A Discontinuous Galerkin Method For General Relativistic Hydrodynamics

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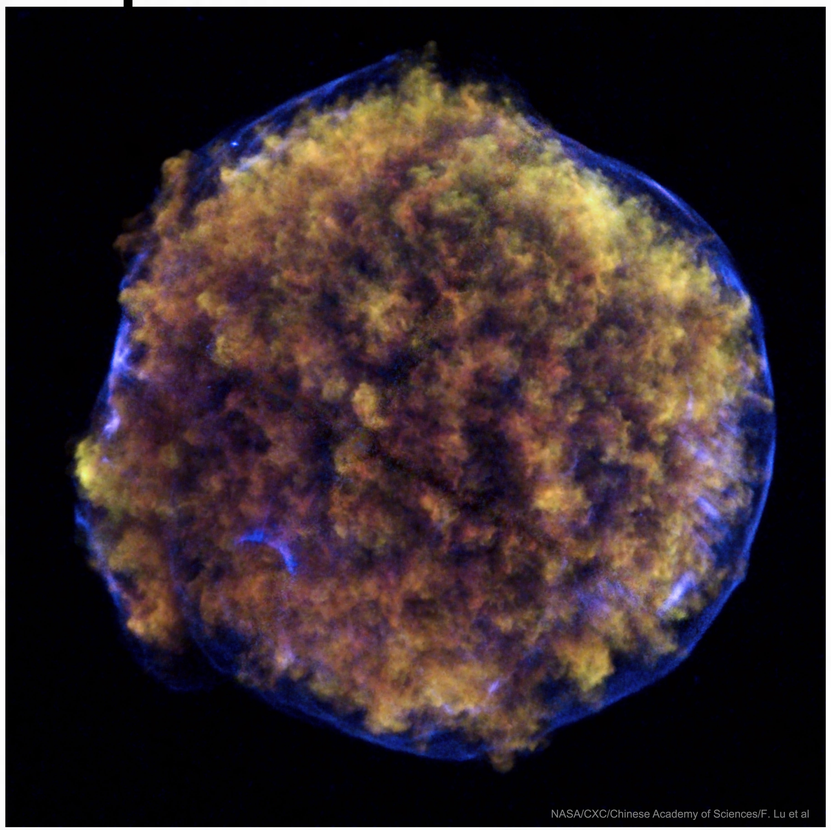
Core-collapse Supernovae

- Understand explosion mechanism
- Predict nucleosynthetic yields
- Gravitational waves
- Nuclear matter
- Neutrinos

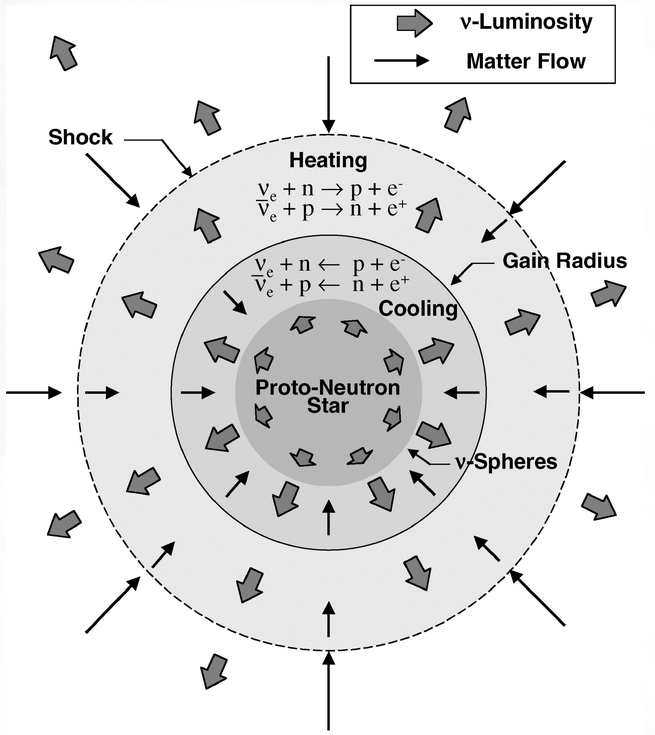


X-Ray remnant of Tycho's SN
(Figure courtesy of APOD)

What are we doing?

- Developing a new code to study core-collapse supernovae: the **t**oolkit for **h**igh-**or**der **n**eutrino-r**ad**iation hydr**o**dynamics (thornado)
- 3+1 (CFA)
  - **Hydrodynamics**
  - Neutrino transport
  - Gravity
- Runge-Kutta Discontinuous Galerkin

$$\partial_t \left(\sqrt{\gamma}\, \boldsymbol{U} \right) + \partial_i \left(\alpha \sqrt{\gamma}\, \boldsymbol{F}^i \left(\boldsymbol{U} \right) \right) = \sqrt{\gamma}\, \boldsymbol{S}$$



Mezzacappa A. 2005.
Annu. Rev. Nucl. Part. Sci. 55:467–515

3+1 GR-Hydro Equations

$$\frac{\partial}{\partial t}\begin{pmatrix}\sqrt{\gamma}\,D\\ \sqrt{\gamma}\,S_j\\ \sqrt{\gamma}\,\tau\end{pmatrix}+\frac{\partial}{\partial x^i}\begin{pmatrix}\sqrt{\gamma}\,\alpha\left(v^i-\alpha^{-1}\,\beta^i\right)D\\ \sqrt{\gamma}\,\alpha\left(P^i{}_j-\alpha^{-1}\,\beta^i\,S_j\right)\\ \sqrt{\gamma}\,\alpha\left(S^i-D\,v^i-\alpha^{-1}\,\beta^i\,\tau\right)\end{pmatrix}$$
$$=\sqrt{\gamma}\begin{pmatrix}0\\ \frac{1}{2}\,\alpha\,P^{ik}\,\partial_j\gamma_{ik}+S_i\,\partial_j\,\beta^i-(\tau+D)\,\partial_j\,\alpha\\ \alpha\,P^{ij}\,K_{ij}-S^j\,\partial_j\,\alpha\end{pmatrix}$$

Geometry

γ : Determinant of spatial three-metric

α : Lapse function

$\boldsymbol{\beta}$: Shift vector

$\boldsymbol{K}$: Extrinsic curvature tensor

Evolved Quantities

D : Conserved rest-mass density

$\boldsymbol{S}$: Conserved momentum-density

τ : Conserved energy-density

Auxiliary Quantities

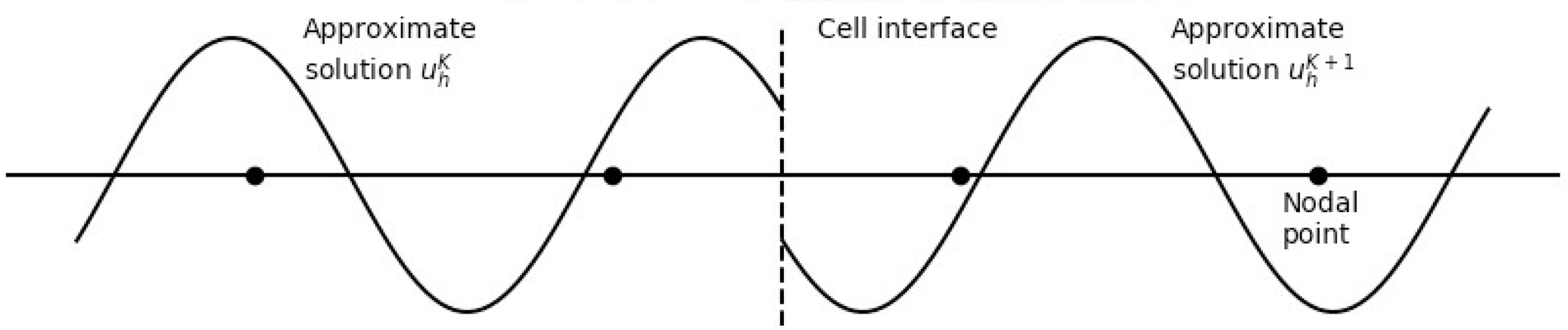
$\boldsymbol{P}$: Pressure tensor

$\overrightarrow{v}$: Fluid three-velocity

Discontinuous Galerkin Method

Approximate solution as a series of polynomials, discontinuous at cell-boundaries

