

A Celebration of Women in Mathematics

In March 1994 I organized a two-day conference at the Massachusetts Institute of Technology in which nine distinguished women mathematicians gave colloquium lectures about their current mathematical research. There are many exciting mathematical meetings where the speakers are all men: this was a meeting where exciting mathematics was presented that was unusual only in that all the speakers were women. From the comments that I heard after the meeting, the conference was clearly very successful. The audience contained a cross-section of the mathematical community ranging from MIT Institute Professors to undergraduates. It was pleasing to hear enthusiasm on all sides for the mathematics that was discussed. I have been told that the meeting was particularly inspiring for many young women in the audience who are in the early stages of their mathematical careers.

The conference was funded by MIT and the Visiting Professorship for Women Program of the National Science Foundation. The Mathematics Department at MIT was a most supportive host for the conference and I thank the many people in the department who contributed to the success of the conference, particularly David Benney and Haynes Miller and the excellent administrative staff of the department. To our pleasure, the conference was opened by Mark Wrighton, the MIT Provost. Mildred Dresselhaus, a distinguished Institute Professor, introduced the second day with entertaining and illuminating remarks about her career at MIT. She made the very pertinent observation that for a woman

to be really helpful to the cause of women in science, it is important that she work from a base of solid scientific achievement. The talks at the meeting were excellent illustrations of such achievements.

This article contains short summaries of the mathematics presented by the speakers:

Joan S. Birman, *Studying Surfaces in Knot Complements*

Ingrid Daubechies, *Wavelets and Applications*

Dusa McDuff, *The Geometry of Symplectic Energy*

Jill P. Mesirov, *Mathematical Theory in Parallel Algorithms*

Cathleen S. Morawetz, *The Wave Equation Revisited*

Jean E. Taylor, *Surface Motion Due to Surface Energy Reduction*

Chuu-Lian Terng, *Soliton Equations and Differential Geometry*

Karen K. Uhlenbeck, *Moduli Spaces and Adiabatic Limits*

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Studying Surfaces in Knot Complements

Joan S. Birman



My talk at the *Celebration of Women in Mathematics* was about applications of the theory of braids to the study of knots in 3-space.

A knot K is a topological circle which is smoothly embedded in 3-space R^3 . Its *knot type* is its equivalence class under isotopies of the pair (R^3, K) . Our interest is in the *knot problem*,

i.e., the problem of how to decide whether two knots in 3-space determine the same knot type. My current research (joint with William Menasco) is directed toward solving that problem algorithmically. The ingredients needed for our proposed algorithm are (i) a concept of a ‘simplest’ representative K_0 of a knot, chosen so that each knot K has at least one and at most finitely many distinct simplest representatives; (ii) a complexity function $c(K)$ which measures how far K is from the same K_0 ; (iii) a method for reducing $c(K)$; and (iv) an understanding of how the finitely many simplest representatives are related to one another. Braid structures provide the tools for attacking these problems.

Choose polar coordinates (z, r, θ) in R^3 . Let A be the z -axis. Then $R^3 - A$ is fibered by half-planes $H_\theta = \{(z, r, \theta) \in R^3 / \theta = \text{constant}\}$. Our knot K is represented as a *closed braid* with respect to the z -axis A if K meets every H_θ transversally, i.e. K is nowhere tangent to an H_θ . For example, if K is the ‘unknot’, an especially simple representative would be the unit circle $U_0 = \{(0, 1, \theta) \in R^3; 0 \leq \theta < 2\pi\}$ in the (r, θ) plane. This is our simplest representative, and it is unique up to isotopy in the complement of the braid axis.

Every smooth representative of every knot type bounds an orientable surface which is em-

