The Mathematics of Paul Erdős

László Babai, Carl Pomerance, and Péter Vértesi

Paul Erdős died September 20, 1996, and a memorial article appears elsewhere in this issue. This feature article gives a cross section of his monumental oeuvre. Most of Erdős's work falls roughly into the following categories:

- number theory
- finite combinatorics (including graph theory)
- combinatorial geometry
- set theory, set-theoretical topology
- constructive theory of functions (approximation theory)
- other areas of classical analysis (polynomials, theory of series, functions of a complex variable)
- probability theory, ergodic theory

The first two areas are represented in Erdős's work by more than 600 articles each, the next three by more than 100 articles each. There are some overlaps in this rough count. A large number of articles fall into a "miscellaneous" category.

In what follows, Pomerance gives a glimpse into the variety of topics Erdős worked on in number theory. Babai discusses (infinite) set theory, finite combinatorics, combinatorial geometry, combinatorial number theory, and probability theory. Vértesi treats approximation theory, with a hint of related work on polynomials.

—László Babai, Organizer

Note: Except where otherwise noted, all photographs in this article are from the collection of Vera T. Sós.

Paul Erdős, Number Theorist Extraordinaire

Carl Pomerance

Nearly half of Paul Erdős's 1,500 papers were in number theory. He was a giant of this century, showing the power of elementary and combinatorial methods in analytic number theory, pioneering the field of probabilistic number theory, making key advances in diophantine approximationand arithmetic functions, and until his death leading the field of combinatorial number theory. Paul Erdős was also a kind and generous man, one who would seek out young mathematicians, work with them, give them ideas, teach them, and in the process make a lifelong friend and colleague. I was one of these lucky ones, but more on that later.

Perhaps the single most famous paper of Erdős is [3], wherein he described an elementary proof of the prime number theorem.

The history of the prime number theorem seems to be punctuated by major developments at half-century intervals and often in two's. At the end of the eighteenth century Gauss and Legendre independently conjectured that the number of primes up to x, denoted $\pi(x)$, is asymptotically $x/\log x$ as $x \to \infty$. (This came some fifty years after Euler had proved that the sum of the reciprocals of the primes is infinite.) In the mid-nineteenth century Chebyshev showed by an elementary method that there are positive constants c_1, c_2 with $c_1 x/\log x < \pi(x) < c_2 x/\log x$ for all large x, and

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Photograph courtesy of Mrs. Anne Davenport.

The Manchester bridge party (1934–1938). Bottom to top: Harold Davenport, Erdős, Chao-Ko, Zilinkas.

Riemann laid down a plan to prove the prime number conjecture of Gauss and Legendre via analytic methods. It is in this paper that Riemann stated what came to be known as the "Riemann hypothesis," one of the most famous and important unsolved problems in mathematics.

At the close of the nineteenth century, de la Vallée Poussin and Hadamard independently succeeded in giving complete proofs of the prime number theorem. (Though roughly following Riemann's plan, they avoided an outright assault on the Riemann hypothesis.) A tour de force for analytic methods in number theory, it was thought by many that an elementary proof of the prime number theorem was impossible. It was thus quite a sensation when Erdős and Atle Selberg actually did come up with elementary proofs in 1948.

Another fifty years have passed. Will there soon be another great advance?

In response to the theory of quantum mechanics, Einstein exclaimed, "God does not play dice with the universe." Though this never happened, I would like to think that Paul Erdős and the great probabilist, Mark Kac, replied, "Maybe so, but something is going on with the primes." In 1939 Erdős and Kac [10] proved one of the most beautiful and unexpected results in mathematics. Their theorem states that the number of prime factors in a number is distributed, as the number varies, according to a Gaussian distribution, a bell curve.

Let $\omega(n)$ denote the number of distinct prime factors of n. In 1917 Hardy and Ramanujan proved that $\omega(n)$ is normally $\log\log n$. What this means is that for each $\varepsilon > 0$, the *density* of the set of natural numbers n with $(1-\varepsilon)\log\log n < \omega(n) < (1+\varepsilon)\log\log n$ is 1. (A set S of natural numbers has density d if the number of members of S up to x, when divided by x, tends to d as $x \to \infty$. So, for example, the odd numbers have density $1/2$, the prime numbers have density 0, and the set of numbers with an even number of decimal digits does not have a density.) Later, Paul Turán, a close friend of Erdős, came up with a greatly simplified proof of the Hardy-Ramanujan theorem by showing that the sum of $(\omega(n) - \log\log n)^2$ for n up to x is of order of magnitude $x \log\log x$. This would later come to be thought of as a variance calculation as in probability theory, but it was not conceived of in this way.

Mark Kac viewed the number theoretic function $\omega(n)$ probabilistically. He reasoned that being divisible by 2, 3, 5, etc., should be thought of as "independent events," and so $\omega(n)$ could be viewed as a sum of independent random variables. Since the sum of $1/p$ for p prime, $p \le x$, is about $\log\log x$, Kac reasoned that this is what is behind the Hardy-Ramanujan-Turán theorem and that in fact a Gaussian distribution should be involved, with standard deviation $\sqrt{\log\log x}$. That is, Kac conjectured that for each real number u, the density of the set of n with $\omega(n) \le \log\log n + u\sqrt{\log\log n}$ exists and is equal to

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-t^2/2}\,dt,$$

the area under the bell curve from $-\infty$ to u.

How the collaboration of Erdős and Kac came about is best left to Kac's own words, as quoted by Peter Elliott in [2]:

"If I remember correctly, I first stated (as a conjecture) the theorem on the normal distribution of the number of prime divisors during a lecture in Princeton in March 1939. Fortunately for me and possibly for Mathematics, Erdős was in the audience, and he immediately perked up. Before the lecture was over he had completed the proof, which I could not have done, not having been versed in the number theoretic methods, especially those related to the sieve."

What Erdős knew quite well was that via the methods of sieves, as developed by Brun early in this century, it could be shown that the primes up to x^ε actually do distribute themselves "independently" among the numbers up to x, and of course no number $n \le x$ can have more than $1/\varepsilon$ prime factors that exceed x^ε.

This result opened the book on probabilistic number theory, the branch of mathematics that studies number theoretic functions, such as $\omega(n)$, via probabilistic methods.

Paul Erdős is equally well known for his remarkable problems. Here are a few in number theory:

1. Among the integers up to x, the set of powers of 2 have the property that the various subset sums are all different, and of course there are $\log_2 x + O(1)$ powers of 2 up to x. Is it true that if S is a set of integers in $[1, x]$ with all subset sums different, then S has at most $\log_2 x + O(1)$ members?

2. Suppose A is a subset of the nonnegative integers such that every nonnegative integer n can be written as $a_1 + a_2$, where a_1, a_2 are in A. Let $r(n)$ be the number of such representations of n. Must the sequence $r(n)$ be unbounded?

3. Suppose B is a subset of the positive integers with the sum of the reciprocals of the members of B being infinite. Must B contain arbitrarily long arithmetic progressions?

