William T. Tutte (1917–2002)

Arthur M. Hobbs and James G. Oxley

William Thomas Tutte (rhymes with "hut") was the leading graph and matroid theorist of his generation. His first mathematical paper appeared in 1940 and his last in 2000. According to a list published in 1969, between 1940 and 1949 there were fifty-five papers published in graph theory, eleven of them by Tutte; in 2000, more than 1,500 books and papers appeared that were classified under 05C (graph theory) on MathSciNet. Like graph theory, matroid theory has grown dramatically from its introduction in 1935 by Hassler Whitney: MathSciNet shows in the period 1990–2000 over 1,000 items were published having the word "matroid" in the title or review. Despite early contributions by Garrett Birkhoff, S. Mac Lane, and B. L. van der Waerden, the first major advances in matroid theory were made by Tutte in his 1948 Cambridge Ph.D. thesis [3], which formed the basis of an important sequence of papers published over the next two decades. Tutte's work in graph theory and matroid theory has been profoundly influential on the development of both the content and the direction of these two fields.

To summarize Tutte's notable contributions, he broke the German Army High Command's code [10] during World War II, he advanced graph theory from a subject with one text (D. König's) toward its present extremely active state, and he developed

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Whitney's definitions of a matroid into a substantial theory.

Tutte's contributions to graph and matroid theory were immense, but his terminology was idiosyncratic, frequently at variance with most other researchers. Hardest of all for a novice approaching Tutte's work is the fact that he often used standard terms in graph and matroid theory in ways that differ from their conventional usage. While Tutte's many theorems are frequently cited today, much of his terminology has been discarded. In the descriptions of his matroid results that appear below, the terminology used follows J. Oxley [2] and, with minor variations, is fairly universally accepted. For his graph theory results, we follow the terminology of D. West [9]. To describe Tutte's results and their significance, we shall need to introduce some of this terminology together with some well-known results.

Background

A *graph* may be viewed as consisting of a set of points in space, called *vertices*, and segments of curves joining pairs of vertices; these curves are called *edges* and share only their ends with other edges and with vertices of the graph. For example, the *complete graph* K_n consists of n vertices and $\binom{n}{2}$ edges, with one edge joining every two distinct vertices. If an edge of a graph joins a vertex to itself, the edge is called a *loop*. If two or more edges join the same pair of vertices, the edges are said to be *parallel*. A graph is *simple* if it has no loops and no parallel edges. The *degree* of a vertex in a graph is the number of ends of edges

meeting that vertex. Thus a loop contributes 2 to the degree of the vertex it meets. Since each edge has two ends, the sum of the degrees of a graph is always even. A graph is *cubic* if every vertex has degree three. A *path* is an alternating sequence of vertices and edges, each edge joining the vertex before it to the vertex after it in the sequence, with no repeated vertices. A *cycle* is formed from a path by adding an edge joining its end vertices. A graph is *connected* if between every two distinct vertices there is a path. A *forest* is a graph with no cycles, and a *tree* is a connected forest. A graph is said to be *Hamiltonian* if it has a cycle that includes all of the vertices of the graph, and such a cycle is called a *Hamiltonian cycle*.

Abstracting from the behavior of linearly independent sets of columns of a matrix, in 1935, Whitney defined a *matroid M* to consist of a finite set E (or $E(M)$) and a collection $\mathcal{I}$ (or $\mathcal{I}(M)$) of subsets of E called *independent sets* with the properties that the empty set is independent; every subset of an independent set is independent; and if one independent set has more elements than another, then an element can be chosen from the larger set to adjoin to the smaller set to produce another independent set. A set that is not independent is called *dependent*, and minimal such sets are called *circuits*. For a matrix A over a field $\mathbb{F}$, if E is the set of column labels on A and $\mathcal{I}$ is the set of subsets of E that label linearly independent sets of columns, then $(E, \mathcal{I})$ is a matroid, $M[A]$. Such a matroid is said to be *representable over the field* $\mathbb{F}$. If G is a graph, E is its set of edges, and $\mathcal{I}$ is the collection of the edge sets of the forests in G, then $(E, \mathcal{I})$ is a matroid, $M(G)$, called the *cycle matroid* of G. Its circuits are the edge sets of the cycles in G. A matroid that is isomorphic to the matroid $M(G)$ for some graph G is called *graphic*. Representable and graphic matroids were introduced by Whitney, and these classes have motivated much of the development of matroid theory ever since.

A graph is *planar* if it can be drawn in the plane without edges crossing, and it is a *plane graph* if it is so drawn in the plane. The drawing separates the rest of the plane into regions called *faces*. Every plane graph G has a *dual graph* G^*, formed by introducing a vertex of G^* for each face of G and joining two vertices of G^* by k edges if and only if the corresponding faces of G share k edges in their boundaries. Whitney defined the dual of a graph abstractly and showed that a graph has such a dual if and only if it is a planar graph.

For arbitrary matroids there is a natural notion of a dual matroid that extends the notion of duality for planar graphs. For a matroid M on E, let $\mathcal{B}$ be the collection of *bases*, that is, maximal independent sets, of M. Clearly $\mathcal{B}$ uniquely determines M. The collection $\{E - B : B \in \mathcal{B}\}$ is the set of bases

of another matroid on E, namely, the *dual M^** of M. Evidently,

$$(1) \qquad (M^*)^* = M.$$

If G is a planar graph and G^* is a dual graph of G, then $(M(G))^* = M(G^*)$. If G is not planar, then $(M(G))^*$ is still defined, but one can show that it is not graphic. So the class of graphic matroids is *not closed under duality*, since the dual of a member of the class need not be in the class. It is easy to show that all bases of a matroid M have the same cardinality; we call this the *rank $r(M)$* of M.

We will define further concepts relating to graphs and matroids as we need them.

Tutte's Rise to Prominence

William Thomas Tutte was born at Fitzroy House in Newmarket, Suffolk, England, on May 14, 1917, and died on May 2, 2002, in Waterloo, Ontario. He attended Trinity College, Cambridge, from 1935 until January 1941, majoring in chemistry and earning his bachelor's degree in 1939 and his master's in 1941. He spent the rest of World War II working at Bletchley Park.

At the end of the war, because of his war work, Tutte was given a Fellowship at Trinity College. The decision to so honor him was justified: With little help from his advisor, Shaun Wylie, he completed a 417-page thesis in three years, graduating in 1948. The thesis is remarkable both for its contents and for its omission of his work on Hamiltonian cycles, which he published while he was writing the thesis.

