Lajos Pukánszky (1928–1996)

Jacques Dixmier, Michel Duflo, András Hajnal, Richard Kadison, Adam Korányi, Jonathan Rosenberg, and Michèle Vergne

Lajos Pukánszky was born in Budapest on November 24, 1928. He defended his Ph.D. thesis, which was written under the direction of Béla Sz. Nagy, in 1955 in Szeged, Hungary. He left Hungary in 1956. After taking several posts in the United States and France, he was a professor at the University of Pennsylvania from 1965 until his retirement. He died February 15, 1996, in Philadelphia. The mathematical community has lost one of its stars.

Niels Pedersen, for whom a memorial article appears in the January 1998 *Notices*, was Pukánszky's de facto student. As that article says, "[Pedersen's] interaction with Pukánszky was especially deep and fruitful. They became good friends and remained so throughout their lives."

Below are five commentaries on aspects of Pukánszky's mathematical life and mathematics.

—the authors

Jonathan Rosenberg

I was privileged to know Lajos Pukánszky since 1976. In many ways his personality was like his mathematics: based on a broad knowledge base, but single-minded, uncompromising, and very deep. Lajos was without a doubt the world's foremost expert on solvable Lie groups. But his magnum opus [8], which occupied an entire issue of the *Annales Scientifiques de l'École Normale Supérieure*, was probably never read by more than a dozen people. It stands as a lofty but isolated peak on the mathematical landscape—admired by many from afar, but scaled only by a few diehards. This was the brilliance, but also the tragedy, of Lajos Pukánszky—that his life's work was so perfect, yet so inaccessible and underappreciated by the mathematical public.

Lajos was a truly cultured person in a sense which we may not see again in future generations. He seemed to know all of European history, philosophy, and literature by heart. I think he identified himself with the protagonist of Thomas Mann's *Doctor Faustus*, a work which he discussed with me quite a number of times. But the best quick summary of his character may be found in the quotation which he attached to the beginning of his big *École Normale* paper. It is a quote within a quote within a quote: Schiller, quoted by Bohr, quoted by Heisenberg. Roughly translated, part of it says "...and in the abyss lies truth." Pukánszky's life work was the probing of this abyss.

Jonathan Rosenberg is professor of mathematics at the University of Maryland. His e-mail address is jmr@math.umd.edu.

This segment of the article was read at a memorial service in Philadelphia in 1996.

András Hajnal and Adam Korányi

Lajos Pukánszky was a good friend of ours. There were long periods when we did not meet, but each time we resumed contact we continued our conversation as if we had stopped just the day before. Between 1953 and 1956 as beginning mathematicians we regularly had lunch together in the restaurant of the University of Szeged and went afterwards to have coffee in an espresso bar; G. Pollák and, during the last year, I. Kovács were also part of the group. Then as later Lajos was a good friend always ready to help, but this did not mean that anyone could persuade him to waste his time. He maintained the strictest standards both in his private life and in his scientific work. He demanded competence in everything. Woe to him who made

András Hajnal is professor of mathematics at Rutgers University. His e-mail address is ahajnal@math.rutgers.edu. *Adam Korányi is professor of mathematics at City University of New York, Lehman College.*

This segment is a slightly abridged English translation of the original as it appeared in the Hungarian journal Matematikai Lapok.

a statement about Thomas Mann but turned out to be unfamiliar with Mann's correspondence with Karl Kerényi. He organized his life so as to be able to do mathematics at the highest possible level. Following a strict schedule, fighting chronic insomnia, he worked, so to speak, beyond his forces even though he was also able to relax and amuse himself among friends. But this he permitted himself only on designated rest days. When we teased him, saying that like Anatole France's Sylvestre Bonnard he was looking for "l'austère douceur du sacrifice",[1] he only smiled and continued with his work.

Coming back to Thomas Mann, who was a frequent subject of conversation in Szeged, we all saw the similarity between Lajos and the hero of Mann's novel *Doctor Faustus*. This not only concerned his personality, it also included the fact that Lajos too was interested, besides mathematics, in music and theology above all. Music was important to him all his life. And he studied theology very seriously: Catholic theology in his youth and Jewish when he was old, even though he was not religious, at least not in the period we knew him. Anyway, we never dared to ask him a direct question about this.

