Interview with Michael Atiyah and Isadore Singer

Martin Raussen and Christian Skau

The interviewers were Martin Raussen, Aalborg University, Denmark; and Christian Skau, Norwegian University of Science and Technology, Trondheim, Norway. This interview took place in Oslo on May 24, 2004, during the Abel Prize celebrations. It originally appeared in the European Mathematical Society Newsletter, September 2004, pages 24-30.

The Index Theorem

Raussen & Skau: *First, we congratulate both of you for having been awarded the Abel Prize for 2004. This prize has been given to you for "the discovery and the proof of the Index Theorem connecting geometry and analysis in a surprising way and your outstanding role in building new bridges between mathematics and theoretical physics". Both of you have an impressive list of fine achievements in mathematics. Is the Index Theorem your most important result and the result you are most pleased with in your entire careers?*

Atiyah: First, I would like to say that I prefer to call it a theory, not a theorem. Actually, we have worked on it for twenty-five years, and if I include all the related topics, I have probably spent thirty years of my life working on the area. So it is rather obvious that it is the best thing I have done.

Singer: I, too, feel that the Index Theorem was but the beginning of a high point that has lasted to this very day. It's as if we climbed a mountain and found a plateau we've been on ever since.

R & S: *We would like you to give us some comments on the history of the discovery of the Index Theorem.*[1] *Were there precursors, conjectures in this direction already before you started? Were there only mathematical motivations or also physical ones?*

Atiyah: Mathematics is always a continuum, linked to its history, the past—nothing comes out of zero. And certainly the Index Theorem is simply a continuation of work that, I would like to say, began with Abel. So of course there are precursors. A theorem is never arrived at in the way that logical thought would lead you to believe or that posterity thinks. It is usually much more accidental, some chance discovery in answer to some kind of question. Eventually you can rationalize it and say that this is how it fits. Discoveries never happen as neatly as that. You can rewrite history and make it look much more logical, but actually it happens quite differently.

Singer: At the time we proved the Index Theorem we saw how important it was in mathematics, but we had no inkling that it would have such an effect on physics some years down the road. That came as a complete surprise to us. Perhaps it should not have been a surprise because it used a lot of geometry and also quantum mechanics in a way, à la Dirac.

R & S: *You worked out at least three different proofs with different strategies for the Index Theorem. Why did you keep on after the first proof? What different insights did the proofs give?*

Atiyah: I think it is said that Gauss had ten different proofs for the law of quadratic reciprocity. Any good theorem should have several proofs, the more the better. For two reasons: usually, different proofs have different strengths and weaknesses, and they generalize in different directions—they are not just repetitions of each other. And that is certainly the case with the proofs that we came up with. There are different reasons for the proofs, they have different histories and backgrounds. Some of them are good for this application, some are good for that application. They all shed light on the area. If you cannot look at a problem from different directions, it is probably

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[1] *More details were given in the laureates' lectures.*


Photograph by Anne Lise Flavik.

Abel Prize winners Michael Atiyah (left) and Isadore Singer.

not very interesting; the more perspectives, the better!

Singer: There isn't just one theorem; there are generalizations of the theorem. One is the families index theorem using K-theory; another is the heat equation proof that makes the formulas that are topological more geometric and explicit. Each theorem and proof has merit and has different applications.

Collaboration

R & S: *Both of you contributed to the Index Theorem with different expertise and visions—and other people had a share as well, I suppose. Could you describe this collaboration and the establishment of the result a little more closely?*

Singer: Well, I came with a background in analysis and differential geometry, and Sir Michael's expertise was in algebraic geometry and topology. For the purposes of the Index Theorem, our areas of expertise fit together hand in glove. Moreover, in a way, our personalities fit together, in that "anything goes": Make a suggestion—and whatever it was, we would just put it on the blackboard and work with it; we would both enthusiastically explore it; if it didn't work, it didn't work. But often enough, some idea that seemed far-fetched did work. We both had the freedom to continue without worrying about where it came from or where it would lead. It was exciting to work with Sir Michael all these years. And it is as true today as it was when we first met in '55—that sense of excitement and "anything goes" and "let's see what happens".

