14-Moment-Inspired Visco-Resistive MHD

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Physical Model

Baryon number, momentum, and energy are conserved:

$$\nabla_\mu N^\mu = 0\,, \tag{1}$$
$$\nabla_\mu T^{\mu\nu} = 0\,, \tag{2}$$

where

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{hydro}} + T^{\mu\nu}_{\mathrm{dissipative}} + T^{\mu\nu}_{\mathrm{EM}}\,. \tag{3}$$

How to define N^μ and $T^{\mu\nu}_{\mathrm{hydro}}$, etc.?

Viscous Hydrodynamics

Fluid-frame projector: $\Delta^{\mu\nu} := g^{\mu\nu} + u^{\mu}\, u^{\nu}$

Decompose momentum into components parallel and perpendicular to four-velocity: $p^{\mu} = E_{\boldsymbol{p}}\, u^{\mu} + p^{\langle\mu\rangle}$, with $p^{\langle\mu\rangle} := \Delta^{\mu}{}_{\nu}\, p^{\nu}$

$$
\begin{aligned}
N^{\mu} &= n\, u^{\mu} + n^{\mu} && (4) \\
T^{\mu\nu} &= (\rho + \rho\,\epsilon)\, u^{\mu}\, u^{\nu} + P\, \Delta^{\mu\nu} \\
&\quad + \Pi\, \Delta^{\mu\nu} + q^{\mu}\, u^{\nu} + q^{\nu}\, u^{\mu} + \pi^{\mu\nu} \\
&\quad + T^{\mu\nu}_{\mathrm{EM}}\,, && (5)
\end{aligned}
$$

where

$$
\begin{aligned}
T^{\mu\nu}_{\mathrm{EM}} &:= \frac{1}{4\pi}\left(F^{\mu\alpha}\, F_{\alpha}{}^{\nu} - \frac{1}{4}\, g^{\mu\nu}\, F^{\alpha\beta}\, F_{\alpha\beta}\right) && (6) \\
&= \frac{1}{2}\, (u^{\mu}\, u^{\nu} + \Delta^{\mu\nu})\, (e_{\alpha}\, e^{\alpha} + b_{\alpha}\, b^{\alpha}) - 2\, u^{(\mu}\, b^{\nu)\alpha}\, e_{\alpha} - (e^{\mu}\, e^{\nu} + b^{\mu}\, b^{\nu}) && (7)
\end{aligned}
$$

Resistive Magnetohydrodynamics

$$\nabla_\mu \mathcal{J}^\mu = 0 \tag{8}$$

$$\mathcal{J}^\mu := \rho_{\rm q}\, u^\mu + J^\mu \tag{9}$$

$$\nabla_\nu \left(J^\mu\, u^\nu\right) = -\frac{1}{\tau}\left(J^\mu - \eta^{-1}\, e^\mu\right) + \frac{\delta_{\rm VB}}{\tau}\, \sqrt{b^2}\, b^{\mu\nu}\, J_\nu \tag{10}$$

Evolution Equations

$$\nabla_\mu \left[\left(\Pi + \frac{\zeta}{\tau_\Pi} \right) u^\mu \right] = S_\Pi \tag{11}$$

$$\nabla_\mu \left[q_\nu \, u^\mu + \frac{\kappa}{\tau_\mathrm{q}} \, T \, \Delta^\mu{}_\nu \right] = (S_\mathrm{q})_\nu \tag{12}$$

$$\nabla_\mu \left[\pi^{\alpha\beta} \, u^\mu + 2 \, \frac{\eta}{\tau_\pi} \, g^{\mu(\alpha} \, u^{\beta)} \right] = (S_\pi)^{\alpha\beta} \tag{13}$$

$$\nabla_\mu \left[J_\nu \, u^\mu \right] = (S_\mathrm{J})_\nu \tag{14}$$

Primitive Variable Recovery

$$V = \left(\rho, W v^i, e, B^i, E^i, J_\nu, q^\mu, \pi^{\mu\nu}, \Pi\right)^T \tag{15}$$

$$U\left(V\right) = \begin{pmatrix} \rho\, W \\ \rho\, h\, W^2\, v_j + \epsilon^{ijk}\, E_j\, B_k + \cdots \\ \rho\, h\, W^2 - p - \rho\, W + \frac{1}{2}\left(E^2 + B^2\right) + \cdots \\ B^i \\ E^i \\ J_\nu\, u^0 \\ q_\nu\, u^0 + \frac{\kappa}{\tau_\mathrm{q}}\, T\, \Delta^0{}_\nu \\ \pi^{\mu\nu}\, u^0 + 2\, \frac{\eta}{\tau_\pi}\, g^{0(\mu}\, u^{\nu)} \\ \left(\Pi + \frac{\zeta}{\tau_\Pi}\right) u^0 \end{pmatrix} \tag{16}$$

Problem: How to compute $V\left(U\right)$?

