Donald C. Spencer (1912–2001)

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Joseph J. Kohn

Donald C. Spencer died on December 23, 2001. He was born on April 25, 1912, in Boulder, Colorado. He was an undergraduate at the University of Colorado (B.A. in 1934) and at the Massachusetts Institute of Technology (B.S. in Aeronautical Engineering in 1936) and a graduate student in mathematics at Cambridge University (Ph.D. in 1939, Sc.D. in 1963). His doctoral thesis, "On a Hardy-Littlewood problem of diophantine approximation and its generalizations", was written under the direction of J. E. Littlewood and initiated Spencer's remarkable mathematical career. Spencer taught at M.I.T. (1939–42), at Stanford (1942–50 and 1963–68), and at Princeton (1950–63 and 1968–78). He was the Eugene Higgins Professor at Princeton (1971–72) and the Henry Burchard Fine Professor at Princeton (1972–78). He received the degree Sc.D. honoris causa from Purdue University (1971) and was the joint recipient with A. C. Schaeffer of the Bôcher Memorial Prize (1948). He was a member of the U.S. National Academy of Sciences (from 1961), a fellow of the American Academy of Arts and Sciences (from 1967), and a recipient of the National Medal of Science (1989). He was the author of numerous important research papers and four influential books (see [1], [2], [3], [4], [5]).

Spencer's mathematical work is truly impressive and spans many fields in which he made fundamental contributions. Before coming to Princeton, he worked in number theory on lattice points and on sequences of integers; in applied mathematics on fluid mechanics; and in the theory of one complex variable on univalent functions, on conformal mappings, and on Riemann surfaces.

Donald C. Spencer

At Princeton, Spencer's research turned to several complex variables and complex manifolds. He collaborated with Kunihiko Kodaira, who is also widely recognized as one of the great mathematical pioneers of the twentieth century. Together, Spencer and Kodaira developed the modern theory of deformations of complex manifolds into a major tool, the basis for a large body of subsequent research. This work has had and continues to have tremendous influence in large segments of mathematics, including the theory of several complex variables, differential geometry, algebraic geometry, and mathematical physics. In fact, Spencer's work with Kodaira was one of the most remarkable mathematical collaborations of the twentieth century; its only parallel is the famous Hardy-Littlewood work. To give a feeling for the excitement generated by these efforts, I quote from the introduction to Kodaira's book (*Complex Manifolds and Deformations of Complex Structures*, Springer-Verlag, 1981):

> In order to clarify this mystery, Spencer and I developed the theory of deformations of compact complex manifolds.

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After retirement in Durango, CO.

> The process of the development was the most interesting experience in my whole mathematical life. It was similar to an experimental science developed by the interaction between experiments (examination of examples) and theory. In this book I have tried to reproduce this interesting experience; however I could not fully convey it. Such an experience may be a passing phenomenon which cannot be reproduced.

Furthermore, in 1987 Kodaira wrote in his letter to the President's Committee on the National Medal of Science:

> Spencer's contributions to mathematics go far beyond his published papers. He exerted tremendous influence on his collaborators and students. His enthusiasm knew no limit and was contagious. In Princeton he was always surrounded by a group of mathematicians who shared his enthusiasm and collaborated in the research of complex analysis (I was one of them). In the 1950s the theory of complex manifolds was developed extensively in Princeton. The driving force behind this development was in fact Spencer's enthusiasm.

At the same time, Spencer introduced the use of potential theory in the study of complex manifolds with boundaries, and in particular formulated the "d-bar-Neumann problem", which has led to very important developments in both several complex variables and partial differential equations. In the last decade of his career, Spencer worked on overdetermined systems of partial differential equations and on pseudogroups.

In all his work Spencer shows remarkable originality and insight. His influence on his students, his collaborators, and his many friends also has had a lasting impact on twentieth-century mathematics. Just as Spencer had an unfailing instinct for how to approach mathematical research, so he also had an unfailing instinct for how to inspire both students and fellow mathematicians. Recently Victor Guillemin, professor of mathematics at M.I.T., wrote the following tribute:

> When I first met Don Spencer, I was a twenty-two-year-old graduate student, and Spencer was in his mid-forties and at the height of his fame. Kodaira-Spencer was the most exciting development of that era in differential geometry, the early sixties equivalent of Seiberg-Witten, and I and a lot of my fellow graduate students were caught up in this excitement and working on Spencer sequences, Spencer cohomology, the Spencer theory of deformations, etc. This was the state of affairs when I first met Don in the fall of 1960; and therefore, not surprisingly, the thing I most remember about him was his incredible kindness and empathy for young people. I have had several older colleagues who have been able, as one says, to "bridge the generation gap", but there is no one I've ever known who had this quality to the extent that Spencer did. Some of us took advantage of it and must at times have bored him silly with the catalogue of our accomplishments and aspirations. But he was unfailing nonjudgmental and sympathetic. More than that, one could feel that his interest in these aspirations was the genuine commodity, and not just feigned to make us feel good. One's ego at 22 is a fragile vessel, and Don instinctively knew this. For that (as for much else) I remember Don as the kindest, nicest person I've ever known.

In 1978 Spencer retired from Princeton and moved to Durango, Colorado. There he became very active in the conservation and ecology movements and also an avid hiker. He soon made many friends and became well known in the area. The city of Durango designated April 25 as "Don Spencer Day". After his retirement, Don Spencer kept in touch with his many friends and colleagues, always inspiring, supportive, and full of enthusiasm.

