André Weil: A Prologue

Anthony W. Knapp

André Weil, one of the truly great mathematicians of the twentieth century, died in Princeton on August 6, 1998. Weil succeeded in being a universal pure mathematician, making notable advances in at least eight of the nineteen areas of pure and applied mathematics recognized by the International Mathematical Union: algebra, number theory and arithmetic algebraic geometry, algebraic geometry, differential geometry and global analysis, topology, Lie groups and Lie algebras, analysis, and history of mathematics. He is known also as cofounder, with Henri Cartan and others, of the Bourbaki group, whose books put a large portion of basic modern mathematics on a sound footing and helped revitalize French mathematics in the wake of two world wars.

Photograph courtesy of Sylvie Weil.

André Weil in Princeton working with his cat Catsou, November 1960.

The breadth of Weil's mathematics is extraordinary. Here are just a few high points: the fundamental Mordell-Weil Theorem for elliptic curves, the construction of the Bohr compactification in the theory of almost periodic functions, the development of harmonic analysis on locally compact abelian groups, a proof of the Riemann hypothesis for curves over finite fields, the introduction of fiber bundles in algebraic geometry, the formulation of the Weil conjectures for the number of points on a nonsingular projective variety, the derivation of a high-dimensional Gauss-Bonnet formula jointly with Allendoerfer, the introduction of the Weil group in class field theory as a tool more useful than the Galois group, a Cauchy integral formula in several complex variables that anticipates the Silov boundary, and the number theory of algebraic groups. In addition, he did foundational work on uniform spaces, characteristic classes, modular forms, Kähler geometry, the use of holomorphic fiber bundles in several complex variables, and the geometric theory of theta functions. Weil's own commentary on his papers may be found in his *Collected Papers.*[1]

A substantial portion of Weil's research was motivated by an effort to prove the Riemann hypothesis concerning the zeroes of the Riemann zeta function. He was continually looking for new ideas from other fields that he could bring to bear on a proof. He commented on this matter in a 1979 interview:[2] Asked what theorem he most wished he had proved, he responded, "In the past it sometimes occurred to me that if I could prove the Riemann hypothesis, which was formulated in 1859, I would keep it secret in order to be able to reveal it only on the occasion of its centenary in 1959. Since 1959, I have felt that I am quite far from it; I have gradually given up, not without regret."[3] An example of his bringing ideas from other fields to bear on a proof is his fascination with the Lefschetz fixed-point formula. In a 1935 paper he used this formula to give new proofs of some structure-theoretic theorems about compact con-

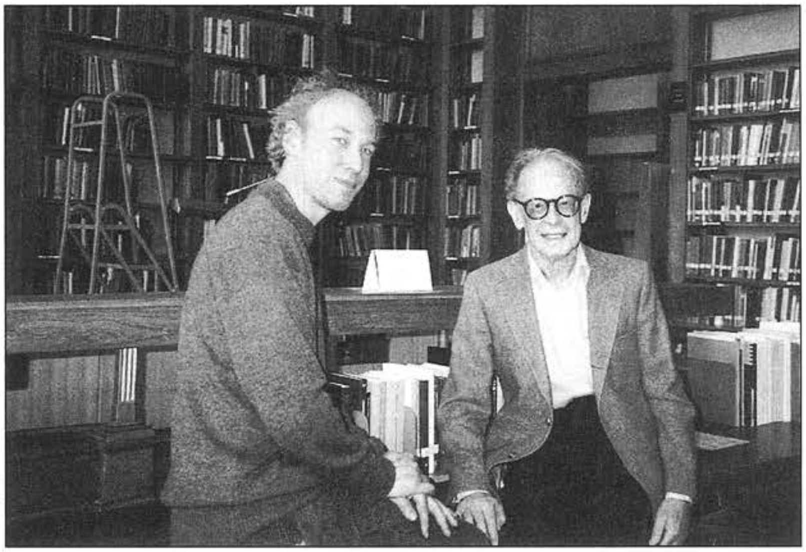
Anthony W. Knapp is editor of the Notices.

