Garrett Birkhoff and the *Survey of Modern Algebra*

Saunders Mac Lane

Garrett Birkhoff became fascinated with finite groups when he was an undergraduate major in mathematics at Harvard. Upon graduation in 1932, at the age of nineteen, he traveled in Europe with a Henry fellowship. There he studied Speiser's book on group theory and van der Waerden on modern algebra. At Cambridge University he was impressed by the elegant group theoretic ideas of Philip Hall. Then Garrett also discovered the idea of a lattice—a poset with both lower and upper bounds (only later did it turn out that this structure had been found by Dedekind, in a little noted study of what he called a "dual group"). While in Cambridge, Garrett also conceived the idea that there could be a real "universal algebra" and realized this idea by proving what is now known as Birkhoff's theorem, characterizing varieties of algebras.

He considered algebras given by a set with specified operations (unary, binary, etc.) which satisfy a given list of identities. All such constitute a "variety". Birkhoff's theorem states that a class of algebras is such a variety if and only if it is closed under the formation of subalgebras, direct products, and homomorphic images. This result became the starting point for the subsequent active development of universal algebra.

After holding a Junior Fellowship at Harvard 1933–36, Garrett became an instructor in 1936. At that time Harvard provided a full year undergraduate course in geometry; Garrett advocated the establishment of a corresponding full year course in algebra, to be called Mathematics 6. He taught the first version of this course in 1937–38, emphasizing Boolean algebra, set theory, vectors and group theory; he prepared notes of his course. I taught a somewhat different version of Mathematics 6 in the next year, 1939–40, after I joined the Harvard faculty in 1938; I also provided typed notes of my version of the course. In the subsequent years Garrett and I combined our preliminary notes to publish with MacMillan in 1941 our joint book, *Survey of Modern Algebra*. It provided a clear and enthusiastic emphasis on the then new modern and axiomatic view of algebra, as advocated by Emmy Noether, Emil Artin, van der Waerden, and Philip Hall. We aimed to combine the abstract ideas with suitable emphasis on examples and illustrations. Groups were started by examples such as the group of symmetries of the square. Vector spaces were introduced by axioms, but with n-tuples of numbers as illustrations. The chapter on matrices began with linear transformations and explained matrix multiplication in terms of the composition of the corresponding linear transformations. The Galois theory was presented with the conceptual ideas of Emil Artin, which made the Galois correspondence (subgroups to subfields) vivid. In brief, the emphasis was axiomatic and abstract, but built on examples.

At that time one of my good midwestern friends told me that our survey "would not fly beyond the Charles River." For a year or two this was perhaps so. But American mathematics, spurred by the in-

Editor's Note: Garrett Birkhoff passed away on November 11, 1996, at the age of 85.

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fluence of refugees from Europe and the urgent needs of war research, was rapidly developing. Throughout mathematics, ideas in their abstract form mattered. Our *Survey* was at hand and provided these ideas with examples. It soon became, and was for many years, the text of choice for an undergraduate course in algebra. We were fortunate to be there, young and enthusiastic, at the time when new views of algebra came to fruition. And these ideas are still there; Garrett and I were both pleased with the recent publication of the fifth edition of *Survey* (A. K. Peters, 1996).

We enjoyed teaching and writing algebra because it was clear, exciting, and fun to present. The book was prepared at a time when both of us were assistant professors, so without tenure. Yes, we did know then that research mattered for tenure, but our joy in teaching was somehow connected with our respective research. Also, the mathematics department at Harvard both emphasized research and expected all faculty members to be steadily active in teaching undergraduates. These responsibilities were in effect combined in our activity. Then and later we took part in the flow of new ideas from discovery to use and to present to students.

Garrett's own research was involved. It was then primarily in lattice theory and universal algebra. His original slim colloquium volume on lattice theory was later expanded to a much more comprehensive version, reflecting the growth in this field. During the war Garrett's interests grew to include hydrodynamics and other applied mathematics—with an occasional pause to prepare revised editions of *Survey*. On the fiftieth anniversary of its publication, the *Mathematical Intelligencer*, in its column "Years Ago", edited by Karen V. H. Parshall, gave a description of *Survey*, complete with pictures of the authors (vol. 14, no. 1, 1992, 26–31 pp.).

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The Many Lives of Lattice Theory

Gian-Carlo Rota

Introduction

Never in the history of mathematics has a mathematical theory been the object of such vociferous vituperation as lattice theory. Dedekind, Jónsson, Kurosh, Malcev, Ore, von Neumann, Tarski, and most prominently Garrett Birkhoff have contributed a new vision of mathematics, a vision that has been cursed by a conjunction of misunderstandings, resentment, and raw prejudice.

The hostility towards lattice theory began when Dedekind published the two fundamental papers that brought the theory to life well over one hundred years ago. Kronecker in one of his letters accused Dedekind of "losing his mind in abstractions," or something to that effect.

