Nathan Jacobson (1910–1999)

Georgia Benkart, Irving Kaplansky, Kevin McCrimmon, David J. Saltman, and George B. Seligman

When a colleague was explaining how a mathematician can be recognized to have reached the summit of recognition by his peers, he used the metaphor, "He has become part of the furniture." That is, his contributions have become a part of the daily vocabulary and working equipment of many of us. Such is certainly the status of Nathan Jacobson. As my fellow authors will show more specifically, he earned his dominance by recasting whole theories of algebraic systems and by insisting on the module-theoretic viewpoint in their study. His expository and research monographs and his ambitious textbooks have indebted a worldwide community to him for strong and articulate leadership. The authors use this opportunity to remind us of some of the ways his ideas have shaped our thought.

"Jake", the name all used, died on December 5, 1999, at the age of eighty-nine. Extensive autobiographical material is to be found in the "Personal History and Commentary" that he wrote in seven installments in his *Collected Mathematical Papers* [B14], published in three volumes by Birkhäuser in 1989. I recommend these passages both for more details on his personal life and for his comments on the development of his mathematical work. In this segment of the present article I provide a sketch of his career.

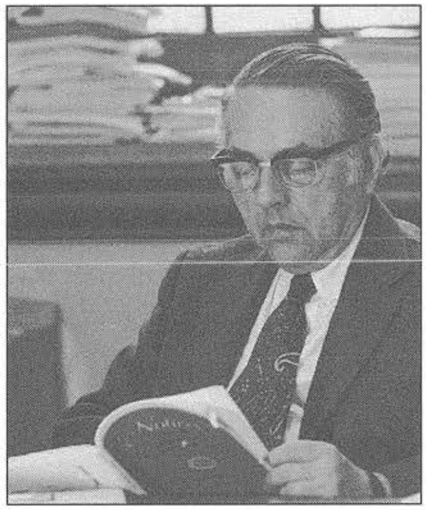
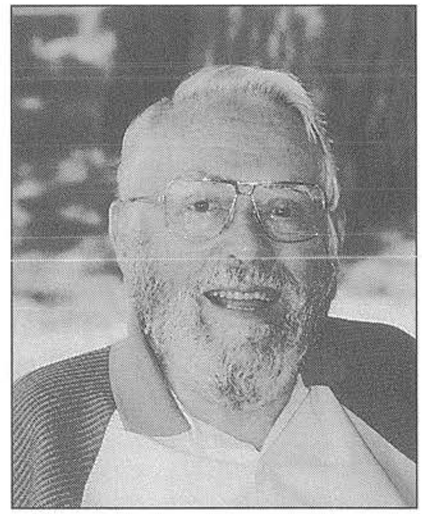
His "official" birth date was September 8, 1910, but Jake maintained that the correct one was October 5. His father emigrated to Nashville, Tennessee, when Jake was five, leaving the family in Poland until he was well enough established to bring them over. The First World War was nearing its end when Jake, his brother, and his mother were able to board a Dutch ship with help from the Hebrew Immigrant Aid Society. After a few months in the rear of his father's Nashville grocery, Jake and his family moved to Birmingham, Alabama, and then, in 1923, to Columbus, Mississippi. Jake graduated from the S. D. Lee High School in Columbus in 1926. He entered the University of Alabama that fall, intending to follow a maternal uncle into law.

While following a pre-law program, he took all mathematics courses available. The notice of his professors was attracted to the extent that in his junior year he was offered a teaching assistantship in mathematics. Two of these professors, Fred Lewis and William P. Ott, were always remembered fondly as having inspired him to turn to a career in mathematics. With their advice he applied for graduate study to Chicago, Harvard, and Princeton, accepting an offer of a "research assistantship" at Princeton. The stipend ($500) fell just a little short of the bill for tuition, room, and board, but the following years saw increases to levels that he described as "a substantial surplus over living expenses."

His dissertation *Non-commutative Polynomials and Cyclic Algebras*, with J. H. M. Wedderburn as advisor, was accepted for the Ph.D. in 1934. How his time in Princeton and subsequently at the Institute for Advanced Study led to what became his leadership in the algebraic theory of Lie algebras is described below by Irving Kaplansky and Georgia Benkart.

Emmy Noether had taken a position at Bryn Mawr. She gave weekly lectures, attended by Jake, at the Institute. She took an interest in Jake's work, but all opportunities for collaboration ended with her sudden death in the spring of 1935. Jake was appointed as her replacement at Bryn Mawr for the following academic year. After a postdoctoral fellowship with Adrian Albert at Chicago in 1936–37, he was appointed to a junior position at

George B. Seligman is professor of mathematics emeritus at Yale University. His e-mail address is selig@math.yale.edu.

Nathan Jacobson...1945 ...1970 ...1997

the University of North Carolina. Jake praised the university's president, Frank Graham, and the department head, Archibald Henderson, for their rejection of the exclusionary practices concerning Jews that barred the doors to many positions.

Although he had been on the faculty for five years, rising to the rank of associate professor, Jake was still subject to the Navy's requirement of special teacher training before being entrusted with teaching in the U.N.C. wartime program for prospective flyers. Fortunately the pedagogical preparation was offered in Chicago. There it enabled Jake to renew and consolidate his relationship with his inseparable helpmeet and companion through fifty-four years of marriage. Florence Dorfman ("Florie") gave up her doctoral research with Albert, but continued in mathematics not only as an educator but also as Jake's reader, supporter, critic, and coauthor. When the children were older, she returned as a highly successful and beloved teacher at Albertus Magnus College. The hospitality of their home is surely among the reasons why the mathematics department at Yale has a reputation for warmth and friendliness.

In 1943 Jake left the Navy and North Carolina for the Army training program and an associate professorship at Johns Hopkins, where he had earlier spent a year as a visitor. It was during his time at Hopkins that he developed much of the general theory of rings that is his most famous achievement. The offer of a tenured associate professorship from Yale that he received and accepted in 1947 represented more than an appreciation of his outstanding research and teaching. The anti-Semitic barrier to senior appointments in the faculty of Yale College had fallen only in 1946, and there were still misgivings about that step in too many quarters; but the time had come when merit could prevail.

