Jean Leray (1906–1998)

Armand Borel, Gennadi M. Henkin, and Peter D. Lax

***Editor's Note:** Jean Leray, called the "first modern analyst" in an article in* Nature, *died November 10, 1998, in La Baule, France. He is best known for his stunning work in partial differential equations, including the first mathematical description of turbulence in fluid flow and an early application of the idea of a function space to solving differential equations. But the work that he did in algebraic topology and several complex variables has also had a huge impact. He was the one who introduced sheaves and spectral sequences into topology, and he was a pioneer in establishing a general theory of residues in several complex variables.*

Leray was born November 7, 1906, in Nantes, France; went to the École Normale Supérieure; and became a professor first in Nancy, then in Paris, and ultimately, starting in 1947, at the Collège de France. He was a member of the Académie des Sciences de Paris, the National Academy of Sciences of the USA, the Royal Society of London, and at least half a dozen other national academies. He received the Malaxa Prize (Romania, 1938) with J. Schauder, the Grand Prix in mathematical sciences (Académie des Sciences de Paris, 1940), the Feltrinelli Prize (Lincei, 1971), and the Wolf Prize (Israel, 1979) with A. Weil.

His Selected Papers *[L97], edited by Paul Malliavin, are in three volumes: on algebraic topology, partial differential equations, and several complex variables respectively. Each has a detailed introduction that includes thorough references; the respective introductions are by Armand Borel, Peter Lax, and Gennadi Henkin. These authors have kindly prepared from their introductions the abridged versions below, which are intended for a broad* Notices *audience.*

Armand Borel

Jean Leray was first and foremost an analyst. His involvement with algebraic topology was initially incidental and became later, at first for nonmathematical reasons, a topic of major interest for him for about ten years. His papers are written with his own notation and conventions, which, for the most part, have not been adopted and are consequently little read today. But it is there that sheaves and spectral sequences originated and were first used, so this work has exerted an immense influence on the further course of algebraic topology and of the emerging homological algebra. It divides naturally into three parts.

Leray-Schauder

Leray's first contacts with topology came through his collaboration with Juliusz Schauder [LS34], following the pattern of earlier work by Schauder: he had considered continuous transformations of a Banach space B of the form

$$(1) \qquad \Phi(x) = x - F(x)$$

where F is a completely continuous operator, defined on B or only on some bounded set, and he had deduced results on elliptic or hyperbolic equations from an extension to that situation of the invariance of domain and of Brouwer's fixed point theorem.

The paper [LS34] introduces, again in analogy with Brouwer, a *topological degree* $d(\Phi, \omega, b)$, where ω is a bounded open set in B and $b \in B$ does not belong to the boundary of ω. The degree has in particular the property that it can be $\neq 0$ only if $b \in \Phi(\omega)$. The paper also defines an *index* $i(\Phi, a)$ at a point a that is isolated in its fiber $\Phi^{-1}\left(\Phi(a)\right)$. It is an integer, which under some technical assumptions is equal to ± 1. If $\Phi^{-1}(b)$ consists of finitely many points, then $d(\Phi, \omega, b)$ is the sum of the $i(\Phi, a)$ for $a \in \Phi^{-1}(b)$. These notions are applied to a family of transformations

$$(2) \qquad \Phi(x, k) = x - F(x, k)$$

depending on a parameter k varying on a closed interval K of the real line. For each $k \in K$, the transformation $F(x, k)$ is as above, defined on $\omega(k)$, and the union of the $\omega(k) \times k$ is bounded in $B \times K$. The goal is to investigate the fixed points of $F(x, k)$, i.e., the zeroes of $\Phi(x, k)$. This is done via a study of $d\left(\Phi(x, k), \omega(k), 0\right)$. It is assumed that for some fixed value k_0 of k the transformation $F(x, k_0)$ has finitely many zeroes. The goal is then to prove, under some conditions, the existence of fixed points for other values of k, some of which depend continuously on k. The results are applied to a variety of functional or differential equations.

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Leray's first paper in algebraic topology [L35] is a sequel to [LS34]. Leray gives a formula for the degree of the composition of two transformations of the type (1) and deduces from it:

a. invariance of the domain under assumptions somewhat more general than those of Schauder,
b. a theorem about the invariance of the number of bounded components in the complement of a closed bounded set C: it is the same for C and C' if there exists a homeomorphism f of C onto C' such that the differences $f(x) - x$ for $x \in C$ form a bounded set.

Alexandroff and Hopf would have liked to include (b) for finite-dimensional spaces in their book (*Topologie*, Springer, 1935), but it was too late when they heard about it. They did acknowledge it in a footnote on page 312, though.

The War Years

Apart from [L35], topology in the work of Schauder and Leray-Schauder is definitely a servant, a tool to prove theorems in analysis. It might well have remained so for Leray had it not been for the Second World War. Leray campaigned as an officer, was captured in 1940, and sent to an officers' camp in Austria, where he stayed until the end of the war. There he and some colleagues created a university, of which he became the director ("recteur"). He feared that if his competence in fluid dynamics and mechanics were known to the Germans, he might be required to work for them, so he turned his minor interest in topology into the major one and presented himself as a topologist. Indeed, during those five years he carried out research only in topology.

His first goal was to set up a theory of transformations and equations which would include Leray-Schauder and would be directly applicable to more general spaces without a reduction to finite-dimensional spaces, in contrast with Schauder's work and [LS34]. He also wanted to avoid simplicial approximations, triangulations, subdivisions of complexes, and quasilinearity of the ambient space. So he had to create a new homology theory.

