Towards Discontinuous Galerkin Methods for General Relativistic Core-Collapse Supernova Simulations

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Outline

Motivation

Methods

Results

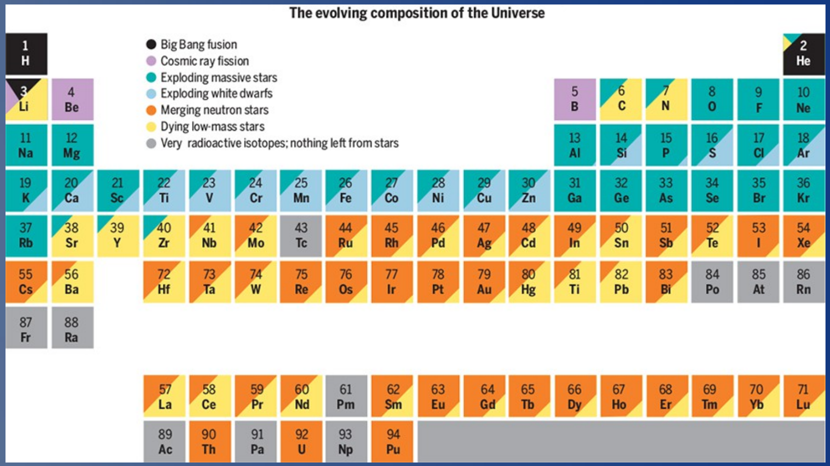
Summary, etc.

Outline

Motivation

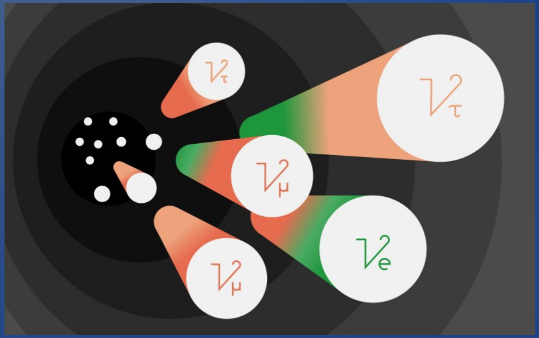
Motivation

- Why study core-collapse supernovae (CCSNe)?
  - Nucleosynthesis
  - Neutron stars
  - Gravitational waves
  - Neutrinos



sciencemag.org

APOD



sanfordlab.org

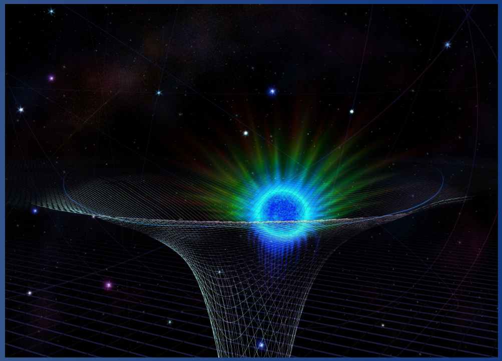
Outline

Methods

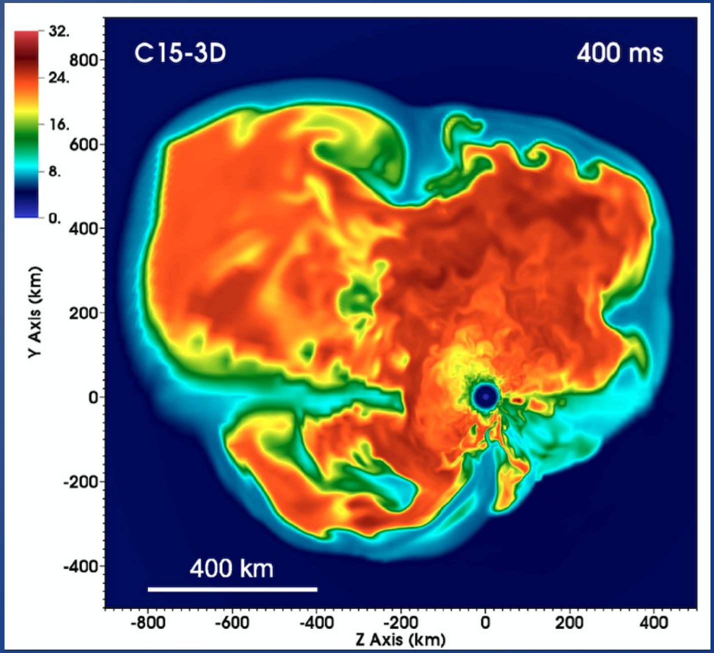
How to study CCSNe?

