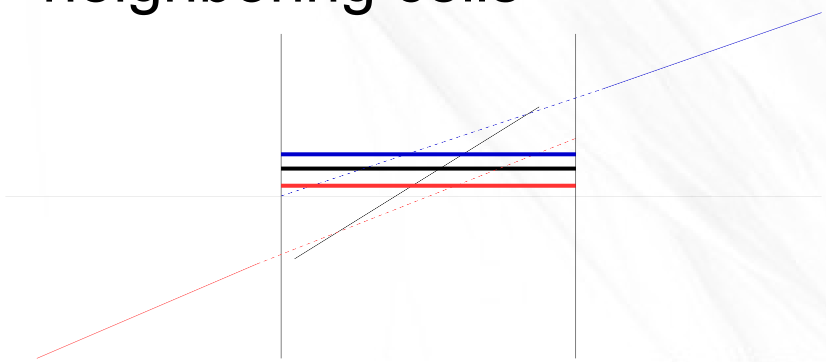
$$\tilde{m} = \mathrm{MinMod}\left(\beta_{\mathrm{TVD}}\frac{\overline{u}_K - \overline{u}_{K-1}}{\Delta x},\; m,\; \beta_{\mathrm{TVD}}\frac{\overline{u}_{K+1} - \overline{u}_K}{\Delta x}\right),$$

$$\mathrm{MinMod}(a,b,c) = \begin{cases} 0, & a,b,c \text{ not all same signs} \\ \min(|a|,|b|,|c|), & \text{else} \end{cases}$$

Troubled-Cell Indicator

Fu & Shu, (2017), J. Comp. Phys. 347, 305

- MinMod limiter activates at smooth extrema
- To alleviate, we detect troubled-cells before applying limiter
- Quantify the difference in solutions in neighboring cells



- Extrapolate solution in neighboring cells into target cell
- Compute cell-averages
- Compute magnitude of differences between extrapolated solution and cell-average
- Compare value to parameter set at run-time

Bound-Preserving Limiter

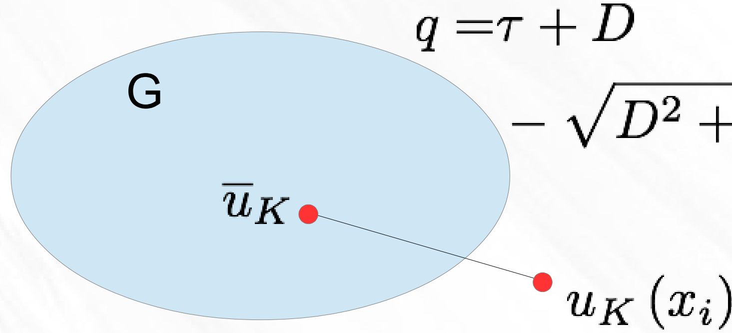
Qin et al., (2016), J. Comp. Phys., 315, 323

- Higher-order methods can exceed physical bounds
  - $P < 0$
  - $\rho < 0$
  - $|v| > c$
- Cell-average is guaranteed to be physical, but **all** quadrature points need to be physical
- Damp point-values towards cell-average (allowed because of CFL condition)

- Define (convex) "set of admissible states", G
- Leads to two conditions:

$$D > 0$$

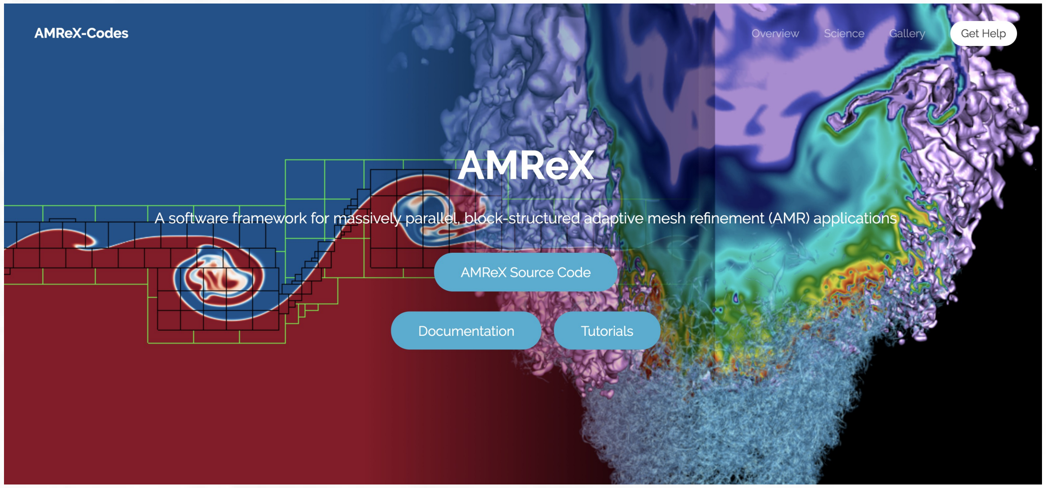
$$q = \tau + D - \sqrt{D^2 + S^i S_i} > 0$$