Until about 1935 the main tools in algebraic topology were the simplicial homology groups, with some new ideas of Čech and Vietoris to define homology for more general spaces. Around 1935 a new type of homology group was introduced independently by several people and soon christened cohomology groups by H. Whitney. They were dual to homology groups but had the great advantage of having a product, adding the degrees, soon called the cup-product. For differentiable manifolds an example was well known: the complex of exterior differential forms with its product and exterior differential, the cohomology of which was related to homology by the de Rham theorems. In fact, one of the proponents of a cohomology theory, J. W. Alexander, had indeed been inspired by the exterior differential calculus (*Annals of Math.* (2) **37** (1936), 698–708). Leray viewed it in this way and remarked in [L50a], section 5, that Alexander was the "first to apply this formalism to the topology of abstract spaces." Following Alexander, he wanted to develop directly a theory akin to cohomology and warned the reader later in many papers that he would call his groups homology groups, since he had little use for the traditional homology groups (I shall use cohomology). While developing his ideas, he was pretty much cut off from current research,[1] so that he started essentially from scratch in his own framework.

The outcome was a three-part "Course in algebraic topology taught in captivity", published in 1945 [L45], previously announced in part in some *Comptes Rendus* notes. In the context of Leray's oeuvre, it has to be viewed as a first step. The concepts appearing there for the first time have either been strongly modified or not survived, so that there is little point in supplying many details here. I shall mainly try to give some basic definitions, in particular that of "form on a space", which Leray viewed as the analogue of differential forms in his theory. In the introduction to the second part of [L45], he states that his forms on the space obey most of the rules of the calculus of Pfaffian forms and that the main interest of the paper seems to him to be its treatment of a problem in topology, alien to any assumptions of differentiability, by computations of that nature.

The starting points are the notions of a complex on a space and of a "couverture". Fix a ground ring L (usually $\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}$, or $\mathbb{Q}$). An *abstract complex* is a free finitely generated graded (by degrees in $\mathbb{N}$) L-module, endowed with a differential d, of square zero, which increases the degree by one. A *complex* K on a space E is an abstract complex, to each element C of which is assigned a subset of E, its *support* $|C|$, with some natural properties. An important example is the cochain complex of the nerve of a finite cover of E, the support of a simplex being the intersection of the subsets labeled by the vertices. Given a closed subspace F of E, let $F.K$ be the quotient of K by the submodule of elements not meeting F, the support of $F.C$ being $F \cap |C|$. The complex K is a *couverture* if xK is acyclic for all x, plus a coherent condition for the generator of $H^0(xK; L)$. This is a condition similar to the validity of the Poincaré lemma. The *forms on* E are the elements of a couverture. Since those are finitely generated by definition, infinitely many

[1] *His main information came from some reprints Heinz Hopf had managed to procure for him, mainly by him and some people around him, in particular his paper on the homology of grouplike manifolds* (Annals of Math. ***42*** *(1941), 28–52) and Gysin's Thesis* (Comm. Math. Helv. ***14*** *(1942), 61–122).*

will be needed to define the cohomology of a space in general. Besides, there is not yet a product. Leray defines the latter geometrically, via the notion of intersection $K \circ K'$ of two complexes K and K'. It is the quotient of $K \otimes K'$ by the intersection of the kernels of the natural maps $K \otimes K' \to xK \otimes xK'$ $(x \in E)$. The union of the couvertures (over L) is a differential graded algebra. By definition, its cohomology is the cohomology ring $H^*(E; L)$ of E with coefficients in L.

To compute it, it is not necessary to use all couvertures: a family stable under intersection, which contains couvertures with arbitrary small supports, suffices. This allows one to see that for a finite polyhedron we get back the usual cohomology or that for a compact space it is equivalent to Čech cohomology. The cohomology is mostly used for locally compact spaces. As was noticed later, it is in that case equivalent to the Alexander-Spanier cohomology with compact supports. The paper gives many properties of these cohomology groups, for which I refer to §§6 to 9 in [B]. Let me just mention: a long exact sequence in cohomology (with compact supports) for a space and a closed subspace; generalizations of Hopf's theorem to compact groups, of the Lefschetz fixed point theorem, and of the Leray-Schauder theory; for manifolds: Poincaré and Alexander duality, the Jordan-Brouwer theorem. Among the new results: on a compact space a cohomology class of strictly positive degree is nilpotent. If E has a closed finite cover such that all nonempty intersections are acyclic, the cohomology of E is isomorphic to that of the nerve of the cover.

The Topology of a Continuous Map

Leray had developed a very special theory, but—granted that his cohomology was essentially Čech cohomology, say for compact spaces—the concrete results did not seem to go drastically beyond those of mainstream algebraic topology (even though a closer examination would have revealed a novel approach and more general assumptions for a number of familiar results), so [L45] did not create such a big impression. However, Leray had other goals. For him, algebraic topology should not only study the *topology of a space*, i.e., algebraic objects attached to a space, invariant under homeomorphisms, but also the *topology of a representation* (continuous map), i.e., topological invariants of a similar nature for continuous maps.

Of course, if one is given a continuous map $f: E \to E^*$, there is always an induced homomorphism in homology or cohomology, but Leray had something much deeper in mind, and the implementation of that idea led him to break entirely new ground. That he had conceived of that development while still in captivity is clear from the footnote in the first page of the third part of [L45]. Also, in a conversation with A. Weil in summer 1945 (see A. Weil, *Collected Papers*, II, p. 526), he had spoken of a homology "with variable coefficients" and it is likely that, as an example, the cohomology groups of the fibers of a continuous map were very much on his mind.