Don was my thesis adviser, mentor, colleague, and friend. I consider myself fortunate to have been so close to such a remarkable human being. Here I will give a brief account of his mathematical contributions to the theory of partial differential equations (P.D.E.).

Don's contributions to P.D.E. can be divided into three groups: (1) equations arising from fluid mechanics, (2) equations arising from complex analysis, and (3) the theory of linear overdetermined systems. His work in (1) is discussed below by Paul Garabedian. The work in (2) is divided into two parts: equations arising from one complex variable and those from several complex variables. Here I will concentrate on the work connected with several complex variables. The contributions in (3) are discussed by Hubert Goldschmidt.

The development of the theory of P.D.E. is closely linked with advances in complex analysis; in fact, Riemann's approach to the study of conformal mapping via the Dirichlet principle led to the systematic development of the theory of elliptic P.D.E. and associated variational problems. The application of these methods to the theory of several complex variables was initiated by Hodge in his theory of harmonic integrals on compact manifolds. It is this work that led H. Weyl to prove the fundamental hypoellipticity theorem, known as Weyl's lemma, which in turn led to the development of the general theory of elliptic P.D.E.

The theory of harmonic integrals on compact manifolds is a crucial ingredient in the Kodaira-Spencer theory of deformations of complex structures. An example of this is the fundamental existence theorem proved by Kodaira, Nirenberg, and Spencer (see [6]), which is described below by P. A. Griffiths.

Spencer's most spectacular contribution to the theory of P.D.E., one which has been a major influence in mathematical research, is setting down the program to generalize harmonic integrals to noncompact manifolds. Spencer called this program the $\bar{\partial}$-Neumann problem. Spencer's contributions were not simply confined to his papers: they were also made through his teaching, lectures, and principally his unique remarkable ability to communicate his ideas and his enthusiasm to so many students and colleagues. His first papers in the direction of the $\bar{\partial}$-Neumann problem were [7] with G. F. D. Duff and [8] with P. R. Garabedian. The $\bar{\partial}$-Neumann problem for (p, q)-forms on a manifold Ω can be formulated as follows. Let $L_2^{p,q}(\Omega)$ denote the space of square-integrable (p, q)-forms on Ω, let $\bar{\partial} : L_2^{p,q}(\Omega) \to L_2^{p,q+1}(\Omega)$ denote the L_2 closure of $\bar{\partial}$, and let $\bar{\partial}^* : L_2^{p,q}(\Omega) \to L_2^{p,q-1}(\Omega)$ denote the L_2 adjoint of $\bar{\partial}$. The $\bar{\partial}$-Neumann problem is: given $\alpha \in L_2^{p,q}(\Omega)$, does there exist a $\varphi \in L_2^{p,q}(\Omega)$ that is in the intersection of the domains of $\bar{\partial}$ and of $\bar{\partial}^*$ with $\bar{\partial}\varphi$ in the domain of $\bar{\partial}^*$ and $\bar{\partial}^*\varphi$ in the domain of $\bar{\partial}$ such that

$$\bar{\partial}\bar{\partial}^*\varphi + \bar{\partial}^*\bar{\partial}\varphi = \alpha?$$

In [9] Spencer formulated this problem for the case when $\Omega \subset X$, where X is a complex manifold, $\bar{\Omega}$ is compact, and Ω has a smooth boundary, and he outlined some of the important applications. The main feature of the $\bar{\partial}$-Neumann problem on manifolds with boundary is that it is a nonelliptic boundary-value problem for an elliptic operator (the boundary values are imposed by the requirement of belonging to the domain of $\bar{\partial}^*$). Spencer and I (see [10]) studied this phenomenon by introducing the operator $\bar{\partial}_\tau = \bar{\partial} + \tau\partial$, setting up the analogous $\bar{\partial}_\tau$-Neumann problem, and studying its behavior for small τ. In [10] we managed to solve the problem on balls in $\mathbb{C}^n$ by use of spherical harmonics, but this case is too special for the important applications.

Receiving National Medal of Science from President George Bush, 1989.

One of the main applications of the $\bar{\partial}$-Neumann problem on $(0,1)$-forms, envisioned by Spencer, is the construction of holomorphic functions with specified properties. It is this approach that C. B. Morrey used in [11] to show that a compact real-analytic manifold M can be embedded real analytically into euclidean space. Thus Morrey wanted to use the $\bar{\partial}_\tau$-Neumann problem with Ω a thin tubular neighborhood of M in the complexification of M; if the $\bar{\partial}_\tau$-Neumann problem could be solved on this Ω, then enough independent holomorphic functions could be constructed to give the required embedding. In [11] Morrey proved the following fundamental "$\frac{1}{2}$ estimate". There exists a constant $C > 0$ such that

$$\|\varphi\|_{\frac{1}{2}}^2 \le C(\|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2 + \|\varphi\|^2)$$

for all $(0,1)$-forms φ whose coefficients are in $C^\infty(\bar{\Omega})$ and which are in the domain of $\bar{\partial}^*$. Here $\|\cdot\|_{\frac{1}{2}}$ denotes the Sobolev $\frac{1}{2}$-norm. Morrey's proof of this estimate works for any complex manifold with smooth strongly pseudoconvex boundary.