4. The set $\{2, 3, 4, 6, 12\}$ is a "covering set", since there are residue classes with these moduli that cover the integers: in particular, 0 mod 2, 0 mod 3, 1 mod 4, 1 mod 6, 11 mod 12 will do. For each k is there a (finite) covering set with distinct moduli, each $> k$?

Each of these problems has its own interesting story, as do hundreds of other Erdős problems. They are often tips of icebergs. For example, problem 2, which is joint with Turán, is related to the famous theorem of Erdős and W. Fuchs [8], which asserts that no matter what sequence is chosen for A, the sum of $r(n)$ for n up to x cannot be of the form $cx + o(x^{1/4}/(\log x)^{1/2})$. (Erdős was justifiably proud of this beautiful result. A few years ago he wrote (in [7]) that the Erdős-Fuchs theorem "certainly will survive the authors by centuries.")

Problem 4 is related to an old problem of Euler, who considered whether an odd number $n > 1$ can be expressed as a sum of a prime and a power of 2. Euler noticed that 127 and 959 cannot be represented, though de Polignac conjectured in 1849 that every odd number $n > 1$ *can* be represented! The problem was revived by Romanoff in 1934 and solved independently by Erdős [4] and van der Corput [1] in 1950. In fact, Erdős showed that there is an infinite arithmetic progression of odd numbers that *cannot* be represented as a sum of a prime and a power of 2. The proof used the covering set $\{2, 3, 4, 8, 12, 24\}$. If problem 4 were to hold, then there would be, for each positive integer k, an infinite arithmetic progression containing no numbers that are a sum of a power of 2 and a number with at most k different prime factors. Erdős was fond of repeating Selfridge's covering set problem: is there a covering set with odd moduli > 1? This is still unsolved.

Erdős in deep thought, around 1960.

While he was alive, Erdős offered money for each of problems 1–4 and many others. For example, problem 3 (which would have the sensational corollary that there are arbitrarily long arithmetic progressions consisting of primes) was worth \$3,000. Erdős liked to joke that his prize money violated the minimum wage law.

Paul Erdős, often through his prizes, inspired many other mathematicians. Endre Szemerédi, for example, earned \$1,000 when he showed a slightly weaker result than problem 3: he showed that if B does not have density 0, then it contains arbitrarily long arithmetic progressions. And Helmut Maier and Gérald Tenenbaum earned money from Erdős when they showed that the density of those numbers n with two divisors a, b with $a < b < 2a$ is 1.

I too owe much to Paul Erdős. At the end of an article in 1956 Erdős gave a brief heuristic argument on why he thought there should be infinitely many Carmichael numbers and, in fact, why they should be plentiful among all numbers. (A composite number n for which $a^n \equiv a \bmod n$ for all a is called a *Carmichael number*. In 1910 Carmichael conjectured there should be infinitely many.) Erdős and I discussed his heuristic argument several times over the years, and he was very pleased when Red Alford, Andrew Granville, and I succeeded recently in making it the backbone of a proof of the infinitude of Carmichael numbers. We were happy to dedicate our paper to Erdős on the occasion of his eightieth birthday.

I like to tell the story of how I first met Paul Erdős, since it is not only a good story, but shows a fundamental quality of Erdős as a man and a mathematician. I was home on April 8, 1974, watching a baseball game on television. I was then an assistant professor at the University of Georgia, less than a couple of years from graduate school, with few theorems but a love of numbers. This was not an ordinary baseball game, but the one in which Hank Aaron of the Atlanta Braves hit his 715th major league home run, thus surpassing the supposedly unbeatable record of 714 that had been set by Babe Ruth some four decades earlier.

I noticed that 714 and 715 have a peculiar property, namely, that their product is also the product of the first 7 primes. The next morning I challenged my colleague David Penney to find an interesting property of 714 and 715. He soon found the same thing I had, but he also posed the problem to his numerical analysis class, where a student came up with another interesting property: the sum of the prime factors of 714 is equal to the sum of the prime factors of 715. Working with another student, Carol Nelson, Penney and I found many other examples of consecutive pairs of numbers with this latter property and were able to come up with a strong heuristic for why there ought to be infinitely many. We wrote up our observations in a light-hearted article that was published several months later in the *Journal of Recreational Mathematics*. Calling 714 and 715 a "Ruth-Aaron pair", we conjectured that such pairs have density 0: that is, the set of n, such that the sum of the prime factors of n is equal to the sum of the prime factors of $n+1$, has density 0.

Paul Erdős, the giant of twentieth-century number theory, was also a reader of the *Journal of Recreational Mathematics.* He did not know me, nor should he have, but he wrote me a letter saying he could prove the conjecture that Ruth-Aaron pairs have density 0 and he would like to visit Georgia and discuss it with me. Much of what I now know in mathematics I learned from Erdős working on this and subsequent joint papers. It is fair to say that I owe my career to this serendipitous collaboration.

I am very grateful to have this chance to write these words in tribute to Paul Erdős. But I am cognizant of the vast amount of his work that did not get mentioned. In fact, I cannot close without giving four more delightful results.

Amicable numbers have been studied since Pythagoras: a pair m, n is *amicable* if the sum of the proper divisors of m is n, and vice versa. The first amicable pair with $m \neq n$ is $220, 284$. Paul Erdős [5] was the first to prove thatthe set of amicable numbers has density 0. It is still not known if there are infinitely many.

Can the product of consecutive integers be a power? This problem, with roots in the eighteenth century, was settled in the negative by Erdős and Selfridge [11] in 1975. It is still not known whether the Erdős-Selfridge theorem can be generalized to arithmetic progressions—the conjecture is that a product of four or more consecutive terms of a coprime arithmetic progression cannot be a power.

An *additive function* $f(n)$ is a real-valued function defined on the natural numbers with the property that $f(mn) = f(m) + f(n)$ whenever m and n are coprime. For example, the number-of-prime-factors function $\omega(n)$, mentioned above in connection with the Erdős-Kac theorem, is additive, as is the function $\log n$. In 1944 Erdős proved that if $f(n)$ is additive and $f(n+1) \geq f(n)$ for all large n, then $f(n) = c \log n$ for some number c. This also holds if one replaces the monotonicity assumption with $f(n+1) - f(n) \to 0$. Others, including Feller, Wirsing, Kátai, and Kovács, have added more to this theory of characterizing the logarithm as an additive function. For example, Wirsing [14] proved in 1970 the long-standing conjecture of Erdős that if $f(n)$ is additive and $f(n+1) - f(n)$ is bounded, then there is a number c such that $f(n) - c \log n$ is bounded.