The author is grateful for help from Armand Borel, Komaravolu Chandrasekharan, Élisabeth Lutz, and Takashi Ono.

[1] Œuvres Scientifiques, Collected Papers, *vol. I–III, Springer-Verlag, New York, 1979.*

[2] Pour la Science, *November 1979.*

[3] *"Autrefois, il m'est quelquefois venu à l'esprit que, si je pouvais démontrer l'hypothèse de Riemann, laquelle avait été formulée en 1859, je la garderais secrète pour ne la révéler qu'à l'occasion de son centenaire en 1959. Comme en 1959, je m'en sentais encore bien loin, j'y ai peu à peu renoncé, non sans regret."*



Photograph courtesy of Sylvie Weil.

Weil with Pierre Deligne in the library at the Institute for Advanced Study, about 1990.

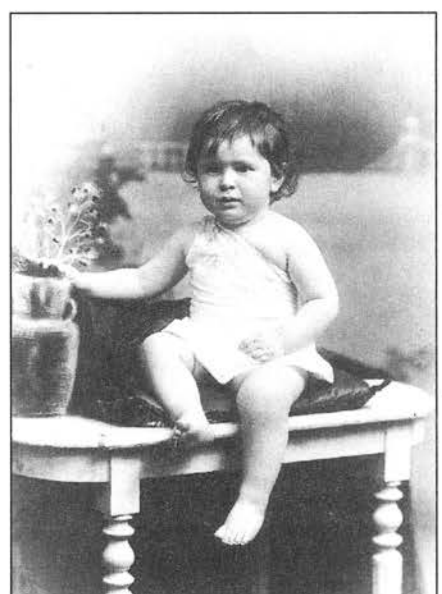
nected Lie groups. Later, in 1949, he evidently had in mind a conjectural analog of the Lefschetz formula, valid over finite fields, when he stated the Weil conjectures,[4] having established them earlier in the one-dimensional case. The last of these conjectures is known as the "Riemann hypothesis over finite fields", and thus he had indeed found a connection between the Lefschetz formula and the Riemann hypothesis.

Weil had high standards, even for prizes. Early in his career he participated in a revolt against the introduction of some medals in France to recognize particular academic accomplishments. He felt that this kind of prize did more harm than good. But he seemed not to be averse to recognition of lifetime achievement. On his own vita at the Institute for Advanced Study, he listed just one prize he had received, the Kyoto Prize of 1994, which is the grandest of all the lifetime-achievement awards. He accepted also, but did not list, the Wolf Prize in Mathematics for 1979, received jointly with Jean Leray, and the AMS Steele Prize of 1980 for lifetime achievement. The citation for the Steele Prize aptly summarized his career to that point: "To André Weil for the total effect of his work on the general course of twentieth century mathematics, especially in the many areas in which he has made fundamental contributions."

The early part of his life, until 1947, is the subject of his autobiography, *The Apprenticeship of a Mathematician*.[5] André Weil was born on May 6, 1906, in Paris. He was a child prodigy, read widely, and became consummately cultured by his late teens. By 1921 he knew Greek, Latin, some German and English, and a little Sanskrit; he would learn more languages later. He was introduced to Jacques Hadamard in 1921, and Hadamard gave him occasional advice from then on. He finished high school and entered the École Normale Supérieure in 1922, attending the Hadamard Seminar from the start. He decided that it was important to read the masters and began to do so in mathematics and all the other subjects that interested him. He began to read Riemann in 1922. In 1925 he passed the Agrégation examination and thereby finished at the École Normale Supérieure.

It was during his period at the École Normale that he was introduced to the *Bhagavad Gita*, the core of an Indian epic poem. This was to become the foundation of his own personal philosophy, and in turn this philosophy was to be a decisive factor in what happened to him in World War II.