The events of his early years at Yale and his visits to Paris and elsewhere are covered in the *Collected Papers,* to which we owe lists of his publications and of his Ph.D. students. Outstanding was the academic year 1956–57, when Adrian Albert organized support, mainly from the research offices of the arms of the Department of Defense, for some ten established and younger algebraists to be at Yale. The university cooperated by partial support for teaching in most cases. Some of Jake's collaborations from that year are [58] and [59] in the list of bibliographic selections.

In July 1961 Jake represented the National Academy of Sciences at the Leningrad Fourth All-Union Congress of Mathematicians of the USSR. After considerable resistance, he agreed to serve as chair of the Yale mathematics department for 1965–68, with assurance that no extension nor reappointment was expected. During his term he succeeded in appointing Abraham Robinson, the founder of nonstandard analysis and an outstanding contributor to both pure and applied mathematics. Another coup was negotiating the return to Yale of our former Ph.D., Robert Langlands.

As president of the American Mathematical Society in 1971–1972, Jake had to mediate between an "activist" faction, particularly in opposition to the Vietnam War, and a "purist" faction, who felt the Society should adhere strictly to scientific aims. Although his personal sentiments were with the activists, he preserved the respect of all parties by offering all a hearing and by following an open and democratic process in discussion and decisions. His term as vice president of the International Mathematical Union (IMU) (1972–74) was more stormy. The issue at the center of contention was the refusal of the Soviet authorities, as represented by L. S. Pontrjagin, the other vice president of the IMU, to permit many outstanding Soviet mathematicians to participate in International Congresses. Beyond that, anti-Semitic and antidissident practices kept promising students from being admitted to universities and senior scholars who had fallen out of favor from being allowed to emigrate. The determination with which Jake protested may be gathered from his comments in the June 1980 issue of the *Notices* in response to a vicious personal attack by Pontrjagin.

His retirement from Yale in 1981 came only after he had earned the honor of carrying the university's mace as senior professor at the commencement ceremonies. Students, colleagues, and fellow scholars gathered to honor him and to present him with their contributions in a volume, *Algebraists' Homage* [AH]. Retirement made it possible for him to accept numerous invitations from around the world. Kevin McCrimmon and David Saltman write of his activity and influence on research in the retirement years.

In February of 1992 he suffered a crippling stroke. The effect on his speech gradually wore off, but his right hand was nearly useless for writing, and he could not walk unaided. With Florie taking on much of the mechanics, he finished the book on division algebras [B16] for publication in 1996, completing the journey he had started with Wedderburn. Meanwhile, Florie was receiving powerful medication. The combination of illness and treatment took her from Jake's side in 1996. No visitor thereafter could fail to be reminded how much she had meant to him.

There was still one happy occasion. He was able to make the trip to Baltimore in January of 1998 to be honored with the Society's Leroy P. Steele Prize for Lifetime Achievement. A photo accompanying this article shows his radiance at that event. His only lament was the absence of Florie. May they now have found reunion.

—George B. Seligman, organizer

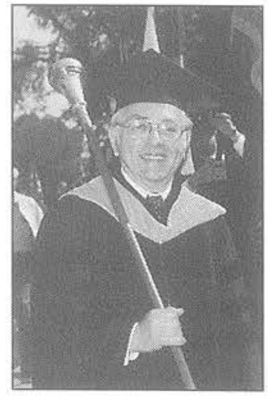
Irving Kaplansky

With the death of Nathan Jacobson ("Jake") the world of mathematics has lost a giant of twentieth-century algebra.

I shall begin by recalling my first contact with Jake. It was in the summer of 1938 at the University of Chicago. With a fresh bachelor's degree, I was attracted by the special program in algebra that summer. I attended Jake's course on continuous groups. This carried me from the definition of a topological space (new to me) to exciting topics at the frontier. Also, in a seminar course conducted by Albert I heard Jake give a talk on locally compact division rings. This kindled in me an interest in locally compact rings that has lasted to this day. Pontrjagin had done the pioneering work by showing that the only connected locally compact division rings are the reals, complexes, and quaternions. The paper [3], joint with Olga Taussky, took a big step forward by studying a general locally compact ring. This laid the foundation for all subsequent work in the field, work that eventually answered all the major questions.

Irving Kaplansky is director emeritus of the Mathematical Sciences Research Institute, Berkeley. His e-mail address is kap@msri.org.

At the time of the summer of 1938, Jake was only four years beyond the doctorate. His thesis advisor at Princeton was Wedderburn. The thesis [1] concerned finite-dimensional associative algebras. Thus there is a remarkable continuity in the passing of the mantle from Wedderburn to Jacobson.



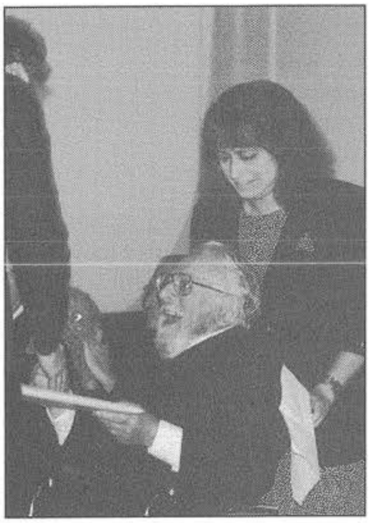
Yale, 1981.

I hope that many readers of this piece will also read the autobiography and (to borrow a word from Halmos) the automathography contained in the three volumes of [B14]. From this we learn that a second major influence on Jake at Princeton was the presence of Hermann Weyl at the newly founded Institute for Advanced Study. Weyl gave a course on Lie groups and Lie algebras for which notes were written by Jake and by Richard Brauer. A second lifelong interest was planted in Jake at that time. It promptly bore fruit in the influential paper [4]. (I believe that this is the first paper to use the term "Lie algebra"; the change from "infinitesimal group" was made in Weyl's lectures.) This elegant paper is probably best known for a lemma (Lemma 2 on page 877): *If A and B are matrices over a field of characteristic* 0 *and A commutes with $AB - BA$, then $AB - BA$ is nilpotent.* I fell in love with this lemma and came back to it repeatedly. Just say "Jacobson's lemma" to just about anyone, and he or she is likely to light up in recognition.

With wife Florie, around 1960.