Atiyah: No doubt: Singer had a strong expertise and background in analysis and differential geometry. And he knew certainly more physics than I did; it turned out to be very useful later on. My background was in algebraic geometry and topology, so it all came together. But of course there are a lot of people who contributed in the background to the buildup of the Index Theorem—going back to Abel, Riemann, much more recently Serre, who got the Abel Prize last year, Hirzebruch, Grothendieck, and Bott. There was lots of work from the algebraic geometry side and from topology that prepared the ground. And of course there are also a lot of people who did fundamental work in analysis and the study of differential equations: Hörmander, Nirenberg.... In my lecture I will give a long list of names[2]; even that one will be partial. It is an example of international collaboration; you do not work in isolation, neither in terms of time nor in terms of space—especially in these days. Mathematicians are linked so much, people travel around much more. We two met at the Institute at Princeton. It was nice to go to the Arbeitstagung in Bonn every year, which Hirzebruch organized and where many of these other people came. I did not realize that at the time, but looking back, I am very surprised how quickly these ideas moved.

R & S: *Collaboration seems to play a bigger role in mathematics than earlier. There are a lot of conferences, we see more papers that are written by two, three, or even more authors—is that a necessary and commendable development or has it drawbacks as well?*

Atiyah: It is not like in physics or chemistry where you have fifteen authors because they need an enormous big machine. It is not absolutely necessary or fundamental. But particularly if you are dealing with areas that have rather mixed and interdisciplinary backgrounds, with people who have different expertise, it is much easier and faster. It is also much more interesting for the participants. To be a mathematician on your own in your office can be a little bit dull, so interaction is stimulating, both psychologically and mathematically. It has to be admitted that there are times when you go solitary in your office, but not all the time! It can also be a social activity with lots of interaction. You need a good mix of both; you can't be talking all the time. But talking some of the time is very stimulating. Summing up, I think that it is a good development—I do not see any drawbacks.

Singer: Certainly computers have made collaboration much easier. Many mathematicians collaborate by computer instantly; it's as if they were talking to each other. I am unable to do that. A sobering counterexample to this whole trend is Perelman's results on the Poincaré conjecture: He worked alone for ten to twelve years, I think, before putting his preprints on the Net.

Atiyah: Fortunately, there are many different kinds of mathematicians, they work on different subjects, they have different approaches and different personalities—and that is a good thing. We do not want all mathematicians to be isomorphic,

[2] *Among those: Newton, Gauss, Cauchy, Laplace, Abel, Jacobi, Riemann, Weierstrass, Lie, Picard, Poincaré, Castelnuovo, Enriques, Severi, Hilbert, Lefschetz, Hodge, Todd, Leray, Cartan, Serre, Kodaira, Spencer, Dirac, Pontrjagin, Chern, Weil, Borel, Hirzebruch, Bott, Eilenberg, Grothendieck, Hörmander, Nirenberg.*

we want variety: different mountains need different kinds of techniques to climb.

Singer: I support that. Flexibility is absolutely essential in our society of mathematicians.

R & S: *Perelman's work on the Poincaré conjecture seems to be another instance in which analysis and geometry apparently get linked very much together. It seems that geometry is profiting a lot from analytic perspectives. Is this linkage between different disciplines a general trend—is it true that important results rely on this interrelation between different disciplines? And a much more specific question: What do you know about the status of the proof of the Poincaré conjecture?*

Singer: To date, everything is working out as Perelman says. So I learn from Lott's seminar at the University of Michigan and Tian's seminar at Princeton. Although no one vouches for the final details, it appears that Perelman's proof will be validated. As to your first question: When any two subjects use each other's techniques in a new way, frequently, something special happens. In geometry, analysis is very important; for existence theorems, the more the better. It is not surprising that some new [at least to me] analysis implies something interesting about the Poincaré conjecture.

Atiyah: I prefer to go even further—I really do not believe in the division of mathematics into specialities; already if you go back into the past, to Newton and Gauss.... Although there have been times, particularly post-Hilbert, with the axiomatic approach to mathematics in the first half of the twentieth century, when people began to specialize, to divide up. The Bourbaki trend had its use for a particular time. But this is not part of the general attitude to mathematics: Abel would not have distinguished between algebra and analysis. And I think the same goes for geometry and analysis for people like Newton.

It is artificial to divide mathematics into separate chunks and then to say that you bring them together as though this is a surprise. On the contrary, they are all part of the puzzle of mathematics. Sometimes you would develop some things for their own sake for a while, e.g., if you develop group theory by itself. But that is just a sort of temporary convenient division of labor. Fundamentally, mathematics should be used as a unity. I think the more examples we have of people showing that you can usefully apply analysis to geometry, the better. And not just analysis; I think that some physics came into it as well: many of the ideas in geometry use physical insight as well—take the example of Riemann! This is all part of the broad mathematical tradition, which sometimes is in danger of being overlooked by modern, younger people who say "we have separate divisions". We do not want to have any of that kind, really.

