Representations of Finite Groups: A Hundred Years, Part II

T. Y. Lam

Recapitulation

The origin of the representation theory of finite groups can be traced back to a correspondence between R. Dedekind and F. G. Frobenius that took place in April of 1896. The present article is based on several lectures given by the author in 1996 in commemoration of the centennial of this occasion.

In Part I of this article we recounted the story of how Dedekind proposed to Frobenius the problem of factoring a certain homogeneous polynomial arising from a determinant (called the "group determinant") associated with a finite group G. In the case when G is abelian, Dedekind was able to factor the group determinant into linear factors using the characters of G (namely, homomorphisms of G into the group of nonzero complex numbers). In a stroke of genius, Frobenius invented a general character theory for arbitrary finite groups, and used it to give a complete solution to Dedekind's group determinant problem. Interestingly, Frobenius's first definition of (nonabelian) characters was given in a rather ad hoc fashion, via the eigenvalues of a certain set of commuting matrices. This work led Frobenius to formulate, in 1897, the modern definition of a (matrix) representation of a group G as a homomorphism $D : G \to \mathrm{GL}_n(\mathbb{C})$ (for some n). With this definition in place, the character $\chi_D : G \to \mathbb{C}$ of the representation is simply defined by $\chi_D(g) = \mathrm{trace}(D(g))$ (for every $g \in G$). The idea of studying a group through its various representations opened the door to a whole new direction of research in group theory and its applications.

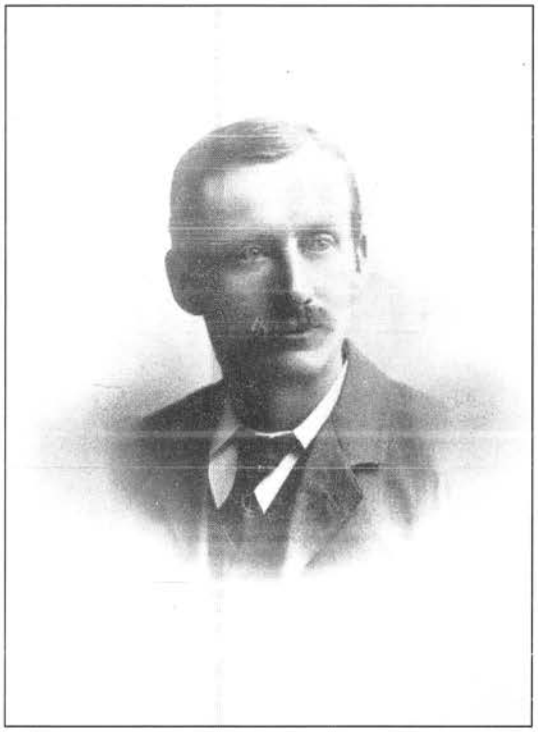
Having surveyed Frobenius's invention of character theory and his subsequent monumental contributions to representation theory in Part I of this article, we now move on to tell the story of another giant of the subject, the English group theorist W. Burnside. This comprises Part II of the article, which can be read largely independently of Part I. For the reader's convenience, the few bibliographical references needed from Part I are reproduced here, with the same letter codes for the sake of consistency. As in Part I, [F: (53)] refers to paper (53) in Frobenius's collected works [F]. Burnside's papers are referred to by the year of publication, from the master list compiled by Wagner and Mosenthal in [B]. Consultation of the original papers is, however, not necessary for following the general exposition in this article.

William Burnside (1852–1927)

Remarks in this section about Burnside's life and work are mainly taken from A. R. Forsyth's obituary note [Fo] on Burnside published in the *Journal of the London Mathematical Society* the year after Burnside's death, and from the forthcoming book

T. Y. Lam is professor of mathematics at the University of California, Berkeley. His e-mail address is `lam@math.berkeley.edu`.

This article, in two parts, is a slightly abridged version of an article appearing concurrently in the Proceedings of the "Aspects of Mathematics" conference *(N. Mok, ed.), University of Hong Kong, H.K., 1998. Part I, concerning the study of group determinants by Ferdinand Georg Frobenius and the resulting theory of characters and group representations, appeared in the March 1998 issue of the* Notices, *pages 361–372.*



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William Burnside.

of C. Curtis on the pioneers of representation theory [Cu_2, Ch. 3].

Born in London of Scottish stock, William Burnside received a traditional university education in St. John's and Pembroke Colleges in Cambridge. In Pembroke he distinguished himself both as a mathematician and as an oarsman, graduating from Cambridge as Second Wrangler in the 1875 Mathematical Tripos. He took up a lectureship in Cambridge after that, and remained there for some ten years, teaching mathematics and acting as coach for both the Math Tripos and for the rowing crews. In 1885, at the instance of the Director of Naval Instruction (a former Cambridge man named William Niven), Burnside accepted the position of professor of mathematics in the Royal Naval College at Greenwich. He spent the rest of his career in Greenwich, but kept close ties with the Cambridge circles, and never ceased to take an active role in the affairs of the London Mathematical Society, serving long terms on its Council, including a two-year term as president (1906–08). In Greenwich he taught mathematics to naval personnel, which included gunnery and torpedo officers, civil and mechanical engineers, as well as cadets. The teaching task was not too demanding for Burnside, which was just fine, as it afforded him the time to pursue an active program of research. Although physically away from the major mathematical centers of England, he kept abreast of the current progress in research throughout his career, and published a total of some 150 papers in pure and applied mathematics. By all accounts, Burnside led a life of steadfast devotion to his science.

