The Work of Harvey Friedman

by Anil Nerode and Leo A. Harrington

Mathematical logician Harvey Friedman was recently awarded the National Science Foundation's annual Waterman Prize, honoring the most outstanding American scientist under thirty-five years of age in all fields of science and engineering. When a mathematician wins such an award, the mathematical community naturally wishes to understand the underlying achievements, and their implications. Friedman continues the great tradition of Frege, Russell, and Gödel. This can be characterized as the exercise of acute philosophical perspective to distill exact mathematical definitions and questions from important foundational issues. These questions in turn give rise to mathematical subjects and theorems of depth and beauty. Friedman's contributions span all branches of mathematical logic (recursion theory, proof theory, model theory, set theory, theory of computation). He is a generalist in an age of specialization, yet his theorems often require extraordinary technical virtuosity. We discuss only a few selected highlights.

Friedman's ideas have yielded radically new kinds of independence results. The kinds of statements proved independent before Friedman were mostly disguised properties of formal systems (such as Gödel's theorem on unprovability of consistency) or assertions about abstract sets (such as the continuum hypothesis or Souslin's hypothesis). In contrast, Friedman's independence results are about questions of a more concrete nature involving, for example, Borel functions or the Hilbert cube.

EDITOR'S NOTE. The accompanying article describes the work of Harvey M. Friedman of Ohio State University, the recipient of this year's Alan T. Waterman Award [June 1984 *Notices*, page 364]. The 1984 Waterman Award was the ninth in a series begun in 1976. The Award was established by Congress in 1975 in commemoration of the twenty-fifth anniversay of the creation of the National Science Foundation and in honor of Waterman, the first director of the Foundation, whose term as director covered twelve years. Three of these awards have gone to mathematicians: Charles L. Fefferman received the first Waterman Award (1976) and William P. Thurston received the fourth one (1979). The award carries a stipend of $150,000, intended to provide three-year support for the recipient. In addition to the biographical information listed in the June *Notices*, it is reported that for part of 1984 Friedman has served as a consultant at AT&T Bell Laboratories and as a visiting scientist at the IBM Watson Research Center in Yorktown Heights.

In [**1**] Friedman showed that Borel determinacy cannot be proved in Zermelo set theory with the axiom of choice (ZC), or any of the usual formal systems associated with at most countably many iterations of the power set operation, after Martin [**17**] proved Borel determinacy from certain reasonable extensions of Zermelo Fraenkel set theory with the axiom of choice (ZFC). Subsequently, Martin [**18**] gave a proof of Borel determinacy using uncountably many iterations of the power set operation.

These results reached their mature form in [**9**] as follows:

THEOREM 1 (FRIEDMAN [**9**]). *It is necessary and sufficient to use uncountably many iterations of the power set operation to prove the following. Every symmetric Borel subset of the unit square contains or is disjoint from the graph of a Borel function. In particular, this assertion is provable in* ZFC *but not in* ZC.

In Friedman [**4, 9**], the classical theorem of Cantor that the unit interval I is uncountable is analyzed. Cantor's argument produces an

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$x \in I$ which is not a term in any given infinite sequence $y_1, y_2, \ldots \in I$. Specifically, there is a Borel diagonalization function $F : \overline{Q} \to I$ such that no $F(y)$ is a coordinate of y (here $\overline{Q} = I^\omega$ is the Hilbert cube). Friedman observes that the value of $F(y)$ depends on the order in which the coordinates of y are given, at least for the F coming from Cantor's argument. Let $x \approx y$ mean that $x, y \in \overline{Q}$ have the same coordinates, and $x \sim y$ mean that y is obtained by permuting finitely many coordinates of x.

THEOREM 2 (FRIEDMAN [4, 9]). *If $F : \overline{Q} \to I$ is a Borel function satisfying the invariance condition $x \approx y \to F(x) = F(y)$, then some $F(x)$ is a coordinate of x. Furthermore, this is provable in* ZC *but not in* ZFC *with the power set axiom deleted (even for the weaker theorem with $\approx$ replaced by $\sim$).*

Subsequently, Friedman [12] gave closely related one-dimensional theorems with the same metamathematical properties. We present two of these.