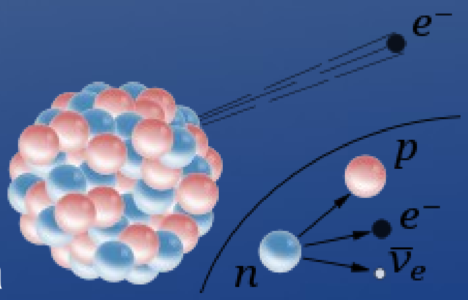
- Simulations!
  - Macrophysics
    - Gravity
    - Hydrodynamics
    - Neutrino transport
  - Microphysics
    - Equation of state
      - $p = p(\rho, T, Y_e)$
    - Weak interactions

phys.org



Lentz et al., 2015





wikipedia

How do **we** study CCSNe?

thornado (https://github.com/endeve/thornado)

- **t**oolkit for **h**igh-**or**der **n**eutrino-r**ad**iation hydr**o**dynamics
- Hydrodynamics
  - Discontinuous Galerkin methods
  - Explicit Runge-Kutta methods
- Neutrino Transport
  - Discontinuous Galerkin methods
  - Implicit/Explicit Runge-Kutta methods
- Gravity
  - Continuous Galerkin (finite-element) methods
- Tabulated microphysics from weaklib (https://github.com/starkiller-astro/weaklib)
- Parallelism via AMReX (https://amrex-codes.github.io/amrex/)

Why GR?

- Results of GR (e.g., Bruenn et al., (2001), Müller at al., (2012))
    - Increased compactness of proto-neutron star
    - Decreased region between gain radius and shock
    - Increased post-shock fluid speed
- Effects
    - Harder neutrino spectra
    - Higher neutrino luminosities
    - Changes in neutrino heating
    - Changes in convection and turbulence

3+1 Hydrodynamics Equations

$$\partial_t \left(\sqrt{\gamma}\,\boldsymbol{U}\right) + \partial_i \left(\alpha \sqrt{\gamma}\,\boldsymbol{F}^i\left(\boldsymbol{U}\right)\right) = \alpha \sqrt{\gamma}\,\boldsymbol{S}\left(\boldsymbol{U}\right)$$

$$\begin{pmatrix} \rho \\ v^j \\ e \\ n_e \end{pmatrix} \longrightarrow \begin{pmatrix} D \\ S_j \\ \tau \\ N_e \end{pmatrix} \qquad \boldsymbol{U} = \begin{pmatrix} D \\ S_j \\ \tau \\ N_e \end{pmatrix} = \begin{pmatrix} \rho\,W \\ \rho\,h\,W^2\,v_j \\ \rho\,h\,W^2 - p - \rho\,W \\ n_e\,W \end{pmatrix}$$

$$W = \left(1 - v^i v_i\right)^{-1/2}$$

$$\rho\,h = \rho + e + p$$

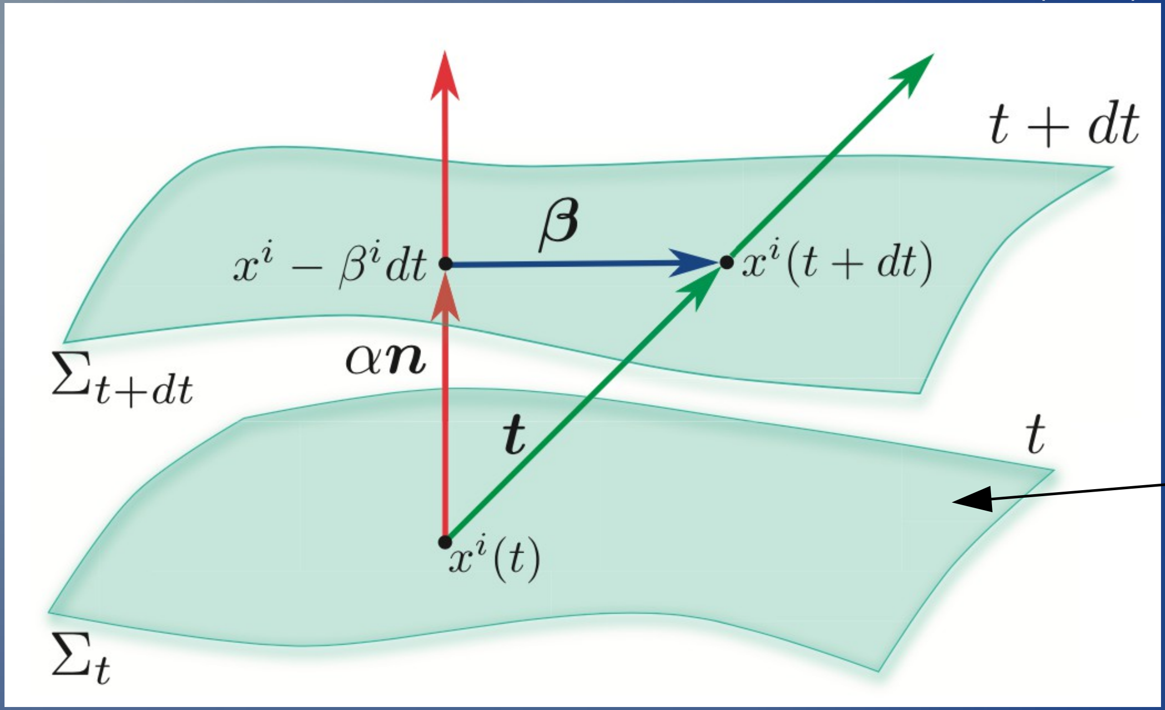
$$P^{ij} = \rho\,h\,W^2\,v^i\,v^j + p\gamma^{ij}$$