AMReX

https://amrex-codes.github.io/amrex/

- Parallel/AMR framework developed primarily from Lawrence Berkeley National Laboratory
- Block-Structured AMR



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Numerical Method - RKDG

Results

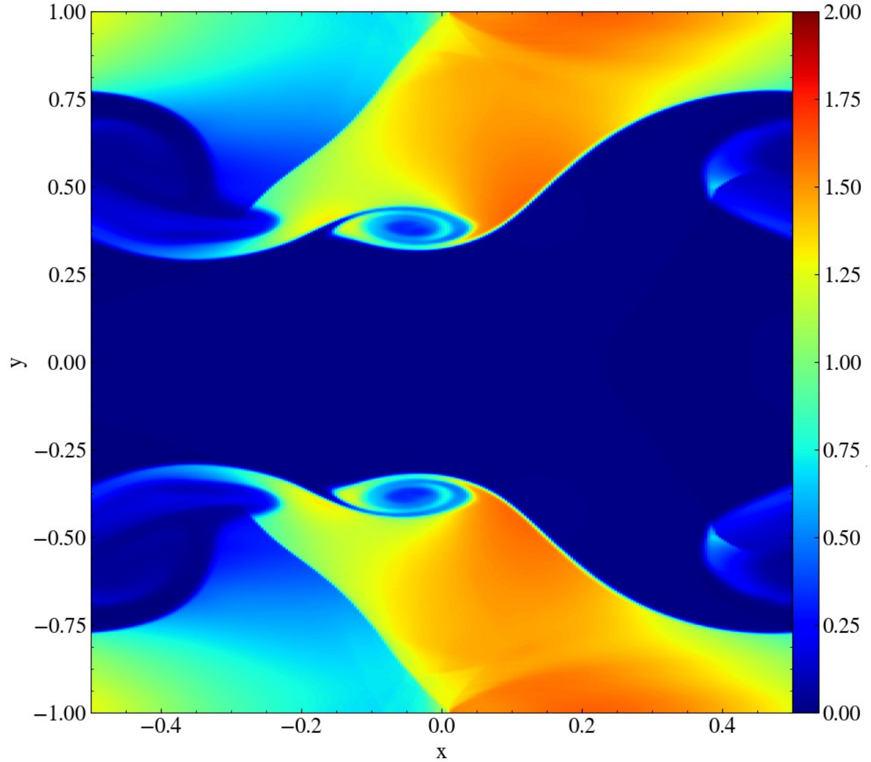
Summary/Future Work

Relativistic Kelvin-Helmholtz Instability

Radice & Rezzolla, (2012), A&A, 547, A26

- Smooth solution
- Turbulent regions*
- 256 x 512
- Third-order method
- SSP-RK3
- Periodic boundary conditions
- HLL Riemann solver
- Characteristic Limiting
- Run with AMReX