The circumstances of World War I affected Weil's way of learning mathematics and later played a role in establishing the goals of Bourbaki. This matter is discussed at length in *The Apprenticeship* and in articles by A. Borel[6] and J. Dieudonné[7] about Bourbaki. Briefly, the problem was that France, unlike Germany, wanted everyone of suitable age, mathematicians included, to go to the front lines in World War I. The result was that after the war there were few mathematicians living in France who were born between 1880 and 1900. Some old masters were still alive, but they could not be said to be fully aware of new developments in mathematics internationally. In addition, their interests, except in the case of Élie Cartan (who was little understood), were rather confined, basically developing the lines of study begun a century



Photograph courtesy of Sylvie Weil.

André Weil, Paris, 1907, age one.

[4]*These are now theorems, due to B. Dwork, A. Grothendieck, M. Artin, and P. Deligne.*

[5]*Birkhäuser, Basel, 1992, translated from the French original (*Souvenirs d'Apprentissage, *Birkhäuser, Basel, 1991) by Jennifer Gage, reviewed elsewhere in this issue of the* Notices.

[6]Notices, *March 1998, 373–380.*

[7]Amer. Math. Monthly *77 (1970), 134–145.*

earlier by Fourier and Cauchy. The only serious window to international mathematics was Hadamard's Seminar, and it was insufficient.

Thus it was that Weil began traveling extensively. During the year 1925–26, he spent six months in Rome, with Vito Volterra as an informal advisor, and he learned algebraic geometry. The most significant event of the year for him mathematically was that he learned of L. J. Mordell's 1922 paper in which it was proved that the abelian group of $\mathbb{Q}$ rational points on an elliptic curve defined over the rationals $\mathbb{Q}$ is finitely generated, and he saw that it was related to some thoughts that he had had a year earlier. In more detail, if $y^2 = x^3 + qx + p$ has rational coefficients and distinct complex roots, Poincaré (1901) had shown that the operation of combining two rational solutions by connecting them with a straight line, finding the third intersection of the line and the curve, and then reflecting that point about the x axis is an abelian group operation, the 0 element being the point at infinity. Poincaré had asked whether this abelian group is finitely generated, and Mordell's theorem answers this question affirmatively.

The next year he spent partly in Germany on a fellowship, with Courant as advisor but learning from E. Noether, M. Dehn, C. L. Siegel, H. Hopf, A. Ostrowski, and O. Toeplitz, among others. He spent one month with G. Mittag-Leffler in Sweden, in theory doing some writing for Mittag-Leffler. Mittag-Leffler, who was the founding editor of *Acta Mathematica*, promised Weil that his thesis could be published in that journal. In the summer of 1927 Weil returned to Paris and the Hadamard Seminar to continue work in earnest on his thesis (thèse d'état) on Diophantine equations. At age twenty-one he was done with the mathematics and the writing in short order. He received his D.Sc. degree in 1928.

In order to describe the result, it is necessary to provide some further background about the work of Poincaré and Mordell. The terminology of the time was a little careless by modern standards about singularities. But modulo that detail, it had already been realized that birational transformations between curves over $\mathbb{Q}$ preserve $\mathbb{Q}$ rational points and therefore two birationally equivalent curves have the same rationality properties. As a consequence, the degree of the curve should not be regarded as an invariant; the fundamental invariant is the genus of the compact Riemann surface obtained by considering all complex solutions in projective space. Elliptic curves are nonsingular of genus 1, thus yield Riemann surfaces that are tori. Both Poincaré and Mordell used parametrizations of an elliptic curve by means of the associated Weierstrass $\wp$ function. Under this parametrization the abelian group operation is simply addition on the torus, with the point at infinity on the curve corresponding to the element 0 on the torus. Mordell found that the operation of division of the parameters by 2, in the special cases of the genus 1 projective curves $u^3 + v^3 = w^3$ and $u^4 + v^4 = w^2$, corresponded to Fermat's method of infinite descent for proving Fermat's Last Theorem for degrees 3 and 4. Mordell was then able to adapt the method of descent to elliptic curves, and a consequence was his theorem that the group is finitely generated.

