`thornado`-Hydro (xCFC)

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thornado

toolkit for high-order neutrino-radiation hydrodynamics

https://github.com/endeve/thornado

- DG
- SSPRK/IMEX[a]
- GR (xCFC)
- Hydro[b] (Valencia)
- Neutrino transport[c] (M1)
- Interfaces to tabulated EoS/Opacities (weaklib: `https://github.com/starkiller-astro/weaklib`)

- Fluid self-gravity via Poseidon: `https://github.com/jrober50/Poseidon`
- GPUs via OpenACC or OpenMP pragmas
- MPI parallelism and AMR via AMReX: `https://github.com/AMReX-Codes/amrex`

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[a] Chu et al. (2019)

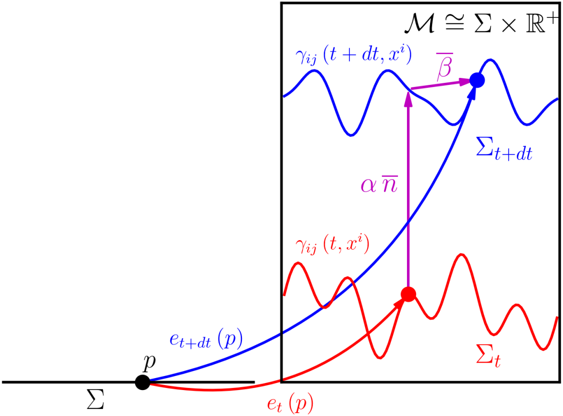
[b] Dunham et al. (2020); Endeve et al. (2019); Pochik et al. (2021)

[c] Laiu et al. (2021); Laiu et al. (2025)

Motivation

- Capturing diffusion limit for neutrino transport without ad-hoc methods
- High-order accuracy on a compact stencil
- An alternative simulation code than the standard FV/FD methods
- An experiment in applied math: are DG methods effective for CCSN simulations?

3+1 Decomposition



$$ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu = -\alpha^2\,dt^2 + \gamma_{ij}\left(dx^i + \beta^i\,dt\right)\left(dx^j + \beta^j\,dt\right)$$

Conformally-Flat Condition

Developed by Wilson et al. (1996), extended by Cordero-Carrión et al. (2009)

$$\gamma_{ij}\left(x\right) = \psi^{4}\left(x\right)\,\overline{\gamma}_{ij}\left(x^{i}\right)$$
$$K = 0,\ \partial_{t}\,K = 0$$
$$\text{(Always and everywhere)}$$

- Exact in spherical symmetry!
- Hyperbolic $\rightarrow$ Elliptic equations
- Good for long-time simulations

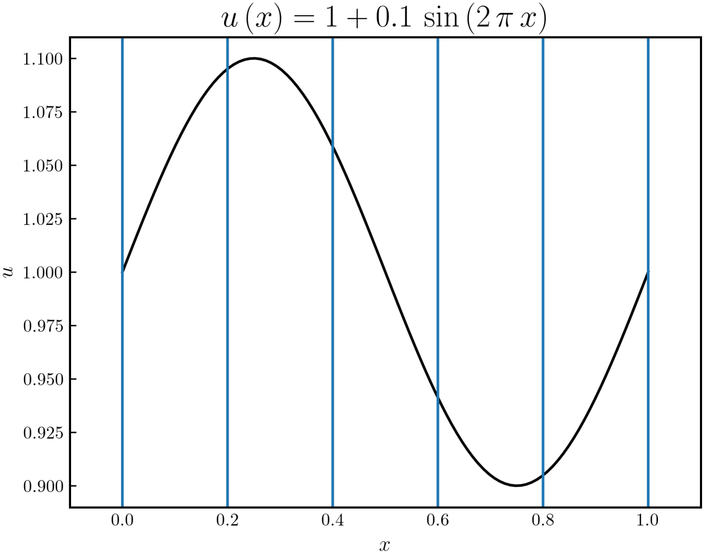
Special case: Schwarzchild spacetime in isotropic coordinates ($G = c = 1$)

$$\alpha = \left(1 + \frac{1}{2}\,\Phi\right)\left(1 - \frac{1}{2}\,\Phi\right)^{-1}$$
$$\psi = 1 - \frac{1}{2}\,\Phi$$
$$\beta^{i} = 0,$$

