

# The Work of Kiyosi Itô

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The *Notices* solicited the following article describing the work of Kiyosi Itô, recipient of the 2006 Gauss Prize. The International Mathematical Union also issued a news release, which appeared in the November 2006 issue of the *Notices*.

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**Kiyosi Itô, 1954, when he was a Fellow at the Institute for Advanced Study, Princeton.**

On August 22, 2006, the International Mathematical Union awarded the Carl Friedrich Gauss Prize at the opening ceremonies of the International Congress of Mathematicians in Madrid, Spain. The prizewinner is Kiyosi Itô. The Gauss prize was created to honor mathematicians whose research has had a profound impact not just on mathematics itself but also on other disciplines.

To understand the achievements of Itô, it is helpful to understand the context in which they were developed. Bachelier in 1900, and Einstein in 1905, proposed mathematical

models for the phenomenon known as Brownian motion. These models represent the random motion of a very small particle in a liquid suspension. Norbert Wiener and collaborators showed in the 1920s that Einstein's model exists as a stochastic process, using the then-new ideas of Lebesgue measure theory. Many properties of the process were established in the 1930s, the most germane for this article being that its sample paths are of infinite variation on any compact time interval, no matter how small. This made the Riemann-Stieltjes integration theory inapplicable. Wiener wanted to use such integrals to study filtering theory and signal detection, important during the second world war. Despite these problems he developed a theory of integrals, known today as

Wiener integrals, where the integrands are non-random functions. This served his purpose but was unsatisfying because it ruled out the study of stochastic differential equations, among other things.

The problem in essence is the following: how can one define a stochastic integral of the form  $\int_0^t H_s dW_s$ , where  $H$  has continuous sample paths and  $W$  is a Wiener process (another name for Brownian motion), as the limit of Riemann-style sums? That is, to define an integral as the limit of sums such as  $\sum_{1 \leq i \leq n} H_{\xi_i} (W_{t_{i+1}} - W_{t_i})$ , with convergence for all such  $H$ . Unfortunately as a consequence of the Banach-Steinhaus theorem,  $W$  must then have sample paths of finite variation on compact time intervals. What Itô saw, and Wiener missed, was that if one restricts the class of potential integrands  $H$  to those that are adapted to the underlying filtration of sigma algebras generated by the Wiener process, and if one restricts the choice of  $\xi_i \in [t_i, t_{i+1})$  to  $t_i$ , then one can use the independence of the increments of the Wiener process in a clever way to obtain the convergence of the sums to a limit. This became the stochastic integral of Itô. One should note that Itô did this in the mathematical isolation of Japan during the second world war and was one of the pioneers (along with G. Maruyama) of modern probability in Japan, which has since spawned some of the world's leading probabilists. Moreover since Jean Ville had named martingales as such only in 1939, and J. L. Doob had started developing his theory of martingales only in the 1940s, Itô was unaware of the spectacular developments in this area that were happening in the U.S., France, and the Soviet Union. Thus modern tools such as Doob's martingale inequalities were unavailable to Itô, and his

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creativity in the proofs, looked at today, is impressive. But the key result related to the stochastic integral was Itô's change of variables formula.

Indeed, one can argue that most of applied mathematics traditionally comes down to changes of variable and Taylor-type expansions. The classical Riemann-Stieltjes change of variables, for a stochastic process  $A$  with continuous paths of finite variation on compacts, and  $f \in C^1$  is of course

$$f(A_t) = f(A_0) + \int_0^t f'(A_s) dA_s.$$

With the Itô integral it is different and contains a "correction term". Indeed, for  $f \in C^2$  Itô proved

$$f(W_t) = f(W_0) + \int_0^t f'(W_s) dW_s + \frac{1}{2} \int_0^t f''(W_s) ds.$$

This theorem has become ubiquitous in modern probability theory and is astonishingly useful. Moreover Itô used this formula to show the existence and uniqueness of solutions of stochastic ordinary differential equations:

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt; \quad X_0 = x_0,$$

when  $\sigma$  and  $b$  are Lipschitz continuous. This approach provided methods with an alternative intuition to the semigroup/partial differential equations approaches of Kolmogorov and Feller, for the study of continuous strong Markov processes, known as diffusions. These equations found applications without much delay: for example as approximations of complicated Markov chains arising in population and ecology models in biology (W. Feller), in electrical engineering where  $dW$  models white noise (N. Wiener, I. Gelfand, T. Kailath), in chemical reactions (e.g., L. Arnold), in quantum physics (P. A. Meyer, L. Accardi, etc.), in differential geometry (K. Elworthy, M. Emery), in mathematics (harmonic analysis (Doob), potential theory (G. Hunt, R. Gettoor, P. A. Meyer), PDEs, complex analysis, etc.), and, more recently and famously, in mathematical finance (P. Samuelson, F. Black, R. Merton, and M. Scholes).

When Wiener was developing his Wiener integral, his idea was to study random noise, through sums of iterated integrals, creating what is now known as "Wiener chaos". However his papers on this were a mess, and the true architect of Wiener chaos was (of course) K. Itô, who also gave it the name "Wiener chaos". This has led to a key example of Fock spaces in physics, as well as in filtering theory, and more recently to a fruitful interpretation of the Malliavin derivative and its adjoint, the Skorohod integral.

Itô also turned his talents to understanding what are now known as Lévy processes, after the renowned French probabilist Paul Lévy. He was able to establish a decomposition of a Lévy process into a drift, a Wiener process, and an integral mixture of compensated compound Poisson

processes, thus revealing the structure of such processes in a more profound way than does the Lévy-Khintchine formula.

In the late 1950s Itô collaborated with Feller's student H. P. McKean Jr. Together Itô and McKean published a complete description of one-dimensional diffusion processes in their classic tome, *Diffusion Processes and Their Sample Paths* (Springer-Verlag, 1965). This book was full of original research and permanently changed our understanding of Markov processes. It developed in detail such notions as local times and described essentially all of the different kinds of behavior the sample paths of diffusions could manifest. The importance of Markov processes for applications, and especially that of continuous Markov processes (diffusions), is hard to overestimate. Indeed, if one is studying random phenomena evolving through time, relating it to a Markov process is key to understanding it, proving properties of it, and making predictions about its future behavior.

Later in life, when conventional wisdom holds that mathematicians are no longer so spectacular, Itô embraced the semimartingale-based theory of stochastic integration, developed by H. Kunita, S. Watanabe, and principally P. A. Meyer and his school in France. This permitted him to integrate certain processes that were no longer adapted to the underlying filtration. Of course, this is a delicate business, due to the sword of Damocles Banach-Steinhaus theorem. In doing this, Itô began the theory of expansion of filtrations with a seminal paper and then left it to the work of Meyer's French school of the 1980s (Jeulin, Yor, etc.). The area became known as *grossissements de filtrations*, or in English as "the expansions of filtrations". This theory has recently undergone a revival, due to applications in finance to insider trading models, for example.

A much maligned version of the Itô integral is due to Stratonovich. While others were ridiculing this integral, Itô saw its potential for explaining parallel transport and for constructing Brownian motion on a sphere (which he did with D. Stroock), and his work helped to inspire the successful use of the integral in differential geometry, where it behaves nicely when one changes coordinate maps. These ideas have also found their way into other domains, for example in physics, in the analysis of diamagnetic inequalities involving Schrödinger operators (D. Hundertmark, B. Simon).

It is hard to imagine a mathematician whose work has touched so many different areas of applications, other than Isaac Newton and Gottfried Leibniz. The legacy of Kiyosi Itô will live on a long, long time.