Singer: The Index theorem was in fact instrumental in breaking barriers between fields. When it first appeared, many old-timers in special fields were upset that new techniques were entering their fields and achieving things they could not do in the field by old methods. A younger generation immediately felt freed from the barriers that we both view as artificial.

Atiyah: Let me tell you a little story about Henry Whitehead, the topologist. I remember that he told me that he enjoyed very much being a topologist: he had so many friends within topology, and it was such a great community. "It would be a tragedy if one day I would have a brilliant idea within functional analysis and would have to leave all my topology friends and to go out and work with a different group of people." He regarded it to be his duty to do so, but he would be very reluctant. Somehow, we have been very fortunate. Things have moved in such a way that we got involved with functional analysts without losing our old friends; we could bring them all with us. Alain Connes was in functional analysis, and now we interact closely. So we have been fortunate to maintain our old links and move into new ones—it has been great fun.

Mathematics and Physics

R & S: *We would like to have your comments on the interplay between physics and mathematics. There is Galilei's famous dictum from the beginning of the scientific revolution, which says that the laws of nature are written in the language of mathematics. Why is it that the objects of mathematical creation, satisfying the criteria of beauty and simplicity, are precisely the ones that time and time again are found to be essential for a correct description of the external world? Examples abound; let me just mention group theory and, yes, your Index Theorem!*

Singer: There are several approaches in answer to your questions; I will discuss two. First, some parts of mathematics were created in order to describe the world around us. Calculus began by explaining the motion of planets and other moving objects. Calculus, differential equations, and integral equations are a natural part of physics because they were developed for physics. Other parts of mathematics are also natural for physics. I remember lecturing in Feynman's seminar, trying to explain anomalies. His postdocs kept wanting to pick coordinates in order to compute; he stopped them, saying: "The laws of physics are independent of a coordinate system. Listen to what Singer has to say, because he is describing the situation without coordinates." Coordinate-free means geometry. It is natural that geometry appears in physics, whose laws are independent of a coordinate system.

Symmetries are useful in physics for much the same reason they're useful in mathematics. Beauty aside, symmetries simplify equations, in physics and in mathematics. So physics and math have in common geometry and group theory, creating a close connection between parts of both subjects.

Second, there is a deeper reason, if your question is interpreted as in the title of Eugene Wigner's essay *The Unreasonable Effectiveness of Mathematics in the Natural Sciences.*[3] Mathematics studies coherent systems which I will not try to define. But it studies coherent systems, the connections between such systems, and the structure of such systems. We should not be too surprised that mathematics has coherent systems applicable to physics. It remains to be seen whether there is an already developed coherent system in mathematics that will describe the structure of string theory. [At present, we do not even know what the symmetry group of string field theory is.] Witten has said that 21st-century mathematics has to develop new mathematics, perhaps in conjunction with physics intuition, to describe the structure of string theory.

Atiyah: I agree with Singer's description of mathematics having evolved out of the physical world; it therefore is not a big surprise that it has a feedback into it. More fundamentally: to understand the outside world as a human being is an attempt to reduce complexity to simplicity. What is a theory? A lot of things are happening in the outside world, and the aim of scientific inquiry is to reduce this to as simple a number of principles as possible. That is the way the human mind works, the way the human mind wants to see the answer.

If we were computers, which could tabulate vast amounts of all sorts of information, we would never develop theory—we would say, just press the button to get the answer. We want to reduce this complexity to a form that the human mind can understand, to a few simple principles. That's the nature of scientific inquiry, and mathematics is a part of that. Mathematics is an evolution from the human brain, which is responding to outside influences, creating the machinery with which it then attacks the outside world. It is our way of trying to reduce complexity into simplicity, beauty, and elegance. It is really very fundamental; simplicity is in the nature of scientific inquiry—we do not look for complicated things.

I tend to think that science and mathematics are ways the human mind looks and experiences—you cannot divorce the human mind from it. Mathematics is part of the human mind. The question whether there is a reality independent of the human mind has no meaning—at least, we cannot answer it.