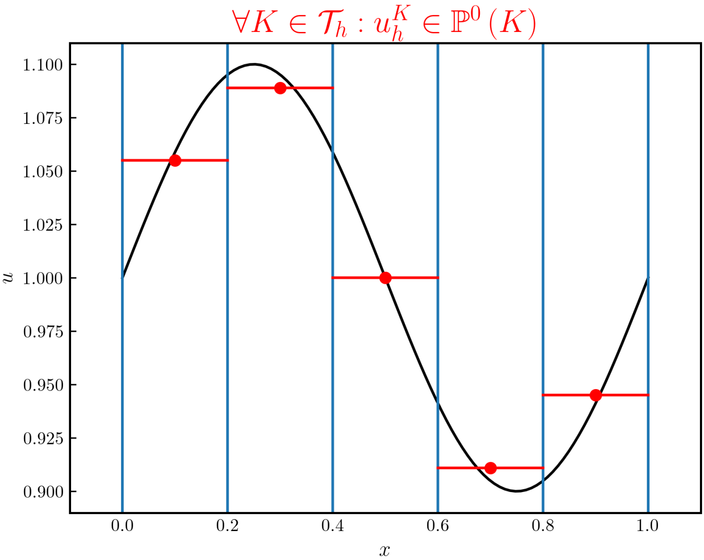
with

$$\Phi\left(r\right) := -\frac{M}{r}$$

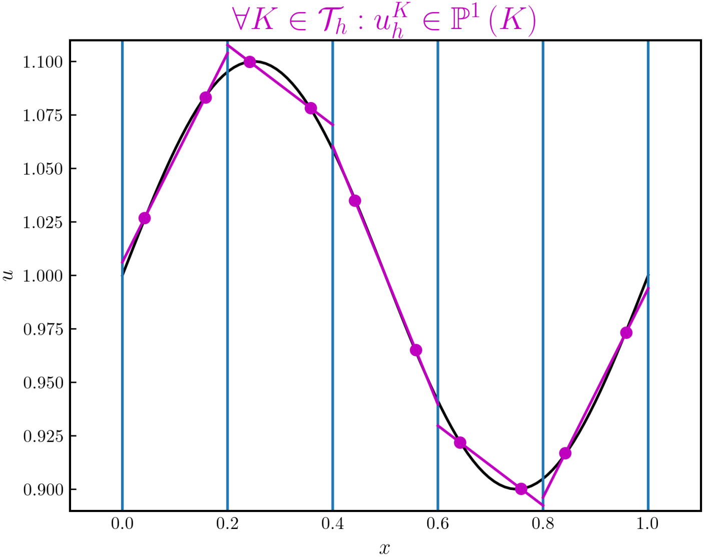
Discontinuous Galerkin (DG) Method



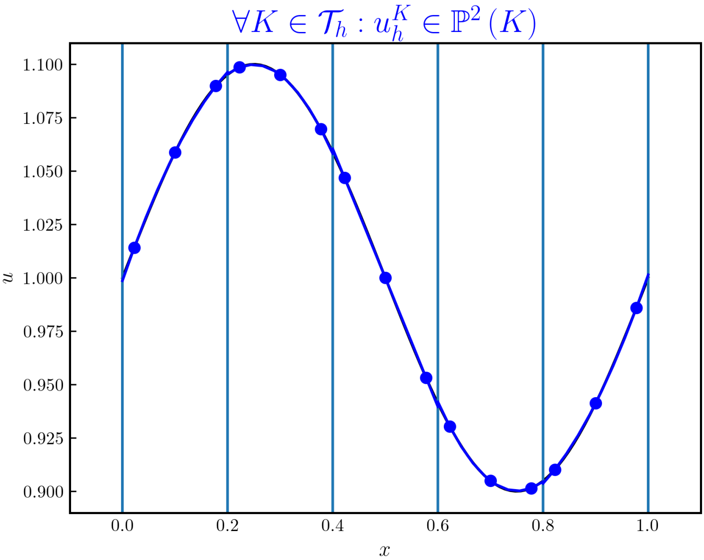
Discontinuous Galerkin (DG) Method



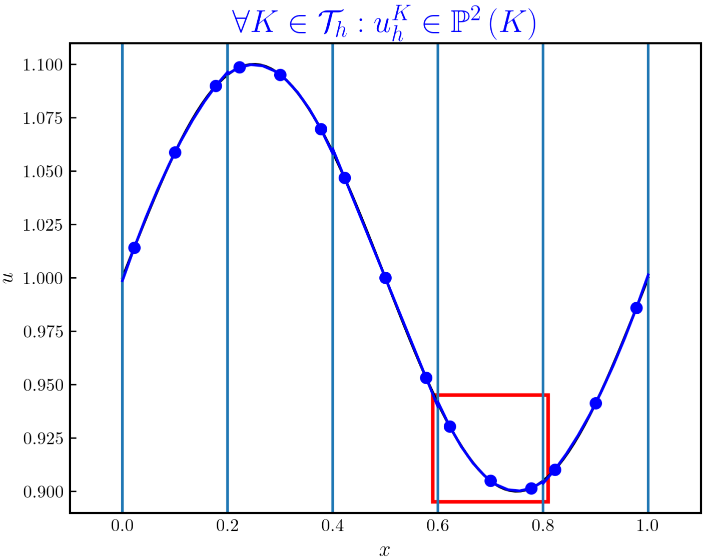
Discontinuous Galerkin (DG) Method



Discontinuous Galerkin (DG) Method

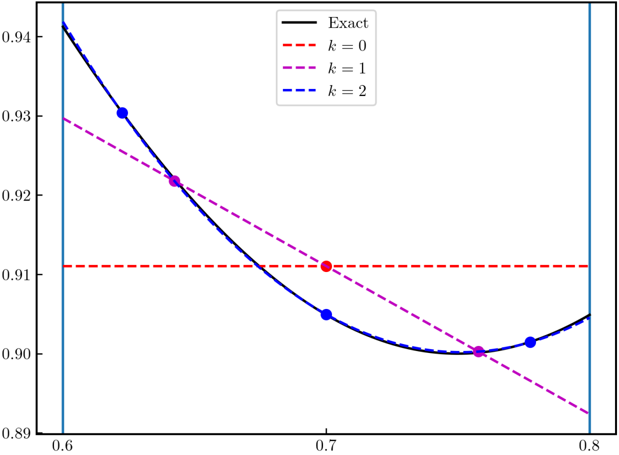


Discontinuous Galerkin (DG) Method



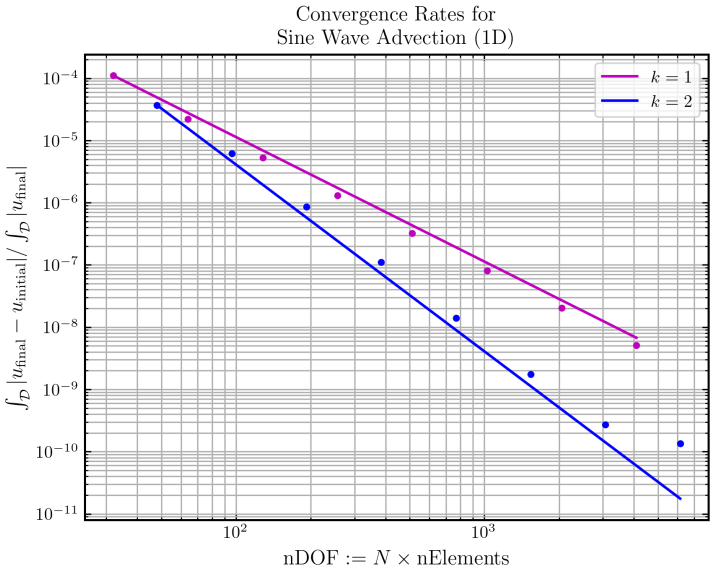
Discontinuous Galerkin (DG) Method

$$\mathbb{P}^k \ni u_h\left(x,t\right) := \sum_{i=1}^{k+1} u_i\left(t\right)\, \ell_i\left(x\right)$$