Burnside's early training was very much steeped in the tradition of applied mathematics in Cambridge.[1] At that time, applied mathematics meant essentially the applications of analysis (function theory, differential equations, etc.) to topics in theoretical physics such as kinematics, elasticity, electrostatics, hydrodynamics, and the theory of gases. So not surprisingly, in the first fifteen years of his career, Burnside's published papers were either in these applied areas, or else in elliptic and automorphic functions and differential geometry. On account of this work, he was elected Fellow of the Royal Society in 1893. Coincidentally, it was also around this time that Burnside's mathematical interests began to shift to group theory, a subject to which he was to devote his main creative energy in his mature years.

After authoring a series of papers entitled "Notes on the theory of groups of finite order" (and others), Burnside published his group theory book [B_1] in 1897, the first in the English language offering a comprehensive treatment of finite group theory. A second, expanded edition with new material on group representations appeared in 1911. For more than half a century, this book was without doubt the one most often referred to for a detailed exposition of basic material in group theory. Reprinted by Dover in 1955 (and sold for \$2.45), Burnside's book is now enshrined as one of the true classics of mathematics. We will have more to say about this book in the next section.[2]

Since group theory was not a popular subject in England at the turn of the century, Burnside's group-theoretic work was perhaps not as much appreciated as it could have been. When Burnside died in 1927, the *London Times* reported the passing of "one of the best known Cambridge athletes of his day".[3] We can blame this perhaps on the journalist's ignorance and lack of appreciation of mathematics. However, even in Forsyth's detailed obituary, which occupied seventeen pages of the *Journal of the London Math Society*, no more than a page was devoted to Burnside's work in group theory, even though Forsyth was fully aware that it was this work that would "provide the most continuous and most conspicuous of his contributions to his science." None of Burnside's greatest achievements that now make him a household

[1] *According to Forsyth [Fo], pure mathematics was then largely "left to Cayley's domain, unfrequented by aspirants for high place in the tripos."*

[2] *Those familiar with Dover publications will know that there are two other Dover reprints of books by "Burnside": one is* Theory of Probability, *and the other is* Theory of Equations. *The former was indeed written by William Burnside; published posthumously in 1928, it was also one of the earliest texts in probability theory written in English. However, the 2-volume work* Theory of Equations *was written (ca. 1904) by Panton and* another *Burnside. William Snow Burnside, professor of mathematics in Dublin, was a contemporary of William Burnside; they published papers in the same English journals, one as "W. S. Burnside" and the other simply as "W. Burnside". An earlier commentary on this was given by S. Abhyankar [Ab, footnote 43, p. 91].*

[3] *The full text of the* London Times *obituary on Burnside was quoted in [Cu_2, Ch. 3].*

name in group theory was even mentioned in Forsyth's article. I can think of two reasons for this. The first is perhaps that Forsyth did not have any real appreciation of group theory. While he was Sadlerian Professor of Mathematics in Cambridge, his major field was function theory and differential equations.[4] We cannot blame him for being more enthusiastic about Burnside's work in function theory and applied mathematics; after all, it was this work that won Burnside membership in the Royal Society. Second, Burnside's achievements in group theory were truly way ahead of his time; the deep significance of his ideas and the true power of his vision only became clear many a year after his death. Today, I do not hear my applied math colleagues talk about Burnside's work in hydrodynamics or the kinetic theory of gases, but I will definitely teach my students Burnside's great proof of the $p^a q^b$ theorem (that any group of order $p^a q^b$ is solvable) in my graduate course in group representation theory! In mathematics, as in other sciences, it is time that will tell what are the best and the most lasting human accomplishments.

Theory of Groups of Finite Order (1897, 1911)

Through his work on the automorphic functions of Klein and Poincaré, Burnside was knowledgeable about the theory of discontinuous groups. It was perhaps this connection that eventually steered him away from applied mathematics and toward research on the theory of groups of finite order. In the early 1890s, Burnside followed closely Hölder's work on groups of specific orders; soon he was publishing his own results on the nature of the order of finite simple groups. Frobenius's early papers in group theory apparently first aroused his interest in finite solvable groups.[5]

The first edition of Burnside's masterpiece *Theory of Groups of Finite Order* appeared in 1897; it was clearly the most important book in group theory written around the turn of the century. While intended as an introduction to finite group theory for English readers, the book happened to contain some of the latest research results in the area at that time. For instance, groups of order $p^a q^b$ were shown to be solvable if either $a \leq 2$ or if the Sylow groups were abelian, and (nonabelian) simple groups were shown to have even order if the order was the product of fewer than six primes.[6] It is clear that Burnside realized that these results were not in their final form, for he wrote in [B_1, 1st ed., p. 344]:

> If the results appear fragmentary, it must be mentioned that this branch of the subject has only recently received attention: it should be regarded as a promising field of investigation than as one which is thoroughly explored.

