Representations of Finite Groups: A Hundred Years, Part I

T. Y. Lam

Introduction

Mathematical ideas in any subject area are often discovered and developed over a period of time, so it is usually not possible to assign a specific date to a discovery. But in a few cases a discovery may have been accompanied by an event of such a unique or peculiar nature that the discovery itself has come to be identified with that event. A well-known instance of this is Hamilton's discovery of the quaternions, which is invariably associated with his famous walk on October 16, 1843, along the Royal Canal in Dublin. His carving of the quaternion equations on a stone of the Brougham Bridge added such an element of romance to the story that the date of 10/16/1843 is indelibly etched in history books of mathematics as the date of birth of the quaternions. Another instance took center stage some fifty years later—this time it was the creation of the theory of representations of finite groups. On April 12, 1896, F. G. Frobenius penned his first letter to R. Dedekind to describe his new ideas on factoring a certain homogeneous polynomial associated with a finite group, called the "group determinant". Two more letters quickly ensued (on April 17 and April 26, 1896), and by the end of April that year, Frobenius was in possession of the rudiments of the character theory of finite groups. It was to take some more time for the idea of group representations to be fully developed, but the famous Frobenius-Dedekind *Briefwechsel* in April 1896 is now hailed by historians as the single most significant event marking the birth of the representation theory of finite groups.

As a student of algebra, I have always been fascinated by the theory of group representations. I dabbled in the subject thirty years ago when I wrote my doctoral dissertation, and have remained a user and admirer of the subject ever since. When I realized that April of 1996 was the one-hundredth anniversary of the discovery of the representation theory of finite groups, the temptation to have some kind of "celebration" of this occasion was great. Purely by chance I got a call in March 1996 from Alan Weinstein, our department's colloquium chairman, who asked me to recommend someone for an unfilled colloquium slot. Before I hung up, I found that I had "volunteered" myself to be colloquium speaker for a talk to commemorate the centennial of group representation theory! I will forever be in shame for suggesting myself as colloquium speaker, but then I got my chance to tell the fascinating stories associated with the birth of representation theory, on April 18, 1996, *almost exactly* one hundred years after Frobenius penned his first famous group-determinant letter to Dedekind. The same talk was repeated with some variations in May at Ohio State University, and then in June of the same year in the "Aspects of Mathematics" Conference at my alma mater, the University of Hong Kong. Due to my administra-

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T. Y. Lam is professor of mathematics at the University of California, Berkeley. His e-mail address is `lam@math.berkeley.edu`.

This article, in two parts, is a slightly abridged version of an article appearing concurrently in the Proceedings of the "Aspects of Mathematics" Conference *(N. Mok, ed.), University of Hong Kong, H.K., 1998. Part II, concerning William Burnside, the Burnside problems, and their influence on contemporary mathematics, will appear in the April 1998 issue of the* Notices.

tive duties at MSRI, the writing of the article was put off for more than a year. A fall sabbatical in 1997 finally enabled me to finish the project, so I am now pleased to offer this leisurely written account of my lecture. A somewhat longer version with more technical details will appear concurrently in the proceedings of the "Aspects" conference, published by the University of Hong Kong. In particular, some of the proofs omitted from this article can be found in the Hong Kong proceedings.

Disclaimer and References

Before we try to tell the reader what this article is, we should perhaps first tell him/her what the article is *not*. A proper coverage of the history of the representation theory of finite groups would take no less than a full-length volume, starting with the pioneering work of Molien, Cartan, Dedekind, Frobenius, Burnside, on to the rewriting of the foundations of the subject by Schur, Noether, then to the pivotal work of Brauer on both the ordinary and the modular representation theory of groups, climaxing perhaps with the monumental classification program of finite simple groups (which certainly would not have been possible without the aid of character theory). Such a major undertaking is best left to the experts, and I was glad to learn that Professor Charles Curtis is preparing such a volume [Cu_2] in the History of Mathematics series of the Society. In my one-hour talk, all I had time for was to give the audience a few snapshots of the big story, focusing on the origin of representation theory to suit the centennial occasion. Thus, we started with some background in nineteenth-century mathematics, surveyed the work of Dedekind, Frobenius, and Burnside, and went on to talk a little about Schur and Noether, after which we simply declared ourselves "saved by the bell". This write-up is an expanded version of my talk[1], but still it is sketchy and anecdotal at best, and is no substitute for the more scholarly writings on the subject in the literature. For the latter, we recommend the articles of Hawkins [H_1-H_4], written from the perspective of a historian, and the work of Curtis [Cu_1, Cu_2] and Ledermann [L_1], written from the viewpoint of mathematicians. For surveys of representation theory in the broader framework of harmonic analysis, we recommend Mackey [Ma] and Knapp [Kn]. The recent article of K. Conrad [Con], complete with detailed proofs and interesting computational examples, also makes good reading for those with pencil and paper in hand.

[1] *To save space, the part about Schur and Noether is not included in the present article. Readers interested in the work of Schur and Noether in representation theory may consult [L_2, La, Cu_2].*

Since most material is drawn from existing sources (loc. cit.), we make no pretense of originality in this article. In writing it up, we did try to strike a balance between the mathematics and the human dimensions of mathematics; some of the remarks of a more interpretative nature about mathematicians and mathematical events were my own. It is hoped that, by mixing history with mathematics, and by telling the story in the chatty style of a colloquium lecture, we are able to present a readable and informative account of the origin of the representation theory of finite groups.

I am much indebted to Charles Curtis, who kindly provided me with various chapters of his forthcoming book [Cu_2], and it is my great pleasure to thank him, Keith Conrad, Hendrik Lenstra, Monica Vazirani, and the editorial staff of the *Notices* for comments, suggestions, and corrections on this article.

Backdrop of Late Nineteenth-Century Group Theory

Before we begin our story, a quick look at the group theory scene in Europe in the last decades of the nineteenth century is perhaps in order. If we regard group theory as originating from the time of Gauss, Cauchy, and Galois, the subject was then already more than half a century old. Budding German mathematician Felix Klein inaugurated his Erlangen Program in 1872, proclaiming group theory as the focal point for studying various geometries; in the same year, Norwegian high school teacher Ludwig Sylow published the first proofs of his now famous theorems in the fifth volume of the *Mathematische Annalen*. Arthur Cayley and Camille Jordan were the reigning group theorists of the day. Among the first treatises in group theory were Jordan's *Traité des Substitutions et des Équations Algébriques* (1870) and Netto's *Substitutionentheorie und Ihre Anwendungen auf die Algebra* (1882). Both books were on the theory of permutation groups, then synonymous with group theory itself. (The only notable exception was the work of von Dyck on groups defined by generators and relations in 1882–83.) One of the most popular algebra texts of the day was Serret's *Cours d'Algèbre Supérieure*, the second volume of which (3rd ed., 1866) contained a good dose of groups of substitutions. Abstract groups were treated only later, perhaps first in text form, in Weber's *Lehrbuch der Algebra*. Authors of group theory papers were not always careful, and in fact were sometimes prone to making mistakes. Otto Hölder apparently started the tradition of writing long papers in group theory, analyzing groups case by case, but was not above forgetting a few. Even the great Arthur Cayley, known to be "thoroughly conversant with everything that had been done in every branch of mathematics" [C: pp. 265–266], bewildered his readers by blithely listing, as late as 1878, *three* groups of order 6 in his paper [Ca] in the first issue of the *American Journal of Mathematics.*

