A Discontinuous Galerkin Method for GR-Hydrodynamics in thornado

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Outline

Background/Motivation

Numerical Method

Results

Summary/Future Work

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Background/Motivation

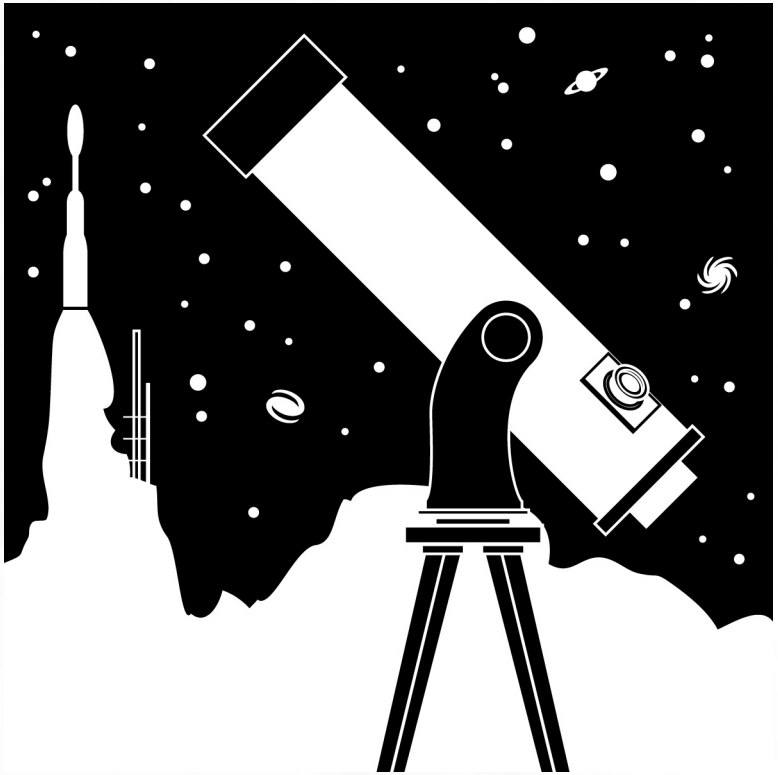
Numerical Method

Results

Summary/Future Work

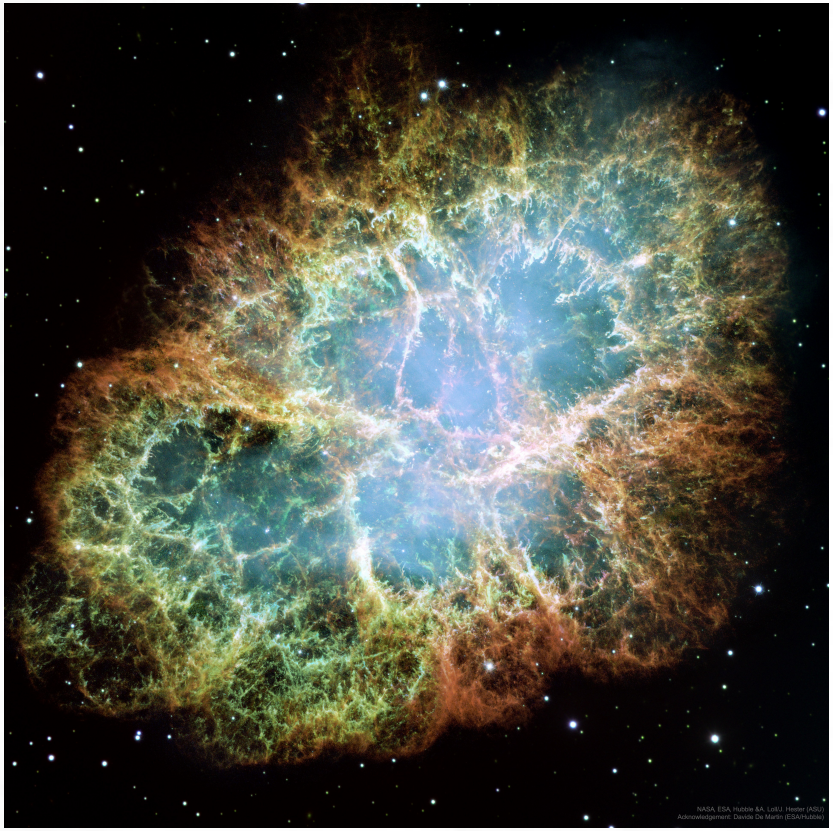
Observations

- Chinese astronomers observe “guest stars” (~200 CE)
- May 1, 1006
  - Cast shadows on the ground at night



Crab

- Observed in 1054 CE
- Image shown is in optical

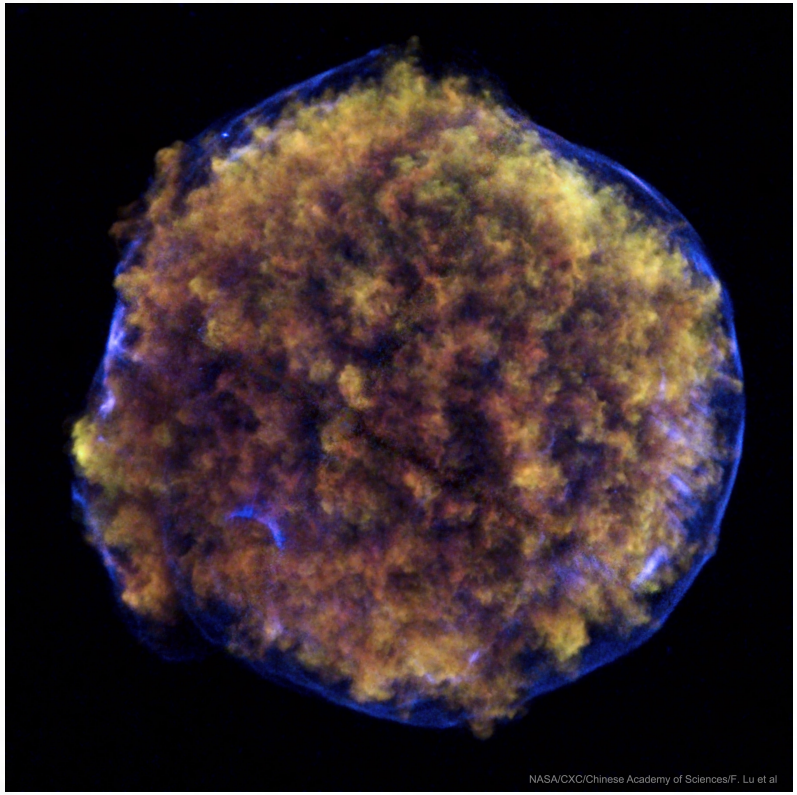


10 ly

Figure from APOD

Tycho's Supernova

- Observed in 1572
- Image shown is in X-Ray



30 ly

Figure from APOD

Cassiopeia A (Cas A)

- Observed in mid-17th century
- Find the pulsar!
- Image shown is in X-Ray (with color-coded elements)