[3] *Comm. Pure App. Math., 13(1), 1960.*

R & S: *Is it too strong to say that the mathematical problems solved and the techniques that arose from physics have been the life blood of mathematics in the past; or at least for the last twenty-five years?*

Atiyah: I think you could turn that into an even stronger statement. Almost all mathematics originally arose from external reality, even numbers and counting. At some point, mathematics then turned to ask internal questions, e.g., the theory of prime numbers, which is not directly related to experience but evolved out of it. There are parts of mathematics about which the human mind asks internal questions just out of curiosity. Originally it may be physical, but eventually it becomes something independent. There are other parts that relate much closer to the outside world with much more interaction backward and forward. In that part of it, physics has for a long time been the life blood of mathematics and inspiration for mathematical work. There are times when this goes out of fashion or when parts of mathematics evolve purely internally. Lots of abstract mathematics does not directly relate to the outside world. It is one of the strengths of mathematics that it has these two and not a single life blood: one external and one internal, one arising as response to external events, the other to internal reflection on what we are doing.

Singer: Your statement *is* too strong. I agree with Michael that mathematics is blessed with both an external and internal source of inspiration. In the past several decades, high-energy theoretical physics has had a marked influence on mathematics. Many mathematicians have been shocked at this unexpected development: new ideas from outside mathematics so effective in mathematics. We are delighted with these new inputs, but the "shock" exaggerates their overall effect on mathematics.

Newer Developments

R & S: *Can we move to newer developments with impact from the Atiyah-Singer Index Theorem? That is, string theory and Edward Witten on the one hand, and on the other hand, noncommutative geometry represented by Alain Connes. Could you describe the approaches to mathematical physics epitomized by these two protagonists?*

Atiyah: I tried once in a talk to describe the different approaches to progress in physics like different religions. You have prophets, you have followers—each prophet and his followers think that they have the sole possession of the truth. If you take the strict point of view that there are several different religions, and that the intersection of all these theories is empty, then they are all talking nonsense. Or you can take the view of the mystic, who thinks that they are all talking of different aspects of reality, and so all of them are correct. I tend

Photograph by Knut Falch/Scanpix.

Abel Prize winners Isadore Singer and Michael Atiyah with Queen Sonja and King Harald of Norway at the royal palace in Oslo, May 2004.

to take the second point of view. The main "orthodox" view among physicists is certainly represented by a very large group of people working with string theory, such as Edward Witten. There are a small number of people who have different philosophies; one of them is Alain Connes, and the other is Roger Penrose. Each of them has a very specific point of view; each of them has very interesting ideas. Within the last few years, there has been nontrivial interaction between all of these.

They may all represent different aspects of reality and, eventually, when we understand it all, we may say "Ah, yes, they are all part of the truth". I think that that will happen. It is difficult to say which will be dominant when we finally understand the picture—we don't know. But I tend to be open-minded. The problem with a lot of physicists is that they have a tendency to "follow the leader": as soon as a new idea comes up, ten people write ten or more papers on it, and the effect is that everything can move very fast in a technical direction. But big progress may come from a different direction; you do need people who are exploring different avenues. And it is very good that we have people like Connes and Penrose with their own independent line from different origins. I am in favor of diversity. I prefer not to close the door or to say "they are just talking nonsense."

Singer: String theory is in a very special situation at the present time. Physicists have found new solutions on their landscape—so many that you cannot expect to make predictions from string theory. Its original promise has not been fulfilled. Nevertheless, I am an enthusiastic supporter of super string theory, not just because of what it has done in mathematics, but also because as a coherent whole, it is a marvelous subject. Every few years new developments in the theory give additional insight. When that happens, you realize how little one understood about string theory previously. The theory of D-branes is a recent example. Often there is mathematics closely associated with these new insights. Through D-branes, K-theory entered string theory naturally and reshaped it. We just have to wait and see what will happen. I am quite confident that physics will come up with some new ideas in string theory that will give us greater insight into the structure of the subject, and along with that will come new uses of mathematics.

Alain Connes's program is very natural—if you want to combine geometry with quantum mechanics, then you really want to quantize geometry, and that is what noncommutative geometry means. Noncommutative geometry has been used effectively in various parts of string theory explaining what happens at certain singularities, for example. I think it may be an interesting way of trying to describe black holes and to explain the Big Bang. I would encourage young physicists to understand noncommutative geometry more deeply than they presently do. Physicists use only parts of noncommutative geometry; the theory has much more to offer. I do not know whether it is going to lead anywhere or not. But one of my projects is to try and redo some known results using noncommutative geometry more fully.

