Gauss and the Arithmetic-Geometric Mean

by David A. Cox

In 1828, after seeing the first works of Abel and Jacobi on elliptic functions, Gauss wrote: "I shall most likely not soon prepare my investigations on transcendental functions which I have had for many years... . Herr Abel has now, I see, anticipated me and relieved me of the burden in regard to one third of these matters..." [**3**, X. 1, page 248]. This is a tantalizing quote—one wonders what Gauss knew that Abel and Jacobi did not. What do the other two thirds "of these matters" refer to? Part of the answer is well known, for Gauss's collected works show that he anticipated much of the later work on modular functions done by Riemann, Klein, and others. But a considerable portion of Gauss's unpublished work in this area dealt with the arithmetic-geometric mean. The agM (this is Gauss's abbreviation) is not widely known today, but for Gauss it lay at the very center of his study of elliptic and modular functions. This article will sketch the theory behind the agM and indicate briefly some of its history and applications.

1. To define the agM we begin with the familiar inequality $(a+b)/2 \geq (ab)^{1/2}$ between arithmetic and geometric means. Notice that from one pair of positive numbers, a and b, we obtain a second pair, $(a+b)/2$ and $(ab)^{1/2}$. If we iterate this process, we obtain sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ defined by

$$(1)\qquad \begin{aligned} a_0 &= a, & b_0 &= b,\\ a_{n+1} &= (a_n+b_n)/2, & b_{n+1} &= (a_n b_n)^{1/2}. \end{aligned}$$

The above inequality enables one to show easily that these sequences converge to a common limit $M(a,b)$, which we define to be the agM of a and b.

One of the most striking features of the agM algorithm is how rapidly it converges. For example, the computation of $M(\sqrt{2},1)$ begins as shown in the table below:

n	a_n	b_n
0	1.414213562373905048802	1.000000000000000000000
1	1.207106781186547524401	1.189207115002721066717
2	1.198156948094634295559	1.198123521493120122607
3	1.198140234793877209083	1.198140234677307205798
4	1.198140234735592207441	1.198140234735592207439

The entries are rounded off to twenty-one decimal places. Since $M(a,b)$ always lies between a_n and b_n, we see that to nineteen decimal places

$$M(\sqrt{2},1) = 1.1981402347355922074.$$

Such accuracy is easily obtained these days, though not without some efort, since we went beyond the usual sixteen digits of double precision. More surprising is the fact that Gauss did these calculations (correctly!) over 180 years ago (see [**3**, III, page 364]).

The theory of the agM begins with its relation to elliptic integrals. The key result is the formula

$$(2)\quad M(a,b)\int_0^{\pi/2}(a^2\cos^2\phi+b^2\sin^2\phi)^{-1/2}\,d\phi = \frac{\pi}{2}.$$

To prove this let $\mu = M(a,b)$, and let $I(a,b)$ denote the above integral. Following Gauss [**3**, III, page 352], we introduce a new variable ϕ' defined by

$$(3)\qquad \sin\phi = \frac{2a\sin\phi'}{a+b+(a-b)\sin^2\phi'}.$$

It can be shown that

$$\begin{aligned}(a^2\cos^2\phi+b^2\sin^2\phi)^{-1/2}\,d\phi\\ = (a_1^2\cos^2\phi'+b_1^2\sin^2\phi')^{-1/2}\,d\phi',\end{aligned}$$

and $I(a,b) = I(a_1,b_1)$ follows. Thus $I(a,b) = I(a_1,b_1) = \cdots = I(\mu,\mu) = \pi/2\mu$ since the a_n's and b_n's converge to μ.

We thus have a very efficient method for computing elliptic integrals, and this is the most common application of the agM. A nice example is a recent paper of Buhler, Gross, and Zagier [**1**] concerned with numerical evidence for the conjectures of Birch and Swinnerton-Dyer for a particular elliptic curve of rank 3 over **Q**. Very delicate computations have to be made regarding the behavior of a certain L-series at $s=1$, and one of the factors of the L-series is the real period

Ω of the elliptic curve (see [9]). Since the curve in question can be written $y^2 = (x-e_1)(x-e_2)(x-e_3)$ with $e_1 < e_2 < e_3$, from (2) one can show that

$$\Omega = 4\int_{e_3}^{\infty} \frac{dx}{y} = \frac{2\pi}{M(\sqrt{e_3-e_1}, \sqrt{e_2-e_1})}.$$

The agM thus gives a wonderfully quick method for finding Ω.