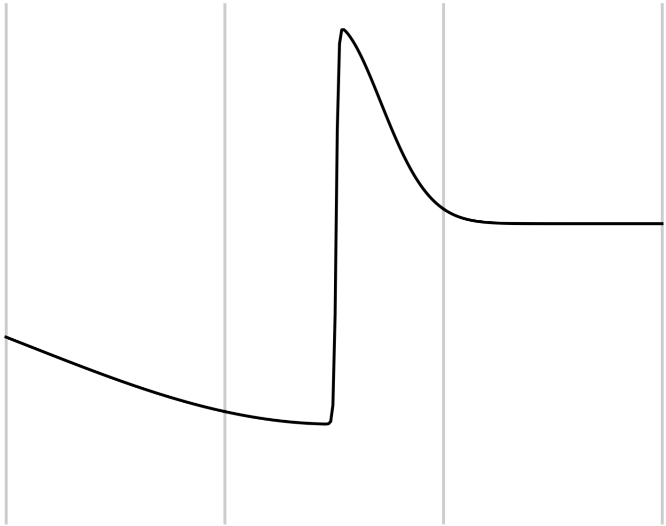
Time Stepping

- For hydro-only, strong-stabililty-preserving Runge–Kutta (SSPRK) methods
  - Convex combinations of forward-Euler timesteps
- For hydro+neutrinos, realizability-preserving ImEx methods (Chu et al. (2019))



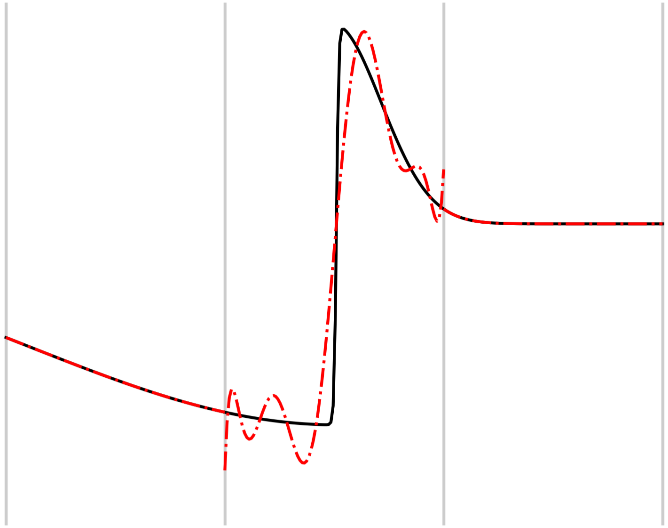
Convergence Rates for
Sine Wave Advection (1D)
$k = 1$
$k = 2$
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
10^{2}
10^{3}
$\int_{\mathcal{D}} |u_{\text{final}} - u_{\text{initial}}| / \int_{\mathcal{D}} |u_{\text{final}}|$
nDOF := $N \times$ nElements

Slope Limiter (MinMod)