Primitive Variable Recovery

Never been solved before, so no literature to follow

Strategy: Combine previous (known) methods for smaller systems and invent methods as needed

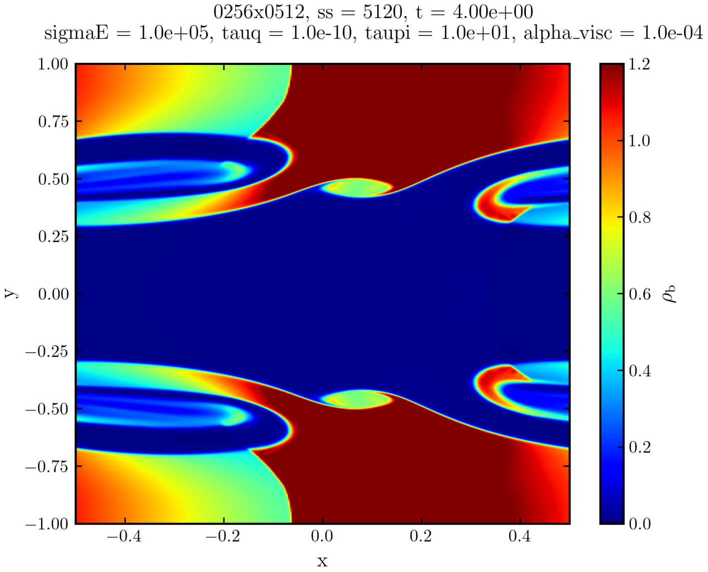
Issues:

- Root-finder converges to incorrect solution
- Wrong temperature recovered for heat flux

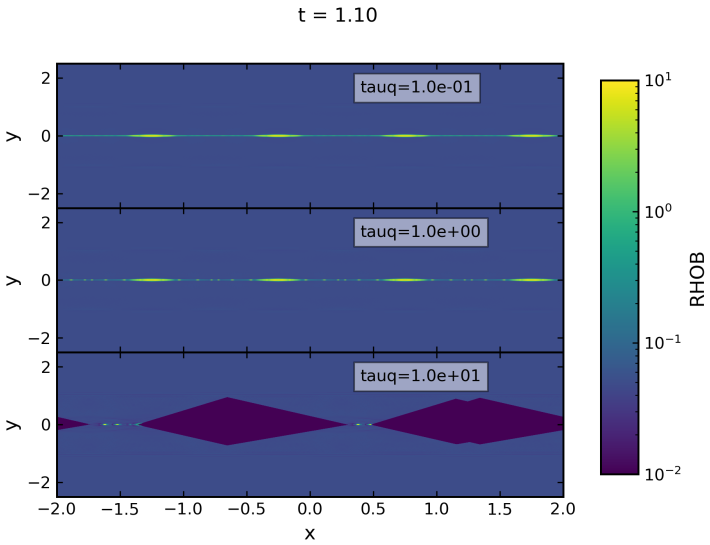
Algorithm

1. Guess v^i and $\rho\, h\, W$ (e.g. $v^i = 0$, $h\, W = 1$)
2. Use these to compute guesses for thermo quantities (e, p, T, etc.)
3. Use thermo to compute dissipation coefficients (ζ, κ, etc.)
4. With these, recover all primitives (incl. updated v^i and $\rho\, h\, W$)
5. Iterate steps 1-4 within Newton–Raphson algorithm
6. If 5 fails to converge, repeat 1-5 using entropy to compute e (more stable, but invalid near shocks)
7. If 6 fails to converge, use ideal (non-viscous, non-resistive) solution

What Works: Viscous, Non-Resistive, No heat-flux



What Doesn't Work: Viscous, Resistive



Tried So Far

- Setting floors
- Use better initial guess (e.g., ideal solution for v^i)
- Iterate over different variables (e.g., Poynting flux, presssure, $1/(h\,W)$)

What to Try Next?

- Check if $e < 0$ with zero heat flux; if so, add energy
- If $e > 0$ without heat flux and $e < 0$ with heat flux, then check T
- If T is bad, define $q^{\nu} = q^{\nu}\left(q^{*}, e\right)$; else, limit heat flux

Summary/Future Work

- Developed solver for visco-resistive MHD equations
- Shown that the model works and shown the influence of visco-resistivity
- Make con2prim algorithm more robust

Boltzmann–Vlasov Equation

Four-momentum vector: $p^\mu = (E_{\boldsymbol{p}}, \boldsymbol{p})^\mu$, with $E_{\boldsymbol{p}} := -u \cdot p$

Faraday tensor: $F^{\mu\nu} := u^\mu\, e^\nu - u^\nu\, e^\mu + \epsilon^{\mu\nu\alpha\beta}\, b_\alpha\, u_\beta$

Distribution function: $f = f\,(x, p)$

$$p^\mu\, \partial_\mu\, f + \left(q\, F^{\mu\nu}\, p_\nu + \Gamma^\mu{}_{\alpha\beta}\, p^\alpha\, p^\beta\right) \partial_{p^\mu}\, f = C\,[f] \tag{17}$$

[Most et al., 2022]

Moments

Moments of Boltzmann–Vlasov equation lead to definitions of N^μ and $T^{\mu\nu}_{\text{hydro}} + T^{\mu\nu}_{\text{dissipative}}$:

$$N^\mu := \int \frac{g\, d^3p}{(2\pi)^3\, E_{\boldsymbol{p}}}\, p^\mu\, f \tag{18}$$

$$T^{\mu\nu}_{\text{hydro}} + T^{\mu\nu}_{\text{dissipative}} := \int \frac{g\, d^3p}{(2\pi)^3\, E_{\boldsymbol{p}}}\, p^\mu\, p^\nu\, f \tag{19}$$

[Denicol et al., 2019]

With $p^\mu = E_{\boldsymbol{p}}\, u^\mu$ and $f = \left(\exp\left(\left(E_{\boldsymbol{p}} - \mu\right)/T\right)\right)^{-1}$, these become the hydro part of the ideal magnetohydrodynamics equations and $T^{\mu\nu}_{\text{dissipative}} = 0$

Bibliography

Denicol, G. S., Molnár, E., Niemi, H., and Rischke, D. H. (2019).
Resistive dissipative magnetohydrodynamics from the Boltzmann-Vlasov equation.
Phys. Rev. D, 99(5):056017.

Most, E. R., Noronha, J., and Philippov, A. A. (2022).
Modelling general-relativistic plasmas with collisionless moments and dissipative two-fluid magnetohydrodynamics.
MNRAS, 514(4):4989–5003.