I took a course in lattice theory from Oystein Ore while a graduate student at Yale in the fall of 1954. The lectures were scheduled at 8 a.m., and only one other student attended besides me—María Wonenburger. It is the only course I have ever attended that met at 8 o'clock in the morning. The first lecture was somewhat of a letdown, beginning with the words: "I think lattice theory is played out" (Ore's words have remained imprinted in my mind).

For some years I did not come back to lattice theory. In 1963, when I taught my first course in combinatorics, I was amazed to find that lattice theory fit combinatorics like a shoe. The temptation is strong to spend the next fifty minutes on the mutual stimulation of lattice theory and combinatorics of the last thirty-five years. I will, however, deal with other aspects of lattice theory, those that were dear to Garrett Birkhoff and which bring together ideas from different areas of mathematics.

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This article is based on an invited address delivered at the Garrett Birkhoff Memorial Conference, Harvard University, April 1, 1997.

Lattices are partially ordered sets in which least upper bounds and greatest lower bounds of any two elements exist. Dedekind discovered that this property may be axiomatized by identities. A lattice is a set on which two operations are defined, called join and meet and denoted by $\vee$ and $\wedge$, which satisfy the idempotent, commutative and associative laws, as well as the absorption laws:

$$a \vee (b \wedge a) = a$$
$$a \wedge (b \vee a) = a.$$

Lattices are better behaved than partially ordered sets lacking upper or lower bounds. The contrast is evident in the examples of the lattice of partitions of a set and the partially ordered set of partitions of a number. The family of all partitions of a set (also called equivalence relations) is a lattice when partitions are ordered by refinement. The lattice of partitions of a set remains to this day rich in pleasant surprises. On the other hand, the partially ordered set of partitions of an integer, ordered by refinement, is not a lattice and is fraught with pathological properties.

Distributive Lattices

A distributive lattice is a lattice that satisfies the distributive law:

$$a \vee (b \wedge c) = (a \wedge b) \vee (a \wedge c).$$

For a long time a great many people believed that every lattice is distributive. This misunderstanding was finally cleared up when Garrett Birkhoff, in the early thirties, proved a fundamental theorem, which we summarize next.

There is a standard way of constructing distributive lattices. One takes all the order ideals of a partially ordered set P. An order ideal is a subset of P with the property that if $x \in P$ and $y \leq x$, then $y \in P$. Union and intersection of order ideals are order ideals. In other words, the set of all order ideals of a partially ordered set is a distributive lattice.

Garrett Birkhoff proved the converse of this statement: every finite distributive lattice is isomorphic to the lattice of order ideals of some partially ordered set. The resulting contravariant functor from the category of partially ordered sets to the category of distributive lattices, known as the "Birkhoff transform", provides a systematic and useful translation of the combinatorics of partially ordered sets into the algebra of distributive lattices.

The definitive generalization of Birkhoff's theorem to arbitrary distributive lattices was obtained in the sixties by Ann Priestley. Briefly, there is a nontrivial extension of the notion of topological space that takes order into account, defined by Leopoldo Nachbin in his thesis. Distributive lattices are represented as lattices of closed order ideals on such ordered topological spaces. Point set topology has been nontrivially extended to ordered topological spaces, but this extension has remained largely unknown. Dieudonné was taken with it after he read the copy of Nachbin's thesis that the author, working in total isolation, sent him from Rio de Janeiro. Dieudonné tried to drum up some interest in ordered topological spaces without success.

It is a miracle that families of sets closed under unions and intersections can be characterized solely by the distributive law and by some simple identities. Jaded as we are, we tend to take Birkhoff's discovery for granted and to forget that it was a fundamental step forward in mathematics.

Modular Lattices

Modular lattices are lattices that satisfy the following identity, discovered by Dedekind:

$$(c \wedge (a \vee b)) \vee b = (c \vee b) \wedge (a \vee b).$$

This identity is customarily recast in user-friendlier ways. Examples of modular lattices are lattices of subspaces of vector spaces, lattices of ideals of a ring, lattices of submodules of a module over a ring, and lattices of normal subgroups of a group. For example, in the lattice of subspaces of a vector space the meet of two subspaces is their set theoretic intersection, and the join of two subspaces is the subspace spanned by the two subspaces. Join and meet of linear varieties in projective space are algebraic renderings of projection and section of synthetic projective geometry. Synthetic projective geometry, relying as it does on axioms of incidence and refusing any appeal to coordinates, is best understood in the language of modular lattices.

But synthetic geometry acquired a bad name after algebraic geometers declared themselves unable to prove their theorems by synthetic methods. The synthetic presentation of geometry has become in the latter half of this century a curiosity, cultivated by Italians and by Professor Coxeter. Modular lattices were dismissed without a hearing as a curious outgrowth of a curiosity.

