A Discontinuous Galerkin Method for General Relativistic Hydrodynamics

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FISK-VANDERBILT Master's-to-Ph.D. BRIDGE PROGRAM

Overview

Ultimate goal: Determine the core-collapse supernova explosion mechanism
Why study this?: It will provide insights into distributions of metals in galaxies, nuclear properties of matter, gravitational waveforms, and more!
Multi-faceted problem including gravity, neutrino transport, and **hydrodynamics**
As a first step we are developing numerical methods to solve the general relativistic hydrodynamics equations using a discontinuous Galerkin (DG) method
 - DG method provides high-order accuracy along with a local approach, making it desirable for problems involving shocks

About the code: THORNADO

The algorithms are being implemented in the Toolkit for High-ORder Neutrino rADiation-hydrOdynamics, THORNADO, which is being developed for simulations of core-collapse supernovae and related phenomena. THORNADO combines continuous and discontinuous finite element (CG and DG) methods for the major physics components in the conformal flatness approximation (CFA).

- Hydrodynamics (DG)
- Neutrino transport (DG)
- Gravity (CG)

3+1 Decomposed GR Hydro Equations in Conservative Form

$$\partial_t\left(\sqrt{\gamma}\,\mathbf{U}\right) + \partial_i\left(\sqrt{\gamma}\,\mathbf{F}^i\right) = \mathbf{S}$$

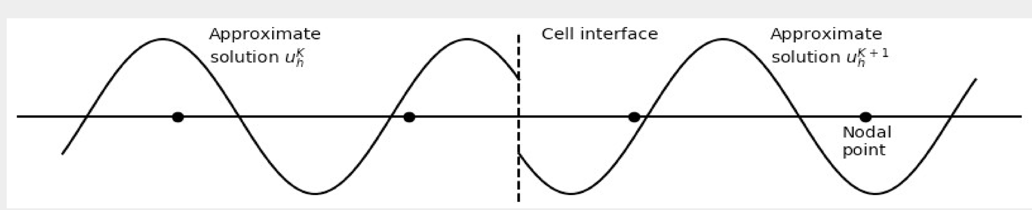
$$\mathbf{U} = \begin{pmatrix} D \\ S_j \\ \tau \end{pmatrix} \equiv \begin{pmatrix} \rho W \\ \rho h W^2 v_j \\ \rho W (h W - 1) - p \end{pmatrix} \qquad \mathbf{F}^i \equiv \begin{pmatrix} D\left(\alpha v^i - \beta^i\right) \\ \alpha S^i{}_j - \beta^i S_j \\ \alpha\left(S^i - D v^i\right) - \beta^i \tau \end{pmatrix}$$

$$W = \left(1 - v^i v_i\right)^{-1/2} \qquad \mathbf{S} \equiv \sqrt{\gamma}\begin{pmatrix} 0 \\ \frac{1}{2}\alpha S^{ik}\partial_j \gamma_{ik} + S_i \partial_j \beta^i - (\tau + D)\partial_j \alpha \\ \alpha S^{ij} K_{ij} - S^j \partial_j \alpha \end{pmatrix}$$