Finally, I cannot resist describing the Erdős "multiplication table theorem." Let $M(n)$ be the number of distinct numbers in the $n \times n$ multiplication table. For example, in the familiar 10×10 multiplication table (at least familiar to those of us who did not grow up with calculators), there are 43 distinct numbers among the 100 entries, and so $M(10) = 43$. Erdős asked about the behavior of $M(n)/n^2$ as $n \to \infty$. What do you think it is? Clearly, since the multiplication matrix is symmetric, we have $\limsup M(n)/n^2 \leq 1/2$. Is $1/2$ the limit? Erdős showed [6] in 1960 that $M(n)/n^2 \to 0$ as $n \to \infty$, a theorem that I find as surprising as it is delightful. (Once one sees the proof, the surprise factor diminishes, though not the delight. As we saw before, most numbers up to n have about $\log\log n$ prime factors, and thus most products in the table have about $2 \log\log n$ prime factors. This is an abnormal number of primes for a number up to n^2, so there are not very many products.) We still do not have an asymptotic formula for $M(n)$ as $n \to \infty$, though from the work of Tenenbaum we have some good estimates.

This last result illustrates a most important point. At first glance one might think of the work of Erdős as a collection of unconnected and ad hoc results. Upon deeper inspection, especially of the proofs, one finds a glorious theory, with many interrelations of ideas and tools. It is this edifice of "Erdős-theory" that Paul Erdős leaves for us, and number theory is much the richer for it.

For more on the number theory of Paul Erdős, see [7, 9, 12, 13].

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Finite and Transfinite Combinatorics

László Babai

The Combinatorial Vein

A hallmark of much of Erdős's work is his unique *combinatorial vision*, which revolutionized several fields of mathematics. Wherever he looked, he found elementary, yet often enormously difficult, combinatorial questions.

Nothing serves as a better illustration of this point than the excitement his questions brought to the simplest concepts of *Euclidean plane geometry*: points, lines, triangles.

Consider a set of k points and t lines in the plane. What would Erdős ask about them? Many things, but perhaps the simplest question is this: what is the maximum number $f(k, t)$ of incidences between the points and the lines? After many years Szemerédi and Trotter (1983) confirmed Erdős's conjecture that the points of a square grid together with a certain set of lines give the optimal order of magnitude. The proof from *The Book* appeared in 1997 (L. Székely).

Another, even simpler, problem of Erdős asks the maximum number $g(n)$ of unit distances that can occur among n points in the plane. In 1946 Erdős proved that $n^{1+c/\log\log n} < g(n) < cn^{3/2}$. The upper bound was improved by Beck, Spencer, Szemerédi, and Trotter to $n^{4/3}$. Erdős conjectured that the lower bound, obtained from the square grid, has the correct order of magnitude. The problem remains wide open; the large gap between the upper and lower bounds offers a continuing challenge.

Let me quote a related open problem Erdős volunteered for the "Math Investigations" column of George Berzsenyi in the student journal *Quantum*: "Let $f(n)$ be the largest integer for which there is a set of n distinct points $x_1, x_2, \dots, x_n$ in the plane for which for every x_i there are $\geq f(n)$ points x_j equidistant from x_i. Determine $f(n)$ as accurately as possible. Is it true that $f(n) = o(n^\epsilon)$ for every $\epsilon > 0$?" Erdős offered \$500 for a proof and "much less" for a counterexample. The estimate $f(n) < cn^{2/5}$ follows from a result of Clarkson et al. (1990).

As these questions indicate, Erdős's combinatorics, as long as finite sets are concerned, is about *asymptotic orders of magnitude*. Asymptotic thinking has been common in number theory (especially in the study of the distribution of prime numbers), which was Erdős's first love. But it seems to be without precedent in combinatorics and in geometry. And even within number theory, Erdős's style brought about a new field, *combinatorial number theory*, an area expounded in hundreds of papers by Erdős (57 of them joint with A. Sárközy).

Combinatorial ideas appear already in Erdős's earliest work on number theory. Erdős was greatly influenced by a question he heard in 1934 from Fourier analyst Simon Sidon on sequences of integers with pairwise different sums. In a paper published in Tomsk (Siberia) in 1938 Erdős considers a multiplicative version of Sidon's problem: what is the maximum number $f(n)$ of positive in-

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Photograph by J. A. Bondy.

Two giants of combinatorics share a passion: Erdős and William T. Tutte play "Go" at Tutte's home in Westmontrose, Ontario, 1985. Another favorite game of Erdős's was Ping-Pong.

Erdős and Vera T. Sós in Princeton, 1985, during a period of intensive collaboration on extremal graph theory.

tegers $a_i \leq n$ such that all the pairwise products $a_i a_j$ are different? Erdős proves that $f(n) = \pi(n) + O(n^{3/4})$ by reducing the problem to the following lemma: If a graph on k vertices has no 4-cycle, then it has at most $O(k^{3/2})$ edges. This result is a precursor of extremal graph theory, another field Erdős set out to create more than a decade later. In showing the near-optimality of his error term, Erdős relies on the near-optimality of his graph theory bound, demonstrated by former fellow student Eszter (Esther) Klein using a projective plane of prime order (which she remarkably rediscovered from scratch).

Master of Patterns

Elementary geometry and Ramsey theory met in one of Erdős's earliest papers (1935), written with fellow undergraduate and lifelong friend, George Szekeres, then a student of chemical engineering. They proved that sufficiently many points in the plane necessarily include k points that form a convex k-gon. Later Erdős dubbed the question the "Happy Ending Problem": proposed by Eszter Klein and Erdős, the problem was first solved by Szekeres, who subsequently married Klein. "The wedding took place just a day after I learned that Vinogradov had proved the odd Goldbach conjecture," Erdős recalled in 1995.

Szekeres, in a remarkable tour de force, even rediscovered Ramsey's theorem, which was then only three years old, for his solution. This work represented a milestone in Erdős's combinatorial thinking. Erdős recognized the vast domain opened up by Ramsey's theorem, this "generalized pigeon hole principle." In the cited 1935 paper with Szekeres, Erdős studied Ramsey numbers for graphs, the first step in building what at the hands of Erdős would become *Ramsey theory,* a large area in finite and transfinite combinatorics. Transfinite Ramsey theory became a fundamental part of modern *set theory.*

One of Erdős's heroes was Georg Cantor; Erdős learned the basics of Cantor's set theory from his father. Erdős loved infinite cardinals and contributed to the birth of very large ones. (We use the term "transfinite" to emphasize that the focus is beyond ω, usually far beyond.)

Although the methods of the finite and the transfinite are almost disjoint (counting is fundamental in the former, well-ordering in the latter), it was Erdős's axiom that if a question makes sense both for finite and for infinite sets, it must be investigated in both domains. This view is especially prominent in his fifty-four (often massive) joint papers with A. Hajnal.