Let $K = \{0,1\}^\omega$ be the Cantor set. Define the shift $s : K \to K$ by $s(x)_n = x_{n+1}$. Define the "square" $x^{(2)}$ for $x \in K$ by $x^{(2)}_n = x_{n^2}$. Let T be the circle group (i.e., [0, 1) with addition modulo 1). We say that $F : K \to K$ is shift invariant if $F(sx) = F(x)$. We say that $F : T \to T$ is doubling invariant if $F(2x) = F(x)$.

THEOREM 3 (FRIEDMAN [12]). *Every shift invariant Borel function on K is somewhere its square. There is a Borel (in fact, continuous) function on T which agrees somewhere with every doubling invariant Borel function on T. Furthermore, these (three) theorems are provable in* ZC *but not in* ZFC *with the power set axiom deleted.*

Friedman [9] develops extensions of Theorem 2, which are proved using uncountably many iterations of the power set operation. Recently, Friedman has extended this idea as follows: Let Q be the rational numbers and $P(Q)$ be the Cantor space of all subsets of Q. Let G be the Baire space of all products defined on ω.

THEOREM 4 (FRIEDMAN [11]). *If $F : G \to G$ is a Borel function such that isomorphic elements go to isomorphic values, then some $F(x)$ is isomorphically imbeddable in x. Furthermore, this theorem is provable in* ZC *but not in* ZFC *with the power set axiom deleted.*

THEOREM 5 (FRIEDMAN [11]). *If $F : P(Q) \to P(Q)$ is a Borel function such that order isomorphic arguments go to order isomorphic values, then some $F(A)$ is isomorphic to an interval in A (even of the form (a, b) where a, b are points in A). Furthermore, this theorem requires uncountably many iterations of the power set operation to prove. In particular, it is provable in* ZFC *but not in* ZC.

Friedman [9] also develops far-reaching extensions of Theorem 2 by combining these ideas with Ramsey's theorem. This leads to independence results from full ZFC, and in fact to theorems about Borel functions on the Hilbert cube for which it is necessary and sufficient to use "large cardinals" to prove.

The large cardinals involved are as follows: A cardinal is inaccessible if $\lambda < \kappa$ implies $2^\lambda < \kappa$, and κ is not the sup of fewer than κ cardinals below κ. We say that κ is a Mahlo cardinal if κ is inaccessible and every closed unbounded subset of κ contains an inaccessible cardinal.

The 1-Mahlo cardinals are the Mahlo cardinals. The $(n+1)$-Mahlo cardinals are the n-Mahlo cardinals in which every closed unbounded subset contains an n-Mahlo cardinal.

The group of all permutations of ω which fix all but finitely many numbers acts on $\overline{Q}$ by permuting coordinates. This group also acts diagonally on any $\overline{Q}^n$ by $g \cdot (x_1, \ldots, x_n) = (gx_1, \ldots, gx_n)$. For $x, y \in \overline{Q}^n$ let $x \sim y$ indicate that x, y are in the same orbit under this action.

THEOREM 6 (FRIEDMAN [9]). *Let $F : \overline{Q} \times \overline{Q}^n \to I$ be a Borel function such that if $x \in \overline{Q}$, $y, z \in \overline{Q}^n$, and $y \sim z$, then $F(x, y) = F(x, z)$. Then there is an infinite sequence $\{x_k\}$ from $\overline{Q}$ such that for all indices $s < t_1 < \ldots < t_n$, $F(x_s, x_{t_1}, \ldots, x_{t_n})$ is the first coordinate of x_{s+1}. In order to prove this for all n, it is necessary and sufficient to use the existence of n-Mahlo cardinals for all n.*

These ideas have been extended in Friedman [11] to much larger cardinals. A measurable cardinal is a cardinal κ which carries a κ-additive {0, 1}-valued measure on all subsets of κ. A Ramsey cardinal is a cardinal κ such that for every set E of finite subsets of κ, there is an unbounded $B \subseteq \kappa$ such that for finite subsets x of B, membership in E depends only on the cardinality of x.

Let G_0 be the Baire space of finitely generated products on ω.

THEOREM 7 (FRIEDMAN [11]). *Let $F : G_0^\omega \to G_0$ be a Borel function mapping pointwise isomorphic arguments to isomorphic values. Then for some $x \in G_0^\omega$, $F(x)$ is isomorphically imbeddable into a coordinate of x. Furthermore, this theorem is provable in* ZFC *but not in* ZC.