$$\boldsymbol{F}^i = \begin{pmatrix} F_D \\ F^i_{S_j} \\ F_\tau \\ F_{N_e} \end{pmatrix} = \begin{pmatrix} D\left(v^i - \alpha^{-1}\beta^i\right) \\ P^i{}_j - \alpha^{-1}\beta^i\,S_j \\ S^i - D\,v^i - \alpha^{-1}\beta^i\,\tau \\ N_e\left(v^i - \alpha^{-1}\beta^i\right) \end{pmatrix}$$

$$\boldsymbol{S} = \begin{pmatrix} 0 \\ \frac{1}{2}P^{ik}\,\partial_j\gamma_{ik} + \alpha^{-1}S_i\,\partial_j\beta^i - \alpha^{-1}\left(\tau + D\right)\partial_j\alpha \\ P^{ij}\,K_{ij} - \alpha^{-1}S^j\,\partial_j\alpha \\ 0 \end{pmatrix}$$

3+1 GR

Rezzolla & Zanotti (2013)



$\gamma_{ij} = \gamma_{ij}(x)$

$\alpha = \alpha(x)$ **Lapse** of proper time

$\beta^i = \beta^i(x)$ How much coordinates have **shifted**

$\alpha, \ \beta^i$ Freely specifiable

$K_{ij} = K_{ij}(\alpha, \beta^i, \gamma_{ij})$ How the 3D surface is curved in the 4D manifold

Conformally-Flat Approximation (CFA)

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

$$\gamma_{ij} = \psi^4 \,\mathrm{diag}\left(\overline{\gamma}_{11}, \overline{\gamma}_{22}, \overline{\gamma}_{33}\right)$$

- Eliminates dynamical degrees of freedom
- Exact in spherical symmetry
- Valid for slowly rotating progenitors (Dimmelmeier et al., (2002))

$$\psi = \psi\left(x\right)$$

$$\partial_t \overline{\gamma}_{ij} := 0$$

$$K^i{}_i = \gamma^{ij} K_{ij} := 0$$

$$\partial_t K^i{}_i := 0$$

"Maximal Slicing"

Gravity Solver

- Solves CFA equations (coupled system of elliptic PDEs)

$$\overline{\nabla}^2 \psi = -\left(2\pi\, \psi^5 E + \frac{1}{8}\psi^{-7}\, \overline{K}_{ij}\overline{K}^{ij} \right)$$

$$\overline{\nabla}^2 \left(\alpha\psi\right) = \alpha\psi \left(\frac{7}{8}\psi^{-8}\overline{K}_{ij}\overline{K}^{ij} + 2\pi\, \psi^4 \left(E + 2S\right) \right)$$

$$\overline{\nabla}^2 \beta^i = 16\pi\, \alpha\psi^4 S^i - \frac{1}{3}\overline{D}^j \overline{D}_k \beta^k + 2\alpha\psi^{-6}\overline{K}^{ij}\overline{D}_j \left(\alpha\psi^6\right)$$

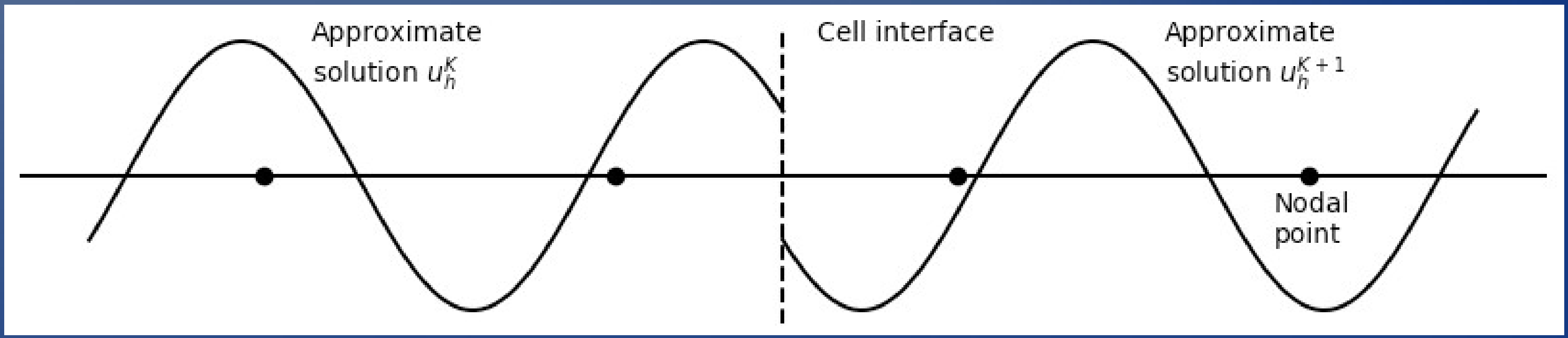
- E, S, S^i are derived from stress-energy tensor of fluid
- Uses hybrid of continuous Galerkin (finite element) and spectral methods

DG Method (1D)

- Discretize computational domain
- Locally approximate solution as polynomial

$$u(x,t) \approx u_h(x,t) := \sum_{i=1}^{N} u_i(t)\,\ell_i(x)$$

- Use weak form to evolve in time with SSP-RK methods
- Apply slope- and bound-preserving-limiters at each stage of SSP-RK algorithm