For the thesis Tutte invented a class of algebraic structures that he called "nets" and that later he learned were equivalent to representable matroids. He developed a theory of "cleavages" of a 2-connected graph into 3-connected parts, thus deepening a result of Whitney. He introduced the dichromatic polynomial, now called the "Tutte polynomial", one of the more important functions in graph and

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matroid theory. In addition, his thesis introduced matroid minors, now the most important and widely studied of matroid substructures. Finally, he developed this theory to the point that he was able to characterize graphic matroids by giving a list of matroids, which he called "gnarls", such that a matroid is graphic if and only if it has no gnarl as a minor.

In 1948 H. S. M. Coxeter helped Tutte get a position at the University of Toronto. Tutte served there until 1962, when the University of Waterloo (founded in 1958) recruited him. Ralph Stanton, head of the mathematics department at that time,
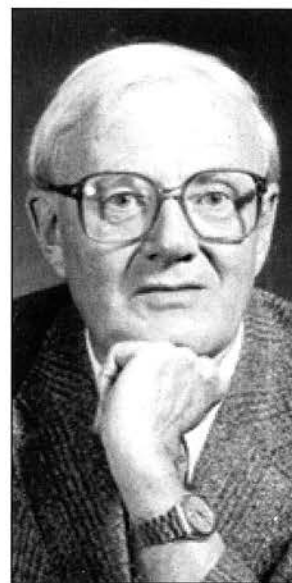
went to Toronto with Gerald Berman to talk to Tutte and his wife, Dorothea (married 1949). The opportunity appealed to Tutte, both because it offered the possibility of advancement and because he and Dorothea enjoyed nature and liked the relatively rural setting of Waterloo. Tutte accepted the position, and he and Dorothea bought a house in the small nearby town of West Montrose, next to a covered bridge. They enjoyed showing their lovely garden and nearby scenery, and they could name every bird and plant in the garden. They liked hiking. Bill often organized hiking trips and even bought wilderness properties for them to hike in as birthday presents for Dorothea.

Mathematical Beginnings and Blanche Descartes

Not long after he started his undergraduate studies at Cambridge, Tutte was introduced by his chess-playing friend R. Leonard Brooks to two of Brooks's fellow mathematics students, Cedric A. B. Smith and Arthur Stone. The four became fast friends, and Tutte came to refer to the group as "the Gang of Four" or "the Four Horsemen" [8]. The Four joined the Trinity Mathematical Society and devoted many hours to studying unsolved mathematics problems together.

They were most interested in the problem of squaring a rectangle or square, that is, of finding squares of integer side-lengths that exactly cover, without overlaps, a rectangle or square of integer side-lengths. If the squares are all of different sizes, the squaring is called *perfect*. While still undergraduates at Cambridge, the Four found an ingenious solution involving currents in the wires of an electrical network. Solving this network gives the required sizes of the squares as the currents in the wires. Their first perfect squared square and the accompanying theory were published in 1940 [4, pp. 10–38]; it was anticipated by one year by a perfect squared square found empirically by R. P. Sprague.

About this work Tutte said, "This [research] soon called for much graph theory. It was linked, through a 'Smith diagram,' with the study of 3-connected planar graphs..., and with Kirchhoff's Laws for electrical circuits.... It was linked through

rotor theory...with graph symmetry.... It was linked through the tree-number...with the theory of graph functions satisfying simple recursion formulae..." [5, p. xix].

The Gang of Four were typically lively undergraduates. They decided to create a very special mathematician, Blanche Descartes, a mathematical poetess. She published at least three papers, a number of problems and solutions, and several poems. Each member of the Four could add to Blanche's works at any time, but it is believed that Tutte was her most prolific contributor.

The Four carefully refused to admit that Blanche was their creation. Visiting Tutte's office in 1968, Hobbs remarked, "Sir, I notice you have two copies of that proceedings. I wonder if I could buy your extra copy?" Tutte replied, "Oh, no, I couldn't sell that. It belongs to Blanche Descartes."

Bletchley Park

At about the time Tutte completed his master's in chemistry in 1941, his tutor, Patrick Duff, recommended him to the famous code-breaking group at Bletchley Park. After training at the Code and Cypher School in London, Tutte was assigned to the research section and there to a group trying to break the German Army High Command's code that the British called "FISH". It was produced by the Lorenz SZ 40/42 teleprinter cypher attachment, which generated code in the 5-bit 32-letter Baudot Teleprinter Code alphabet [10]. Unlike the case of the 3- or 4-wheel Enigma machine, the British did not have a Lorenz code machine, but the clear 12-word preamble in the early FISH transmissions suggested that it had 12 wheels [10], [6]. Its code was produced letter-by-letter by a mod 2 addition of a letter from the message and a letter from a key produced by the machine. Thus, the message letter R (code 01010) might be added to the key letter A (11000), producing the code letter D (10010).

On August 30, 1941, a clerk in Athens, Greece, sent a Lorenz-coded message of about 4,000 characters. Apparently the receiving office in Vienna didn't quite get it, for they asked for a repeat. Proper protocol would require that the message be sent either exactly as before or with a new setting on the wheels of the machine, but the clerk recoded it with the same initial setting as before and sent it again with some variations in abbreviations, spacing, and punctuation. Even in such circumstances breaking the code is not easy, but eventually John Tiltman found 3,976 characters of the key produced by the machine with that wheel setting. Tutte sought patterns within this key, and after a few months' work he discovered a repetitive pattern of length $574 = 14 \cdot 41$. This indicated that one of the wheels had 41 teeth. It took still more time to fully find the design of the Lorenz

machine, but he and the others in his section worked out the placement and number of teeth of the 12 wheels within the machine. Their decoding efforts were paralleled by German efforts to improve the code, so there was a constant struggle between the German encrypters and the British codebreakers. The rest of the story is too long for this article. Suffice it to say that Tutte's work led to the development of the first working electronic computer, Colossus, to run decoding algorithms. By the end of the war, ten Colossus machines were in use [10].

Fragments and Hamiltonian Graphs

A summary of Tutte's work is simplified by the fact that, in the commentaries on his selected works [4] and in other published lectures [7], [8], he gave not only many useful historical facts about the context in which his work was done but also numerous insights into his thinking when he was developing his many theorems. We draw heavily from these works in what follows.

A graph with more than k vertices is k-connected if it cannot be disconnected by removing $k - 1$ vertices. A cycle with exactly three edges is a *triangle*. If every face of a plane graph is a triangle, the graph is a *plane triangulation*. A cycle C in a plane graph G is *separating* if G has vertices both inside and outside C.