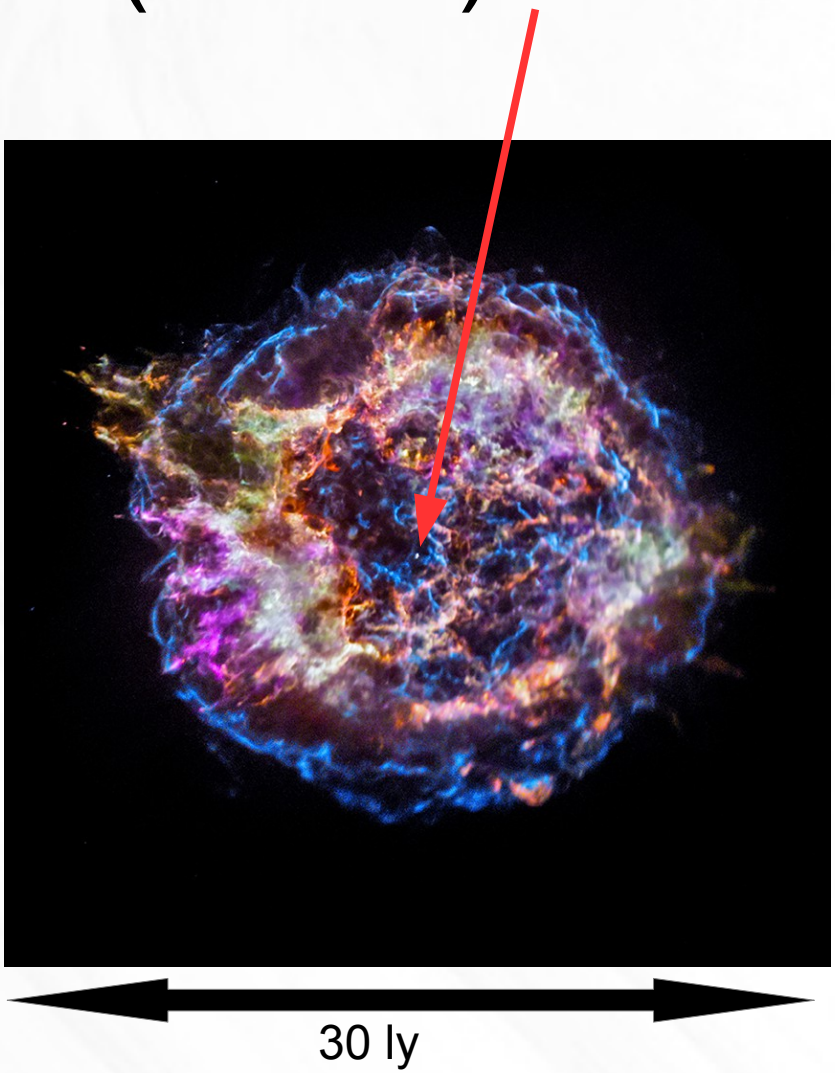


Figure from APOD

Light Curve

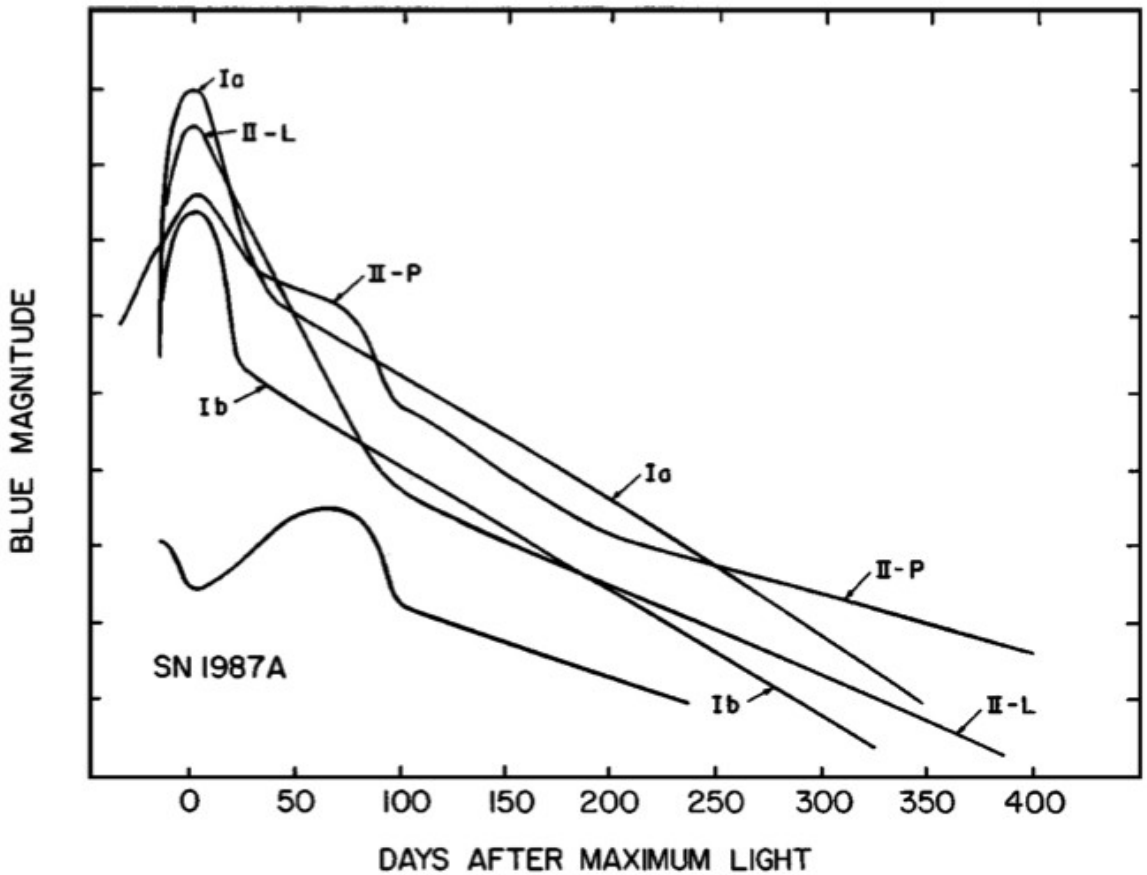
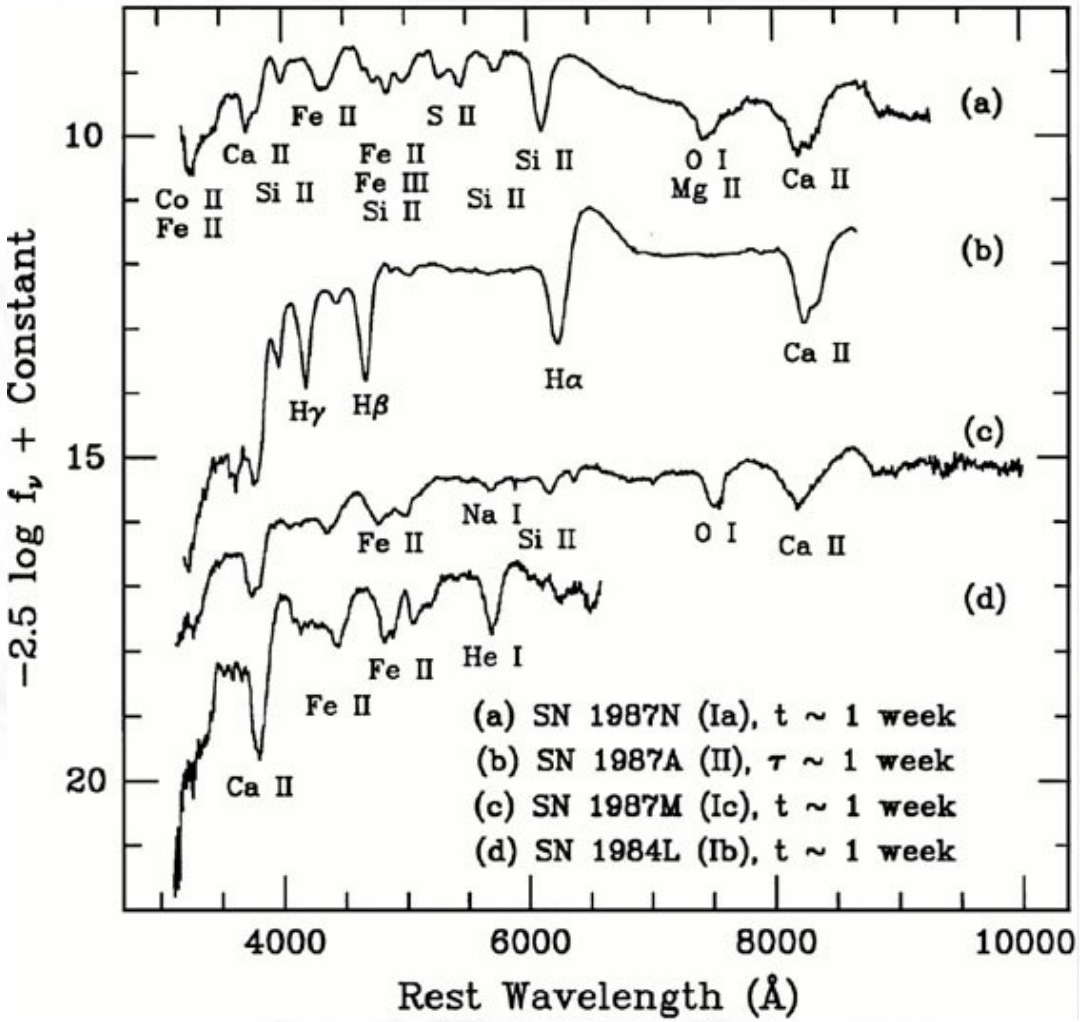


Figure 3 Schematic light curves for SNe of Types Ia, Ib, II-L, II-P, and SN 1987A. The curve for SNe Ib includes SNe Ic as well, and represents an average. For SNe II-L, SNe 1979C and 1980K are used, but these might be unusually luminous.