Burnside was right on target in his perception that much more was in store for this line of research. However, Burnside's assessment at that time of the possible role of groups of linear substitutions was a bit tentative. Since the theory of permutation groups occupied a large part of [B_1] while groups of linear substitutions hardly received any attention, Burnside felt obliged to give an explanation to his readers. In the preface to the first edition of [B_1], he wrote:

> My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

Little did he know that, just as his book was going to press, Frobenius on the continent had just made his breakthroughs in the invention of group characters, and was in fact writing up his first memoir [F: (56)] on the new representation theory of groups! As it turned out, the next decade witnessed some of the most spectacular successes in applying representation theory to the study of the structure of finite groups—and Burnside himself was to be a primary figure responsible for these successes. There was no question that Burnside wanted his readers to be brought up to date on this exciting development. When the second edition of [B_1] came out in 1911 (fourteen years after the first), it was a very different book, with much more definitive results and with six brand new chapters introducing his readers to the methods of group representation theory! In the preface to this new edition, Burnside wrote:

> In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good. In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions.

[4] *His 2-volume work on the former and 6-volume work on the latter were quite popular in his day.*

[5] *A group G is* solvable *if it can be constructed from abelian groups via a finite number of group extensions. In case G is finite, an equivalent definition is that the composition factors of G are all of prime order.*

[6] *Burnside showed that the order must be 60, 168, 660, or 1,092.*

With these words, Burnside brought the subject of group theory into the twentieth century, and he went on to present 500 pages of great mathematics in his elegant and masterful style. Though later authors have found an occasional mistake in $[B_1]$, and there are certainly some typographical errors,[7] Burnside's book has remained as valuable a reference today as it has been throughout this century. I myself have developed such a fondness for Burnside's book that every time I walk into a used book store and have the good fortune to find a copy of the Dover edition on the shelf, I would buy it. By now I have acquired seven (or is it eight?) copies. True book connoisseurs would go instead for the first edition of Burnside's book, because of its scarcity and historical value. Apparently, a good copy could command a few hundred dollars in the rare book market.

Burnside's Work in Representation Theory

After reading Frobenius's papers [F: (53), (54)], Burnside saw almost immediately the relevance of Frobenius's new theory to his own research on finite groups. What he tried to do first was to understand Frobenius's results in his own way. In the 1890s Burnside had also followed closely the work of Sophus Lie on continuous groups of transformations, so, unlike Frobenius, he was conversant with the methods of Lie groups and Lie algebras. Given a finite group G, he was soon able to define a Lie group from G whose Lie algebra is the group algebra $\mathbb{C}G$ endowed with the bracket operation $[A, B] = AB - BA$. Analyzing the structure of this Lie algebra, he succeeded in deriving Frobenius's principal results both on characters and on the group determinant. (For more details on this, see $[Cu_2]$ and $[H_2]$.) He published these results in several parts, in [B: 1898a, 1900b], etc. Burnside certainly was not claiming that he had anything new; he wrote in [B: 1900b]:

> The present paper has been written with the intention of introducing this new development to English readers. It is not original, as the results arrived at are, with one or two slight exceptions, due to Herr Frobenius. The modes of proof, however, are in general quite distinct from those used by Herr Frobenius.

Actually, while Burnside's methods were different from Frobenius's, what he did was close in spirit to what was done by Molien $[M_1]$ in 1893. Both used the idea of the regular representation, the only difference being, that Molien worked with $\mathbb{C}G$ as an associative algebra (or "hypercomplex system") while Burnside worked with $\mathbb{C}G$ as a Lie algebra. However, Molien's paper $[M_1]$ was understood by few, which was perhaps why it did not receive the recognition that it deserved. Burnside, for one, was apparently frustrated by the exposition in Molien's paper. In a later work [B: 1902f], in referring to $[M_1]$, Burnside lamented openly:

> It is not, in fact, very easy to find exactly what is and what is not contained in Herr Molien's memoir.

Later, the methods of both Molien and Burnside were superseded by those of Emmy Noether [N]. As was noted in the section "Factorization of $\Theta(G)$ for Modern Readers" in Part I of this article, a quick application of the theorems of Maschke and Wedderburn à la Noether yields all there is to know about representations at the basic level.