Using the "$\frac{1}{2}$ estimate", I proved (see [12]) that the solution of the $\bar{\partial}$-Neumann problem exists for manifolds with strongly pseudoconvex smooth boundaries and that, locally near the boundary, the solution φ gains one derivative on α (in the interior it automatically gains two derivatives). Hörmander, in [13], proved existence using a different approach. He used Morrey's techniques to obtain L_2 estimates of φ with weights that are singular at the boundary. In this way he was able to prove existence in L_2 without assuming smoothness of the boundary or strong pseudoconvexity; all that is needed is weak pseudoconvexity. Because of interior ellipticity, Hörmander's solution gains two derivatives locally in the interior, but there is no control near the boundary.

The $\bar{\partial}$-Neumann problem and the mathematics that grew out of it have had an enormous impact on the theory of several complex variables, on the theory of P.D.E., on analysis, and recently on algebraic geometry. In particular, Hörmander's solution has engendered "the L_2 methods", which have been a very powerful tool. Hörmander has recently written a paper (see [14]), dedicated to the memory of D. C. Spencer, which exposes this approach and its consequences. The approach, based on regularity near the boundary, has led to numerous developments. Some of these are the study of subelliptic estimates; the calculus of pseudodifferential operators; the study of the operators $\bar{\partial}_b$ and $\Box_b$ on CR manifolds and on the Heisenberg group; and analysis on weakly pseudoconvex manifolds: global regularity and irregularity, regularity of biholomorphic and proper mappings, multiplier ideals, etc. Multiplier ideals, which arose from the study of estimates for the $\bar{\partial}$-Neumann problem, are currently used effectively in Kähler geometry, algebraic geometry, and the study of degenerate elliptic P.D.E.

Don played an essential role in all this during the 1950s and 1960s. Apart from his own contributions, he communicated his ideas and enthusiasm to most mathematicians who worked on these questions, students and colleagues alike. These included: A. Andreotti, M. Ash, P. E. Conner, H. Grauert, L. Hörmander, S. H. C. Hsiao, N. Kerzman, J. J. Kohn, M. Kuranishi, C. B. Morrey, L. Nirenberg, H. Rossi, and W. J. Sweeney. All of us found Don ready at any time to listen to us, encourage us, challenge us, and make us appreciate the importance of the enterprise. In the 1970s, 1980s, and 1990s the number of people working on these questions increased dramatically, and I am sure that Don had a direct influence on many of these also.

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Phillip A. Griffiths

Don Spencer and his friend and collaborator Kunihiko Kodaira were the principal founders of deformation theory—a central part of modern algebraic geometry and indeed of modern mathematics. The development of Kodaira-Spencer theory, as it is now called, took place during the middle and late 1950s. With the benefit of hindsight we may see it as arising from a natural confluence of Spencer's previous interests and some of the major mathematical developments at the time.

On the one hand, through his work with Schaeffer [17] and Schiffer [18], [19], [20], Spencer was intimately familiar with moduli of Riemann surfaces: firstly the by-then-classical theory of Teichmüller for compact Riemann surfaces and secondly the more general case where the Riemann surface may have a boundary (to which Spencer and his collaborators made significant contributions). Central to the former theory are the quadratic differentials on the Riemann surface. As will be seen below, the search for their higher-dimensional analogues was a significant issue.

On the other hand, three aspects of the general mathematical environment of the time were particularly relevant. Namely, this was a period of intense activity in the areas of

(i) harmonic integrals,

(ii) sheaf theory, and

(iii) several complex variables.

The first was in the tradition of Riemann, Hodge, and Weyl, among others, and was a subject to which each of Kodaira and Spencer individually made significant contributions (cf. [1], [2], [3]). Much of this work was concerned with what is now the "standard" linear elliptic theory on compact

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manifolds, including those with boundary, and especially the Kähler case.

The second was a very general theory, which for present purposes may be thought of as providing a systematic framework for analyzing the obstruction to globally piecing together solutions to a problem that has local solutions. This framework introduces the higher cohomology groups $H^q(X, \mathcal{J})$ of a sheaf $\mathcal{J}$ on a space X. For X an n-dimensional compact complex manifold and $\mathcal{J}$ the sheaf associated to a holomorphic vector bundle, the duality theorem of Kodaira and Serre is

$$H^q(X, \mathcal{J})^* \cong H^{n-q}(X, \Omega_X^n(\mathcal{J}^*)). \tag{1}$$

For X a compact Riemann surface, taking $q = 0$ and $\mathcal{J} = (\Omega_X^1)^{\otimes 2}$ to be the sheaf of quadratic differentials, (1) gives the natural isomorphism

$$\begin{pmatrix} \text{dual space to the} \\ \text{quadratic differentials} \\ \text{on } X \end{pmatrix} \cong H^1(\Theta_X), \tag{2}$$

where Θ_X is the sheaf of holomorphic vector fields on X. This observation was to provide a key hint about how to measure variation of complex structure in higher dimensions.