Given a cycle C in a graph G, a *fragment of C*, or a *C-fragment*, consists either of an edge (and its ends) not in C but joining two vertices of C, or of a component Q of $G - V(C)$ together with all edges (and their ends) from vertices of Q to vertices of C. The vertices a C-fragment shares with C are called its *vertices of attachment*.

Tutte introduced fragments in his thesis. He used these tools in several of his graph theory papers, including his 1960 paper "Convex representations of graphs" and his important paper "How to draw a graph" [4, pp. 364–88]. He used a generalization of fragments to matroids in proving his characterization of graphic matroids (1959).

The question of whether a graph has a Hamiltonian cycle is one of the prime exemplars of an NP-complete problem and is a special case of the well-known traveling salesman problem. Consequently, finding significant classes of graphs that certainly are or are not Hamiltonian is important. In 1884 P. G. Tait conjectured that every 3-connected planar cubic graph has a Hamiltonian cycle. If true, a proof of this conjecture would have also proved the famous Four Color Conjecture.

In his occasional free time at Bletchley Park, Tutte thought about Tait's conjecture. He discovered a Hamiltonian graph with the remarkable quality that it had an edge through which all Hamiltonian cycles passed. He then used three copies of

that graph to construct a counterexample to Tait's conjecture [4, pp. 47–50].

In 1931 H. Whitney had shown that a plane triangulation with no separating triangle is Hamiltonian. In 1956 Tutte extended Whitney's theorem by showing that every connected planar graph G with at least one cycle has a cycle C, all of whose fragments have at most three vertices of attachment. A nontrivial consequence of this is the following theorem of Tutte.

Theorem 1. *Every 4-connected planar graph is Hamiltonian.*

Excluded-Minor Theorems

The notion of a graph minor was introduced in the 1930s and early 1940s by K. Wagner and H. Hadwiger. Tutte's thesis generalized this notion to matroids. If M is a matroid on a set E and $\mathcal{I}$ is the collection of independent sets, then for an element e of E, there are two natural matroids on $E - \{e\}$: the *deletion $M \backslash e$* of e and the *contraction M/e* of e. The first has as its independent sets all members of $\mathcal{I}$ that are subsets of $E - \{e\}$. To specify the second, let B_e be $\{e\}$ when $\{e\}$ is independent, and let B_e be $\emptyset$ otherwise. Then M/e has as its independent sets all subsets I of $E - \{e\}$ for which $I \cup B_e \in \mathcal{I}$. These operations of deletion and contraction are natural extensions of their namesakes in graph theory. Since an element e of the cycle matroid $M(G)$ of a graph G is just an edge of G, the matroids $M(G) \backslash e$ and $M(G)/e$ are the cycle matroids of, respectively, the graph $G \backslash e$ that is obtained from G by removing the edge e and the graph G/e that is obtained from G by identifying the ends of e and then deleting e. Both the graph and the matroid operations of deletion and contraction commute with themselves and with each other, so for a matroid M or a graph G, it makes sense to talk about a matroid or a graph of the form $M \backslash X/Y$ or $G \backslash X/Y$, where X and Y are disjoint subsets of E. Such a matroid or graph is called a *minor* of M or of G. (In the case of graphs, we also allow the deletion of vertices of degree zero in constructing a minor.) Several basic classes of matroids and graphs have the property that every minor of a matroid or graph in the class is also in the class. Indeed, the remarks above establish that every minor of a graphic matroid is graphic. We say that the class of graphic matroids is *minor-closed*. For another example, the class of planar graphs is minor-closed.

For any class $\mathcal{M}$ of matroids or graphs that is minor-closed, there is a list of *excluded minors*, that is, those matroids or graphs not in $\mathcal{M}$ such that every deletion and every contraction is in $\mathcal{M}$.

A matroid is *binary* if it is representable over the 2-element field $GF(2)$. A matroid is *regular* if it is representable over $\mathbb{R}$ by a *totally unimodular*
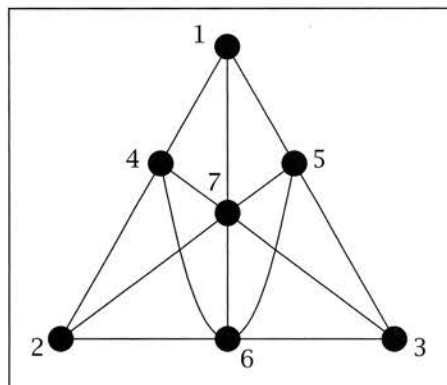
Figure 1. The Fano matroid F_7.

matrix A, that is, by a matrix all of whose subdeterminants are in $\{0, 1, -1\}$. In fact, regular matroids are precisely the matroids that are representable over all fields (see [2, Theorem 6.6.3]), so every regular matroid is binary.

The class of graphic matroids is not closed under duality. By contrast, the classes of regular matroids and of binary matroids are both closed under duality. Indeed, if $[I_r \mid D]$ is an $r \times n$ matrix representing M over a field $\mathbb{F}$, then M^* is represented over $\mathbb{F}$ by the $(n - r) \times n$ matrix $[-D^T \mid I_{n-r}]$, where the order of the labels on the columns of the last matrix matches the order on the original matrix. We observe that duality is a natural generalization of orthogonality, since every row of $[I_r \mid D]$ is orthogonal to every row of $[-D^T \mid I_{n-r}]$.

Every graphic matroid is regular. To see this, let G be a graph, orient its edges arbitrarily, and then take the vertex-edge incidence matrix of the resulting directed graph. It is not difficult to show (see [2, Proposition 5.1.3]) that the resulting matrix A is totally unimodular and that $M[A] = M(G)$. After Tutte recognized the link between the structures he studied in his thesis and Whitney's matroids, his "main preoccupation...was the question: 'When is a matroid graphic?'" [7, p. 100]. Using the observations above, he divided this problem into three subproblems:

    (i) When is a matroid binary?
    (ii) When is a binary matroid regular?
    (iii) When is a regular matroid graphic?

The first of these questions is the simplest and has numerous answers. One of those given by Tutte involves excluded minors. If A is a matrix over a field $\mathbb{F}$ and e labels a column of A, then $M[A] \backslash e$ is represented over $\mathbb{F}$ by the matrix that is obtained by deleting the column labelled e. If e labels a zero column, then $M[A]/e = M[A] \backslash e$; otherwise, by elementary row operations, we can transform A into a matrix A' in which e labels a standard basis vector. It follows from basic linear algebra that $M[A'] = M[A]$; that is, these elementary row operations do not alter the matroid. Now $M[A]/e$ is the matroid that is represented by the matrix that is obtained from A' by deleting both the column labelled by e and the row containing the unique nonzero entry of e. We deduce from these observations that the class of $\mathbb{F}$-representable matroids is minor-closed. In particular, the class of binary matroids is minor-closed. By observing that if the

matrix A above is totally unimodular, then so is the matrix A', we deduce that the class of regular matroids is also minor-closed.