bedded in 3-space, and so also an orientable surface S of minimum genus. For example, every representative U of the unknot bounds a disc \mathcal{D} , our standard representative bounding the unit disc \mathcal{D}_0 in the plane $z = 0$. This disc is foliated radially by the arcs $\{\mathcal{D}_0 \cap H_\theta, 0 \leq \theta < 2\pi\}$. These arcs also define a *flow* on \mathcal{D}_0 as one pushes the H_θ 's forward through the fibration of $R^3 - A$. See Figure 1a. What happens to this foliation and the associated flow when we replace U_0 with a nonstandard closed-braid representative U of the unknot, say one which passes n times around the axis A before closing up to a circle? Clearly U will still bound a disc, say \mathcal{D} , which will in general admit a singular foliation, the leaves of the foliation being the components of $\{\mathcal{D} \cap H_\theta, 0 \leq \theta < 2\pi\}$. It can be shown that the disc \mathcal{D} can be assumed to have a particularly nice foliation: the leaves are all arcs, the singularities (if any) are all saddle points, and near each point of $A \cap \mathcal{D}$ the foliation is radial. Figure 1b shows the foliation in a neighborhood of two points in $A \cap \mathcal{D}$. It is immediately clear that if this foliation is to be extended to all of \mathcal{D} , then singularities must occur.

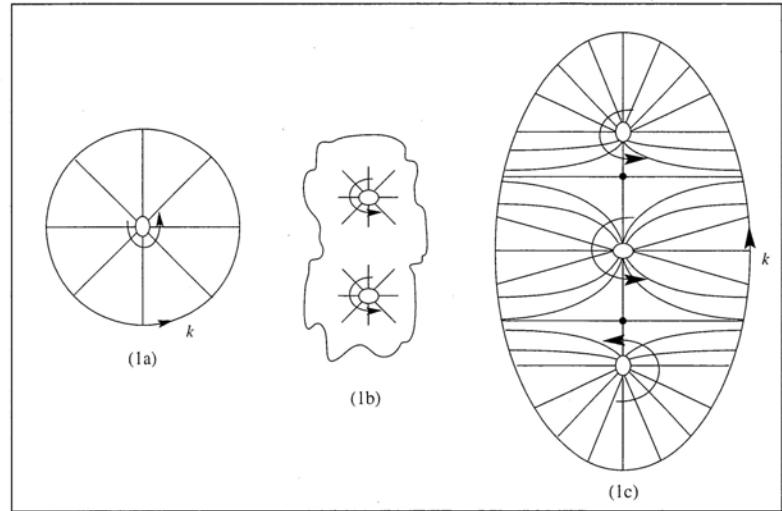


Figure 1

The foliations which we study can be shown to decompose the surface S into a union of *tiles*, each of which is a 2-cell which contains exactly one singularity of the foliation. In particular, S is a union of three types of foliated tiles, depicted in Figures 2a, 2b, and 2c, each of which has a canonical embedding in 3-space relative to our cylindrical coordinates. Two of the tile types intersect K nontrivially, but the third is entirely in the interior of S . These tiles are all depicted as they appear when viewed on the positive side of

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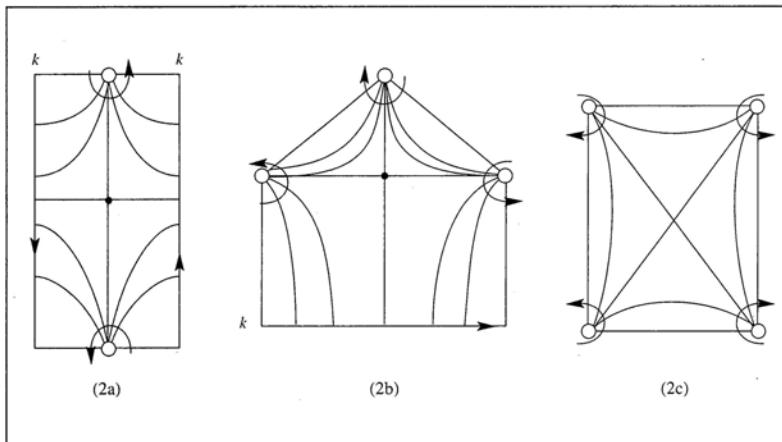


Figure 2

S. The black dots show the points where A pierces S . There is a well-defined sense of the flow about each pierce-point, clockwise or counterclockwise. The orientation of K is consistent with the sense of the flow around the vertices. Notice that counterclockwise flows are preferred, reflecting the fact that the linking number of K with A is $n > 0$.

Returning to our example and pulling \mathcal{D} back (with its foliation) to its preimage D under the embedding we are able to replace \mathcal{D} , which is difficult to visualize because of its complicated embedding in 3-space, by the foliated model disc D . A simple example is given in Figure 1c. The disc is pierced 3 times by the braid axis and is foliated by two tiles of the type illustrated in Figure 2a. We leave it as an exercise for the reader to reconstruct the embedded disc and the closed braid representative of its boundary from the foliated model disc of Figure 1c.