THEOREM 8 (FRIEDMAN [11]). *Let $F : G_0^\omega \to G_0$ be a Borel function mapping pointwise isomorphic arguments to isomorphic values. Then for some $x \in G_0^\omega$, for all subsequences y of x, $F(y)$ is isomorphically imbeddable into a coordinate of y. This is provable in* ZFC + *"there is a measurable cardinal," but not in* ZFC + *"there is a Ramsey cardinal."*

In Paris-Harrington [21] an interesting example of a theorem stated in finite set theory but not provable in finite set theory, is given. Also see [**16**], [**23**], [**24**].

THEOREM 9 (PARIS-HARRINGTON [**21**]). *For each k, r, s, there is an n so large that if all subsets of $[1, n]$ of size k are colored with r colors, then there is a set $E \subseteq [1, n]$ such that all subsets of E of size k have the same color and the size of E is at least s and the minimum of E. Furthermore, this is provable in finite set theory augmented with definition by recursion on ω, but not in finite set theory.*

Subsequently, Friedman [**7**] found an interesting example of a finite theorem which is conceptually even clearer and independent of much stronger systems. Just as Theorem 9 is based on Ramsey's theorem, Friedman's work is based on the following theorem of J. B. Kruskal [**17**].

A tree is a nonempty partial ordering with a least element, such that the set of predecessors of any element is linearly ordered. If T_1, T_2 are finite trees then $h : T_1 \to T_2$ is said to be a homeomorphic imbedding if $a \leq_{T_1} b$ iff $h(a) \leq_{T_2} h(b)$, and $h(\inf(a, b)) = \inf(h(a), h(b))$. We write $T_1 \leq T_2$ if such an h exists.

THEOREM 10 (KRUSKAL [**19**], NASH-WILLIAMS [**20**]). *If $T_1, T_2, \ldots$ are finite trees then for some $i < j$, $T_i \leq T_j$.*

The formal system ATR is obtained from finite set theory by introducing countably infinite sets with the principle of definition by transfinite recursion on countable well orderings. This system goes just beyond what is referred to as predicative analysis.

THEOREM 11 (FRIEDMAN [**7, 13**]; SIMPSON [**22**]). *For all k there is an n so large that for all finite trees $T_1, \ldots, T_n$ and $\operatorname{card}(T_i) \leq i$, there are $i_1 < \ldots < i_k$ such that $T_{i_1} \leq \ldots \leq T_{i_k}$. This theorem, as well as Kruskal's Theorem 10 above, are provable in ZFC without the power set axiom, but not in ATR. Also see* [**16, 23, 24**].

THEOREM 12 (FRIEDMAN [**8, 13**]). *Theorem 11 for $k = 12$ is provable in ZFC without the power set axiom in a few pages, but any proof in ATR must use at least $2^{[1000]}$ pages.*

Friedman [**10**] has extended Kruskal's theorem in an interesting way so that the theorem has yet stronger metamathematical properties. He considers the class $\mathrm{Tr}_\infty(n)$ of finite trees with n distinct labels, and defines $T_1 \leq_r T_2$ if and only if there is an $h : T_1 \to T_2$ which is a label-preserving homeomorphic imbedding, preserving left to rightness, with the additional crucial condition that if b is an immediate successor of **a** in T_1 and $h(a) < c < h(b)$ in T_2, then $l(c) \geq l(h(b))$ ($l(c)$ is the label of c). The subscript "r" means "restricted."

THEOREM 13 (FRIEDMAN [**10**], SIMPSON [**22**]). *For every $T_1, T_2, \ldots \in \mathrm{Tr}_\infty(n)$, there are $i < j$ such that $T_1 \leq_r T_2$. For all k, n there is an m so large that for all $T_1, \ldots, T_m \in \mathrm{Tr}_\infty(n)$ with each $\operatorname{card}(T_i) \leq i$, there are $i_1 < \ldots < i_k$ such that $T_{i_1} \leq_r T_{i_2} \leq_r \ldots \leq_r T_{i_k}$. These theorems are provable in the single set quantifier comprehension axiom system with the full scheme of induction (Π^1_1-CA), but not in Π^1_1-CA with set induction instead of the full scheme of induction (Π^1_1-CA$_0$).*

Two of the most fundamental concepts in mathematical logic are the model theoretic concept of translatability and the proof theoretic concept of relative consistency. Friedman [**6**] proves that under surprisingly general conditions, these concepts coincide. Specifically, let S, T be finitely axiomatized first-order theories. We say that S is translatable into T if there are first-order definitions of the symbols in S in terms of those in T, such that every axiom of S, when so translated, becomes a theorem of T.