The first publications by Leray in that new direction are [L46a] and [L46b], which introduce first versions of sheaves, cohomology with respect to sheaves, and the spectral sequence of a continuous map.

In [L46a] a sheaf $\mathcal{B}$ on the space E associates to each *closed* subset F of E a module (or algebra) over the given ground ring L and to each inclusion $F \supset F'$ a homomorphism $\mathcal{B}(F) \to \mathcal{B}(F')$ with a natural transitivity property. The sheaf $\mathcal{B}$ is *normal* if $\mathcal{B}(F)$ is the inductive limit of the $\mathcal{B}(F')$ for $F' \supset F$. A basic example is the q-th cohomology sheaf $\mathcal{B}^q_E$ of E, which assigns $H^q(F; L)$ to F. (It is normal, since the cohomology has compact supports.) Normality is always assumed. As a further example, the sheaf $\mathcal{B}$ of germs of continuous functions is obtained in this setup by letting $\mathcal{B}(F)$ be the set of equivalence classes of continuous functions defined in open neighborhoods of F, two such functions being equivalent if they coincide on some neighborhood of F.

A form on E with coefficients in the sheaf $\mathcal{B}$ is a finite linear combination $\sum b_i X_i$, where the X_i belong to the basis of a couverture and $b_i \in \mathcal{B}(|X_i|)$. It is asserted that the constructions and results of [L45] extend to that case, whence the definition of the cohomology group (or ring if $\mathcal{B}$ is a sheaf of rings) $H^*(E; \mathcal{B})$ of E with respect to $\mathcal{B}$. Now let $\pi : E \to E^*$ be a continuous map. By definition, the transform $\pi(\mathcal{B})$ of a sheaf on B by π is the sheaf $F^* \mapsto \mathcal{B}(\pi^{-1}(F^*))$, the direct image in this setup. The q-th cohomology sheaf of π is, by definition, $\pi(\mathcal{B}^q_E)$, which assigns $H^q\left(\pi^{-1}(F^*); L\right)$ to F^* (the q-th right-derived functor of the direct image functor, in today's parlance).

The (p,q)-cohomology group of π is $H^p\left(E; \pi(\mathcal{B}^q_E)\right)$. The cohomology ring of π, which I shall denote $H^*(\pi)$, is the direct sum of the (p,q)-cohomology groups, with the product inherited from those on the cohomology of E^* and of the closed subsets of E.

The next *Comptes Rendus* note [L46b] is devoted to the structure of $H^*(\pi)$. By this is meant a procedure allowing one to relate it to the cohomology of E. I shall not try to describe it (see §11 of [B] for some details). One recognizes in it a number of constructions soon to be codified in the notion of a spectral sequence. There is a filtration of $H^*(E; L)$, and the successive quotients are arrived at by a sequence of approximations, starting from subquotients of $H^*(\pi)$ and using the action of differentials on representative forms. Applications to fibre bundles are given in this and the two following *Comptes Rendus* notes (*C. R. Acad. Sci. Paris* **223** (1946), 395–397, 412–415). The last one in particular describes the real cohomology ring of the quotient G/T of a compact simple group G by a maximal torus T when G is *classical*.

In 1947 various improvements were contributed by H. Cartan, J.-L. Koszul, and Leray himself. The analysis of [L46b] led Koszul to what we now call the spectral sequence of a filtered differential graded ring, a notion soon adopted by Leray (under some more general assumptions) and called later by him "spectral ring". Cartan suggested allowing complexes to be differential graded *algebras*, not necessarily free, finitely generated. Cartan and Leray independently introduced the notion of a fine complex (Leray's terminology), i.e., stable under partitions of unity associated to finite covers. Then $H^*(E;L)$ could be defined as the cohomology of just one fine couverture, a considerable conceptual simplification. In [L50a] Leray gives his final exposition of the theory (always cohomology with compact supports of locally compact spaces). He also introduces the cohomology of E with respect to a differential graded sheaf (now called hypercohomology) and shows it to be the abutment of a spectral sequence in which an early term (E_2 nowadays) is the cohomology of E with respect to the derived sheaf $\mathcal{H}\mathcal{B}$ of $\mathcal{B}$, the "fundamental theorem of sheaf theory".

During that period Leray had pursued his work on fibre bundles, in particular, homogeneous spaces of compact connected Lie groups. In his last paper on this topic [L50c], among other results Leray determines the real cohomology ring of G/T, where G is now any compact semisimple group and T a maximal torus, and establishes the Hirsch formula giving the Poincaré polynomial of G/H when H has the same rank as G.

After 1950, as before the war, algebraic topology played only a subservient role in Leray's work and appeared mainly in his theory of residues and in one paper on fixed point theorems [L59c].

Leray's framework is sheaf cohomology and spectral sequences for cohomology with compact supports of locally compact spaces, and his theory has proved to be a very powerful instrument for those spaces. But Leray's ideas penetrated other parts of topology and of mathematics as well. For this, various generalizations of his theory were needed, and we list them briefly.

H. Cartan produced three versions of sheaf theory between 1947 and 1950, of increasing generality. The last one [C] is valid over any regular space. The definition of sheaf is modified in a point of capital importance: a sheaf on the space X now assigns to each *open* subset a module, or ring (in Leray, and in the first two versions of Cartan, closed subsets were used). Injective resolutions are introduced; the fundamental theorem of sheaf theory is proved in full generality (and became a fundamental tool in the construction of derived categories). In 1950 a spectral sequence in singular homology or cohomology, also for general spaces, was introduced by J.-P. Serre, was applied to a very broad (and new) type of fibration, and was used in particular to study homotopy groups of spheres.