Figure Credit: Wheeler, J. C., & Harkness, R. P. 1990, RPPh, 53, 1467

Spectra



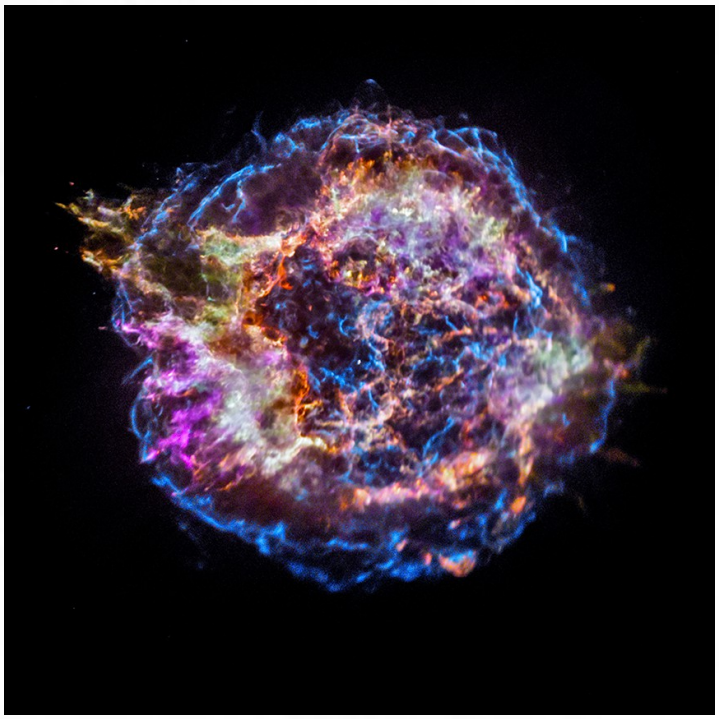
https://ned.ipac.caltech.edu/level5/March03/Filippenko/Filippenko2.html

Interpretation/Theory

- Fritz Zwicky (1898-1974)
  - Type I and Type II SNe
  - Much more than “spherical bastards”!
- Baade and Zwicky (1934)
  - Postulated Type II SNe are transitions to neutron stars
- Colgate and White (1966)
  - Neutrino-driven explosion (1D)
- Bethe and Wilson (1985)
  - Delayed neutrino-driven explosion (1D)
- Modern picture
  - Delayed neutrino-driven mechanism with multi-D effects

Motivation: Core-Collapse Supernovae

- Explosion mechanism
- Metal distribution
  - Solar system is ~3-4 ly in diameter
- Gravitational waves
- Neutrinos
- Nuclear matter



30 ly

Cas A, color-coded by chemical composition. Figure from APOD

Outline

Background/Motivation - CCSNe

Numerical Method

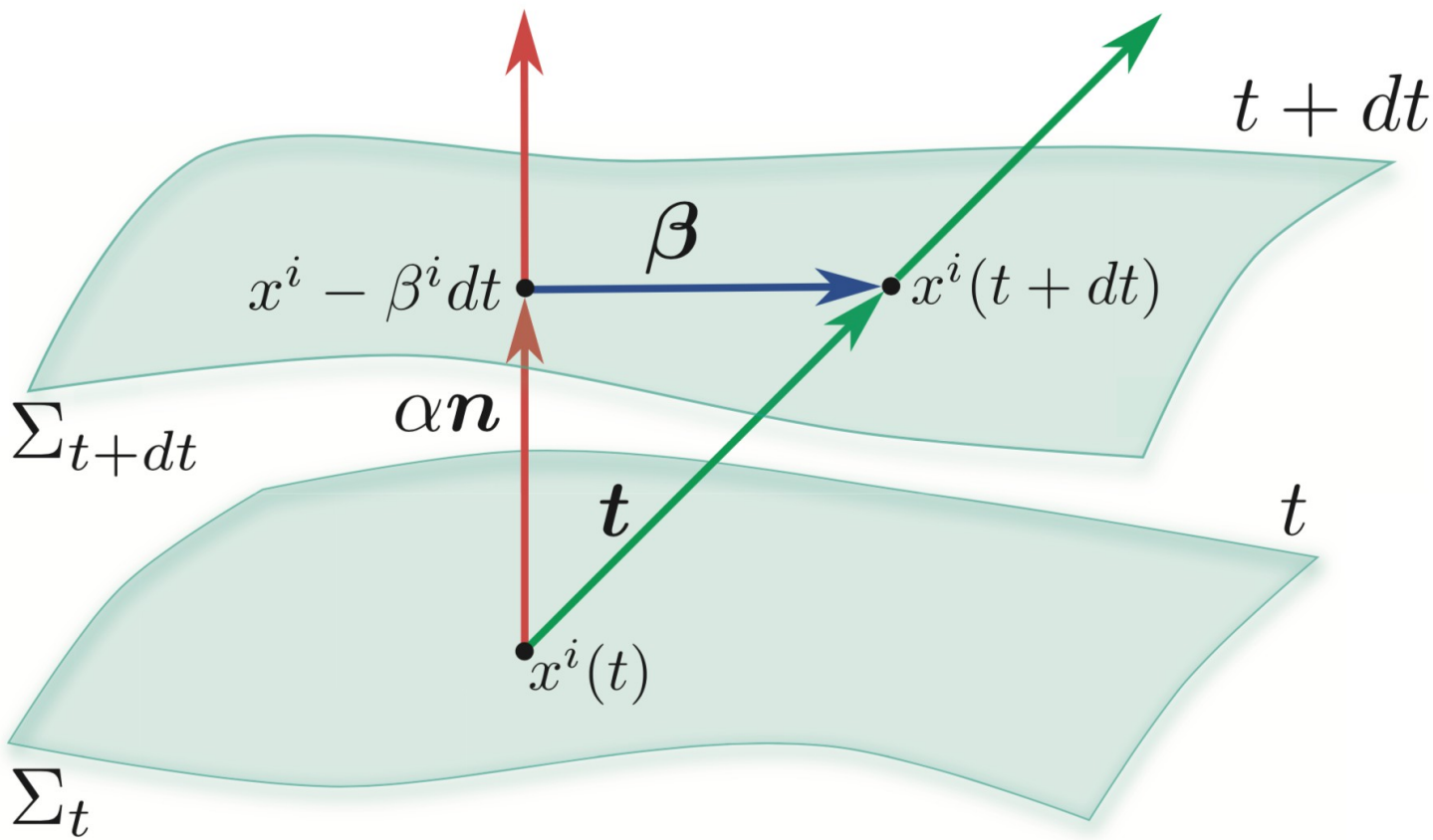
Results

Summary/Future Work

What Are We Doing?

- Developing new code to study CCSNe: **t**oolkit for **h**igh-**or**der **n**eutrino-r**ad**iation hydr**o**dynamics (`thornado`)
- 3+1 (CFA)
  - **Hydrodynamics**
  - Neutrino Transport
  - Gravity
- Runge-Kutta Discontinuous Galerkin (RKDG)

3+1 Decomposition



Rezzolla & Zanotti, *Relativistic Hydrodynamics,* 2013, Oxford

3+1 GR-Hydro Equations

$c = 1$

$$\partial_t \left(\sqrt{\gamma}\,\boldsymbol{U}\right) + \partial_i \left(\alpha\,\sqrt{\gamma}\,\boldsymbol{F}^i\left(\boldsymbol{U}\right)\right) = \sqrt{\gamma}\,\boldsymbol{S}$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\sqrt{\gamma}\,D\\ \sqrt{\gamma}\,S_j\\ \sqrt{\gamma}\,\tau\end{pmatrix} + \frac{\partial}{\partial x^i}\begin{pmatrix}\alpha\,\sqrt{\gamma}\left(v^i - \alpha^{-1}\,\beta^i\right)D\\ \alpha\,\sqrt{\gamma}\left(P^i{}_j - \alpha^{-1}\,\beta^i\,S_j\right)\\ \alpha\,\sqrt{\gamma}\left(S^i - D\,v^i - \alpha^{-1}\,\beta^i\,\tau\right)\end{pmatrix}$$
$$= \sqrt{\gamma}\begin{pmatrix}0\\ \frac{1}{2}\,\alpha\,P^{ik}\,\partial_j\gamma_{ik} + S_i\,\partial_j\,\beta^i - (\tau + D)\,\partial_j\,\alpha\\ \alpha\,P^{ij}\,K_{ij} - S^j\,\partial_j\,\alpha\end{pmatrix}$$