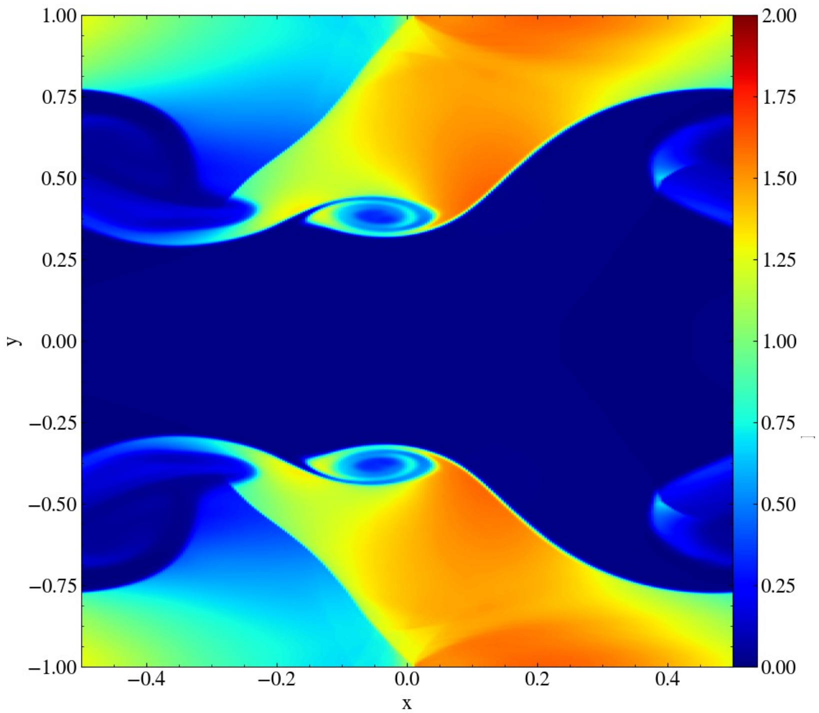
Primitive Mass-Density

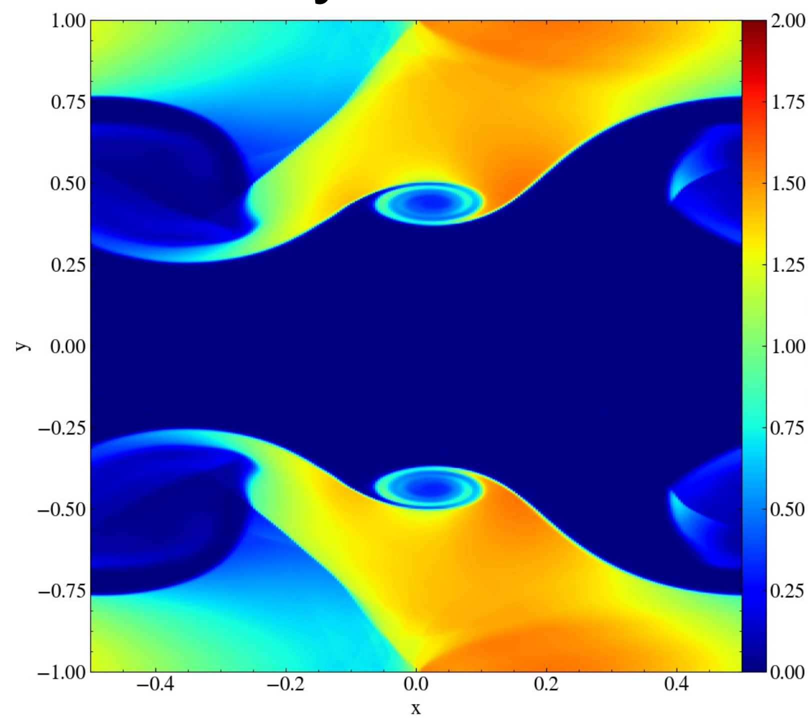


* Assuming ideal EOS

HLL vs. HLLC Riemann Solvers

Primitive Mass-Density





HLL

HLLC

Approximates Riemann fan with only two waves

nX = 256
nY = 512
nNodes = 3
SSP-RK3

Approximates Riemann fan with two waves and one contact wave
Mignone & Bodo (2005), MNRAS, 364, 126