- Locally high-order accurate
- Slope limiters
- Bound-preserving limiters

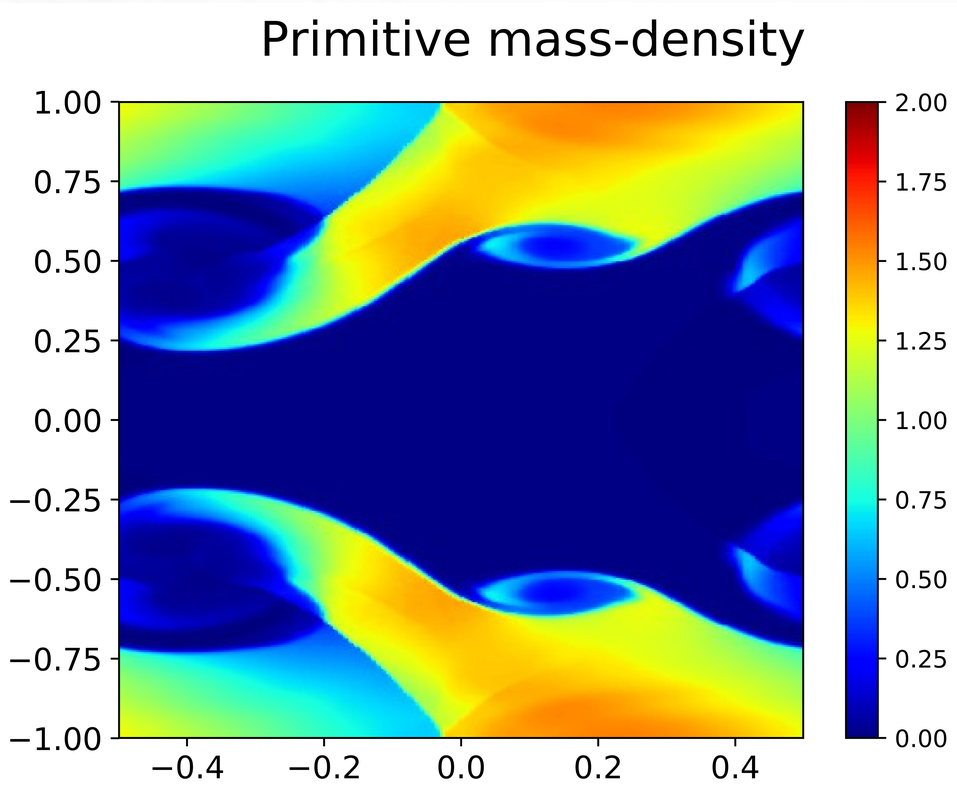


Results

Relativistic Kelvin-Helmholtz Instability

- Initial conditions from: Beckwith & Stone (2011), ApJS, 193:6
  - Instability seeded in vertical component of velocity
- Periodic boundary conditions in both dimensions

nX = 128, nY = 256
nNodes = 3
t = 3.0
SSP-RK3



Special Relativistic 2D Riemann Problem

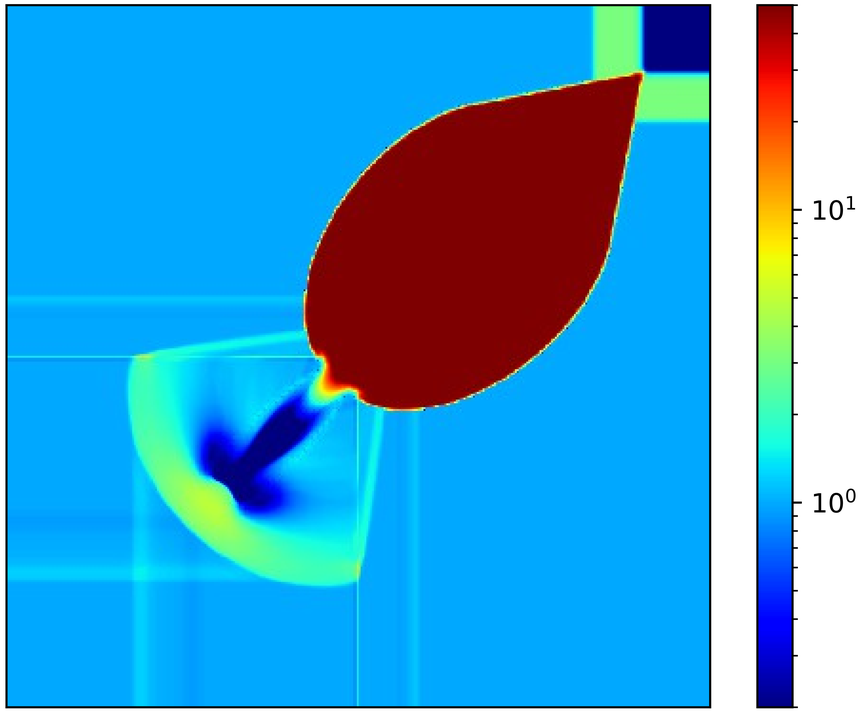
- Strong shocks in multiple dimensions

$$
\begin{aligned}
(\rho, v_x, v_y, p)^{\mathrm{NE}} &= (0.1, 0, 0, 0.01), \\
(\rho, v_x, v_y, p)^{\mathrm{NW}} &= (0.1, 0.99, 0, 1), \\
(\rho, v_x, v_y, p)^{\mathrm{SW}} &= (0.5, 0, 0, 1), \\
(\rho, v_x, v_y, p)^{\mathrm{SE}} &= (0.1, 0, 0.99, 1).
\end{aligned}
$$

- Del Zanna & Bucciantini (2002), A&A, 390:1177
- Free expansion

nX = nY = 256
nNodes = 3
t = 0.4
SSP-RK3

ℓog_{10}(Pressure)