Geometry

γ : Determinant of spatial three-metric

α : Lapse function

$\boldsymbol{\beta}$: Shift vector

$\boldsymbol{K}$: Extrinsic curvature tensor

Evolved Quantities

D : Conserved rest-mass density

$\boldsymbol{S}$: Conserved momentum-density

τ : Conserved energy-density

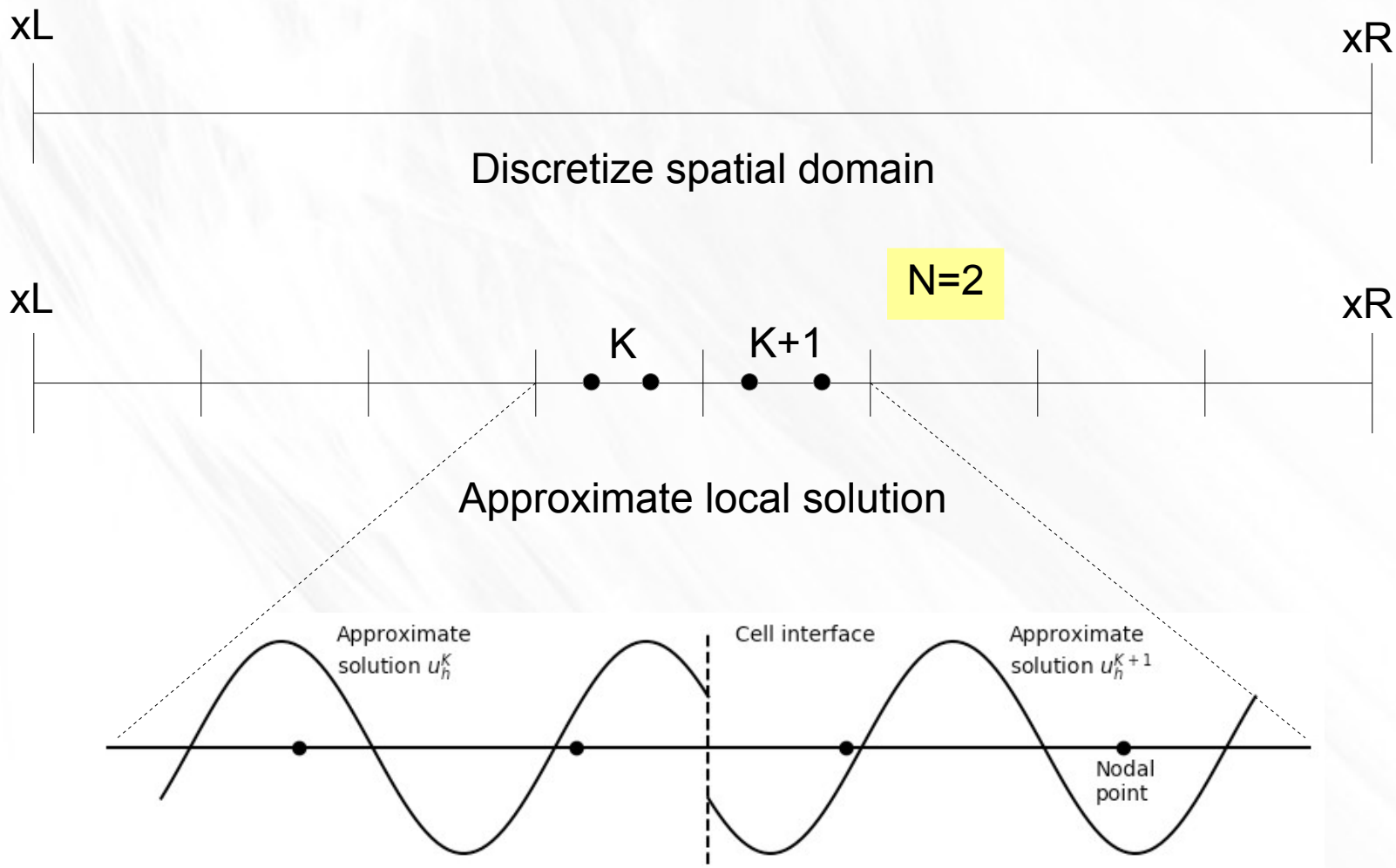
Auxiliary Quantities

$\boldsymbol{P}$: Pressure tensor

$\vec{v}$: Fluid three-velocity

Numerical Method

Discontinuous Galerkin (DG)



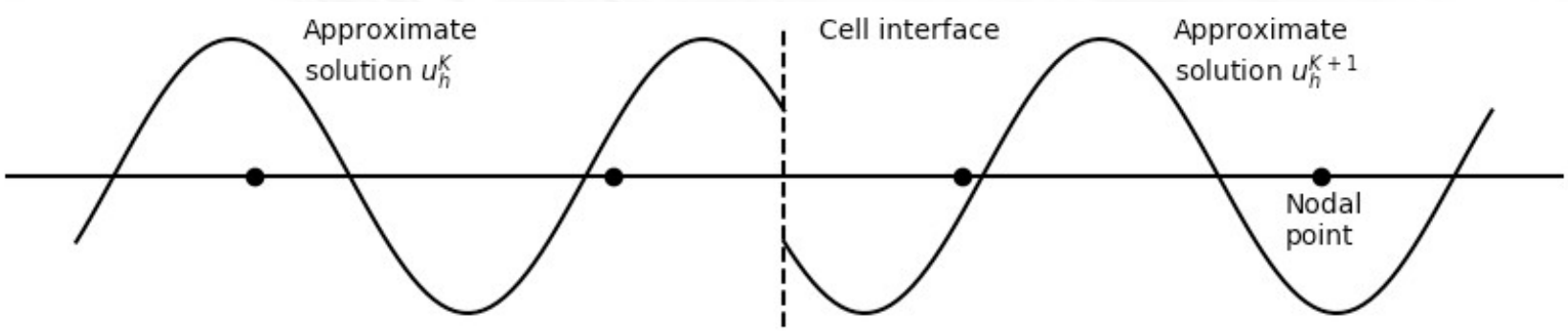
Numerical Method

Discontinuous Galerkin (DG)

- Discretize spatial domain
- Locally approximate solution as polynomial

$$u(x,t) \approx u_h(x,t) \equiv \sum_{i=1}^{N} u_h(x_i,t)\,\ell_i(x)$$

- Use weak form to evolve solution in time with SSP-RK methods
- Apply slope and bound-preserving limiters at each stage of SSP-RK algorithm



Weak Form (1D)

Assume three-metric explicitly independent of time

$$\partial_t \left(\sqrt{\gamma}\,\boldsymbol{U}\right) + \partial_x \left(\alpha\,\sqrt{\gamma}\,\boldsymbol{F}\left(\boldsymbol{U}\right)\right) = \sqrt{\gamma}\,\boldsymbol{S}$$

$$\partial_t \left(\sqrt{\gamma}\,\boldsymbol{U}\right)\,\ell_i\left(x\right) + \partial_x \left(\alpha\,\sqrt{\gamma}\,\boldsymbol{F}\left(\boldsymbol{U}\right)\right)\,\ell_i\left(x\right) = \sqrt{\gamma}\,\boldsymbol{S}\,\ell_i\left(x\right), \quad i = 1, \cdots, N$$

$$\int_K \partial_t \left(\sqrt{\gamma}\,\boldsymbol{U}\right)\,\ell_i\left(x\right)\,dx + \int_K \partial_x \left(\alpha\,\sqrt{\gamma}\,\boldsymbol{F}\left(\boldsymbol{U}\right)\right)\,\ell_i\left(x\right)\,dx = \int_K \sqrt{\gamma}\,\boldsymbol{S}\,\ell_i\left(x\right)\,dx$$

$$\int_K \partial_t \left(\sqrt{\gamma}\,\boldsymbol{U}\right)\,\ell_i\left(x\right)\,dx + \left[\alpha\,\sqrt{\gamma}\,\ell_i\left(x\right)\,\hat{\boldsymbol{F}}\left(\boldsymbol{U}\right)\right]_{x_L}^{x_H} - \int_K \alpha\,\boldsymbol{F}\left(\boldsymbol{U}\right)\,\frac{\partial \ell_i\left(x\right)}{\partial x}\,dV = \int_K \sqrt{\gamma}\,\boldsymbol{S}\,\ell_i\left(x\right)\,dx$$

$$dV = \sqrt{\gamma}\,dx$$

$$\int_K \frac{\partial \boldsymbol{U}}{\partial t}\,\ell_i\,dV = -\left\{\left[\alpha\,\sqrt{\gamma}\,\ell_i\,\hat{\boldsymbol{F}}\right]_{x_L}^{x_H} - \int_K \alpha\,\boldsymbol{F}\,\frac{d\ell_i}{dx}\,dV - \int_K \boldsymbol{S}\,\ell_i\,dV\right\}$$

Numerical flux obtained with approximate Riemann solver

Weak form

PDE → ODE

$$u = \sum_{j=1}^{N} u\left(x_j, t\right)\, \ell_j\left(x\right)$$

$$\int_K \frac{\partial u}{\partial t}\, \ell_i\, dV = -\left\{ \left[\alpha \sqrt{\gamma}\, \ell_i\, \hat{F}\right]_{x_L}^{x_H} - \int_K \alpha\, F\, \frac{d\ell_i}{dx}\, dV - \int_K S\, \ell_i\, dV \right\}, \quad i = 1, \cdots, N$$

$$\int_K \frac{\partial u}{\partial t}\, \ell_i\left(x\right)\, dV = \int_K \frac{\partial}{\partial t} \left(\sum_{j=1}^{N} u\left(x_j, t\right)\, \ell_j\left(x\right) \right) \ell_i\left(x\right)\, dV = \int_K \frac{\partial}{\partial t} \left(\sum_{j=1}^{N} u_j\left(t\right)\, \ell_j\left(x\right) \right) \ell_i\left(x\right)\, dV$$

$$= \sum_{j=1}^{N} \left[\frac{du_j}{dt} \int_K \ell_i\left(x\right)\, \ell_j\left(x\right)\, dV \right]$$

$$M_{ij} \equiv \int_K \ell_i\left(x\right)\, \ell_j\left(x\right)\, dV$$

$$= \sum_{j=1}^{N} M_{ij}\, \frac{du_j}{dt} \Longrightarrow \boldsymbol{M}\, \frac{d\boldsymbol{u}}{dt}, \quad \boldsymbol{u} = \left(u_1\left(t\right), \cdots, u_N\left(t\right)\right)^T$$

$$\boldsymbol{\ell} = \left(\ell_1\left(x\right), \cdots, \ell_N\left(x\right)\right)^T$$

$$\frac{d\boldsymbol{u}}{dt} = -\boldsymbol{M}^{-1} \left\{ \left[\alpha \sqrt{\gamma}\, \boldsymbol{\ell}\, \hat{F}\right]_{x_L}^{x_H} - \int_K \alpha\, F\, \frac{d\boldsymbol{\ell}}{dx}\, dV - \int_K S\, \boldsymbol{\ell}\, dV \right\}$$