R & S: *If you should venture a guess, which mathematical areas do you think are going to witness the most important developments in the coming years?*

Atiyah: One quick answer is that the most exciting developments are the ones that you cannot predict. If you can predict them, they are not so exciting. So, by definition, your question has no answer.

Ideas from physics, e.g., quantum theory, have had an enormous impact so far, in geometry, some parts of algebra, and in topology. The impact on number theory has still been quite small, but there are some examples. I would like to make a rash prediction that it will have a big impact on number theory as the ideas flow across mathematics—on one extreme number theory, on the other physics, and in the middle geometry: the wind is blowing, and it will eventually reach to the farthest extremities of number theory and give us a new point of view. Many problems that are worked upon today with old-fashioned ideas will be done with new ideas. I would like to see this happen: it could be the Riemann hypothesis, it could be the Langlands program, or a lot of other related things. I had an argument with Andrew Wiles in which I claimed that physics will have an impact on his kind of number theory; he thinks this is nonsense, but we had a good argument.

I would also like to make another prediction, namely that fundamental progress on the physics/mathematics front, string theory questions, etc., will emerge from a much more thorough understanding of classical four-dimensional geometry, of

Einstein's equations, etc. The hard part of physics in some sense is the nonlinearity of Einstein's equations. Everything that has been done at the moment is circumventing this problem in lots of ways. They haven't really got to grips with the hardest part. Big progress will come when people by some new techniques or new ideas really settle that. Whether you call that geometry, differential equations, or physics depends on what is going to happen, but it could be one of the big breakthroughs.

These are of course just my speculations.

Singer: I will be speculative in a slightly different way, though I do agree with the number theory comments that Sir Michael mentioned, particularly theta functions entering from physics in new ways. I think other fields of physics will affect mathematics—such as statistical mechanics and condensed matter physics. For example, I predict a new subject of statistical topology. Rather than count the number of holes, Betti numbers, etc., one will be more interested in the distribution of such objects on noncompact manifolds as one goes out to infinity. We already have precursors in the number of zeros and poles for holomorphic functions. The theory that we have for holomorphic functions will be generalized, and insights will come from condensed matter physics as to what, statistically, the topology might look like as one approaches infinity.

Continuity of Mathematics

R & S: *Mathematics has become so specialized, it seems, that one may fear that the subject will break up into separate areas. Is there a core holding things together?*

Atiyah: I like to think there is a core holding things together, and that the core is rather what I look at myself; but we tend to be rather egocentric. The traditional parts of mathematics, that evolved—geometry, calculus and algebra—all center on certain notions. As mathematics develops, there are new ideas, which appear to be far from the center going off in different directions, which I perhaps do not know much about. Sometimes they become rather important for the whole nature of the mathematical enterprise. It is a bit dangerous to restrict the definition to just whatever you happen to understand yourself or think about. For example, there are parts of mathematics that are very combinatorial. Sometimes they are very closely related to the continuous setting, and that is very good: we have interesting links between combinatorics and algebraic geometry and so on. They may also be related to, e.g., statistics. I think that mathematics is very difficult to constrain; there are also all sorts of new applications in different directions.

It is nice to think of mathematics having a unity; however, you do not want it to be a straitjacket. The center of gravity may change with time. It is not necessarily a fixed rigid object in that sense; I think it should develop and grow. I like to think of mathematics having a core, but I do not want it to be rigidly defined so that it excludes things that might be interesting. You do not want to exclude somebody who has made a discovery saying: "You are outside, you are not doing mathematics, you are playing around." You never know! That particular discovery might be the mathematics of the next century; you have got to be careful. Very often, when new ideas come in, they are regarded as being a bit odd, not really central, because they look too abstract.

Singer: Countries differ in their attitudes about the degree of specialization in mathematics and how to treat the problem of too much specialization. In the United States I observe a trend toward early specialization driven by economic considerations. You must show early promise to get good letters of recommendations to get good first jobs. You can't afford to branch out until you have established yourself and have a secure position. The realities of life force a narrowness in perspective that is not inherent to mathematics. We can counter too much specialization with new resources that would give young people more freedom than they presently have, freedom to explore mathematics more broadly, or to explore connections with other subjects, such as biology these days in which there is lots to be discovered.