His early papers on Lie algebras were also noteworthy for launching the theory of Lie algebras of characteristic $p > 0$. Thus far there had only been one novel example of a simple algebra: the Witt algebra. In [24] he broadened this to a family of algebras. Once again we find his name attached to an object, for they came to be called the Witt-Jacobson algebras. At first blush it might seem that Jake was overoptimistic in wondering whether all the simple ones were now at hand [23, page 481]. But when the classification finally came, the answer was that one had only to modify the Witt-Jacobson algebras in the way that Cartan did in his infinite simple pseudo-groups. In my own study of Lie algebras I cut my teeth reading these papers.

I have now reached the time period when he launched his general structure theory for rings in [31] and [32]. Let A be a ring with unit element. Let J be the intersection of the maximal left ideals in A. There is no apparent reason why J should be a two-sided ideal, but it is. There is no apparent reason why J should be a left-right symmetric, but it is. J is of course the *Jacobson radical.* When it vanishes, A is called *semisimple.* (Warning: Others say "semiprimitive", reserving "semisimple" for the Artinian case.) Now the famous Wedderburn



Receiving the AMS Leroy P. Steele Prize, Baltimore, 1998.

With Dick and Alice Shafer at the Steele Prize ceremony.

structure theorems survive, in a somewhat weakened form. A semisimple ring is a subdirect sum of primitive ones, and a primitive ring resembles matrices over a division ring, with the matrices allowed to be infinite.

This splendid theory works. Over the years there have been repeated uses of it to settle problems not stated in terms of the theory.

The Colloquium volume [B5] includes his account of his structure theory. It was definitive when it appeared. It remains indispensable today; I think it will continue to be indispensable for a long time. Late in life [B16] he returned to the basic classical topic of finite-dimensional division algebras and presented a remarkable new view of this venerable subject.

There are three great classes of algebras: associative, Lie, and Jordan. The date of his associative book is 1956. Just six years later came his Lie algebra book [B6]. It set a high standard for the fairly numerous books that have followed. Among other things, I find the abundance of challenging exercises to be a big plus. After six more years came his book [B7] on Jordan algebras, completing his trio on the three classes of algebras. He did the hat trick! Again, this book was polished, eminently readable, and definitive at the time. But subsequent dramatic developments, above all at the hands of McCrimmon and Zelmanov, have transformed the subject.

It is amazing but true that in addition to writing these three books Jake found the time to write an algebra textbook not once, but twice. I am referring to [B2], [B3], [B4] and [B10], [B12]. The citation for the Steele Prize for Lifetime Achievement (*Notices* **45** (1998), 508) said that the first is superseded by the second. I disagree. I am glad that we have both; they will both be studied and enjoyed for a long time.

Let me return to the debts I owe him. After his structure theory of rings appeared, I ventured to begin a steady stream of correspondence with him about this and about locally compact rings. He was always prompt in replying, and his replies were always helpful. He gently tolerated my often naive stabs. It was like doing a second Ph.D. thesis. This climaxed in his visit to Chicago in the summer of 1947, during which his course on rings was a preview of the forthcoming book. Polynomial identities and central polynomials surfaced at that time. This tale has been told twice—as he remembered it and as I did. I shall not repeat it here. But let me record how indebted I am to him for this inspiration. And I would like also to thank him again for the overly generous footnote [33, page 702] in which he gave me credit for extending his commutativity theorem from $x^n = x$ to $x^{n(x)} = x$.

Let me pay tribute to his wife Florence ("Florie"). Not only did she offer him support through a long and happy marriage, she was a joint author [40]. Jake's final three years were saddened by the loss of Florie. Friends, students, and colleagues are mourning the loss of both. We will always remember the hospitality they were always ready to offer and their outgoing, charming personalities.

In closing I would like to mention three more gems: (1) His inauguration of the fertile concept of triple systems [39]. (2) His reduction of Hermitian forms to quadratic forms [21]. Every linear algebraist should put this into his or her armory. (3) This last is due to his student Glennie [G]: the amazing identity satisfied by special Jordan algebras.

Georgia Benkart

It was spring 1934, and Nathan Jacobson was just finishing his doctoral dissertation on division algebras at Princeton under J. H. M. Wedderburn. Richard Brauer, who had been designated Hermann Weyl's research assistant at the newly established Institute for Advanced Study, was delayed in arriving until the fall, so Jacobson was asked to bridge the gap and write up Weyl's lecture notes on continuous groups. This proved to be a momentous event for Lie theory as well as the start of young Jacobson's distinguished writing career.

Weyl felt that it would be of interest to study Lie algebras over arbitrary fields without recourse to the group or to the algebraic closure of the field. Jacobson, who was well versed in Wedderburn's similar investigations on associative algebras, readily took to the task. His first paper on the subject, "Rational methods in Lie algebras" [4], which appeared in 1935, acknowledged Weyl's profound influence. It rederives the well-known theorems of Lie and Engel on solvable and nilpotent Lie algebras by using methods from elementary linear algebra that set the stage for "rationalizing" other parts of the theory.

A beautiful example of the rationalizing process involves Jacobson's notion of a weakly closed subset S in a finite-dimensional associative algebra

Georgia Benkart is professor of mathematics at the University of Wisconsin-Madison. Her e-mail address is benkart@math.wisc.edu.

A. *Weakly closed* means that for each ordered pair of elements $a, b \in S$, there is a scalar $\gamma(a,b)$ so that $ab + \gamma(a,b)ba \in S$. If every element a of S is *nilpotent* ($a^k = 0$ for some k), then the associative subalgebra S^* of A generated by S is *nilpotent* ($(S^*)^m = 0$ for some m). Jacobson perfectly phrased this lovely little gem so that it can be invoked for Lie and Jordan algebras and Lie superalgebras. It is noteworthy as one of the few general results that apply over any field, even fields of prime characteristic.

One crowning achievement of nineteenth-century mathematics was the classification by Cartan and Killing of the finite-dimensional simple Lie algebras over an algebraically closed field $\mathbb{F}$ of characteristic zero. These Lie algebras are (up to isomorphism):

a) $\mathfrak{sl}_n(\mathbb{F})$, the *special linear* Lie algebra of $n \times n$ matrices over $\mathbb{F}$ of trace 0 for $n \geq 2$;
b) $\mathfrak{so}_n(\mathbb{F})$, the *orthogonal* Lie algebra of $n \times n$ matrices x over $\mathbb{F}$ such that $x^t = -x$ for $n \geq 5$, t denoting transpose;
c) $\mathfrak{sp}_n(\mathbb{F})$, the *symplectic* Lie algebra of $n \times n$ matrices x over $\mathbb{F}$ such that $x^t J + Jx = 0$ for $n \geq 4$. Here n must be even, and J is the $n \times n$ matrix of a nondegenerate skew-symmetric bilinear form.
d) one of 5 exceptional Lie algebras $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2$.