As far as representation of groups is concerned, there was not much in evidence. In his work in the 1870s and 1880s, Klein certainly used matrices to realize groups, but he did this only for a few specific groups, and there was no hint at a possible theory. In number theory, the Legendre symbol $(\frac{a}{p})$ (p an odd prime) perhaps provided the first instance of a "character". This symbol takes values in $\{\pm 1\}$, and is multiplicative in the variable a. Gauss used similar symbols in dealing with Gauss sums and with binary quadratic forms, but allowed these symbols to take roots-of-unity values. In Dirichlet's work on primes in an arithmetic progression, the Dirichlet L-series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

figured prominently a "mod k character" χ, which is multiplicative in n, and zero when n is not relatively prime to k. The abstract definition of an (abelian) character we owe to Richard Dedekind. In one of his supplements to Dirichlet's lectures in number theory [D], c. 1879, Dedekind formally defined a *character* on a finite abelian group G to be a homomorphism from G to the multiplicative group of nonzero complex numbers. Under the pointwise multiplication of functions, the characters of G form a group $\hat{G}$ (called the *character group*), with cardinality equal to $|G|$, the cardinality of the group G itself. Orthogonality relations among characters were proved, and were included in book form in Band II of Weber's *Lehrbuch.* The stage was now set for the discovery of the general character theory of arbitrary finite groups.

Dedekind and the Group Determinant

To a modern student of mathematics, it would have been perfectly natural to extend the definition of a character by taking homomorphisms D of a group G into $\mathrm{GL}_n(\mathbb{C})$ (the group of invertible $n \times n$ complex matrices) and defining $\chi_D(g) = \mathrm{trace}(D(g))$ $(g \in G)$ to get a character. This, however, was *not* an obvious step for the mathematicians in the nineteenth century. Thus the discovery of the notion of characters for general groups was to take a rather circuitous route, through something which Dedekind called the *group determinant.*

A last descendant of Gauss's famous school in Göttingen, Richard Dedekind (1831–1916) was undisputedly the dean of abstract algebra in Germany toward the end of the nineteenth century. Though he preferred a teaching position at a local institute in his hometown of Braunschweig[2] to a chair in a more prestigious university, the mathematical influence he exerted was perhaps a close second to that of Karl Weierstrass. Dedekind's greatest contributions were in the area of number theory. In contemplating the form of the discriminant of a normal number field with a normal basis, Dedekind arrived at a similar determinant in group theory. Given a finite group G, let $\{x_g : g \in G\}$ be a set of commuting indeterminates, and form a $|G| \times |G|$ matrix whose rows and columns are indexed by elements of G, with the (g,h) entry given by $x_{gh^{-1}}$. (One could have taken the (g,h) entry to be x_{gh} (as Dedekind first did), but the two matrices would have differed only by a permutation of columns.) The determinant of $(x_{gh^{-1}})$ was christened the "group determinant" of G; following Dedekind, we denote it by $\Theta(G)$.

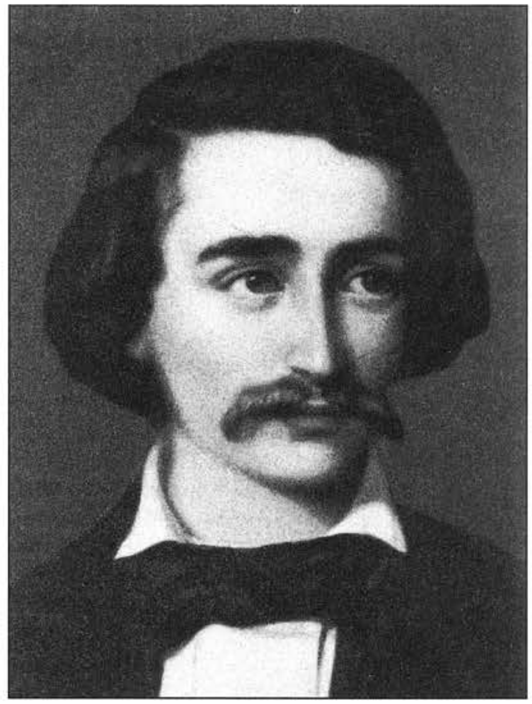


Image (reproduced from oil painting) courtesy of Heiko Harboth.

Richard Dedekind.

In the case of an *abelian* group G, $\Theta(G)$ factors completely into linear forms over $\mathbb{C}$ through the characters of G, as follows:

$$\Theta(G) = \prod_{\chi \in \hat{G}} \left(\sum_{g \in G} \chi(g) x_g \right), \tag{4.1}$$

where $\hat{G}$ is the character group of G. The proof is pretty easy. Indeed, for a fixed $\chi \in \hat{G}$, multiply the gth row of the determinant by $\chi(g)$ and add up all the rows. On the hth column, we get

$$\sum_{g \in G} \chi(g) x_{gh^{-1}} = \left(\sum_{g' \in G} \chi(g') x_{g'} \right) \chi(h).$$

Thus, $\Theta(G)$ is divisible by $\sum_{g \in G} \chi(g) x_g$ for each character χ. Since there are $|\hat{G}|$ different characters and these give rise to different linear forms, we obtain (4.1). The factorization of $\Theta(G)$ was certainly not without precedent. In the case of a cyclic group, the "group matrix" $(x_{gh^{-1}})$ is just a circulant matrix, and the factorization of its determi-

[2] *Now the Technical University of Braunschweig.*


Photograph courtesy of the Institut Mittag-Leffler.

Ferdinand Georg Frobenius.

nant in terms of $|G|$-th roots of unity was well known to nineteenth-century mathematicians.

In the case of a general group G, we can form an "abelianization" $G/[G, G]$, where $[G, G]$ is the (normal) subgroup generated by the commutators in G. The above proof would still give (at least) $|G/[G, G]|$ linear factors of $\Theta(G)$, corresponding to the characters of $G/[G, G]$. However, these will no longer exhaust the group determinant. For instance, if $G = [G, G]$, this would give only the trivial factor $\sum_{g \in G} x_g$. Like most nineteenth-century mathematicians, Dedekind was well grounded in computations. He computed explicitly $\Theta(G)$ for the first nonabelian group S_3 and found that, besides the linear factors $\sum_{g \in G} x_g$ and $\sum_{g \in G} \operatorname{sgn}(g) x_g$ corresponding to the trivial character and the sign character of $G/[G, G]$, $\Theta(G)$ has a remaining squared factor of an irreducible quadratic. He also made similar computations with the quaternion group of order 8, and made the curious observation that, if the scalar field is extended from $\mathbb{C}$ to suitable "hypercomplex systems" (or "algebras" in current terminology), both of his examples of $\Theta(G)$ would factor into linear forms as in the abelian case. Dedekind worked sporadically on this problem in 1880 and 1886, but did not arrive at any definitive conclusions. In a letter to Frobenius dated March 25, 1896, largely concerning Hamiltonian groups, Dedekind mentioned on the side his earlier excursions into the group determinant, including his factorization (4.1) in the abelian case and his thoughts on the possible role of hypercomplex systems in the general case. A follow-up letter dated April 6, 1896, contained the two nonabelian examples he had worked out, along with some conjectural remarks to the effect that the number of linear factors of $\Theta(G)$ should be equal to $|G/[G, G]|$. Feeling, however, that he himself could not achieve anything with the problem, Dedekind invited Frobenius to look into this matter. As it turned out, it was these two letters written by Dedekind that would become the catalyst for the creation of the character theory for abstract nonabelian groups.