From several complex variables, important were results such as the direct-image theorem of Grauert and, especially, the Cauchy-Riemann operator $\overline{\partial}$. A complex manifold may be given either by a coordinate covering $\{U_\alpha, z_\alpha\}$ with complex analytic glueing data

$$\begin{cases} z_\alpha = f_{\alpha\beta}(z_\beta) & \text{in } U_\alpha \cap U_\beta \\ f_{\alpha\beta}(f_{\beta\gamma}(z_\gamma)) = f_{\alpha\gamma}(z_\gamma) & \text{in } U_\alpha \cap U_\beta \cap U_\gamma \end{cases} \tag{3}$$

or by giving the $\overline{\partial}$-operator

$$\overline{\partial} : A^0(X) \to A^{0,1}(X)$$

satisfying the integrability conditions

$$\overline{\partial}^2 = 0. \tag{4}$$

Here, we are given an almost complex structure

$$\begin{cases} J : T(X) \to T(X) \\ J^2 = -\mathrm{Id} \end{cases}$$

and $A^{p,q}(X)$ denotes the space of global smooth differential forms of type (p, q) relative to J. The equivalence of the definitions amounts to proving that (X, J) satisfying (4) has a covering by holomorphic coordinate charts; in effect it must be shown that there are enough local solutions to

$$\overline{\partial} f = 0.$$

This was proved in the real-analytic case by Eckmann-Frölicher and in the C^∞ case by Newlander-Nirenberg. These two approaches to complex manifolds are reflected respectively by the Čech and Dolbeault methods of representing sheaf cohomology; they will also be reflected in the two equivalent approaches to Kodaira-Spencer's deformation theory.

Before turning to that theory, I would call attention to the important series of papers [7], [8], [6], [9] by Kodaira and Spencer and the papers [21], [22] by Spencer which systematically applied sheaf cohomological methods to issues related to divisors and arising from classical algebraic geometry. Essentially, they introduced the exponential sheaf sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$$

($\exp f = e^{2\pi\sqrt{-1}f}$) and drew conclusions from its exact cohomology sequence when the identification

$$H^1(\mathcal{O}_X^*) \cong \begin{Bmatrix} \text{divisor class} \\ \text{group on } X \end{Bmatrix}$$

is made. Particularly noteworthy was their proof of the Lefschetz $(1, 1)$ theorem

$$\begin{Bmatrix} \text{fundamental classes} \\ \text{of divisors} \end{Bmatrix} \cong Hg^1(X)$$

where

$$Hg^1(X) = H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z})$$

is the Hodge group of integral $(1, 1)$ classes. I remember Don relating to me how he and Kodaira excitedly presented their proof to Lefschetz, who grumpily remarked that, first, he couldn't understand the fancy sheaf theory and, second, in any case *he* had proved the theorem. (The latter needs a caveat in that Lefschetz's argument for the semisimplicity of monodromy acting in the homology of a Lefschetz pencil was incomplete—only with Hodge's work was the result established. To this day all proofs of the $(1, 1)$ theorem require Hodge theory.)

Deformation theory had been on Spencer's mind for some time prior to his work with Kodaira (cf. [18], [17], [20]). As he explained it to me, the issue was that they did not know what should play the role in higher dimension of quadratic differentials in the Teichmüller theory on Riemann surfaces. The breakthrough came with (2). With that major "hint" everything began to fall into place, leading to the papers [11], [5], [10], [12] (and relatedly [13], [14], [23]), which brought deformation theory into the core of complex algebraic geometry.

Before describing their theory, I want to say that deformation theory seems to some extent to have "been in the air". In particular, the important work by Frölicher-Nijenhuis independently established the rigidity theorem stated below, and their papers on the calculus of vector-valued differential forms influenced the work of Kodaira-Spencer as well as that of many others working in related areas.

Turning to the Kodaira-Spencer deformation theory, intuitively a deformation of a compact complex manifold X is given by a family $\{X_t\}_{t\in B}$ of compact complex manifolds containing $X = X_{t_0}$ as a member. In case X is a smooth complex projective algebraic variety given by homogeneous polynomial equations

$$X = \{P_\lambda(x_0, \dots, x_N) = 0\},$$

one may imagine deforming X by varying the polynomials to $P_\lambda(x_0, \dots, x_N, t)$, with $P_\lambda(x_0, \dots, x_N, 0) = P_\lambda(x_0, \dots, x_N)$, and setting

$$X_t = \{P_\lambda(x_0, \dots, x_N, t) = 0\},$$

provided that X_t remains of constant dimension for $|t| < \epsilon$ (in which case it will be smooth). More formally, a *deformation* of X is given by

$$\begin{array}{c} \mathcal{X} \\ \downarrow \pi \\ B \end{array} \tag{5}$$

where $\mathcal{X}$ and B are complex manifolds and π is a smooth (i.e., the differential π_* has maximum rank) proper holomorphic mapping. The fibres $X_t = \pi^{-1}(t)$ of (5) are then compact, complex manifolds, and we assume that $X_{t_0} \cong X$ for a reference point $t_0 \in B$. The Kodaira-Spencer theory is local, and so we may think of B as an open neighborhood of the origin in $\mathbb{C}^m$. For the most part, understanding the case $m = 1$ will suffice, and so unless stated otherwise we will restrict to this situation and denote by t a coordinate on B.

Kodaira and Spencer sought to understand as fully as possible the properties of local deformations (5). They asked questions such as

1. When do deformations (5) exist?
2. What properties of X are stable under deformation?
3. How do the cohomology groups $H^q(X_t, \mathcal{J}_t)$ behave in a family (5)?

Here, we may think of $\mathcal{J}$ as the sheaf on $\mathcal{X}$ associated to a holomorphic vector bundle there, and then $\mathcal{J}_t$ denotes the restriction of $\mathcal{J}$ to X_t.