When the underlying field $\mathbb{F}$ is not algebraically closed, it is possible to describe the simple Lie algebras L over $\mathbb{F}$ that upon extension to the algebraic closure $\bar{\mathbb{F}}$ are isomorphic to $\mathfrak{sl}_n(\bar{\mathbb{F}})$, $\mathfrak{so}_n(\bar{\mathbb{F}})$ for $n \neq 8$, or $\mathfrak{sp}_n(\bar{\mathbb{F}})$. Jacobson's ground-breaking papers of 1937–38 [9], [14], [15] showed that L is the Lie algebra $(A, [\cdot, \cdot])/\mathbb{F}1$ constructed from a simple associative algebra A whose center is $\mathbb{F}1$ (or what is now called a *central simple associative algebra*), or it is the set of skew elements of a central simple associative algebra with involution. This work began Jacobson's general program on "forms of algebras" [89] that ultimately led to his classification of the forms of the Lie algebra $\mathfrak{g}_2$ using composition algebras [19] and to the classification of forms of simple Jordan algebras [40], [58].

For any sort of algebra A (associative, Lie, Jordan, etc.), the linear transformations $D : A \to A$ that satisfy the "derivative property" $D(ab) = D(a)b + aD(b)$ are said to be *derivations*. Derivations are very natural objects to study [11], [28] especially in Lie theory, because the adjoint transformation $ad\ x : L \to L$, given by $ad\ x(y) = [x, y]$, of a Lie algebra L is always a derivation. This statement is equivalent to the Jacobi identity. In general, the composition D_1D_2 of two derivations need not be a derivation; however, the set of all derivations is a Lie algebra under the commutator product $[D_1, D_2] = D_1D_2 - D_2D_1$. If the underlying field has characteristic $p > 0$, then it is a consequence of Leibniz's formula,

$$D^p(ab) = \sum_{k=0}^{p} \binom{p}{k} D^{p-k}(a)D^k(b),$$

that $D^p(ab) = D^p(a)b + aD^p(b)$. In other words, D^p is a derivation. It was Jacobson's great insight that the property of being closed under p-powers conveys important structural information. This idea led him to introduce the notion of a *restricted* Lie algebra [11].

Rather than present the general abstract definition, let us assume for simplicity that the center

$$Z(L) = \{z \in L \mid [z, x] = 0 \text{ for all } x \in L\}$$

of the Lie algebra L is zero. In that case, L is restricted if for each $x \in L$, the mapping $(ad\ x)^p$, which is a derivation of L, in fact equals $ad\ y$ for some y in L. Usually the element y is written $x^{[p]}$ to indicate its dependence on both x and the p-power.

The Lie algebras associated to algebraic groups (the analogues of Lie groups over arbitrary fields) are always restricted, so the characteristic p versions of the Lie algebras in (a) through (d) are restricted. They are simple too, except when $p \mid n$ for $\mathfrak{sl}_n(\mathbb{F})$, where it is necessary to factor out scalar multiples of the identity matrix. However, they are not the only finite-dimensional simple Lie algebras over algebraically closed fields of characteristic $p > 0$. The Witt algebra, which is the derivation algebra of the truncated polynomial algebra $\mathbb{F}[x \mid x^p = 0]$, provides an example, as do the Jacobson-Witt algebras, which are the derivations of $\mathbb{F}[x_1, \ldots, x_m \mid x_i^p = 0]$. The latter algebras were discovered and investigated by Jacobson in the early 1940s as part of his efforts to develop a Galois theory for purely inseparable field extensions using derivations rather than automorphisms [28]. His work set the stage for Albert and Frank [AF], [F], who constructed simple Lie algebras from the Jacobson-Witt algebras. Kostrikin and Šafarevič [KS] extended the ideas in [AF] and [F] by identifying four unifying families of simple Lie algebras that live in the Jacobson-Witt algebra. These four are called the *Cartan-type* Lie algebras because they correspond to Cartan's four infinite families (Witt, special, Hamiltonian, contact) of infinite-dimensional complex Lie algebras.

Kostrikin and Šafarevič conjectured that over an algebraically closed field of characteristic $p > 5$ a finite-dimensional *restricted* simple Lie algebra is classical (as in (a) through (d) above) or of Cartan type. Almost one hundred years after the classification of the simple Lie algebras of characteristic zero, Block and Wilson [BW] in 1988 succeeded in proving this conjecture. If the notion of Cartan-type Lie algebras is expanded to include the simple algebras arising from Cartan-type algebras that are twisted by an automorphism, then one can formulate the *Generalized Kostrikin-Šafarevič Conjecture* by erasing the restrictedness assumption in the statement above. In the absence of

Books of Nathan Jacobson

[B1] *The Theory of Rings*, Mathematical Surveys, No. II, Amer. Math. Soc., Providence, RI, 1943.
[B2] *Lectures in Abstract Algebra, Vol. 1, Basic Concepts*, Van Nostrand, Princeton, NJ, 1951; Springer-Verlag reprint, 1975.
[B3] *Lectures in Abstract Algebra, Vol. 2, Linear Algebra*, Van Nostrand, Princeton, NJ, 1953; Springer-Verlag reprint, 1975.
[B4] *Lectures in Abstract Algebra, Vol. 3, Theory of Fields and Galois Theory*, Van Nostrand, Princeton, NJ, 1964; Springer-Verlag reprint, 1975.
[B5] *Structure of Rings*, Colloquium Publications, vol. 37, Amer. Math. Soc., Providence, RI, 1956 and 1964.
[B6] *Lie Algebras*, Interscience-Wiley, New York-London, 1962; Dover reprint, 1979.
[B7] *Structure and Representations of Jordan Algebras*, Colloquium Publications, vol. 39, Amer. Math. Soc., Providence, RI, 1968.
[B8] *Lectures on Quadratic Jordan Algebras*, Tata Institute of Fundamental Research, Bombay, 1969.
[B9] *Exceptional Lie Algebras*, Lecture Notes in Pure and Appl. Math., Dekker, New York, 1971.
[B10] *Basic Algebra I*, Freeman, New York, 1974; second edition, 1985.
[B11] *PI-Algebras. An Introduction*, Springer-Verlag, Berlin-New York, 1975.
[B12] *Basic Algebra II*, Freeman, New York, 1980; second edition, 1989.
[B13] *Structure Theory of Jordan Algebras*, Lecture Notes in Math., University of Arkansas, 1981.
[B14] *Collected Mathematical Papers*, volumes 1–3, Birkhäuser, Boston, 1989.
[B15] A. Adrian Albert, *Collected Mathematical Papers*, 2 vols. (R. Block, N. Jacobson, M. Osborn, D. Saltman, and D. Zelinsky, eds.), Amer. Math. Soc., Providence, RI, 1993.
[B16] *Finite Dimensional Division Algebras over Fields*, Springer-Verlag, Berlin-New York, 1996.