Discontinuous Galerkin Method

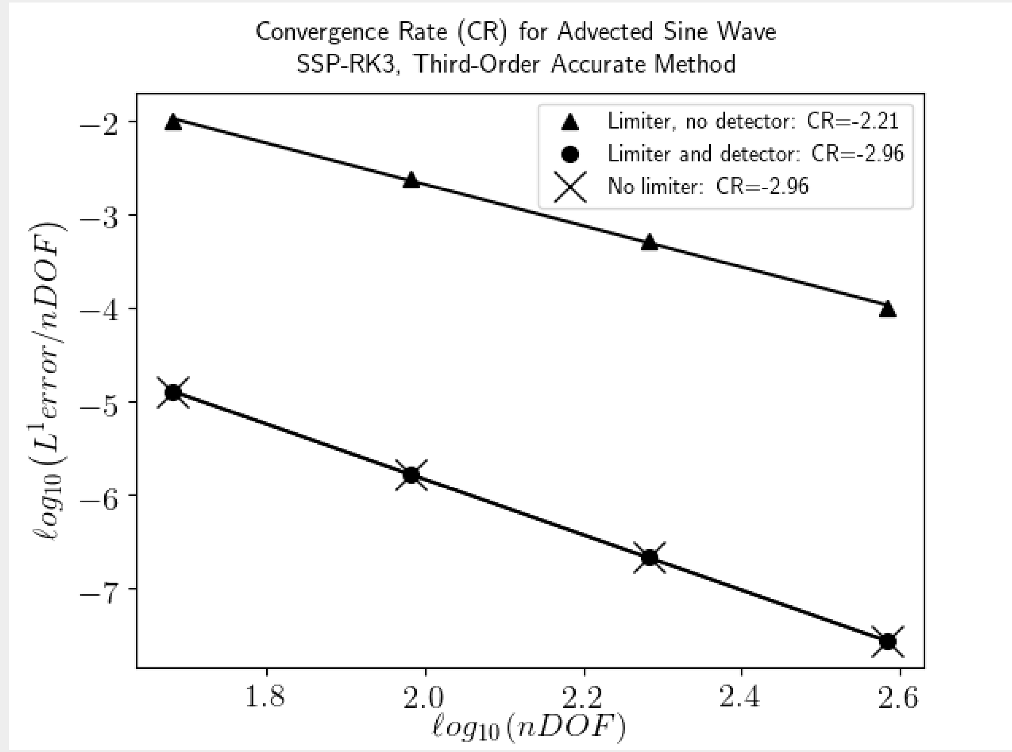
Discretize the spatial domain into local elements, K
Approximate the actual solution in each K, $\mathbf{u}^K$, with a polynomial approximation, $\mathbf{u}_h^K$, which is required to satisfy

$$\int_K \left(\frac{\partial \mathbf{u}_h^K}{\partial t} - \mathbf{F}^i \frac{\partial v}{\partial x^i}\right)\sqrt{\gamma}\, d^3x + \int_{\tilde{K}}\left(\sqrt{\gamma}\,\hat{\mathbf{F}}^i v\Big|_{x^i_H} - \sqrt{\gamma}\,\hat{\mathbf{F}}^i v\Big|_{x^i_L}\right) d^2\tilde{x} = \int_K \mathbf{S}\sqrt{\gamma}\, d^3x$$