The kind of data which we can see in the foliation includes: (i) properties of the graph and subgraphs formed by the singular leaves; (ii) the number and types of tiles which meet at each vertex; (iii) the sense of the flow near the vertices (clockwise or counterclockwise); (iv) the sign of the outward-drawn normal to S near a singularity of the foliation. It turns out that this data (and other combinatorial data) tells us how to modify K so as to reduce $c(K)$. Similar methods also apply to the study of other surfaces which have a special position relative to that of K , e.g., closed incompressible tori in the complement of K .

This work is still in progress. At this writing we have learned a great deal about the closed braid representatives of knots and have an algorithm in a very special case: knots of braid index at most 3, which are represented by a closed 3-braid. We have partial results for closed braid representatives of the unlink and other special cases.

Wavelets and Applications

Ingrid Daubechies



The relatively new development of wavelet theory in the last ten years or so represents a synthesis of ideas from mathematics, physics, engineering, and computer science, some with roots going back to the start of the century. The goal of the talk was to introduce wavelets, to explain both their mathematical and algorithmical aspects, and to illustrate these by several applications. In 1 dimension, wavelets are typically defined as the dilates and translates of one function,

$$\psi^{a,b}(x) = |a|^{-1/2} \psi\left(\frac{x-b}{a}\right),$$

where ψ is often required to have some smoothness and some decay, say $|\psi(x)| \leq C(1+|x|)^{-1-\varepsilon}$, $|\dot{\psi}(\xi)| \leq C(1+|\xi|)^{-1-\varepsilon}$. Moreover, ψ should satisfy $\int \psi(x) dx = 0$. It then turns out that any square integrable function f (and, in fact, many other functions as well) can be “decomposed” into such wavelets,

$$f(x) = C_\psi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle f, \psi^{a,b} \rangle \psi^{a,b}(x) a^{-2} da db,$$

a fact long known to harmonic analysts as Calderón’s formula and to quantum physicists as the resolution of the identity for the $ax + b$ -group. The interest of such a decomposition is that it views f as a superposition of “building blocks” $\psi^{a,b}$, each of which is well localized in “time and frequency”; a typical example is $\psi(x) = (1-x^2)e^{-x^2/2}$, where $|\psi^{a,b}(x)|$ is mostly concentrated in the region $|x-b| \leq 2|a|$, and $|\dot{\psi}^{a,b}(\xi)|$ mostly in $|a|/4 \leq |\xi| \leq 4|a|$. By cutting the domain of integration in a, b into different parts, one effectively carves f into different pieces, which may have very different properties. This is done in (e.g.) Littlewood-Paley type arguments.

There are also wavelet families where the dilation parameter a and the translation parameter b are assigned discrete values only. One

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widely used choice is $a = 2^{-j}$, $b = 2^{-j}k$, with $j, k \in \mathbb{Z}$. It is then also possible to choose ψ in such a way that the $\psi_{j,k} = \psi^{2^{-j}, 2^{-j}k}$ constitute an orthonormal basis for $L^2(\mathbb{R})$. Instead of the integral representation above, we now have

$$f = \sum_{j,k} \langle f, \psi_{j,k} \rangle \psi_{j,k},$$

which can again be understood in L^2 -sense, or in many other functional spaces. Harmonic analysis methods can be used to show that such $\psi_{j,k}$ provide unconditional bases for a variety of Banach spaces, such as $L^p(\mathbb{R})$, $1 < p < \infty$, the Hardy space $H^1(\mathbb{R})$ and its dual BMO, the Sobolev spaces $W^s(\mathbb{R})$, the Besov spaces $B_{p,s}^q(\mathbb{R})$, the Hölder spaces $C^s(\mathbb{R})$, etc. That is, for each of these spaces, there exists a criterion that decides whether f lies in the space or not, solely by looking at the behavior of the absolute values $|\langle f, \psi_{j,k} \rangle|$ of the wavelet coefficients. This unconditionality of wavelet bases for many functional spaces makes them a special and powerful tool.