The next stage of Burnside's work consists of his detailed investigation into the nature of irreducible representations and their applications. Recall that, for Frobenius, the irreducibility (or "primitivity") of a representation was originally defined by the irreducibility of its associated determinant. Since this was clearly a rather unwieldy definition, a more direct alternative definition was desirable. Burnside [B: 1898a] and Frobenius [F: (56)] had both given definitions for the irreducibility of a representation in terms of the representing matrices, although, as Charles Curtis pointed out to me, these early definitions seemed to amount to what we now call *indecomposable* representations. By 1898, E. H. Moore (and independently A. Loewy) had obtained the result that any finite group of linear substitutions admits a nondegenerate invariant hermitian form, and in 1899, Maschke used Moore's result to prove the "splitting" of any subrepresentation of a representation of a finite group (now called "Maschke's Theorem"). With these new results, whatever confusion that might have existed in the definition of the irreducibility of a representation became immaterial. By 1901 (if not earlier), Burnside was able to describe an irreducible representation in no uncertain terms [B: 1900b, p. 147]: "A group G of finite order is said to be represented as an irreducible group of linear substitutions on m variables when G is simply or multiply isomorphic with the group of linear substitutions, and when it is impossible to choose m' $(< m)$ linear functions of the variables which are transformed among themselves by every operation of the group." Aside from its long-windedness, this *is* basically the definition of irreducibility of a matrix representation we use today.

With credit duly given to Maschke (and Frobenius), Burnside went on to prove the "complete reducibility" of representations of finite groups in [B: 1904c], and used it to give a self-contained ac-

[7] *The most glaring one appeared in the General Index, p. 510, where Burnside "blew" his own great theorem with an amusing entry "Groups of order $p^a q^b$, where p, q are primes, are simple" (which even survived the 1955 Dover edition). Who did the proofreading? Oh, dear ....*

count of basic character theory in [B: 1903d], independently of the continuous group approach he initiated in [B: 1898a]. One year later in [B: 1905b], Burnside arrived at a beautiful characterization of an irreducible representation that is still in use today:

Theorem 4.1. A representation $D : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ is irreducible if and only if the matrices in $D(G)$ span $\mathbb{M}_n(\mathbb{C})$.

Burnside proved this result for arbitrary (not just finite) groups (and used it later in [B: 1905c] for possibly infinite groups). Subsequently, Frobenius and Schur extended this theorem to "semigroups" of linear transformations. With this hindsight, we can state Burnside's main result in the following ring-theoretic fashion: *a subalgebra A of $\mathbb{M}_n(\mathbb{C})$ has no nontrivial invariant subspaces in $\mathbb{C}^n$ if and only if $A = \mathbb{M}_n(\mathbb{C})$*. Stated in this form, Burnside's result is of current interest to workers in operator algebras and invariant subspaces. Some generalizations to an infinite-dimensional setting have been obtained, for instance, in Chapter 8 of [HR]. We should also point out that Burnside's result holds in any characteristic; the only necessary assumption is that the ground field be algebraically closed (see [L, p. 109]).

Burnside had a keen eye for the arithmetic of characters (which he called "group characteristics"); many of his contributions to character theory were derived from his unerring sense of the arithmetic behavior of the values of characters. The following are some typical samples of his results proved in this spirit:

1. Every irreducible character χ with $\chi(1) > 1$ has a zero value.

2. The number of real-valued irreducible characters of a group G is equal to the number of real conjugacy classes[8] in G.

3. (Consequences of (2)) If $|G|$ is even, there must exist a real-valued irreducible character other than the trivial character (and conversely). If $|G|$ is odd, then the number of conjugacy classes in G is congruent to $|G|$ (mod 16).

4. If χ is the character of a faithful representation of G, then any irreducible character is a constituent of some power of χ. (This result is usually attributed to Burnside, although, as Hawkins pointed out in [H_3, p. 241], it was proved earlier by Molien. A quantitative version of the result was found later by R. Brauer.)

Burnside's most lasting result in group representations is, of course, his great $p^a q^b$ theorem, which we have already mentioned. Again, the approach he took to reach this theorem was purely arithmetic. Using arguments involving roots of unity and Galois conjugates, he proved the following result in [B: 1904a]:

Theorem 4.2. Let $g \in G$ and χ be an irreducible character of G. If $\chi(1)$ is relatively prime to the cardinality of the conjugacy class of g, then $|\chi(g)|$ is equal to either 0 or $\chi(1)$.

Combining this result with the Second Orthogonality Relation (given in the display box in Part I, p. 368), he obtained a very remarkable sufficient condition for the *nonsimplicity* of a (finite) group:

Theorem 4.3. If a (finite) group G has a conjugacy class with cardinality p^k where p is a prime and $k \geq 1$, then G is not a simple group.

This sufficient condition for nonsimplicity is so powerful that, from it, Burnside obtained immediately the $p^a q^b$ theorem, a coveted goal of group theorists for more than ten years.[9] It is interesting to point out that, in [B_1, 2nd ed., p. 323], after proving the powerful theorem (4.2), Burnside simply stated the $p^a q^b$ theorem as "Corollary 3":

Theorem 4.4. For any primes p, q, any group G of order $p^a q^b$ is solvable.

The proof is so easy and pleasant by means of Sylow theory that we have to repeat it here. We may assume $p \neq q$. By induction on $|G|$, it suffices to show that G is not a simple group. Fix a subgroup Q of order q^b (which exists by Sylow's Theorem), and take an element $g \neq 1$ in the center of Q. If g is central in G, G is clearly nonsimple. If otherwise, $C_G(g)$ (the centralizer of g in G) is a proper subgroup of G containing Q. Then the conjugacy class of g has cardinality $[G : C_G(g)] = p^k$ for some $k \geq 1$, so G is again nonsimple by (4.3), as desired!