Poincaré knew for curves of genus g that the appropriate thing to consider is not points on the curve but unordered sets of g points on the curve, repetitions allowed. The main theorem of Weil's thesis is a theorem of "finite generation" in this setting, with the additional generality that $\mathbb{Q}$ may be replaced by any number field (finite extension of $\mathbb{Q}$). More specifically, a curve C of genus g, being a compact Riemann surface, maps into its "Jacobian variety" $J(C)$, which is a certain torus of complex dimension g, and the map is canonical up to a translation. Fixing base points, we can then map unordered sets of g points on C to the sum of the images in $J(C)$. The Jacobi Inversion Theorem says that this map is onto, and it is one-one for the most part. Weil sets up and proves a version of the infinite descent argument used by Mordell for genus 1.

Finding a committee to approve the thesis was not so easy, since there were essentially no number theorists in France, but he succeeded anyway. Mittag-Leffler had died meanwhile, but his successor at the *Acta*, N. Nörlund, honored Mittag-Leffler's promise and published a paper about the thesis in 1928. When the results in the *Acta* are specialized to genus 1, what is obtained is an extension of Mordell's theorem from elliptic curves over the rationals to elliptic curves over number fields. For this corollary, known as the Mordell-Weil theorem, Weil published a simpler proof in *Bulletin des Sciences Mathématiques* in 1930.[8] Weil made clear in this paper that the argument can be framed in terms of the geometric definition of addition and does not require the use of the Weierstrass $\wp$ function.

Weil spent the year 1928–29 doing his compulsory military service, ending up as a lieutenant in the reserves. He jumped at the chance to have a job in India and took a post as the professor in mathematics at Aligarh Muslim University in northern India starting in early 1930 and lasting until early 1932. In his research he tried his hand at other fields. He combined the idea of ergodicity with von Neumann's work on unitary operators on Hilbert spaces and came up with the ergodic theorem in the L^2 sense. He tried to generalize Poincaré's theorem on the rotation number to a class of differential equations, but did not make much

[8]*Listed as 1929 in* Collected Works.

progress. He worked in several complex variables and extended the Cauchy integral to certain pseudoconvex domains; according to *The Apprenticeship*, this work, published in 1935, led to solving a problem posed by Bergmann and played an indispensable role in later research by Oka.

Weil spent an uneventful 1932–33 in a teaching position at the Université de Marseille and then obtained an appointment at the Université de Strasbourg, where he remained until 1939, rising to the rank of professor. In Strasbourg he was with his longtime friend Henri Cartan and was finally back in a mathematical atmosphere. Weil and some of his friends scattered throughout France started a seminar in Paris in 1933–34. G. Julia lent his name to the enterprise so that a room could be found, and the seminar became the "Séminaire Julia". It concentrated on a different theme each year. After World War II it was reborn as the Séminaire Bourbaki.

Weil describes the birth of the Bourbaki group as follows: He and Henri Cartan were teaching "differential and integral calculus" in late 1934 from the standard book by Goursat. Cartan was forever asking Weil for the best way to treat a given section of the curriculum, and Weil had his own questions for Cartan. Weil proposed that the two of them should get together with their friends who were teaching the same thing at other universities and decide these matters together. Out of this proposal came regular meetings in Paris of Cartan and Weil with Delsarte, Chevalley, Dieudonné, and "a few others", and the group soon called itself Bourbaki. As described in the articles by Dieudonné and Borel, their goal was to organize basic mathematics for a France otherwise cut off from many international developments in mathematics by the circumstances of World War I that were mentioned above. At first these mathematicians were making organizational decisions, and then they viewed themselves as writing textbooks. Weil acknowledges that later they had to be writing at a more advanced level, and perhaps it would be appropriate to say that they were working on something more like an encyclopedia than a series of texts.[9] Success was not immediate. Indeed, finished books did not appear in profusion for a number of years. But mathematics in France was indeed reestablished by soon after World War II, and in that sense a principal goal of Bourbaki had been realized. One cannot pretend that the Bourbaki volumes were the only factors in the reestablishing of mathematics in France, but to the extent that leading mathematicians in France and elsewhere educated soon after World War II learned from the Bourbaki writings, the project has to be reckoned a success.