Ferdinand Georg Frobenius (1849–1917)

Eighteen years Dedekind's junior, Frobenius himself had achieved great fame by 1896. He got his mathematical education at the famous Berlin University, under the tutelage of illustrious teachers such as E. Kummer, L. Kronecker, and K. Weierstrass. He wrote a thesis under Weierstrass in 1870 on the series solution of differential equations, and thereafter taught briefly in the Gymnasium and at the University. The University of Berlin was traditionally a feeder school for faculty positions at the Polytechnicum in Zürich (now the Eidgenössische Technische Hochschule), so it was not surprising that Frobenius moved to Zürich in 1875 to accept a professorial appointment there.

During his seventeen-year tenure at E.T.H., Frobenius made a name for himself by contributing to a wide variety of mathematical topics, especially in linear differential equations, elliptic and theta functions in one and several variables, determinant and matrix theory, and bilinear forms. His preference for dealing with algebraic objects was increasingly apparent by the late 1880s, when he began to make his influence felt also in finite group theory. In 1887 he published[3] the first proof of the Sylow theorems for abstract groups (rather than for permutation groups): his inductive proof for the existence of a Sylow group using the class equation is the one still in use today. The same year saw another great group theory paper of his [F: (36)]; this one offered his penetrating analysis of double-cosets in a finite group, and contained the famous Cauchy-Frobenius Counting Formula, now ubiquitous in combinatorics. Unbeknownst to Frobenius, all of this group-theoretic work was preparing him for the greatest gift he would bestow on mathematics: the theory of group characters, which he was soon to invent.

Careerwise the early 1890s was a time of change for Frobenius. With the death of Leopold Kronecker in December 1891, a chair became vacant at the University of Berlin. It was hardly a surprise to anyone that the call went to the university's former favorite son, F. G. Frobenius. Then forty-three and at the height of his creative powers, Frobenius was clearly a worthy successor to Kronecker. It was just not as clear if Kronecker himself would have

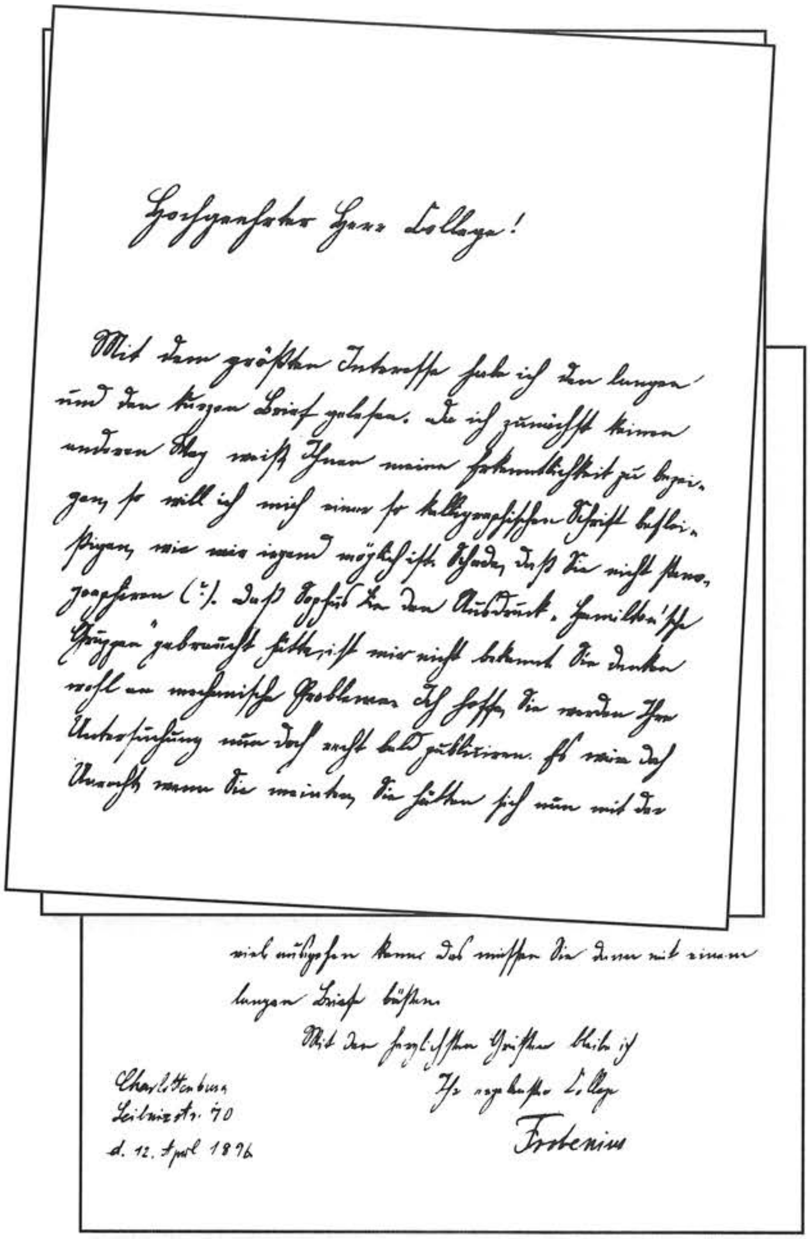
[3] *There seemed to have been an inordinate delay in the publication of [F: (35)]. (This citation means Frobenius's paper (35) in his* Collected Works *[F].) Frobenius communicated this paper to the* Crelle Journal *in March of 1884, but the paper came out only in 1887. By this time some of Frobenius's subsequent papers providing various generalizations of Sylow's Theorems had already come out in* S'Ber. Akad. Wiss. Berlin.

approved of the choice. An ardent believer in his motto "God created the integers, all the rest is the work of men", Kronecker had severely criticized almost everyone engaged in mathematical pursuits involving real or transcendental numbers. His attacks on function theorists were so unsparing and vociferous that at one point even the old Professor Karl Weierstrass was reduced to tears. Kronecker would probably have balked at the idea that his successor be a student of Weierstrass, but then obviously the choice wasn't his.

Frobenius was born in Charlottenburg, a suburb of Berlin. Seventeen years was a long time to be away from home, and in those days people had distinct leanings toward spending their lifetime in their birthtown. Thus, with the call from Berlin, Frobenius was happy to take his family back to Germany in 1893, and settled down in his new home at 70 Leibnizstrasse, Charlottenburg. In the same year, he was elected to membership in the prestigious Prussian Academy of Sciences. With Kronecker and Kummer both dead and his former teacher Weierstrass reaching eighty, Frobenius was to become one of the main torchbearers for the Berlin school of mathematics from that time on.