To study these questions they used both of the ways discussed above of describing a complex manifold. We shall use the first local coordinate method to define the basic invariant of (5), the *Kodaira-Spencer mappings*

$$\rho_t : T_t B \to H^1\left(\Theta_{X_t}\right). \tag{6}$$

Intuitively, these measure the first-order variation of the complex structure. Since the projection mapping π in (5) has maximal rank, shrinking B if necessary we may cover $\mathcal{X}$ by coordinate neighborhoods $\mathcal{U}_\alpha$ with local coordinates (z_α, t) such that

$$\pi(z_\alpha, t) = t.$$

Thus around a point of X_{t_0} the deformation is trivial; i.e., is biholomorphic to a product. We may then expect that the global nontriviality is measured cohomologically. In overlaps we will have (cf. (3))

$$\begin{cases} \text{(i)} & z_\alpha = f_{\alpha\beta}(z_\beta, t) \\ \text{(ii)} & f_{\alpha\beta}(f_{\beta\gamma}(z_\gamma, t), t) = f_{\alpha\gamma}(z_\gamma, t). \end{cases} \tag{7}$$

If we think of the complex manifold as being described by the glueing data (7)(i), the vector fields

$$\theta_{\alpha\beta}(t) =: \frac{\partial f^i_{\alpha\beta}(z_\beta, t)}{\partial t} \frac{\partial}{\partial z^i_\alpha} \tag{8}$$

describe the variation of the glueing data and hence the variation of the complex structure. Here, $\theta_{\alpha\beta}(t) \in C^1(\{U_\alpha(t)\}, \Theta_{X_t})$ is a Čech cochain for the tangent sheaf Θ_{X_t} relative to the open covering $U_\alpha(t) = \mathcal{U}_\alpha \cap X_t$ of X_t. Differentiation of (7)(ii) with respect to t shows that the Čech coboundary

$$\delta\{\theta_{\alpha\beta}(t)\} = 0,$$

and hence there is defined a cohomology class

$$\{\theta_{\alpha\beta}(t)\} \in H^1\left(\Theta_{X_t}\right)$$

which gives the Kodaira-Spencer mapping (6). A basic result of the theory is

(9) *The family (5) is locally trivial if, and only if, all ρ_t are zero.*

Behind this result are a number of subtleties: to show that ρ_t is well defined, to understand the situation when $\rho_{t_0} \neq 0$ but $\rho_t = 0$ for $t \neq t_0$ (jumping of structure), etc.

The other way of describing the family (5) may be expressed as follows. First, the Kodaira-Spencer mappings (6) give, in a precise sense, the obstruction to lifting $\partial/\partial t$ locally over the base to a holomorphic vector field on $\mathcal{X}$. If $\partial/\partial t$ lifts to a holomorphic vector field v with $\pi_*(v) = \partial/\partial t$, then the holomorphic flow of v gives biholomorphisms

$$X_{t_0} \xrightarrow{\sim} X_t \tag{10}$$

and therefore a trivialization

$$\mathcal{X} \cong X_{t_0} \times B.$$

Now, as may be seen using a partition of unity, there is no obstruction to lifting $\partial/\partial t$ as a C^∞ vector field. The resulting diffeomorphisms (10) may be used to transport the complex structures on X_t back to X_{t_0}. Analyzing this carefully, on X we may express the Cauchy-Riemann operator $\bar{\partial}_t$ for X_t as

$$\bar{\partial}_t = \bar{\partial} + \theta(t),$$

where $\theta(t)$ is a Θ_X-valued (0, 1) form having the local expression

$$\theta(t) = \theta(t)^j_{\bar{i}} \partial/\partial z_j \otimes d\bar{z}^i,$$

where

$$\theta(t) = \theta_1 t + \theta_2 t^2/2 + \cdots$$

is a convergent series in t. The integrability condition

$$\overline{\partial}_t^2 = 0$$

gives a series of relations

$$(11) \qquad \begin{cases} \overline{\partial}\theta_1 = 0 \\ \overline{\partial}\theta_2 + \frac{1}{2}[\theta_1, \theta_1] = 0 \\ \vdots \end{cases}$$

The first one implies that θ_1 defines a Dolbeault cohomology class

$$\{\theta_1\} \in H^{0,1}_{\overline{\partial}}(\Theta_X).$$

The second implies that the Dolbeault cohomology class

$$\{[\theta_1, \theta_1]\} \in H^{0,2}_{\overline{\partial}}(\Theta_X)$$

defined by the bracket $[\theta_1, \theta_1]$ should be zero. Collectively the equations (11) are equivalent to

$$\overline{\partial}_t \theta(t) = 0$$

so that

$$\{\theta(t)\} \in H^{0,1}_{\overline{\partial}_t}(\Theta_{X_t}).$$

The relation between the two approaches is expressed by the result

(12) *Under the Dolbeault isomorphism*

$$H^1(\Theta_{X_t}) \cong H^{0,1}_{\overline{\partial}_t}(\Theta_{X_t})$$

one has

$$\rho_t(\partial/\partial t) = \{\theta(t)\}.$$

This shows quite clearly how the Kodaira-Spencer classes give the variation of complex structure.

The second method is more amenable to using the methods of linear elliptic partial differential equations for deformation theory. For example, using the continuity of the eigenvalues in the discrete spectrum of an elliptic operator, Kodaira and Spencer showed

(13) *(Upper semi-continuity): For t close to t_0,*

$$\dim H^q(X_t, \mathcal{J}_t) \le \dim H^q(X_{t_0}, \mathcal{J}_{t_0}).$$

If $H^{q-1}(X_t, \mathcal{J}_{t_0}) = H^{q+1}(X_t, \mathcal{J}_{t_0}) = 0$, then equality holds.