At first sight, (4.4) may not look like such a deep result. However, for many years, it defied the group theorists' effort to find a purely group-theoretic proof. It was only in 1970 that, following up on ideas of J. G. Thompson, D. Goldschmidt [Go] gave the first group-theoretic proof for the case when p, q are *odd* primes. Goldschmidt's proof used a rather deep result in group theory, called Glauberman's $Z(J)$-theorem. A couple of years later, Matsuyama [Mat] completed Goldschmidt's work by supplying a group-theoretic proof for (4.4) in the remaining case $p = 2$. Slightly ahead of [Mat], Bender [Be] also gave a proof of (4.4) *for all p, q*, using (among other things) a variant of the notion of the "Thompson subgroup" of a p-group. While these group-theoretic proofs are not long, they involve technical ad hoc arguments, and certainly do not come close to the compelling simplicity and the striking beauty of Burnside's original character theoretic proof. As for the more general The-

[8] *A conjugacy class is said to be* real *if it is closed under the inverse map.*

[9] *As we indicated in the previous section, in his earlier papers Burnside had proved many special cases of this theorem without using representation theory; much of this was summarized in the first edition of [B_1].*

orem 4.3, which led to the $p^a q^b$ theorem, a purely group-theoretic proof has not been found to date.

Burnside: Visionary and Prophet

If one traces Burnside's group-theoretic work back to its very beginning, it should be clear that one of his main objectives from the start was to understand finite simple groups. Burnside knew the role of finite simple groups from the work of Galois and from the Jordan-Hölder Theorem, but in the 1890s, there was very little to go on. The only known finite simple groups were Galois's alternating groups A_n ($n \geq 5$), Jordan's projective special linear groups $\mathrm{PSL}_2(p)$ ($p \geq 5$), some of the Mathieu groups, and Cole's simple group of order 504. The latter group is now recognized as $\mathrm{PSL}_2(8)$, but finite fields were hardly known in the early 1890s, so in 1893, Cole [Co] had to construct this group "by bare hands" as a permutation group of degree nine.[10]

By 1892, Hölder found all simple groups of order ≤ 200. In another year, with his construction of the simple group of order 504, Cole pushed Hölder's work to the order $660 = |\mathrm{PSL}_2(11)|$, and in another two years, Burnside further extended this work to the order $1{,}092 = |\mathrm{PSL}_2(13)|$. He also proved some of the earliest theorems on the orders of simple groups, showing, for instance, that if they are even, they must be divisible by 12, 16, or 56. With the aid of these theoretical results, which he summarized in the first edition of [B$_1$], Burnside was confident that Hölder's program could be pushed to at least the order 2,000. He observed with premonition, however, that: "As the limit of the order is increased, such investigations as these rapidly become more laborious, as a continually increasing number of special cases have to be dealt with." Clearly, stronger general results would be desirable! With what we can now appreciate as truly uncanny foresight, Burnside finished the first edition of his book with the following closing remark:[11]

> No simple group of odd order is at present known to exist. An investigation as to the existence or nonexistence of such groups would undoubtedly lead, whatever the conclusion might be, to results of importance; it may be recommended to the reader as well worth his attention. Also, there is no known simple group whose order contains fewer than three different primes.…Investigation in this direction is also likely to lead to results of interest and importance.

Thus, in one single paragraph, Burnside managed to lay down two of the most important research problems in finite group theory to be reckoned with in the next century.

"…there is no known simple group whose order contains fewer than three different primes.… Investigation in this direction is also likely to lead to results of interest and importance."

The second one was not to remain open for very long. As we know, Burnside's tour de force[B: 1904a] solved this problem: the solvability of groups of order $p^a q^b$ (Theorem 4.4) implied their nonsimplicity (and conversely, of course). The proof of this great result depended critically on the newly invented tools of character theory. However, the first problem, equivalent to the solvability of groups of odd order, proved to be very difficult. Burnside's odd-order papers [B: 1900c] were clearly aimed at solving this problem, and he obtained many positive results. For instance, he showed that odd-order groups of order $<40{,}000$ were solvable, and ditto for odd-order transitive permutation groups of degree either a prime, or < 100. The fact that some of the proofs involved character theoretic arguments prompted Burnside to make the following prescient comment at the end of the introduction to [B: 1900c]:

> The results obtained in this paper, partial as they necessarily are, appear to me to indicate that an answer to the interesting question as to the existence or nonexistence of simple groups of odd composite order may be arrived at by a further study of the theory of group characteristics.

By the time he published the second edition of [B$_1$], it was clear that Burnside was morally convinced that odd-order groups should be solvable. Short of making a conjecture, he summarized the situation by stating (in Note M, p. 503) that: "The contrast that these results shew between groups of odd

[10]*Five years later, Burnside gave a presentation of Cole's group by generators and relations, and observed that it is isomorphic to $PSL_2(8)$ in [B: 1898d].*

[11]*Notes to §260 in [B$_1$, 1st ed., p. 379].*

and even order suggests inevitably that simple groups of odd order do not exist."