Weil talking with Goro Shimura (left) in Princeton, October 22, 1987. Also in picture: Chikako Shimura (back to camera) and Vyjayanthi Chari (half hidden).

Photograph by C. J. Mozzochi, provided courtesy of Armand Borel, with permission of the photographer.

The first draft of each Bourbaki volume was always written by an expert. Dieudonné qualified as an expert on integration and produced a first draft of a volume on integration about 1937; this developed measures and then integrals in the same kind of progression that P. Halmos later used in his own book on measure theory. Weil too was an expert on this subject. Already in 1934 Weil had begun work on his celebrated advanced book on integration on groups; that book was sent to a publisher in 1937 but did not appear until 1940. An account in Weil's *Collected Papers*, vol. I, pp. 547–549, explains why he took it upon himself to write his own draft for Bourbaki, basing measure theory on locally compact spaces. The account in *The Apprenticeship* shows that he put considerable effort into this Bourbaki volume.[10] Bourbaki volumes contained sections of "Historical Notes", and it is known that Weil played an important role in the writing of these.

While at Strasbourg, Weil had two advanced students, Élisabeth Lutz and Jacques Feldbau, who worked on lower-level French theses for the "diplôme d'études supérieures".[11] Jacques Feldbau was interested in topology, obtained a problem through Weil from Ehresmann, published one paper under his own name and another under the pseudonym Jacques Laboureur, and subsequently died in a concentration camp. Élisabeth Lutz counts herself as a student of Weil from 1934 to 1938. In 1935 she began working on aspects of elliptic

[9] *Indeed, one of the principles of the Bourbaki was always to go from the general to the particular. This principle, reasonable in an encyclopedia, is exactly the opposite of one of the generally accepted principles for teaching in the U.S. from the 1950s onward.*

[10] *The published version of this volume contained an acknowledged mistake that needed to be addressed later: namely, it dealt with integration in such a way that integration could not be usefully applied in probability theory. This matter is addressed in the commentary in* Collected Papers, *vol. I, pp. 547–549, and in Borel's article on the Bourbaki, p. 376.*

[11] *This kind of thesis became a "thèse de troisième cycle" after World War II.*

Photograph courtesy of Sylvie Weil.

André and Eveline Weil at the ICM at Harvard, 1950.

curves over p-adic fields. An elliptic curve over $\mathbb{Q}$ can be put in the form $y^2 = x^3 - Ax - B$ with A and B integers; recall that the abelian group of rational points is finitely generated. In her published paper on the subject,[12] Lutz makes two observations as a consequence of her analysis: first, that any $\mathbb{Q}$ rational point (x, y) of finite order on such a curve has integer coordinates and, second, that either y equals 0 or y^2 divides $4A^3 - 27B^2$. This result is now called the Nagell-Lutz Theorem. It implies that the torsion subgroup of $\mathbb{Q}$ rational points is effectively computable. It remains unknown, and it was a source of concern to Weil, whether the whole group of $\mathbb{Q}$ rational points is effectively computable. Weil describes Lutz's work, and its relationship to his own research, in his *Collected Papers*, vol. I, pp. 534–535. Perhaps as a testament to Weil's standards, Lutz's work was sufficient only for the lower-level French thesis. Lutz wrote a doctoral thesis (thèse d'état) after World War II on a different p-adic topic with a different advisor.