Wavelet bases are also linked with fast algorithms. Every wavelet base is derived from a multiresolution analysis (provided ψ has some decay and some smoothness). This means that there exists an auxiliary function ϕ , the *scaling function*, such that

$$\phi(x) = \sum_n c_n \phi(2x - n), \quad \psi(x) = \sum_n d_n \phi(2x - n);$$

moreover, $\int \phi(x) dx = 1$. Because of this latter property, ϕ can be used to write a resolution of the identity, i.e.,

$$f(x) = \lim_{J \rightarrow \infty} 2^J \int f(y) \phi(2^J(y - x)) dy$$

for continuous f . In particular, the integrals $2^J \int f(y) \phi(2^J y - k) dy$ can be approximated by the sampled values $f(2^{-J}k)$. (In practice, this approximation may have to be refined.) By the recurrence relations above, one can then, by means of a fast algorithm, compute for successive $j < J$ the wavelet coefficients $\langle f, \psi_{j,k} \rangle = 2^{j/2} \int f(y) \psi(2^j x - k) dx$ and the scaling coefficients $\langle f, \phi_{j,k} \rangle$. In fact, algorithms of this type were already known to electrical engineers as *subband filtering* (with exact reconstruction).

A combination of these fast algorithms on one hand, and a thorough and deep understanding of the mathematical properties of the underlying functional analytic tool on the other hand, has led to a variety of applications. A few examples discussed in the talk were compression of large, dense matrices (Beylkin, Coifman, and Rokhlin); theorems showing that wavelets give asymptotically optimal denoising and estimation techniques for many different types of noisy

signals (Donoho—this application exploits that wavelets are an unconditional basis in many spaces); image compression and manipulation from edge information encoded in wavelet coefficients (Mallat).

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The Geometry of Symplectic Energy

Dusa McDuff

The talk was divided into three parts.

The first part was an elementary introduction to symplectic geometry—the geometry introduced on a smooth $2n$ -manifold M by a closed 2-form ω which is nondegenerate (that is, its top exterior power ω^n is a volume form). The basic example of a symplectic manifold is Euclidean space \mathbb{R}^{2n} with its standard form

$$\omega_0 = dx_1 \wedge dx_2 + \cdots + dx_{2n-1} dx_{2n}.$$

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One important concept is that of the **symplectic gradient** X_H of a function $H : M \rightarrow \mathbf{R}$. This vector field is characterized by the equation

$$\iota(X_H)\omega = \omega(X_H, \cdot) = dH,$$

and, in contrast to the gradient with respect to a Riemannian metric, points tangent to the level surfaces of H . Moreover, the flow which it generates (called the Hamiltonian flow) preserves the symplectic form ω . Thus, if M is the unit 2-sphere $S^2 \subset \mathbf{R}^3$ with its usual area form, and if H is the height function (given by the third coordinate function), the symplectic gradient X_H is tangent to the horizontal circles $H = \text{const}$ and generates a rotation about the north-south axis. Here, one can let the Hamiltonian depend on time $t \in [0, 1]$. One then gets a family X_H^t of vector fields which integrates up to a family ϕ_t , $0 \leq t \leq 1$, of symplectomorphisms. The map ϕ_1 is called the time-1 map of H_t . Essentially, any symplectomorphism (i.e., diffeomorphism which preserves ω) is the time 1-map of such a flow.

Symplectomorphisms have important geometric properties, which are just beginning to be understood. One striking result is Gromov's non-squeezing theorem (1985), which states that if there is a symplectomorphism which embeds a $2n$ -ball of radius r into a cylinder of radius R , then $r \leq R$. Here, a cylinder is a product of a 2-dimensional disc of radius R with another symplectic manifold (M, ω) of dimension $2n - 2$. Gromov proved this result when M is Euclidean space \mathbf{R}^{2n-2} with its usual symplectic form, and it has been proved for all manifolds M by Lalonde-McDuff.

This leads to the notion of a **capacity**. We define the capacity $c(B)$ of a ball of radius r to be πr^2 , and, for an arbitrary subset U in a symplectic $2n$ -manifold, define $c(U)$ to be the supremum of the capacities of the symplectic $2n$ -balls which embed symplectically in U . Thus, the nonsqueezing theorem implies that the capacity of the cylinder $B^2(r) \times \mathbf{R}^{2n-2}$ is πr^2 .