Though already well versed in group theory, before 1896 Frobenius had never heard of the definition of the group determinant.[4] However, he was a great expert in determinant theory, and he had actually dealt with somewhat similar determinants in his earlier work in theta functions and in linear algebra. As a result, Dedekind's problem of factoring the group determinant caught his immediate attention. He was struck by Dedekind's factorization of $\Theta(G)$ in the abelian case, but was not convinced that hypercomplex numbers would provide the right tool for its generalization. Thus, he set out to investigate the factorization of $\Theta(G)$ just over the complex field. He was amazingly quick in coming to grips with this problem. Working with almost feverish intensity, he invented in less than a month the general character theory of finite groups, and applied this new-found theory to solve the factorization problem for the group determinant. He reported his findings in three long letters to Dedekind on April 12, 17, and 26 of 1896. These letters, together with others in the Frobenius-Dedekind correspondence currently held in the archives of the Technical University of Braunschweig, are now the first written record of the invention of the character theory of finite groups.

[4]*Dedekind had not published any of his findings on this topic.*


Hochgeehrter Herr College!

Mit der herzlichsten Grüßen bleibe ich
Ihr ergebenster College
Frobenius

Charlottenburg
Leibnizstr. 70
d. 12. April 1896

The top portion is part of the first page of Frobenius's April 12, 1896, letter to Dedekind. The letter began with "Hochgeehrter Herr College!", a common salutation between colleagues in Frobenius's time. The lower portion is part of the last page, on which Frobenius signed off as "Ihr ergebenster College, Frobenius", and wrote in the left margin his home address, "Charlottenburg, Leibnizstr. 70", dating the letter "d. 12. April 1896". The letter was written on six large sheets of paper, with four pages per sheet.

I want to thank Clark Kimberling for kindly supplying me with a copy of this letter. The mathematical community owes much to Kimberling, who rediscovered the letters written by Frobenius to Dedekind (and various other Dedekind correspondences) among the papers left in the estate of Emmy Noether. The interesting circumstances surrounding the recovery of these letters are reported in Kimberling's Web page at the URL `http://www.evansville.edu/~ck6/bstud/dedek.html`.

—T. Y. L.

Since the Frobenius letters have already been analyzed in detail in the writings of Hawkins and Curtis (loc. cit.), we will try to approach them from a different angle. Assuming we are talking to a modern audience, we will first discuss how the group determinant can be factored by using the current tools of representation theory. With this hindsight we will then return to Frobenius's work, and explain how he solved the factorization problem for $\Theta(G)$ in 1896 and invented the theory of group characters in the meantime.

There is actually also a strategic reason for our approach. Although it was the group determinant which first led Frobenius to the invention of group characters, the modern theory of group representations is no longer developed through group determinants. In fact, few current texts on representation theory even touch upon this subject, so it is rather likely that modern students of representation theory have never heard of the group determinant. The following section explaining a part of Frobenius's work in terms of the modern methods of representation theory will therefore serve as a useful link between the old approach and the new.

Factorization of $\Theta(G)$ for Modern Readers

Actually, what we are going to do in this section is not all that "modern". Everything we shall say here was known to Emmy Noether, as the reader can easily verify by reading her account of the group determinant in her fundamental paper on representation theory [N: §23, pp. 685–686]. In fact, true to form, Noether considered more generally "system-matrices" and "system-determinants" over possibly nonsemisimple algebras. It suffices for our purposes to work with the group algebra $\mathbb{C}G$: this is the algebra consisting of finite formal linear combinations $\sum_{g\in G} a_g g$ ($a_g \in \mathbb{C}$), which are added and multiplied in the natural way.

As we have noted in an earlier section, a *representation* of a group G means a group homomorphism $D : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$; the number n is called the *dimension* (or the *degree*) of the representation. The representation D is said to be *irreducible* if no (nontrivial) subspace of $\mathbb{C}^n$ is invariant under the action of $D(G)$. Each representation D (irreducible or not) gives rise to a *character* $\chi_D : G \longrightarrow \mathbb{C}$, defined by

$$\chi_D(g) = \mathrm{trace}(D(g)) \quad (\text{for any } g \in G).$$

Two n-dimensional representations D, D' are said to be *equivalent* if there exists a matrix $U \in \mathrm{GL}_n(\mathbb{C})$ such that $D'(g) = U^{-1}D(g)U$ for all $g \in G$. In this case, clearly $\chi_D = \chi_{D'}$. Conversely, if $\chi_D = \chi_{D'}$ and G is a finite group, a basic result in representation theory guarantees that D and D' are equivalent.

Let us now broaden our view of the group determinant by introducing a determinant for any representation of a finite group G, as follows. Given a representation $D : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$, we take a set of indeterminates $\{x_g : g \in G\}$ as before, and let

$$\Theta_D(G) = \det\left(\sum_{g\in G} x_g D(g)\right). \tag{6.1}$$

We note the following three facts:

1. If we think of $\sum_{g\in G} x_g g$ as a "generic" element $\mathbf{x}$ of the group algebra $\mathbb{C}G$, the matrix $\sum_{g\in G} x_g D(g)$ above is just $D(\mathbf{x})$ upon extending D to a $\mathbb{C}$-algebra homomorphism $\mathbb{C}G \longrightarrow \mathbb{M}_n(\mathbb{C})$. Indeed, it is often convenient to think of the representation D as being "given" by this algebra homomorphism.

2. $\Theta_D(G)$ depends only on the equivalence class of the representation D, since a conjugation of the representing matrices will not change the determinants.

3. In the case where D is the *regular representation* (so that $D(g)$ is the permutation matrix associated with the left multiplication of g on G), $\Theta_D(G)$ is precisely the group determinant $\Theta(G)$. In fact, on the hth column, the matrix $x_{g'}D(g')$ has an entry $x_{g'}$ on the $g'h$th row and zeros elsewhere. Therefore, on the hth column, $\sum_{g'\in G} x_{g'}D(g')$ has exactly the entry $x_{gh^{-1}}$ on the gth row.

Clearly $\Theta_{D_1\oplus D_2}(G) = \Theta_{D_1}(G)\Theta_{D_2}(G)$. Therefore, to compute $\Theta(G)$, we can first "break up" the regular representation into its irreducible components. This is a standard procedure in the representation theory of finite groups, which utilizes the fundamental structure theorem on $\mathbb{C}G$ due to Maschke and Wedderburn. According to this result,

$$\mathbb{C}G \cong \mathbb{M}_{n_1}(\mathbb{C}) \times \cdots \times \mathbb{M}_{n_s}(\mathbb{C}) \tag{6.2}$$

for suitable n_i's (such that $\sum_i n_i^2 = |G|$). The projection from $\mathbb{C}G$ onto $\mathbb{M}_{n_i}(\mathbb{C})$ provides the ith irreducible complex representation D_i, and, using a little bit of ring theory, one sees from (6.2) that the regular representation is equivalent to $\bigoplus_i n_i D_i$. Next we make the following observation.