Time-Stepping Algorithm

Strong-Stability-Preserving Runge-Kutta (SSP-RK)

Cockburn & Shu, (2001) J. Sci. Comp., Vol. 16, No. 3

$$\frac{du}{dt} = L(u)$$

1.) $u^{(0)} = u^n;$

2.) For $i = 1, \cdots, N_s$ compute the intermediate functions:

$$u^{(i)} = \Lambda\,\Pi\left(\sum_{j=0}^{i-1} \alpha_{ij}\, w^{ij}\right), \quad w^{ij} = u^{(j)} + \frac{\beta_{ij}}{\alpha_{ij}}\,\Delta t^n\, L\left(u^{(j)}\right)$$

3.) $u^{n+1} = u^{N_s}$

Limiters

Forward-Euler: $N_s = 1, \quad u^{n+1} = u^n + \Delta t^n\, L(u^n)$

Slope Limiter

MinMod Limiter

- Higher-order methods can develop unphysical oscillations
- Map solution from nodal to modal representation
- Compare slope in target cell to derivatives in neighbors via cell-averages

Legendre Polynomials

$$u_h(x,t) = \sum_{i=1}^{N} u_h(x_i,t)\,\ell_i(x) = \sum_{n=0}^{N-1} c_n(t)\,P_n(x)$$

Dimensionless parameter with value between 1 and 2

$$\tilde{c}_1 = \mathrm{MinMod}\left(\beta_{\mathrm{TVD}}\frac{\overline{u}_K - \overline{u}_{K-1}}{\Delta x},\ c_1,\ \beta_{\mathrm{TVD}}\frac{\overline{u}_{K+1} - \overline{u}_K}{\Delta x}\right),$$

$$\mathrm{MinMod}(a,b,c) = \begin{cases} 0, & a,b,c \text{ not all same signs} \\ \mathrm{sign}(a)\min(|a|,|b|,|c|), & \text{else} \end{cases}$$

Troubled-Cell Indicator

Fu & Shu, (2017), J. Comp. Phys. 347, 305

- MinMod limiter activates at smooth extrema
- To alleviate, we detect troubled-cells before applying limiter
- Quantify the difference in solutions in neighboring cells
- Extrapolate solution in neighboring cells into target cell
- Compute magnitude of differences between extrapolated solution and cell-average
- Compare value to constant parameter

Bound-Preserving Limiter

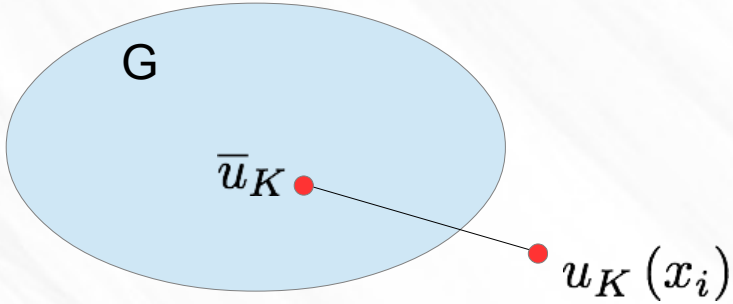
Qin et al., (2016), J. Comp. Phys., 315, 323

- Higher-order methods can exceed physical bounds
  - P < 0
  - ρ < 0
  - |v| > c
- Cell-average is guaranteed to be physical, but **all** nodal points need to be physical
- Define "set of admissible states", G
- Leads to two conditions:

$$D > 0$$

$$q = \tau + D - \sqrt{D^2 + S^i S_i} > 0$$

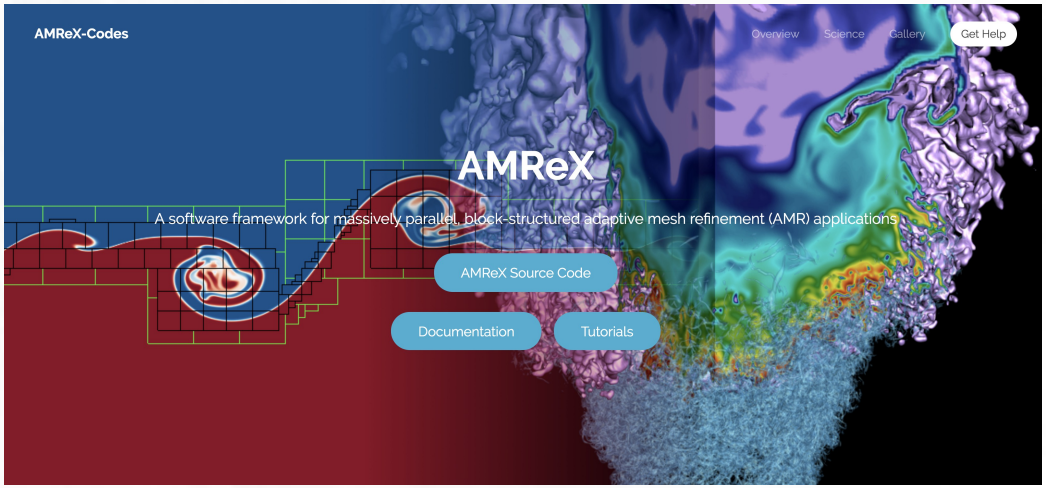
- Damp point-values towards cell-average (allowed because of CFL condition)



AMReX

https://amrex-codes.github.io/amrex/

- Parallel/AMR framework developed primarily from Lawrence Berkeley National Laboratory
- Block-Structured AMR



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Background/Motivation - CCSNe

Numerical Method - RKDG

Results

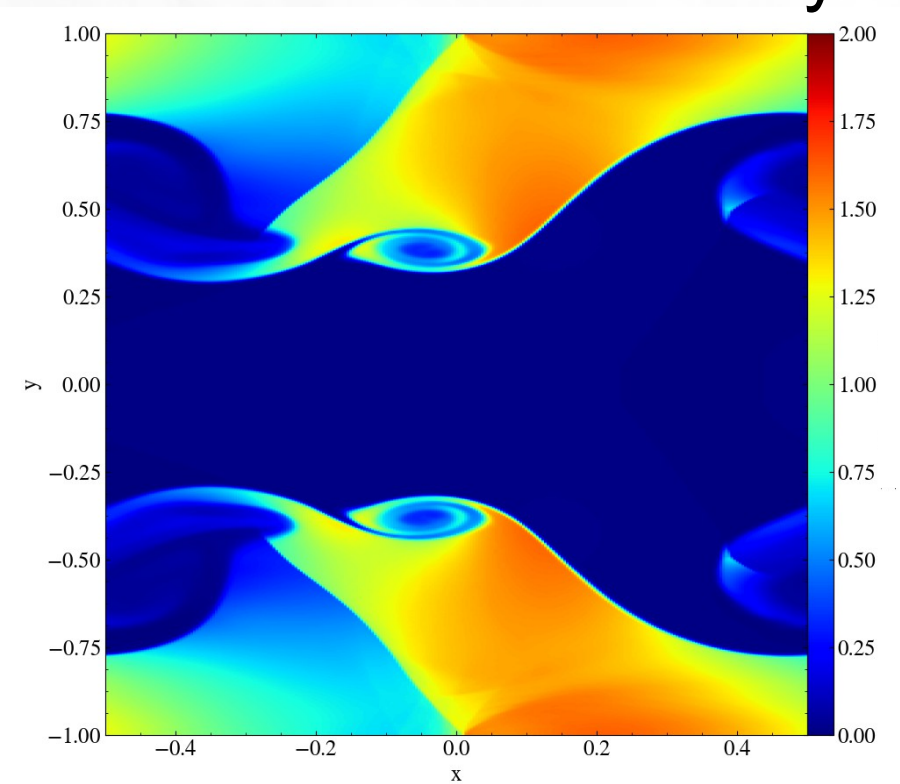
Summary/Future Work

Relativistic Kelvin-Helmholtz Instability

Radice & Rezzolla, (2012), A&A, 547, A26

- Smooth solution
- Turbulent regions*
- 256 x 512
- Third-order method
- SSP-RK3
- HLL Riemann solver
- Characteristic Limiting
- Run with AMReX