When I was young the job market was good. It was important to be at a major university, but you could still prosper at a smaller one. I am distressed by the coercive effect of today's job market. Young mathematicians should have the freedom of choice we had when we were young.

R & S: *The next question concerns the continuity of mathematics. Rephrasing slightly a question that you, Professor Atiyah, are the originator of, let us make the following gedanken experiment: if, say, Newton or Gauss or Abel were to reappear in our midst, do you think they would understand the problems being tackled by the present generation of mathematicians—after they had been given a short refresher course? Or is present-day mathematics too far removed from traditional mathematics?*

Atiyah: The point that I was trying to make there was that really important progress in mathematics is somewhat independent of technical jargon. Important ideas can be explained to a really good mathematician, such as Newton or Gauss or Abel, in conceptual terms. They are in fact coordinate-free—more than that, technology-free and in a sense jargon-free. You don't have to talk of ideals, modules or whatever—you can talk in the common language of scientists and mathematicians. The really important progress mathematics has made within two hundred years could easily be understood by people such as Gauss and Newton and Abel. Only a small refresher course in which they were told a few terms—and then they would immediately understand.

Actually, my pet aversion is that many mathematicians use too many technical terms when they write and talk. They were trained in a way that, if you do not say it 100 percent correctly, like lawyers, you will be taken to court. Every statement has to be fully precise and correct. When talking to other people or scientists, I like to use words that are common to the scientific community, not necessarily just to mathematicians. And that is very often possible. If you explain ideas without a vast amount of technical jargon and formalism, I am sure it would not take Newton, Gauss, and Abel long—they were bright guys, actually!

Singer: One of my teachers at Chicago was André Weil, and I remember his saying: "If Riemann were here, I would put him in the library for a week, and when he came out he would tell us what to do next."

Communication of Mathematics

***R & S:** Next topic: communication of mathematics. Hilbert, in his famous speech at the International Congress in 1900, in order to make a point about mathematical communication, cited a French mathematician who said: "A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street." In order to pass on to new generations of mathematicians the collective knowledge of the previous generation, how important is it that the results have simple and elegant proofs?*

Atiyah: The passing of mathematics on to subsequent generations is essential for the future, and this is only possible if every generation of mathematicians understands what they are doing and distills it out in such a form that it is easily understood by the next generation. Many complicated things get simple when you have the right point of view. The first proof of something may be very complicated, but when you understand it well, you readdress it, and eventually you can present it in a way that makes it look much more understandable—and that's the way you pass it on to the next generation! Without that, we could never make progress—we would have all this messy stuff. Mathematics does depend on a sufficiently good grasp, on understanding of the fundamentals so that we can pass it on in as simple a way as possible to our successors. That has been done remarkably successfully for centuries. Otherwise, how could we possibly be where we are? In the 19th century, people said: "There is so much mathematics, how could anyone make any progress?" Well, we have—we do it by various devices, we generalize, we put all things together, we unify by new ideas, we simplify lots of the constructions—we are very successful in mathematics and have been so for several hundred years. There is no evidence that this has stopped: in every new generation, there are mathematicians who make enormous progress. How do they learn it all? It must be because we have been successful communicating it.

Singer: I find it disconcerting speaking to some of my young colleagues, because they have absorbed, reorganized, and simplified a great deal of known material into a new language, much of which I don't understand. Often I'll finally say, "Oh; is that all you meant?" Their new conceptual framework allows them to encompass succinctly considerably more than I can express with mine. Though impressed with the progress, I must confess impatience because it takes me so long to understand what is really being said.

***R & S:** Has the time passed when deep and important theorems in mathematics can be given short proofs? In the past, there are many such examples—e.g., Abel's one-page proof of the addition theorem of algebraic differentials or Goursat's proof of Cauchy's integral theorem.*

Atiyah: I do not think that at all! Of course, that depends on what foundations you are allowed to start from. If we have to start from the axioms of mathematics, then every proof will be very long. The common framework at any given time is constantly advancing; we are already at a high platform. If we are allowed to start within that framework, then at every stage there are short proofs.

One example from my own life is this famous problem about vector fields on spheres solved by Frank Adams, for which the proof took many hundreds of pages. One day I discovered how to write a proof on a postcard. I sent it over to Frank Adams and we wrote a little paper which then would fit on a bigger postcard. But of course that used some K-theory; not that complicated in itself. You are always building on a higher platform; you have always got more tools at your disposal that are part of the lingua franca which you can use. In the old days you had a smaller base: if you make a simple proof nowadays, then you are allowed to assume that people know what group theory is, you are allowed to talk about Hilbert space. Hilbert space took a long time to develop, so we have got a much bigger vocabulary, and with that we can write more poetry.