He left Hungary in the turmoil following the 1956 uprising, together with one of us. His purpose was to get access to the great mathematical centers of the world. It followed from his character that he cultivated those parts of mathematics that he judged to be of the most central importance, which were also those that required the greatest amount of knowledge. He worked alone, but beginning in 1953, he corresponded with Jacques Dixmier and regularly discussed his mathematical projects with him. The dedication of his article written for the retirement of Dixmier is the line of Horace, "O et praesidium et dulce decus meum."[2]

He did not like to travel, and above all he hated to be in the limelight. Pleading ill health, he did not go to the international conference organized by Niels Vigand Pedersen in Copenhagen for his sixtieth birthday. (But back in 1964 he did come from Los Angeles to visit the two of us in Berkeley, where we happened to be at the time.)

Beginning in 1994, the three of us again lived fairly near each other. Usually we met in New York, where Lajos insisted on a strict ritual: meeting at 2:00 p.m. in the Metropolitan Museum of Art in front of Michelangelo's portrait, later dinner in the Hungarian restaurant Mocca, from where he rushed away by taxi to get his train. He was his old self; even his memory was almost the same as before, although he was not able anymore to tell, as in Szeged, exactly what he had been doing on which day of which year.

[1] *"the austere sweetness of sacrifice".*

[2] *"O my safety and my sweet honor."*

Photograph courtesy of Béla Pukánszky.

Lajos Pukánszky

He had long been sick, but he did not complain much. He worked as before. Four chapters—that is, a very large part—of his projected new book were found fully completed after his death. He suffered from severe anemia, and his physicians were unable to find its cause or its cure. In the last two or three weeks of his life he was very weak; we found out about this only later. It seems he could not go on anymore, and we did not watch him carefully enough. Now there is only his memory for us to guard.

Michèle Vergne

I met Pukánszky for the first time in 1970. After 1980 I never saw him again, but I have kept an image of him in my memory. The news of his deliberate death affected me. I do not want to believe that I shall never again see his thin, worried silhouette in Paris.

I would like to relate some memories of the years 1970–76. In 1970 I was young, I felt like a nobody, and I was suffering from it. He appeared to me as a somebody. One of my first research articles consisted in giving a simpler proof of the "Pukánszky irreducibility criterion". This was work undertaken under the aegis of Dixmier, in an active group made up of Nicole Berline, Pierre Bernat, Michel Duflo, Monique Levy-Nahas, Mustapha Raïs, Rudolf Rentschler, Pierre Renouard, and others. Representation theory of solvable Lie groups was

Michèle Vergne is Directeur de Recherches at the Centre National de la Recherche Scientifique (CNRS). Her e-mail address is `vergne@dmi.ens.fr`.

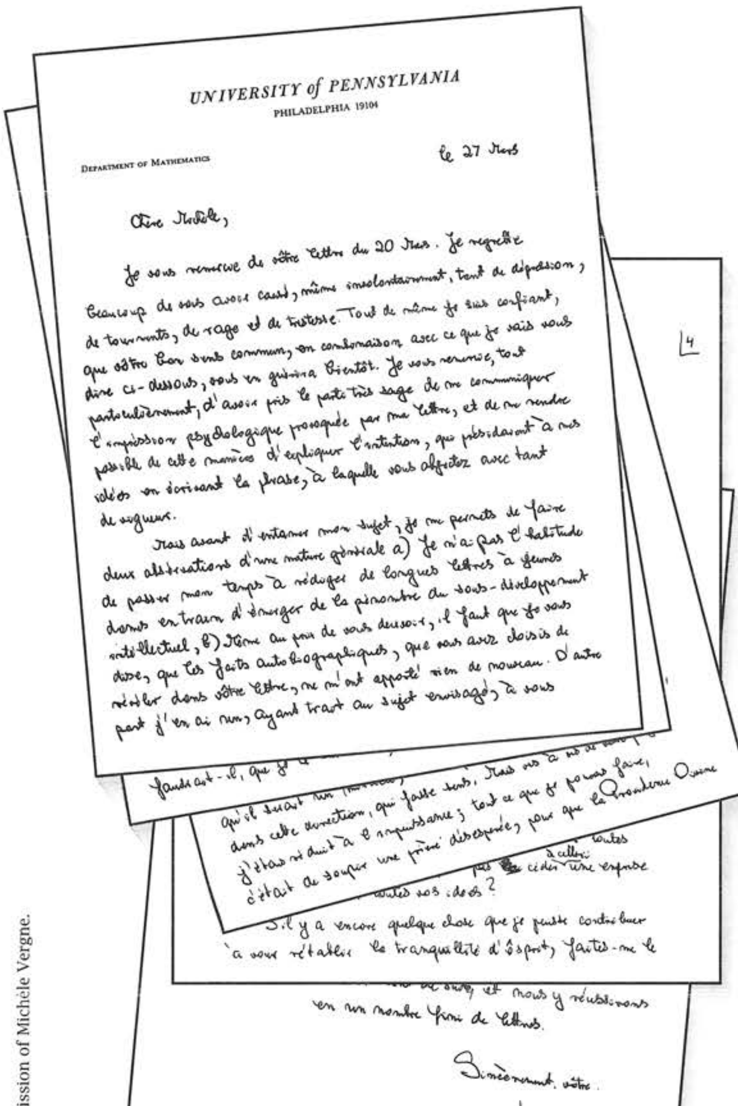
in full bloom at this time. Pukánszky, through his many articles on representations of nilpotent and solvable Lie groups, contributed to this flowering. I had studied his work on this subject. I had my small place in the middle of the "Dixmier family", but it seemed to me that I did not count for much. Then, how happy I was when Pukánszky would came to Paris! He surrounded me with a completely refined and exaggerated kindness, but so satisfying since the tributes were directed toward me.