Burnside was not to see a solution of the odd-order problem in his lifetime; in fact, there was not much progress on the problem to speak of for at least another forty-five years. Then, with Brauer's new idea of studying simple groups via the centralizers of involutions gradually taking hold, new positive results began to emerge on the horizon. Finally, building on work of M. Suzuki, M. Hall and themselves, Feit and Thompson succeeded in proving the solvability of all odd-order groups in 1963. Their closely reasoned work [FT] of 255 pages occupied a single issue of the *Pacific Journal of Mathematics*. Burnside was proved to be right not only in "conjecturing" the theorem, but also in predicting the important role that character theory would play in its proof. Indeed, Chapter V of the Feit-Thompson paper, almost 60 pages in length, relies almost totally on working with characters and Frobenius groups. Feit and Thompson received the Cole Prize for this work in 1965, and Thompson was awarded the Fields Medal in 1970 for his subsequent work on minimal simple groups. All of this work culminated later in the classification program of finite simple groups in the early 1980s.

"The contrast that these results shew between groups of odd and even order suggests inevitably that simple groups of odd order do not exist."

The spectacular successes of this program have apparently exceeded even Burnside's dreams, for he had stated on page 370 in the first edition of [B_1] that "a complete solution of this latter problem is not to be expected." That was, however, in the "dark ages" of the 1890s. Burnside would probably have felt very differently if he had known the $p^a q^b$ theorem, the odd-order theorem, and the existence of some of the sporadic simple groups found in the 1960s and 1970s. Today I think it is generally agreed that the classification program of finite simple groups could not have been possible without the pioneering efforts of Burnside.

Another well-known group-theoretic problem that came from Burnside's work in 1902–05 concerns the structure of torsion groups (groups all of whose elements have finite order). There are two (obviously related) versions of this problem, which may be stated as follows:

Burnside Problem (1). Let G be a finitely generated torsion group. Is G necessarily finite?

Burnside Problem (2). Let G be a finitely generated group of finite exponent N (that is, $g^N = 1$ for any $g \in G$). Is G necessarily finite?

Working in the setting of representation theory, Burnside was able to give an affirmative answer to (2) *in the case of complex linear groups*. In fact, his methods showed that, if G is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ for some n, and G has exponent N, then $|G| \leq N^3$. This result was proved by a trace argument, obviously inspired by Burnside's then ongoing work on characters. Burnside also showed that the answer to (2) is yes for any group G with exponent $N \leq 3$. Later, Schur gave an affirmative answer to (1) for any $G \subseteq \mathrm{GL}_n(\mathbb{C})$, and Kaplansky extended Schur's result to $G \subseteq \mathrm{GL}_n(k)$ for any field k; details of the proofs can be found in [L, §9].

Progress on Burnside's Problems (1) and (2) was at first very slow. A positive solution for (2) was furnished for $N = 4$ by I. N. Sanov in 1948, and for $N = 6$ by M. Hall in 1958. For $N \geq 72$, P. S. Novikov announced a negative answer to (2) in 1959; however, the details were never published. Finally, for N odd and ≥ 4381, the negative answer to (2) appeared in the joint work of P. S. Novikov and S. I. Adian in 1968. For small values of $N \notin \{2,3,4,6\}$ or N even, apparently not much is known. In particular, the cases $N = 5$, 8 seem to be still open. As for Problem (1), the answer turned out to be much easier. In 1964, E. S. Golod produced for every prime p an *infinite* group on two generators in which every element has order a finite power of p; this disposed of Burnside's Problem (1) in the negative.

This was, however, not the end of the story. Since the 1930s, group theorists have considered another variant of the Burnside Problems, which we can formulate as follows. For given natural numbers r and N, let $B(r,N)$ be the "universal Burnside group" with r generators and exponent N; in other words, $B(r,N)$ is the quotient of the free group on r generators by the normal subgroup generated by all Nth powers. Burnside's Problem (2) above amounts to asking whether $B(r,N)$ is a finite group. The following variant of this problem is called

The Restricted Burnside Problem. For given natural numbers r and N, are there only finitely many finite quotients of $B(r,N)$?

The point is that, even if the universal group $B(r,N)$ is infinite, one would hope that there are only finitely many ways of "specializing" it into finite quotients (and therefore a unique way to specialize it into a *largest possible* finite quotient). In

1959, A. I. Kostrikin announced a positive solution to this problem for all prime exponents; much of his work (and that of the Russian school) is reported in his subsequent book *Around Burnside*. After the partial negative solution of the Burnside Problem (2) was known, the interest in the Restricted Burnside Problem intensified. The breakthrough came in the early 1990s when E. Zelmanov came up with an affirmative solution to this problem, *for all r and all N*. Surprisingly (to others if not to experts), Zelmanov's solution depends heavily on the methods of Lie algebras and Jordan algebras. Another ingredient in Zelmanov's solution is the classification of finite simple groups: some consequences of the classification theorem were used in reducing the general exponent case to the case of prime power exponent via the earlier results of Hall and Higman. Zelmanov's main work was then to affirm the Restricted Burnside Problem first for $N = p^k$ with p odd [Z_1], and then for the (much harder) case $N = 2^k$ [Z_2]. For this work, Zelmanov received the Fields Medal in 1994. Looking back, I think it is quite remarkable that Burnside's work in representation theory and the open problems he proposed actually spawned the later work of two Fields Medalists. What a tremendous legacy to mathematics!