The passage to open subsets in the definition of sheaves opened the way to the introduction of sheaves in several complex variables (Cartan, Serre), in algebraic geometry over $\mathbb{C}$ (Kodaira, Spencer, Serre), and over any algebraically closed groundfield (Serre). These generalizations and applications go far beyond Leray's own contributions. Still, the sources of those groundbreaking ideas are the notes [L46a], [L46b], and they are so original that no earlier work by someone else can be viewed as a precursor.

Peter D. Lax

Jean Leray was one of the leading mathematicians of the twentieth century. A large part of his interests center on partial differential equations, especially those arising in mathematical physics. His investigations, some of them going back more than sixty years, still set the agenda of research in the fields in which he worked. The methods he introduced have found their uses in far-flung areas of mathematics.

Leray's papers are well organized; each distinct result has a chapter of its own, and the chapters are divided into short sections devoted to particular technical aspects of the argument. Since a priori estimates lie at the heart of most of his arguments, many of Leray's papers contain symphonies of inequalities; sometimes the orchestration is heavy, but the melody is always clearly audible.

Within the subject of partial differential equations, Leray studied both stationary problems, mostly governed by elliptic equations, and time-dependent problems, governed by parabolic and hyperbolic equations. His 1933 dissertation [L33], in the *Journal de Mathématiques Pures et Appliqués*, deals with stationary problems, using an abstract and extended version of Erhardt Schmidt's method of deformation and bifurcation. A wealth of applications is presented, including the existence of

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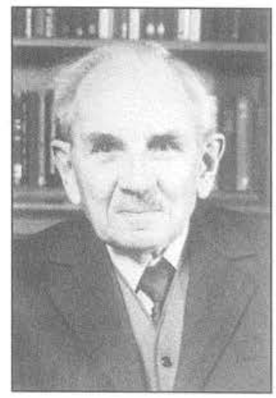

Photographs of Jean Leray courtesy of the Académie des Sciences Archives and Jean Leray relatives.

steady rotating fluids in three dimensions that satisfy the Navier-Stokes equation.

In 1934 Leray and Schauder [LS34] devised the epoch-making method bearing their name, using deformations to prove the existence of solutions for various classes of equations. This method extends Brouwer's notion of the degree to mappings of infinite-dimensional spaces of form **I** plus compact map. Like its finite-dimensional counterpart, the degree remains invariant under continuous deformations at every point that is not the image of a boundary point. To apply this principle in a concrete situation, two sets of a priori estimates have to be made: one showing the compactness of the one-parameter family of mappings employed, the other showing that all points on a sphere of radius R are mapped into points outside of a sphere. In addition, one has to verify for a particular value of the parameter that the degree of the mapping is nonzero. Leray and Schauder gave a number of applications of their method to solve the Dirichlet problem for various classes of quasilinear second-order elliptic equations; the norm they employ is the Hölder norm.

Ever since its appearance the Leray-Schauder degree has been one of the most powerful methods for dealing with nonlinear problems. A quick search of *Mathematical Reviews* disclosed 591 references to papers that make use of it.

Leray returned to elliptic problems again and again. In a 1935 paper in *Commentarii Matematici Helvetici* Leray used degree theory to construct steady ideal fluid flow in the plane around an obstacle and its wake. In a technically formidable paper in 1939 he showed how to use degree theory to construct solutions of boundary value problems for second-order fully nonlinear elliptic equations in two variables, including the Monge-Ampère equation. In the 1960s, in collaboration with J.-L. Lions, he examined results of Vishik and of Minty and Browder from the point of view of degree theory in finite-dimensional space. In the 1970s he and Y. Choquet-Bruhat used a fixed point theorem to solve the Dirichlet problem for second-order elliptic equations in divergence form.

We turn now to Leray's studies of time-dependent problems. In a paper [L34] that appeared in *Acta Mathematica* in 1934 Leray investigates the existence, uniqueness, and smoothness of solutions of the initial value problem for the Navier-Stokes equation in three-dimensional space. Physicists sometimes deride such existential pursuits by mathematicians, saying that they stop just when things are getting interesting, but what Leray found about existence, smoothness, and uniqueness of solutions was far more interesting for the physics of fluids than anything thought of before. He showed that in three space dimensions smooth initial data give rise to solutions that are smooth for a finite time; these solutions may be continued beyond this time only as generalized (weak) solutions of the Navier-Stokes equations. Leray calls these *turbulent* solutions. He shows that if two solutions, one regular and the other turbulent, have the same initial values, then they are equal; but it is still not known if turbulent solutions are uniquely determined by their initial data. Leray's results suggest a scenario for the occurrence of turbulence in fluid flow as the breakdown of smooth solutions as well as the possibility of the branching of weak solutions into different time histories.

Leray shows that in order for a solution to become turbulent at time T the maximum velocity $V(t)$ must blow up like $const/\sqrt{T-t}$ as t approaches T. No such solutions have been found so far. Leray suggested that there may be singular similarity solutions of the form

$$u_i(x,t) = (T-t)^{-1/2}U_i((T-t)^{-1/2}x),$$

u_i denoting the components of velocity. Clearly a solution of this form becomes singular as t approaches T. However, recently Necas, Ruzicka, and Sverak (1996) have shown that the equations that must be satisfied by the functions U_i have no solution of class L^3 in the whole three-dimensional space. Even more recently, Tai-peng Tsai [Tsa] has shown that no similarity solution, unless identically zero, has locally finite energy and locally finite rate of energy dissipation.

