Mikio Sato, a Visionary of Mathematics

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Like singularities in mathematics and physics, ideas propagate, and the speed of propagation depends highly on the energy put into promoting them. Mikio Sato did not spend a great deal of time or energy popularizing his ideas. We can hope that his receipt of the 2002/2003 Wolf Prize[1] will help make better known his deep work, which is perhaps too original to be immediately accepted. Sato does not write a lot, does not communicate easily, and attends very few meetings. But he invented a new way of doing analysis, "Algebraic Analysis", and created a school, "the Kyoto school".

Born in 1928[2], Sato became known in mathematics only in 1959–1960 with his theory of hyperfunctions. Indeed, his studies had been seriously disrupted by the war, particularly by the bombing of Tokyo. After his family home burned down, he had to work as a coal delivery man and later as a school teacher. At age 29 he became an assistant professor at Tokyo University. He studied mathematics and physics, on his own.

To understand the originality of Sato's theory of hyperfunctions, one has to place it in the mathematical landscape of the time. Mathematical analysis from the 1950s to the 1970s was under the domination of functional analysis, marked by the success of the theory of distributions. People were essentially looking for existence theorems for linear partial differential equations (PDE), and most of the proofs were reduced to finding "the right functional space", to prove some *a priori* estimate and apply the Hahn-Banach theorem.

It was in this environment that Mikio Sato defined hyperfunctions in 1959–1960 as boundary values of holomorphic functions, a discovery that allowed him to obtain a position at Tokyo University thanks to the clever patronage of Shokichi Iyanaga, an exceptionally open-minded person and a great friend of French culture. Next, Sato spent two years in the USA, in Princeton, where he unsuccessfully tried to convince André Weil of the relevance of his cohomological approach to analysis.

Sato's method was radically new, in no way using the notion of limit. His hyperfunctions are not limits of functions in any sense of the word, and the space of hyperfunctions has no natural topology other than the trivial one. For his construction, Sato invented local cohomology in parallel with Grothendieck. This was truly a revolutionary vision of analysis.

But besides its evident originality, Sato's approach had deep implications since it naturally led to microlocal analysis, as I will try to explain.

The theory of linear PDE with variable coefficients was in its early beginnings in the years 1965–1970 and under the shock of Hans Lewy's example showing that the first order linear equation $(-\sqrt{-1}\partial_1 + \partial_2 - 2(x_1 + \sqrt{-1}x_2)\partial_3)u = v$ had no solution, even a local solution, in the space of distributions[3]. The fact that an equation had no

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[1] *Translation of a paper appearing in* La Gazette des Mathématiciens ***97*** *(2003) on the occasion of Sato's receiving the 2002/2003 Wolf Prize.*

[2] *In 1990, Sato gave an interview to Emmanuel Andronikof who unfortunately passed away in 1994. I have made use of his notes, which were edited by A. D'Agnolo. I also have benefited from the scientific comments of J-B. Bost and A. Chambert-Loir, whom I warmly thank.*

[3] *The slightly simpler equation* $(\partial_1 + \sqrt{-1}x_1\partial_2)u = v$ *does not have any solution in the space of germs at the origin of distributions in* $\mathbb{R}^2$ *either, nor even in the space of germs of hyperfunctions.*

Mikio Sato (left) with Pierre Schapira, around 1972.

solution was quite disturbing at that time. People thought that it was a defect of the theory, that the spaces one had considered were too small to admit the solutions. Of course, often just the opposite is true and one finds that the occurrence of a cohomological obstruction heralds interesting phenomena: the lack of a solution is the demonstration of some deep and hidden geometrical phenomena. In the case of the Hans Lewy equation, the hidden geometry is "microlocal", and this equation is microlocally equivalent to an induced Cauchy-Riemann equation on a real hypersurface of the complex space.

In mathematics, as in physics, in order to treat phenomena in a given (affine) space, one is naturally led to compute in the dual space. One way, the most commonly used in analysis, is via the Fourier transform. This transform, far from being of a local nature, is not easily adapted to calculus on manifolds. By contrast, Sato's method is perfectly suited for this case: you can complexify a real analytic manifold and, instead of looking at the behavior at infinity of the Fourier transform, you look "where the boundary values come from". In technical terms, one regards the cotangent bundle (more precisely, $\sqrt{-1}$-times the cotangent bundle) as the conormal bundle to the real space in the complex space. This is how Sato defines the analytic wave front set of hyperfunctions (in particular, of distributions), a closed conic subset of the cotangent bundle, and he shows that if a hyperfunction u is a solution of the equation $Pu = 0$, then its wave front set is contained in the real part of the characteristic variety of the operator P. This is the starting point of microlocal analysis, invented by Sato, a kind of revolution in analysis.