Garrett once described to me his first meeting with von Neumann. After exchanging a few words they quickly got down to their common interest in lattice theory, and von Neumann asked Garrett, "Do you know how many subspaces you get when you take all joins and meets of three subspaces of a vector space in general position?" Garrett immediately answered, "Twenty-eight!", and their collaboration began at that moment.

The free modular lattice with three generators, which indeed has twenty-eight elements, is a beautiful construct that is presently exiled from textbooks in linear algebra. Too bad, because the elements of this lattice explicitly describe all projective invariants of three subspaces.

One of Garrett's theorems on modular lattices states that the free modular lattice generated by two linearly ordered sets (or chains) is distributive. This result has been shamelessly restated without credit in disparate mathematical languages.

The core of the theory of modular lattices is the generalization of the theory of linear dependence of sets of vectors in a vector space to sets of linear subspaces of any dimension. Dilworth, Kurosh, Ore, and several others defined an extended concept of basis, and they established invariance of dimension and exchange properties of bases. The translation of their results into coordinate language is only now being carried out.

Two recent developments in modular lattices are:

First, the discovery of 2-distributive lattices by the Hungarian mathematician Andras Huhn. A 2-distributive lattice is a lattice that satisfies the identity

$$a \vee (x \wedge y \wedge z) =$$
$$(a \wedge (x \vee y)) \vee (a \wedge (x \vee y)) \vee (a \vee (y \wedge z)).$$

This improbable identity implies that the lattice is modular and much more. It has been shown by Bjarni Jónsson, J. B. Nation, and several others that 2-distributive lattices are precisely those lattices that are isomorphically embeddable into the lattice of subspaces of a vector space over any field whatsoever, subject only to cardinality restrictions. Thus, 2-distributive lattices come close to realizing the ideal of a universal synthetic geometry, at least for linear varieties. They have a rich combinatorial structure.

Second, the theory of semiprimary lattices. These lattices were given their unfortunate name by Reinhold Baer, but, again, only recently has their importance been realized in the work of such young mathematicians as Franco Regonati and Glenn Tesler. Examples of semiprimary lattices are the lattice of subgroups of a finite Abelian group and the lattice of invariant subspaces of a nilpotent matrix. Semiprimary lattices are modular, and hence every element is endowed with a rank or dimension. However, the elements of semiprimary lattices are additionally endowed with a finer type of rank, which is a partitition of an integer, or a Young shape, as we say in combinatorics. For the lattice of subgroups of an Abelian group such a partition comes from the structure theorem for finite Abelian groups; for the invariant subspaces of nilpotent matrices the partition comes from the Jordan canonical form.

This finer notion of dimension leads to a refinement of the theory of linear dependence. One major result, due to Robert Steinberg, is the following. Consider a complete chain in a semi-primary lattice. Two successive elements of the chain differ by one dimension, but much more is true. As we wind up the chain, we fill a Young shape with integers corresponding to the positions of each element of the chain, and thus every complete chain is made to correspond to a standard Young tableau.

Now take two complete chains in a semiprimary lattice. It is easy to see that a pair of complete chains in a modular lattice determines a permutation of basis vectors. In a semiprimary lattice each of the two chains is associated with a standard Young tableau, hence we obtain the statement and proof of the Schensted algorithm, which precisely associates a pair of standard Young tableaux to every permutation.

Lattice of Ideals

Dedekind outlined the program of studying the ideals of a commutative ring by lattice-theoretic methods, but the relevance of lattice theory in commutative algebra was not appreciated by algebraists until the sixties, when Grothendieck demanded that the prime ideals of a ring should be granted equal rights with maximal ideals. Those mathematicians who knew some lattice theory watched with amazement as the algebraic geometers of the Grothendieck school clumsily reinvented the rudiments of lattice theory in their own language. To this day lattice theory has not made much of a dent in the sect of algebraic geometers; if ever it does, it will contribute new insights. One elementary instance: the Chinese remainder theorem. Necessary and sufficient conditions on a commutative ring are known that insure the validity of the Chinese remainder theorem. There is, however, one necessary and sufficient condition that places the theorem in proper perspective. It states that the Chinese remainder theorem holds in a commutative ring if and only if the lattice of ideals of the ring is distributive.