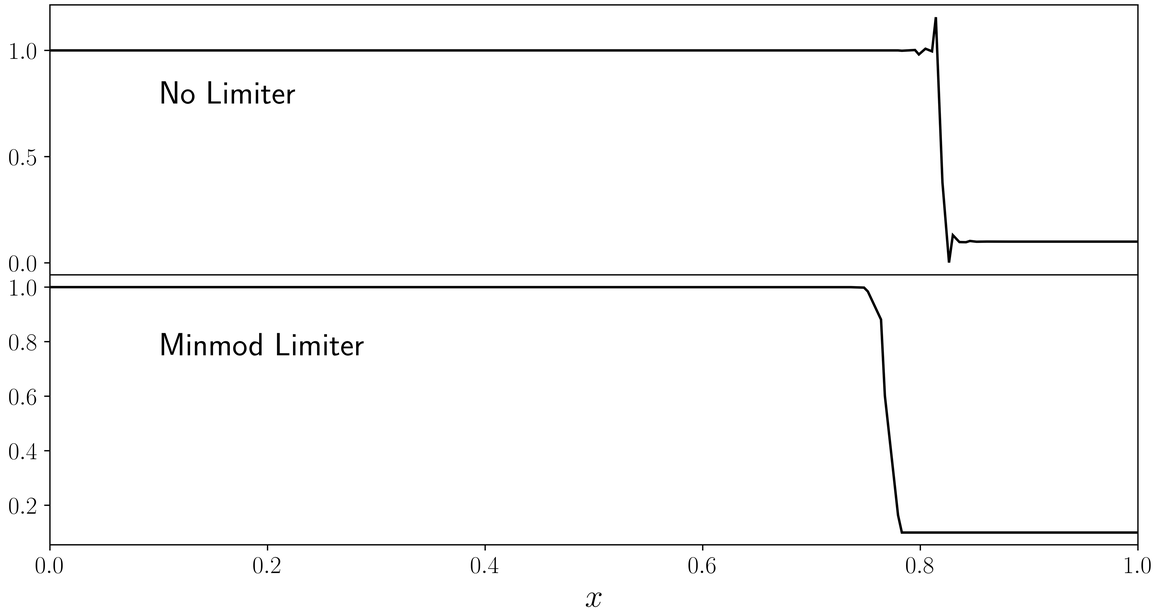


Limiters

Near shocks, polynomial representation can lead to oscillatory behavior in solution

To mitigate oscillations, degree of polynomial is truncated, slope is modified

We also use a shock detector to avoid limiting in smooth regions of solution



Additionally, we use a bound-preserving limiter to avoid non-physical situations

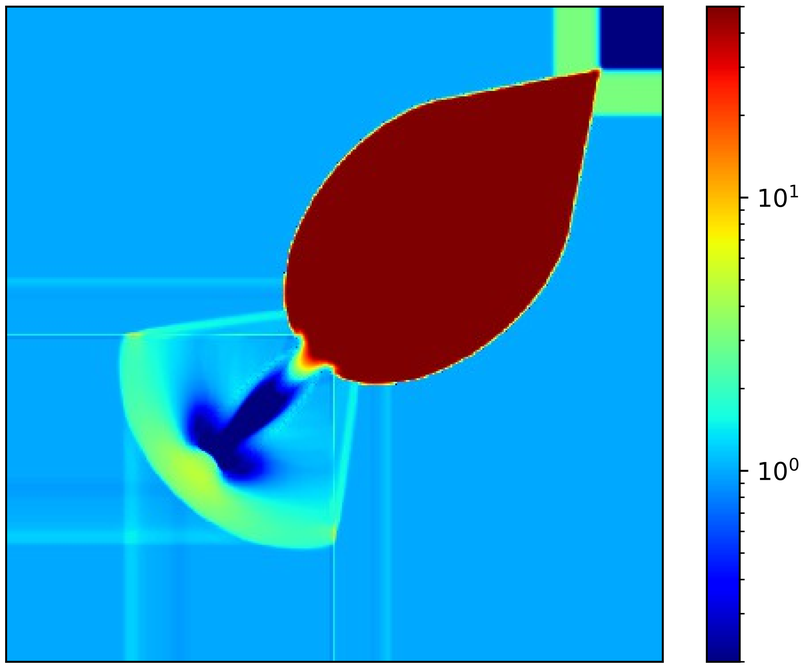
Component-Wise vs. Characteristic Limiting

- Could limit component-wise, but components are coupled
- Better is to limit characteristic variables
  - Construct local characteristic decomposition with cell-averages
  - Compute slopes in characteristic variables
  - Limit with MinModB limiter
  - Recover components from limited characteristic variables

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}} \longrightarrow \boldsymbol{R}\,\boldsymbol{\Lambda}\,\boldsymbol{R}^{-1}$$

Characteristic Limiting

ℓog_{10}(Pressure)



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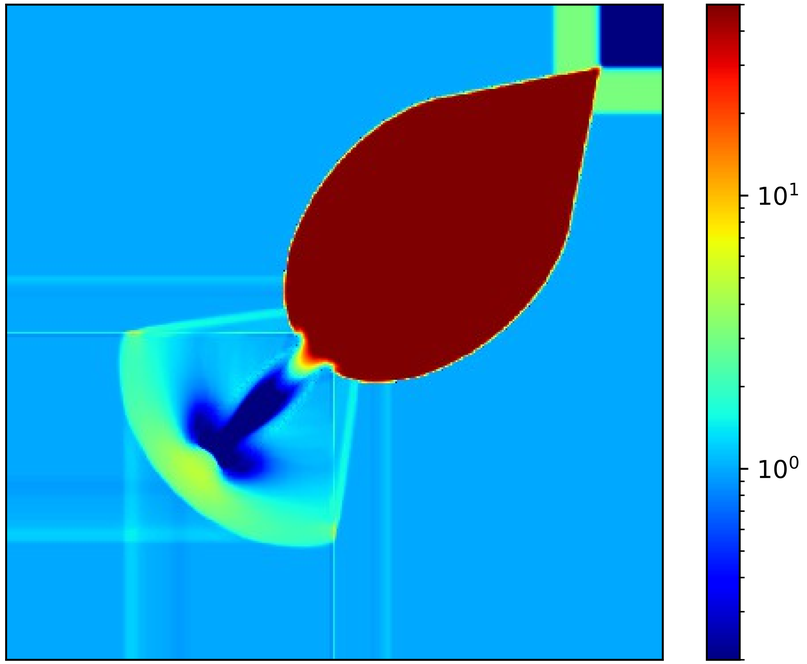
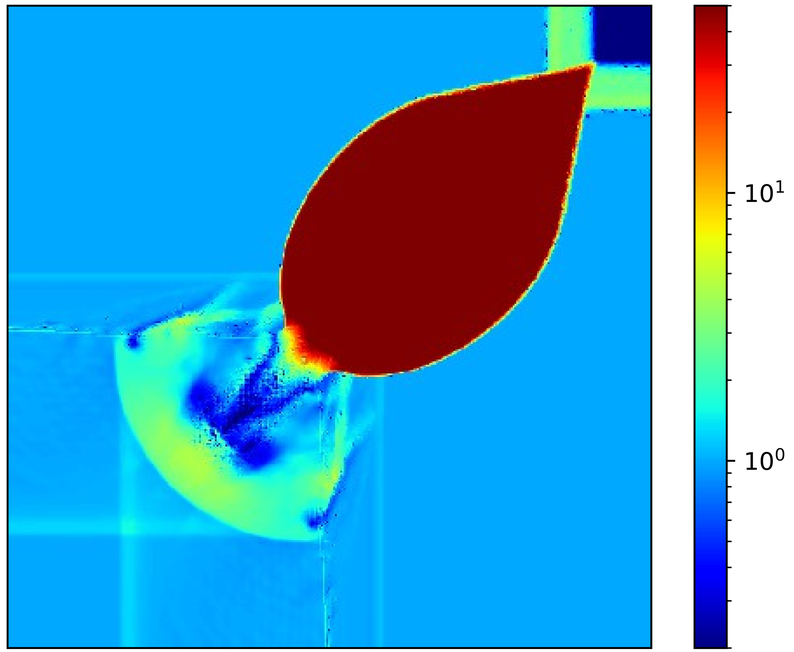
Characteristic Limiting