As a corollary of this together with (9), they deduced the following result, which was proved independently by Frölicher and Nijenhuis.

(14) *If $H^1(X, \Theta_X) = 0$, then the family (5) is locally trivial.*

The result (13) is also a consequence of Grauert's direct-image theorem.

Some of the deepest results in the theory come by using nonlinear elliptic theory. The joint paper [5] with Louis Nirenberg was in response to the question: *Given a class $\{\theta_1\} \in H^1(\Theta_X)$, does there exist a family (5) with*

$$\rho_{t_0}(\partial/\partial t) = \{\theta_1\}?$$

From the discussion above, we see that what is required is to construct a convergent series

$$\theta(t) = \theta_1 t + \theta_2 t^2/2 + \cdots$$

such that (11) is satisfied. The first equation is automatic, but the second and higher ones present cohomological obstructions: e.g., if

$$\{[\theta_1, \theta_1]\} \neq 0 \text{ in } H^2(\Theta_X),$$

then we cannot find θ_2 satisfying the second equation in (11), and so forth. The main result in [5] is

(15) *If $H^2(\Theta_X) = 0$, then there exists a family (5) such that the Kodaira-Spencer mapping*

$$\rho_{t_0} : T_{t_0}B \to H^1(\Theta_X)$$

is an isomorphism.

These methods were extended by Kuranishi to prove the existence of a local universal family (5) where B is an analytic subvariety in an open neighborhood of the origin in $H^1(\Theta_X)$ with

$$\dim B \ge \dim H^1(\Theta_X) - \dim H^2(\Theta_X).$$

Another stability result from Kodaira and Spencer's last paper [12] on deformations of complex structure is

(16) *If X is a Kähler manifold, then the X_t are also Kähler for t close to t_0.*

As shown later by Hironaka, this result is false globally: the limit of Kähler manifolds may not be Kähler.

Before turning to some other aspects of Kodaira-Spencer deformation theory and the work it stimulated, I would like to mention one particular aspect of the one-variable work of Schiffer and Spencer that has had significant impact in current algebraic geometry. Namely, the projectivized tangent space $\mathbb{P}H^1(\Theta_X)$ to the local deformation of a smooth algebraic curve is, in a natural way, the image space for the bicanonical mapping

$$(17) \qquad \varphi_{2K} : X \to \mathbb{P}H^1(\Theta_X),$$

which one may think of as using a basis for the quadratic differentials as homogeneous coordinates. Mathematically, if for a point $p \in X$ we let $\Theta_X(p)$ denote the sheaf of vector fields with a first order pole at p, then there is an exact sheaf sequence

$$(18) \qquad 0 \to \Theta_X \to \Theta_X(p) \to \Theta_X(p)\big|_p \to 0\,.$$

In terms of a local coordinate z centered at p, sections of $\Theta_X(p)$ are

$$\theta = \left(\frac{a}{z} + b_0 + b_1 z + \cdots\right)\partial/\partial z$$

and $\Theta_X(p)\big|_p \cong \mathbb{C}$ is a skyscraper sheaf supported at p, and the right-hand mapping in (18) is

$$\theta \mapsto a.$$

The bicanonical mapping (17) then sends p to the line in $H^1(\Theta_X)$ spanned by

$$(19) \qquad \delta\left(\frac{a}{z}\frac{\partial}{\partial z}\right), \quad a \neq 0,$$

where δ is the coboundary map

$$\Theta_X(p)\big|_p \xrightarrow{\delta} H^1(\Theta_X)$$

in the exact cohomology sequence of (18). By definition, $\varphi_{2K}(p)$ is the *Schiffer variation* associated to p. From (19) we see that

$$\varphi_{2K}(p) = 0 \text{ in } \mathbb{P}H^1(\Theta_X(p)).$$

As explained to me by Don, this has the following intuitive geometric meaning: To first order, the deformation with tangent $\varphi_{2K}(p)$ arises by leaving unchanged the complex structure on $X \setminus \{p\}$ and changing it at p by a δ-function. This can all be made mathematically precise, and Schiffer variations have had a variety of important applications in contemporary algebraic geometry, including results on the global moduli of Riemann surfaces and on the higher Chow groups of algebraic cycles by Collino and others.

In addition to variation of complex structures, Kodaira and Spencer also considered deformations of holomorphic principal bundles [11] (and their associated vector bundles) and the relative case of deformations of a compact complex codimension-one submanifold within a fixed ambient complex manifold [15], [10] (the higher codimension case being done by Kodaira). In all these cases there is a Kodaira-Spencer map encoding the first-order variation, and results (including an existence theorem) analogous to those given above were established. In [11] a rich set of examples illustrating the general theory is worked out in detail.

Soon after the publication of the fundamental papers [11] there was a flood of results applying the general theory. Among these were the rigidity of simply connected homogeneous Kähler manifolds (Bott) and the rigidity of hyperbolic symmetric compact complex manifolds having no unit disc factor in their universal coverings (Calabi-Vesentini). In the case of a compact Kähler manifold, a variation of complex structure leads to a variation of Hodge structure, which has had rich applications in algebraic geometry. Moreover, the Kodaira-Spencer framework quickly spawned a number of deformation theories of other structures, including discrete subgroups of semisimple Lie groups (Calabi, Weil, and Matsushima, among others), and deformation theory of algebras (Gerstenhaber). Over the years these theories have been refined, expanded, extended, and applied. (I shall not attempt any summary.)