Tutte's excluded-minor answer to (i) is contained in the next theorem. It involves a particular uniform matroid where, for $0 \le r \le n$, the *uniform matroid* $U_{r,n}$ is the matroid on $\{1, 2, \ldots, n\}$ whose independent sets consist of all subsets of $\{1, 2, \ldots, n\}$ with at most r elements.

Theorem 2. *A matroid is binary if and only if it has no minor isomorphic to $U_{2,4}$.*

The answers to Tutte's subproblems (ii) and (iii) were again given in terms of excluded minors but were considerably more difficult to derive. The *Fano matroid*, F_7, is the matroid that is represented over $GF(2)$ by the matrix consisting of the seven nonzero vectors of length 3, that is, by the matrix

$$A = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

Geometrically, F_7 can be represented as in Figure 1. There the independent sets consist of all sets of at most three points such that no three are collinear, where the curved line through 4, 5, and 6 indicates that $\{4, 5, 6\}$ is dependent.

It is straightforward to show that the Fano matroid is not regular but that all of its proper minors are, so F_7 is an excluded minor for the class of regular matroids. There is another excluded minor for the class of regular matroids that is easily derived from F_7, namely, its dual F_7^*.

Tutte noted that deletion, contraction, and duality are related by the following beautiful identity:

$$(2) \qquad (M \backslash e)^* = M^*/e.$$

Combining this with (1), we get

$$(3) \qquad (M/e)^* = M^* \backslash e.$$

We conclude, since the class of regular matroids is closed under duality, that F_7^* is another excluded minor for this class. Tutte answered subproblem (ii) by proving the following.

Theorem 3. *A binary matroid is regular if and only if it has no minor isomorphic to F_7 or F_7^*.*

On combining this with the answer to subproblem (i), one obtains the following.

Corollary 4. *A matroid is regular if and only if it has no minor isomorphic to $U_{2,4}$, F_7, or F_7^*.*

Tutte developed a geometry in which the circuits of the matroid were "points", and he used this in the development of a homotopy theorem for matroids by which he proved Theorem 3. He

commented [8, p. 7]: "I do not find the homotopy theorem in the later literature. Perhaps it is mentioned with a warning that it is terribly long and then the author tells of some shorter, slicker proof of the excluded minor conditions. That is the way of Mathematics." Theorem 3 is a brilliant result and was always recognized as such, but, as Tutte suggests, most readers were frightened off by the proof technique. The best currently known proof was given by A. M. H. Gerards in 1989 and relies only on some elementary ideas of linear algebra. But the latter proof did not appear till some thirty years after Tutte's landmark result.

Tutte attacked subproblem (iii) by analogy with the proof by fragments of K. Kuratowski's theorem that a graph can be drawn in the plane if and only if it does not have K_5 or $K_{3,3}$ as a minor. Here $K_{3,3}$ is the three-houses, three-utilities graph in which every house is joined to every utility by an edge. Tutte's work on (iii) culminated in the following very pleasing generalization of Kuratowski's theorem.

Theorem 5. *A regular matroid is graphic if and only if it has no minor isomorphic to $M^*(K_5)$ or $M^*(K_{3,3})$.*

Theorems 2, 3, and 5, done in the context of nets, first appeared in Tutte's Ph.D. thesis [3] a decade before their publication in a journal. These theorems were the cornerstones of Tutte's "Lectures on Matroids" presented in 1964 at the first matroid theory conference, which was organized by Jack Edmonds and was held at the National Bureau of Standards. They are among Tutte's best-known contributions to matroid theory.

Tutte's work with graph minors was eventually taken up by his former student Neil Robertson in collaboration with P. D. Seymour. Beginning in the 1980s, they have produced an extraordinary sequence of papers describing the structure of graphs. Perhaps their most powerful result so far is the Graph-Minor Theorem.

Theorem 6. *In every infinite set of graphs, there are two such that one is a minor of the other.*

To illustrate the usefulness of this theorem, consider the very old question of characterizing graphs that can be drawn in a surface of genus g without crossing edges. Kuratowski's theorem established that there are exactly two excluded minors for the plane. Around 1980 D. Archdeacon, H. H. Glover, J. P. Huneke, and C. S. Wang found that there are thirty-five excluded minors for the graphs that can be drawn in the projective plane. But the number of excluded minors grows quickly with the genus, and it is not obvious that the list is even finite for other values of g. To see that the list is finite, suppose we have an infinite list $\mathcal{L}$ of excluded minors for a surface. Then by the Graph-Minor Theorem one member of $\mathcal{L}$ is a minor of another. Since this is impossible by the minimality of excluded minors, $\mathcal{L}$ must be finite.

A matroidal conjecture related to the Graph-Minor Theorem is given at the end of the section below on connectivity in matroids.

Colorings, Flows, and the Tutte Polynomial

Perhaps Tutte's most long-lasting legacy will be the polynomial that bears his name. Tutte's work in this area began with Chapter 5 of his thesis. It was published in terms of graphs in his 1947 paper "A ring in graph theory" [4, pp. 55–69]. In a connected graph G, a *spanning tree* is a tree that contains all of the vertices of G. The spanning trees of G coincide with the edge-sets of bases of the matroid $M(G)$. Let $b(G)$ denote the number of spanning trees of G. In their 1940 paper "The dissection of rectangles into squares" [4, pp. 10–38], the Gang of Four had noted a formula that implies that, for all nonloop edges e of G,

$$(4) \qquad b(G) = b(G \backslash e) + b(G/e).$$

Recall that $G \backslash e$ is obtained from G by deleting e, while G/e is obtained from G by identifying the ends of e and then deleting e. Tutte wrote [7, p. 53]: "When I was doing my Ph.D. research I began to collect other functions of graphs that satisfied similar recursions." For a graph G and a positive integer λ, let $P(G; \lambda)$ be the number of colorings of the vertices of G using $\{1, 2, \ldots, \lambda\}$ such that if two vertices are joined by an edge, then they receive different colors. The smallest positive integer n for which $P(G; n)$ is nonzero is called the *chromatic number* of G. George D. Birkhoff introduced $P(G; \lambda)$ in 1912–13. R. M. Foster made the elementary observation that if e is an edge of G, then

$$(5) \qquad P(G; \lambda) = P(G \backslash e; \lambda) - P(G/e; \lambda).$$

Using this, it is easily shown that the function P is a polynomial in λ; it is called the *chromatic polynomial* of G.