Under the influence of the revolutionary atmosphere of May 1968, I was a "leftist". In principle I should have loathed Pukánszky, since he was the very representative of the decadent bourgeoisie targeted by the "permanent" antiestablishment activity of May 1968. Certain students had abandoned their studies to promote revolution by working in the factories. Claude Chevalley, my thesis advisor, had gone to Vincennes, an interdisciplinary noncompetitive university that accepted workers. The people alone were pure. These things were burning convictions for me. Despite everything I allowed myself to be invited by Lajos Pukánszky to excellent restaurants, I walked with him in Paris, I found everything amusing, and he would always tell me that I acted like a child. I found his company amusing, his conversation penetrating. He said to me, "You must be really feverish in order to find life so much fun." But he was feverish too. He was nervous, agitated, worried, and thin as a rail. He was an insomniac, spent whole nights without being able to sleep; his hotels were always too noisy. This difficulty in living made me feel for him. He spoke excellent French, which he embellished by quoting from Molière, Racine, La Bruyère, Anatole France. He was necessarily right whenever one disputed a definition or a grammatical point. He returned with the dictionary and a long discourse. This was sometimes amusing, sometimes irritating. He often repeated to me about mathematical work, "It is not necessary to hope in order to undertake, nor to succeed in order to persevere," and his persistence in inciting my perseverance put me in a really bad mood.

Starting in 1967, Lajos Pukánszky was interested in difficult problems about representations of solvable Lie groups. He had immediately understood that the orbit method proposed by Kirillov in 1962 could also be very fruitful in the study of representations of solvable Lie groups. He was particularly interested in groups that are not of type I, for which he produced factor representations resembling packets of coadjoint orbits.

In 1974 I was supposed to explain in the Séminaire Bourbaki the recent work of Auslander-Kostant and Pukánszky on the irreducible unitary representations of solvable Lie groups. This lecture and article were difficult to prepare, and I sent a preliminary version of the article to Pukánszky to ask his opinion. He wrote me a long letter on this subject in which he gave me advice on editing it, not always kind, but certainly completely fair. "I think that the host of details that you display here is going to bore rather than enlighten the novice." He added some corrections, many suggestions. I protested about certain parts of his letter; the phrase "you have come quite a long way since our last meeting" especially irritated me. Here is a portion of his response that shows what importance he attached to the study of groups from the point of view of C^*-algebras.

"I remember to this day very clearly the questions that you asked me in Williamstown, but I have


UNIVERSITY of PENNSYLVANIA
PHILADELPHIA 19104

DEPARTMENT OF MATHEMATICS

le 27 Mars

Chère Michèle,

Je vous remercie de votre lettre du 20 mars. Je regrette beaucoup de vous avoir causé, même involontairement, tant de dépression, de tourments, de rage et de tristesse. Tout de même je suis confiant, que votre bon sens commun, en combinaison avec ce que je vais vous dire ci-dessous, vous en guérira bientôt. Je vous remercie, tout particulièrement, d'avoir pris le parti très sage de me communiquer l'impression psychologique provoquée par ma lettre, et de me rendre possible de cette manière d'expliquer l'intention, qui présidait à mes idées en écrivant la phrase, à laquelle vous objectez avec tant de vigueur.