There are a few somewhat different ways of defining "S is consistent relative to T," considered in [**6**]. We focus attention on the following one: Let EFA (exponential function arithmetic) be the standard weak system of arithmetic based on $0, 1, <, =, +, \cdot$, and exponentiation, where induction is applied to bounded formulas only. The quantifier complexity of a formula is a standard measure of the number of alternations of quantifiers that are present.

We say that S is consistent relative to T if for some fixed n, the following is provable in EFA. If there is an inconsistency proof in S then there is an inconsistency proof in T which uses only formulas whose quantifier complexity is at most n more than the greatest quantifier complexity of formulas used in the given inconsistency proof in S.

It is straightforward to see that translatability implies relative consistency.

THEOREM 14 (FRIEDMAN [**6**]). *Let S, T be finitely axiomatized theories containing* EFA *(and a weak theory of finite sequences of objects other than natural numbers, if applicable). Then S is translatable into T if and only if S is consistent relative to T.*

Friedman [**5**] has initiated an interesting new branch of model theory called Borel model theory. A totally Borel model is a structure whose domain is $\mathbb{R}$ and every relation that is definable (in the language considered) over the structure is Borel. Friedman [**5**] considers the three fundamental quantifiers Q_m (almost all in the sense of Lebesgue measure), Q_c (almost all in the sense of Baire category), and Q_{ω_1} (uncountably many), in addition, of course, to the usual quantifiers $\forall, \exists$.

Friedman proves the following completeness and duality theorem.

THEOREM 15 (FRIEDMAN [**5**], STEINHORN [**25**]). *Let φ be a sentence based on Q_m (or Q_m and $\forall, \exists$).*

Then φ is true in all totally Borel models if and only if φ can be proved using, roughly speaking, the following principles: singletons are of measure 0, subsets of measure 0 are of measure 0, the union of two sets of measure 0 is of measure 0, $\mathbb{R}$ is not of measure 0, and almost all vertical cross sections of a two-dimensional set are of measure 0 if and only if almost all horizontal cross sections are of measure 0 (Fubini's theorem). If Q_c is used instead of Q_m, then this is true if "measure 0" is replaced by "meager." As a consequence we have that φ is true in all totally Borel structures for Q_m (or $(Q_m, \forall, \exists)$) if and only if φ^* is true in all totally Borel structures for Q_c (or $(Q_c, \forall, \exists)$) where φ^* is obtained from φ by replacing Q_m by Q_c (duality).

Friedman has obtained a number of fundamental results in intuitionistic set theory. The usual axioms for ZF are extensionality, pairing, union, infinity, foundation, power set, comprehension, and finally replacement. The axiom scheme of collection is an alternative to the axiom scheme of replacement, but it is a fundamental theorem of set theory that these are equivalent.

However, if we use intuitionistic logic instead of ordinary logic, then the proof that replacement implies collection breaks down. Thus we let ZFIR be ZF with intuitionistic logic formulated with the scheme of replacement, and ZFIC be ZF with intuitionistic logic formulated with the scheme of collection.

THEOREM 16 (FRIEDMAN [**2**]; FRIEDMAN-ŠČEDROV [**15**]). ZFIR *does not imply* ZFIC. *It is provable in a weak system of arithmetic that ordinary* ZFC *is consistent if and only if* ZFIC *is consistent.*

Two basic desirable properties of intuitionistic formal systems (which almost never hold for ordinary formal systems) are the disjunction property, which asserts that if a disjunction $A \vee B$ is provable then one of the disjuncts is provable; and the numerical existence property, which asserts that if $(\exists n)(An)$ is provable then for some n, $A\bar{n}$ is provable. DP trivially follows from NEP.

Friedman proves the following highly surprising theorem via a mysterious application of Gödel self-reference.

THEOREM 17 (FRIEDMAN [**3**]). *Let T be a recursively axiomatized intuitionistic formal system subject to, roughly, the same weak hypotheses commonly used in Gödel's incompleteness theorems. Then T has the numerical existence property if and only if T has the disjunction property.*

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