2D Riemann Problem

Del Zanna & Bucciantini, (2002), A&A, 390, 1177

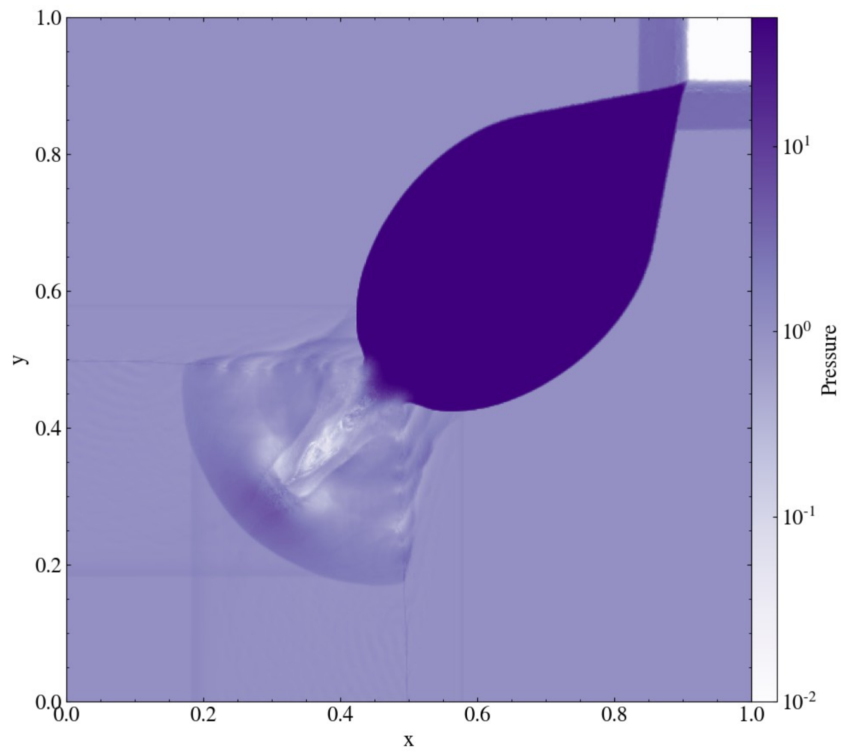
- Highly relativistic
  - W ~ 7
- Contact waves
- 512 x 512
- Homogeneous boundary conditions
- HLL Riemann solver
- Component-wise limiting
- Run with AMReX

Pressure

2D Riemann Problem

Del Zanna & Bucciantini, (2002), A&A, 390, 1177

- Highly relativistic
  - W ~ 7
- Contact waves
- 512 x 512
- Third-order method
- SSP-RK3
- HLL Riemann solver
- Component-wise limiting
- Run with AMReX



