14-Moment-Inspired Visco-Resistive MHD

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July 28, 2025

Physical Model

Baryon number, momentum, and energy are conserved:

$$\nabla_\mu N^\mu = 0\,, \tag{1}$$
$$\nabla_\mu T^{\mu\nu} = 0\,, \tag{2}$$

where

$$T^{\mu\nu} = T^{\mu\nu}_{\text{hydro}} + T^{\mu\nu}_{\text{dissipative}} + T^{\mu\nu}_{\text{EM}}\,. \tag{3}$$

How to define N^μ and $T^{\mu\nu}_{\text{hydro}}$, etc.?

Viscous Hydrodynamics

Fluid-frame projector: $\Delta^{\mu\nu} := g^{\mu\nu} + u^\mu\, u^\nu$

Decompose momentum into components parallel and perpendicular to four-velocity: $p^\mu = E_{\boldsymbol{p}}\, u^\mu + p^{\langle\mu\rangle}$, with $p^{\langle\mu\rangle} := \Delta^\mu{}_\nu\, p^\nu$

$$
\begin{aligned}
N^\mu &= n\, u^\mu + n^\mu && (4)\\
T^{\mu\nu} &= (\rho + \rho\,\epsilon)\, u^\mu\, u^\nu + P\,\Delta^{\mu\nu} \\
&\quad + \Pi\,\Delta^{\mu\nu} + q^\mu\, u^\nu + q^\nu\, u^\mu + \pi^{\mu\nu} \\
&\quad + T^{\mu\nu}_{\mathrm{EM}}\,, && (5)
\end{aligned}
$$

where

$$
\begin{aligned}
T^{\mu\nu}_{\mathrm{EM}} &:= \frac{1}{4\pi}\left(F^{\mu\alpha}\, F_\alpha{}^\nu - \frac{1}{4}\, g^{\mu\nu}\, F^{\alpha\beta}\, F_{\alpha\beta}\right) && (6)\\
&= \frac{1}{2}\,(u^\mu\, u^\nu + \Delta^{\mu\nu})\,(e_\alpha\, e^\alpha + b_\alpha\, b^\alpha) - 2\, u^{(\mu}\, b^{\nu)\alpha}\, e_\alpha - (e^\mu\, e^\nu + b^\mu\, b^\nu) && (7)
\end{aligned}
$$

Resistive Magnetohydrodynamics

$$\nabla_\mu \mathcal{J}^\mu = 0 \tag{8}$$

$$\mathcal{J}^\mu := \rho_\mathrm{q}\, u^\mu + J^\mu \tag{9}$$

$$\nabla_\nu \left(J^\mu\, u^\nu\right) = -\frac{1}{\tau} \left(J^\mu - \eta^{-1}\, e^\mu\right) + \frac{\delta_\mathrm{VB}}{\tau} \sqrt{b^2}\, b^{\mu\nu}\, J_\nu \tag{10}$$

Evolution Equations

$$\nabla_\mu \left[\left(\Pi + \frac{\zeta}{\tau_\Pi} \right) u^\mu \right] = S_\Pi \tag{11}$$

$$\nabla_\mu \left[q_\nu \, u^\mu + \frac{\kappa}{\tau_\mathrm{q}} \, T \, \Delta^\mu{}_\nu \right] = (S_\mathrm{q})_\nu \tag{12}$$

$$\nabla_\mu \left[\pi^{\alpha\beta} \, u^\mu + 2 \, \frac{\eta}{\tau_\pi} \, g^{\mu(\alpha} \, u^{\beta)} \right] = (S_\pi)^{\alpha\beta} \tag{13}$$

$$\nabla_\mu \left[J_\nu \, u^\mu \right] = (S_\mathrm{J})_\nu \tag{14}$$

Primitive Variable Recovery

$$V = \left(\rho, W v^i, e, B^i, E^i, J_\nu, q^\mu, \pi^{\mu\nu}, \Pi\right)^T \tag{15}$$

$$U\left(V\right) = \begin{pmatrix} \rho\, W \\ \rho\, h\, W^2\, v_j + \epsilon^{ijk}\, E_j\, B_k + \cdots \\ \rho\, h\, W^2 - p - \rho\, W + \frac{1}{2}\left(E^2 + B^2\right) + \cdots \\ B^i \\ E^i \\ J_\nu\, u^0 \\ q_\nu\, u^0 + \frac{\kappa}{\tau_{\mathrm{q}}}\, T\, \Delta^0{}_\nu \\ \pi^{\mu\nu}\, u^0 + 2\, \frac{\eta}{\tau_\pi}\, g^{0(\mu}\, u^{\nu)} \\ \left(\Pi + \frac{\zeta}{\tau_\Pi}\right) u^0 \end{pmatrix} \tag{16}$$

Problem: How to compute $V\left(U\right)$?