Lemma 6.3. *Each $\Theta_{D_i}(G)$ is an irreducible polynomial over $\mathbb{C}$, and it is not proportional to $\Theta_{D_j}(G)$ for each $j \neq i$.*

Proof. The crucial point here is that, if we write $D_i(\mathbf{x}) = (\lambda_{jk}(\mathbf{x}))$, then the linear forms $\lambda_{jk}(\mathbf{x})$'s are linearly independent over $\mathbb{C}$. In fact, suppose $\sum_{j,k} c_{jk}\lambda_{jk}(\mathbf{x}) = 0$ where $c_{jk} \in \mathbb{C}$. Since $D_i : \mathbb{C}G \longrightarrow \mathbb{M}_{n_i}(\mathbb{C})$ is *onto*, we can find suitable values of the x_g's in $\mathbb{C}$ such that $D_i(\mathbf{x})$ becomes a matrix unit $E_{j_0k_0}$. Plugging in these values of x_g's into $\sum_{j,k} c_{jk}\lambda_{jk}(\mathbf{x}) = 0$, we see that each $c_{j_0k_0} = 0$. Having proved the linear independence of the $\lambda_{jk}(\mathbf{x})$'s, we can then extend them to a basis of the space of all linear forms in $\{x_g : g \in G\}$. This basis will now serve as new variables for the polynomial ring $\mathbb{C}[x_g : g \in G]$, and in terms of these new variables, it is well known that $\det(\lambda_{jk}(\mathbf{x}))$ is irreducible.

To prove the last statement in (6.3), it suffices to see that $\Theta_{D_i}(G)$ actually determines the representation D_i. To this end, think of $\Theta_{D_i}(G)$ as a polynomial in x_1. Since $D_i(1) = I_{n_i}$, x_1 appears only in the diagonal of $D_i(\mathbf{x})$. Writing $D_i(g) = (a_{jk}(g))$, we have $\lambda_{jj}(\mathbf{x}) = \sum_{g \in G} a_{jj}(g) x_g$, and so

$$\begin{aligned} \Theta_{D_i}(G) &= \prod_{j=1}^{n_i} \lambda_{jj}(\mathbf{x}) + \cdots \\ &= x_1^{n_i} + \sum_{g \in G \setminus \{1\}} \chi_{D_i}(g) x_1^{n_i - 1} x_g + \cdots . \end{aligned} \tag{6.4}$$

Thus, this irreducible factor determines the character χ_{D_i}, and, as we have observed before, χ_{D_i} determines D_i, as desired. □

In view of the above, it follows that

$$\Theta(G) = \prod_{i=1}^{s} \Theta_{D_i}(G)^{n_i} \tag{6.5}$$

is the *complete* factorization of the group determinant into irreducibles over $\mathbb{C}$. Here, since the representation D_i has dimension n_i, the degree of the irreducible factor $\Theta_{D_i}(G)$ is n_i—the same as the multiplicity with which $\Theta_{D_i}(G)$ appears in $\Theta(G)$. Also, from (6.2) s is seen to be the $\mathbb{C}$-dimension of $Z(\mathbb{C}G)$ (the center of $\mathbb{C}G$), which is given by the number of conjugacy classes of G. We shall return to this point a little later in the next section.

From (6.4) and (6.5), we see clearly that the factorization of $\Theta(G)$ is intimately linked to the irreducible characters of G.

Frobenius's First Definition of (Irreducible) Characters

Of course, the efficient treatment of the factorization of $\Theta(G)$ given in the last section was based on a lot of hindsight. The pioneers of mathematics did not have hindsight, and must bank on only serendipity and sheer determination. As we all know, the first step in any new direction of mathematics is very often the most difficult one to take. Frobenius knew that he needed to invent a new character theory to factor the group determinant, but unlike us he started essentially without a clue. It will thus be very instructive for us to see how he actually managed to find the first light in a pitch-dark tunnel.

As we have pointed out before, "group representation" was not in the vocabulary of the nineteenth-century mathematician, so the modern definition of "character" was inaccessible to Frobenius in 1896. Instead, Frobenius first arrived at the definition of characters by working with a certain commutative $\mathbb{C}$-algebra which he later recognized to be $Z(\mathbb{C}G)$, the center of the group algebra. In order to explain his ideas quickly, it is again easier to take advantage of what modern readers already know, although we will try to make relevant comments on the points where Frobenius had difficulty due to the lack of modern machinery at his disposal. The theoretical underpinning of the exposition below is the notion of a commutative semisimple algebra over $\mathbb{C}$.

Let g_j $(1 \le j \le s)$ be a complete set of representatives for the conjugacy classes of a finite group G (with $g_1 = 1$), and let $C_j \in \mathbb{C}G$ be the "class sums" (sums of group elements conjugate to g_j). It is well known (and easy to prove) that these C_j's give a $\mathbb{C}$-basis for $Z(\mathbb{C}G)$, with structure constants $\{a_{ijk}\}$ defined by the equation:

$$C_j C_k = \sum_i a_{ijk} C_i. \tag{7.1}$$

Here, up to a multiple (given by the size of the ith conjugacy class), a_{ijk} is the number of ordered triples $(x, y, z) \in G^3$ such that $x \sim g_j, y \sim g_k$, $z \sim g_i$, and $z = xy$. (Here, "$\sim$" means conjugacy in G.) Frobenius set up these numbers a bit differently by working with an equation $xyw = 1$ instead of $xy = z$; the difference is only notational. The point is that he was extremely familiar with these constants, which count the number of solutions of such equations in groups. Now we bring in something a bit more modern, namely, the Wedderburn decomposition (6.2). Taking the centers in this decomposition, we get

$$Z(\mathbb{C}G) = \mathbb{C}\epsilon_1 \times \cdots \times \mathbb{C}\epsilon_s \tag{7.2}$$

for suitable central idempotents $\epsilon_i \in \mathbb{C}G$ with $\epsilon_i \epsilon_j = 0$ for $i \neq j$. From (7.2), we know that $Z(\mathbb{C}G)$ is (commutative and) semisimple. Frobenius was not equipped with all this modern jargon, so instead he had to do a lot of ad hoc calculations with the counting numbers $\{a_{ijk}\}$ to check what we now know as the trace condition for semisimplicity. Anyway, Frobenius did this, so he could use this semisimplicity information, if only implicitly.

Starting with (6.2), let $D_i : \mathbb{C}G \longrightarrow \mathbb{M}_{n_i}(\mathbb{C})$ be the projection map giving the ith irreducible representation, and let χ_i be the corresponding character $(\chi_i(g) = \operatorname{trace}(D_i(g)))$. Since D_i maps center to center, we have

$$D_i(C_j) = c_{ij} I_{n_i} \quad \text{for suitable } c_{ij} \in \mathbb{C}. \tag{7.3}$$

Computing traces, we get $h_j \chi_i(g_j) = n_i c_{ij}$, where h_j is the cardinality of the jth conjugacy class. Therefore,

$$c_{ij} = \frac{h_j \chi_i(g_j)}{n_i} = \frac{h_j \chi_i(g_j)}{\chi_i(1)}. \tag{7.4}$$

From (7.3) we have $C_j = \sum_i c_{ij} \epsilon_i$; in particular,

$$C_j \epsilon_i = c_{ij} \epsilon_i. \tag{7.5}$$

Thus $\{\epsilon_1, \ldots, \epsilon_s\}$ is a basis of $Z(\mathbb{C}G)$ consisting of common eigenvectors for the (commuting) left multiplication operators by $\{C_1, \ldots, C_s\}$. The eigenvalues of the left multiplication operator by C_j are the c_{ij}'s as given in (7.4).