The theory of ideals in polynomial rings was given an abstract setting by Emmy Noether and her school. Noetherian rings were defined, together with prime and primary ideals, and fundamental factorization theorems for ideals were proved. It does not seem outrageous to go one step further in Dedekind's footsteps and extend these theorems to modular lattices. This program was initiated by Oystein Ore and developed by Morgan Ward of Caltech and by his student, Bob Dilworth. Dilworth worked at this program on and off all his life, and in his last paper on the subject, published in 1961, he finally obtained a lattice theoretic formulation of the Noetherian theory of ideals. I quote from the introduction of Dilworth's paper:

> The difficulty [of the lattice theory of ideals] occurred in treating the Noether theorem on decompositon into primary ideals. ... In this paper, I give a new and stronger formulation for the notion of a "principal element" and...prove a [lattice theoretic] version of the Krull Principal Ideal Theorem. Since there are generally many non-principal ideals of a commutative ring which are "principal elements" in the lattice of ideals, the [lattice theoretic] theorem represents a considerable strengthening of the classical Krull result.

Forgive my presumptuousness for making a prediction about the future of the theory of commutative rings, a subject in which I have never worked. The theory of commutative rings has been torn by two customers: number theory and geometry.

Our concern here is the relationship between commutative rings and geometry, not number theory. In the latter part of this century algebra has so overwhelmed geometry that geometry has come to be viewed as a mere "façon de parler". Sooner or later geometry in the synthetic vein will reassert its rights, and the lattice theory of ideals will be its venue. We intuitively feel that there is a geometry, projective, algebraic, or whatever, whose statements hold independently of the choice of a base field. Desargues's theorem is the simplest theorem of such a "universal" geometry. A new class of commutative rings remains to be discovered that will be completely determined by their lattice of ideals. Von Neumann found a class of noncommutative rings that are determined by their lattices of ideals, as we will shortly see, but the problem for commutative rings seems more difficult. A first step in this direction was taken by Hochster. Algebraic geometry done with such rings might be a candidate for "universal geometry".

Commutative rings set the pace for a wide class of algebraic systems in the sense of Garrett Birkhoff's universal algebra. The lattice of congruences of an algebraic system generalizes the lattice of ideals, and this analogy allows us to translate facts about commutative rings into facts about more general algebraic systems. An example of successful translation is the Chinese remainder theorem in its lattice theoretic formulation, which has been proved for general algebras. The work of Richard Herrmann and his school has gone far in this direction. In view of the abundance of new algebraic structures that are being born out of wedlock in computer science, this translation is likely to bear fruit.

Linear Lattices

Having argued for modular lattices, let me now argue against them.

It turns out that all modular lattices that occur in algebra are endowed with a richer structure. They are lattices of commuting equivalence relations. What are commuting equivalence relations?

Two equivalence relations on a set are said to be independent when every equivalence class of the first meets every equivalence class of the second. This notion of independence originated in information theory and has the following intuitive interpretation. In the problem of searching for an unknown element, an equivalence relation can be viewed as a question whose answer will tell to which equivalence class the unknown element belongs. Two equivalence relations are independent when the answer to either question gives no information on the possible answer to the other question.

Philosophers have gone wild over the mathematical definition of independence. Unfortunately, in mathematics philosophy is permanently condemned to play second fiddle to algebra. The pairs of equivalence relations that occur in algebra are seldom independent; instead, they satisfy a sophisticated variant of independence that has yet to be philosophically understood—they commute.

Two equivalence relations are said to commute when the underlying set may be partitioned into disjoint blocks and the restriction of the pair of equivalence relations to each of these blocks is a pair of independent equivalence relations. In other words, two equivalence relations commute when they are isomorphic to disjoint sums of independent equivalence relations on disjoint sets.

Mme. Dubreil found in her 1939 thesis an elegant characterization of commuting equivalence relations. Two equivalence relations on the same set commute whenever they commute in the sense of composition of relations, hence the name.

The lattice of subspaces of a vector space is an example of a lattice that is naturally isomorphic to a lattice of commuting equivalence relations on the underlying vector space viewed as a mere set. Indeed, if W is a subspace of a vector space V, one defines an equivalence relation on the set of vectors in V by setting $x \equiv_W y$ whenever $x - y \in W$. Meet and join of subspaces are isomorphic to meet and join of the corresponding equivalence relations in the lattice of all equivalence relations on the set V. The lattice of subspaces of a vector space V is isomorphic to a sublattice of the lattice of all equivalence relations on the set V, in which any two equivalence relations commute.

Similar mappings into lattices of commuting equivalence relations exist for the lattice of all ideals of a ring and the lattice of all submodules of a module. Mark Haiman has proposed the term "linear lattice" for lattices of commuting equivalence relations.

Schützenberger found an identity satisfied in certain modular lattices that is equivalent to Desargues's theorem. Not long afterwards, Bjarni Jónsson proved that every linear lattice satisfies Schützenberger's identity. At that time the problem arose of characterizing linear lattices by identities. This brings us to two notable theorems Garrett proved in universal algebra.

The first of Birkhoff's theorems characterizes categories of algebraic systems which can be defined by identities. These are precisely those categories of algebraic systems that are closed under the three operations of products, subalgebras, and homomorphic images. For example, groups and rings can be characterized by identities, but fields cannot, because the product of two fields is not a field. There are algebraic systems which are known to be definable by identities because they have been shown to satisfy the three Birkhoff conditions but for which the actual identities are not known.