During this period Weil's mathematics flourished. He spent the year 1936–37 in the U.S. at the Institute for Advanced Study, returning thereafter to France. He married Eveline in late 1937. By the spring of 1939 he had decided that he would leave France. As a reserve lieutenant, he would have to serve if there were hostilities. Perhaps the effect of World War I on French mathematics would be repeated if there were another war. In *The Apprenticeship* he tells how his Indian philosophy played a role in his decision. It was not just his right to disobey unjust laws, but his duty. He and Eveline accepted an invitation from Lars Ahlfors to visit the Ahlfors family in Finland. A little later he was jailed in Finland as a spy, was spared execution[13] because of efforts by Nevanlinna, and was deported to Sweden. He spent time in jails successively in Sweden, England, and finally France, ending up in Rouen in February 1940. The time in jail in Rouen was a particularly productive one for him mathematically; it was when he proved the Riemann hypothesis for curves over finite fields, edited the proof sheets for his integration book, and made progress on his draft of a Bourbaki volume on integration. He realized the need to have solid foundations for algebraic geometry over any field, and filling this need was ultimately to result in his book *Foundations of Algebraic Geometry*. Charges were finally brought against him (he was accused of failure to report for duty), and a formal kind of trial was conducted on May 3, 1940. Instead of spending five years in jail, he asked to rejoin the army and permission was granted. After several moves in France and England, he finally arrived in Marseille. From there he went through the difficult task of reuniting with his wife and was told he had a job in the U.S. Following a circuitous route, he sailed in January 1941 and eventually succeeded in getting to New York City with Eveline.

His list of publications hardly reflects the turmoil in his life at this time. There was in fact no job in the U.S., but he did have some support from the Rockefeller Foundation. He spent part of 1941 in Princeton and was at Haverford College for 1941–42, with most of his support from the Rockefeller Foundation. It was there that he did his joint work with Allendoerfer on generalizing the Gauss-Bonnet formula. For 1942–43 he was an assistant professor at Lehigh University, with most of his salary again paid by the Rockefeller Foundation. He described his role as "to serve up predigested formulae from stupid textbooks and to keep the cogs of this diploma factory turning smoothly." He was able to continue work on his book *Foundations*, albeit more slowly; this was also the year in which he interacted with S. S. Chern. The year 1943–44 was the low point. The Rockefeller Foundation money had run out, and Weil's job at Lehigh was to teach "the elements of algebra and analytic geometry" for fourteen hours a week to army recruits who were being kept busy before being shipped elsewhere. Afterward he and Eveline vowed never to mention the name "Lehigh" again.

In January 1944, exasperated, Weil prepared to quit his job, regardless of the consequences. He wrote to Hermann Weyl, asking for help, and Weyl arranged for Weil to be awarded a Guggenheim Fellowship even though the deadline for applications had passed. He took a position as professor at the Universidade de São Paulo in Brazil, a center for algebraic geometry, in January 1945 and remained there until 1947. He was able to visit Paris in 1945.

Weil was appointed professor at the University of Chicago in 1947 and kept that position until 1958. He was on leave in Paris for 1957–58 and then became a professor at the Institute for Advanced Study starting in 1958. He remained at the Institute until his retirement in 1976. He traveled once more to India in 1967, and he made at least three trips to Japan—one in 1955 for an international

[12] *Sur l'équation $y^2 = x^3 - Ax - B$ dans les corps* $\mathfrak{p}$-*adiques*, J. Reine Angew. Math. ***177** (1937), 238–247.*

[13] *This is Weil's version. An article by O. Pekonen (L'affaire Weil à Helsinki en 1939,* Gaz. Math., *No. 52 (1992), 13–20) gives further historical background but disputes that Weil was about to be executed. In a postscript at the end of the Pekonen article, Weil points out that Pekonen provides no facts that contradict Weil's version.*

conference on algebraic number theory, another in 1961 for a second conference, and the last in 1994 to receive the Kyoto Prize.