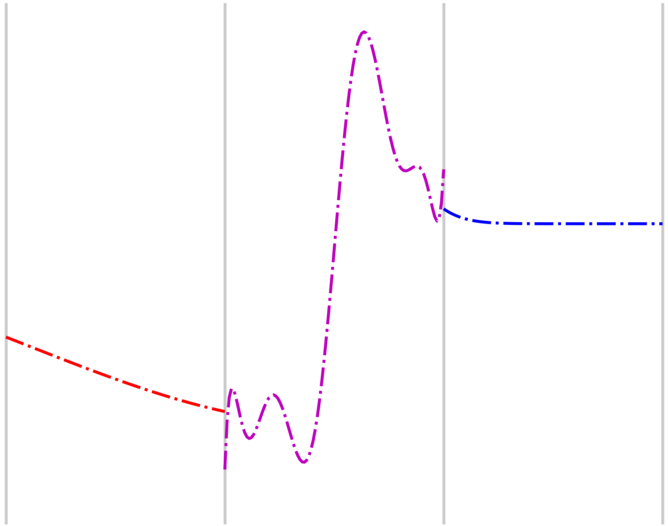


Slope Limiter (MinMod)

$k = 9$

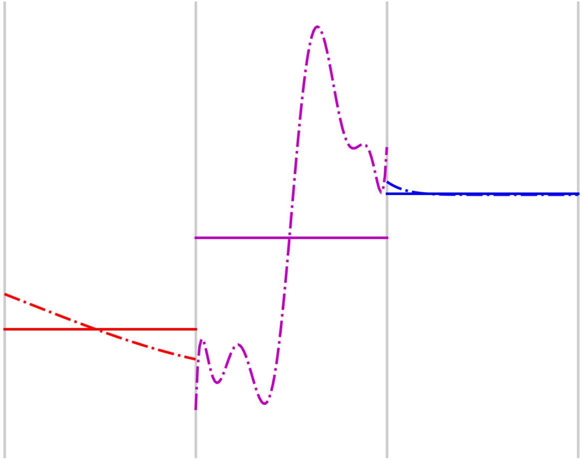


Slope Limiter (MinMod)



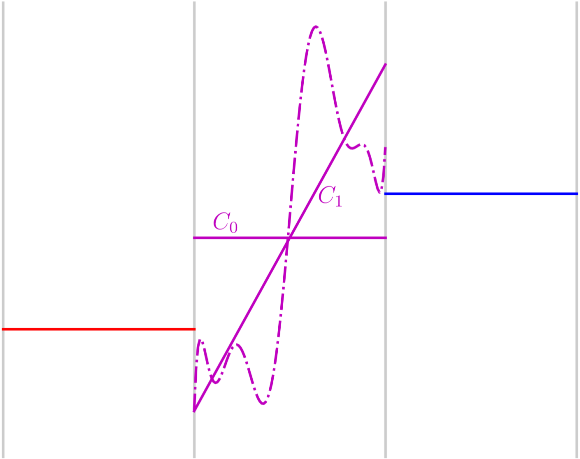
Slope Limiter (MinMod)

$$\bar{u}\left(t\right) := \frac{1}{V_h\left(t\right)} \int_K u_h\left(x,t\right)\, \sqrt{\gamma_h}\left(x,t\right)\, dx,\;\; V_h\left(t\right) := \int_K \sqrt{\gamma_h}\left(x,t\right)\, dx$$



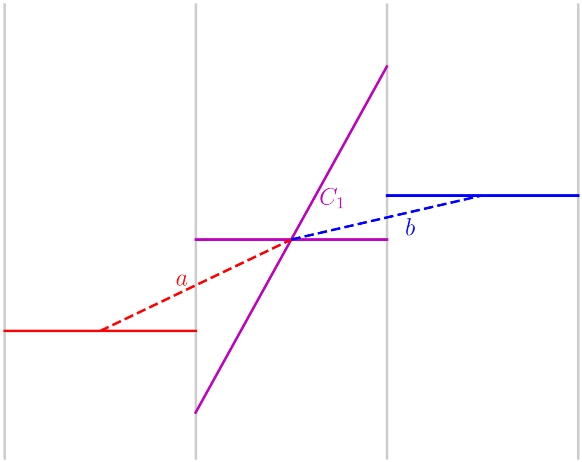
Slope Limiter (MinMod)

$$u_h\left(x,t\right)=\sum_{i=1}^{k+1}u_i\left(t\right)\,\ell_i\left(x\right)=\sum_{n=0}^{k}C_n\left(t\right)\,P_n\left(x\right)$$



Slope Limiter (MinMod)

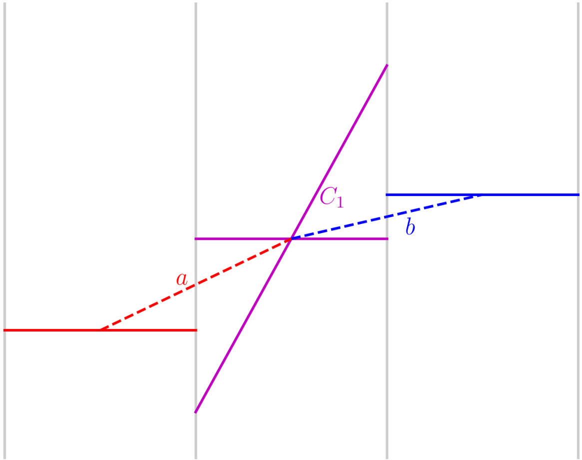
$$a := \bar{u}_K - \bar{u}_{K-1} \qquad b := \bar{u}_{K+1} - \bar{u}_K$$