The second of Birkhoff's theorems states that a category of algebraic systems is endowed with "free algebras" if and only if it is closed under products and subalgebras.

The category of linear lattices is closed under products and sublattices, so that the free linear lattice on any set of generators exists. A thorough study of free linear lattices, revealing their rich structure, was carried out by Gelfand and Ponomarev in a remarkable series of papers. Their results are so stated as to apply both to modular and to linear lattices. The free linear lattice in n generators is intimately related to the ring of invariants of a set of n subspaces in general position in projective space. Gelfand has conjectured that the free linear lattice in four generators is decidable. Recently an explicit set of generators for the ring of invariants of a set of four subspaces in projective space has been given by Howe and Huang; Gelfand's conjecture is the lattice theoretic analog and is thus probably true.

It is not known whether linear lattices may be characterized by identities. Haiman has proved that linear lattices satisfy most of the classical theorems of projective geometry, such as various generalizations of Desargues's theorem, and he proved that not even these generalized Desarguian conditions suffice to characterize linear lattices.

The deepest results to date on linear lattices are due to Haiman, who in his thesis developed a proof theory for linear lattices. What does such a proof theory consist of? It is an iterative algorithm performed on a lattice inequality that splits the inequality into subinequalities by a tree-like procedure and eventually establishes that the inequality is true in all linear lattices, or else it automatically provides a counterexample. A proof theoretic algorithm is at least as significant as a decision procedure, since a decision procedure is merely an assurance that the proof theoretic algorithm will eventually stop.

Haiman's proof theory for linear lattices brings to fruition the program that was set forth in the celebrated paper "The logic of quantum mechanics", by Birkhoff and von Neumann. This paper argues that modular lattices provide a new logic suited to quantum mechanics. The authors did not know that the modular lattices of quantum mechanics are linear lattices. In light of Haiman's proof theory, we may now confidently assert that Birkhoff and von Neumann's logic of quantum mechanics is indeed the long-awaited new "logic" where meet and join are endowed with a logical meaning that is a direct descendant of "and" and "or" of propositional logic.

Lattice Theory and Probability

One of the dramas of present-day mathematics is the advent of noncommutative probability. Lattice theoretically, this drama is a game played with three lattices: the lattice of equivalence relations, Boolean algebras, and various linear lattices that are threatening to replace the first two.

Classical probability is a game of two lattices defined on a sample space: the Boolean σ-algebra of events and the lattice of Boolean σ-subalgebras.

A σ-subalgebra of a sample space is a generalized equivalence relation on the sample points. In a sample space the Boolean σ-algebra of events and the lattice of σ-subalgebras are dual notions, but whereas the Boolean σ-algebra of events has a simple structure, the same cannot be said of the lattice of σ-subalgebras. For example, we understand fairly well measures on a Boolean σ-algebra, but the analogous notion for the lattice of σ-subalgebras—namely, entropy—is poorly understood.

Stochastic independence of two Boolean σ-subalgebras is a strengthening of the notion of independence of equivalence relations. Commuting equivalence relations also have a stochastic analog, which is best expressed in terms of random variables. We say that two σ-subalgebras, Σ_1 and Σ_2, commute when any two random variables X_1 and X_2 defining the σ-subalgebras Σ_1 and Σ_2 are conditionally independent. Catherine Yan has studied the probabilistic analog of a lattice of commuting equivalence relations: namely, lattices of nonatomic σ-subalgebras, any two of which are stochastically commuting. There are stochastic processes where all associated σ-subalgebras are commuting in Yan's sense, for example, Gaussian processes.

In a strenuous tour de force, Catherine Yan has developed a proof theory for lattices of nonatomic commuting σ-subalgebras. Her theory casts new light on probability. It is also a vindication of Dorothy Maharam's pioneering work in the classification of Boolean σ-algebras.

The portrait of noncommutative probability is at present far from complete. Von Neumann worked hard at a probabilistic setting for quantum mechanics. His search for a quantum analog of a sample space led him to the discovery of continuous geometries. These geometries are similar to projective spaces, except that the dimension function takes all real values between zero and one. Von Neumann characterized continuous geometries as modular lattices and showed that noncommutative rings can be associated with continuous geometries which share properties of rings of random variables, in particular that there is the analog of a probability distribution.

Sadly the applications of continuous geometries have hardly been explored; allow me to stick my neck out and mention one possible such application. It is probable that some of the attractive q-identities that are now being proved by representation theoretic methods can be given a "bijective" interpretation in continuous geometries over finite fields. I have checked this conjecture only for the simplest q-identities.