Pressure

Characteristic Limiting

Pressure

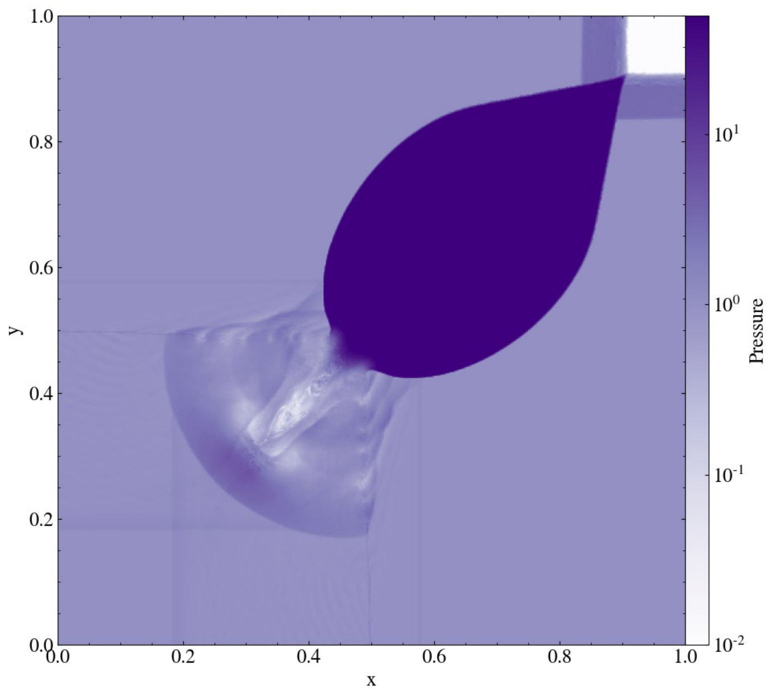
Component-Wise Limiting

Characteristic Limiting

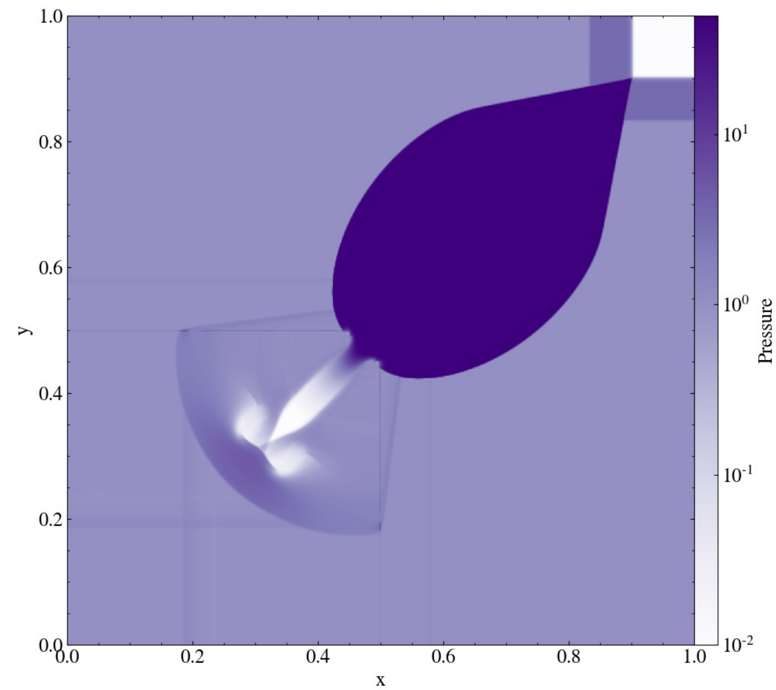
nX = nY = 512
nNodes = 3
SSP-RK3

Characteristic Limiting

Pressure



Component-Wise Limiting

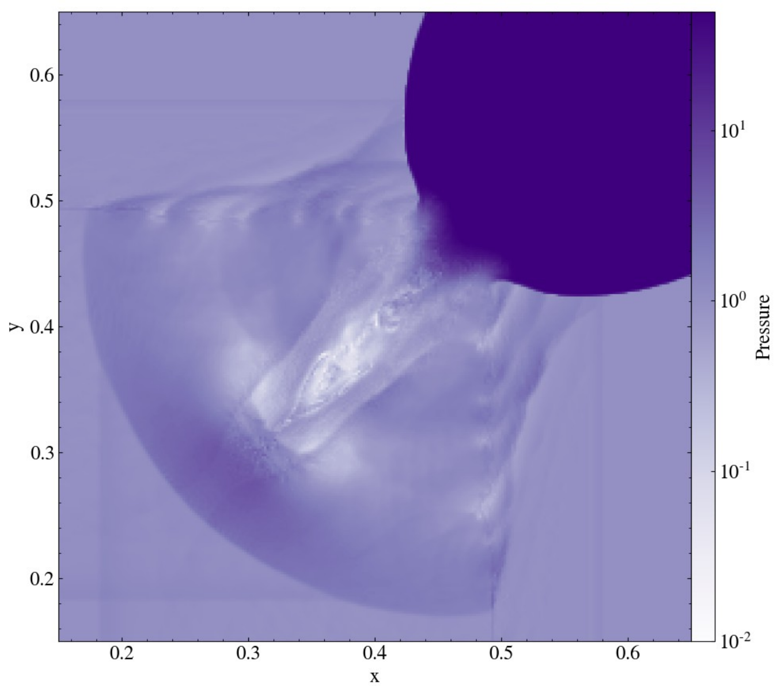


Characteristic Limiting

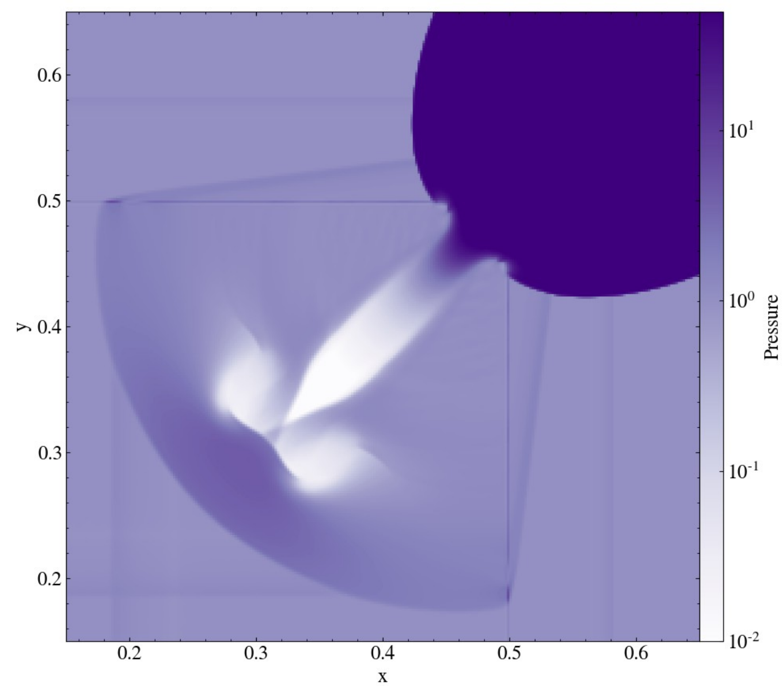
nX = nY = 512
nNodes = 3
SSP-RK3

Characteristic Limiting

Pressure



Component-Wise Limiting



Characteristic Limiting

nX = nY = 512
nNodes = 3
SSP-RK3

2D, Non-Relativistic SASI

- Physically motivated
- Curvilinear coordinates