In the course of constructing his possibly turbulent solutions Leray used a host of concepts and methods of functional analysis that have since become an indispensable part of the arsenal of analysts: the weak compactness of bounded sequences in L^2, and a weakly convergent sequence is strongly convergent if and only if the limit of the norms is the norm of the limit. Leray defined the weak derivative of an L^2 function in the modern sense, as well as the concept of an L^2 vector field that is divergence free in the weak sense. He used mollifiers to show that a weak derivative is a strong derivative.

Despite much effort, remarkably little has been learned in the last sixty years about the smoothness of the weak solutions constructed by Leray. Scheffer (1976) was the first to study the size of the singular set in space-time; subsequently Caffarelli, Kohn, and Nirenberg (1982) have shown that the one-dimensional Hausdorff measure of the singular set is zero. In particular, the singularities cannot lie along a smooth curve. More recently, simplified derivations of the CKN result have been given by Fang-Hua Lin and Chun Liu (1996), as well as by Gang Tian and Zhouping Xin [TiX].

There has been some advance in existence theory. In 1951 Eberhardt Hopf showed that the Navier-Stokes equations have weak solutions with prescribed initial values in smoothly bounded domains in three-dimensional space, with zero velocity at the boundary. Hopf's proof makes use of the same functional analytic machinery as Leray's, but it is simpler in some details; in particular, instead of mollification he uses a Galerkin procedure to construct approximate solutions. A different approach to existence theory was taken by Fujita and Kato (1964); they used fractional powers of operators and the theory of semigroups.

Our knowledge of smooth solutions has advanced. Leray had shown that if the initial data are sufficiently smooth and tend to zero sufficiently fast near infinity, then a unique smooth solution exists in a time interval [0,T]; the size of this interval may depend on the viscosity γ. Ebin and Marsden (1970), Swann (1971), and Kato (1972) have shown that in domains without boundaries T may be chosen to be independent of the size of viscosity and that as γ tends to zero these solutions with fixed initial data tend to the solution of the inviscid incompressible Euler equations. No comparable result is known for flows in a domain with boundaries.

Leray showed that in the absence of a driving force in the interior or on the boundary, solutions of the Navier-Stokes equation tend to zero as t tends to ∞ and that they regain regularity after a finite time. Much work has been done since on the behavior of driven viscous flows as $t \to \infty$, such as the finiteness of the Hausdorff dimension of the so-called attractor set; see, e.g., Babin and Vishik [BV], Constantin and Foias [CF], Témam [Té], Ladyzhenskaya [Lad], and the literature quoted there.

Major effort has been devoted to devising and implementing effective computational schemes for calculating Navier-Stokes flows, steady and time dependent. Curiously, although for many classes of partial differential equations computations have, in von Neumann's prophetic words, "provided us with those heuristic hints which are needed in all parts of mathematics for genuine progress," computations have so far failed to shed much light on whether there are regular solutions that become turbulent.

After World War II Leray turned his attention to time-dependent hyperbolic partial differential equations. As pointed out long ago by Friedrichs and Lewy, the key to the initial value problem is furnished by energy inequalities. Leray [L53] derived these by multiplying the n^{th} order equation $a(x,D)u = 0$ by mu, where $m(x,D)$ is an $(n-1)^{st}$-order hyperbolic differential operator whose characteristics separate those of a; a natural choice is $m = a_\tau$, $\tau = \partial/\partial t$. The product $(mu)au$ is integrated over a domain in (x,t) space bounded by initial and final surfaces. Integration by parts produces integrals over the bounding spacelike surfaces whose integrands are quadratic forms in the $(n-1)^{st}$ derivatives of u. A criterion of Gårding shows that these energy integrals are positive definite.

In 1958 Calderon showed how energy estimates can be derived by employing singular integral (pseudodifferential) operators as symmetrizers of hyperbolic operators.

In the 1960s Leray became interested in hyperbolic equations with multiple characteristics. A typical example is

$$u_{tt} + u_x = 0\;;$$

this equation has solutions of the form $u = e^{-inx+\sqrt{in}t}$, which shows that solutions do not depend boundedly in the C^N norm on their initial data at $t = 0$, no matter how large N is. It follows that the initial value problem cannot be solved for all C^N initial data. The same conclusion holds for all hyperbolic operators $a(x,D)$ with multiple characteristics unless restrictions, called the Levi-Lax condition and given in [Lax], are placed on the allowable lower-order terms; see Mizohata [Mi]. In the 1960s Ohya had discovered that if the coefficients of $a(x,D)$ and the prescribed initial data are not only C^∞ but in an appropriate Gevrey class, then the initial value problem has a solution that belongs to a Gevrey class.

The importance of Gevrey classes in this context is that they are *not* quasianalytic, i.e., that they contain functions with arbitrarily prescribed compact support. Therefore it is possible to define domains of dependence and domains of influence for Gevrey class solutions. Leray [LO67], in collaboration with Ohya, generalized Ohya's result considerably, including even quasilinear equations and systems of n^{th}-order equations.