Now orient the edges of G arbitrarily and let H be an additive abelian group. An *H-flow* on G is an assignment of nonzero members of H to the edges of G such that at every vertex the total flow into the vertex equals the total flow out from the vertex, where all calculations are done in H. Tutte showed [4, pp. 55–69] that the number of such H-flows depends not on the specific group H but only on the order of H. In particular, if $|H| = m$, then the number of H-flows equals the number of $\mathbb{Z}_m$-flows. We call the latter flows *m-flows* and let $F(G; m)$ be the number of such flows. Tutte showed that

$$(6) \qquad F(G; m) = F(G/e; m) - F(G \backslash e; m).$$

The polynomial F is called the *flow polynomial* of G.

These observations led Tutte in [4, pp. 55–69] to discuss functions f on graphs that are invariant under isomorphism:

$$(7) \qquad f(G_1) = f(G_2) \quad \text{if } G_1 \cong G_2,$$

and that satisfy the rule

$$(8) \quad f(G) = f(G \backslash e) + f(G/e)$$
$$\text{for all nonloop edges } e \text{ of } G.$$

He also considered functions satisfying the additional condition that

$$(9) \qquad f(H + K) = f(H)f(K),$$

where $H + K$ is the graph that is the union of the disjoint graphs H and K. Tutte proved that a graph function satisfying (7)–(9) is uniquely determined once its value is known for all k on the graphs consisting of a single vertex and k loops.

A *cut edge* of a connected graph is an edge whose removal disconnects the graph. Tutte's paper [4, pp. 55–69] also introduced a 2-variable polynomial, which Tutte called the *dichromatic polynomial* and which is now known as the *Tutte polynomial*. This polynomial satisfies (8) provided e is not a cut edge of G. It also satisfies (7) and (9), and, indeed, it is the universal graph invariant satisfying these three conditions in a sense that will shortly be made precise.

Much of the theory described above extends to matroids and was developed for representable matroids in Tutte's Ph.D. thesis. However, Tutte never published his work for matroids, and it was not until 1969 that H. H. Crapo published a fully general matroid form of this theory.

In a matroid M on a set $E(M)$, if $X \subseteq E(M)$, then $M \backslash (E(M) - X)$ is a matroid on the set X; its rank is $r(X)$. The *Tutte polynomial* $T(M; x, y)$ of M is

$$\sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}.$$

This polynomial has the attractive property that

$$T(M; x, y) = T(M^*; y, x),$$

where M^* is the dual of the matroid M. Moreover, for a graph G with $k(G)$ components, the chromatic polynomial and the flow polynomial are related to the Tutte polynomial via

$$P(G; \lambda) = \lambda^{k(G)} (-1)^{|V(G)|-k(G)} T(M(G); 1 - \lambda, 0)$$

and

$$F(G; m) = (-1)^{|E(G)|-|V(G)|+k(G)} T(M(G); 0, 1 - m).$$

An element e of a matroid is a *loop* if $\{e\}$ is a circuit of the matroid. The Tutte polynomial has the properties that

$$(10) \qquad f(M_1) = f(M_2) \quad \text{if } M_1 \cong M_2,$$

$$(11) \quad f(M) = f(M \backslash e) + f(M/e)$$
$$\text{if } e \text{ is not a loop in } M \text{ or } M^*,$$

and

$$(12) \quad f(M) = f(M \backslash e) f(M \backslash (E(M) - \{e\}))$$
$$\text{if } e \text{ is a loop of } M \text{ or } M^*.$$

In his thesis, Tutte showed that the Tutte polynomial is a universal matroid invariant satisfying (10)–(12) in the sense that if the values of f on the two one-element matroids $U_{1,1}$ and $U_{0,1}$ are x and y respectively, then $f(M) = T(M; x, y)$ for all matroids M. Tutte never published this result, and it was rediscovered by T. H. Brylawski and published in 1972.

The applications of the Tutte polynomial extend well beyond graphs into such diverse areas as coding theory, percolation theory, electrical network theory, statistical mechanics, and knot theory. Tutte essentially confined his interest to spanning trees and chromatic and flow polynomials, but in view of the importance of polynomial invariants for knots, the link between one such invariant and the Tutte polynomial is worth describing. Consider a link diagram, as illustrated in Figure 2(a), where the crossings alternate between under and over as the link is traversed. Such a link diagram is called *alternating*. Now the faces in such a link diagram can be colored black and white in such a way that adjacent faces receive different colors and the infinite face is colored white (see Figure 2(b)). This coloring is called the *Tait coloring* of the diagram D. Let the graph $S(D)$ have vertices corresponding to the black faces and an edge joining two such vertices when they are the opposite faces of a crossing (see Figure 2(c)). In 1985 Vaughn Jones introduced a single-variable polynomial for links that now bears his name. For this and other contributions to knot theory, Jones was awarded a Fields Medal in 1990. If L is an alternating link, D is the corresponding link diagram, and $S(D)$ is the graph constructed as above, then as M. B. Thistlethwaite showed, the Jones polynomial of L is given, up to an easily derived factor, by an evaluation of the Tutte polynomial of $S(D)$ along the hyperbola $xy = 1$.

Tutte further developed the ideas introduced in [4, pp. 55–69] in his paper "A contribution to the theory of chromatic polynomials" [4, pp. 157–68]. This paper contained two interesting conjectures. The first, which was settled by F. Jaeger in 1975, asserted that there is a fixed integer n such that every graph without a cut edge has an n-flow. Jaeger proved this with $n = 8$, and subsequently Seymour proved it for $n = 6$. Tutte's second conjecture, the 5-Flow Conjecture, is that every graph

without a cut edge has a 5-flow. The *Petersen graph* P_{10} is obtained from the dodecahedron by identifying antipodal vertices and then replacing each pair of parallel edges by a single edge. It has no 4-flow, so 5 is the best possible number in Tutte's conjecture. The 5-Flow Conjecture has now been open for almost fifty years.