Mais avant d'entamer mon sujet, je me permets de faire deux observations d'une nature générale a) Je n'ai pas l'habitude de passer mon temps à rédiger de longues lettres à jeunes dames en train d'émerger de la pénombre du sous-développement intellectuel, b) Même au prix de vous decevoir, il faut que je vous dise, que les faits autobiographiques, que vous avez choisis de révéler dans votre lettre, ne m'ont apporté rien de nouveau. D'autre part j'en ai un, ayant trait au sujet envisagé, à vous

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S'il y a encore quelque chose que je puisse contribuer à vous rétablir la tranquillité d'esprit, faites-moi le

et nous y réussirons en un nombre fini de lettres.

Sincèrement, vôtre.

L. Pukánszky

Letter from Pukánszky to Michèle Vergne, part of which is quoted on the following page.

not thought about them until now. Consequently in connection with this article, I limit myself to making you know that I would have no respect for a mathematician who did not dare ask questions for fear of seeming naive. But a thing of which I am reminded is that you did not show a particularly favorable opinion about the usefulness of C^*-algebras, or any interest in factors that are not of type I, etc. I was a bit saddened by this, since I have been convinced, ever since my study of Godement's work soon after it appeared, that these things are absolutely indispensible for understanding representation theory for general Lie groups, in a way that becomes evident sooner or later to each person who is interested in this study. Furthermore, I was under the impression that you were only repeating current opinion, which is based on reasoning roughly as follows: I see X and Y who have become great men and do not know what a continuous factor is. So why is it necessary for me to know it?

"Since the conference in question, I have spoken about my recent work with several colleagues who are professionals in the theory of C^*-algebras, and despite some compliments that I have received, I have had to notice that their efforts to understand this work have been crowned by absolute failure.

"In view of that, my reaction to your letter of December 20, in which, you recall, you made an assessment of the results of my recent note (in which C^*-algebras play a decisive role), was a mixture of surprise and quite sharp disbelief. Finally, for lack of a better explanation, I came to believe that the views expressed there were sincere, but that they were probably due to the influence of somebody else's opinion (but whose?). Yet in this case the conclusion that a very profound change is operating in the set of your views of the importance of the above objects has become inescapable. Whence surely 'you have come quite a long way since our last meeting.' But at that time, there was nothing in the world that I desired less than that you would persist. In fact, I figured that it would take a miracle for you to be able to say something in this direction that would make sense. But, with respect to your project, I was completely powerless; all that I could do was sigh a desperate prayer that Divine Providence would prevent you from leaving the confines of type I phenomena.

"Finally when I received your most recent letter and your article, one glance was enough for me to see that the miracle had taken place: everything relative to the feared domain was in its place in perfect order, organized in a truly professional manner. At this point the only regrettable thing, as I have already explained to you, was that you have not harvested the fruit whose filled tree you planted before your audience in a marvelous way. To be sure, there were errors, but they could be fixed locally. ...If your current interest does not permit you to persist on this road, I am confident that despite your certainly quite advanced age you will be able to return to it perhaps as soon as next year."

I continued to correspond with him for a long time. He sent me very long letters in perfect French. His attitude, which was a mixture of exaggerated politeness and mocking skepticism, made me react quickly. I responded at length, explaining in great detail all my concerns, my ups, my downs. I needed him, his support, his recognition. He always gave me his attention without stinting. Beneath his irony he was always full of tact and kindness. He never spoke of himself; I think that he sensed himself too particular and did not want to impose on others his day-to-day problems. As for me, I was never preoccupied with his concerns and needs (and he never spoke of them). I have tried to say here how tactfully and how charmingly he knew how to maintain a deep and friendly relationship. Thanks to his letters, I keep an unfailing memory of him.

Richard Kadison

I first heard of Lajos through his early work on "rings of operators", or "von Neumann algebras" as they are now known. Is Singer and I had been trying an idea we had to produce two nonisomorphic factors of type III. It seemed like a good idea, but we got stopped by some technical order-of-choice difficulty. That was in 1956. We filed our work on this away—we had other things to attend to—with the thought of coming back to it in a while. Later in 1956 I received a reprint from Lajos—my first knowledge of him. He had had the same idea and pushed it through successfully. Where Is and I had displayed one of our factors in a general algebraic-analytic way, Lajos displayed it in a very specific analytic-ergodic-theoretic manner. He was able to navigate his way around our difficulty with clever and powerful analytic techniques. We took note of him.