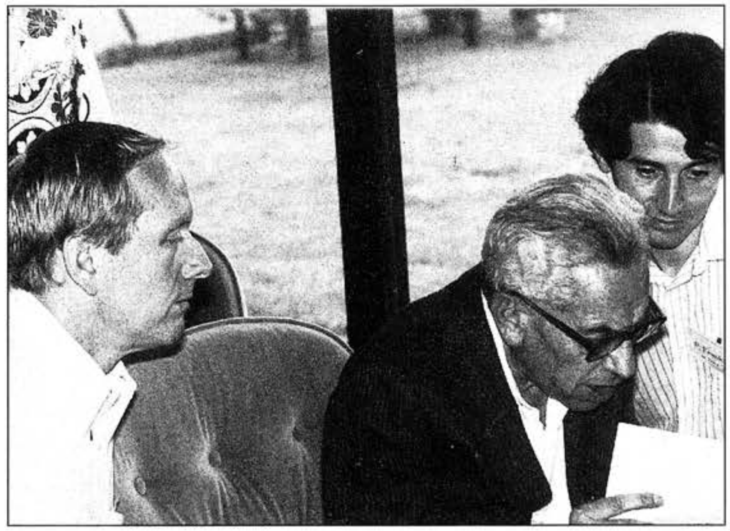
The *chromatic number* of a graph is the smallest number of colors that can be assigned to the vertices of the graph such that adjacent vertices receive different colors. Until the mid-1950s this concept was mostly discussed in the limited context of the 4-color conjecture. Erdős's work, which includes dozens of papers entitled "Chromatic graph theory," played a major role in establishing the true depth of the concept. Together with Hajnal, Erdős also pioneered the extension of this notion to set systems (no member of the set system should be monochromatic), creating one of the powerful unifying concepts of modern combinatorics.

One of the great early successes of Erdős's probabilistic method in finite combinatorics was his proof in 1957 that for any k, m there exists a graph of chromatic number k without cycles of length $\leq m$, putting an end to a long quest by the best combinatorial minds.

While it was relatively easy to get rid of short *odd cycles*, the short even cycles proved to be much harder to eliminate. A surprising explanation of this phenomenon came in a milestone paper by Erdős and Hajnal, "Chromatic number of graphs and set systems" (1966). Corollary 5.6, one of the paper's five dozen results, asserts: If the chromatic number of a graph is $\geq \aleph_1$, then the graph must contain a 4-cycle! Buried in this paper, which is alephs all over, is an important result of finite combinatorics: Erdős's cited result on large chromatic finite graphs without short cycles is generalized to set systems.

Erdős was fascinated by the global nature of the chromatic number. A striking expression of this is his 1962 result that to every k there is an $\epsilon > 0$ such that for all $n > k$ there exist k-chromatic graphs on n vertices such that all of their subgraphs on $\leq \epsilon n$ vertices are 3-colorable. As usual, he pursued the idea for infinite graphs as well and found that it led to a wealth of questions and surprising answers. With Hajnal, Erdős showed (1968) that there exist graphs of uncountable chromatic number on $(2^{\aleph_0})^+$ vertices such that all subgraphs on $\leq 2^{\aleph_0}$ vertices are countably colorable. (Here α^+ denotes the successor cardinal of α.) In the nicely shaping landscape, however, as in virtually all areas of Erdős's inquiry into the transfinite, "independence raised its ugly head" after Cohen's seminal work: a number of related questions turned out to be independent of ZFC (Zermelo-Fraenkel set theory with the axiom of choice), even under the Generalized Continuum Hypothesis. Among these is the question of the existence of an $\aleph_2$-chromatic graph with $\aleph_2$ vertices such that all subgraphs with $\leq \aleph_1$ vertices are $\aleph_0$-colorable (Baumgartner, 1984; Foreman and Laver, 1988).

If combinatorics is the art of *finding patterns under virtually no assumption*, Erdős was the master of this art. Here is a simple example. A set system $\{A_1, \dots, A_k\}$ is called a *sunflower with k petals* (or a Δ-system in the original terminology of Erdős and Rado) if all pairwise intersections $A_i \cap A_j$ are equal to $\bigcap_{i=1}^{k} A_i$. Erdős and Rado recognized the significance of this simple pattern in 1960 and showed that for any k and r, any sufficiently large family of sets of size r contains a sunflower with k petals. Erdős offered \$1,000 for deciding whether C^r is sufficiently large to guarantee $k = 3$ petals



Photograph by Michio Kano.

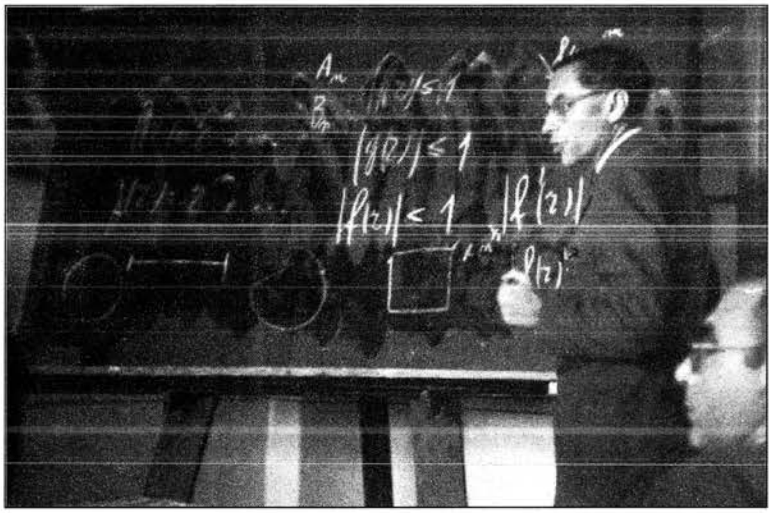
Ronald L. Graham (left), Erdős, Peter Frankl finishing a paper on "anti-Ramsey graphs" at a conference in Hakone, Japan, 1990. The interests shared by Graham and Frankl include solving problems of Erdős, helping Erdős's influence spread in the Orient, and juggling (both are world-class jugglers).

for some (large) C. The problem remains wide open to date. At the hands of Frankl and others, sunflowers have become a powerful tool in the structural theory of set systems; Razborov used them in a profound lower-bound proof in the theory of Boolean circuits.

Ramsey theory provides the ultimate in the quest for simple patterns. Assume we color each r-subset of a set S of cardinality κ red or blue. We say that a subset $H \subseteq S$ is *homogeneous* if all r-subsets of H have the same color. The Erdős-Rado symbol $\kappa \to (\alpha, \beta)^r$ means that regardless of the coloring, there must be either a red-homogeneous subset of size α or a blue-homogeneous subset of size β. We omit β if $\beta = \alpha$. Ramsey's theorem states that $\aleph_0 \to (\aleph_0)^r$ for every finite r. Its finite version says that $N \to (k)^r$ for sufficiently large finite $N = N(k, r)$. The estimation of the quantities $N(k, r)$ is a major problem area. Here is an example of a tantalizing gap: it is known that $n \to (c_1 \log\log n)^3$ (Erdős-Rado, 1952) and $n \not\to (c_2 \sqrt{\log n})^3$ (from the 100-page "giant triple paper" by Erdős, Hajnal, and Rado (1965)).