Singer: Often enough one can distill the ideas in a complicated proof and make that part of a new language. The new proof becomes simpler and more illuminating. For clarity and logic, parts of the original proof have been set aside and discussed separately.

Atiyah: Take your example of Abel's Paris memoir: his contemporaries did not find it at all easy. It laid the foundation of the theory. Only later on, in the light of that theory, we can all say: "Ah, what a beautifully simple proof!" At the time, all the ideas had to be developed, and they were hidden, and most people could not read that paper. It was very, very far from appearing easy for his contemporaries.

Individual Work Style

R & S: *I heard you, Professor Atiyah, mention that one reason for your choice of mathematics for your career was that it is not necessary to remember a lot of facts by heart. Nevertheless, a lot of threads have to be woven together when new ideas are developed. Could you tell us how you work best, how do new ideas arrive?*

Atiyah: My fundamental approach to doing research is always to ask questions. You ask "Why is this true?" when there is something mysterious or if a proof seems very complicated. I used to say—as a kind of joke—that the best ideas come to you during a bad lecture. If somebody gives a terrible lecture—it may be a beautiful result but with terrible proofs—you spend your time trying to find better ones; you do not listen to the lecture. It is all about asking questions—you simply have to have an inquisitive mind! Out of ten questions, nine will lead nowhere, and one leads to something productive. You constantly have to be inquisitive and be prepared to go in any direction. If you go in new directions, then you have to learn new material.

Usually, if you ask a question or decide to solve a problem, it has a background. If you understand where a problem comes from, then it makes it easy for you to understand the tools that have to be used on it. You immediately interpret them in terms of your own context. When I was a student, I learned things by going to lectures and reading books—after that I read very few books. I would talk with people; I would learn the essence of analysis by talking to Hörmander or other people. I would be asking questions because I was interested in a particular problem. So you learn new things because you connect them and relate them to old ones, and in that way you can start to spread around.

If you come with a problem, and you need to move to a new area for its solution, then you have an introduction—you have already a point of view. Interacting with other people is of course essential: if you move into a new field, you have to learn the language, you talk with experts; they will distill the essentials out of their experience. I did not learn all the things from the bottom upward; I went to the top and got the insight into how you think about analysis or whatever.

Singer: I seem to have some built-in sense of how things should be in mathematics. At a lecture, or reading a paper, or during a discussion, I frequently think, "that's not the way it is supposed to be." But when I try out my ideas, I'm wrong 99% of the time. I learn from that and from studying the ideas, techniques, and procedures of successful methods. My stubbornness wastes lots of time and energy. But on the rare occasion when my internal sense of mathematics is right, I've done something different.

R & S: *Both of you have passed ordinary retirement age several years ago. But you are still very active mathematicians, and you have even chosen retirement or visiting positions remote from your original work places. What are the driving forces* for *keeping up your work? Is it wrong that mathematics is a "young man's game" as Hardy put it?*

Atiyah: It is no doubt true that mathematics is a young man's game in the sense that you peak in your twenties or thirties in terms of intellectual concentration and in originality. But later you compensate for that by experience and other factors. It is also true that if you haven't done anything significant by the time you are forty, you will not do so suddenly. But it is wrong that you have to decline, you can carry on, and if you manage to diversify in different fields this gives you a broad coverage. The kind of mathematician who has difficulty maintaining the momentum all his life is a person who decides to work in a very narrow field with great depths, who, e.g., spends all his life trying to solve the Poincaré conjecture—whether you succeed or not, after ten to fifteen years in this field you exhaust your mind; and then, it may be too late to diversify. If you are the sort of person that chooses to make restrictions to yourself, to specialize in a field, you will find it harder and harder—because the only things that are left are harder and harder technical problems in your own area, and then the younger people are better than you.

You need a broad base, from which you can evolve. When this area dries out, then you go to that area—or when the field as a whole, internationally, changes gear, you can change too. The length of the time you can go on being active within mathematics very much depends on the width of your coverage. You might have contributions to make in terms of perspective, breadth, interactions. A broad coverage is the secret of a happy and successful long life in mathematical terms. I cannot think of any counterexample.