A little later that year I was giving a talk to the AMS at a New York meeting, and I mentioned that work of Lajos. After the talk two chaps came up to me and asked if I knew Lajos personally, since I had mentioned his name. I did not, but nonetheless they asked me if I would help them get him a special visa. It seems that he had taken part in the

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Richard Kadison is Kuemmerle Professor of Mathematics at the University of Pennsylvania. His e-mail address is `liming@math.upenn.edu`.

This segment is adapted from remarks made at a memorial service at the University of Pennsylvania in 1996. The setting was the department's Common Room, where the faculty, staff, and a few of Pukánszky's friends were gathered; the paragraphs are best read with that setting in mind.

Hungarian uprising, crushed quickly by Soviet tanks. Lajos had fled to a refugee camp (in Yugoslavia, I think). Those chaps had been trying to get him a visa so that they could hire him at a recently created institute (RIAS, the Research Institute for Advanced Studies) in Baltimore. Of course I was willing to write a letter. It may have added an infinitesimal extra breath to their sails, and Lajos wound up at RIAS in 1957, where he worked as a research associate for three years.

Early in that period Lajos visited me at Columbia. We had a wonderful day together talking mathematics. In 1960 Lajos took an assistant professorship at the University of Maryland. Leon Cohen had become chair of the Maryland department. I remember that he was eager to keep Lajos there; he asked me to use whatever influence I might have to get Lajos to stay. I was fond of Leon, but of course I would not meddle in that sort of decision. The lure of the West Coast was too strong, and Lajos accepted a visiting assistant professorship at Stanford in 1961. In 1962 he was hired by UCLA to a regular assistant professorship. The following year they promoted him to tenure and an associate professorship. In 1964 our group in functional analysis was forming here at Penn. Lajos was recruited as a full professor and returned to the East Coast. How's that for a meteoric rise?

Lajos spent his first year on leave, accepting an invitation to Paris. He worshipped Dixmier, and Dixmier had high regard for Lajos's mathematics. I remember meeting Lajos in Paris that spring. We had dinner together at some place on Boulevard St. Germain. The meal was poor, but the company was splendid. Lajos had learned French that year and insisted on our speaking French together that evening. It was definitely a case of "the blind leading the blind"!

I will not go into all the details of how the speakers for the Nice Congress for our section were chosen. Dixmier ran it; it was the first time that our area was included in an explicit way. Dixmier, one of the fairest and most democratic people I know, canvassed a very large number of important functional analysts by mail, asking them to cast a vote. Pukánszky was the leading candidate for those speaking positions by a large margin.

Many of you will remember Lajos as quiet and reclusive. His friends knew that he had a lively, dry, and wry sense of humor. Sitting at lunch with a good-sized group (about twenty years ago), one of our very bright young research instructors was emphasizing the point that someone had won the Putnam Exam Prize. This instructor seemed to feel that that was a major mathematical credential. An active debate ensued. Lajos listened silently. After a number of minutes and much discussion, Lajos asked a question: "What does doing mathematical puzzles seated on top of a locomotive going 100 miles an hour have to do with being a mathematician?" That seemed to end the debate.

Lajos remained a professor here until his retirement a few years ago. He will be remembered with affection, reverence, and respect by those of us who knew him.

Jacques Dixmier and Michel Duflo

Pukánszky's impressive mathematical œuvre bears witness to a considerable cumulative effort. It is centered in the theory of unitary representations. Despite how focused this mathematics is, Pukánszky showed an immense mathematical knowledge; he used in his papers not only the entire arsenal of functional analysis and the theory of Lie groups but also some quite varied tools: connections, resolution of singularities, partial differential equations, division of distributions, homology, and others.

Pukánszky's early work was on von Neumann algebras and related subjects. The article [1] made an early name for him: in it one finds the construction, by quite an ingenious method, of two nonisomorphic factors of type III. Although this result was later greatly extended, it represented at the time a major advance. Pukánszky established also some properties of maximal abelian subalgebras of type III factors that were known previously only for factors of type II_1. This article had the honor of being reviewed by F. J. Murray in *Mathematical Reviews.*

Most of Pukánszky's subsequent work was devoted to unitary representations of Lie groups. He began by studying tensor products of representations in the context of the inhomogeneous Lorentz group and of $SL(2,\mathbb{R})$. Then he classified the irreducible unitary representations of the covering group of $SL(2,\mathbb{R})$, finding also an explicit Plancherel formula for this group [2].