- Point-mass gravitational potential
- Perturbed with $\ell = 1$ mode instability
- 256x128
- Third-order method
- SSP-RK3

Mass-Density

- Run with AMReX

2D, Non-Relativistic SASI

- Physically motivated
- Curvilinear coordinates
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- 256x128
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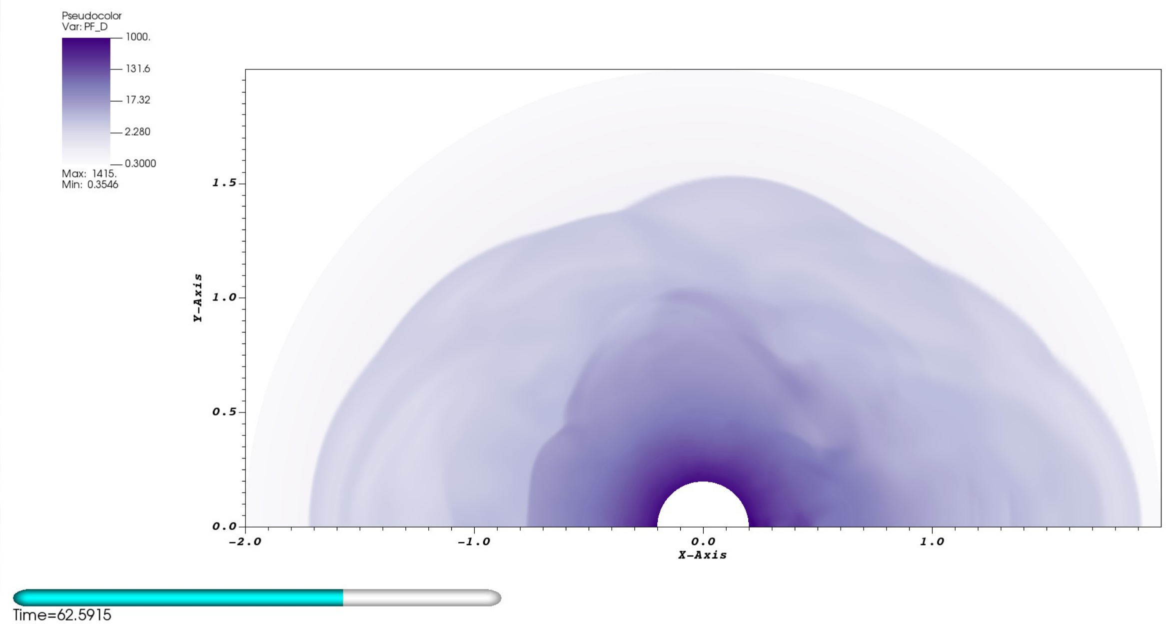
Mass-Density



- Run with AMReX

GR Standing Accretion Shock

- Physically-motivated
- Curvilinear coordinates
- Stationary background spacetime
- 256 elements
- Third-order method
- HLLC Riemann solver
- Characteristic-limiting
- Run with AMReX