ℓog_{10}(Pressure)



Component-Wise Limiting

ℓog_{10}(Pressure)



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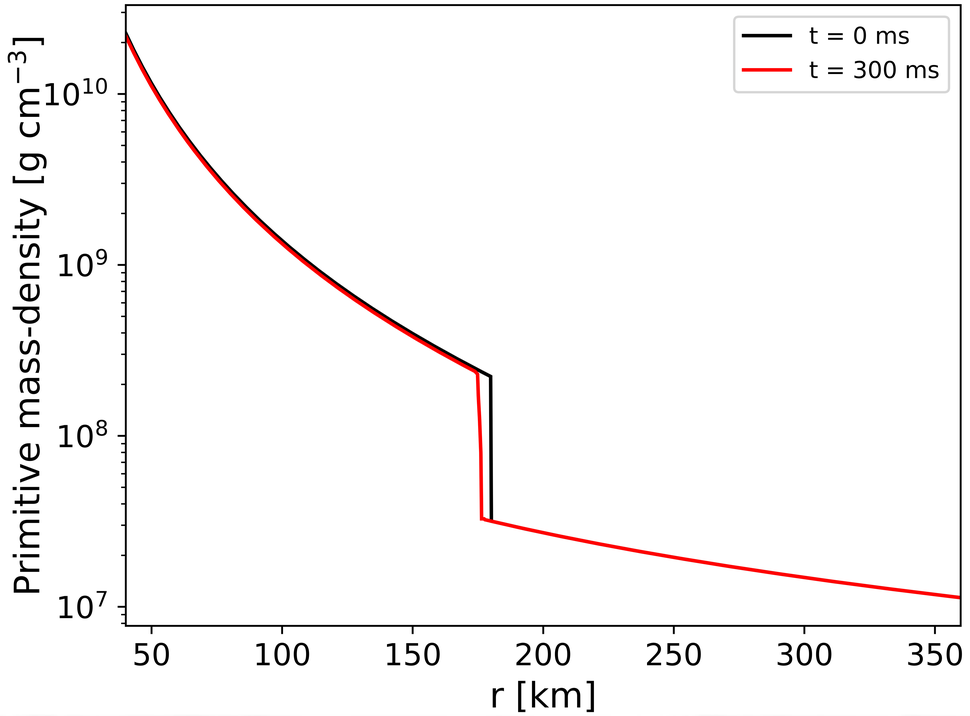
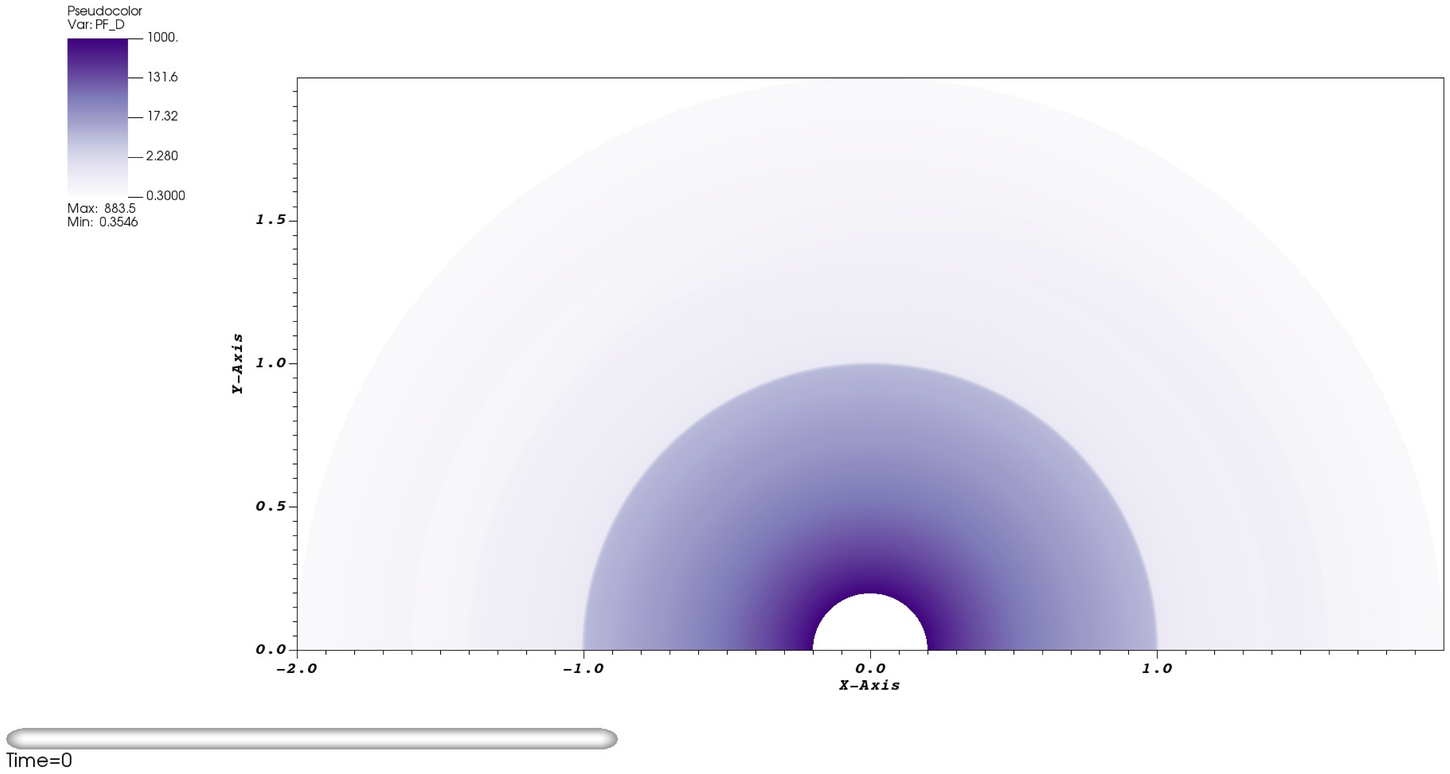
Relativistic Standing Accretion Shock Instability

- Curvilinear coordinates
- Initial conditions from solving relativistic Bernoulli equation
- Extrapolate solution for inner boundary, constant inflow for outer
- Point-mass potential, stationary spacetime



2D Standing Accretion Shock Instability (Preliminary)