Leray's formulation of analytical problems in geometric terms is very much in the spirit of Poincaré, although for Poincaré function spaces were a promised land he saw but did not enter. Like Poincaré, Leray chose to work mostly on problems that came from physics. In marked contrast, the founding members of the Bourbaki movement, most of them Leray's contemporaries, sought inspiration not in nature but in mathematics itself. That Leray remained faithful to nature had a profound effect on postwar French mathematics. For it was his achievements, prestige, and influence that assured a rightful place for his outlook; he was the intellectual guide of the present distinguished French school of applied mathematics. More than that, he provided that balance between the concrete and the abstract that is so essential for the health of mathematics.

Gennadi M. Henkin

The works of Jean Leray in the 1950s and 1960s twice radically changed the direction of the development of contemporary complex analysis.

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René Deheuvels (1953)
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Philippe-A. Dionne (1962)
Jean Vaillant (1964)
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Solange Delache (1968)
Claude Wagschal (1973)
Dominique Schiltz (1987)

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Indeed, before the 1950s the theory of functions of several complex variables was based, in general, on traditional constructive methods.

One can mention here a series of works of K. Oka and H. Cartan, who in the period 1936–50, using the Cauchy-Weil formula (1935), solved the "fundamental problems" (the problems of P. Cousin, K. Weierstrass, H. Poincaré, E. Levi, and A. Weil). At the same time in the 1940s Leray, in connection with the study of the topology of continuous mappings and fiber spaces, developed so-called "sheaf theory" (1946, 1950).

Developing the ideas of Leray and Oka, H. Cartan (1950) introduced coherent analytic sheaves. After this it was found (by H. Cartan and J.-P. Serre and later H. Grauert and R. Remmert) that the methods of sheaf theory allowed one not only to reduce constructive methods (integral formulas of the Cauchy-Weil type) to a minimum in the Oka-Cartan theory but also to give a far-reaching generalization of this theory.

Thus, in the 1950s the constructive analytical methods of integral representations were practically driven out of multidimensional complex analysis and were replaced by algebraic methods of sheaf theory. The weakness of sheaf theory is that it does not provide quantitative estimates for solutions of the "fundamental problems".

At the same time in the 1950s Leray systematically brought to the Cauchy problem sharply advanced, necessary analytical methods, in particular residue theory on complex manifolds. In connection with this theory he introduced into consideration the highly general Cauchy-Leray integral formula. This formula led to progress not only for the Cauchy problem but also for a series of other important problems of complex analysis and differential equations that apparently could not be solved if one had to rely on the nonconstructive methods of sheaf theory.

Thus, thanks to Leray, the constructive methods of residue theory and of integral representations occupied again a first-rank place in complex analysis of several variables.

Holomorphic Cauchy Problem

The connection between multidimensional complex analysis and the Cauchy problem has been more apparent in formulas for elementary solutions of elliptic and hyperbolic equations with constant coefficients found in increasing generality in works of G. Herglotz (1926, 1928), L. Fantappie (1943), I. Petrowski (1945), and Leray (1953). Namely, these formulas express elementary solutions $u(x)$ of a homogeneous hyperbolic operator $P\left(-i\frac{d}{dx}\right)$ of an arbitrary order in terms of abelian integrals on the surface

$$\{\xi \in \mathbb{C}P^n : \ P(\xi) = 0, \ x \cdot \xi = 0\}.$$

Starting from the Herglotz-Petrowski-Leray formula (1953), Leray began in [L56] the study of the Cauchy problem for equations with variable coefficients. He stated his program of investigations in the following way in the introduction to [L57]:[2]

> Nous proposons d'étudier globalement le problème linéaire de Cauchy dans le cas complexe, puis dans le cas réel et hyperbolique, en supposant les données analytiques. Notre principal but est la proposition suivante: les singularities de la solution appartiennent aux caractéristiques issues des singularités des données ou tangentes à la variété qui porte les données de Cauchy. C'est l'extension aux équations aux dérivées partielles de la propriété fondamentale des solutions des équations différentielles ordinaires, linéaires et analytiques: leurs singularités sont des singularités des données.

However, the global Cauchy problem (both in the complex and the real domain) turned out to be a theme so large, difficult, and interesting that in spite of the efforts of Leray himself and his successors (Y. Hamada, C. Wagschal, J. Vaillant, D. Schiltz, D. Agnolo, P. Schapira, E. Leichtnam, B. Sternin, V. Shatalov, ...) the formulated problem is not yet solved completely. One of the most brilliant and unfinished ideas of Leray is contained in the work [L56].

Several deep steps in the realization of this program were done in the fundamental series of Leray's papers entitled "Problème de Cauchy I, II,

[2] *"We propose to study globally the linear Cauchy problem in the complex case, then in the real hyperbolic case, assuming the given data to be analytic. Our main goal is the following proposition: the singularities of the solution belong to the characteristics stemming from singularities of the data or tangents to the variety carrying the Cauchy data. This is the extension to partial differential equations of the fundamental property of solutions of ordinary differential equations that are linear and analytic: their singularities are singularities of the data."*

III, IV, VI". The Leray paper entitled "Problème de Cauchy V" has not been published, but in [L56] and in Leray papers in 1962, 1963, and 1964 there are some indications of the ideas of this work.