restrictedness, Strade's p-envelopes, which are restricted Lie algebras, save the day and enable the classification to be carried out (see [SW], [St]). It is impossible to imagine how the classification might have been achieved without Jacobson's notion of a restricted Lie algebra and his guiding light, for Jacobson kindled in his students and grandstudents a great interest in the classification problem. As a result, he and his descendants—Curtis, Seligman, Wilson, Gregory—and I all have been involved in this enterprise.

Although after the mid-1950s Jacobson devoted much of his research to associative and Jordan algebras, he wrote two books, *Lie Algebras* [B6] and *Exceptional Lie Algebras* [B9], and supervised a number of graduate students in Lie theory. *Lie Algebras* transformed the beautiful classification picture of Cartan and Killing into highly understandable text. Its status as a "classic" having been confirmed by its 1979 republication in the Dover series, *Lie Algebras* still remains the best basic reference for restricted Lie algebras and for an exposition of the famous embedding result known as the Jacobson-Morosov theorem. This book, like Jacobson's papers, has a timeless quality, and one must marvel at just how readable his works are even now, over sixty years after many of them were written.

The nonassociative algebra of octonions (or Cayley numbers) is responsible for most of the exceptional phenomena in Lie and Jordan theory. Just as the complex numbers $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$ are the double of the real numbers $\mathbb{R}$, and the quaternions $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ are the double of the complex numbers, so the octonions $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$ can be regarded as the double of the quaternions. The derivation algebra $\mathrm{Der}(\mathbb{O})$ of the octonions is the simple exceptional Lie algebra $\mathfrak{g}_2$ ([19] or [B6]). The space of 3×3 Hermitian matrices $H_3(\mathbb{O})$ with entries in $\mathbb{O}$ is the exceptional 27-dimensional simple Jordan algebra, now called the Albert algebra, and its derivation algebra $\mathrm{Der}(H_3(\mathbb{O}))$ is the simple exceptional Lie algebra $\mathfrak{f}_4$. The Lie algebras of types $\mathfrak{e}_6$, $\mathfrak{e}_7$, and $\mathfrak{e}_8$ can be constructed using the octonions as well. This is the tale told in *Exceptional Lie Algebras* by an author whose own contributions to that story are immense.

My only class with Jacobson was an exceptional Lie algebras course—it was truly an exceptional Lie algebras course. About twelve years after I took this class, a colleague at Wisconsin, who knew what the book meant to me, brought me nine copies of *Exceptional Lie Algebras* that he had found on sale in New York. They are long gone to good homes, as inquiring minds wanted to know, and there is no better place to start.

As president of the American Mathematical Society and vice president of the International Mathematical Union, Jacobson was extraordinarily busy during my graduate years at Yale. Yet he was a calm, reassuring mentor who never seemed rushed and who always had time to talk. We, his thirty-three Ph.D. students who felt his gentle guidance and experienced his gracious kindness, owe him a special debt that perhaps can be repaid only in kind by emulating his behavior with our own graduate students.

I last saw "Jake" about a year ago, when I briefly stopped in New Haven en route to a colloquium in Boston. Knowing that I had arrived from Princeton, he was eager to reminisce about the exciting early days of the Institute there. He also had just received a copy of Kevin McCrimmon's new book, *A Taste of Jordan Algebras* [McC2]. What delighted him most about the book, dedicated to "Jake" and his wife Florie, was that the contributions of each of them had been acknowledged. That is exactly how he wanted it to be. The mathematical community, their family, and their friends will miss them both very much.

Kevin McCrimmon

I would like to say a few words about Jake's legacy for Jordan algebras. Jordan algebras were

Kevin McCrimmon is professor of mathematics at the University of Virginia. His e-mail address is `kmm4m@virginia.edu`.

introduced by P. Jordan as an attempt to provide an algebraic setting for quantum mechanics that enjoyed all the properties of the usual model yet did not presuppose an underlying associative algebra. In the usual interpretation, quantum-mechanical observables are represented by operators on Hilbert space, but only *hermitian* operators are physically observable. A linear algebra is called a *Jordan algebra* if it satisfies the identities

$$\text{(J1)} \qquad xy = yx,$$

$$\text{(J2)} \qquad (x^2y)x = x^2(yx).$$

If A is an associative algebra, then the vector space A, together with the "anti-commutator" multiplication $x \cdot y := \frac{1}{2}(xy + yx)$, forms a Jordan algebra, denoted A^+. A Jordan algebra J is called *special* if it arises as a Jordan subalgebra of some associative algebra, $J \subseteq A^+$. The most important example is the subspace $H(A, *)$ of $*$-hermitian elements $x^* = x$ with respect to an involution $*$ on A (i.e., an involutive antiautomorphism). Any symmetric bilinear form $\langle \cdot , \cdot \rangle$ on a vector space V over a field F somewhat accidentally gives rise to a special Jordan algebra (a "spin factor") on the space $F \cdot 1 \oplus V$ by having 1 act as identity element and having vectors multiply by $v \cdot w := \langle v, w \rangle 1$; this is a Jordan subalgebra of the Clifford algebra of the bilinear form.