Historically, this computational method was known before the agM was. This seems impossible, but Lagrange, in his 1785 paper on calculating elliptic integrals [5], defined the sequences (1) and showed how to use them to compute integrals. While he noted that the sequences had a common limit, he did nothing with this observation. Only six years later, Gauss, then fourteen, independently discovered the sequences (1) and defined the agM for the first time (although he did not see the connection with elliptic integrals until 1799). In this context it is amusing to note that the authors of [1] only recently realized that the agM was the best way to compute the elliptic integrals in question.

A more unusual computational application of the agM was discovered by Salamin in 1973. Consider Legendre's relation between complete elliptic integrals of the first and second kinds:

$$KE' + K'E - KK' = \pi/2,$$

where

$$K = K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2\phi)^{-1/2}\, d\phi,$$

$$E = E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2\phi)^{1/2}\, d\phi$$

and $K' = K(k'), E' = E(k')$ for $k' = (1-k^2)^{1/2}$. From (2) one easily sees that

$$K = \pi/2M(1, k').$$

Also, using the substitution (3), one can show that

$$E = \left(1 - \sum_{n=0}^{\infty} 2^{n-1} c_n^2\right) K,$$

where $c_n = (a_n^2 - b_n^2)^{1/2}$, and a_n and b_n are as in (1) with $a = 1$ and $b = k'$. (This formula was known to Gauss—see [3, III, page 353].) Setting $k = k' = 1/\sqrt{2}$, the above equations give us a formula for π:

$$\pi = \frac{2M(\sqrt{2}, 1)^2}{1 - \sum_{n=1}^{\infty} 2^{n+1} c_n^2}.$$

This might seem a strange way to compute π, but Salamin shows that it is very efficient (see [8]). Recently, some computer scientists in Japan have used this method to compute π to 10,013,395 decimal places, a world record (see [11]).

Purely theoretical applications of (2) are also known. For example, Reyssat recently proved in [7] that $M(a, b)$ is transcendental whenever a and b are distinct, positive, algebraic numbers. He has also proved that $M(\sqrt{2}, 1)$ and π are algebraically independent. (More generally, $M(1, \sqrt{1-\lambda})$ and π are algebraically independent whenever $-1 \le \lambda < 1$ is an algebraic number such that the elliptic curve $y^2 = x(x-1)(x-\lambda)$ has complex multiplication.)

The final application we give concerns the arc length of the lemniscate $r^2 = \cos 2\theta$. As is well known, the arc length is given by

$$4\int_0^1 (1 - z^4)^{-1/2}\, dz = 4\int_0^{\pi/2} (2\cos^2\phi + \sin^2\phi)^{-1/2}\, d\phi.$$

Using (2) we see that the arc length equals $2\pi/M(\sqrt{2}, 1)$. Gauss, in his development of lemniscatic functions, used the notation $\tilde{\omega} = 2\int_0^1 (1 - z^4)^{-1/2}\, dz$. Then the formula for arc length can be stated as

$$M(\sqrt{2}, 1) = \pi/\tilde{\omega}.$$

To see the importance of this equation we turn to the 98th entry in Gauss's mathematical diary. It is dated May 30, 1799 and reads as follows: "We have established that the arithmetic-geometric mean between 1 and $\sqrt{2}$ is $\pi/\tilde{\omega}$ to the eleventh decimal place; the demonstration of this fact will surely open an entirely new field of analysis" [3, X. 1, page 542]. Thus the first evidence Gauss had for (2) was this coincidence of two numbers! The applications just given support the truth of Gauss's claim of a "new field of analysis".