The Four Color Problem, that every loopless planar graph has chromatic number at most four, was a dominating influence on graph theory for more than a century following its introduction by F. Guthrie in 1852. In his 1966 paper "On the algebraic theory of graph colorings", following a suggestion of O. Veblen, Tutte developed what he called a "geometrical version of the Four Color Problem." The ideas he introduced were further developed by Crapo and G.-C. Rota in their 1970 consideration of the critical problem for matroids. Let G be a graph. Since we are interested in coloring vertices so that every two adjacent vertices are colored differently, we may assume that G is simple. Now suppose that $M(G)$ has rank r. Take a binary representation of $M(G)$, that is, a matrix A over $GF(2)$ such that $M[A] \cong M(G)$. Then we are effectively viewing $M(G)$ as being embedded in the r-dimensional vector space $V(r, 2)$. In a 1966 paper Tutte showed that the graph G has a 4-coloring if and only if there are $(r - 1)$-dimensional subspaces H_1 and H_2 of the embedding space $V(r, 2)$ such that $H_1 \cap H_2$ avoids the edges of G. He defined a k-*block* to be a set F of nonzero vectors in $V(r, 2)$ of rank exceeding k such that F meets every subspace of $V(r, 2)$ of dimension $r - k$. Hence a k-block is a special type of embedded matroid. A k-block is *minimal* if no proper subset of it is also a k-block. For instance, Tutte showed that the minimal 1-blocks are the cycle matroids of odd cycles.

A *tangential* k-*block* is a k-block for which no proper minor without loops is also a k-block. Because the Petersen graph P_{10} has no 4-flow, $M^*(P_{10})$ is a 2-block. Indeed, it is a tangential 2-block. Evidently, K_5 is not 4-colorable, and $M(K_5)$ is a tangential 2-block. Moreover, the Fano matroid is a tangential 2-block. Tutte conjectured that the only tangential 2-blocks are F_7, $M(K_5)$, and $M^*(P_{10})$. He proved this conjecture for tangential 2-blocks of rank at most 6, and B. T. Datta, in 1976 and 1981, proved the cases of rank 7 and 8. The most significant advance toward the resolution of this conjecture was made in 1981 by Seymour, who proved that a tangential 2-block that is not isomorphic to F_7 or $M(K_5)$ has a graphic dual. He used the Four Color Theorem and his 1980 decomposition of regular matroids in his proof of this result. A consequence of this theorem is that Tutte's Tangential 2-Block Conjecture is equivalent to the following 1966 variant of the 5-Flow Conjecture known as Tutte's 4-Flow Conjecture.
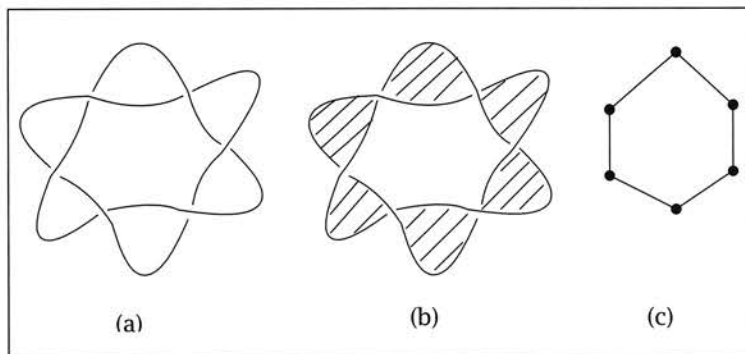

Figure 2.

Conjecture 7. *Suppose that a graph G without cut edges has no 4-flow. Then G has a subgraph contractible to P_{10}.*

Although this conjecture remains open in general, in 1999 N. Robertson, D. Sanders, P. Seymour, and R. Thomas settled it in the important special case when G is a cubic graph.

Connectivity in Matroids

Tutte developed a theory of connection in graphs [4, pp. 229–43] that he subsequently generalized to matroids. This theory has had very important implications in the burgeoning field of matroid structure theory. Concerning the definition of k-connection given earlier, Tutte writes in [4, p. 226] that he now refers to this property as "vertical k-connection [where, in this context, 'vertical' is the adjective from the noun 'vertex'] having come to think of another kind of connection as more natural." Tutte's theory of connection, which is defined in the next paragraph, is preserved under duality. By contrast, the dual of a vertically 3-connected graph need not be vertically 3-connected. For example, adding an edge in parallel to one of the edges of the complete graph K_4 produces a vertically 3-connected graph whose planar dual has a degree-2 vertex and so is certainly not vertically 3-connected.

If m is a positive integer, Tutte [4, pp. 499–522] defined $\{X, E(M) - X\}$ to be an m-*separation* of M if both $|X|$ and $|E(M) - X|$ are at least m and $\xi(X, E(M) - X) \leq m$, where

$$\xi(X, E(M) - X) = r(X) + r(E(M) - X) - r(M) + 1.$$

Thus, for example, if A is a matrix in block form $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$, and X_1 and X_2 label the columns of $\begin{bmatrix} A_1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ A_2 \end{bmatrix}$, respectively, then $\{X_1, X_2\}$ is a 1-separation of $M[A]$. This last kind of separation was considered by Whitney. But Tutte went beyond 1-separations, defining a matroid to be n-*connected* if, for all m with $1 \leq m \leq n - 1$, the matroid has no m-separation. In particular, for a graph G with at least four vertices, $M(G)$ is 3-connected if and only if G is both (vertically)

3-connected and simple. In his 1961 paper "A theory of 3-connected graphs" [4, pp. 229–43], Tutte proved a graph result which, for us, is most usefully stated in the matroid language above. An n-wheel W_n is the graph that is obtained from a cycle C_n with n edges by adding a new vertex v and then joining v to every vertex of C_n. The cycle C_n is called the *rim* of the wheel. Note that, in particular, $W_3 \cong K_4$.

Theorem 8. *Let G be a graph with at least four vertices and no isolated vertices. Suppose that $M(G)$ is 3-connected but that, for all edges e of G, neither $M(G) \backslash e$ nor $M(G)/e$ is 3-connected. Then G is isomorphic to a wheel.*

Some five years later Tutte extended the last result from graphic matroids to matroids in general. The extension required the introduction of a new class of matroids. This class is constructed from the wheels using the following general method. If C is a circuit of a matroid M such that $E(M) - C$ is a circuit of M^*, then $|C| = r(M)$ and there is a new matroid M' that has $\mathcal{B}(M) \cup \{C\}$ as its set of bases. We say that M' has been obtained from M by *relaxing* C. In particular, the rim C_n of the wheel W_n is a circuit of $M(W_n)$ whose complement is a circuit of $M^*(W_n)$. The matroid $M(W_n)$ is also called an n-wheel. Relaxing C_n in $M(W_n)$ gives the n-whirl W^n. In his very important paper "Connectivity in matroids" [4, pp. 499–522] Tutte proved the following:

Theorem 9. *Let M be a 3-connected matroid with at least four elements. If, for all elements e of M, neither $M \backslash e$ nor M/e is 3-connected, then M is a wheel or a whirl.*

A consequence of the last two theorems is that every nontrivial 3-connected matroid or graph can be built up one element at a time from a wheel or a whirl so that all intermediate matroids or graphs are 3-connected, where the two allowable building-up operations are the reverses of deletion and contraction.