Weak Form (1D)

$$\partial_t \boldsymbol{U} + \frac{1}{\sqrt{\gamma}} \partial_r \left(\alpha \sqrt{\gamma}\, \boldsymbol{F} \right) = \boldsymbol{Q}\left(\boldsymbol{U} \right)$$

- Multiply by test function (Lagrange polynomial)
- Integrate over element

Time-dependent metric term incorporated here

$$\int_K \left(\partial_t \boldsymbol{U} \right) \ell_i \sqrt{\gamma}\, dr = -\left\{ \left[\alpha \sqrt{\gamma}\, \ell_i \left(r \right) \, \widehat{\boldsymbol{F}} \right]_{r_L}^{r_U} - \int_K \alpha\, \boldsymbol{F} \frac{d\ell_i}{dr} \sqrt{\gamma}\, dr - \int_K \boldsymbol{Q}\, \ell_i \left(r \right) \, \sqrt{\gamma}\, dr \right\}$$

Numerical flux obtained with approximate Riemann solver (e.g., HLL)

Substituting $\boldsymbol{U}_h \left(r, t \right) = \sum_{i=1}^{N} \boldsymbol{U}_i \left(t \right) \ell_i \left(r \right)$ approximation in yields system of ODEs in

$$\overline{\boldsymbol{U}}_h = \left(\boldsymbol{U}_1, \cdots, \boldsymbol{U}_N \right)^T$$

$$\frac{d}{dt} \overline{\boldsymbol{U}}_h = L\left(\overline{\boldsymbol{U}}_h \right)$$

Time-Stepping Algorithm

Strong-Stability-Preserving Runge-Kutta (SSP-RK)

Cockburn & Shu, (2001) J. Sci. Comp., Vol. 16, No. 3

$$\frac{d}{dt}\overline{\boldsymbol{U}}_h = L\left(\overline{\boldsymbol{U}}_h\right)$$

- Convex combinations of forward-Euler steps

1. $\overline{\boldsymbol{U}}_h^{(0)} = \overline{\boldsymbol{U}}_h^{n}$
2. For $i = 1, \cdots, N_s$ compute the intermediate functions:

$$\overline{\boldsymbol{U}}_h^{(i)} = \Lambda\Pi\left(\sum_{j=0}^{i-1}\alpha_{ij}w_h^{ij}\right), \quad w_h^{ij} = \overline{\boldsymbol{U}}_h^{(j)} + \frac{\beta_{ij}}{\alpha_{ij}}\Delta t^n\, L\left(\overline{\boldsymbol{U}}_h^{(j)}\right)$$

3. $\overline{\boldsymbol{U}}_h^{(n+1)} = \overline{\boldsymbol{U}}_h^{N_s}$

$$\alpha_{ij}, \beta_{ij} > 0, \; \sum_{j=0}^{i-1}\alpha_{ij} = 1$$

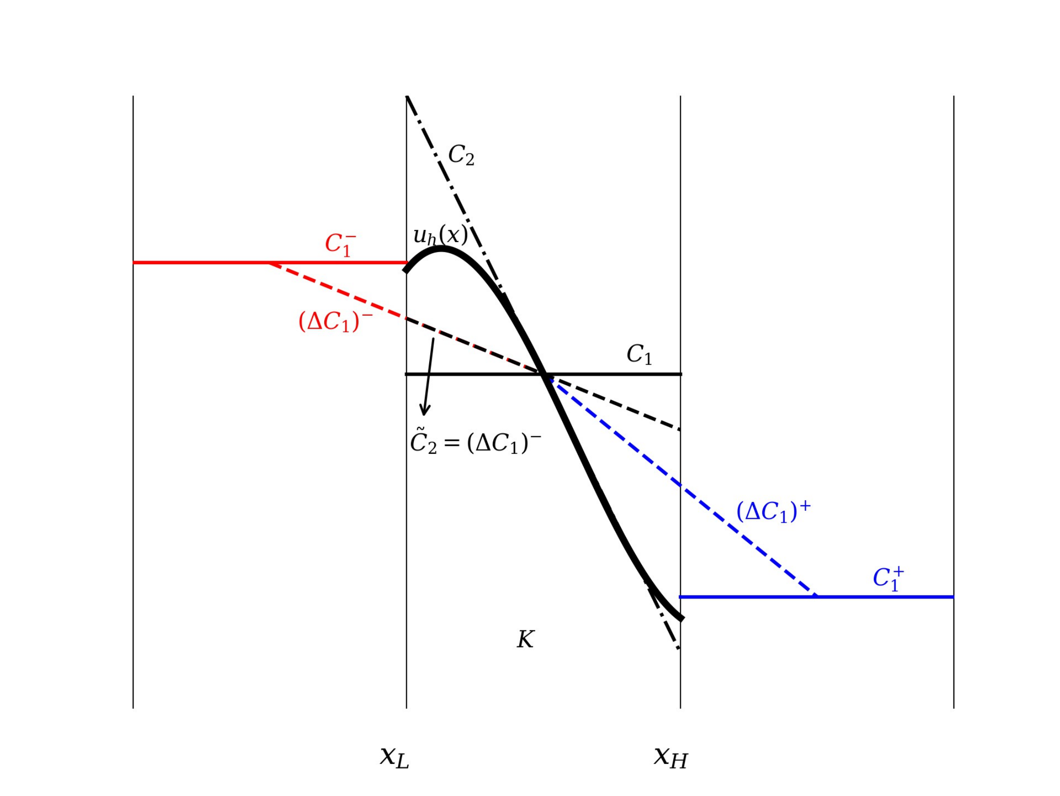
Limiters

Slope-Limiter

- Higher-order polynomials can develop unphysical oscillations
- MinMod limiter
  - Map to spectral representation
  - Filter out high-order terms
  - Map back