2. The surprising fact is that we have barely scratched the surface of the agM. Most people who have heard about it know more or less what has been discussed so far—i.e., the theory of the agM over $\mathbf{R}$. There is also a complete theory over $\mathbf{C}$! This makes sense because the agM is so closely related to elliptic integrals, and, as happens with the latter, the real power of the subject becomes apparent only when we use complex numbers.

We will assume $a, b \in \mathbf{C}^*$ and $a \neq \pm b$. The problem with algorithm (1) is that there seems to be no way to distinguish between the two choices for b_{n+1}. We thus get an uncountable number of sequences, and it is not clear that any of them converge. Fortunately, a distinguished choice of b_{n+1} can be made: since we want the sequences of (1) to have a common limit, it makes sense to let b_{n+1} be the square root of $a_n b_n$ closest to a_{n+1}. More precisely, we call b_{n+1} the *right choice* if $|a_{n+1} - b_{n+1}| \le |a_{n+1} + b_{n+1}|$, and if equality occurs, we require $\mathrm{Im}(b_{n+1}/a_{n+1}) > 0$. Given our restrictions on a and b, there is always a unique right choice for b_{n+1}.

We could thus always require b_{n+1} to be the right choice in (1). But Gauss saw that more is possible. The following example is taken from his notebooks [3, III, page 379]:

n	a_n	b_n
0	3.0000000	1.0000000
1	2.0000000	– 1.7320508
2	.1339746	1.8612098i
3	.0669873 + .9306049i	.3530969 + .3530969i
4	.2100421 + .6418509i	.2836903 + .6208239i
5	.2468676 + .6313374i	.2470649 + .6324002i
6	.2469962 + .6318688i	.2469962 + .6318685i

Note that b_1 is the wrong choice and that the succeeding b_n's are the right choice. The agM algorithm seems to converge nicely.

Surprisingly, it turns out that *any* pair of sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ as in (1) converge to a common limit. But the common limit is *nonzero* if and only if b_{n+1} is the right choice for all but finitely many n (see [2, Proposition 2.1] for a proof). Thus there are only countably many nonzero limits. This leads to the following definition:

DEFINITION. Given $a, b \in \mathbf{C}^*, a \neq \pm b$, a number $\mu \in \mathbf{C}^*$ is a *value of* $M(a,b)$ if it is the common limit of sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ satisfying (1). μ is the *simplest value* if b_{n+1} is the right choice for all $n \geq 0$.

The next question is how to relate the different values of $M(a,b)$. The answer is as follows:

THEOREM. *Given $a, b \in \mathbf{C}^*$ with $a \neq \pm b$ and $|a| \geq |b|$, let μ and λ be the simplest values of $M(a,b)$ and $M(a+b, a-b)$, respectively. Then all values μ' of $M(a,b)$ are given by the formula*

$$1/\mu' = d/\mu + ic/\lambda,$$

where d and c are arbitrary, relatively prime integers satisfying $d \equiv 1 \bmod 4$ and $c \equiv 0 \bmod 4$.

This theorem is basically due to Gauss. He never stated it in this form, but the bits and pieces of proof he left behind show that he knew what was going on. The first complete proofs appeared in 1928 after the last volume of Gauss's collected works was published (see [4] and [10].) We will only sketch some of the ideas involved in the proof (see [2] for the details).

Though the statement of the theorem is elementary, the proof uses some sophisticated mathematics: specifically, theta functions and modular forms (a basic reference is [6]). We will use Gauss's notation for the Jacobi theta functions. For $\tau \in \mathfrak{H} = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$, set $q = e^{\pi i \tau}$ and define

$$p(\tau) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \vartheta_{00}(\tau, 0),$$

$$q(\tau) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} = \vartheta_{01}(\tau, 0).$$