Slope Limiter (MinMod)

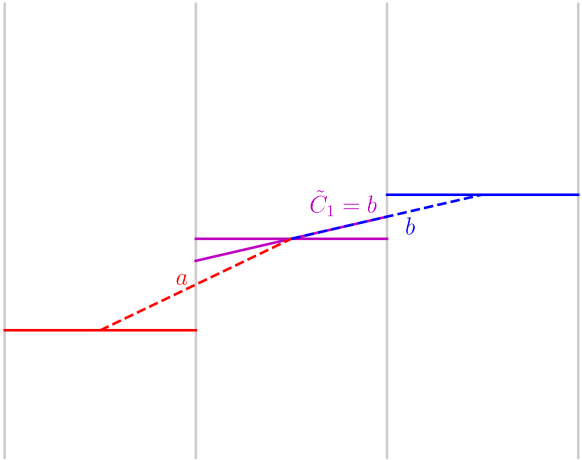
$$\tilde{C}_1 := \operatorname{minmod}(a, b, C_1),$$

$$\operatorname{minmod}(a, b, C_1) := \begin{cases} \operatorname{sgn}(a) \times \min(|a|, |b|, |C_1|), & \operatorname{sgn}(a) = \operatorname{sgn}(b) = \operatorname{sgn}(C_1) \\ 0, & \text{else} \end{cases}$$



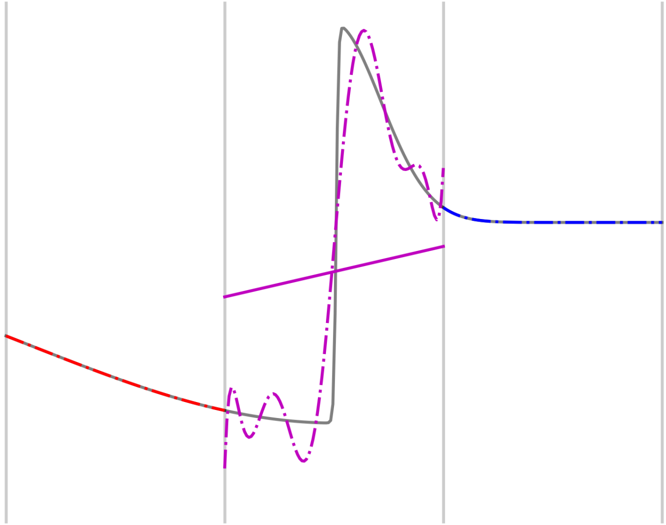
Slope Limiter (MinMod)

$$\sum_{n=1}^{k} C_n \, P_n = u_h \to \tilde{u}_h := \Lambda_{\mathrm{SL}}\left(u_h\right) := C_0 \, P_0 + \tilde{C}_1 \, P_1$$

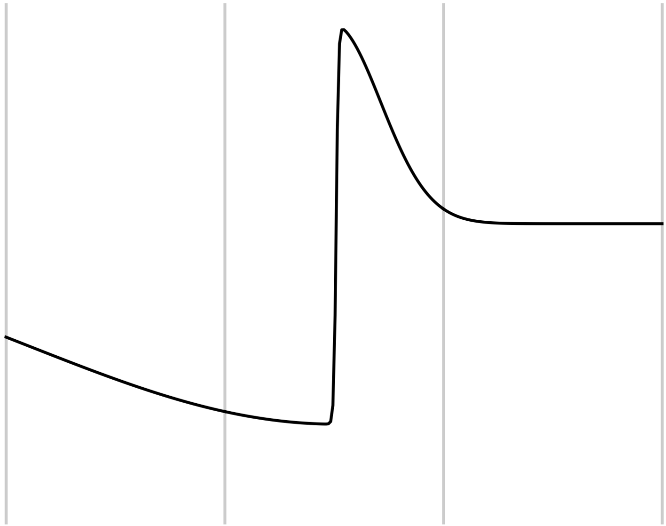


Slope Limiter (MinMod)

$k = 9$
$\tilde{k} = 1$

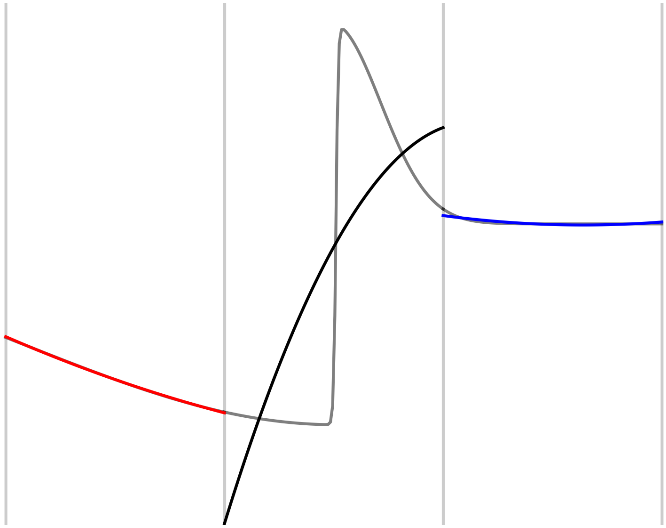


Troubled Cell Indicator (Fu & Shu, 2017)