AMReX[1]

nR = 256
nθ = 128
nNodes = 3
t = 80
SSP-RK3
64 MPI ranks
8 Compute nodes

[1]https://github.com/AMReX-Codes/amrex

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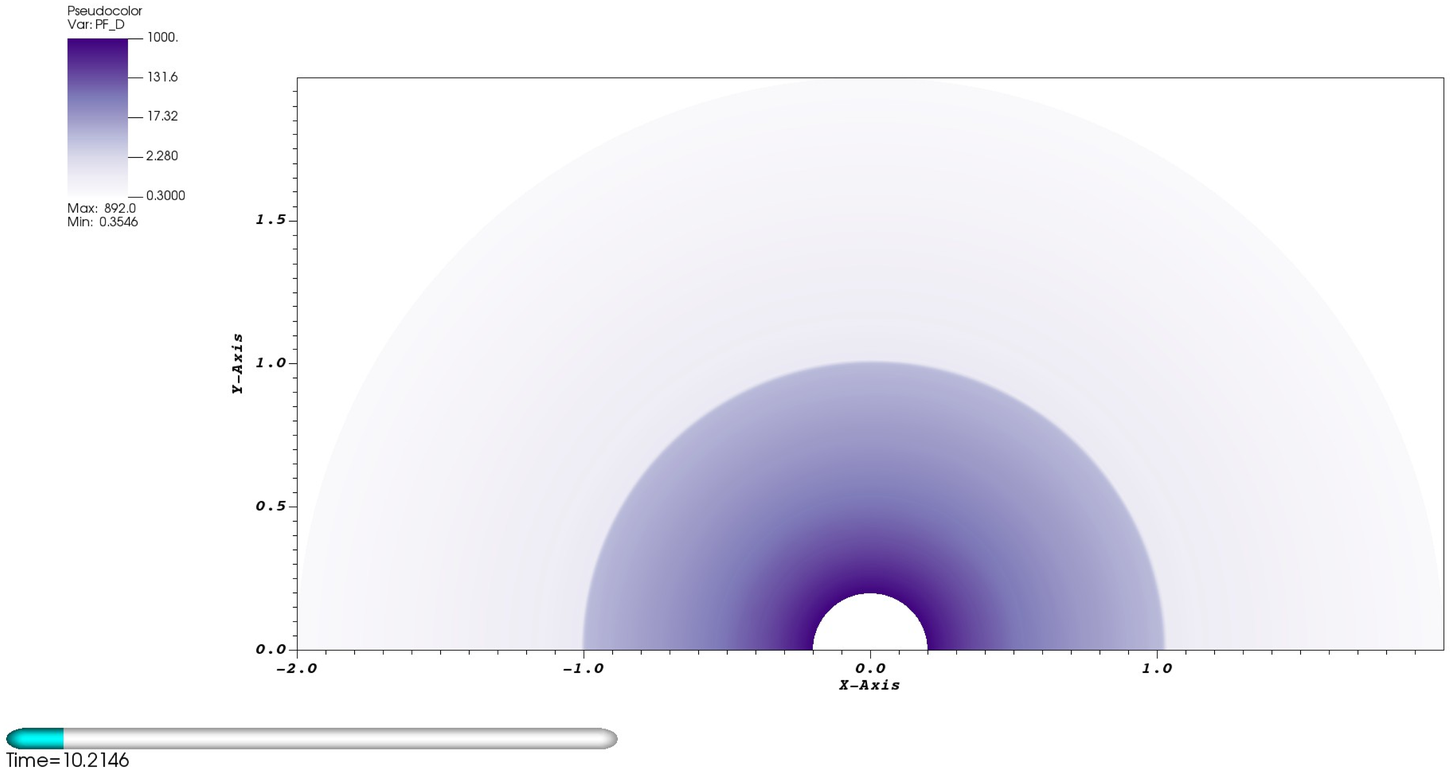

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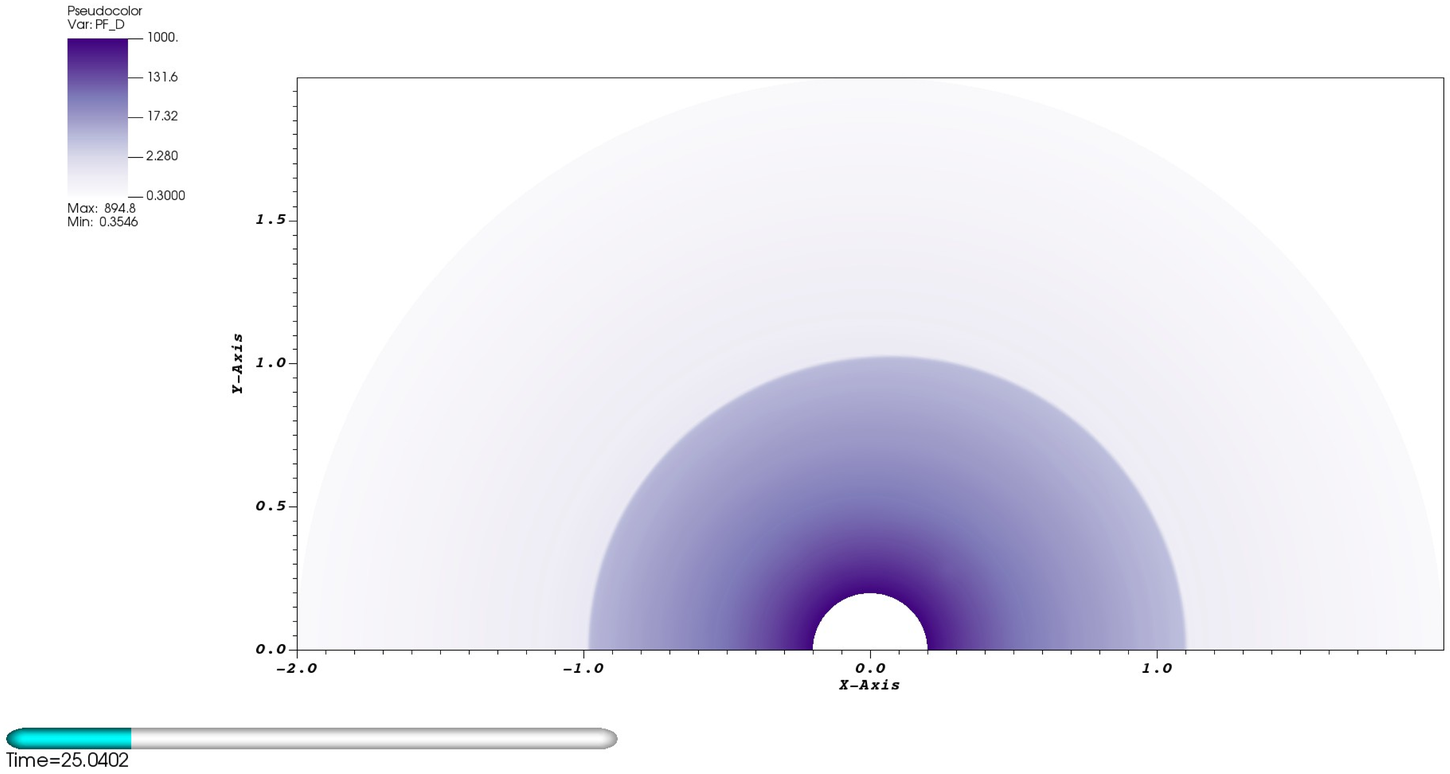