Partition calculus, the term Erdős and Rado (1956) used for transfinite Ramsey theory, started with a result of Erdős that $\kappa \to (\kappa, \aleph_0)^2$, included in a 1941 paper by Dushnik and Miller. Shortly afterwards, Erdős proved the basic result that $(2^\lambda)^+ \to (\lambda^+)^2$ (1942) and noted that by a result of Sierpiński this bound is tight: $(2^\lambda) \not\to (\lambda^+)^2$.

The fact that innocuous problems of transfinite combinatorics lead to *inaccessible* cardinals was a stunning discovery made in a 1943 paper by Erdős and Tarski, especially famous for its footnotes. Regarding the simplest of partition relations, $\kappa \to (\kappa)^2$, they recognized that it cannot hold unless κ is *strongly inaccessible* (κ is not the sum of



Paul Erdős talks about functions of a complex variable in Hungary, 1959. Alfréd Rényi is looking on. The landmark Erdős–Rényi papers "On the evolution of random graphs" were born around this time. Erdős and Rényi worked together on a wide variety of other subjects as well, including the distribution of prime numbers, complex functions, the theory of series, probability theory, statistics, information theory, and statistical group theory.

fewer, smaller cardinals, and $\alpha < \kappa$ implies $2^{\alpha} < \kappa$), and it does hold if κ is *measurable* (admits a nontrivial $< \kappa$-additive (0,1)-measure defined on all subsets of κ). Out of these observations, the *theory of large cardinals*, a vital component of modern set theory, was born. Cardinals satisfying $\kappa \to (\kappa)^2$ are called *weakly compact cardinals.*

Another important class of large cardinals grew out of Erdős's first joint paper with then graduate student András Hajnal (1958). Erdős and Hajnal proved that measurable cardinals satisfy the partition relation $\kappa \to (\kappa)^{<\omega}$ (all finite sets are colored). This relation defines what are called *Ramsey cardinals.* Amazing consequences of the weaker relation $\kappa \to (\omega_1)^{<\omega}$ to *descriptive set theory* were found in the mid-1960s by F. Rowbottom, J. H. Silver, R. Solovay, and others. In recognition of Erdős's pioneering role in defining large cardinals via partition relations, the cardinals satisfying the relation $\kappa \to (\omega_1)^{<\omega}$ are commonly referred to as *Erdős cardinals.*

We stated Ramsey's theorem for two colors; the generalization to a finite number of colors is immediate. It is clear, however, that no homogeneous subset is to be expected unless the set is "large" compared to the number of colors. Nevertheless, one of a short list of *canonical structures* exists regardless of the number of colors! In the simplest case ($r = 2$) there are only three types of canonical structure: a homogeneous set (all pairs have the same color), a multicolored set (each pair has a different color), and the min-coloring: the color of $\{i, j\}$ is $\min(i, j)$ (assuming the set is well ordered). This very useful fact is the Canonical Ramsey Theorem of Erdős and Rado (1950).

In a series of papers starting in 1973 Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus laid the foundations of *Euclidean Ramsey Theory*, melding Ramsey theory to the geometry of real n-space. The typical question is this: given a geometric configuration K, is it true for all r that any r-coloring of n-space contains a monochromatic copy of K, assuming $n \geq n_0(r)$? If K has this property, we say that K is a *Ramsey configuration.* Erdős et al. have shown that the bricks are Ramsey and sets that cannot be inscribed in a sphere are not Ramsey. A major problem left open in their work was settled by Frankl and Rödl in 1986: all triangles are Ramsey.

Much of Erdős's work concerns the paradigm that "*density implies pattern.*" The most famous of Erdős's solved prize problems (\$1,000) asserts that a sequence of integers of positive upper density contains arbitrarily long arithmetic progressions. Proposed by Erdős and Turán in 1936, this conjecture was confirmed in 1975 in "a masterpiece of combinatorial thinking" [4] by Endre Szemerédi, a disciple of Erdős and one of the most formidable problem solvers of our time. Subsequently H. Fürstenberg gave an ergodic proof. Szemerédi's proof builds on his Regularity Lemma, which has far-reaching consequences in graph theory; the method of Fürstenberg's proof gave a new direction to ergodic theory. This is but one of the long list of examples demonstrating the profound relevance of the problems championed by Erdős.

A great many problems in combinatorial number theory have a flavor similar to the problem of arithmetic progressions. Young Erdős was captivated by Sidon's 1934 problem, which asks how dense a set of integers can be if all pairwise sums are different[1]. If we denote by $A(n)$ the number of elements $\leq n$ in such a sequence, it is clear that $A(n) \leq c_1 n^{1/2}$, and "greedy" choice results in a sequence with $A(n) \geq c_2 n^{1/3}$. Only very recently (1997) did Erdős protégé Imre Ruzsa succeed in substantially reducing this sixty-year-old gap. Ruzsa has shown the existence of a Sidon sequence such that $A(n) \geq n^{\alpha - \epsilon}$ with $\alpha = \sqrt{2} - 1 \approx .41$.

In extremal graph theory the fundamental Erdős-Stone-Simonovits theorem (1946, 1966) considers the minimum edge density of graphs that will

[1] *One can hardly overestimate the influence of Sidon's problems on Erdős's career. It is remarkable how the twenty-year-old Erdős's irresistible insistence on mathematical communication virtually compelled Sidon, a reclusive man employed by an insurance company, to reveal his remarkable thoughts to the eager youth. A classic anecdote: One afternoon, when Erdős and Turán showed up at Sidon's doorstep, Sidon opened the door a crack and greeted the two with these words: "Please visit another time and* especially *another person."*

force the appearance of a fixed "pattern" subgraph H. It turns out that asymptotically, the density depends solely on the chromatic number of H! This in particular implies that the set of critical limiting densities is well ordered. Erdős asked whether this fact generalizes from graphs to systems of r-sets ("r-hypergraphs") for $r \geq 3$; his second (and last) \$1,000 award went to Frankl and Rödl for their negative answer, "Hypergraphs don't jump" (1984).

Combinatorics and Probability

Erdős was not versed in probability theory at the time he arrived in Princeton in 1938. He was not even familiar with the central limit theorem. Yet he deeply understood it in a flash when he first heard about it in a lecture by Mark Kac; by the end of the talk he had completed the proof of Kac's conjecture on the normal (Gaussian) distribution of the number of prime divisors of integers. While this result gave birth to probabilistic number theory, Erdős went on and made major contributions to probability theory itself, especially the theory of random walks and Brownian motion. He worked with Kac, K. L. Chung, Dvoretzky, Kakutani, among others, in these areas. Erdős's best-known result in probability theory is a full asymptotic expansion of the *law of the iterated logarithm* (1942).

A "statistical view" of mathematical objects was one of Erdős's key innovations in many areas of mathematics. Following work by Goncharov in the 1940s, Erdős and Turán developed *statistical group theory*, a study of the distribution of various sets of parameters associated with a group, in a series of seven highly technical papers between 1965 and 1976. They showed, for instance, that the logarithms of the orders of elements in the symmetric group S_n are asymptotically normally distributed.