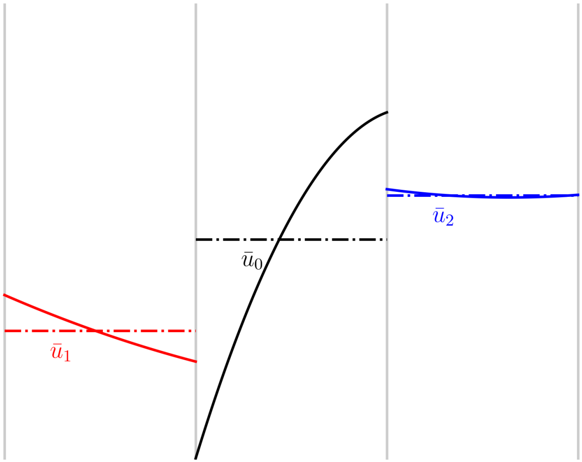


Troubled Cell Indicator (Fu & Shu, 2017)

$k = 2$

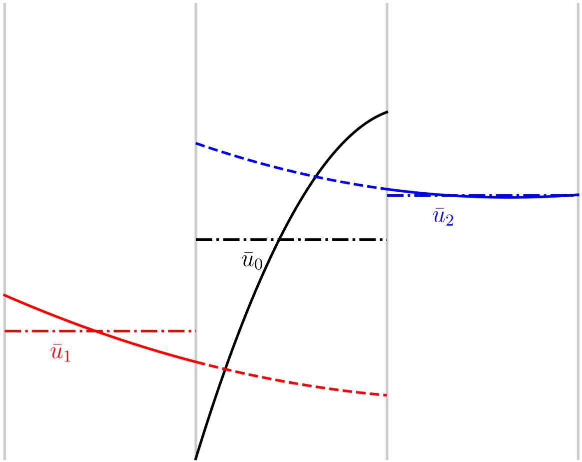


Troubled Cell Indicator (Fu & Shu, 2017)

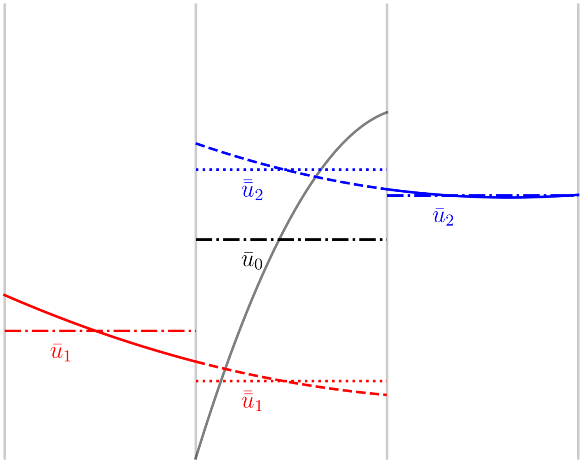


Troubled Cell Indicator (Fu & Shu, 2017)

$u_h\left(x,t\right)=\sum_{i=1}^{k+1}u_i\,\ell_i\left(x\right)$, $u_h\left(x,t\right)=\sum_{i=1}^{k+1}u_i\,\ell_i\left(x\right)$

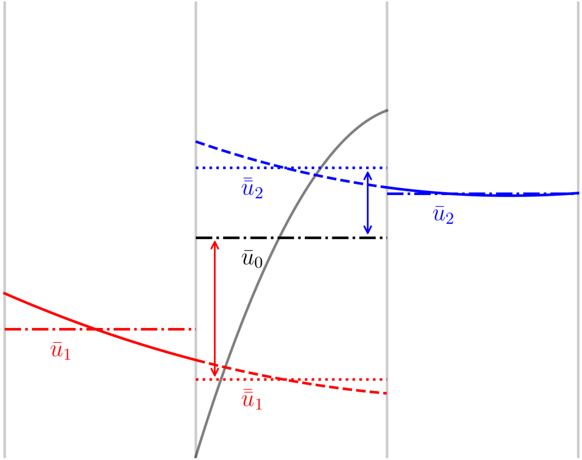


Troubled Cell Indicator (Fu & Shu, 2017)



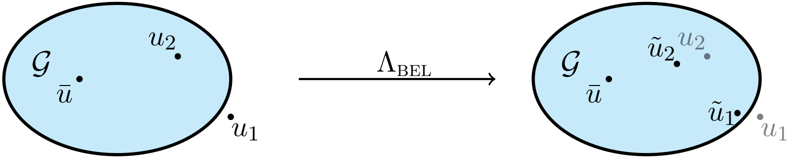
Troubled Cell Indicator (Fu & Shu, 2017)

$$I_K := \left(\max_{j \in \{0,\cdots,2^d\}} (|\bar{u}_j|) \right)^{-1} \sum_{j=1}^{2^d} |\bar{u}_0 - \bar{\bar{u}}_j|$$