A Jordan algebra is called *exceptional* if it is not special; Jordan was seeking an exceptional Jordan model for quantum mechanics. In 1934 Jordan, J. von Neumann, and E. Wigner made a complete classification of finite-dimensional formally real Jordan algebras and showed they were direct sums of five types of simple algebras: spin factors and hermitian $n \times n$ matrices $H_n(\mathbb{R}), H_n(\mathbb{C}), H_n(\mathbb{H})$ over the reals $\mathbb{R}$, the complexes $\mathbb{C}$, or Hamilton's quaternions $\mathbb{H}$, together with a totally unexpected $H_n(\mathbb{O})$ over Cayley's octonions $\mathbb{O}$ (but only for $n = 3$). This latter 27-dimensional Jordan algebra has since become the celebrated exceptional Jordan algebra (often referred to as the *Albert algebra*).

The exceptional Jordan algebra proved to be an invaluable ingredient in explicit constructions of the exceptional Lie algebras, especially those over arbitrary fields. These Lie constructions and related foundational work Jake did with Albert and others on classifying finite-dimensional nonassociative algebras have already been discussed by Georgia Benkart. I will concentrate on the new insights, new concepts, and new tools he brought to Jordan algebras.

Universal Gadgets

In his very first paper on Jordan algebras [31] in 1948, describing the isomorphisms between the special simple Jordan rings classified by Albert in 1946, Jake used results of Ancochea and Kalisch showing that for certain special Jordan algebras $J_i \subseteq A_i$ any Jordan isomorphism $J_1 \to J_2$ extended to an isomorphism or anti-isomorphism of the given associative envelopes $A_1 \to A_2$. In his 1949 papers [40] (with his wife Florie) and [39], he introduced the *universal specialization* μ in the *universal special envelope* U of J (analogous to the well-known universal associative enveloping algebra of Lie theory). This was introduced to reduce *Jordan* homomorphisms (called *specializations*) $J \to A^+$ to *associative* homomorphisms $U \to A$ of

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the universal gadget. It was characterized by its universal property that *all* specializations $J \xrightarrow{\varphi} A$ (not just the isomorphisms) were reduced to associative homomorphisms of $U \xrightarrow{\tilde{\varphi}} A$ by factoring through the universal specialization,

$$\begin{tikzcd} J \arrow[r, "\varphi"] \arrow[d, "\mu"'] & A \\ U \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd} \qquad \text{or better} \qquad \begin{tikzcd} J \arrow[r, "\varphi"] \arrow[d, "\mu"'] & A^+ \\ U \arrow[r, "\tilde{\varphi}"] & A \arrow[u] \end{tikzcd}$$

the right vertical arrow in the second diagram indicating the forgetful map.

This was especially effective, since U is finite dimensional when J is (unlike in the Lie case), so that the homomorphisms of U were well understood from the associative theory. This strategy made the extension of Jordan specializations to the associative envelope *automatic*. There was of course work to be done in describing the universal gadget for any particular Jordan algebra, but then the entire question of Jordan specializations was reduced to the study of this one associative algebra.

The study of specialization is essentially the study of all Jordan "modules". Jake also introduced a universal gadget for multiplication specializations (corresponding to Jordan "bimodules") and showed how it related to a certain "meson algebra" introduced by physicists.

Triple Products

To avoid messy factors $\frac{1}{2}$ in Jordan products, we can introduce the *brace product* (or 2-tad) $\{x, y\} := 2xy$. In 1949 [39] Jake recognized the importance of the so-called 3-tad $\{x, y, z\}$ defined by

$$2\{x, y, z\} := \{\{x, y\}, z\} + \{\{z, y\}, x\} - \{y, \{x, z\}\}.$$

In associative algebras these products take the simple form $\{x, y\} = xy + yx$ and $\{x, y, z\} = xyz + zyx$. *Jordan triple systems* are algebraic structures closed under a triple product behaving like $\{x, y, z\}$. Jordan algebras are of course closed under their triple products, but certain subspaces might be closed under the triple but not the bilinear product, so Jordan triple systems were a wider class of algebraic structures (as Jake had shown for Lie triple systems). Once more an unexpected connection appeared between Jordan and Lie theory: 3-graded Lie algebras $L = L_1 \oplus L_0 \oplus L_{-1}$ lead naturally to *Jordan pairs* $(V_+, V_-) = (L_1, L_{-1})$—a pair of spaces acting on each other, but not on themselves, as Jordan triple systems—via the Jordan triple product $\{x^+, y^-, z^+\} = [[x^+, y^-], z^+]$. Jordan triples and pairs are now seen as important features of the mathematical landscape, with Jordan algebras as especially exemplary members of this family.

The U-Operator and the Fundamental Formula

One particular case of the Jordan triple product occurs when the two outside variables coincide, leading to an important *quadratic product*,

$$U_x y = \frac{1}{2}\{x, y, x\}.$$

This is equal to xyx in special algebras. Jake introduced these operators and the U-notation (its origins are obscure) and conjectured the *Fundamental Formula* in operator terms,

$$U_{U_x y} = U_x U_y U_x.$$

This is easy to verify in associative algebras, since on an element z it becomes $(xyx)z(xyx) = xyxzxyx = x(y(xzx)y)x$. After hearing Jake lecture on this, I. G. Macdonald went home and proved the conjecture. Moreover, using a deep theorem of Shirshov on two-generated Jordan algebras, he went on to establish a general principle that *any* Jordan polynomial identity in three variables that is linear in one of them will hold in all Jordan algebras as soon as it holds in all associative algebras.

The U-operator and its Fundamental Formula have completely recast our view of the Jordan landscape: we have slowly come to realize that the fundamental product in a Jordan system is the quadratic product $U_x y$, not the bilinear product $\{x, y\}$ or the trilinear product $\{x, y, z\}$, which results by polarizing $x \to (x, z)$ in the quadratic expression $U_x y$. This basic product is as associative as such a product can be: unital Jordan algebras are described axiomatically by

- $U_1 = 1_J$,
- $U_x\{y, x, z\} = \{x, y, U_x z\}$,
- $U_{U_x y} = U_x U_y U_x$.

Jake used the U-operator to obtain the basic facts about inverses. He showed that the proper definition of x invertible is that U_x be an invertible operator, with $U_{x^{-1}} = (U_x)^{-1}$. There is no corresponding result for the bilinear multiplication.