In the setting of general algebraic geometry, the Kodaira-Spencer theory became quickly absorbed, adapted, and greatly extended by Grothendieck and his school. Together with the complementary study of global moduli, most especially of marked algebraic curves initiated by David Mumford, one sees that some nearly fifty years after its inception, deformation theory, both local and global, is absolutely central in modern algebraic geometry and its interfaces with other areas, including string theory (quantum cohomology) and mirror symmetry (Calabi-Yau manifolds).

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Hubert Goldschmidt

I first met Don Spencer when I was a sophomore at Princeton University during the winter of 1960–61. I first encountered his vision of mathematics through his *Advanced Calculus* book [6] written with H. Nickerson and N. Steenrod, which had such an influence on a whole generation of undergraduates at Princeton, and then through an informal seminar held in his office. Don was to be my thesis advisor for my senior thesis. His enthusiasm and romantic view of mathematics, his passion for the subject permeated his teachings: research in mathematics was something to be enjoyed. His boundless energy was legendary. He also taught us to persevere in one's vision, not to be led astray and not to deviate from one's path.

I was fortunate to have been associated with Don as his student, colleague, and collaborator; to have been led by him to the forefront of an exciting new field at its very outset; and to have witnessed all its beautiful developments over the years.

In 1960 Kodaira and Spencer began extending their theory of deformation of complex structures to other classes of geometric structures. First, in [3] Kodaira considered structures corresponding to Élie Cartan's primitive pseudogroups of biholomorphic transformations. Then Kodaira and Spencer [4] studied deformations of a class of G-structures which generalize foliations on a manifold.

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Eugenio Calabi, Halsey Royden, Don Spencer, and Phillip Griffiths hiking near Palo Alto, CA, in the 1960s.

In his seminal paper [7], Spencer undertook a program to extend these results to structures corresponding to the pseudogroups of Lie and Élie Cartan. He had a complete vision about how to approach the theory of overdetermined systems of partial differential equations through a whole series of new tools and to use them to study the theory of deformations of pseudogroup structures. Most of it is described in [7]; it would take many years to fully understand, to develop and refine everything which he formulated there, and to carry out his program. Spencer's ideas exerted a profound influence on most mathematicians who worked in this field, in particular, on M. Kuranishi, R. Bott, S. Sternberg, D. Quillen, V. Guillemin, B. Malgrange, A. Kumpera, Ngô Van Quê, and me.

The pseudogroups of Lie and Cartan arise as transformation groups of geometric structures and are defined by systems of partial differential equations. Spencer's program consisted in first studying the geometric structures in terms of the associated systems of partial differential equations. Subsequently, it would lead to a deformation theory of structures on manifolds defined by pseudogroups which would incorporate all the fundamental mechanisms of the theory of deformation of complex structures. In particular, an extension of the Newlander-Nirenberg theorem to elliptic pseudogroups would be required to obtain coordinates for the deformed structures and to prove the existence of local deformations.

The first step was carried out by studying the system of linear partial differential equations associated to a pseudogroup in terms of Ehresmann's theory of jets and constructing a resolution of the sheaf of vector fields which are the infinitesimal transformations of the pseudogroup. This resolution was introduced in order to interpret

Ph.D. Students of Donald C. Spencer

Arthur Grad (1948)
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Robert Hermann (1956)
Joseph J. Kohn (1956)
Bruce L. Reinhart (1956)
Patrick X. Gallagher (1959)
Alan L. Mayer (1961)
Richard T. Bumby (1962)
Phillip A. Griffiths (1962)
Michael E. Ash (1963)
Joseph E. D'Atri (1964)
W. Stephen Piper (1966)
William J. Sweeney (1966)
Roger A. Horn (1967)
Frank W. Owens (1967)
Barry MacKichan (1968)
Laird E. Taylor (1968)
Sherman H. C. Hsiao (1972)
Robert J. V. Jackson (1972)
Robert E. Knapp (1972)
Charles Rockland (1972)
Constantin N. Kockinos (1973)
Marshall W. Buck (1974)
Suresh H. Moolgavkar (1975)
John E. Lindley (1977)
Jack F. Conn (1978)

infinitesimal deformations as elements of a cohomology group. Here, Spencer introduced fundamental tools for the theory of overdetermined systems of partial differential equations, which marks the beginning of a new era in their study. His approach—now called Spencer theory—was more in the spirit of Lie, with a full use of modern concepts; the equations are studied directly, in contrast to the existing Cartan-Kähler theory, where they are recast within the framework of exterior differential systems. It was soon realized that this work was not limited to equations arising from pseudogroups, and a systematic study of the formal theory of overdetermined systems incorporating these methods was undertaken. In fact, the methods introduced in [7] turned out to provide most of the crucial elements for the study of overdetermined systems both from the formal point of view and within the context of real-analytic systems, and we now evoke several of their aspects:

1. To a system of linear equations or to a linear differential operator, Spencer associated complexes of differential operators, the so-called naive and sophisticated Spencer sequences. Intrinsic constructions of these sequences were later given by Bott, Quillen, and Goldschmidt. The cohomology of these complexes, which is now called the Spencer cohomology of the operator (or of the equations), is a crucial ingredient in the study of overdetermined systems; in fact, it was later shown to be equal to the cohomology arising from the compatibility conditions for the given differential operator. Quillen also provided the proof of Spencer's assertion made in [7]: if the system is elliptic, the associated sophisticated Spencer sequence is an elliptic complex. This result is in fact critical for the deformation theory of structures defined by elliptic pseudogroups.