Bound-Enforcing Limiter (Λ_{BEL})

- Proposed in Qin et al. (2016)
- Define $\mathcal{G}$ as the set of physically-admissible states; i.e., positivite density and energy density, subluminal fluid velocity (for an ideal EoS, $\mathcal{G}$ is convex)

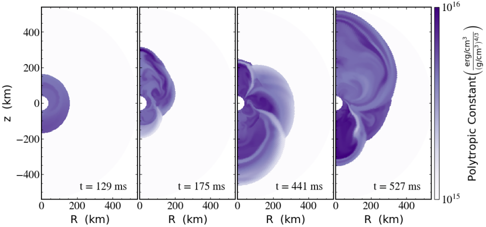


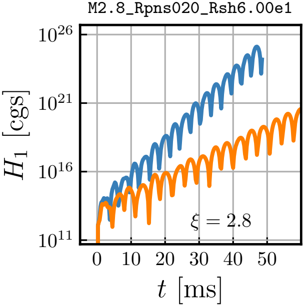
$u_h = u_1\,\ell_1 + u_2\,\ell_2$

$\tilde{u}_h := \Lambda_{\text{BEL}}\left(u_h\right) = \tilde{u}_1\,\ell_1 + \tilde{u}_2\,\ell_2$

Standing Accretion Shock Instability

Used thornado to investigate the role of GR on the SASI[1]



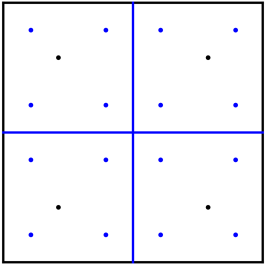


[1] Dunham et al. (2020, 2024)

Mesh Refinement with AMReX

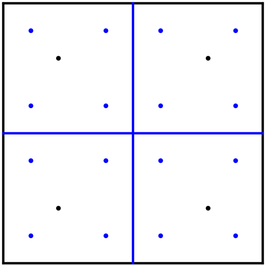
https://github.com/AMReX-Codes/amrex

Interpolation



Given $U_h\left(x,t\right)=\sum\limits_{i\in\mathcal{N}}U_i\left(t\right)\,\ell_i\left(x\right)$, how to compute $u_h^{(j)}\left(x,t\right)=\sum\limits_{i\in\mathcal{N}}u_i^{(j)}\left(t\right)\,\ell_i^{(j)}\left(x\right)$, or vice-versa? ($\mathcal{N}$ is the set of all grid-points for any given element.)

Conservative Interpolation



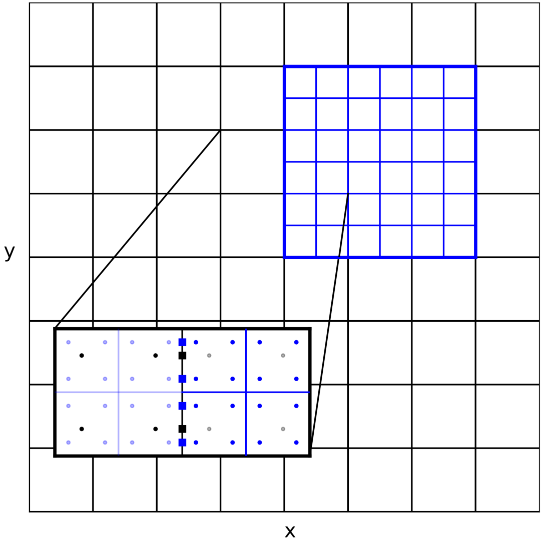
- Schaal et al. (2015)

Given a solution on a coarse element K, U_h, define the solution on the jth ($j \in \{1, \cdots, 2^d\}$) fine element $k^{(j)}$, $u_h^{(j)}$ via the minimization of an L^2-projection:

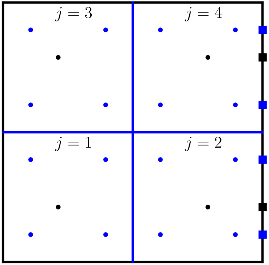
$$\forall i \in \mathcal{N} : \min_{\tilde{u}_i^{(j)}} \int_{k^{(j)}} \left(\tilde{U}_h - \tilde{u}_h^{(j)} \right)^2 d^d x,$$

where $\tilde{U} := \sqrt{\gamma}\, U$. For the geometry fields, we use the same procedure but do not pre-multiply with $\sqrt{\gamma}$.