The natural question is what do theta functions have to do with the agM? The answer lies in the observation that if

$$a = \mu p(\tau)^2, \qquad b = \mu q(\tau)^2, \tag{4}$$

for $\mu \in \mathbf{C}^*$ and $\tau \in \mathfrak{H}$, then μ is a value of $M(a,b)$. To see why this is true, consider the theta identities

$$p(\tau)^2 + q(\tau)^2 = 2p(2\tau)^2, \qquad p(\tau)q(\tau) = q(2\tau)^2.$$

It follows that $p(2\tau)^2$ and $q(2\tau)^2$ are the respective arithmetic and geometric means of $p(\tau)^2$ and $q(\tau)^2$. Setting $a_n = \mu p(2^n\tau)^2$ and $b_n = \mu q(2^n\tau)^2$, we get sequences satisfying (1), and the formulas for $p(\tau)$ and $q(\tau)$ show that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ converge to μ.

To exploit this, set $k'(\tau) = q(\tau)^2/p(\tau)^2$, and note that solutions of (4) are equivalent to solutions of $k'(\tau) = b/a$ with $\mu = a/p(\tau)^2$. Thus the set

$$R = \{p(\tau)^2/a : k'(\tau) = b/a\}$$

consists of reciprocals of values of $M(a,b)$.

The function $k'(\tau)$ has some remarkable properties. Recall the action of $\mathrm{SL}(2,\mathbf{Z})$ on $\mathfrak{H}$ via linear fractional transformations, and set

$$\Gamma_2(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbf{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \bmod 4,\ b \equiv 0 \bmod 2 \right\}.$$

Then $k'(\tau)$ is a modular function for $\Gamma_2(4)$. Thus it is invariant under $\Gamma_2(4)$ and, in fact, induces a biholomorphism $\mathfrak{H}/\Gamma_2(4) \tilde{\to} \mathbf{C} - \{0, \pm 1\}$. Our restrictions on a and b ensure that $b/a \in \mathbf{C} - \{0, \pm 1\}$, so that if we fix one $\tau \in \mathfrak{H}$ with $k'(\tau) = b/a$, if follows that

$$R = \{p(\gamma\tau)^2/a : \gamma \in \Gamma_2(4)\}.$$

The next crucial fact is that $p(\tau)^2$ is a modular form of weight one for $\Gamma_2(4)$. Hence,

$$p(\gamma\tau)^2 = (c\tau + d)p(\tau)^2, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2(4).$$

If we set $\mu = a/p(\tau)^2$, then a typical element of R can be written

$$p(\gamma\tau)^2/a = (c\tau + d)p(\tau)^2/a = d/\mu + ic/\lambda,$$

where $\lambda = i\mu/\tau$. It is easy to show that d and c can be any relatively prime integers with $d \equiv 1 \bmod 4$ and $c \equiv 0 \bmod 4$. Though λ may seem mysterious, it is possible to show (using theta identities) that

$$a + b = \lambda p(-1/2\tau)^2, \qquad a - b = \lambda q(-1/2\tau)^2.$$

From (4) we see that λ is a value of $M(a+b, a-b)$.

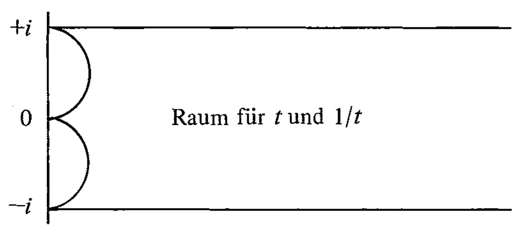
These last few paragraphs illustrate nicely the interaction between the agM and theta functions, modular functions, and modular forms. The formula for the values of $M(a,b)$ given by the theorem now makes sense. Of course, we are not finished with the proof. The next step is to show that τ can be chosen so that μ and λ are simplest values. This follows from a lemma (due to Gauss) which states that $\mu = a/p(\tau)^2$ is the simplest value ("einfachste Mittel"—see [**3**, III, page 477]) whenever τ lies in a certain fundamental domain of $\Gamma_2(4)$. The final step in the proof is to show that R contains the reciprocals of all values of $M(a,b)$.