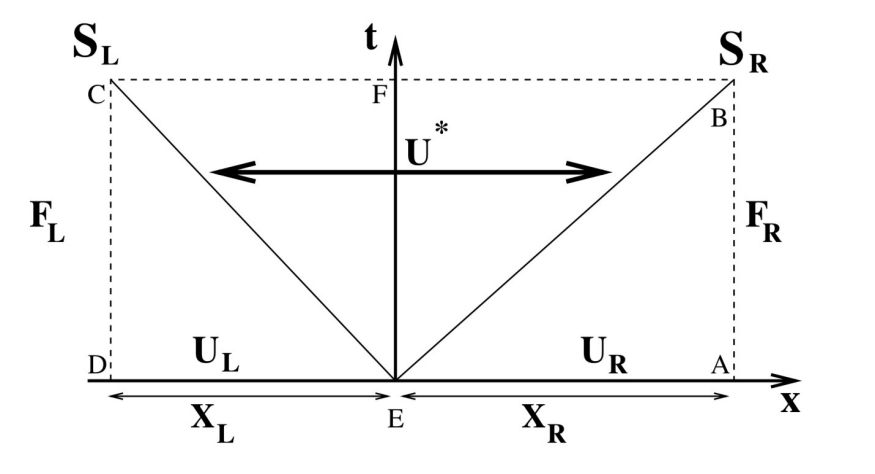
Primitive Mass-Density



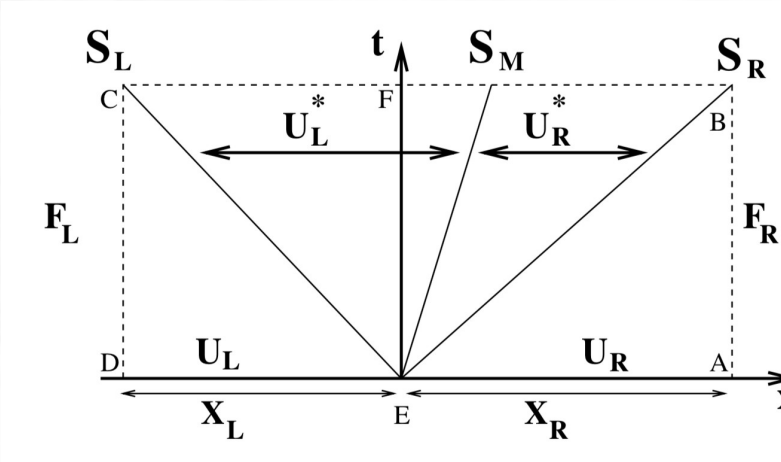
* Assuming ideal EOS

HLL vs. HLLC Riemann Solvers

Batten et al., 1997, SIAM-JSC, 18, 1553



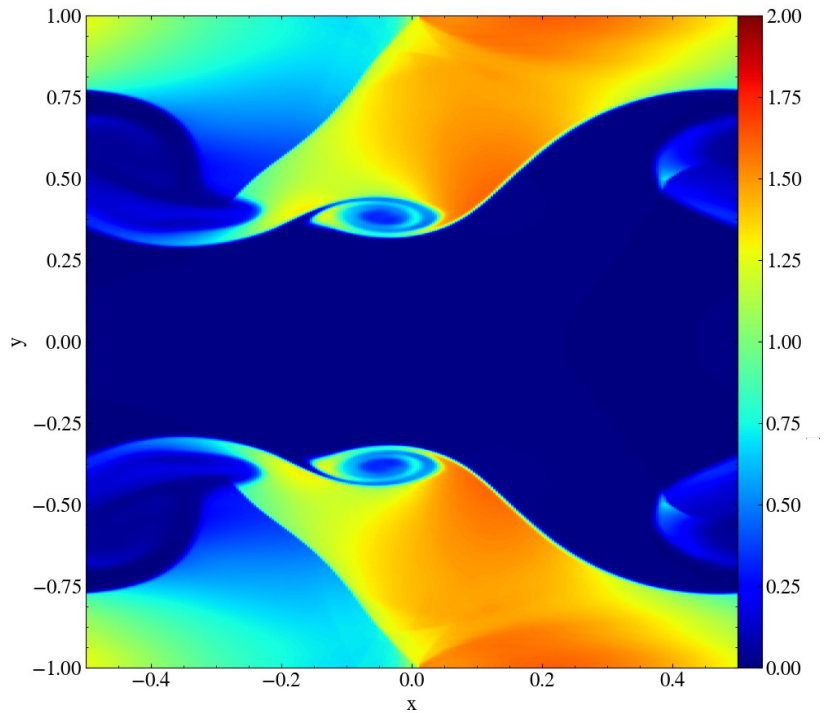
HLL



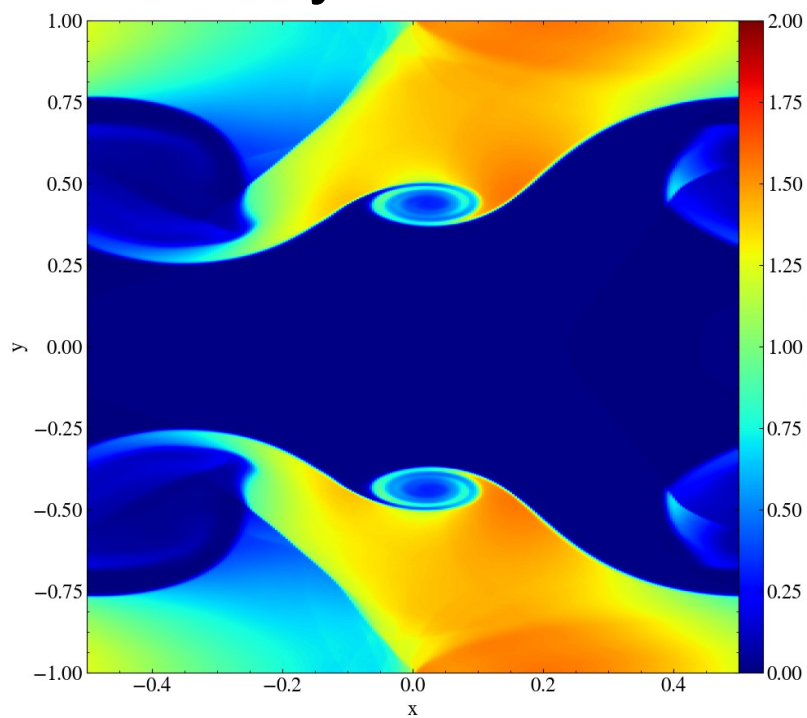
HLLC

HLL vs. HLLC Riemann Solvers

Primitive Mass-Density



HLL



HLLC

Approximates Riemann fan with only two waves

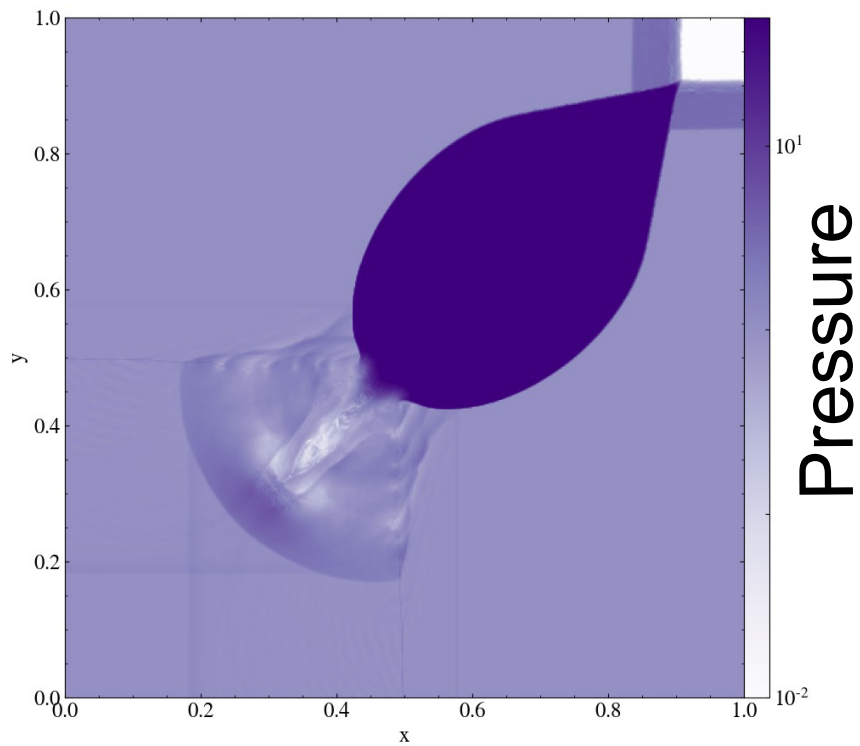
nX = 256
nY = 512
nNodes = 3
SSP-RK3

Approximates Riemann fan with two waves and one contact wave
Mignone & Bodo (2005), MNRAS, 364, 126

2D Riemann Problem

Del Zanna & Bucciantini, (2002), A&A, 390, 1177

- Highly relativistic
  - W ~ 7
- Contact waves
- 512 x 512
- Third-order method
- SSP-RK3
- HLL Riemann solver
- Component-wise limiting
  - Coupled variables
- Run with AMReX



Characteristic Decomposition (Proof of concept: Linear case)

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = \boldsymbol{0}$$

$$= \frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}} \frac{\partial \boldsymbol{U}}{\partial x}$$

$$= \frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{R}\,\boldsymbol{\Lambda}\,\boldsymbol{R}^{-1} \frac{\partial \boldsymbol{U}}{\partial x} \qquad \boldsymbol{\Lambda} = \mathrm{diag}\left(\lambda_1, \lambda_2, \cdots, \lambda_N\right)$$