Singer: I became a graduate student at the University of Chicago after three years in the U.S. Army during World War II. I was older and far behind in mathematics. So I was shocked when my fellow graduate students said, "If you haven't proved the Riemann hypothesis by age thirty, you might as well commit suicide." How infantile! Age means little to me. What keeps me going is the excitement of what I'm doing and its possibilities. I constantly check [and collaborate!] with younger colleagues to be sure that I'm not deluding myself—that what we are doing is interesting. So I'm happily active in mathematics. Another reason is, in a way, a joke. String theory needs us! String theory needs new ideas. Where will they come from, if not from Sir Michael and me?

Atiyah: Well, we have some students....

Singer: Anyway, I am very excited about the interface of geometry and physics and delighted to be able to work at that frontier.

History of the EMS

***R & S:** You, Professor Atiyah, have been very much involved in the establishment of the European Mathematical Society (EMS) around 1990. Are you satisfied with its development since then?*

Atiyah: Let me just comment a little on my involvement. It started an awful long time ago, probably about thirty years ago. When I started trying to get people interested in forming a European Mathematical Society in the same spirit as the European Physical Society, I thought it would be easy. I got mathematicians from different countries together and it was like a mini-UN: the French and the Germans wouldn't agree; we spent years arguing about differences, and—unlike in the real UN, where eventually at the end of the day you are dealing with real problems of the world and you have to come to an agreement sometime—in mathematics, it was not absolutely essential. We went on for probably fifteen years before we founded the EMS.

On the one hand, mathematicians have much more in common than politicians. We are international in our mathematical life; it is easy to talk to colleagues from other countries. On the other hand, mathematicians are much more argumentative. When it comes to the fine details of a constitution, then they are terrible; they are worse than lawyers. But eventually—in principle—the good will was there for collaboration.

Fortunately, the timing was right. In the meantime, Europe had solved some of its other problems. The Berlin Wall had come down—so suddenly there was a new Europe to be involved in the EMS. This very fact made it possible to get a lot more people interested in it. It gave an opportunity for a broader base of the EMS, with more opportunities and also relations to the European Commission and so on.

Having been involved with the setup, I withdrew and left it to others to carry on. I have not followed in detail what has been happening except that it seems to be active. I get my newsletter, and I see what is going on.

Roughly at the same time as the collapse of the Berlin Wall, mathematicians in general—both in Europe and in the United States—began to be more aware of their need to be socially involved and that mathematics had an important role to play in society. Instead of being shut up in their universities doing just their mathematics, they felt that there was some pressure to get out and get involved in education, etc. The EMS took on this role at a European level, and the EMS congresses—I was involved in the one in Barcelona—definitely made an attempt to interact with the public. I think that these are additional opportunities over and above the old-fashioned role of learned societies. There are a lot of opportunities both in terms of the geography of Europe and in terms of the broader reach.

Europe is getting ever larger: when we started we had discussions about where were the borders of Europe. We met people from Georgia, who told us very clearly that the boundary of Europe is this river on the other side of Georgia; they were very keen to make sure that Georgia is part of Europe. Now, the politicians have to decide where the borders of Europe are.

It is good that the EMS exists; but you should think rather broadly about how it is evolving as Europe evolves, as the world evolves, as mathematics evolves. What should its function be? How should it relate to national societies? How should it relate to the AMS? How should it relate to the governmental bodies? It is an opportunity! It has a role to play!

Apart from Mathematics...

***R & S:** Could you tell us in a few words about your main interests besides mathematics?*

Singer: I love to play tennis, and I try to do so two to three times a week. That refreshes me, and I think that it has helped me work hard in mathematics all these years.

Atiyah: Well, I do not have his energy! I like to walk in the hills, the Scottish hills—I have retired partly to Scotland. In Cambridge, where I was before, the highest hill was about this [gesture] big. Of course you have got even bigger ones in Norway. I spent a lot of my time outdoors, and I like to plant trees, I like nature. I believe that if you do mathematics, you need a good relaxation that is not intellectual—being outside in the open air, climbing a mountain, working in your garden. But you actually do mathematics meanwhile. While you go for a long walk in the hills or you work in your garden, the ideas can still carry on. My wife complains, because when I walk she knows I am thinking of mathematics.

Singer: I can assure you, tennis does not allow that!

***R & S:** Thank you very much on behalf of the Norwegian, the Danish, and the European Mathematical Societies!*