The foundations of a beautiful statistical theory of combinatorial structures were laid in the landmark study by Erdős and Rényi of the "*evolution of random graphs*" in a series of seven papers between 1959 and 1968.

Let us construct a "random" graph with n vertices and m edges by picking the edge set uniformly at random from the set of $\binom{\binom{n}{2}}{m}$ possibilities. Erdős and Rényi observed the typical behavior of these graphs as a function of $m = m_n$ and determined very sharp thresholds for various monotone properties to become "typical." For instance, connectedness occurs around $m_n = n \log n/2$; in fact, they prove that if $(m_n/n) - (\log n/2) \to c$, then the probability of connectedness approaches $e^{-e^{-c}}$.

The most striking discovery of Erdős and Rényi was a *phase transition* which occurs around $m_n = n/2$: suddenly, a *giant component* appears. If $m_n < (1 - \epsilon)n/2$, then typically all connected components of the graph are of size $O(\log n)$ and have very simple structure. But when $m_n > (1 + \epsilon)n/2$, the largest component has size $> c(\epsilon)n$, while all other components remain of logarithmic size and are absorbed into the giant component as m_n increases. Béla Bollobás took the lead in a 1984 paper in uncovering the fine structure of this phase transition, the study of which has yielded a series of remarkable insights and is continuing to this day.

Much of Erdős's work had an impact on the *theory of computing*, a field in which Erdős never took an interest [2]. Richard M. Karp writes: "The Erdős-Rényi papers on random graphs exerted major influence on my work. The beautiful scenario of the successive stages in the evolution of random graphs, progressing in an essentially inevitable way, has stimulated me to find other stochastic processes, associated with algorithms, which unfold in the same kind of inevitability. Researchers have exhibited such processes in connection with many problems related to graphs, Boolean formulas and other structures. Specific results related to random graphs have been applied to hashing, storage allocation, load balancing and other problems relevant to algorithms and computer systems."

Probabilistic Proof of Existence

Among the numerous techniques Paul Erdős taught us, perhaps the *probabilistic method* has been the most influential. This method establishes the existence of certain objects by selecting an object at random from a certain probability space and proving that the object has the desired properties with positive (usually overwhelming) probability. While Erdős was not the first to employ an idea of this type, it was he who recognized its vast scope and developed it into a powerful technique.

Erdős first demonstrated the power of this method in 1947 by proving that his Ramsey bound with Szekeres, $n \to (c \log n)^2$, is tight apart from the constant c; i.e., there exists a graph on n vertices without homogeneous subsets (clique or independent set) of size $c_1 \log n$. The probability bound is obtained by generously overestimating, via simple counting, the number of graphs which do *not* have the desired property.

This *non-constructive proof of existence* immediately raised the challenge of an *explicit construction*. Frankl (1977) was the first, via the sunflower technique, to construct explicit graphs without homogeneous subsets of size n^ϵ; an elegant alternative proof was given by Frankl and Wilson using the linear algebra method (1981). Their bounds on homogeneous subsets are, however, still far from logarithmic.

Another great success of the probabilistic method was Erdős's cited result on the existence of graphs with large chromatic number and without short cycles (1959). In this case a mere random choice alone will not suffice; the random graph obtained from a carefully chosen distribution needs to be modified in order to satisfy the conditions.

The *derandomization* of this result required major effort and was eventually successful in simultaneous work by Margulis and Lubotzky-Phillips-Sarnak (1988). A key ingredient is the theory of diophantine equations of the form $x^2 + 4q^2(y^2 + z^2 + w^2) = n$ (Ramanujan conjecture, solved for this case by Eichler (1954) and Igusa (1959)).

Erdős applied the probabilistic method in many other contexts, in combinatorics as well as in number theory, geometry, and analysis. Let me state a beautiful example from analysis: In a 1959 paper with A. Dvoretzky, Erdős demonstrated the existence of a power series of the form $\sum_1^\infty e^{i\alpha_n} n^{-1/2} z^n$ with real α_n that diverges on the whole unit circle.

To derandomize another probabilistic proof of Erdős in graph theory, Graham and Spencer (1971) invoked André Weil's character sum estimates, which imply, that, in some sense, "quadratic residues are random." Recently (1996) Kollár, Rónyai, and Szabó employed the elements of commutative algebra to derandomize, for infinitely many values of the parameters, a result of Erdős in extremal graph theory.

Our limited experience thus indicates that algebraic tools of considerable depth may hold the key to replacing probabilistic proofs of existence by explicit construction. In most cases, however, the Erdős-style proofs of existence cannot currently be matched by explicit constructions, a challenge that continues to grow with the increasing number of applications of the probabilistic method [1].

Why should we care about derandomizing probabilistic proofs of existence? The combinatorist may find the challenge and the beauty of the question inspiring. The reasons, however, run considerably deeper in the theory of computing. The central objective of that area is to show the intrinsic computational difficulty of *explicit functions* (the difficulty of computing a random function being evident).

The probabilistic method is most often used to demonstrate the existence of objects that are actually present in abundance. For instance, a random graph is very likely to have the right Ramsey parameter. Is the method doomed to fail when searching for rare objects? A coloring problem for set systems led Erdős and Lovász to the discovery of the Local Lemma (1974), a powerful tool to detect events of low but nonzero probability. Informally the lemma asserts that the intersection of a set of events, none of which correlates with more than a small number of others, is nonempty. This will demonstrate the existence of certain exponentially rare objects. We note that naive sampling will not encounter these objects. Algorithmic versions of this result, which actually find "the needle in the haystack," were obtained by J. Beck and subsequently by N. Alon (1991).

Erdős not only set up derandomization challenges but also invented an important derandomization tool in a 1973 paper with John Selfridge: the "method of conditional expectations." The method has since been extended and resulted in the derandomization of large classes of randomized algorithms.

For decades hardly anyone other than Erdős recognized the significance of the probabilistic method. The situation changed with the 1974 publication of *Probabilistic Methods in Combinatorics* by Erdős and Joel Spencer. This thin volume had a major impact on all areas of discrete mathematics.

References. We refer as [M4] to bibliography item [4] of the memorial article by this writer appearing elsewhere in this issue. Most papers of Erdős cited above can be found either in *The Art of Counting* [M4] or the bibliographies of the survey articles in [M12]. An enormous amount of relevant material appears in various chapters of the monumental *Handbook of Combinatorics* ([3] below). Further references:

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Approximation Theory

Péter Vértesi

Paul Erdős wrote more than a hundred papers that are closely related to the approximation of functions. It is difficult to outline the wealth of these results in such a short survey; the selection necessarily reflects my taste.

As a Ph.D. student of Lipót (Leopold) Fejér, Erdős began to work on problems closely related to interpolation and orthogonal polynomials: mean convergence, quadrature processes, investigations of normal point systems, and generally to reveal connections between the distribution of certain

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node systems and the behavior of the generated processes. These questions have been in the main stream of classical approximation theory.