Some of my teachers and mentors have always urged me to "read the masters": they taught me that the great insight of the masters, implicit or explicit in their original writing, is not to be missed at any cost. In closing this section, I think I'll pass on this cogent piece of advice to our younger colleagues, using Burnside's book [B_1] again as a case in point. There is so much valuable information packed into this classic that sometimes it is left to later generations to unearth the "treasures" that the great master (knowingly or sometimes even unknowingly) left behind. In §§184–185 in the second edition of [B_1], Burnside discussed the characters of transitive permutation representations of a group G by making a "table of marks" (their character values), and showed how to "compound" such marks and resolve the results into integral combinations of the said marks. More than a half century later, L. Solomon resurrected this idea in [So], and formally constructed the commutative Grothendieck ring of the isomorphism classes of finite G-sets, which he appropriately christened the "Burnside ring" of the group. Today this Burnside ring $\mathfrak{B}(G)$ is an important object not only in representation theory, but also in combinatorics and topology (especially homotopy theory).[12] Some of the connections between $\mathfrak{B}(G)$ and the group G itself found by later authors are rather amazing. For instance, A. Dress [Dr] has shown that *G is a solvable group if and only if the Zariski prime spectrum of $\mathfrak{B}(G)$ is connected*, and there is even a similar characterization of minimal simple groups G in terms of $\mathfrak{B}(G)$. When I saw Louis Solomon in April 1997 at an MSRI workshop on the interface of representation theory and combinatorics, I asked him if the term "Burnside ring" originated with his paper [So]. He confirmed this, but added emphatically, "It is all in Burnside!"

A Tale of Two Mathematicians

As I contemplated and wrote about the career and work of F. G. Frobenius and W. Burnside, I could not help noticing the many interesting parallels between these two brilliant mathematicians. There was as much difference in style between them as one would expect between a German and an Englishman, and yet there were so many remarkable similarities in their mathematical lives that it is tempting for us to venture a direct comparison.

Burnside was three years Frobenius's junior, and survived him by ten, so they were truly contemporaries. Coincidentally, they were elected to the highest learned society of their respective countries in the same year, 1893: Frobenius to the Prussian Academy of Sciences, and Burnside to the Royal Society of England. Mathematically, both started with analysis and found group theory as the subject of their true love in their mature years. Both got into group theory via the Sylow Theorems, and published their own proofs of these theorems for abstract groups: Frobenius in 1887, and Burnside in 1894. Other group theory papers of Burnside in the period 1893–96 also in part duplicated results obtained earlier by Frobenius. Obviously, Frobenius had the priority in all of these, and Burnside felt embarrassed about not having checked the literature sufficiently before he published his own work. Burnside learned a valuable lesson from this experience, and from that time on, he was to follow Frobenius's publications very closely. In his subsequent papers, he made frequent references to Frobenius's work, always referring to him politely as "Herr Frobenius" or "Professor Frobenius". In Burnside's group theory book [B_1], Frobenius received more citations than any other author, including Jordan and Hölder. Frobenius was, however, less enthused about Burnside's work, at least at the outset. In his May 7, 1896, letter to Dedekind, Frobenius wrote,[13] after mentioning an 1893 paper of Burnside on the group determinant:

> This is the same Herr Burnside who annoyed me several years ago by quickly rediscovering all the theorems I had published on the theory of groups, in the same order and without exception: first my proof of Sylow's Theorems, then the theorem on groups with square-free orders, on groups of order

[12] *A good reference for this topic is [CR], where the entire last chapter is devoted to the study of Burnside rings and their modern analogues, the representation rings.*

[13] *English translation following [H_3, p. 242].*

$p^a q$, on groups whose order is a product of four or five prime numbers, etc. etc.

If the above sentiment was expressed in 1896, we can imagine how Frobenius felt later when he saw Burnside's papers [B: 1898a, 1900b], etc., in which Burnside re-derived practically all of Frobenius's results on the group determinant, group characters, and orthogonality relations! At least once or twice (e.g., on page 269 of the second edition of [B_1]), Burnside had stated that in [B: 1898a] he had "obtained independently the chief results of Professor Frobenius' earlier memoirs." For an expert analysis of this claim of Burnside, we refer the reader to [H_2, p. 278].

It was perhaps a stroke of fate that the group-theoretic work of Frobenius and Burnside remained perennially intertwined: they were interested in the same problems, and in many cases they strived to get exactly the same results. The following are some interesting comparisons.

1. Both Frobenius and Burnside worked on the question of the *existence of normal p-complements in finite groups*, and each obtained significant conditions for the existence of such complements. Their conditions are different, and the results they obtained are both standard results in finite group theory today. Frobenius's result seems stronger here, since it gives a necessary and sufficient condition, while Burnside's result offers only a sufficient condition.