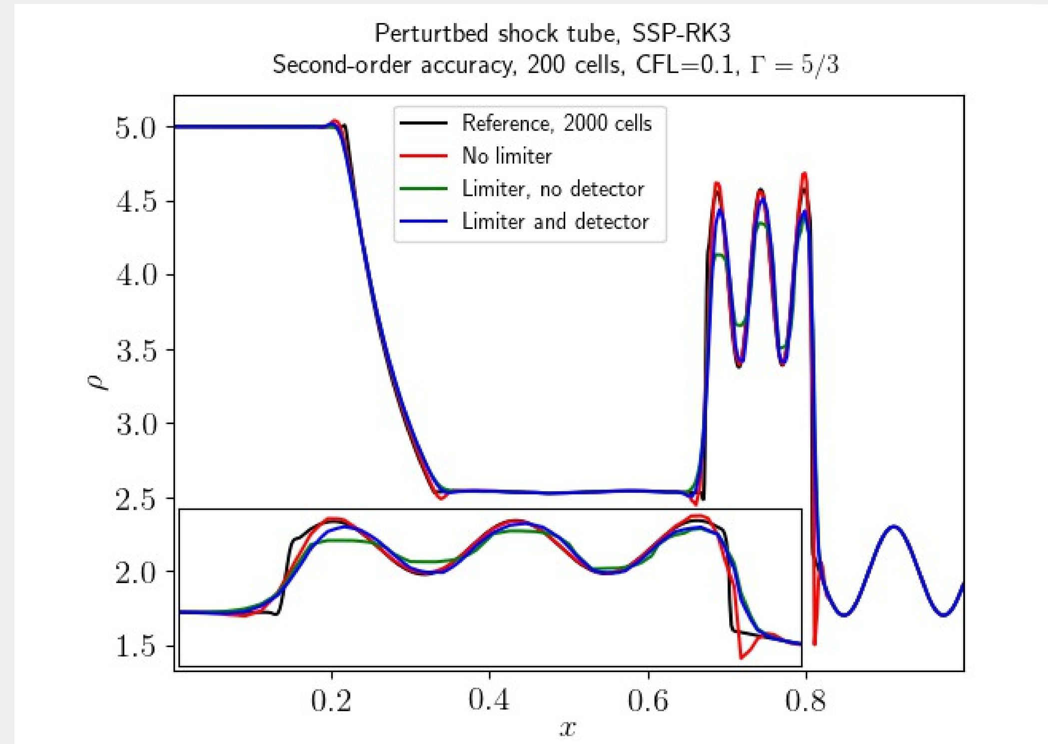


for each element K and each *test function* v. Since the polynomials are local they are in general discontinuous at the cell interfaces (see figure)
This sets up a Riemann problem at each interface, which we solve using the HLLC numerical flux, $\hat{\mathbf{F}}^i$
The discontinuities can lead to spurious oscillations in the solution, so we use a slope limiter to damp them (see slope limiter box)
 - However, in smooth regions the slope limiter unnecessarily causes a loss in accuracy, so we also employ a troubled-cell indicator so that the slope limiter only activates in areas where a discontinuity is present, i.e. shocks

Slope Limiter/Troubled-Cell Indicator



If using the limiter but not the troubled-cell indicator (triangles) we don't get third-order accuracy. This is because the limiter is activating--unnecessarily--in smooth regions (the peaks of the sine wave). The effect of the troubled-cell indicator is to only activate the limiter near discontinuities. Since this is a smooth problem there are no discontinuities and the limiter *never* activates, which is why using the limiter with the troubled-cell indicator results in third-order accuracy. The lines are straight-line fits, and their slope is defined as the convergence rate (CR).



Main: Fluid mass-density in the comoving frame at a time 0.35 for a Riemann problem with sinusoidal perturbations ahead of the shock [3]. We see that if no limiters are used (red line) the three peaks near 0.7<x<0.8 are well-resolved, but there is a large overshoot around x=0.8. When we use the slope limiter, but not the discontinuity detector (green line), we see that the overshoot around x=0.8 is gone, **but the three peaks are not well-resolved**. Finally, when we use the slope limiter *and* the troubled-cell indicator (blue line) we again see that the overshoot around x=0.8 is gone, **but now the three peaks are better resolved**. We also note that for this problem we made use of the *positivity limiter*, which ensures positivity of the density (among other things), without which the simulation crashes. The black line is a high-resolution reference for comparison. Inset: blow-up of three peaks.

Standing Accretion Shock Instability (SASI)

What is the SASI?: An instability in the supernova shockwave that eventually disrupts the stellar envelope. It is characterized by global sloshing and spiral motions in the post-shock flow.
Why study the SASI?: The SASI is widely believed to be a key instability that aids the neutrino-driven supernova explosion.

The supernova explosion mechanism is a multi-physics problem, but details of important hydrodynamics phenomena can be studied with simplified models such as the one described here