However, within a very short time, Erdős began to formulate his own problems and outlined new paths to search, such as the closer investigation of the Lebesgue function of Lagrange interpolation, questions on the optimal Lebesgue constant, and rough and fine theories of different approximating tools. The first paper in his series "Problems and results on the theory of interpolation" appeared in 1958, followed by periodic updates (1961, 1968, 1976, 1980, 1983, 1991).

Over the years Erdős obtained (mainly with coauthors) fundamental and very strong theorems. We may mention the 1980 result on the a.e. (almost everywhere) divergence of Lagrange interpolation on an arbitrary system of nodes[2], the Erdős condition on convergent interpolatory processes (with A. Kroó and J. Szabados, 1989), and the results on a.e. divergence of the arithmetic mean of Lagrange interpolation based on Chebyshev nodes (with G. Grünwald, 1937; an error in their proof was eliminated in a paper with G. Halász, 1991).

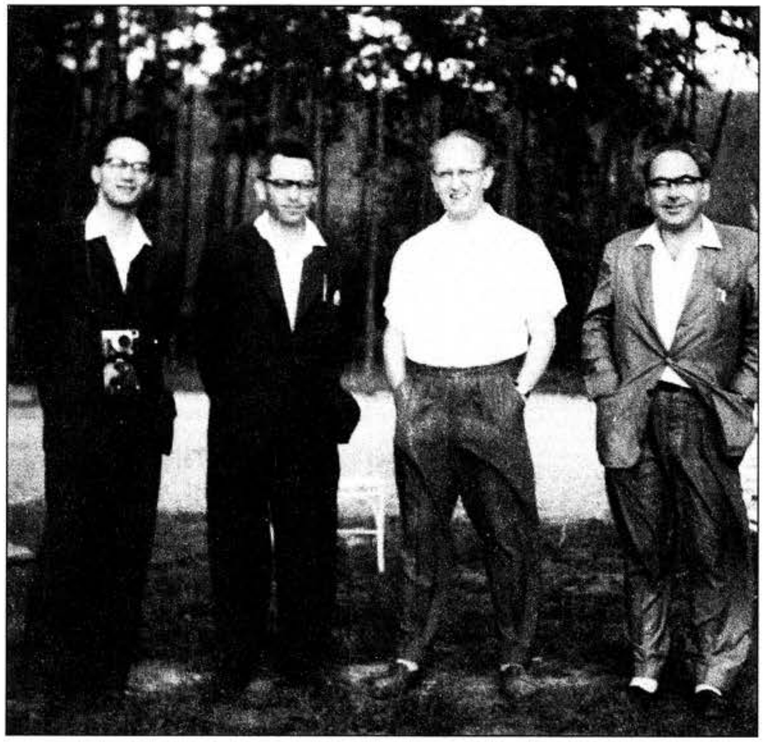
Questions in approximation theory are closely related to the behavior of polynomials. So it is no surprise that Erdős wrote many papers dealing with the related problems about polynomials (Remez and other inequalities, the distribution of roots, length of polynomials, geometry of polynomials, etc.). Rather than going into the details of this subject, I conclude this survey with two influential "appetizers" and refer the interested reader to the monographs [4, 5].

Erdős's work invariably attracted a great deal of attention and continues to influence the work of many mathematicians. This survey includes several lists of authors inspired by specific results of Erdős. References to their papers can be found in the bibliographies of the works listed in our "References."

One of the most often-quoted results in approximation theory appeared in a 1937 paper by Erdős and Paul Turán in the *Annals of Math.* In this inaugural opus of their 3-piece series "On interpolation" the young authors proved the remarkable positive result that for *any* continuous function f the Lagrange interpolation polynomials *converge in mean* to f if the interpolation is taken over the roots of the system of orthogonal polynomials with respect to *any weight function.* More precisely they proved that *for every $f \in C$ and weight w*

$$(1) \quad \int_{-1}^{1} \{f(x) - L_n(f, w, x)\}^2 w(x)\, dx \le \sqrt{6} E_{n-1}(f).$$

[2]*Note by the organizer: The coauthor of this striking result is P. Vértesi.*



Erdős enjoyed working with several mathematicians on entirely different problems simultaneously. Left to right: G. Grätzer, Erdős, Paul Turán, and Alfréd Rényi at Dobogókő, Hungary, 1959. Turán was one of Erdős's closest friends and his first major collaborator. Erdős and Turán worked together on a variety of subjects in number theory, classical analysis, combinatorics, and statistical group theory.

Here $X = \{x_{kn};\ 1 \le k \le n;\ n \in \mathbf{N}\} \subset I := [-1, 1]$ is an *interpolatory matrix* (i.e., for fixed n, x_{kn} are different); $L_n(f, X, x) \in \mathcal{P}_{n-1}$ is the n-th Lagrange interpolatory polynomial based on the nodes $\{x_{kn}\}$, $1 \le k \le n$; w is a weight on $[-1, 1]$, i.e., $w \ge 0$ and $\int_I w > 0$; if $\{x_{kn} = x_{kn}(w)\}$ where $x_{kn}(w)$, $1 \le k \le n$ are the roots of the n-th orthonormal polynomial $(p_n(w))$ with respect to w, $n \in \mathbf{N}$ (i.e., $\int_I p_n(w) p_m(w) w = \delta_{nm}$), then $L_n(f, w)$ replaces $L_n(f, X)$; finally, $E_n(f) = \min_{P \in \mathcal{P}_n} \|f - P\|$ where $\|\ldots\|$ stands for the maximum norm on I. Note that by Weierstrass's Theorem, the right-hand side converges to 0 as $n \to \infty$.

To appreciate this mean-convergence theorem, we state a fundamental negative result of G. Faber (1914), which says that *for every $X \subset I$ there is an $f \in C$ with*

$$(2) \qquad \limsup_{n \to \infty} \|L_n(f, X, x)\| = \infty.$$

A natural question that challenged many mathematicians was to replace the exponent 2 with a larger one. Such results are known for special matrices. For instance, for the case of $X = T = \left\{\cos \frac{2k-1}{2n} \pi\right\}$ (Chebyshev matrix) Erdős and Feldheim proved in 1936 that

$$\lim_{n\to\infty}\int_{-1}^{1}|f(x)-L_n(f,T,x)|^p\frac{1}{\sqrt{1-x^2}}\,dx=0$$

holds for any continuous f with *arbitrary* $p>0$. The corresponding trigonometric case is due to J. Marcinkiewicz. As it turned out almost forty (!) years later, however, generally the exponent 2 cannot be improved. This nice result is due to P. Nevai. Similar problems were considered by, among others, R. Askey, V. M. Badkov, B. Della Vecchia, G. Freud, G. Mastroianni, B. Muckenhoupt, D. S. Lubinsky, A. K. Varma, and Y. Xu (cf. [2]).