Primitive Variable Recovery

Never been solved before, so no literature to follow

Strategy: Combine previous (known) methods for smaller systems and invent methods as needed

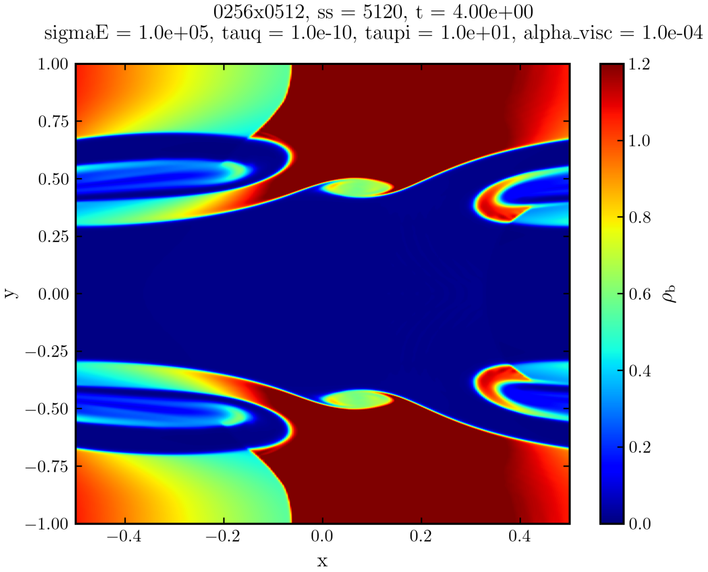
Issues:

- Root-finder converges to incorrect solution
- Wrong temperature recovered for heat flux

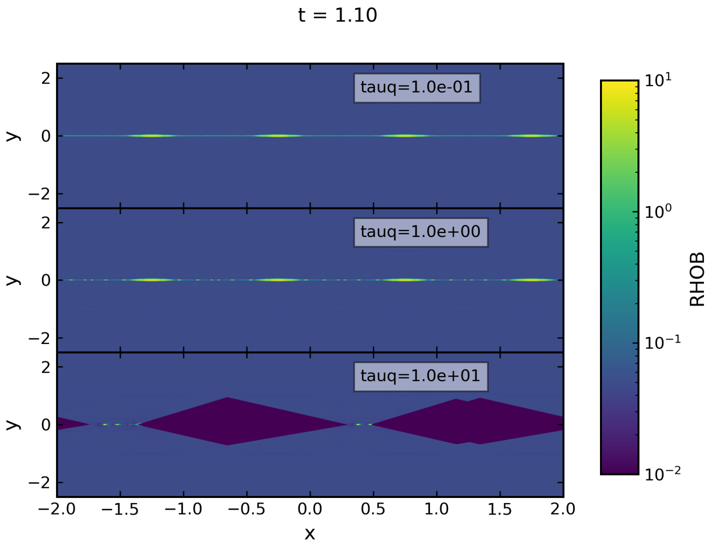
Algorithm

1. Guess v^i and $\rho\, h\, W$ (e.g. $v^i = 0$, $h\, W = 1$)
2. Use these to compute guesses for thermo quantities (e, p, T, etc.)
3. Use thermo to compute dissipation coefficients (ζ, κ, etc.)
4. With these, recover all primitives (incl. updated v^i and $\rho\, h\, W$)
5. Iterate steps 1-4 within Newton–Raphson algorithm
6. If 5 fails to converge, repeat 1-5 using entropy to compute e (more stable, but invalid near shocks)
7. If 6 fails to converge, use ideal (non-viscous, non-resistive) solution

What Works: Viscous, Non-Resistive, No heat-flux



What Doesn't Work: Viscous, Resistive



Tried So Far

- Setting floors
- Use better initial guess (e.g., ideal solution for v^i)
- Iterate over different variables (e.g., Poynting flux, presssure, $1/(h\,W)$)

What to Try Next?

- Check if $e < 0$ with zero heat flux; if so, add energy
- If $e > 0$ without heat flux and $e < 0$ with heat flux, then check T
- If T is bad, define $q^\nu = q^\nu \left(q^*, e\right)$; else, limit heat flux

Summary/Future Work

- Developed solver for visco-resistive MHD equations
- Shown that the model works and shown the influence of visco-resistivity
- Make con2prim algorithm more robust

Boltzmann–Vlasov Equation

Four-momentum vector: $p^\mu = (E_{\boldsymbol{p}}, \boldsymbol{p})^\mu$, with $E_{\boldsymbol{p}} := -u \cdot p$

Faraday tensor: $F^{\mu\nu} := u^\mu \, e^\nu - u^\nu \, e^\mu + \epsilon^{\mu\nu\alpha\beta} \, b_\alpha \, u_\beta$

Distribution function: $f = f\,(x, p)$

$$p^\mu \, \partial_\mu \, f + \left(q \, F^{\mu\nu} \, p_\nu + \Gamma^\mu{}_{\alpha\beta} \, p^\alpha \, p^\beta \right) \partial_{p^\mu} \, f = C\,[f] \tag{17}$$

[?]

Moments

Moments of Boltzmann–Vlasov equation lead to definitions of N^μ and $T^{\mu\nu}_{\text{hydro}} + T^{\mu\nu}_{\text{dissipative}}$:

$$N^\mu := \int \frac{g\, d^3p}{(2\pi)^3\, E_{\boldsymbol{p}}}\, p^\mu\, f \tag{18}$$

$$T^{\mu\nu}_{\text{hydro}} + T^{\mu\nu}_{\text{dissipative}} := \int \frac{g\, d^3p}{(2\pi)^3\, E_{\boldsymbol{p}}}\, p^\mu\, p^\nu\, f \tag{19}$$

[?]

With $p^\mu = E_{\boldsymbol{p}}\, u^\mu$ and $f = \left(\exp\left(\left(E_{\boldsymbol{p}} - \mu\right)/T\right)\right)^{-1}$, these become the hydro part of the ideal magnetohydrodynamics equations and $T^{\mu\nu}_{\text{dissipative}} = 0$

Bibliography

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