The formula $\lambda = i\mu/\tau$ has some nice consequences. By abuse of notation set $\mu = M(a,b)$ and $\lambda = M(a+b, a-b)$. If we let $c = (a^2-b^2)^{1/2}$, then $M(a+b, a-b) = M(a,c)$ since a and c are the arithmetic and geometric means of $a+b$ and $a-b$. Thus $\lambda = i\mu/\tau$ implies that

$$\tau = M(a,b)/M(a,c). \tag{5}$$

This shows that τ can be explicitly computed in terms of a and b. An especially nice case is when $a = \sqrt{2}$ and $b = 1$. Then $c = 1$, giving us $\tau = i$!

3. Gauss's notes on the agM span thirty years and contain some amazing facts about modular function theory. In particular, he basically knew everything mentioned in the above sketch. (With regard to $p(\tau)^2$ being a modular form, he even computed the q-expansions at the cusps—see [**3**, X. 1, page 224].) Another example of what he knew is his famous drawing of a fundamental domain:



Note that Gauss worked in the right half-plane instead of the upper half-plane. His theta functions were thus power series in $e^{-\pi t}$, where $\mathrm{Re}(t) > 0$. In the third volume (1876) of his collected works, the above picture is reproduced inaccurately; the editor did not know its real meaning (see [**3**, III, page 478]). Only in the eighth volume (1900), edited by Fricke and Stäckel, do we find a correct drawing (see [**3**, VIII, page 105]). Another surprise found in Gauss's notes is his use of the reduction theory of positive-definite quadratic forms to analyze $p(\tau)^2$ (see [**3**, X. 1, page 224]). What he does is closely related to finding fundamental domains for subgroups of $\mathrm{SL}(2,\mathbf{Z})$.

While these fragments are quite compelling, we must be careful not to read too much into them. Gauss did not have a general theory of modular functions, and in some respects he is closer to Euler than to us. But is clear that he knew a tremendous amount about certain basic modular functions and forms.

The final topic we explore is the chronology of Gauss's discoveries about the agM. We've seen that everything began on May 30, 1799 when he noticed that $M(\sqrt{2},1) = \pi/\tilde{\omega}$. How long did Gauss take to get from this observation to the agM of complex numbers? We find part of the answer in his mathematics diary. The 101st and 102nd entries show that Gauss had proofs of both $M(\sqrt{2},1) = \pi/\tilde{\omega}$ and the more general formula (2) by December 1799 (see [**3**, X. 1, page 544]). But the 109th entry, dated June 3, 1800, reads as follows: "Between two given numbers a and b there are infinitely many means both arithmetic-geometric and harmonic-geometric, the observation of whose mutual connection has been a source of happiness for us" [**3**, X. 1, page 550]. (The harmonic-geometric mean of a and b is $M(a^{-1}, b^{-1})^{-1}$.) This quote is amazing: The infinitely many means and their mutual connection must refer to the theorem discussed above. Yet only six months have passed! Is it possible to discover so much mathematics in so short a period of time?

Unfortunately, such a question is beyond the scope of this article. Our goal will be more modest: We will show that Gauss, in June 1800, was aware of the theta functions and knew the formula (5) giving τ in terms of the agM. Our evidence comes from the 108th entry of the mathematical diary. It is also dated June 1800 and announces a complete theory of elliptic functions ("sinus lemniscatici universalissime accepti"—see [**3**, X. 1, page 548]). To see how this relates to the agM, we need to review Gauss's work on lemniscatic functions.

In January 1797 Gauss defined $\mathrm{sl}\,\phi$ and $\mathrm{cl}\,\phi$, the lemniscatic sine and cosine, as follows:

$$\mathrm{sl}\,\phi = x \Leftrightarrow \int_0^x (1-z^4)^{-1/2}\,dz = \phi,$$
$$\mathrm{cl}\,\phi = \mathrm{sl}(\tilde{\omega}/2 - \phi).$$