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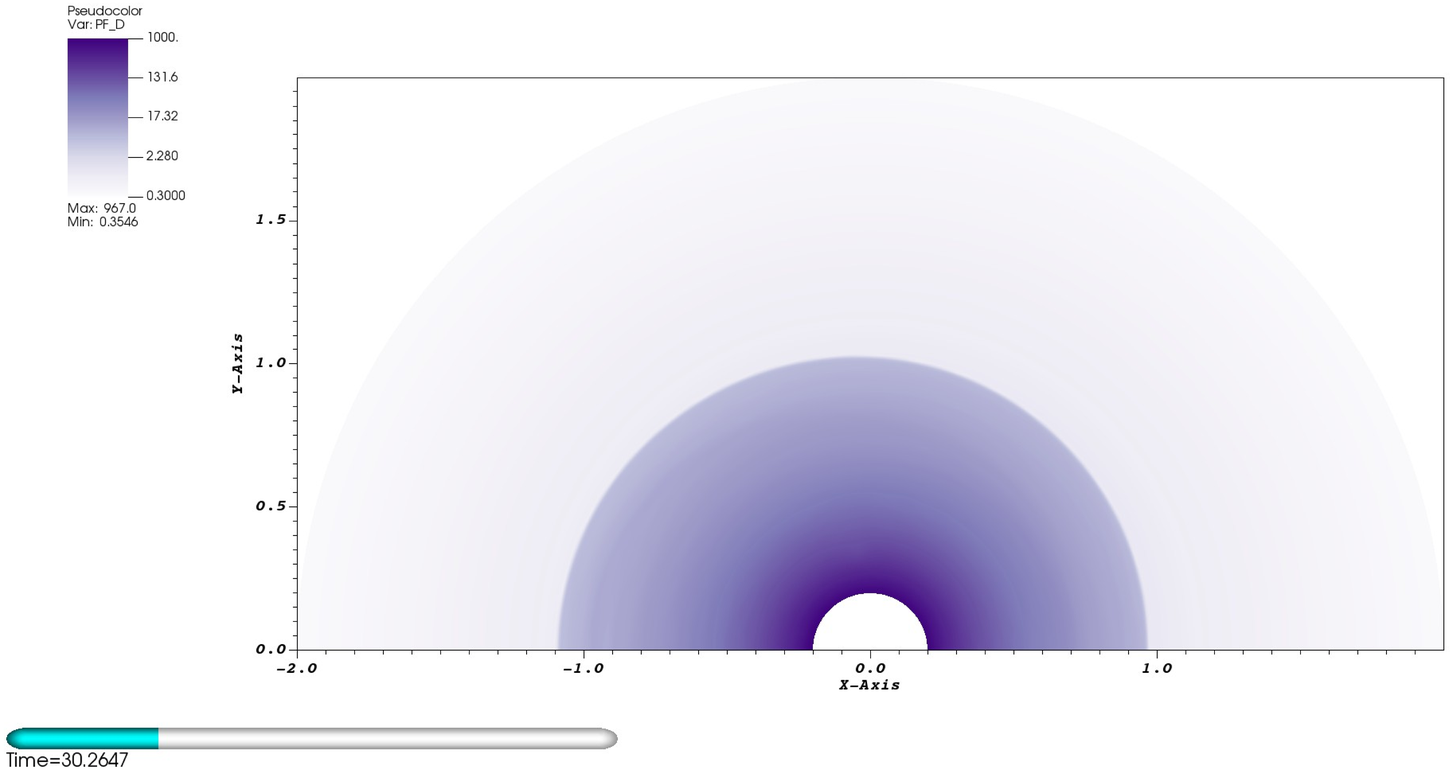

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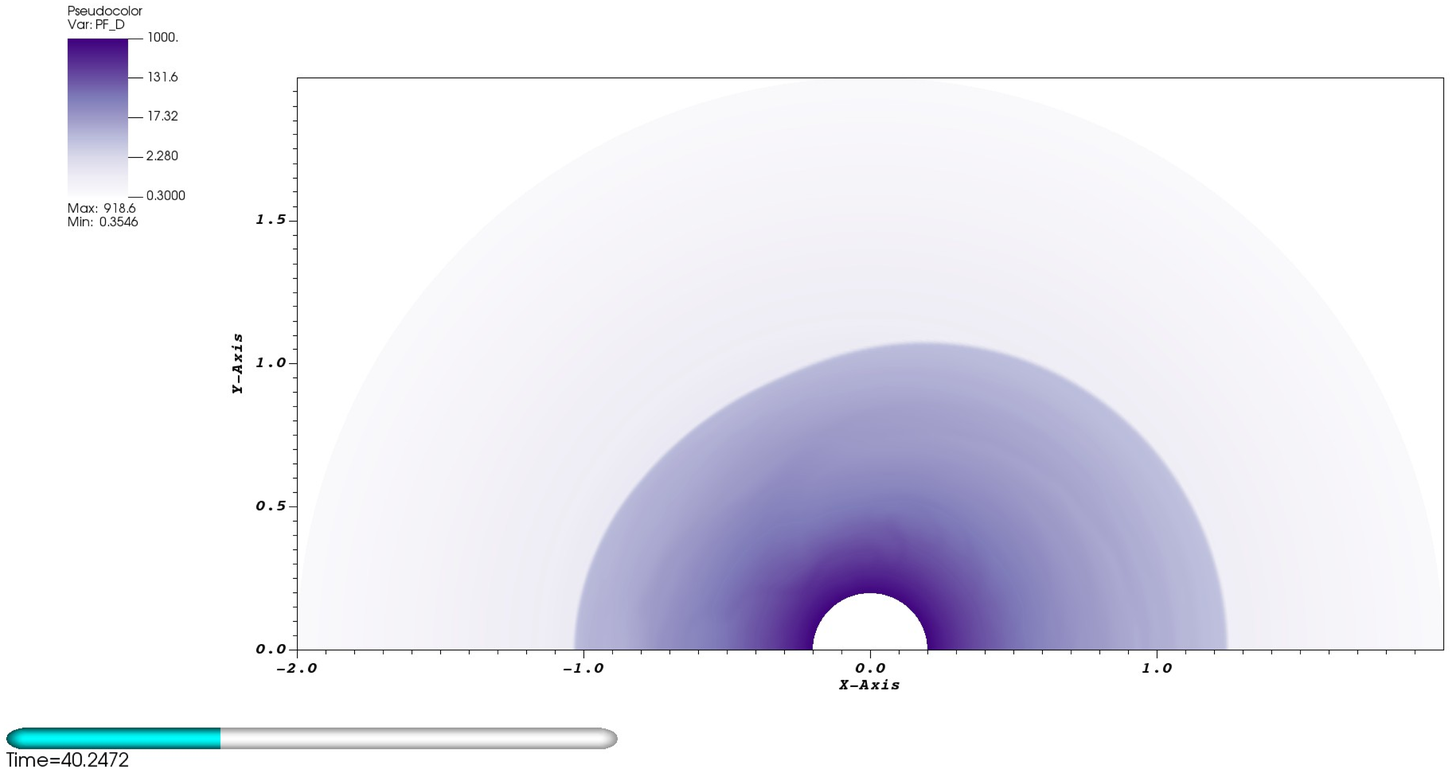