2. *On transitive groups of prime degree*: a topic of great interest to Frobenius. Here, Burnside had the scoop, as he proved in [B: 1900c] that any such group is either doubly transitive or metacyclic, from which it follows that there are no simple groups of odd (composite) order and prime degree. The paper [B: 1900c] appeared heel-to-heel following Burnside's paper [B: 1900b] on "group-characteristics", and represented the first applications of group characters to group theory proper, a fact acknowledged by Frobenius himself.

3. *On Frobenius groups*: Burnside had been keenly interested in these groups, and devoted pages 141–144 in [B_1, 1st ed.] and subsequently [B: 1900a] to their study. He was obviously trying to prove that the Frobenius kernel is a subgroup, and by 1901, he was able to prove this in case the Frobenius complement has even order or is solvable. If one assumes the Feit-Thompson Theorem, this would give a de facto proof of the desired conclusion in all cases. Maybe this was one of the reasons that fueled Burnside's belief that odd order groups are solvable? We do not know for sure. Anyway, on the Frobenius group problem, it was Frobenius who had the scoop, as he proved that the Frobenius kernel is a subgroup in *all* cases in 1901. Frobenius's great expertise with induced characters gave him the edge in this race.

4. *Solvability of $p^a q^b$ groups*: This was clearly a common goal that both Frobenius and Burnside had very much hoped to attain. If m is the exponent of p modulo q, Burnside furnished a positive solution in case $a < 2m$ [B_1, 1st ed., p. 345], and Frobenius later relaxed Burnside's hypothesis to $a \le 2m$. The truth of the result in *all* cases was proved by Burnside in 1904 (Theorem 4.4 above); here, Burnside's acumen with the arithmetic of characters gave him the winning edge.

Since Frobenius and Burnside worked on many common problems and obtained related results on them, it is perhaps not surprising that posterity sometimes got confused about which result is due to which author. One of the most conspicuous examples of this is the famous counting formula, which says that, with a finite group G acting on a finite set S, the average number of fixed points of the elements of G is given by the number of orbits of the action (see box). Starting in the mid-1960s, more and more authors began to refer to this counting formula as "Burnside's Lemma". According to P. M. Neumann [Ne], S. Golomb and N. G. de Bruijn first made references and attributions to Burnside for this result in 1961 and 1963–64, after which the name "Burnside's Lemma" began to take hold. While Burnside did have this result in his group theory book [B_1, p. 191], he had basically little to do with the lemma. In his paper "A lemma that is not Burnside's" [Ne], Neumann reported that Cauchy was the first to use the idea of the said lemma in the setting of multiply transitive groups, and it was Frobenius who formulated the lemma explicitly in [F: (36), p. 287], and who first understood its importance in applications. Neumann's recommended attribution "Cauchy-Frobenius Lemma" was lauded by de Bruijn in a quotation at the end of [Ne]; however, in his group theory book [NST] with Stoy and Thompson, Neu-

"A lemma that is not Burnside's"

$$\#(G\text{-orbits on } S) = \frac{1}{|G|} \sum_{g \in G} \pi(g)$$

where $\pi(g)$ is the number of points in S fixed by g. According to R. C. Read [PR, p. 101], "this lemma has been likened to the country yokel's method of counting cows, namely count the legs and divide by four." Ironically, it is sometimes easier to count legs than to count cows! For instance, to count the number of different necklaces a jeweler can make using six beads of two colors (say green and white), we can get the answer, 13, by applying the above formula to the dihedral group of twelve elements acting on a set of $2^6 = 64$ "formal necklaces". Burnside deserved credit for popularizing the Cauchy-Frobenius formula by including it in his book. Later, a far-reaching generalization of this formula known as Pólya's Fundamental Theorem became a major landmark in the field of enumerative combinatorics.

From the viewpoint of representation theory, π is the character of the permutation representation associated with the action of G on S. In case this action is doubly transitive, Burnside showed in his book that π is the sum of the trivial character and an irreducible character of G.

mann somehow decided to refer to the result as "Not Burnside's Lemma"!

Another case in point is the theorem, mentioned already in Part I, that the degree of an irreducible (complex) representation of a group G divides the order of G. Some authors have attributed this theorem to Burnside, but again it was Frobenius who first proved this result, as one can readily check by reading the last page of his classical group determinant paper [F: (54)].[14] Burnside supplied a proof of this result in his own terms, but the theorem was definitely Frobenius's. Issai Schur, a student of Frobenius, proved later that the degree of an irreducible representation divides the index of the center of G, and N. Itô was to prove eventually that this degree in fact divides the index of any abelian normal subgroup.

With such little tales on attributions, we conclude our discussion of the life and work of Frobenius and Burnside. Although their work was so closely linked, there seemed to have been no evidence that they had either met, or even corresponded with each other. Would the history of the representation theory of finite groups be any different if these two great mathematicians had known each other, or if there had been a *Briefwechsel* between them, like that between Frobenius and Dedekind?

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[14] *The result was also proved independently by Molien, as pointed out by Hawkins [H_2, p. 271].*

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