- Evolved over several characteristic SASI development timescales

GR Standing Accretion Shock

- Physically-motivated
- Curvilinear coordinates
- Stationary background spacetime
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- Third-order method
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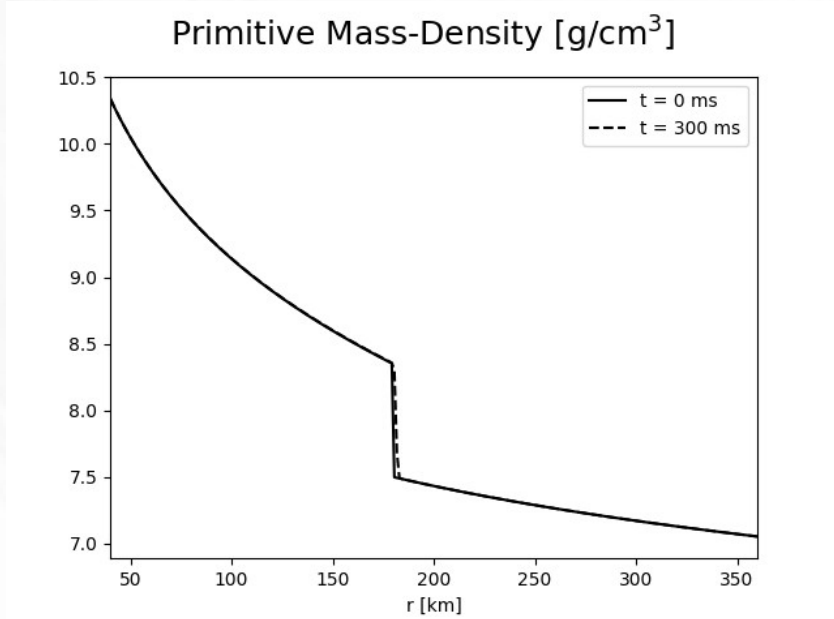

- Evolved over several characteristic SASI development timescales

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Motivation/Background - CCSNe

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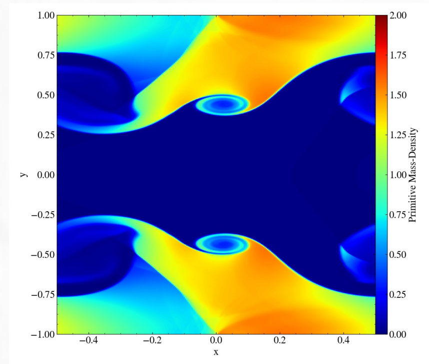
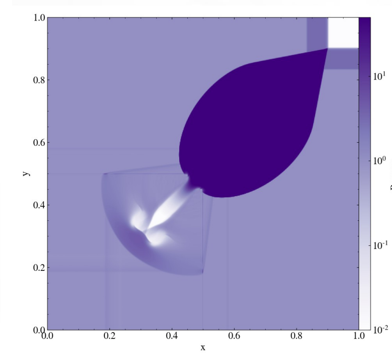
Results – KHI/2DRP/SASI/GR-SAS

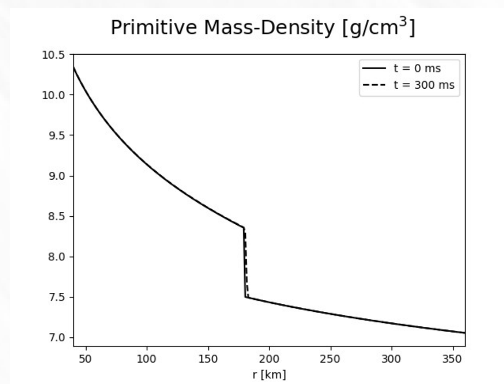
Summary/Future Work

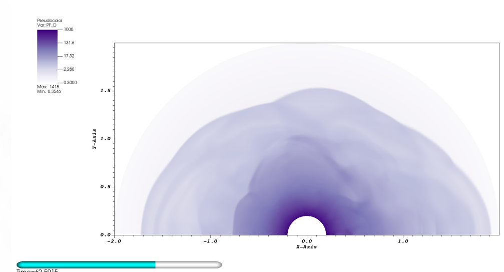
Summary

- Developing Multi-D DG-GR Hydro solver
- Implemented characteristic slope-limiting
- Implemented bound-preserving limiter
- Incorporated AMReX
- Successfully run several difficult test problems









Future Work

- MPI scaling tests
- Implement and test 3D components
- Run 2D and 3D, GR-SASI problem
  - Investigate effects of GR-hydro
- AMR
- Couple with CFA gravity solver (Nick Roberts)
- Couple with neutrino-transport solver (Ran Chu, Zach Elledge)

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  - NSF GP-1806692
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Thank you!
Questions?

Bonus Material

GR-SAS Initial conditions

- M_PNS = 1.4 Msun
- R_PNS = 40 km
- Mdot = 0.3 Msun/s
- R_shock = 180 km