Of course, at this time other mathematicians (especially L. Hörmander) and physicists (e.g., D. Iagolnitzer) had the intuition that the cotangent space was the natural space for analysis, and in fact this intuition arose much earlier (in the work of J. Hadamard, F. John, and J. Leray, in particular). Indeed, pseudo-differential operators did exist before the wave front set. But Sato was the first to make the objects of analysis, such as distributions, live in the cotangent space, and for that purpose he constructed a key tool of sheaf theory, the microlocalization functor, that is, the "Fourier-Sato" transform of the specialization functor. This is also the origin of the microlocal theory of sheaves of [3]. In 1973 Sato and his two students, M. Kashiwara and T. Kawai, published a treatise on the microlocal analysis of PDE [8]. Certainly this work had a considerable impact, although most analysts did not understand a single word. Hörmander and his school then adapted the classical Fourier transform to these new ideas, leading to the now popular theory of Fourier-integral operators.

Already in the 1960s, Sato had the intuition of $\mathcal{D}$-module theory, of holonomic systems, and of the b-function (the so-called Bernstein-Sato b-function). He gave a series of talks on these topics at Tokyo University but had to stop for lack of combatants. His ideas were reconsidered and systematically developed by Masaki Kashiwara in his 1969 thesis ([1], [2]). As its name indicates, a $\mathcal{D}$-module is a module over the (sheaf of) ring(s) $\mathcal{D}$ of differential operators, and a module over a ring essentially means "a system of linear equations" with coefficients in this ring. The task is now to treat (general) systems of linear PDE. This theory, which also simultaneously appeared in Moscow in a more algebraic framework developed by Gelfand's student J. Bernstein, quickly had considerable success in several branches of mathematics. In 1970–1980, Kashiwara obtained almost all the fundamental results of the theory, in particular those concerned with holonomic modules, such as his constructibility theorem, his index theorem for holomorphic solutions of holonomic modules, the proof of the rationality of the zeroes of the b-function, and his theory of regular holonomic modules.

The mathematical landscape of 1970–1980 had thus considerably changed. Not only did one treat equations with variable coefficients, but one treated systems of such equations and moreover one worked microlocally, that is, in the cotangent bundle, the phase space of the physicists. But there were two schools in the world: the C^∞ school issuing from classical analysis and headed by Hörmander, who developed the calculus of Fourier integral operators[4], and the analytic school that Sato established, which was almost nonexistent outside Japan and France.

France was a strategic place to receive Sato's ideas since they are based on those of both Jean Leray and Alexandre Grothendieck. Like Leray, Sato

[4] *Many names should be quoted at this point, in particular those of V. Maslov and Yu. Egorov.*

understood that singularities have to be sought in the complex domain, even for the understanding of real phenomena. Sato's algebraic analysis is based on sheaf theory, a theory invented by Leray in 1944 when he was a prisoner of war, clarified by Cartan, and made extraordinarily efficient by Grothendieck and his formalism of derived categories and the *six opérations*.

Sato, motivated by physics as usual, then tackled the analysis of the S-matrix in light of microlocal analysis. With his two new students, M. Jimbo and T. Miwa, he explicitly constructed the solution of the n-points function of the Ising model in dimension 2 using Schlesinger's classical theory of isomonodromic deformations of ordinary differential equations. This naturally led him to the study of KdV-type nonlinear equations. In 1981, with his wife Yasuko Sato, he interpreted the solutions of the KP-hierarchies as points of an infinite Grasmannian manifold and introduced his famous τ-function. These results would be applied to other classes of equations and would have a great impact in mathematical physics in the study of integrable systems and field theory in dimension 2.

In parallel with his work in analysis and in mathematical physics, Sato obtained remarkable results in group theory and in number theory.

He introduced the theory of "prehomogeneous vector spaces", that is, of linear representations of complex reductive groups with a dense orbit. The important case where the complement of this orbit is a hypersurface gives good examples of b-functions.