His book *Foundations* was published in 1946, and the frequency of his papers increased. In 1949 alone he published his paper on the Weil Conjectures, as well as articles on fiber spaces in algebraic geometry, on theta functions, and on differential geometry. He was a speaker at the International Congress of Mathematicians (ICM) in Cambridge, MA, in 1950 and again in Amsterdam in 1954, and he was an hour speaker at the ICM in Helsinki in 1978. He was a member of the Académie des Sciences (Paris), a foreign member of the Royal Society (London), and a foreign member of the National Academy of Sciences of the U.S.A.

Between lectures September 8, 1955, at International Symposium on Algebraic Number Theory in Tokyo. Left to right: Teiji Takagi, Richard Brauer (back to camera), Shokichi Iyanaga, André Weil, and Emil Artin.

Photograph by Asako Hatori, provided courtesy of Shokichi Iyanaga, with permission of the photographer.

A list of the courses he taught at Chicago and the Institute appears in Volume I of his *Collected Works*. He had four successful Ph.D. students while at the University of Chicago: Arnold S. Shapiro (1950), Frank D. Quigley (1953), Norman T. Hamilton (1955), and David Hertzig (1957). For Weil's 1959–60 course on adeles and algebraic groups, Michel Demazure and Takashi Ono served as notetakers and converted the notes into a form that eventually became a book. Ono says of this experience that although he had already obtained his Ph.D. in Japan, "I always think of him as my real advisor and am proud of being his nonofficial student all my life."

Books by André Weil

- *Arithmétique et Géométrie sur les Variétés Algébriques*, Hermann, Paris, 1935.
- *Sur les Espaces à Structure Uniforme et sur la Topologie Générale*, Hermann, Paris, 1937.
- *L'intégration dans les Groupes Topologiques et Ses Applications*, Hermann, Paris, 1940; second edition, 1953.
- *Foundations of Algebraic Geometry*, Colloquium Publications, vol. 29, Amer. Math. Soc., New York City, 1946; second edition, Providence, RI, 1962.
- *Sur les Courbes Algébriques et les Variétés Qui s'en Déduisent*, Hermann, Paris, 1948.
- *Variétés Abéliennes et Courbes Algébriques*, Hermann, Paris, 1948; second edition of this and *Sur les Courbes Algébriques et les Variétés Qui s'en Déduisent* published together under the collective title *Courbes Algébriques et Variétés Abéliennes*, 1971.
- *Introduction à l'Étude des Variétés Kählériennes*, Hermann, Paris, 1958.
- *Basic Number Theory*, Springer-Verlag, New York, 1967; second edition, 1974; third edition, 1995.
- *Dirichlet Series and Automorphic Forms*, Lecture Notes in Mathematics, vol. 189, Springer-Verlag, New York, 1971.
- *Elliptic Functions according to Eisenstein and Kronecker*, Springer-Verlag, Berlin, 1976.
- *Œuvres Scientifiques, Collected Works*, vols. I–III, Springer-Verlag, New York, 1979.
- *Number Theory for Beginners*, with the collaboration of Maxwell Rosenlicht, Springer-Verlag, New York, 1979.
- *Adeles and Algebraic Groups*, Birkhäuser, Boston, 1982; based on notes by M. Demazure and T. Ono in 1959–60.
- *Number Theory: An Approach through History from Hammurapi to Legendre*, Birkhäuser, 1984.
- *Souvenirs d'Apprentissage*, Birkhäuser, Basel, 1991; English translation by Jennifer Gage, *The Apprenticeship of a Mathematician*, Birkhäuser, Basel, 1992; translated also into German, Italian, and Japanese.

In addition, there were several sets of unpublished lecture notes. The full list of Weil's publications through 1978 appears in his *Collected Works*.