Lebesgue estimated the difference $L_n(f)-f$ by

$$|L_n(f,X,x)-f(x)|\le\{\lambda_n(X,x)+1\}E_{n-1}(f). \tag{3}$$

Here the *n-th Lebesgue function* $\lambda_n(X,x)$ is defined as $\lambda_n(X,x):=\sum_{k=1}^{n}|\ell_{kn}(X,x)|$, where the $\ell_{kn}\in\mathcal{P}_{n-1}\setminus\mathcal{P}_{n-2}$ are the (unique) fundamental polynomials corresponding to X (i.e., $\ell_{kn}(X,x_{jn})=\delta_{kj},\ 1\le k,\ j\le n,\ n\in\mathbf{N}$). Relation (3) shows that the Lebesgue function $\lambda_n(X,x)$ and the *Lebesgue constant* $\Lambda_n(X):=\|\lambda_n(X,x)\|$ play a fundamental role concerning the convergence-divergence behavior of Lagrange interpolation.

In the seminal first paper in the "Problems ..." series (1958), Erdős proved that *for any fixed* $X\subset[-1,1]$, *real* $\varepsilon>0$, $A>0$, *the measure of the set for which*

$$\lambda_n(X,x)\le A,\qquad x\in\mathbf{R},\quad n\ge n_0(A,\varepsilon), \tag{4}$$

is less than ε.

The basic ideas of this work were used, developed, and completed by Erdős and many (co)authors (G. Halász, D. Newman, J. Knoppenberger, J. Szabados, A. K. Varma, P. Vértesi, Y. G. Shi) in a series of papers. These papers resulted in more or less "best possible" theorems on the behavior of $\lambda_n(X,x)$ and similar expressions, and they gave far-reaching generalizations of the Faber theorem and the Grünwald-Marcinkiewicz result (see the papers highlighted in the fourth paragraph of this survey and further references in [1]).

It is natural to investigate the sequence

$$\Lambda_n^*:=\min_{X\subset I}\Lambda_n(X),\qquad n\in\mathbf{N}. \tag{5}$$

In the rather difficult second paper of the "Problems ..." series (1961) Erdős, improving on a joint result with P. Turán (1961), obtained the bound

$$|\Lambda_n^*-\frac{2}{\pi}\log n|\le c,\qquad n\ge n_0, \tag{6}$$

but the famous Bernstein-Erdős conjectures on the optimal matrix X^* for which $\Lambda_n(X^*)=\Lambda_n^*$ and on the behavior of $\lambda_n(X^*,x)$ were proved only in 1978 (T. Kilgore, C. deBoor, A. Pinkus, L. Brutman (cf. [1])). The bound (6) also attracted much interest; a long list of papers on this subject is cited in [1].

Let us now consider functions f satisfying the Lipschitz condition $|f(x)-f(y)|\le c|x-y|^\alpha$ with some constant c; $Lip(\alpha)$ denotes the class of such functions. For $f\in Lip(\alpha)$, $0<\alpha<1$, (3) yields the bound

$$\|L_n(f,X)-f\|\le cn^{-\alpha}\Lambda_n(X). \tag{7}$$

In their 1955 joint paper "On the role of the Lebesgue function in the theory of Lagrange interpolation" Erdős and Turán established the following surprising facts.

Let us suppose that $\Lambda_n(X)\sim n^\beta$ $(\beta>0)$. *Then if* $\alpha>\beta$, *we have uniform convergence for any* $f\in Lip(\alpha)$; *if* $\alpha<\frac{\beta}{\beta+2}$, *then for some* $f_1\in Lip(\alpha)$, $\|L_n(f_1,X)\|$ *is unbounded as* $n\to\infty$. *However, if* $\frac{\beta}{\beta+2}<\alpha<\beta$, *then both convergence and divergence can happen.*

This means that in the third case the convergence-divergence behavior of $L_n(f,X)$ is not determined by the order of $\Lambda_n(X)$ alone; we have to take a closer look at the matrix X itself. Erdős and Turán refer to the interval $[\beta/(\beta+2),\beta]$ as the domain of a *finer theory* and point out a number of analogous situations for further study.

This is an extremely influential work. Over the years dozens of papers tried to settle corresponding questions (rough and fine theory) for the trigonometric case, other operators, Hermite-Fejér interpolation, etc. (cf. [1]).

Now here are two results on polynomials. According to the Bernstein-Markov inequality,

$$|p_n'(x)|\le\min\left(\frac{n}{\sqrt{1-x^2}},n^2\right)\cdot\|p_n\|,\qquad |x|\le1,\quad p_n\in\mathcal{P}_n. \tag{8}$$

However, as Erdős has shown in a short paper (1940), we can do better if we restrict the zeros of the polynomial. Namely, *if* $p_n\in\mathcal{P}_n$ *has no root in* $(-1,1)$, *then*

$$|p_n'(x)|\le\min\left(\frac{4\sqrt{n}}{(1-x^2)^2},\frac{en}{2}\right)\cdot\|p_n\|,\qquad |x|\le1. \tag{9}$$

This result was one of the starting points of investigations on polynomials with restricted zeros and initiated many interesting general problems (cf. the works of P. Borwein, T. Erdélyi, M. von Golitschek, G. G. Lorentz, Y. Makovoz, A. Máté, J. Szabados, A. K. Varma, and others mentioned in the monograph [3]).

Let me close this survey with comments on another short paper by Erdős, "On the distribution of roots of orthogonal polynomials." In this 1972 paper, which represents a bridge between approximation theory and polynomials, Erdős showed that certain weights w with *infinite* support have the so-called arcsine distribution; i.e., the distribution of the contracted zeros of the corresponding orthogonal polynomials is similar to the root-distribution of the Chebyshev polynomials ("Erdős-type weights," as they are referred to today[3]).

During the last fifteen to twenty years investigations of so-called weighted approximations on $\mathbb{R}$ (approximating $f(x)w(x)$ by $p_n(x)w(x)$, $p_n \in \mathcal{P}_n$, $x \in \mathbf{R}$) have been very intensive; many approximating tools and formulae were developed (mainly for Erdős- and Freud-type weights) by G. Freud, A. Levin, D. S. Lubinsky, H. N. Mhaskar, P. Nevai, E. A. Rahmanov, E. B. Saff, J. L. Ullman, V. Totik, R. S. Varga, and their students (cf. the monograph [5] and the references therein). The next stage should be the investigation of the previously mentioned problems concerning *weighted* interpolation. The first steps have already been done; they clearly mark that Paul Erdős's ideas are very much alive.

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[3]*Let* $w(x) = e^{-Q(x)}$. *If* $Q(x) = |x|^{\alpha}$, $\alpha > 1$, $x \in \mathbf{R}$ *(Freud-weight), then* w *is* not *arcsine; on the other hand for* $Q_k(x) = \exp(\exp(\ldots\exp(|x|^{\alpha})\ldots))$ *($k \geq 1$ times),* $\alpha > 0$, $x \in \mathbf{R}$ *(Erdős-weight),* w *is* arcsine *(cf. Erdős-Freud (1974)).*