Let G be a connected real Lie group. The classes of irreducible unitary representations of G form a set denoted $\widehat{G}$. The determination of $\widehat{G}$ for G noncompact semisimple was begun about 1947 and has been pursued to the present day. If one wants to study $\widehat{G}$ for arbitrary G, one of the first things to do, therefore, is to consider the case where G is solvable. In 1962 Kirillov's orbit method completely settled the case that G is nilpotent: $\widehat{G}$ is then identified with $\mathfrak{g}^*/G$, where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}^*$ is the dual of $\mathfrak{g}$ with the coadjoint representation. In [3] Pukánszky makes a major addition to Kirillov's results: he shows that, for G nilpotent, in order to calculate the charac-

Jacques Dixmier is professor of mathematics, retired from Université Paris VI. Michel Duflo is professor of mathematics at the Université Paris VII and the Ecole Normale Supérieure in Paris. His e-mail address is `Michel.Duflo@ens.fr`.

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ter of a representation associated to an orbit $\Omega \subset \mathfrak{g}^*$, one can use the measure on Ω induced by the symplectic structure, up to a constant factor that depends only on the dimension of Ω. He then finds that the Plancherel measure on $\widehat{G} = \mathfrak{g}^*/G$ is defined by a rational differential form. The book [4], based on a course given in Paris in 1964–65, expounds the entire Kirillov theory with some additions and with proofs that are simplified or new. This very clear book has attracted a number of young researchers to the theory of unitary representations.

Starting in 1967, in a series of papers that do honor to his tenacity, Pukánszky passed from the nilpotent case to the solvable case. This passage encounters a host of new difficulties.

Let G be a simply connected solvable Lie group, $\mathfrak{g}$ its Lie algebra. One says that G is *exponential* if the map exp is a bijection of $\mathfrak{g}$ onto G. For such a group Bernat had determined $\widehat{G}$ by extending Kirillov's method. In the construction of the irreducible representations of G, one must choose, for a linear form l on $\mathfrak{g}$, a subalgebra of $\mathfrak{g}$ *subordinate* to l. This choice presents no difficulty for G nilpotent. For G exponential, Bernat's method is complicated. Pukánszky gives in [5] a very clear method, introducing a geometric condition that has continued to play a large role in the subsequent development and was soon called the *Pukánszky condition*.

In [6] Pukánszky again considered the case where G is exponential. Let T be in $\widehat{G}$, corresponding to an orbit Ω. Among the many results of this article, let us note only the one that permits, when Ω is closed, the calculation of the character of T, in the following form. If $\phi \in C_c^\infty(G)$, T_ϕ is a Hilbert-Schmidt operator, and

$$\mathrm{Tr}(T_\phi^* T_\phi) = \int_\Omega \psi(l)\, d\nu(l),$$

where $d\nu$ is the canonical measure on Ω and where ψ is obtained in the following three steps:

1) one multiplies $\widetilde{\phi} * \phi$ by

$$\lambda(\exp l) = \Delta(\exp l)^{\frac{1}{2}} \prod_{\alpha \in \mathcal{F}} \frac{\exp\left(\frac{1}{2}\alpha(l)\right) - \exp\left(-\frac{1}{2}\alpha(l)\right)}{\alpha(l)},$$

where $\mathcal{F}$ is a certain set of roots of G and where Δ is the modular function of G;

2) one transports the resulting function to $\mathfrak{g}$ by means of the exponential map; and

3) one takes the Fourier transform to obtain a function on $\mathfrak{g}^*$.

However, for technical reasons, Pukánszky must suppose that $\mathfrak{g}$ is algebraic.

In [7] G is still connected, simply connected, and solvable. The Lie algebra $\mathfrak{g}$ is assumed algebraic, but G is not assumed exponential. (This situation is therefore not far from the case of solvable groups of type I, for which $\widehat{G}$ had been determined a little earlier by Auslander and Kostant.) Thanks to analysis even more difficult than in [6], Pukánszky shows that if $T \in \widehat{G}$ corresponds to a closed orbit, the character of T may be calculated almost as in the exponential case. As exp is no longer bijective, it is necessary to suppose that the function denoted ϕ above has its support in a certain open subset of G defined in terms of the roots of G and independent of the choice of T.