In the introduction to the article "Problème de Cauchy I" [L57] Leray describes his idea of uniformization of the solution of the Cauchy problem in the following brief and expressive way:[3]

> Ce premier article étudie la solution $u(x)$ du problème de Cauchy près de la variété S qui porte les données de Cauchy. Si S n'est caractéristique en aucun de ses points, alors, $u(x)$ est holomorphe près de S, vu le théorème de Cauchy-Kowalewski, et nos théorèmes n'énoncent rien de neuf. Mais nous admettons que S soit caractéristique en certains de ses points: il s'agit d'un cas sans analogue en théorie des équations différentielles ordinaires, en théorie des équations aux dérivées partielles ce cas joue un rôle fondamental, parce qu'il est celui où $u(x)$ présente les singularités les plus simples: $u(x)$ peut être uniformisé et, sauf des cas exceptionnels, est algébroide.

In the work of Gårding-Kotake-Leray (1964) developing [L57], the authors obtained an asymptotic expansion of the solution of the Cauchy problem in the neighborhood of characteristic points. The Leray uniformization method was applied with success to nonlinear systems in work of Y. Choquet-Bruhat (1966).

A fundamental concept in the Leray program (1957, 1963) is the so-called unitary solution of the Cauchy problem. Denote by ξ^* the hyperplane in $\mathbb{C}^n$ or the point in $(\mathbb{C}P^n)^*$ defined by the equation

$$\xi^* : \ \xi \cdot x = \xi_0 + \xi_1 \cdot x_1 + \cdots + \xi_n \cdot x_n = 0.$$

Let $a(x, \xi)$ be a polynomial of degree m with respect to ξ, independent of ξ_0, with coefficients that are holomorphic with respect to $x \in \Omega$. Let $g(x, \xi)$ be the *principal part* of $a(x, \xi)$, i.e., the term homogeneous in ξ of degree m such that $a(x, \xi) - g(x, \xi)$ is a polynomial in ξ of degree $< m$. A *unitary solution* for the operator $a\left(x, \frac{\partial}{\partial x}\right)$ is, by definition, a solution $U(\xi, y)$ of the Cauchy problem

$$a\left(y, \frac{\partial}{\partial y}\right) U(\xi, y) = 1,$$

where the function $U(\xi, y)$ has a zero of order m on the surface $\xi \cdot y = 0$. Due to zero homogeneity with respect to ξ, the function $U(\xi, y)$ is a function of $y \in \Omega$ and $\xi^* \in (\mathbb{C}P^n)^*$. Let $a^*\left(x, \frac{\partial}{\partial x}\right)$ be the adjoint operator for $a\left(x, \frac{\partial}{\partial x}\right)$, and let $U^*(\xi, y)$ be a unitary solution corresponding to $a^*\left(x, \frac{\partial}{\partial x}\right)$.

The Leray uniformization result [L57] can be applied to describing, in general, the singularities of the multivalued function $U(\xi, y)$ in the neighborhood of *characteristic points* (y, ξ), those with $\xi \cdot y = 0$ and $g(y, \xi) = 0$. The uniformization of unitary solutions of the Cauchy problem is used in an essential way in the fundamental work of Leray [L62] for defining the singular part of the "elementary solution" for a hyperbolic operator. For such an operator $a\left(x, \frac{\partial}{\partial x}\right)$, of degree m, a theorem of J. Hadamard (1923) and I. Petrowski (1937) states the global existence and uniqueness of the *elementary solution* $E(x, y)$ of the equation $a\left(x, \frac{\partial}{\partial x}\right) E(x, y) = \delta(x - y)$ with condition $\operatorname{supp} E \subset \mathcal{E}(y)$, where $\mathcal{E}(y)$ is the union of all timelike paths originating from y. The formulated existence and uniqueness result gives no precise information about the singularities of $E(x, y)$. Such information can be obtained from the following formula for $E(x, y)$ given in [L62]:

$$E(x, y) = \mathcal{L}(U^*(\xi, y)),$$

where $U^*(\xi, y)$ is a unitary solution of the operator a^* adjoint to a and $\mathcal{L}$ denotes a generalized Laplace transform defined in [L62]. The Leray formula, applied to a homogeneous operator with constant coefficients $a(\partial/\partial x)$, turns into the Herglotz-Petrowski-Leray formula. For this case,

$$U^*(\xi, y) = \frac{1}{m!}(\xi \cdot y)^m / a(\xi).$$

From the Leray formula it follows that $E(x, y)$ as a function of x is holomorphic outside of the characteristic conoid $K(y)$, the union of all bicharacteristics originating at y. In addition, the principal part of the singularity of $E(x, y)$ can be computed on the conoid $K(y)$.

The work of Leray [L62] was generalized for the case of nonstrictly hyperbolic equations in works of Atiyah-Bott-Gårding (1970, 1973) and was used by them for the development of the Petrowski lacunas theory (1945) for hyperbolic differential operators. For further results on the holomorphic Cauchy problem and applications,

[3] *"This first article studies the solution $u(x)$ of the Cauchy problem close to the variety S carrying the Cauchy data. If S is characteristic at none of its points, then $u(x)$ is holomorphic near S, by the Cauchy-Kovalevsky theorem, and our theorems say nothing new. But we allow that S is characteristic at certain of its points. This is a case without an analog in the theory of ordinary differential equations; in the theory of partial differential equations this case plays a fundamental role because this is the one for which $u(x)$ presents the simplest singularities: $u(x)$ can be uniformized and, save for some exceptional cases, is algebroidal." (An algebroidal function in a domain D is a function on D that satisfies a monic polynomial equation with coefficients that are holomorphic on D.)*

see [L97], [DS], [Lei], [Shap], [StSh], [V], and references therein.

Theory of Residues on Complex Manifolds

The results of Leray on the Cauchy problem turned out to be closely connected to multidimensional residue theory. Multidimensional residue theory started actually with H. Poincaré's (1887) work, in which Poincaré introduced the 1-form-residue of any rational 2-form in $\mathbb{C}^2$.