An important result developed by Tutte during his consideration of separation and connection in matroids appears in the 1965 paper "Menger's Theorem for matroids". The significance of this result has only recently been realized more than thirty years after its publication. We follow J. Geelen and G. Whittle (2002) in describing how this result can be used and its link to Menger's Theorem. Let G be a graph with edge set E and suppose $X \subseteq E$. The *vertex boundary* of X is the set of vertices that meet both X and $E - X$. Now let X and Y be disjoint subsets of E, each having vertex boundary of size k. We can *contract X onto Y* if it is possible to find a minor of G with edge set $X \cup Y$ such that the vertex boundary of X has size k in the minor. Menger's Theorem for graphs implies that X can

be contracted onto Y if and only if G has no vertex cut of size less than k that separates X and Y. Rather than describe Tutte's matroid result in full generality, we consider it in the special case of a rank-r matroid M represented over a field $\mathbb{F}$. Effectively, we can view $E(M)$ as a multiset of elements of the vector space $V(r, \mathbb{F})$. For a subset X of $E(M)$, the *subspace boundary* is the intersection of the spans of X and $E(M) - X$. The rank of this subspace boundary is $\xi(X, E(M) - X) - 1$. Now suppose that X and Y are disjoint subsets of $E(M)$ such that both $\xi(X, E(M) - X)$ and $\xi(Y, E(M) - Y)$ equal k. Tutte's theorem asserts that the subspace boundary of X can be contracted onto the subspace boundary of Y provided that M has no smaller separations separating X from Y, that is, provided $\xi(X', Y') \geq k$ for all partitions $\{X', Y'\}$ of $E(M)$ with $X' \supseteq X$ and $Y' \supseteq Y$. This result has played an important role in two 2002 papers of Geelen and Whittle, and of Geelen, Gerards, and Whittle which attack what are currently the two central problems in matroid theory:

(i) Rota's 1971 conjecture that for all finite fields $\mathbb{F}$, the set of excluded minors for representability over $\mathbb{F}$ is finite; and

(ii) the conjecture that for all finite fields $\mathbb{F}$, every infinite sequence of matroids representable over $\mathbb{F}$ contains two matroids, one of which is a minor of another. This conjecture is inspired by Robertson and Seymour's Theorem 6 above.

Further Contributions to Graph Theory

Space prevents our thoroughly explaining Tutte's many other contributions to graph theory. Here we summarize some of these other results.

Factors of Graphs

While he was working on his thesis, Tutte also studied the problem of characterizing all graphs G that have a subgraph having exactly one edge at each of the vertices of G. Such a subgraph is called a 1-*factor*. Tutte successfully solved this problem [4, pp. 93–7], thus generalizing J. Petersen's 1891 result for cubic graphs.

Theorem 10. *Given a graph G and a subgraph H of G, let $o(H)$ be the number of components of H having an odd number of vertices. Then G has a 1-factor if and only if $o(G \backslash X) \leq |X|$ for every subset $X \subseteq V(G)$.*

In 1952 Tutte generalized the 1-factor theorem by characterizing for each function f from the vertices of graph G to the set of nonnegative integers, those graphs G which have a subgraph whose degree at each vertex v of G is $f(v)$. These subgraphs are now called "f-factors".

Tree Packing

For a given integer k, which graphs G have k edge-disjoint spanning trees? One obvious condition is that, given any partition of the vertices into m cells, each spanning tree must include enough edges between cells to connect them together; this minimum number is $m - 1$. So for k trees there must be $k(m - 1)$ such edges. In 1961 Tutte [4, pp. 216–25] and independently and simultaneously C. St. J. A. Nash-Williams proved that this condition also suffices. The resulting theory of edge-disjoint packings of graphs with trees has found applications in matroid theory, rigidity of frameworks, electrical circuit theory, and survivability of networks.

Chromatic Polynomials and the Golden Number

During the decade 1879–1890, the Four Color Problem was thought to have been settled by A. B. Kempe (rhymes with "hemp"), using an ingenious device now called a "Kempe chain". In 1890 P. J. Heawood found a flaw in Kempe's proof and showed that Kempe's idea proved the five color theorem. He extended the work to higher genus surfaces. With the problem wide open again, G. D. Birkhoff introduced the function $P(G; \lambda)$ as a tool for attacking it, and in 1946 he and D. C. Lewis published a very long paper that has formed the basis for much subsequent work.

The Four Color Problem was finally settled by K. Appel and W. Haken in 1976, using Kempe chains and a discharging method developed from Euler's polyhedron formula. But the proof is not fully satisfying, because it relies on the treatment of more than 10,000,000 cases by a computer. A later proof by Robertson, Sanders, Seymour, and Thomas is easier to follow but still relies in an essential manner on the use of the computer. Thus work on the Four Color Theorem continues today, with the hope of finding a humanly readable proof.

Birkhoff's chromatic polynomials represented a different approach to the Four Color Theorem. Sometime around the beginning of 1968, Ruth Bari gave Tutte a copy of her thesis, which contained the chromatic polynomials of about 100 planar graphs. Later, Dick Wick Hall sent his collection of 900 chromatic polynomials of planar graphs to Tutte. Tutte asked Gerald Berman to use his newly developed computer program for finding zeros of polynomials to find the zeros of these functions. Examining the resulting stack of 11×15 fanfold printouts, which was many inches thick, Tutte noticed that almost all of the polynomials had a zero near 1.618. Moreover, the chromatic polynomials of the more complicated plane triangulations had zeros closely approximating 2.618. Recognizing that this number is approximately 1 plus the golden mean $\tau = (1 + \sqrt{5})/2$, a zero of $x^2 - x - 1$, he initiated a study of chromatic polynomials evaluated at this number. He discovered that $|P(T; \tau + 1)| \leq \tau^{5-k}$ for any 2-connected plane triangulation T with k vertices [4, pp. 571–8]. In [4, pp. 581–95] he went further, showing that $\tau + 1$ is never a zero of a chromatic polynomial of a plane 2-connected triangulation and that

$$P_T(\tau + 2) = \sqrt{5} \cdot \tau^{3(k-3)} \cdot P_T^2(\tau + 1).$$

To find a number $(\tau + 2)$ close to 4 at which chromatic polynomials of plane triangulations are positive was particularly exciting, since the Four Color Theorem can be stated as "$P(G; 4) > 0$ for every plane graph G."

Enumeration of Graphs

A *near-triangulation* of the plane is a plane graph in which every face but one is a triangle. The remaining face may or may not be a triangle and, in Tutte's research, is taken as the infinite face. A triangulation is *strict* if it has no separating cycle with exactly two edges, and a strict triangulation is *simple* if it has no separating triangle.