We have gotten the above conclusions much more quickly than Frobenius did, since he had to summon up the main results from his earlier paper [F: (51)] on commuting operators to show the existence and independence of the eigenvectors, and the independence proof depended critically on the aforementioned semisimplicity property of $Z(\mathbb{C}G)$. His paper [F: (51)], the first in the famous trilogy of 1896 papers [F: (51), (53), (54)] in *S'Ber. Akad. Wiss. Berlin*, was in turn inspired by the earlier work of Weierstrass, Dedekind, and Study on commutative hypercomplex systems. With modern techniques, however, all of Frobenius's work can be done as above in a few lines.

This work having been done, the eigenvalues c_{ij} can now be used to *define* the character values $\chi_i(g_j)$ via the equation (7.4). (Of course, one has to know $n_i = \chi_i(1)$ first, but this is a relatively minor problem.[5]) Circuitous as it looks, this was exactly how Frobenius in [F: (53)] first defined the characters χ_i as class functions on G! After defining the χ_i's, Frobenius promptly obtained the First and Second Orthogonality Relations between the (irreducible) characters in [F: (53)] (see box).

$$\sum_{g\in G} \chi_i(g)\overline{\chi_j(g)} = \delta_{ij}|G|$$

$$\sum_i \chi_i(g)\overline{\chi_i(h)} = \delta_{g,h}|C_G(g)|$$

These First and Second Orthogonality Relations among the irreducible characters χ_i's, proved by Frobenius in his inaugural paper [F: (53)], have remained a benchmark of the character theory of finite groups. Here the δ_{ij}'s are the usual Kronecker deltas, and $\delta_{g,h}$ is 1 if g, h are conjugate in G, and 0 otherwise; $C_G(g)$ denotes the centralizer of g in G.

Although today we have a much easier approach to characters (via representations), the original approach taken by Frobenius is by no means forgotten. Nowadays Frobenius's results above simply survive in the following form:

Theorem 7.6. The structure constants $\{a_{ijk}\}$ and the character table $(\chi_i(g_j))$ determine each other.

Indeed, suppose the a_{ijk}'s are given. Then the a_{1jk}'s determine the h_j's, and the above work determined the χ_i's. Conversely, if the χ_i's are given, a calculation using the Second Orthogonality Relation leads to an explicit formula expressing a_{ijk} in terms of the various character values.

Frobenius's Theorem (7.6) above has remained a deeply significant result in character theory. Inherent in its proof is the result that the $\mathbb{Q}$-span of the values of an irreducible character χ is always an algebraic number field, nowadays called the *character field* of χ. And the explicit expression of the a_{ijk}'s in terms of character values has various interesting applications to the construction and study of finite simple groups; a pertinent reference for this is Higman's article [Hi].

Frobenius realized from the start that the characters of a group are objects of a highly arithmetic nature. He observed in [F: (53), §2, Eq. (15)] that the constants c_{ij} are all algebraic integers[6], and showed later in [F: (54), §12] that the character values are also algebraic integers. Using all this in conjunction with the First Orthogonality Relation, he deduced the important arithmetical result that each character degree n_i divides $|G|$.

Frobenius's Group Determinant Paper

Having published the group character paper [F: (53)], Frobenius was finally ready to demonstrate to the world the applications he had in mind for Dedekind's factorization problem for the group determinant $\Theta(G)$. This he did in the final paper [F: (54)] of the 1896 series. Since he did not have any of our modern techniques at his disposal, the factorization of $\Theta(G)$ took another giant step.

First Frobenius wrote down the factorization of $\Theta(G)$ as follows:

$$\Theta(G) = \prod_{i=1}^{t} \Phi_i^{e_i}, \tag{8.1}$$

where the Φ_i's are distinct (homogeneous) irreducibles, say, of degree f_i. After a scaling, we may assume that each Φ_i has a term $x_1^{f_i}$; this determines the Φ_i's uniquely (up to their order of appearance). The job is to describe the Φ_i's and to determine the exponents e_i in (8.1). If we take the modern approach to $\Theta(G)$ and assume the work

[5] *Frobenius was somewhat vague about this problem, which caused Hawkins [H3: p. 239] to remark that in [F: (53)] "the characters are never completely defined." But so much information is available in [F: (53)] that this problem can be resolved one way or another. For instance, once we know the ratios $\chi_i(g_j)/\chi_i(1)$ for all j, $\chi_i(1)$ can be determined from the First Orthogonality Relation.*

[6] *An efficient modern proof is as follows. Since the ring $\sum_i \mathbb{Z}C_i$ is a finitely generated abelian group, each C_i is integral over $\mathbb{Z}$. Applying this to (7.5), we see that the same is true for each c_{ij}.*

we did in the earlier section on its factorization, the following information is at hand:

(1) The number t of distinct irreducible factors in (8.1) equals the number s of conjugacy classes in G.

(2) For all i, f_i (the degree of Φ_i) equals the multiplicity e_i in (8.1).

For Frobenius, however, each of these statements was to require a proof. (1) was not too hard; he took care of it using the orthogonality relations he developed in [F: (53)] (see box). But (2) turned out to be a real challenge! Of course (2) was confirmed by all the examples known to Frobenius and Dedekind. But Frobenius was a cautious man, and any cautious man (or woman) knows that a few overly simplified examples in mathematics could be totally misleading! So at first Frobenius was not ready to believe that $e_i = f_i$. This proved to be a fortuitous circumstance for students of the history of mathematics, for they have here a unique opportunity to observe directly, through letters written by Frobenius to Dedekind, how Frobenius went about attacking (and sometimes *not* attacking) this difficult problem.[7] He first proved (2) in the case of linear factors ($f_i = 1$), which was not hard; then he managed to settle the case of quadratic factors ($f_i = 2$), which was very hard. He wrote to Dedekind to ask for help or for possible counterexamples; in the meantime, he computed some examples of cubic factors to confirm (2). He confided to Dedekind how he would sometimes try to "attain the goal of proving $e_i = f_i$" by occupying himself with totally unrelated activities, such as going with his wife to the trade exhibition, and then to the art exhibition, by reading a novel at home, or else by ridding his fruit trees of caterpillars. Showing a nice sense of humor, he went on to write in his June 4, 1896, letter to Dedekind:

> I hope you will not give away the trade secret to anyone. My great work *On the Methods of Mathematical Research* (with an appendix on caterpillar catching), which makes use of it, will appear after my death.