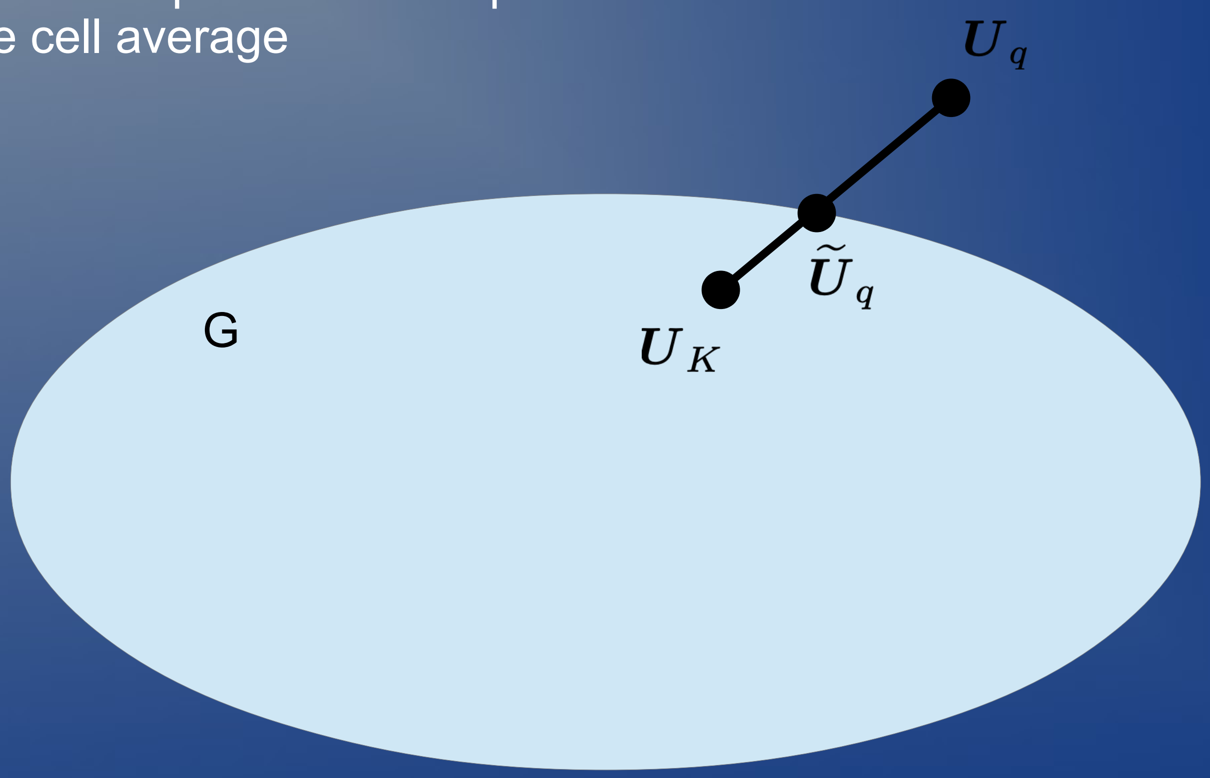
$$u_h\left(x,t\right)=\sum_{i=1}^{N}u_i\left(t\right)\ell_i\left(x\right)=\sum_{n=1}^{N}c_n\left(t\right)P_n\left(x\right)$$

Pochik et al., (2020)



Bound-Preserving Limiter

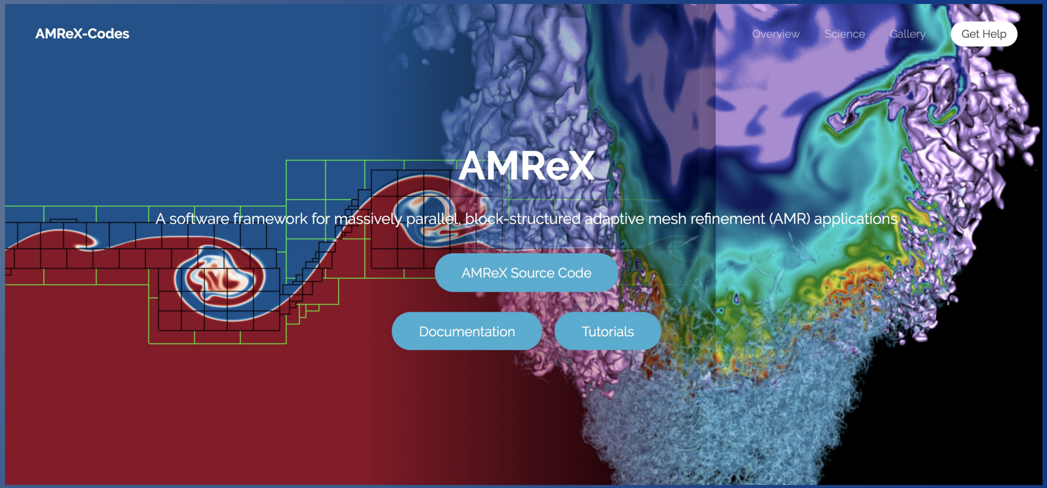
- The cell-averages are guaranteed to be physical, but not the nodal points (Qin et al. (2016))
- “Set of admissible states”, G, forms a convex set
- Unphysical nodal points are damped toward the cell average