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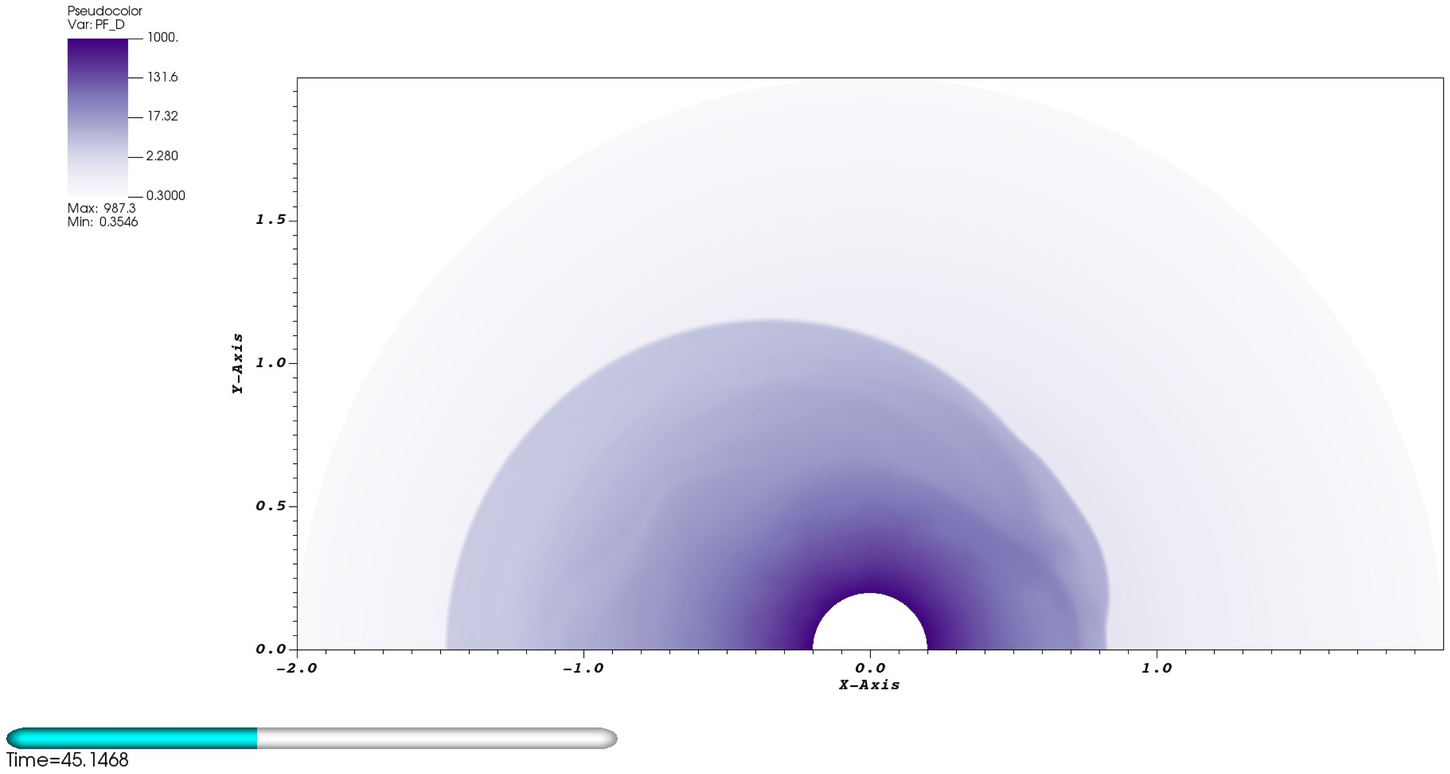

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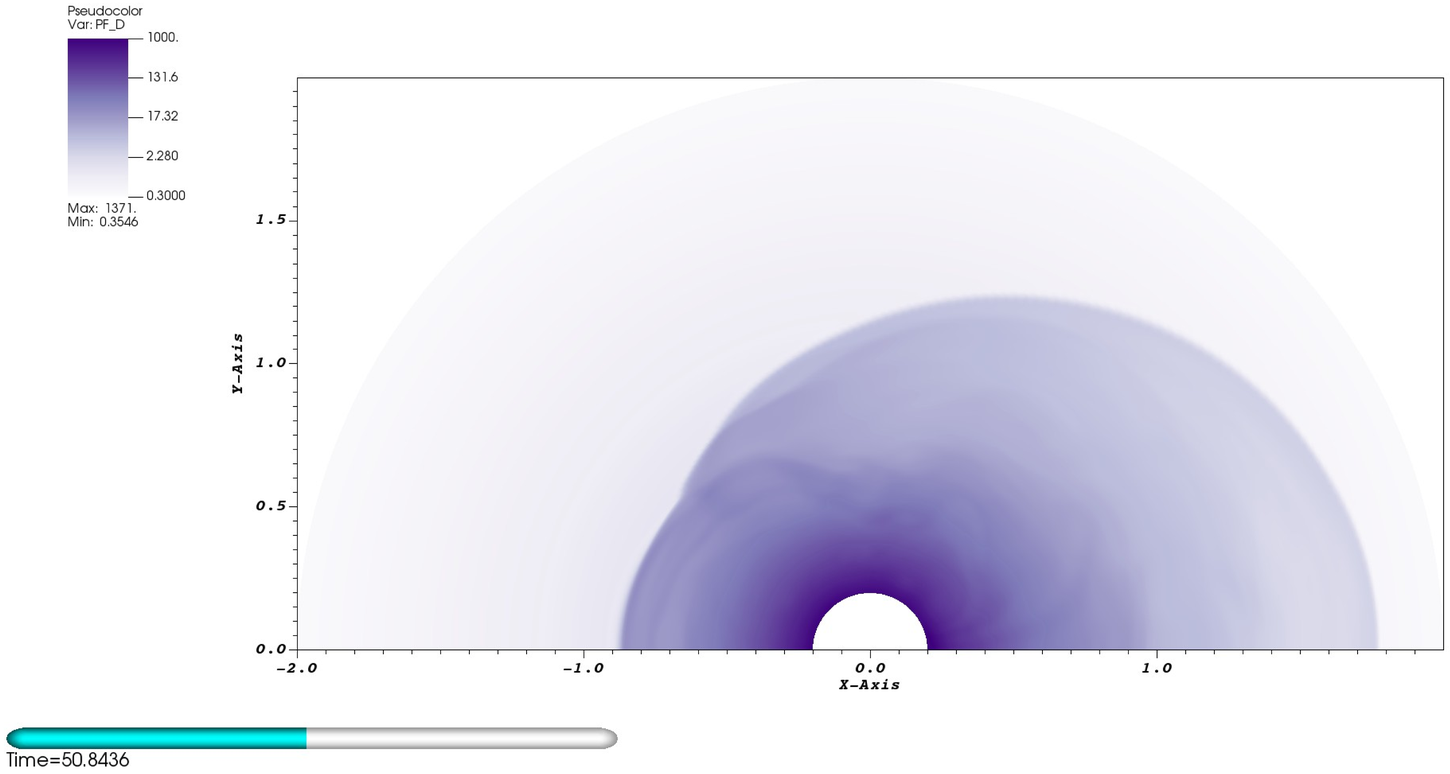