Frobenius's promised book never appeared, but apparently his "methods of mathematical research" are still widely practiced among math professors and their graduate students today. Frobenius's skirmishes with the $e_i = f_i$ problem lasted five months, but ended on a happy note: he finally managed to prove it in full generality toward the end of 1896. This enabled him to write up his paper [F: (54)] on the group determinant. In Section 9 of this paper, he wrote:

> The power exponent, wherein a prime factor is contained in the group determinant, is equal to the degree of that factor,

declaring it the "Fundamental Theorem of the theory of group determinants." This was certainly the crown jewel of his monumental work in character theory in 1896. Frobenius's proof was an amazing display of technical wizardry, occupying four and a half pages of the *Sitzungsberichte*. Today, of course, it is much easier to prove this Fundamental Theorem as we did in the earlier section on the factorization of $\Theta(G)$. The approach used in that section also showed clearly how the irreducible factors of $\Theta(G)$ correspond to the irreducible characters χ_i: up to a permutation, the Φ_i in (8.1) is simply the $\Theta_{D_i}(G)$ in (6.4), therefore corresponding to the character $\chi_i := \chi_{D_i}$ (and of course $e_i = f_i = n_i$). The equation (6.4) showed that the coefficient of $x_1^{n_i-1}x_g$ in Φ_i is $\chi_i(g)$ for $g \neq 1$. More generally, the other coefficients can be determined explicitly too. Frobenius proceeded by first extending each χ_i by induction, from a unary function to an n-ary function (for any $n \geq 1$); each $\chi_i(g_1, \ldots, g_n)$ is a polynomial function of the values of χ_i. (For instance, to start the induction, $\chi_i(g,h) = \chi_i(g)\chi_i(h) - \chi_i(gh)$.) With these "$n$-characters" defined, Frobenius then determined Φ_i by the following remarkable formula [F: (54), §3, Eq. (15)]:

$$(8.2)\quad n_i! \cdot \Phi_i = \sum \chi_i(g_1, g_2, \ldots, g_{n_i}) x_{g_1} x_{g_2} \ldots x_{g_{n_i}},$$

where the summation is over all n_i-tuples of elements of G. This computes all coefficients of Φ_i as polynomial functions of the ordinary character values $\{\chi_i(g) : g \in G\}$. So far, group theorists have not made use of these "higher" characters in any substantial way; possibly, a lot more can be done here.

Before we leave group determinants, we should mention a couple of rather surprising recent developments in the subject. It is well known that the characters of a group are not sufficient to determine the group; for instance, the dihedral group and the quaternion group of order 8 happen to have the same character tables. Nevertheless, Formanek and Sibley [FS] have shown that the group determinant $\Theta(G)$ does determine G, and Hoehnke and Johnson [HJ] have shown that the 1-, 2-, and 3-characters of G (mentioned above) also suffice to determine G. These newly discovered facts might have surprised the forefathers of the theory of group determinants.

So far we have only discussed group determinants in characteristic zero (namely, over the complex numbers). In several papers in 1902 and 1907, L. E. Dickson had studied the group determinant over fields of characteristic $p > 0$. We refer the

[7] *Our account of Frobenius's exploits here follows the excellent documentary of Hawkins in [H_3, H_4].*

reader to Conrad's paper [Con] for a good survey on Dickson's work.

The Harvest: 1897–1917

As early as in the introduction to his first group character paper [F: (53)], Frobenius had expressed his belief that this new character theory would lead to essential enrichment and significant advancement of finite group theory. In the twenty remaining years of his life, he was to write, with seemingly unstoppable energy, some fifteen more papers in group theory (not to mention numerous papers in other areas), further developing the theory of group characters and group representations, and applying these to the theory of finite groups. We shall give only a summary of this part of the story here.

1. The first significant development after the trilogy of the 1896 papers was that Frobenius was able to introduce formally the notion of group representations and relate it to the group determinant; he did this again following the suggestion of Dedekind. It is of historical interest to see how Frobenius formulated this definition, so we quote directly from the source [F: (56), §2]:

> Let $\mathfrak{H}$ be an abstract group, $A, B, C\ldots$ be its elements. One associates to the element A the matrix (A), to the element B the matrix (B), etc., in such a way that the group $\mathfrak{H}'$ is isomorphic[8] to the group $\mathfrak{H}$, that is, $(A)(B) = (AB)$. Then I say that the substitutions or the matrices $(A), (B), (C), \ldots$ *represent* the group $\mathfrak{H}$.

Though a bit clumsy to the modern reader, this is essentially the definition of group representation as we know it today. Frobenius also pointed out for the first time, in [F: (56), §4, Eq. (5)], that the characters he defined in [F: (53)] are given by traces of the representing matrices of irreducible (or, in his own term, "primitive") representations. For Frobenius, the irreducibility of a representation D was *defined* by the irreducibility of its determinant $\Theta_D(G)$. The notion of irreducibility was to undergo several reworkings and reformulations in the years to come.

Highly significant is the fact that, in [F: (56)], Frobenius explicitly acknowledged the contributions of Molien's papers [M_1, M_2], which had come to his attention through Eduard Study. Molien's powerful method of analyzing the group algebra as a hypercomplex system was inspired by the Lie algebra methods of W. Killing and É. Cartan. To a considerable extent, it anticipated the later work of Maschke, Wedderburn, and Noether; it is also much closer to one of the ways of studying representation theory today. Molien's understanding of the notion of semisimplicity (and his ability to use it efficiently) was the benchmark of his work, though this work was not widely recognized by his contemporaries. Frobenius, however, did not hesitate to praise it, and referred to [M_1] as an "excellent work" ([F: (56), p. 92]). Upon learning that Molien was only a Privatdozent in Dorpat, Frobenius even wrote to the influential Dedekind to see if he could help advance Molien's career. Nevertheless, Molien's work remained in relative obscurity; today he is remembered mainly through his generating function formula in the theory of polynomial invariants. Fortunately for modern readers, an excellent analysis of Molien's contributions to representation theory is available from Hawkins's paper [H_2].

2. In two subsequent papers [F: (57), (58)], Frobenius introduced the "composition" (now called tensor product) of characters, and developed the relationship between the characters of a group and those of its subgroups. From the latter work came the all-important notion of *induced representations.* It is truly a stroke of genius that, within only a couple of years of his invention of character theory, he came up with the brilliant reciprocity law for induced representations, which now bears his name. The two papers [F: (57), (58)] were to provide some of the most powerful tools for the many applications of representation theory to the structure theory of groups to be found in the twentieth century.

Nowadays we have the techniques of group algebras, tensor products, Hom-functors, etc., which make everything easy and "natural". But in mathematics, "naturalness" is only a function of time. What is natural to us today was simply nonexistent at the end of the nineteenth century. To prove the main facts about induced representations and compositions of characters, Frobenius could resort to only one tool, the group determinant. For modern readers, it is actually quite amazing to see how Frobenius turned the group determinant into a veritable workhorse of representation theory, and used it in paper after paper to get new miles in the subject! While most (if not all) of Frobenius's group determinant proofs have now been superseded by easier modern ones, in my opinion they remain a most fitting testament to the formidable power and consummate skill of a nineteenth-century mathematician.