AMReX

https://amrex-codes.github.io/amrex/

- Parallel/AMR framework developed primarily from Lawrence Berkeley National Laboratory
- Block-Structured AMR

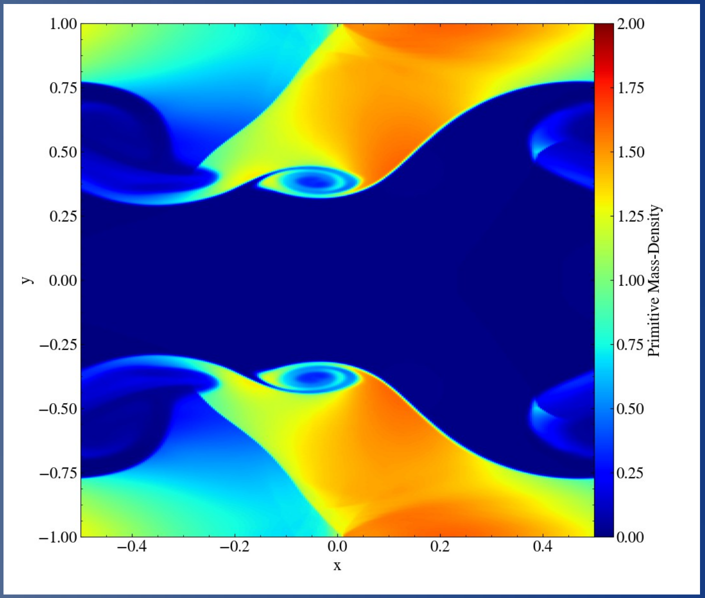


Outline

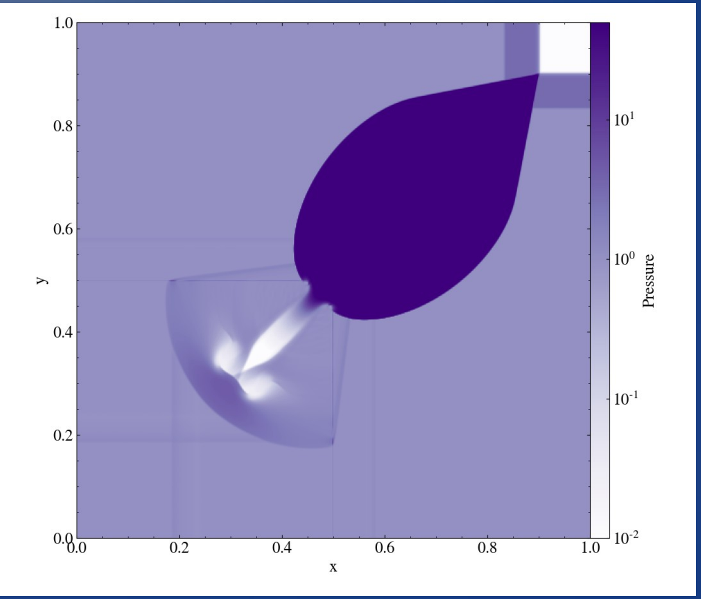
Results

Results

- Kelvin—Helmholtz instability
- Initial conditions from Radice & Rezzolla (2012)
- 256 x 512 elements
- 3rd order methods
- Run with AMReX