The triumph of von Neumann's ideas on quantum probability is his hyperfinite factor, which unlike Hilbert space has a modular lattice of closed subspaces. For a long time I have wondered why quantum mechanics is not done in the hyperfinite factor rather than in Hilbert space. Philosophically, probability in a hyperfinite factor is more attractive than ordinary probability, since the duality between events and σ-subalgebras is replaced by a single modular lattice that plays the rọle of both. On several occasions I have asked experts in quantum mechanics why the hyperfinite factor has been quietly left aside, and invariably I received evasive answers. Most likely, physicists and mathematicians needed some fifty years of training to grow accustomed to noncommutative probability, and only now are the tables beginning to turn after the brilliant contributions to noncommutative geometry and noncommutative probability by Alain Connes and Dan Virgil Voiculescu.

Other Directions

It is heartening to watch every nook and cranny of lattice theory coming back to the fore after a long period of neglect. One recent instance: MacNeille, a student of Garrett's, developed a theory of completion by cuts of partially ordered sets, analogous to Dedekind's construction of the real numbers. His work was viewed as a dead end until last year, when Lascoux and Schützenberger, in their last joint paper, showed that MacNeille's completion neatly explains the heretofore mysterious Bruhat orders of representation theory.

Two new structures that generalize the concept of a lattice should be mentioned in closing. First, Tits buildings. It is unfortunate that presentations of buildings avoid the lattice theoretic examples, which would display the continuity of thought that leads from lattices to buildings.

Second, Δ-matroids, due to Kung, and developed by Dress, Wentzel, and several others. Garrett Birkhoff realized that Whitney's matroids could be cast in the language of geometric lattices, which Garrett first defined in a paper that appeared right after Whitney's paper in the same issue of the *American Journal.* Roughly, Δ-matroids are to Pfaffians as matroids are to determinants. Δ-matroids call for a generalization of lattices that remains to be explored.

These developments, and several others that I have not mentioned, are a belated validation of Garrett Birkhoff's vision, which we learned in three editions of his *Lattice Theory*, and they betoken Professor Gelfand's oft-repeated prediction that lattice theory will play a leading role in the mathematics of the twenty-first century.

Garrett Birkhoff and Applied Mathematics

David M. Young

Introduction

Garrett Birkhoff contributed to many areas of mathematics during his long and distinguished career. He is, of course, very well known for his work in algebra and in lattice theory. However, in this article we will focus on his work in applied mathematics, including the numerical solution of elliptic partial differential equations, reactor calculations and nuclear power, and spline approximations. We will also give a very brief discussion of his work on fluid dynamics. Additional information on Birkhoff's work in applied mathematics can be found in many of the publications listed below; see especially [11].

The author gratefully acknowledges the contributions of Richard Varga and Carl de Boor. Varga contributed the section entitled "Reactor Calculations and Nuclear Power", and de Boor contributed the section entitled "Spline Approximations".

The Numerical Solution of Elliptic Partial Differential Equations

In this section we describe two aspects of Birkhoff's work on the numerical solution of elliptic partial differential equations (PDE), his role in the automation of "relaxation methods", and his work on the dissemination of information on the numerical selection of elliptic PDE. Additional work of Birkhoff in this area is described in the section entitled "Reactor Calculations and Nuclear Power".

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The Automation of Relaxation Methods

With the advent of high-speed computers in the 1940s Birkhoff became very interested in their possible use for obtaining numerical solutions to problems involving elliptic PDE. Many such problems could be "reduced", by the use of finite difference methods, to the solution of a (usually) large system of linear algebraic equations where the matrix was very sparse. However, because of the relatively low speeds and the very limited memory sizes of computers which were then available, the direct solution of such systems was usually out of the question.

On the other hand, many such large linear systems were actually being solved by R. V. Southwell and his associates in England using relaxation methods and without using computers; see [39]. Relaxation methods involve first choosing an initial guess for the unknown solution, u, at each grid point and then computing at each point the "residual", i.e., a number which measures the amount by which the linear equation for that point fails to be satisfied. One can eliminate, or "relax", the residual at a given grid point by suitably modifying the value of u at that point. (If one "overcorrects" or "overrelaxes", then the sign of the residual is changed.) Of course the residuals at nearby grid points are also changed when the value of u at a particular grid point is changed. By repeated use of relaxation a skilled person could soon achieve a situation where all of the residuals were very small and where the values of u at the grid points provided a satisfactory solution to the problem.

In the late 1940s when I asked Birkhoff for a thesis topic, he suggested that I work on the "automation" of relaxation methods. Actually there was already a systematic iteration procedure available, namely, the "Liebmann method" [34] (which is a special case of the Gauss-Seidel method). However, the Liebmann method is often exceedingly slow. Another method that was available at the time was Richardson's method [36]. This method involves the use of a number of parameters. However, at the time it was not obvious how the parameters should be chosen. (It was discovered later that by a suitable choice of the parameters, which could be found using Chebyshev polynomials, one could obtain very rapid convergence; see, e.g., [38] and [43].)