<u>About the Initial Conditions</u>

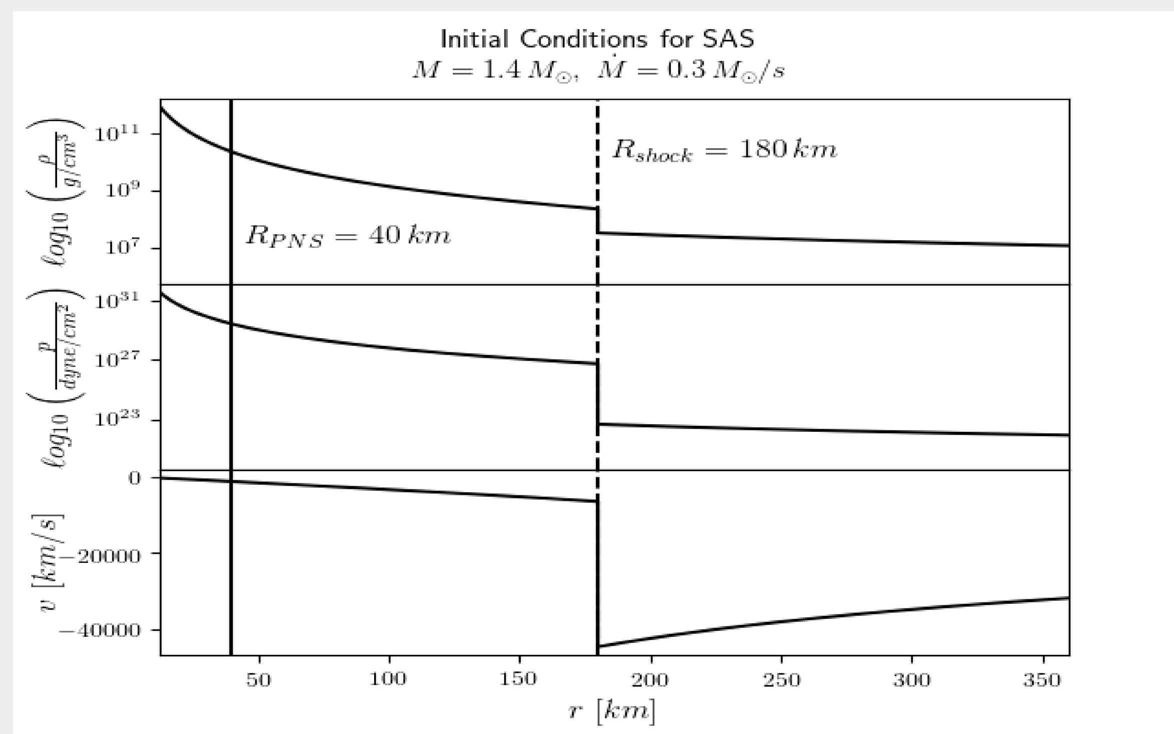
To generate these initial conditions we start with the 3+1 decomposed GR hydro equations (see box), assuming:

- spherical symmetry
- conformally-flat, steady-state spacetime metric
- perfect fluid
- ideal gas (polytropic) equation of state $p = (\Gamma - 1)\, e$
- initial gauge (coordinate) choice: zero shift-vector

With this we can derive two conservation laws:

Conservation of mass:
$$\partial_r\left(\sqrt{\gamma}\,\alpha\,\rho\, W\, v\right) = 0$$
Conservation of energy:
$$\partial_r\left(\sqrt{\gamma}\,\alpha\,\rho\, h\, W^2 v\right) = -\sqrt{\gamma}\,\rho\, h\, W^2\, v\, \partial_r \alpha$$

The pre-shock flow is assumed to be cold, so p~0. For numerical reasons we then artificially add in a small pre-shock pressure by assuming a constant Mach number of 100.



Initial conditions for the spherically-symmetric standing accretion shock problem. The top panel shows the comoving mass-density in g/cm^3, the middle panel shows the comoving pressure in dyne/cm^2, and the bottom panel shows the fluid three-velocity in km/s.The dashed vertical line shows the initial location of the shock and the solid vertical line shows the location of the surface of the proto-neutron star (PNS). The lower radial limit is three times the Schwarzschild radius.

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Future Work

- Implement DG-GR hydro solver and study SASI-driven flows in GR
- Upgrade to be fully three-dimensional
- Couple with CG gravity solver and develop DG-GR neutrino transport solver
- Use for core-collapse supernova simulations

References

[1] Fu, G., & Shu, C. W. (2017). A new troubled-cell indicator for discontinuous Galerkin methods for hyperbolic conservation laws. Journal of Computational Physics, 347, 305–327. http://doi.org/10.1016/j.jcp.2017.06.046
[2] Blondin, J. M., & Mezzacappa, A. (2006). The Spherical Accretion Shock Instability in the Linear Regime. The Astrophysical Journal, 642(1), 401–409. http://doi.org/10.1086/500817
[3] Del Zanna, L., & Bucciantini, N. (2002). An efficient shock-capturing central-type scheme for multidimensional relativistic flows. I. Hydrodynamics, 1186, 1177–1186. http://doi.org/10.1051/0004-6361:20020776

Background image from APOD: https://apod.nasa.gov/apod/ap170305.html