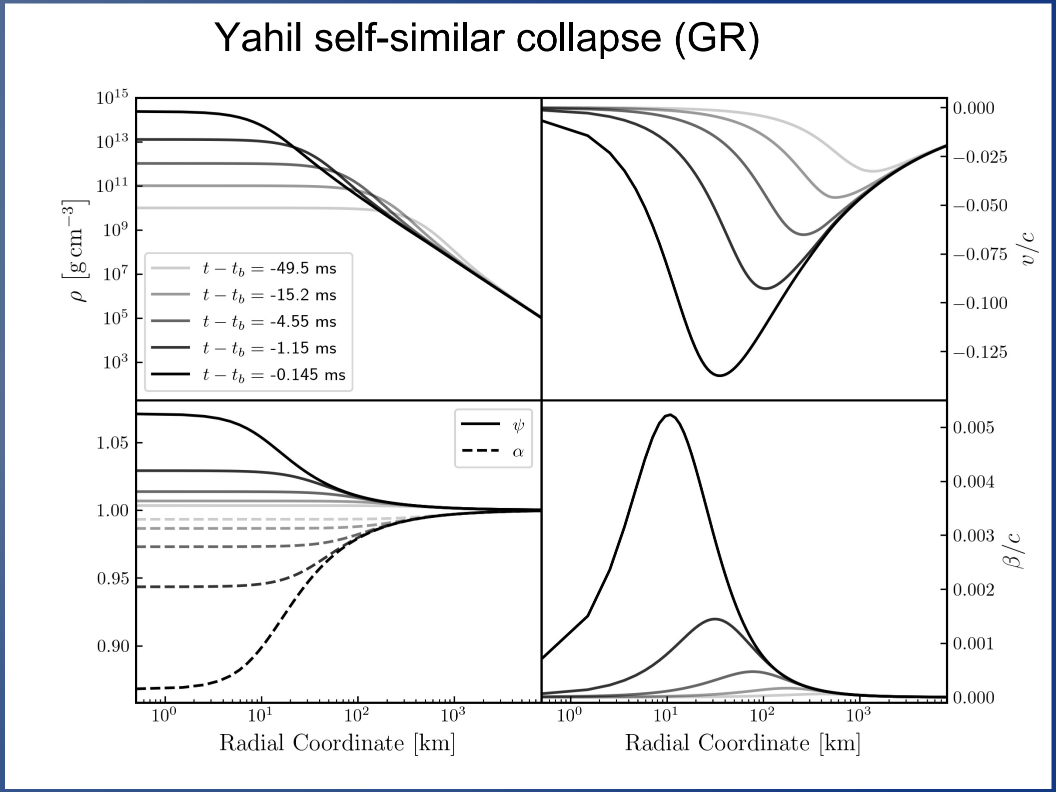


- 2D Riemann Problem
- Initial conditions from Del Zanna & Bucciantini (2002)
- 512 x 512 elements
- 3rd order methods
- Lorentz factor ~ 7
- Run with AMReX



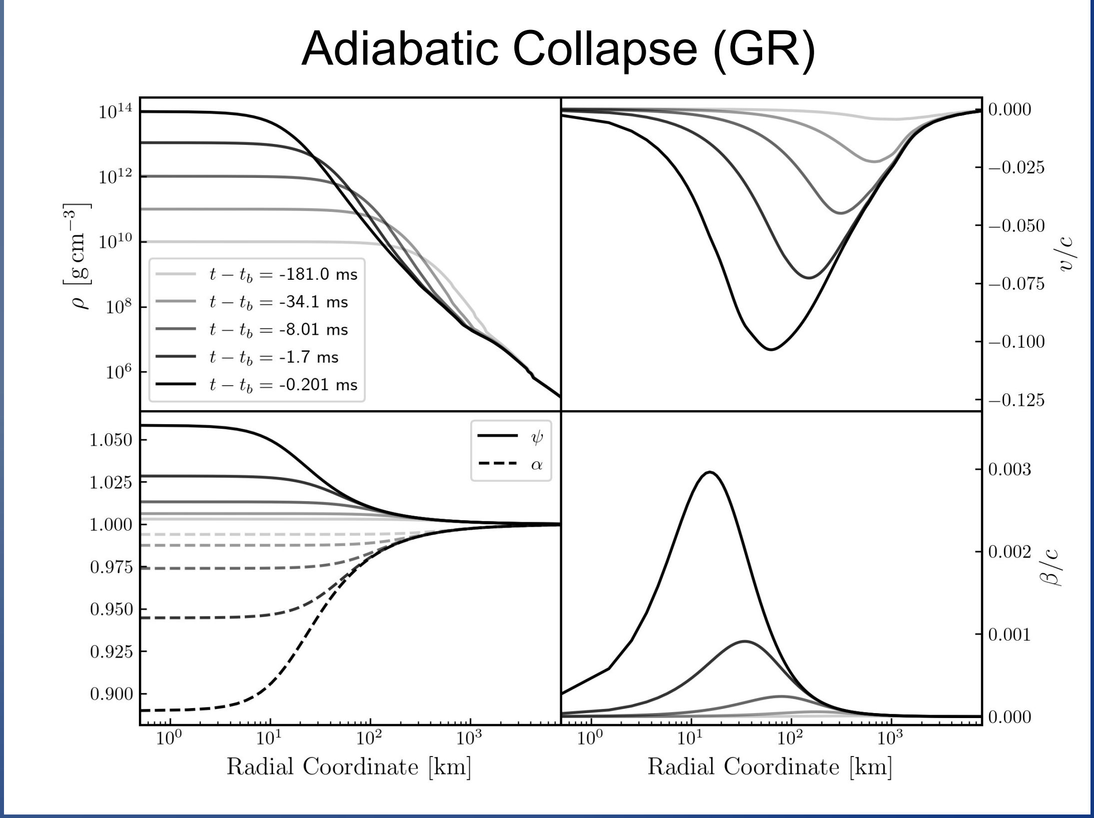
Results

- Polytropic self-similar collapse (Yahil, A. (1983))
  - $p = K\,\rho^{\Gamma}$
  - $\Gamma = 1.3$
- Simulates homologous collapse of stellar core
- Evolved from t–tb=-150 ms until central density exceeds 1e15 g/cm^3
- 3rd order methods
- Geometric grid
  - 256 elements
  - 1e5 km
  - 1 km inner cell width