2. Essential information about a system of differential equations is contained in its δ-cohomology (now also called Spencer cohomology) groups. In fact, they are the cohomology groups of the so-called δ-complex, which first appeared as a subcomplex of the Spencer sequence. Spencer asserted that this cohomology is finite-dimensional and foresaw the implications of the vanishing of certain groups. Mumford pointed out to Spencer in 1961 that the dual of the δ-complex is a Koszul complex; this remark gave a direct proof of Spencer's assertion. This led Guillemin and Sternberg to conjecture that Cartan's notion of involutiveness, as reformulated by Matsushima, is equivalent to the vanishing of the δ-cohomology. This equivalence was proved by Serre using commutative algebra and would provide the link between Spencer's theory and Cartan's methods. It is a remarkable fact that Cartan's notion of involutiveness can be recast in terms of this cohomology. Techniques from homological algebra and commutative algebra could now provide tools for the study of overdetermined systems and for new proofs of the celebrated Cartan-Kuranishi prolongation theorem.

3. In view of proving the existence of solutions of real-analytic systems, Spencer formulated the so-called δ-estimate in terms of the δ-complex. For a real-analytic system satisfying appropriate conditions, it would give the convergence of power series solutions and thus the existence of local solutions. The δ-estimate was subsequently proved by Ehrenpreis, Guillemin, and Sternberg, and later by Sweeney. Malgrange realized that this estimate is essentially equivalent to the "privileged neighborhood theorem" of Grauert and used this theorem together with the method of majorants to prove a direct existence theorem for analytic systems (see [5]).

These methods would give rise to the basic existence theorems for overdetermined systems of partial differential equations which guarantee the existence of sufficiently many formal solutions for an arbitrary system, linear or nonlinear. In the case of real-analytic systems, the results mentioned in (3) would then provide us with the existence of local solutions.

In [8] Spencer formulated a generalization of the $\bar{\partial}$-Neumann problem for overdetermined elliptic systems on small convex domains in the hope of proving the exactness of the sophisticated Spencer sequence associated to such equations. That this exactness holds in general for such systems is now called the Spencer conjecture. The Spencer-Neumann problem was solved, and the exactness of the Spencer sequence was proved in several special cases by Spencer's students MacKichan, Sweeney, and Rockland (see [1], Chapter X). By adapting Henri Cartan's argument for the exactness of the Dolbeault sequence, in [7] Spencer also proved the exactness (in the C^∞-sense) of the Spencer sequence under the assumption that the elliptic operator has real-analytic coefficients.

This last result implies that the Spencer sequence of an elliptic pseudogroup, whose equations have real-analytic coefficients, is exact and gives rise to an interpretation of the infinitesimal deformations as elements of a cohomology group, just as in the case of complex structures. Another aspect of the methods of [7] for studying deformation theory is the recasting of the notion of an almost-structure (corresponding to the pseudogroup) as a section of one of the vector bundles appearing in a nonlinear sequence corresponding to the naive or to the sophisticated Spencer sequence. The integrability condition for the almost-structure is then formulated in terms of this sequence. The second fundamental theorem for a pseudogroup asserts the existence of coordinates for an almost-structure satisfying this requisite integrability condition. For elliptic pseudogroups whose equations have real-analytic coefficients, the second fundamental theorem was proved by Malgrange in [5]. The proof of this generalization of the Newlander-Nirenberg theorem was inspired by the one which Spencer gave for the exactness (in the C^∞-sense) of the sophisticated linear Spencer sequence and requires all of the mechanism developed by Spencer. This completed Spencer's program for studying deformations of structures initiated in [7] and showed that indeed Spencer had introduced all the essential components for the study of the deformation theory of pseudogroup structures.

In [5] Malgrange introduced the formalism of differential calculus à la Grothendieck into the study of pseudogroups and reworked the Spencer mechanism for studying deformations of structure. Also, he clarified the correspondence between the linear Lie equations and their finite forms. Many of Spencer's ideas concerning pseudogroups now appeared in a rigorous form. Malgrange's work was utilized by Goldschmidt and Spencer in their paper [2] to study the second fundamental theorem for an arbitrary pseudogroup. They introduced the nonlinear Spencer cohomology of a transitive pseudogroup, whose vanishing implies the validity of the second fundamental theorem for the pseudogroup. By means of a spectral sequence argument involving the nonlinear sequences, they showed that this nonlinear Spencer cohomology depends only on the Lie algebras of all formal infinitesimal transformations of the pseudogroup. Many of the main ingredients of their solution of the "integrability problem" for flat pseudogroups appear here. In fact, their results concerning Spencer cohomology allowed them to use Guillemin's Jordan-Hölder decompositions and Galois theory type methods to provide a proof of the second fundamental theorem for all transitive pseudogroups on Euclidean space containing the translations, the so-called flat pseudogroups.

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Enrico Bombieri

The earliest mathematical works of Don Spencer, reflecting the influence of his advisor J. E. Littlewood and Littlewood's collaborator, G. H. Hardy, were in analytic number theory. A particularly important problem on which Spencer worked was to show that a progression-free sequence of positive integers cannot have positive density. The paper [1], written jointly with Raphaël Salem, disproved in a clever fashion a then widely held conjecture. The particular problem has been of lasting interest, with major progress by Szemerédi and quite recently by Gowers, who was awarded a Fields Medal in part for his work on it. In addition to this

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