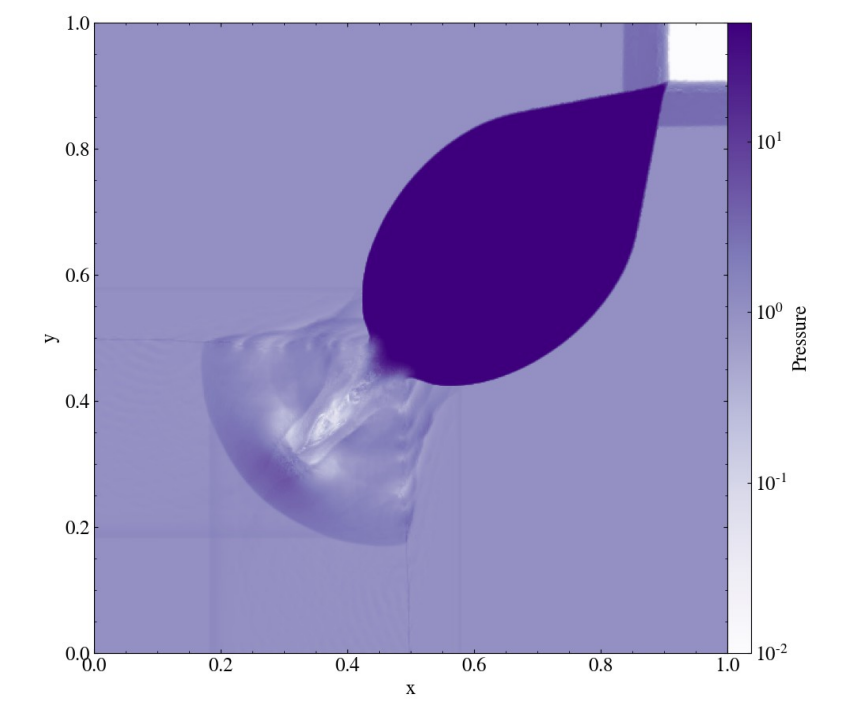
$$= \frac{\partial \left(\boldsymbol{R}^{-1}\boldsymbol{U}\right)}{\partial t} + \boldsymbol{\Lambda} \frac{\partial \left(\boldsymbol{R}^{-1}\boldsymbol{U}\right)}{\partial x}$$

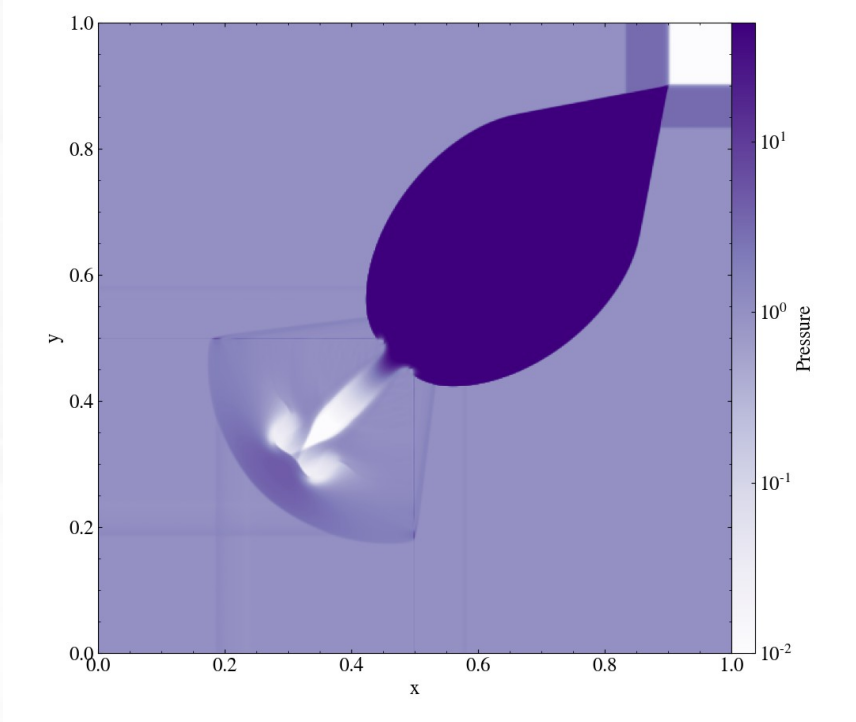
$$= \frac{\partial \boldsymbol{W}}{\partial t} + \boldsymbol{\Lambda} \frac{\partial \boldsymbol{W}}{\partial x}$$

Decoupled!

Characteristic Limiting

Pressure





Component-Wise Limiting

Characteristic Limiting

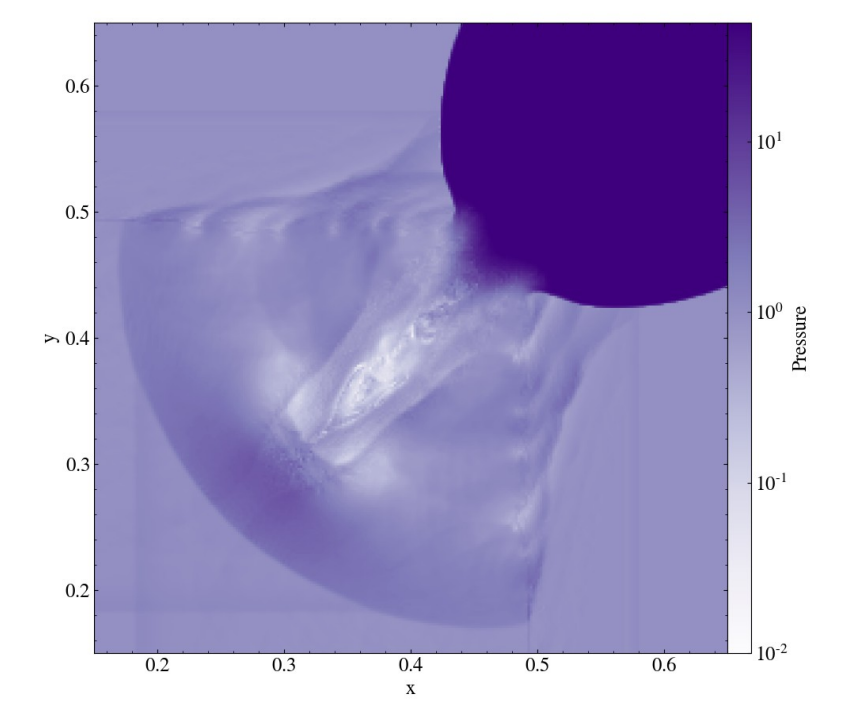
Limiting physical variables

nX = nY = 512
nNodes = 3
SSP-RK3

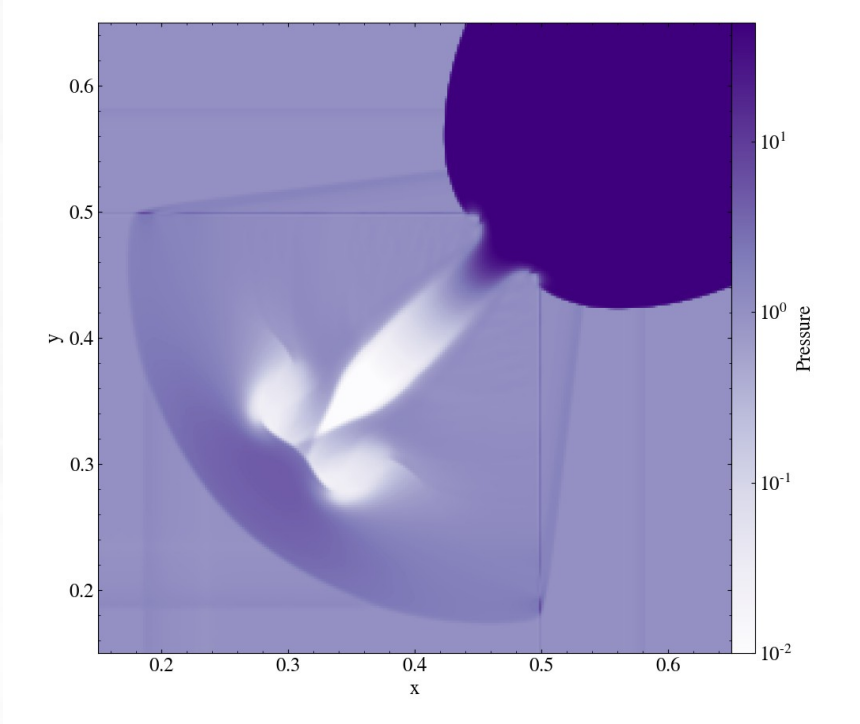
Limiting (decoupled) characteristic variables