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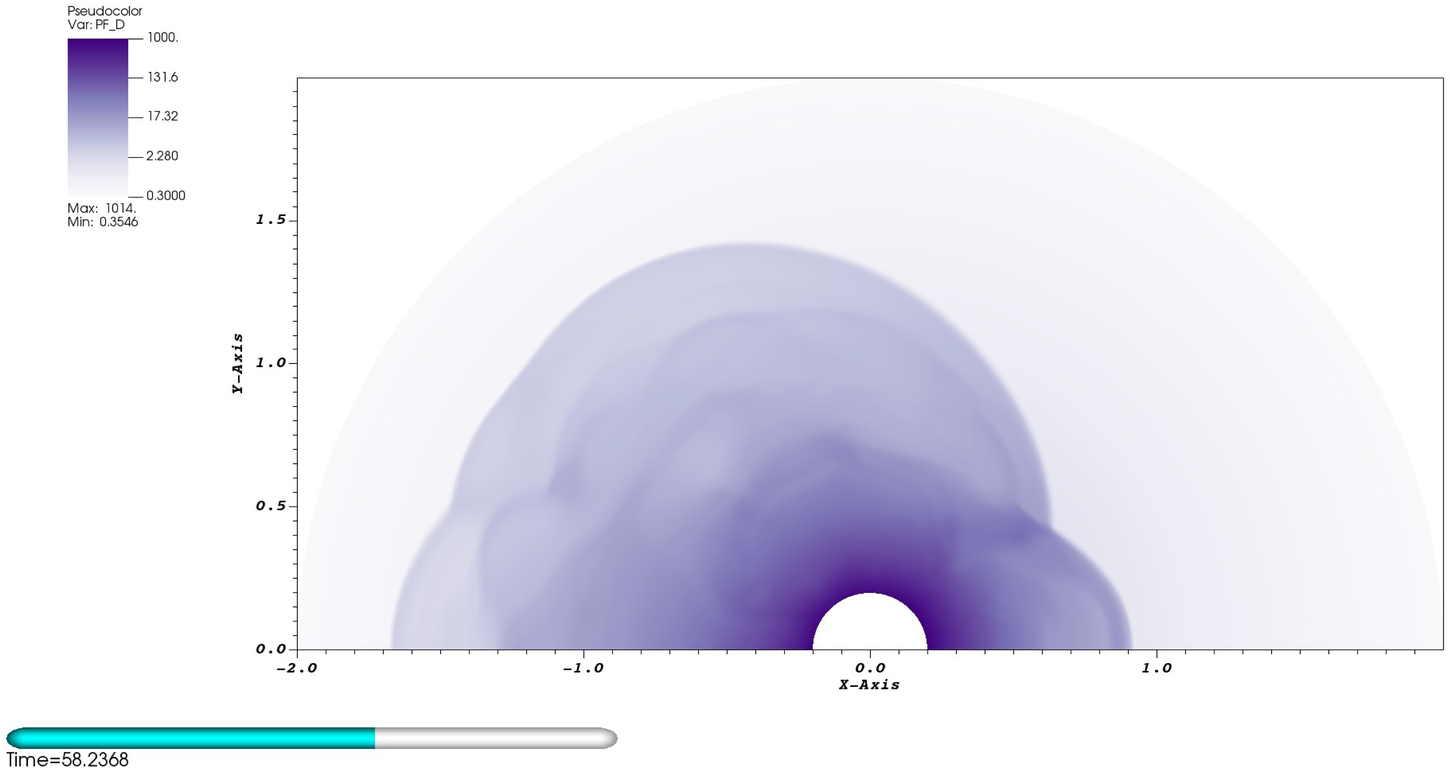

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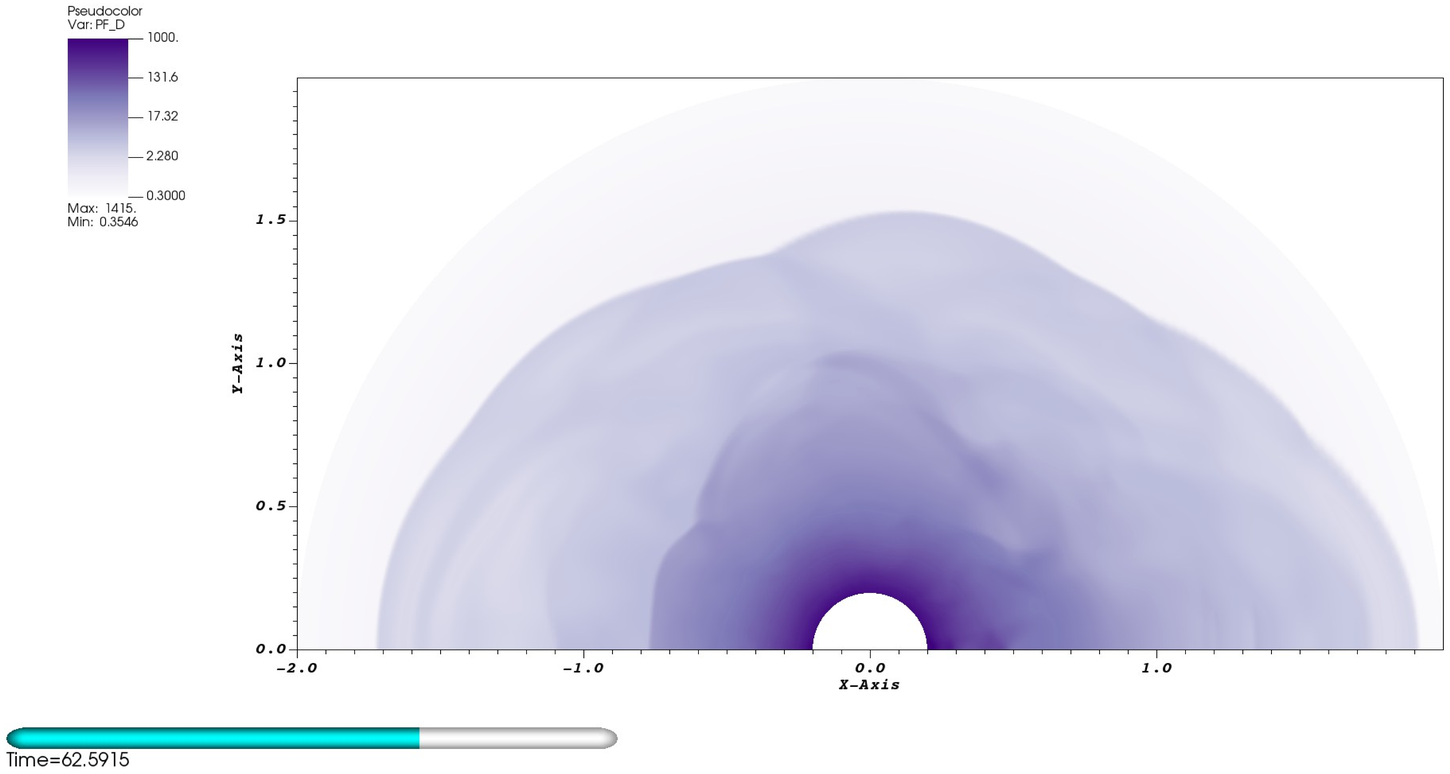

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Acknowledgements

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I would also like to acknowledge Vanderbilt University for helping to fund this research

Conclusion

- Implemented DG method to solve GR-hydro equations
- Code has been validated with several challenging test-problems
  - Compares well with other published results
- Characteristic limiting is necessary for higher order methods
- Future Work
  - Couple GR-hydro with adaptive mesh refinement (AMReX)
    - Run multi-D GR SAS problem
      - Investigate effect of SASI turbulence
  - Couple with CFA gravity solver (Nick Roberts)
  - Couple with neutrino transport solver (Zach Elledge, Ran Chu)

Characteristic Decomposition (Proof of concept: Linear case)

$$
\begin{aligned}
&\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = \mathbf{0} \\
=&\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}} \frac{\partial \boldsymbol{U}}{\partial x} \\
=&\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{R}\,\boldsymbol{\Lambda}\,\boldsymbol{R}^{-1} \frac{\partial \boldsymbol{U}}{\partial x} \\
=&\frac{\partial \left(\boldsymbol{R}^{-1}\,\boldsymbol{U}\right)}{\partial t} + \boldsymbol{\Lambda} \frac{\partial \left(\boldsymbol{R}^{-1}\,\boldsymbol{U}\right)}{\partial x} \\
=&\frac{\partial \boldsymbol{W}}{\partial t} + \boldsymbol{\Lambda} \frac{\partial \boldsymbol{W}}{\partial x} = \mathbf{0}, \\
&\boldsymbol{\Lambda} \equiv \mathrm{diag}\left[\lambda_1, \lambda_2, \cdots, \lambda_n\right]
\end{aligned}
$$