In [8] G denotes an arbitrary simply connected solvable group. Pukánszky associates to each orbit a family of semifinite factor representations parametrized by a torus. If all orbits are locally closed (as is the case if G is of type I), these representations are enough to carry out a central decomposition of the regular representation. But if certain orbits are not locally closed, a generalization of orbits is essential: Pukánszky introduces *quasi-orbits*, which are kinds of packets of orbits, on which G acts ergodically. As in the Auslander-Kostant theory, if Ω is a quasi-orbit, there exists a canonical principal fiber bundle $B(\Omega) \to \Omega$, whose structure group is a torus and which is a G-space. The closures of the G-orbits in $B(\Omega)$ are called *generalized orbits* in a later paper. It is to the generalized orbits that Pukánszky associates semifinite factor representations that he calls central and that permit the decomposition of the regular representation. One of the striking corollaries is that the regular representation is entirely of type I or entirely of type II; this dichotomy was absolutely unforeseen. The introduction of [8] contains conjectures, some of which anticipate deep results of Connes concerning the relationship between Lie group representations and injective factors.

Let G be a locally compact group. Let $R(G)$ be the von Neumann algebra on $L^2(G)$ generated by the left translations. It has been known for a long time (Godement and Segal) that, for G unimodular, $R(G)$ is semifinite and that a canonical trace can be constructed on $R(G)$. In the announcements of [8] Pukánszky says that if G is connected solvable (not necessarily unimodular), $R(G)$ is semifinite and that a "quasicanonical" trace can be constructed on $R(G)$. Dixmier proved in 1969 that $R(G)$ is semifinite for G any connected Lie group whatsoever. However, Dixmier's proof was incomplete, and in the difficult memoir [9] Pukánszky established the results needed to complete this proof.

Pukánszky completes the article [8] in [10]. Let G continue to be a simply connected solvable group. For every $T \in \widehat{G}$, let $\ker T$ be the kernel of T in the C^*-algebra of G. Let Prim G be the set of primitive ideals in $C^*(G)$. Let $\widehat{G}_{cent}$ be the set of quasi-equivalence classes of central representations of G. Then $T \mapsto \ker T$ is a bijection of $\widehat{G}_{cent}$

onto Prim G. In [11], to which we are going to return, it is proved that the central representations are exactly the traceable factor representations. The results of [8, 10, 11] furnish, from a certain point of view, a complete description of harmonic analysis on G.

The article [11] marks the start of a new cycle in Pukánszky's work, attacking now arbitrary connected Lie groups; such a program was probably his ambition from the beginning.

Thus, let G be a connected Lie group. Let $G^\cap$ be the set of quasi-equivalence classes of factor representations of G. The map $T \mapsto \ker T$ is a surjection δ of $G^\cap$ onto Prim G. Let $G^\cap_{tr}$ be the set of elements of $G^\cap$ that are traceable. Then δ induces a bijection of $G^\cap_{tr}$ onto Prim G. This result (which, according to Guichardet, is false for an arbitrary locally compact group), obtained after a rather formidable proof, is a major accomplishment in representation theory. (A part of the above work is summarized in [13] and [17].)

Let G be a separable locally compact group. One says that G is a CCR group if, for every $T \in \widehat{G}$ and every $\phi \in L^1(G)$, $T(\phi)$ is compact. In [12] Pukánszky gives a geometric characterization, in terms of orbits, of simply connected CCR groups. Such a group, assumed to have no semisimple direct factor, has a cocompact radical (but this condition is far from being sufficient), whence the title of the article. The result is a consequence of more general theorems. A factor representation T of G is said to be GCCR (G for "generalized") if every element of $T(C^*(G))$ is "compact" in the sense of the factor generated by $T(G)$. This said, under the assumption that G has no semisimple direct factor, the following conditions are equivalent:

(i) every point of Prim G is closed,

(ii) every traceable factor representation of G is GCCR,

(iii) every irreducible traceable representation of G is CCR,

(iv) the radical of G is cocompact and its roots are purely imaginary,

(v) the orbits of G satisfy a certain geometric condition (whose description is too long to be given explicitly here).

Pukánszky characterizes also, among all simply connected Lie groups with cocompact radical, those that are of type I.