Flux Corrections



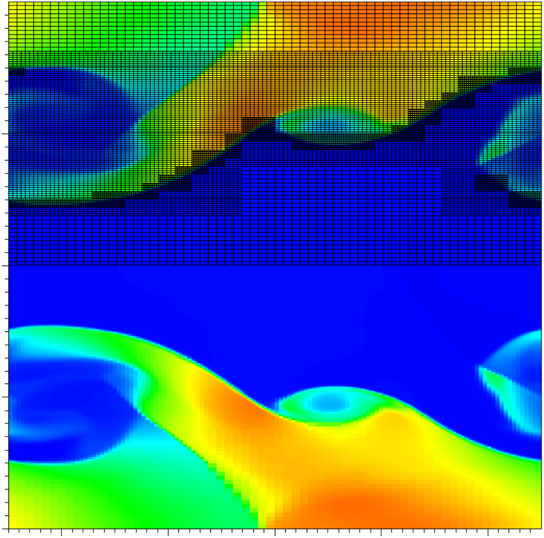
Flux Corrections



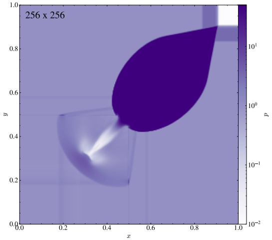
The numerical flux across, say, the x^1 interface on the right side of an element for a 2D problem with no gravity and Cartesian coordinates is given by

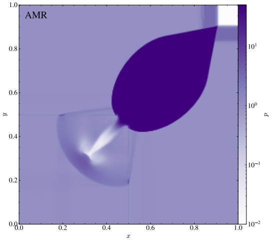
$$\int_{K^2} \widehat{F}^1 \left(U_h^-, U_h^+\right) \ell_{k_1} \left(x^1_{\mathrm{H}}\right) \ell_{k_2} \left(x^2\right) dx^2$$
$$= \sum_{j \in \{2,4\}} \int_{k^2_{(j)}} \widehat{F}^1 \left(U_h^-, U_h^+\right) \ell_{k_1} \left(x^1_{\mathrm{H}}\right) \ell_{k_2} \left(x^2\right) dx^2$$
$$:= \sum_{j \in \{2,4\}} \int_{k^2_{(j)}} \widehat{F}^1 \left(u_h^{(j),-}, u_h^{(j),+}\right) \ell_{k_1} \left(x^1_{\mathrm{H}}\right) \ell_{k_2} \left(x^2\right) dx^2$$

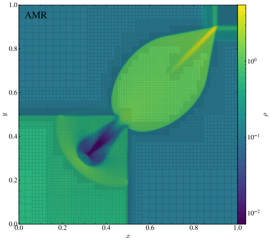
Kelvin–Helmholtz Instability

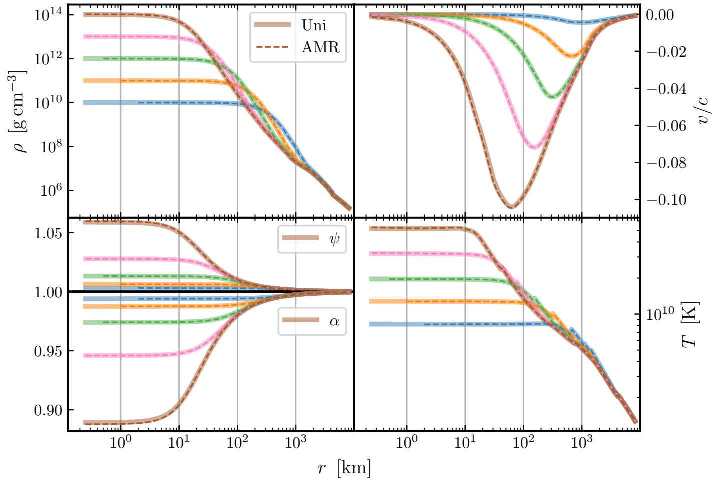


2D Blast Wave (Del Zanna & Bucciantini, 2002)

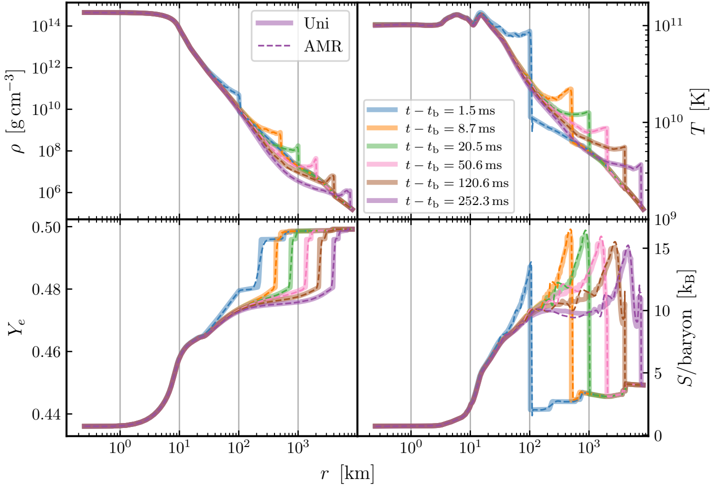







10^{14}
10^{12}
10^{10}
10^{8}
10^{6}
ρ [g cm^{-3}]
Uni
AMR
0.00
−0.02
−0.04
−0.06
−0.08
−0.10
v/c
1.05
1.00
0.95
0.90
ψ
α
10^{10}
T [K]
10^{0}
10^{1}
10^{2}
10^{3}
r [km]

Adiabatic Collapse (Post-Bounce Phase)



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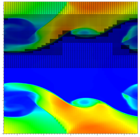
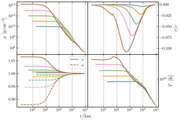
Summary

Can run multi-D pure hydro problems in GR with AMR





Can run 1D hydro+self-gravity problems in GR with AMR

Working on coupling GR transport to existing hydro+gravity modules