In 1962 Sato also discovered how to deduce the Ramanujan conjecture on the coefficients of the modular form Δ from Weil's conjectures concerning the number of solutions of polynomial equations on finite fields. His ideas allowed M. Kuga and G. Shimura to treat the case of compact quotients of the Poincaré half-space, and it would be necessary to wait another ten years for P. Deligne to definitely prove that Weil's conjectures imply the Ramanujan-Petersson conjecture.

While M. Sato and J. Tate shared the 2002/2003 Wolf Prize, they also share a famous conjecture[5] in number theory concerning the repartition of Frobenius angles. Let P be a degree 3 polynomial with integer coefficients and simple roots. Hasse showed that for any prime p that does not divide the discriminant of P, the number of solutions of the congruence $y^2 = P(x) \pmod{p}$ is like $p - a_p$, with $|a_p| \leq 2\sqrt{p}$. When writing $a_p = 2\sqrt{p}\cos\theta_p$ with $0 \leq \theta_p \leq \pi$, the Sato-Tate conjecture predicts that the repartition of the angles θ_p follows the probability law with density $(2/\pi)\sin^2\theta d\theta$. Note that Tate was led to this conjecture by studying algebraic cycles and Sato by computing numerical data.

Sato's most recent works are essentially unpublished and have been presented in seminars attended only by a small group of people. They treat an algebraic approach to nonlinear systems of PDE, in particular holonomic systems, of which theta functions are examples of solutions!

Looking back, forty years later, we realize that Sato's approach to mathematics is not so different from that of Grothendieck, that Sato did have the incredible temerity to treat analysis as algebraic geometry, and that he was also able to build the algebraic and geometric tools adapted to his problems.

His influence on mathematics is, and will remain, considerable.

[5] *For recent developments on this conjecture, see [4].*

References

[1] M. Kashiwara, Algebraic study of systems of partial diffential equations, Thesis, Tokyo Univ. (1970), translated by A. D'Agnolo and J-P. Schneiders, *Mémoires Soc. Math. France*, vol. 63, 1995.

[2] ——, $\mathcal{D}$-modules and microlocal calculus, *Translations of Mathematical Monographs*, vol. 217, Amer. Math. Soc., Providence, RI, 2003.

[3] M. Kashiwara and P. Schapira, Sheaves on manifolds, *Grundlehren der Math. Wiss.* **292** (1990), Springer.

[4] B. Mazur, Controlling our errors, *Nature* **443**, 7 September 2006, pages 38-40 (`http://www.nature.com/nature/journal/v443/n7107/full/443038a.html`).

[5] M. Sato, Theory of hyperfunctions, I & II, *Jour. Fac. Sci. Univ. Tokyo*, **8** (1959–1960), 139–193, 487–436.

[6] ——, $\mathcal{D}$-modules and nonlinear systems, in *Integrable Systems in Quantum Field Theory and Statistical Mechanics*, Adv. Stud. Pure Math., vol. 19, Academic Press, Boston, MA, 1989, pp. 417–434.

[7] ——, The KP hierarchy and infinite-dimensional Grassmann manifolds, *Theta functions—Bowdoin 1987, (Brunswick, ME, 1987), Proc. Sympos. Pure Math.*, vol. 49, Part 1, Amer. Math. Soc., Providence, RI, 1989, pp. 51–56.

[8] M. Sato, T. Kawai, and M. Kashiwara, Microfunctions and pseudo-differential equations, *Hyperfunctions and pseudo-differential equations*, Komatsu (ed.), Proceedings Katata 1971, *Lecture Notes in Math.*, vol. 287, Springer-Verlag, 1973, pp. 265–529.

[9] M. Sato and T. Kimura, A classification of irreducible prehomogeneous vector spaces and their relative invariants, *Nagoya Math. J.* **65** (1977), 1–155.

[10] M. Sato, T. Miwa, and M. Jimbo, Holonomic quantum fields, I–V, *Publ. Res. Inst. Math. Sci.* **16** (1980), 531–584; **15** (1979), 871–972; **15** (1979), 577–629; **15**, (1979), 201–278; **14** (1978), 223–267.

[11] M. Sato and Y. Sato, Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold, Nonlinear Partial Differential Equations in Applied Science (Tokyo, 1982), *North-Holland Math. Stud.*, vol. 81, North-Holland, Amsterdam, 1983, 259–271.

[12] M. Sato and T. Shintani, On zeta functions associated with prehomogeneous vector spaces, *Ann. of Math.* **100** (1974), 131–170.