Characteristic Limiting

Pressure



Component-Wise Limiting



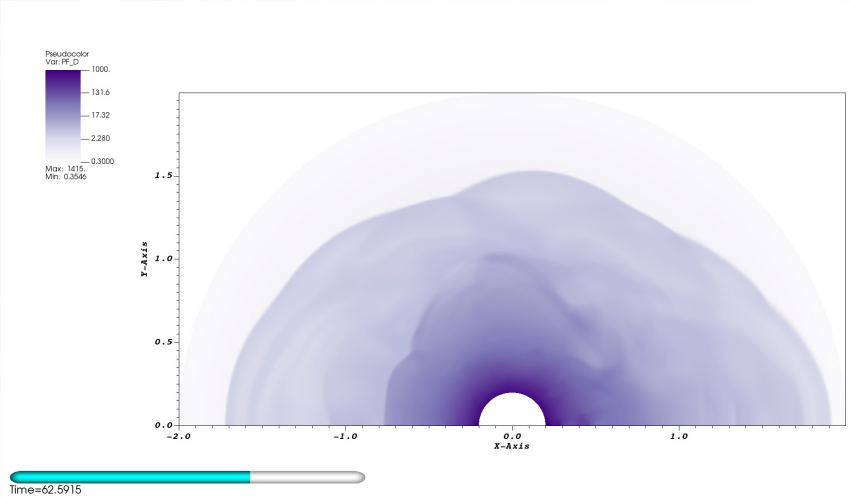
Characteristic Limiting

nX = nY = 512
nNodes = 3
SSP-RK3

2D, Non-Relativistic SASI

- Physically motivated
- Curvilinear coordinates
- Point-mass gravitational potential
- Perturbed with $\ell = 1$ mode instability
- 256x128
- Third-order method
- SSP-RK3

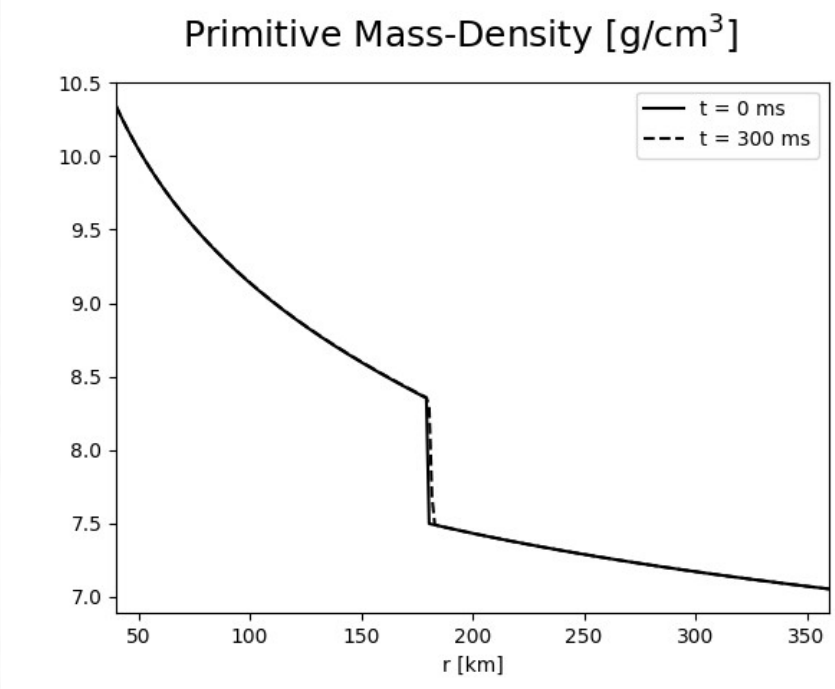
Mass-Density



- Run with AMReX

GR Standing Accretion Shock

- Physically-motivated
- Curvilinear coordinates
- Stationary background spacetime
- 256 elements
- Third-order method
- HLLC Riemann solver
- Characteristic-limiting
- Run with AMReX



- Evolved over several characteristic SASI development timescales

Outline

Background/Motivation - CCSNe

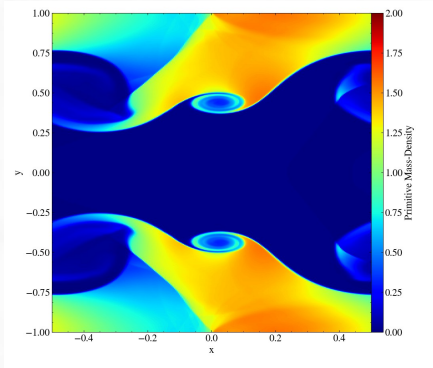
Numerical Method - RKDG

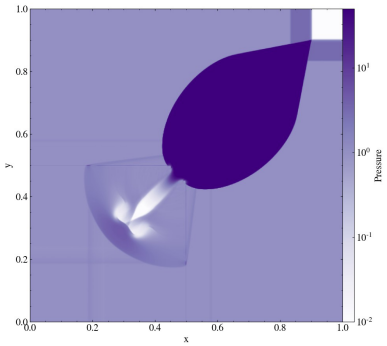
Results – KHI/2DRP/SASI/GR-SAS

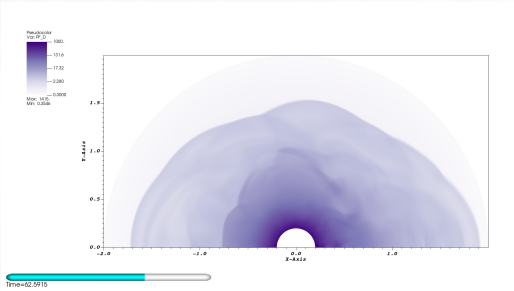
Summary/Future Work

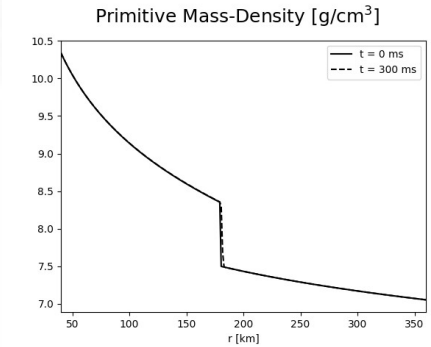
Summary

- Developing Multi-D DG-GR Hydro solver
- Implemented characteristic slope-limiting
- Implemented bound-preserving limiter
- Incorporated AMReX
- Successfully run several challenging test problems





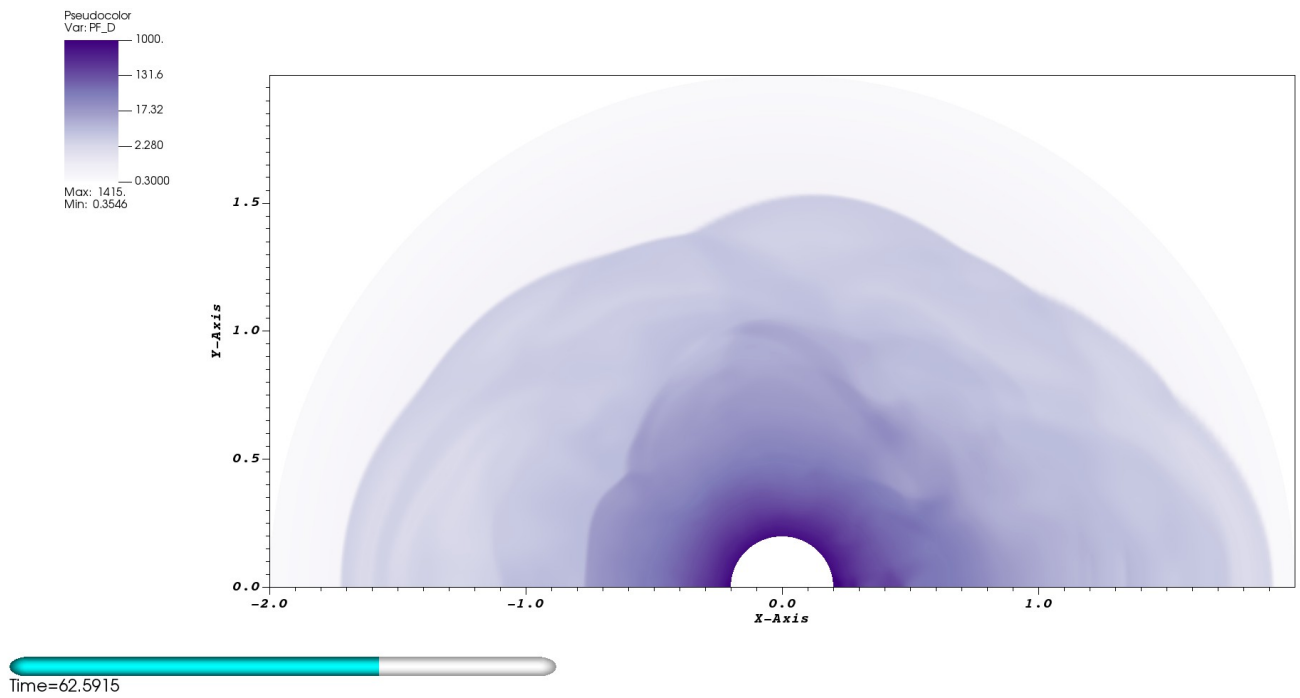




Future Work

- MPI scaling tests
- Implement and test 3D components
- Run 2D and 3D, GR-SASI problem
  - Investigate effects of GR-hydro
- AMR
- Couple with CFA gravity solver (Nick Roberts)
- Couple with neutrino-transport solver (Ran Chu, Zach Elledge)

Acknowledgements

- I would like to acknowledge the NSF for helping to fund this research
  - NSF GP-1505933
  - NSF GP-1806692
- I would also like to acknowledge Vanderbilt University for helping to fund this research

Thank you!
Questions?