3. Frobenius's computations of the characters of some specific groups have had a profound impact in representation theory, starting with the characters of the projective unimodular groups $PSL_2(p)$, which he already computed in his inaugural paper [F: (53)] in character theory. Years later, this work blossomed into the amazingly rich subject of representation theory of finite groups

[8] *In Frobenius's time, this term did not preclude the mapping $A \mapsto (A)$ from being many-to-one.*

of Lie type.[9] Frobenius already observed as early as in [F: (53), end of §8] that the character values of the symmetric groups are all rational integers. Shortly thereafter, in [F: (60), (61)], he single-handedly opened up the investigation into the representation theory of the symmetric groups S_n and the alternating groups A_n. His classification and analysis of the characters (and therefore the representations) of S_n anticipated the work of Rev. Alfred Young, and laid firm foundations for much of the future work on symmetric functions in the new century. In [F: (60)], Frobenius built certain generating functions from the character values of S_n, and determined these generating functions. Thus, at least in principle, he managed to compute the characters of S_n on any given conjugacy class. The most memorable case of this computation is Frobenius's determinantal formula for the character degrees of S_n: for an irreducible character χ_λ corresponding to a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of n (where $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$), Frobenius showed (cf. [F: (60), §3, Eq. (6)]) that

$$\begin{aligned}\chi_\lambda(1) &= n!\det\left(\frac{1}{(\lambda_i - i + j)!}\right)_{1\le i,j\le r} \\ &= \frac{n!\,\Delta(\mu_1, \mu_2, \dots, \mu_r)}{\mu_1!\mu_2!\cdots\mu_r!},\end{aligned} \tag{9.1}$$

where $\mu_i = \lambda_i + r - i$ and $\Delta(\mu_1, \mu_2, \dots, \mu_r)$ is the Vandermonde determinant with the parameters μ_i. The same formula was obtained independently by Young, but Frobenius seemed to have the priority here. Much later, the Frobenius-Young determinantal formula for character degrees (for S_n) was to be given another equivalent combinatorial form in terms of the "hook-lengths" $h_{ij}(\lambda)$ in the Ferrers diagram of the partition λ: the Frame-Robinson-Thrall hook-length formula recasts the character degrees in the form

$$\chi_\lambda(1) = \frac{n!}{\prod_{i,j} h_{ij}(\lambda)}. \tag{9.2}$$

Today the representation theory of S_n lies at the heart of algebra and combinatorics, and impacts many branches of pure and applied mathematics.

4. Even before his work in character theory, Frobenius had taken a keen interest in finite solvable groups, and had published two papers on them in 1893 and 1895, focusing on the existence and structure of their subgroups. At the turn of the century, his interest in the subject was heightened by his newly invented theory of group characters. He was to write three more papers in the solvable group series, and a handful of other papers on multiply transitive groups, some of them using character theory. One of his most spectacular results (in [F: (63), p. 199]) is now a staple in any graduate course in the representation theory of finite groups:

Theorem 9.3. If G is a finite group acting transitively on a set such that no element in $G \setminus \{1\}$ fixes more than one point, then the set of fixed-point-free elements of G together with the identity forms a (normal) subgroup K of G. (If $K \subsetneq G$, G is called a *Frobenius group*, and K is called its *Frobenius kernel*. Any one-point stabilizer of the action is called a *Frobenius complement*.)

A century later, Frobenius's proof of this theorem using induced characters and the idea of the kernel of an irreducible representation has not lost its magic and charm. Even more remarkably, no purely group-theoretic proof of this beguilingly simple statement has been found to date, so Frobenius's original argument in [F: (63)] has remained the *only* known proof of (9.3)! Years later, Frobenius's Theorem inspired the Brauer-Suzuki theory of exceptional characters, and Zassenhaus classified the doubly transitive Frobenius groups, linking them to the classification of finite near-fields. The theory of Frobenius groups also helped launch the distinguished career of Fields Medalist J. G. Thompson, who proved in his Chicago thesis (1959) the long-standing conjecture that Frobenius kernels are *nilpotent* groups.

5. With his student Issai Schur, Frobenius introduced the notion of the *index* (or *indicator*):

$$s(\chi) := \frac{1}{|G|}\sum_{g\in G}\chi(g^2) \tag{9.4}$$

of an irreducible character χ, and showed that $s(\chi)$ takes values in $\{1, -1, 0\}$. In this Frobenius-Schur theory, the χ's fall into three distinct types: $s(\chi) = 1$ if χ comes from a real representation, $s(\chi) = -1$ if χ does not come from a real representation but is real-valued, and $s(\chi) = 0$ if χ is not real-valued. The Frobenius-Schur indices contain important information about a group G which goes beyond the character table of G: for instance, the number of square roots of an element $g \in G$ can be computed via the indices in (9.4) by the expression $\sum_\chi s(\chi)\chi(g)$, a fact quite important in group theory. In connection with their work on characters of the first type, Frobenius and Schur also proved the interesting result that any complex orthogonal representation of a finite group is equivalent to a real orthogonal representation.

Frobenius and Number Theory

A close kinship between number theory and group theory is provided by the fact that any normal extension of number fields K/F gives rise to a finite Galois group $G = \mathrm{Gal}(K/F)$. Thus, the applicability of Frobenius's character theory to number theory should come as no surprise. The true interaction

[9] *There is a bit of irony here, since Frobenius was known to have a great disdain for the work of Sophus Lie.*

between the two theories, however, did not take place during Frobenius's lifetime, and had to wait until the 1920s, when algebraic and analytic number theory became more fully developed.

The idea of using representations of Galois groups in number theory first emerged in Artin's work in 1923. For any character χ on a Galois group $G = \mathrm{Gal}(K/F)$ as in the last paragraph, Artin introduced what is now called the *Artin L-function* $L(s, \chi, K/F)$ associated with χ. This is a function in a complex variable s ($|s| > 1$), which encodes both information about χ and about the primes in F and K. For instance, when χ is the trivial character (respectively the regular character) of G, $L(s, \chi, K/F)$ is the Dedekind zeta function of F (respectively of K). (The Dedekind zeta function of a number field is, in turn, a direct generalization of the Riemann zeta function for the rationals.) Artin's theory of L-functions made use of Frobenius's work in two ways. First, Artin showed that, in the case when G is abelian and $\chi(1) = 1$, his L-functions coincide with the L-functions studied earlier by Hecke. This required the full force of Artin's Reciprocity Law, which Artin established by using ideas of Tchebotarëv's proof of a conjecture of Frobenius (now called Tchebotarëv's Density Theorem). Second, Artin showed that, in the non-abelian case, Frobenius's induced characters provided the perfect means to relate the Artin L-functions to the (abelian) Hecke L-functions. Later, Brauer completed Artin's work by proving that any character of G is an integral combination of characters induced from 1-dimensional characters of suitable subgroups of G. With this powerful induction theorem, Brauer proved that $L(s, \chi, K/F)$ extends to a meromorphic function in $\mathbb{C}$, and that the quotient of the Dedekind zeta functions $\zeta_K(s)/\zeta_F(s)$ is an entire function. In this work (for which Brauer received the Society's Frank Nelson Cole Prize in 1949), the interplay between character theory and number theory came to its fruition. Later, the representations of Galois groups became an important topic in the theory of modular forms, but that is another story.

Coda

About a hundred years ago Dedekind posed to Frobenius the problem of factoring a certain determinant associated with a finite group. The solution of this abstract problem led Frobenius to the invention of character theory, and subsequently the representation theory of finite groups. Today, these theories provide basic tools for various branches of algebra, and their generalizations to the case of topological and Lie groups play an important role in harmonic analysis. In the meantime, group characters and representations have come to be used extensively in many applied fields, such as spectroscopy, crystallography, quantum mechanics, molecular orbital theory, and ligand field theory. These amazingly diverse applications, made possible by the purely theoretical work of Dedekind and Frobenius which predated them by decades, seem to provide another striking instance of the great "unreasonable effectiveness" of mathematics.

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