Notice the strong analogy with $\sin\phi$ and $\cos\phi$. The addition law came quickly, and by March 1797 he had defined $\mathrm{sl}\,\phi$ and $\mathrm{cl}\,\phi$ for complex ϕ, and he knew the double periods $2\tilde{\omega}$ and $2i\tilde{\omega}$. In October 1798 Gauss saw how to express $\mathrm{sl}(\phi)$ as a quotient of theta functions $P(\phi)/Q(\phi)$, where

$$Q(\psi\tilde{\omega}) = 2^{-1/4}\left(\tfrac{\pi}{\tilde{\omega}}\right)^{1/2}(1 + 2e^{-\pi}\cos 2\psi\pi + 2e^{-4\pi}\cos 4\psi\pi + 2e^{-9\pi}\cos 6\psi\pi + \cdots), \tag{6}$$

and $P(\psi\tilde{\omega})$ is given by a similar formula. Thus six months before the observation of May 30, 1799, Gauss was well aware of one side of the equation $M(\sqrt{2},1) = \pi/\tilde{\omega}$. Notice also that the general form of the theta function is easily discerned from the above formula.

We can now see the real significance of the diary entry of May 30, 1799, for when Gauss noticed that $M(\sqrt{2},1) = \pi/\tilde{\omega}$, he must have realized instantly that there was a deep connection between the agM and the lemniscatic functions. This is the "entirely new field of analysis" that his diary refers to.

It remains to see how Gauss got from the lemniscatic functions to general elliptic functions. The goal is to construct a function $S(\phi)$ which inverts the elliptic integral

$$\int ((1-z^2)(1+\mu^2 z^2))^{-1/2}\,dz.$$

Note that $\mu = 1$ corresponds to the lemniscatic case. Gauss's idea is to generalize formula (6), using the agM. Specifically, he first sets

$$\tilde{\omega} = \frac{\pi}{M(\sqrt{1+\mu^2},1)}, \qquad \tilde{\omega}' = \frac{\pi}{M(\sqrt{1+\mu^2},\mu)},$$
$$\tau = i\frac{\tilde{\omega}'}{\tilde{\omega}} = i\frac{M(\sqrt{1+\mu^2},1)}{M(\sqrt{1+\mu^2},\mu)},$$

and $Q(\phi)$ is defined analogously to (6); i.e.,

$$Q(\psi\tilde{\omega}) = (1+\mu^2)^{-1/4}(\pi/\tilde{\omega})^{1/2} \cdot (1 + 2e^{\pi i\tau}\cos 2\psi\pi + 2e^{4\pi i\tau}\cos 4\psi\pi + \cdots).$$

A similar formula gives $P(\phi)$, and then Gauss sets $S(\phi) = P(\phi)/Q(\phi)$. The function $S(\phi)$ inverts the above integral, satisfies the appropriate differential equation, and has periods $2\tilde{\omega}$ and $2i\tilde{\omega}'$. This is the theory that Gauss announced in June 1800.

The formulas for P and Q are a classical method for producing elliptic functions. A standard difficulty is showing that such functions invert all elliptic integrals—this is called the uniformization problem. The wonderful thing about Gauss's theory is that it solves this problem explicitly: The uniformizing variable τ is computed by using the agM! The formula for τ also substantiates two of our earlier claims. First, we see that Gauss did know formula (5) for τ in June 1800, and second, we see clearly that the agM was essential to Gauss's study of elliptic functions.

The full story of Gauss's work on the agM is quite complicated. No short survey article can convey the real richness of Gauss's mathematical thought. We have only sampled some of the highlights. For more of the details the reader should consult [2] or [**3**, X.2].

There is one final point to consider. In §1 we saw numerous applications of the agM over **R**. Far fewer are known over **C**, partly because most people simply aren't aware that the agM of complex numbers can be defined. It is our hope that once this theory becomes better known, the applications will follow. Some of them could be very exciting.

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The article above is the tenth in the series of Special Articles published in the *Notices*. The author, David A. Cox, was an undergraduate at Rice University, where he earned his B.A. degree in 1970. He then went to Princeton University to study étale homotopy theory with Eric Friedlander. After receiving his Ph.D. degree in 1975, he taught one year at Haverford College and four years at Rutgers University. In 1979 he went to Amherst College in Amherst, Massachusetts, where he is currently Associate Professor of Mathematics.

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