Joining the enumerationists, Tutte said, "It occurred to me once that it might be possible to get results of interest in the theory of map-colourings without actually solving the [Four Colour] Problem. For example, it might be possible to find the average number of 4-colourings, on vertices, for planar triangulations of a given size."

"One would determine the number of triangulations of $2n$ faces, and then the number of 4-coloured triangulations of $2n$ faces. Then one would divide the second number by the first to get the required average" [7, p. 114].

To attack this problem, he eliminated all symmetry of a plane triangulation or near-triangulation by rooting the graph. He chose a root vertex v, a root edge A incident with v, and a root face F incident with A. He then set up generating functions for the numbers of rooted plane triangulations and related graphs, with the coefficients depending on the number of vertices in the graph. He found recursions satisfied by these functions and solved the recursions, obtaining some very nice results. For example, if a_n is the number of distinct rooted triangulations or near-triangulations on $2n$ vertices and if

$$g(x) = \sum_{n=0}^{\infty} a_n x^n,$$

then

$$g(x) = 2 \sum_{n=0}^{\infty} \frac{(4n + 1)! \, x^n}{(n + 1)! \, (3n + 2)!}.$$

Similarly, Tutte found that the number of rooted 2-connected plane cubic graphs with $2n$ vertices is

$$p_n = \frac{2^n (3n)!}{(n + 1)! \, (2n + 1)!}.$$

Moreover, he found that the average number of Hamiltonian cycles in a rooted 2-connected plane cubic graph is

$$\frac{3(2n)!\,[(2n+2)!]^2}{2^{n+2}[(n+1)!]^2(n+2)!(3n)!} \sim \frac{8\sqrt{3}}{\sqrt{\pi n}}\left(\frac{32}{27}\right)^n.$$

Turning to colorings, he said [7, p. 125], "I now knew the number of general rooted triangulations with $2n$ faces. Why not also find the number of λ-colored ones, for an arbitrary positive integer λ?" Letting N denote a general rooted plane near-triangulation, he considered

$$f(z,\lambda) = \sum_N z^{r(N)} P(N;\lambda),$$

where $r(N)$ is the number of nonroot triangular faces of N. He found a differential equation satisfied by a related function H in a variable $t = z^2$. He said [7, pp. 127-8], "The coefficients of successive powers of t in H can be calculated from this equation. I am tempted to say that the problem of finding the number of 4-coloured rooted triangulations with $2n$ faces is now solved; it is the appropriate coefficient in the power series H defined as that solution of [the differential equation] in which the coefficients of t^0 and t^1 are zero. But we still need an asymptotic approximation. For that we can perhaps now look to the theory of differential equations."

What we have said here only touches the surface of his work in this area, represented by more than twenty-five papers written over a period of more than thirty years.

Reconstruction

Let G be an unlabelled graph without loops and with at least three vertices. For each vertex v of G we make a card showing $G - v$, formed by erasing v and every edge incident with v. Given only the resulting deck of $|V(G)|$ cards, can we reconstruct G? For example, if G is a path of length two, then two of the three cards show two vertices joined by an edge, and the third card shows two isolated vertices. Since each edge appears in all but two of the cards (and so exactly once in this deck of three cards), we conclude that there are two edges, and the vertex removed to form the third card meets both edges. Hence G is a path. In 1941 P. J. Kelly and S. M. Ulam conjectured that any loopless graph with at least three vertices can be reconstructed from the deck of vertex-deleted subgraphs; this is known as the *Reconstruction Conjecture*.

Label the vertices of a graph G with the integers $1, 2, ..., n$. The *adjacency matrix* $A(G)$ is an $n \times n$ matrix in which entry (i, j) is 1 if the vertices labelled i and j are adjacent and is 0 otherwise. For a loopless graph, the main diagonal of $A(G)$ is all zeros. For such a graph, the *characteristic polynomial* of G is $\det(A(G) - \lambda I_n)$, and its *spectrum* is the set of zeros, including multiplicities, of the characteristic polynomial. These are independent of the labelling of the vertices of G.

Since the Reconstruction Conjecture has proved to be extremely intractable, the question has arisen, "Which graph properties are reconstructible?" Even this problem remained open for the chromatic polynomial, the chromatic number, the characteristic polynomial, the spectrum, and the number of Hamiltonian cycles of a graph until, in [4, pp. 528-47] and in a 1979 paper entitled "All the King's Horses", Tutte showed that all five of these properties are reconstructible by showing that the Tutte polynomial is reconstructible.

Conclusion

As a result of William Thomas Tutte's contributions to graph theory and matroid theory, both subjects have many results bearing his name. With his code-breaking efforts, he made what has been called the "greatest intellectual feat of the whole war" [1]. Among his many honors, he was awarded the Tory Medal in 1975, the Killam Prize in 1982, and the CRM-Fields Prize in 2001. He was elected a Fellow of the Royal Society of Canada (1958), a Fellow of the Royal Society of London (1987), and an Officer of the Order of Canada (2001).

Acknowledgments

References

[1] B. Fox, Colossal adventures, *New Scientist* (May 10, 1997), quoting from interview with Tony Sale.

[2] J. G. Oxley, *Matroid Theory*, Oxford University Press, New York, 1992.

[3] W. T. Tutte, *An Algebraic Theory of Graphs*, Ph.D. thesis, Cambridge University, 1948.

[4] _____ , *Selected Papers of W. T. Tutte. Volumes I and II* (D. McCarthy and R. G. Stanton, eds.), Charles Babbage Research Center, Winnipeg, 1979.

[5] _____ , *Graph Theory*, Cambridge University Press, New York, 2001 (republication of a 1984 text).

[6] _____ , Fish and I, *Coding Theory and Cryptography* (Annapolis, MD, 1998), Springer, Berlin, 2000, pp. 9-17.

[7] _____ , *Graph Theory As I Have Known It*, Oxford University Press, Oxford, 1998.

[8] _____ , The coming of the matroids, *Surveys in Combinatorics, 1999* (J. D. Lamb and D. A. Preece, eds.), London Math. Soc. Lecture Notes, vol. 267, Cambridge University Press, Cambridge, 1999, pp. 1-14.

[9] D. West, *Introduction to Graph Theory*, second edition, Prentice-Hall, Inc., Upper Saddle River, NJ, 2001.

[10] S. Wylie, Breaking Tunny and the birth of Colossus, *Action This Day* (M. Smith and R. Erskine, eds.), Bantam Press, London, 2001, pp. 317-341.