Once more, while the U-operator and Fundamental Formula were proving their worth algebraically, they popped their heads up again in differential geometry in work of Koecher: U_x arises naturally out of the inversion map $j(x) = -x^{-1}$ by

$$U_x = (\partial j \mid_x)^{-1}$$

for $\partial j \mid_x$ the usual differential (best linear approximation) of the nonlinear map j at the point x. This allowed T. A. Springer [Sp] to base an entire theory and classification of Jordan algebras on the operation of inversion. Another illustration of how U_x arises from inversion is the Hua identity, which can be written as

$$U_x(y) = x - (x^{-1} - (x - y^{-1})^{-1})^{-1}$$

or as

$$x^{-1} + (y^{-1} - x)^{-1} = (x - U_x y)^{-1}$$

whenever x, y, and $y^{-1} - x$ are invertible. This identity is relatively easy to derive by an argument appealing to Zariski density and change of unit element. It was Jake who showed how to change units in Jordan algebras (passing from J with unit 1 to an "isotope" with unit u, for any invertible u) and demonstrated the power of this method in Jordan theory.

Generic Norms

Jake made frequent use of the concept of generic norm. This is a generalization of the determinant for matrices, more generally of the "reduced norm" on finite-dimensional associative algebras. He showed that the generic norm could be defined for any finite-dimensional *power-associative algebra*, i.e., an algebra in which each element generates an associative subalgebra, so that the usual rules of powers apply, though the algebra as a whole need not be associative [69]. The key idea was that the "generic element" satisfied a *generic minimum polynomial*

$$x^n - \sigma_1(x)x^{n-1} + \ldots + (-1)^n \sigma_n(x)1 = 0$$

in which each σ_i is a homogeneous polynomial function of degree i. Here σ_1 is the *generic trace* and σ_n the *generic norm*. For ordinary associative matrix algebras these are just the usual trace and determinant, and the generic minimum polynomial is the characteristic polynomial. The key player here was the generic norm $N(\cdot)$, since the minimum polynomial could be recovered as

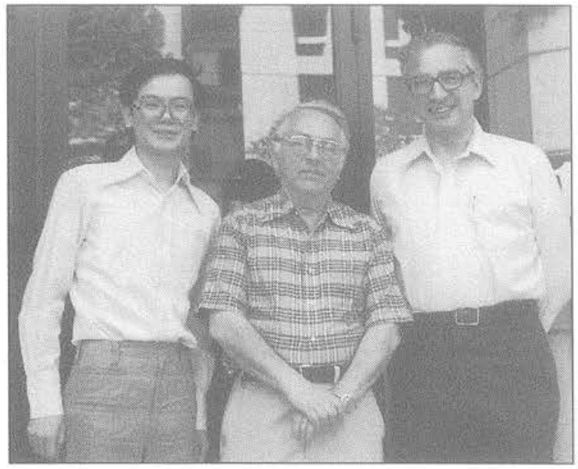
$$N(\lambda 1 - x) = \lambda^n - \sigma_1(x)\lambda^{n-1} + \ldots + (-1)^n \sigma_n(x)1$$

for an indeterminate λ.

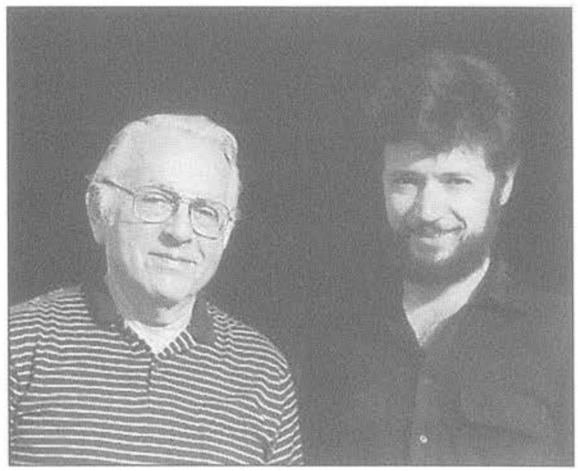
Any quadratic form Q is the generic norm of a Jordan algebra of "degree 2", $x^2 - T(x)x + Q(x)1 = 0$. Only certain very special cubic forms, such as with the Albert algebra, arise as generic norms of "degree 3" Jordan algebras, $x^3 - T(x)x^2 + S(x)x - N(x)1 = 0$. We no longer say, with Cartan, that the exceptional Lie group E_6 arises as a group of transformations preserving a certain cubic form on 27-dimensional space; we say that E_6 arises as the group of linear transformations on the 27-dimensional Albert algebra that preserve the generic norm, or equivalently, the surface $N(x) = 1$. Similarly, F_4 arises not as the isotropy group of a point on the cubic surface, but as the isotropy group of the identity element, or better, as precisely the automorphism group of the Albert algebra.

Inner Ideals

David Topping [T] introduced *quadratic ideals* (subspaces $B \subseteq J$ closed under the quadratic product $U_B J \subseteq B$) in his study of Jordan algebras of self-adjoint operators on Hilbert space. Jake recognized immediately the utility of these as an analogue of one-sided ideals for general Jordan algebras and developed a structure theory for Jordan rings with descending chain condition (d.c.c.) on quadratic ideals that was completely analogous to the Artin-Wedderburn theory for associative rings with d.c.c. on left ideals [73]. Like Athena springing full grown from the mind of Zeus, this theory sprang full grown from the mind of Jake.



Yale, about 1980. Left to right: Ying Cheng, Jacobson, and Walter Feit.



Jacobson at his 75th birthday celebration, 1985, with Efim Zelmanov.

This was the first truly ring-theoretic approach to Jordan algebras: the requisite idempotents (for no theory seemed possible without a rich supply of idempotents) arose from the minimal quadratic ideals instead of elements algebraic over a field. After Jake's paper, one could paraphrase Archimedes and say, "Give me the Fundamental Formula and I can move the world." I mimicked Jake's entire paper [McC1] to get a theory of *quadratic Jordan rings* based entirely on the product $U_x y \approx xyx$, which had no need of a scalar $\frac{1}{2}$ and hence was applicable not only to fields of characteristic 2 but also to arithmetic situations, such as algebras over the integers.

Later Jake rechristened these B's *inner ideals.* The Jordan product xyx does not have a left or right like the associative product xy; it has an *inside* and an *outside.* An *inner ideal* is a subspace B closed under inner multiplication by J, $U_B J \subseteq B$, while an *outer ideal* is closed under outer multiplication, $U_J B \subseteq B$. If there is a scalar $\frac{1}{2}$ available, these outer ideals are the same thing as *ideals* (two-sided, both inner and outer).