Largely as a result of the stimulus, encouragement, and many useful suggestions provided by Birkhoff, I was able to develop a method which is now called the "successive overrelaxation" (SOR) method and which is described in [41, 42]. (The SOR method was developed independently by Frankel [31], who called it the "extrapolated Liebmann" method.) The SOR method provides an order-of-magnitude improvement in convergence as compared to the Gauss-Seidel method for many linear systems corresponding to the numerical solution of elliptic PDE. Thus, for a class of problems corresponding to the Dirichlet problem the number of iterations required for convergence with the SOR method is proportional to h^{-1}, where h is the grid size, as compared with h^{-2} as required with the Gauss-Seidel method.

The SOR method, with generalizations, modifications, and extensions (see, e.g., Varga [40]), was used extensively for engineering and scientific computations for many years. Eventually it was superseded by other methods, such as preconditioned conjugate gradient methods and methods based on the use of Chebyshev polynomials.

Further discussion of Birkhoff's role in the automation of relaxation methods can be found in [11].

Dissemination of Information on the Numerical Solution of Elliptic PDE

Birkhoff was very active in the dissemination of information on the numerical solution of elliptic PDE. This activity included the preparation of a book with Robert Lynch (see [16]) and playing a leading role in the arranging of two conferences on "Elliptic Problem Solvers". The first of these conferences was held in Santa Fe in 1980 and led to a publication; see [37]. The second conference was held in Monterey in 1982 and also led to a publication; see [20].

The book with Lynch provides an excellent survey of many topics, including formulations of typical elliptic problems and classical analysis, difference approximations, direct and iterative methods, variational methods, finite element methods, integral equation methods, and a description of the ELLPACK software package. The book contains a wealth of information and is recommended reading for anyone interested in working in this area.

The two conferences provided, among other things, forums for discussions about the ELLPACK software package that was being developed at Purdue University by John Rice and his associates. Contributions to ELLPACK were made by a number of other institutions. For example, several iterative programs were contributed by The University of Texas.

David Kincaid and David Young, who directed the development at The University of Texas of the ITPACK software package for solving large sparse linear systems by iterative methods, regard Birkhoff as the "godfather" of the project. For several years he had been patiently but seriously suggesting that such a package be developed. The implementation of his idea was delayed in part by uncertainty as to how to choose the iteration parameters, such as omega for the SOR method, and how to decide when to terminate the iteration process. Eventually, as described in the book by Hageman and Young [32] and in the paper by Kincaid et al. [33], these and other obstacles were largely overcome and the ITPACK software package was completed.

Reactor Calculations and Nuclear Power

Garrett Birkhoff was intimately associated with reactor computations which played an essential role in the design of nuclear power reactors. This arose primarily from his role as a consultant to the Bettis Atomic Power Laboratory from 1955 through the early 1960s.

As a brief background, analytical models of nuclear reactors were brand new in the early 1950s, unlike the case of analytical fluid dynamics, which had enjoyed two hundred years of development. Fortuitously, high-powered digital computers were also making their appearance in the early 1950s. Because building full-scale nuclear reactors was both expensive and very time consuming, it was prudent and farsighted then to look to digital computers to numerically solve the associated nuclear reactor models. Even more fortuitous was the simultaneous emergence in 1950 of David M. Young's thesis [41], which contained an analytic treatment of the SOR iterative method for numerically solving second-order elliptic boundary problems.

In that exciting period when nuclear reactors were first being considered for naval ships, Bettis hired in 1954 five new Ph.D.s—Harvey Amster, Elis Gelbard, and Stanley Stein in physics, and Jerome Spanier and Richard Varga in mathematics—all of whom made contributions to various aspects of nuclear reactor theory. There is no doubt that detailed discussions with the energetic consultant, Garrett Birkhoff, helped solidify many of their emerging ideas. Garrett loved the challenge of working in new research areas, and his enthusiasm was infectious!

But Garrett's contributions to reactor theory and reactor computations were much more than just the random discussions of a consultant with Bettis people. Three solid contributions of his stand out. Early on he saw the relevance of nonnegative matrices (or, more generally, operators which leave a cone invariant) to nuclear reactor theory, and this can be seen in his publications [3] and [21]. In the latter paper the now well-known terms *essentially nonnegative* and *essentially positive* matrices, as well as *supercritical*, *critical*, and *subcritical* multiplicative processes, were first introduced. Second, while SOR-type iterative methods were being used for solving reactor problems at Bettis, alternating-direction (implicit), or ADI, iterative methods were similarly used for solving reactor problems at the Knolls Atomic Power Laboratory. The superiority of ADI iterative methods over the SOR method had been shown by Peaceman and Rachford [35] and by Douglas and Rachford [30], both for special Laplace-type problems in a rectangle. Garrett observed, in a classroom lecture at Harvard University, that the *commuting nature* of certain matrices may not hold in regions other than a rectangle, a property implicitly used in [30] and in [35]. This observation was the impetus for two research papers, [22] and [28], where many positive and negative results for such ADI schemes were presented.