Leray [L59a] developed a general residue theory on complex manifolds and applied it to the investigation of concrete integrals depending on parameters arising from solving the Cauchy problem. F. Pham (1967) developed Leray's investigation in a more general context: namely, one can consider the integral $I(t) = \int_{x\in\gamma} \omega(x,t)$ of a rational (algebraic) differential p-form $\omega(x,t)$ depending algebraically on a parameter $t \in T$, with respect to a p-cycle γ on an algebraic manifold X, where γ does not intersect the singularity $S(t)$ of the p-form $\omega(x,t)$.

It was proved that the integral $I(t)$ is a (multivalued) analytic function of the parameter t outside of an analytic manifold $L \subset T$, called the "Landau manifold". For the case considered by Leray, the singularities of $\omega(z,t)$ have the form of poles on the hypersurface $S(t)$ depending linearly on t. To the Landau manifold corresponds a manifold L of such values t when $S(t)$ has a singular (double quadratic) point. For this case Leray (1959), applying the Picard-Lefchetz formula and a residue formula, proved the following:

Let $p = n = \dim_{\mathbb{C}} X$. Then going around the manifold L along a simple loop, beginning and ending in the point $t_0 \in T \setminus L$, the integral $I(t_0)$ turns into

$$I(t_0) + (-1)^{\frac{(n-1)(n-2)}{2}}(2\pi i)N\int_e \operatorname{Res}\omega(x,t),$$

where e is the so-called $(n-1)$-dimensional "vanishing cycle" on $S(t_0)$ and N is a linking index of e with γ. Hence, Leray [L59a] obtained explicit formulas for the singular part $I(t)$ in the neighborhood of L. The only singularities of this integral that can appear are poles, algebraic singularities of the second order, and logarithmic singularities.

Further, Leray (1967), generalizing the work of N. Nilsson (1964), applied the residue theory to the investigation of singularities of integrals of the large class of multivalued analytic forms whose singularities form algebraic submanifolds.

Leray (1956, 1959), developing on the one hand the Herglotz-Petrowski-Leray (1953) formula and on the other hand the theory of the analytic Fantappie functionals (1943), found a formula called by him the Cauchy-Fantappie formula, which led to fundamental progress in analysis.

We formulate here only two direct applications of the Leray formulas to the theory of analytic functionals.

Let D be a linearly concave domain in $\mathbb{C}P^n$ in the sense that for every $z \in D$ there exists a projective hyperplane $\mathbb{C}P^{n-1}_{\xi(z)} = \{w \in \mathbb{C}P^n : \xi(z)\cdot w = 0\}$ depending continuously on z, passing through the point z, and contained in D. Suppose $\{w_0 = 0\}$ is contained in D. The set of projective hyperplanes contained in D forms in the dual space $(\mathbb{C}P^n)^*$ the open set D^*. Let

$$M = \{z \in \mathbb{C}P^n : \widetilde{P}_1(z) = \cdots = \widetilde{P}_r(z) = 0\}$$

be an algebraic subset of $\mathbb{C}P^n$ of dimension k, where the homogeneous polynomials $\widetilde{P}_1, \widetilde{P}_2, \dots, \widetilde{P}_r$ are such that rank[grad $\widetilde{P}_1, \dots,$ grad $\widetilde{P}_r$] $= n-k$ almost everywhere on M. Let $\mathcal{H}^*(K)$ denote the space of linear functionals on the space $\mathcal{H}(K)$ of holomorphic functions on $K = \mathbb{C}P^n \setminus D$. For the functional $\mu \in \mathcal{H}^*(K)$ we define the *Cauchy-Fantappie indicatrix* as the function

$$f(\xi) = \mathcal{F}\mu(\xi) = \left\langle \mu, \frac{z_0}{\xi\cdot z}\right\rangle, \quad \xi \in D^*.$$

We have $f \in \mathcal{H}(D^*, \mathcal{O}(-1))$, where $\mathcal{O}(l)$ denotes the line bundle over $(\mathbb{C}P^n)^*$ whose sections are homogeneous functions of $(\xi_0, \xi_1, \dots, \xi_n)$ of degree l. The main result of the theory of analytic functionals of L. Fantappie (1943), A. Martineau (1962, 1967), and L. Aizenberg (1966) can be formulated as follows:

> The mapping $\mu \mapsto \mathcal{F}\mu$ realizes an isomorphism of the space $\mathcal{H}^*(K)$ and the space $\mathcal{H}(D^*, \mathcal{O}(-1))$.

The main application of analytic functionals according to L. Fantappie (1943, 1956) consists of different methods of integration of partial differential equations with constant coefficients, including an explicit solution of the Cauchy problem. This application can be deduced from the following:

> The functional $\mu \in H^*(K)$ has support on $K \cap M$ if and only if its Cauchy-Fantappie indicatrix $f = \mathcal{F}\mu$ satisfies the system of differential equations
>
> $$\widetilde{P}_j\left(\frac{d}{d\xi}\right) f(\xi) = 0, \quad j = 1, 2, \dots, r.$$

This last statement (Henkin (1995)) can be interpreted as a variant of the Ehrenpreis (1960, 1970) and Palamodov (1961, 1967) "fundamental principle" for systems with constant coefficients. For further results on residue theory on complex manifolds and applications, see [L97], [A], [BGVY], [BP], [D], [H], [Tsi], and the references therein.

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