The articles [14, 15, 16], and [18] are entirely geometric. Let G be a simply connected solvable Lie group, $\mathfrak{g}$ be its Lie algebra, and Ω be a coadjoint orbit. Pukánszky shows in [14] that the canonical mapping $\Lambda\pi_1(\Omega) \to H_*(\Omega, \mathbb{Z})$ is bijective, Λ denoting the exterior algebra. Let ω_Ω be the canonical symplectic form on the orbit Ω, g be in Ω, G_g be its stabilizer in G, G_{go} be its identity component, $\mathfrak{g}_g$ be its Lie algebra, and χ_g be the character of G_{go} with differential $-2i\pi g|\mathfrak{g}_g$. If $a, b \in G_g$, let α, β be the corresponding elements of $\pi_1(\Omega) \simeq G_g/G_{go}$. Then $\chi_g(aba^{-1}b^{-1}) = \exp 2i\pi(\alpha \wedge \beta, [\omega_\Omega])$, where $[\omega_\Omega]$ is the image of ω_Ω in $H^2(\Omega)$. This reproves the theorem of Kostant saying that χ_g extends to a character of G_g if and only if $[\omega_\Omega]$ is integral. The principal application is to the quasi-orbits studied in [8]. Let Ω' be such a quasi-orbit. Then one has an isomorphism $\Lambda\pi_1(\Omega') \to H_*(\Omega', \mathbb{Z})$. In general there does not exist a canonical 2-form on Ω', but Pukánszky constructs "admissible" elements ω of $Z^2(\Omega')$; here "admissible" means that ω, restricted to each coadjoint orbit Ω containing Ω', is equal to ω_Ω. Recall that in [8] Pukánszky introduced a principal fiber bundle $B(\Omega') \xrightarrow{\tau} \Omega'$ whose structure group is the dual $\widehat{\Pi}$ of a certain subgroup Π of $\pi_1(\Omega')$; $B(\Omega')$ is a G-space, and the generalized orbits are the closures of the G-orbits in $B(\Omega')$. Pukánszky constructs a remarkable section of τ, and one of the consequences is that if $[\omega]$ can be chosen integral, the restriction of τ to a generalized orbit is a bijection onto Ω'.

In [15] Pukánszky introduces Hamiltonian G-foliations. These are generalizations of Kostant's transitive Hamiltonian G-spaces. The generalized orbits are Hamiltonian G-foliations, and Pukánszky characterizes among the Hamiltonian G-foliations those that are isomorphic to generalized orbits. The moment map is defined in this context. However, as is shown by examples, many phenomena in the transitive case do not extend to the general case.

The article [16] has as its goal the generalization of certain results of N. V. Pedersen (Pukánszky's de facto student). Let G, g, Ω be as above. One supposes that ω_Ω is integral, which allows the construction of a fibration by complex lines L on Ω and of all the machinery of Kostant (prequantization, quantization). One supposes that $\mathfrak{g}$ admits a real G-invariant polarization. Then the quantization defines an isomorphism between $\mathcal{E}^1$ (a certain subalgebra of the Poisson algebra of Ω) and the Lie algebra of differential operators of order ≤ 1 on a certain space of sections of L. In Pukánszky's generalization, ω_Ω is no longer assumed integral, and thus it is necessary to replace Ω by a "standard sheet" of Ω. Then there exists a complex polarization of $\mathfrak{g}$ for which the announced result of Pedersen, suitably modified, remains true. Pukánszky deduces from this that if Ω is simply connected, there exist global Darboux coordinates on Ω (a result that Pedersen had established for G exponential). The article [18] gives a quite different proof for the existence of Darboux coordinates. It is based on a theorem valid when G is an arbitrary Lie group: in the presence of a certain ideal $\mathfrak{m}$ of $\mathfrak{g}$, Ω becomes a principal fiber bundle with structure group $\mathfrak{m}^\perp$, canonically isomorphic to a subbundle of the cotangent bundle $T^*\Omega$.

As a result of his remarkable mathematics, Pukánszky very soon attained an international reputation. His subsequent work lived up to the high early expectation. He gave an invited address at the International Congress of Mathematicians in Nice in 1970. In 1988 a conference entitled "The Orbit Method in Representation Theory" was held at the University of Copenhagen in honor of his sixtieth birthday. The proceedings of the conference were published by Birkhäuser as volume 82 of the Progress in Mathematics series.

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