Garrett was also very much interested in *semi-discrete* approximations of time-dependent problems, such as the heat-conduction equation; here "semi-discrete" means that time remains a continuous variable while other variables, usually the space variables, are discretized. This was researched in his paper [28], where Padé approximations to the function $\exp(z)$ were connected with time-stepping schemes for parabolic-like partial differential equations.

In no uncertain terms, Garrett Birkhoff, through his own research and his collaboration with others, left an indelible mark on nuclear reactor theory.

Spline Approximation

Birkhoff materially influenced the early development of spline theory and practice through his consulting work for General Motors Research. This work started in 1959 when General Motors decided that perhaps widespread use of nuclear energy was not just around the corner and needed some other useful problems for some of the members of its Nuclear Engineering Department to work on. One of the problems posed was the mathematical representation of automobile surfaces in order to exploit the recently developed numerically controlled milling machines for the cutting of dies needed for the stamping of outer and inner panels. The idea was to determine the free parameters in a suitably flexible mathematical model so as to fit closely to measurements taken from the finished physical model of the car. There was also the hope that eventually the design process itself could be carried out entirely on computers.

Birkhoff was quick to recommend the use of cubic splines (i.e., piecewise cubic polynomial functions with two continuous derivatives) for the representation of smooth curves. He was familiar with their use in naval design through his contact with the David Taylor Model Basin, and he also knew of their use at Boeing through a report written by MacLaren. Furthermore, in joint work with Henry Garabedian (see [14]) he developed what we would now call a four-mode, twelve-parameter C^1 macro finite element consisting of eight harmonic polynomial pieces, as a bivariate generalization of cubic spline interpolation, capable of interpolating a C^1 surface to a given rectangular mesh of cubic splines. This method eventually led de Boor to the now standard method of bicubic spline interpolation.

Subsequently, W. J. Gordon of General Motors Research developed the technique of spline blending for fitting smooth surfaces to an arbitrary (rectangular) smooth mesh of curves. This method too has become standard. Some mathematical aspects of blending are taken up in [15].

Birkhoff observed that the cubic spline is a good approximation to the draftsman's (physical) spline only when the latter is nearly flat. He contributed to the mathematical understanding of a more accurate model of the latter; see [26]. His insight into mechanics also made it obvious to him that a cubic spline which vanishes at all its modes must necessarily have exponential growth in at least one direction. The resulting paper [12] on the error in cubic spline interpolation was the first one to demonstrate and make use of the exponential decay of the fundamental functions of spline interpolation for "reasonable" breakpoint sequences.

The survey paper [13] provides a very good record for the many and wide-ranging suggestions concerning interpolation and approximation to univariate and bivariate data which Birkhoff made in those early days.

Somewhat later, in [4], a paper on local spline approximation by moments, Birkhoff proposed what is probably the first spline quasi-interpolant, i.e., a method of approximation that is local, stable, and aims only at reproducing all polynomials of a certain degree (rather than at matching function values).

Birkhoff's method is now treated as a special case of the de Boor-Fix quasi-interpolant. Already the above-mentioned survey contains detailed ideas about the use of splines in the numerical solution of integral and differential equations. The case of eigenvalue calculations for second-order ordinary differential equations via the Rayleigh-Ritz method is worked out in detail in [27], while the use of tensor-product splines in the numerical solution of partial differential equations is examined in [29] and in other work by Schultz. Since rectangular meshes cannot handle all practically important situations, Birkhoff also investigated splines on triangular meshes in [8, 1, 17]. The theme of multivariate interpolation was taken up one more time, but this time by Birkhoff the algebraist in [9].

Numerical Fluid Dynamics

In this section a very brief discussion of Birkhoff's work in numerical fluid dynamics will be given. For additional information the reader should see his two books, which are cited below, as well as his survey article [10].

Birkhoff worked extensively in numerical fluid dynamics, especially from the middle 1940s to the late 1950s. He was greatly influenced by the work of John von Neumann in fluid dynamics and in the then-emerging field of high-speed computing.

In 1981 Birkhoff was invited to give the John von Neumann lecture at the SIAM meeting in Troy, New York. This lecture led to the publication of a very informative survey article in numerical fluid dynamics; see [10].

It seems truly unfortunate that Birkhoff will not be around to witness the many advances in numerical fluid dynamics which will undoubtedly take place in the next twenty-five to fifty years and which in many cases will benefit from his ideas.

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