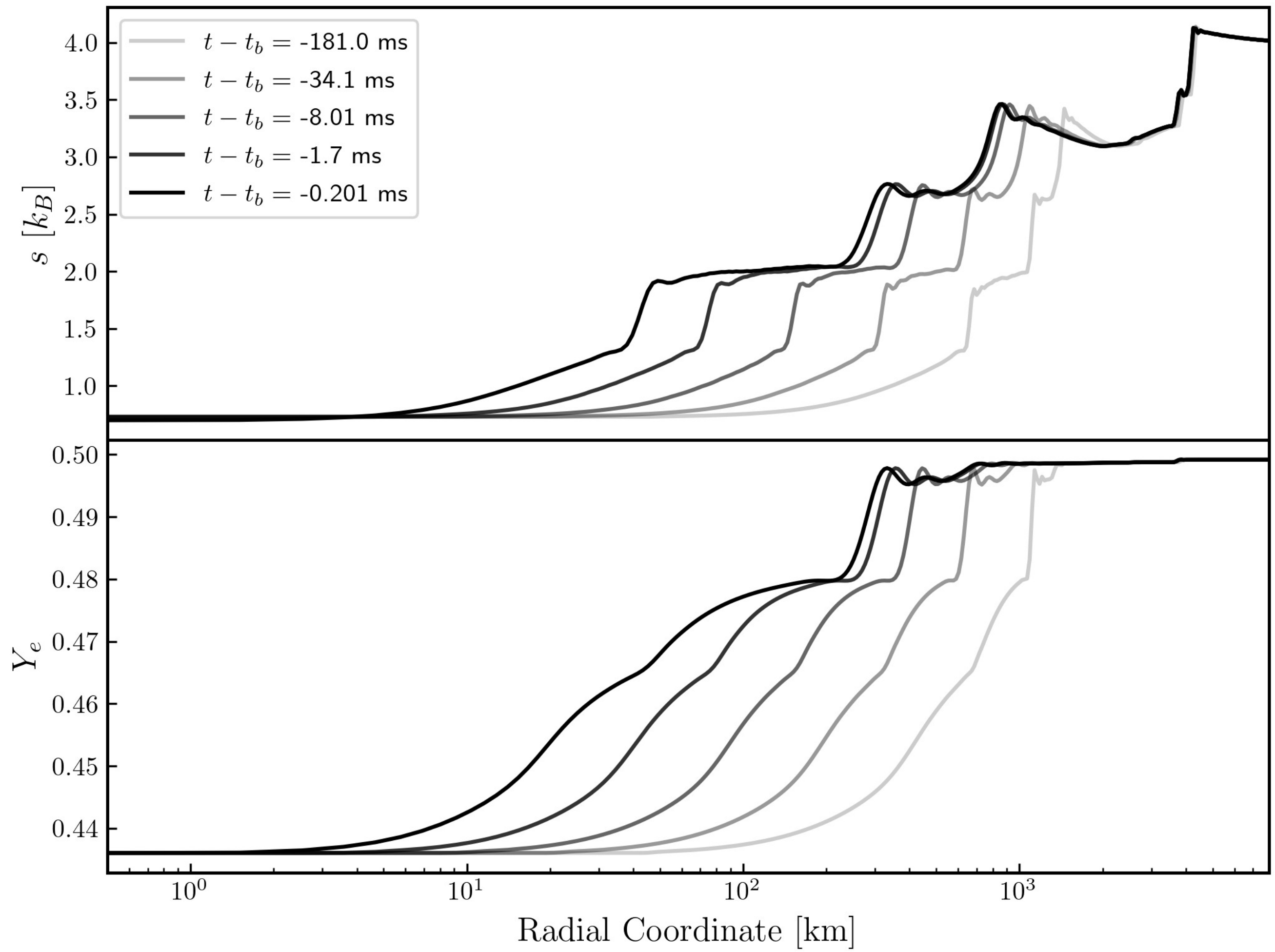
Results

- Adiabatic collapse of 15 Solar mass progenitor (Woosley & Heger (2007))
- SFHo EOS (Steiner et al., (2013a))
- 3rd order SSPRK method
- 2nd order DG method
- Geometric grid
  - 256 elements
  - 8e3 km
  - 1 km inner cell width



Compare to Newtonian results in Pochik et al., (2020)

Results



$t - t_b =$ -181.0 ms
$t - t_b =$ -34.1 ms
$t - t_b =$ -8.01 ms
$t - t_b =$ -1.7 ms
$t - t_b =$ -0.201 ms
s $[k_B]$
4.0
3.5
3.0
2.5
2.0
1.5
1.0
Y_e
0.50
0.49
0.48
0.47
0.46
0.45
0.44
10^0
10^1
10^2
10^3
Radial Coordinate [km]

Outline

Summary, etc.

Summary

- Developing thornado for CCSN simulations
    - Relativistic hydrodynamics
    - Relativistic gravity
    - Tabulated, nuclear EOS
- Evolved polytropic self-similar collapse
- Evolved adiabatic collapse of realistic 15 Solar mass progenitor with nuclear EOS

Ongoing/Future Work

- Evolve adiabatic collapse further
- Multi-D test simulations
- Incorporate neutrino transport solver
- Port/optimize hydro solver on GPUs
- Implements mesh refinement via AMReX
- CCSN simulations!

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