The final achievement of the classical age in Jordan algebra was Jake's structure theory for algebras with *capacity* [B13]. The d.c.c. leads to minimal inner ideals, which generate division idempotents, so Jake just started from a decomposition of the unit 1 into a finite sum of division idempotents. Serendipitously, this was just what was needed when Efim Zelmanov ushered in the new age in Jordan theory with his classification of simple Jordan algebras of arbitrary dimension.

And so the torch was passed to a new generation.

David J. Saltman

Nathan Jacobson has had an important and deep influence on the theory of central simple algebras despite the fact that he wrote on the subject only at isolated points of his career. His contributions can be divided into distinct time periods, separated by periods when he published in other areas. His influence was not only exercised by means of research papers but also through his exposition of known results in monographs.

To begin with, Jake's Ph.D. thesis was in the area of central simple algebras [1]. In that work he traced the connection between cyclic algebras and twisted polynomial rings. The deepest part concerned the so-called *Schur index*, which is the degree of the division algebra associated to a central simple algebra. In his thesis Jake shows that the Schur index of a cyclic algebra can be computed by knowing the factorization of a polynomial in the twisted polynomial ring. Though this work dates to 1934, the point of view is still being exploited and generalized, for example, in the work of Louis Rowen.

The next phase of Jake's work in central simple algebras came in two papers, [11] and [12], in 1937. In the then-current theory of central simple algebras, an important place was occupied by the so-called Noether-Skolem Theorem, which concerned automorphisms of these algebras. Jacobson began the investigation of derivations and their place in the theory. His first important result was analogous to the automorphism case. He proved that every derivation of a central simple algebra, trivial on the center, was inner. His next result had no automorphism version: Jake proved that every derivation on the center extended to the whole algebra, a result clearly false for automorphisms. These observations turned out to be crucial in the theory, then just begun, of so-called p-algebras, which are central simple algebras of prime characteristic p and degree a power of p. Jake could rewrite cyclic algebras of degree p in terms of derivations instead of automorphisms. It turns out that this approach (but not the crossed-product approach) extends beautifully to Azumaya algebras over rings. Furthermore, behind Jacobson's derivation extension result lay a useful fact that has become key in the theory, namely, that the Brauer group map is surjective over purely inseparable extensions.

The next paper on central simple algebras, of particular note, is on another topic still. In [15] Jacobson began his study of central simple algebras with involution, a subject he concerned himself with until the end of his career. An *involution* of A is an anti-automorphism of order 2, and for the moment an involution will always be the identity on the center F of A. If $A = \mathrm{End}_F(V)$, then involutions correspond to similarity classes of symmetric or antisymmetric nonsingular bilinear forms on V. In the symmetric case (that of a quadratic form) the involution is called *orthogonal*, a concept that generalizes to an arbitrary central simple A.

Given a quadratic form, one can define the useful and important Clifford algebra and even Clifford subalgebra. In [70] Jake showed that one could define an even Clifford algebra for an orthogonal

David J. Saltman is professor of mathematics at the University of Texas, Austin. His e-mail address is `saltman@fireant.ma.utexas.edu`.

Ph.D. Students of Nathan Jacobson

Charles L. Carroll Jr. (1945)
Eugene Schenkman (1950)
Charles W. Curtis (1951)
William G. Lister (1951)
Henry G. Jacob (1953)
George B. Seligman (1954)
Morris Weisfeld (1954)
Bruno Harris (1956)
Earl J. Taft (1956)
Dallas W. Sasser (1957)
Maria J. Woneburger (1957)
Tae-il Suh (1961)
Herbert F. Kreimer Jr. (1962)
Charles M. Glennie (1963)
David A. Smith (1963)
Dominic C. Soda (1964)
Harry P. Allen (1965)
Eugene A. Klotz (1965)
Kevin M. McCrimmon (1965)
Joseph C. Ferrar (1966)
Daya-Nand Verma (1966)
Lynn Barnes Small (1967)
John R. Faulkner (1969)
Samuel R. Gordon (1969)
Michel Racine (1971)
Jerome M. Katz (1972)
Ronald Infante (1973)
Louis H. Rowen (1973)
Georgia M. Benkart (1974)
David J. Saltman (1976)
Robert A. Bix (1977)
Leslie Hogben (1978)
Craig L. Huneke (1978)

involution on an arbitrary central simple A. This was the important first result in a continuing long program of many people (e.g., J. P. Tignol) that has extended many parts of the theory of quadratic forms to involutions.

While on the subject of involutions, let me jump ahead to one of Jake's last published results. An involution on A, still trivial on the center, which is not orthogonal is called *symplectic.* If $S \subset A$ is the space of elements fixed by a symplectic involution, then there is a form p on S, called the *Pfaffian*, whose square is a form of the determinant restricted to S. In [79] Jake showed that the function field L of the zero set of the Pfaffian was a so-called "generic $\frac{1}{2}$ splitting field". That is, $A \otimes_F L$ has Schur index 2 in the nontrivial cases, and any other field with this property is a specialization of L. This was the first (and best understood) example of what one calls generalized Brauer-Severi varieties.

The last subject of Jake's interest we highlight is reduced norms. If A is a central simple algebra with center F and $\bar{F}$ is the algebraic closure of F, then $A \otimes_F \bar{F}$ is isomorphic to a ring of matrices $M_n(\bar{F})$. Such an example has the well-known determinant map $d : M_n(\bar{F}) \to \bar{F}$. However, this map can be defined on A itself and is then called the *reduced norm* $n : A \to F$. This is a polynomial map and can be thought of as a polynomial on A. Jake showed that this polynomial carries a surprising amount of information about the algebra-structure of A. Namely, he showed that a linear map $f : A \to B$ that preserves 1 and preserves the norm, must be an isomorphism or anti-isomorphism. This result is a special case of the Jordan theory in [76], where it arises, in the above form, only after combining other results.

The book mentioned above, [B16], was Jake's last publication and will very likely influence the field significantly. In [B11] Jake wrote a monograph giving a complete exposition of Amitsur's noncrossed product result. His clear, careful, and streamlined approach was a huge influence on mathematicians following up on Amitsur's result.

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