The second part introduced Hofer's idea of the **energy** of a symplectomorphism ϕ . We measure the size of a Hamiltonian function H by

$$\text{Totvar } H = \max_{x \in M} H(x) - \min_{x \in M} H(x),$$

and define the length of the path ϕ_t , $0 \leq t \leq 1$, generated by H_t to be

$$\mathcal{L}(\phi_t) = \int_0^1 \text{Totvar } H_t dt.$$

The energy $e(\phi)$ is then defined to be the infimum of the lengths of the paths ϕ_t which have time-1 map equal to ϕ . Note that, although one uses the first derivative of H in order to generate the flow, the size of H depends only on its

values, not its derivative. Therefore, it is *a priori* not clear that the energy of a nontrivial map is always positive. That this is the case follows from Hofer's **energy-capacity inequality**, which states that if U is an open set which is disjoined by ϕ , i.e.,

$$\phi(U) \cap U = \emptyset,$$

then $e(\phi) \geq c(U)$. As an example, consider the function $H = y$ on \mathbf{R}^2 with $\omega_0 = dx \wedge dy$. The flow generated by H is translation in the x direction:

$$(x, y) \mapsto (x + t, y),$$

and it is easy to check that the energy required to disjoin the rectangle $[0, 1] \times [0, a]$ is precisely the area a .

The third part of the talk outlined a geometric proof of the energy-capacity inequality, which is due to Lalonde-McDuff. The idea is to use a flow ϕ_t whose time-1 map disjoins a ball in M of radius r to construct an embedded ball of radius r in a cylinder $B^2(R) \times M$ where

$$\pi R^2 = \mathcal{L}(\phi_t) + \pi r^2 / 2 + \epsilon.$$

The nonsqueezing theorem then implies that

$$\pi r^2 \leq \mathcal{L}(\phi_t) + \pi r^2 / 2 + \epsilon,$$

and hence, taking the infimum over all such paths ϕ_t that

$$c(U) = \sup \pi r^2 \leq 2e(\phi).$$

A refinement of this argument gives (a slightly modified version of) the full energy-capacity inequality.

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Mathematical Theory in Parallel Algorithms

Jill P. Mesirov

Suppose we would like to understand a physical phenomenon, for example, how air flows past an airplane wing, or how a protein folds. We might perform real experiments and get data from measurements of actual phenomena in the

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laboratory, e.g., via a wind tunnel. However, it would be difficult inferring the behavior in unknown cases from this data. On the other hand, we might also try to build a mathematical model, perhaps adding some simplifying assumptions, which characterizes the physical phenomenon. Using the model we can then do simulations. Historically, these calculations were done by hand, but now we use computers to do them. We can test the model's predictions by comparing the results of simulations against real experimental data obtained for stereotypical examples. If the predictions match the experimental data, then we gain confidence in our models and may use them to predict behavior in cases where it is unknown.

The mathematical model/simulation paradigm described above is used to help design the cars we drive, the medicines we take, and the airplanes in which we fly. This approach often represents a more cost-effective way of doing some "experiments" via computer simulation. Its success has created an ever-increasing demand for higher-performance computers to allow for real time, interactive design, and graphics. Mathematics plays an important role in this process, both in modeling physical phenomena, and in designing efficient algorithms for high-performance parallel computers.

For the purpose of our discussion, a parallel computer consists of a collection of processors, each with its own local memory, connected by a network. A processor can only operate on data in its own local memory. If it has to use data that is located in another processor's memory, then that data must be sent over the network. This type of action is referred to as "communications". When designing algorithms for such distributed-memory parallel computers, we seek to minimize the time spent in communications. This makes the algorithms more efficient in using the processing power of the machine. In this lecture we described an interesting interplay between mathematics and the design of efficient parallel algorithms in the context of N-body algorithms for both fluid flow and molecular dynamics applications. We also showed some videos of our simulations which the laboratories of the future may produce in real time.

The two cases we considered require different communication patterns. For the fluid flow problem, we showed how rotated and translated Gray codes can be used to construct timewise edge independent Hamiltonian paths on hyper-

cubes, yielding an optimal global communication pattern for machines with this kind of network. In molecular dynamics codes for protein dynamics, the force law is a local one. Thus, one is led to solving a local, i.e., on subcubes of side $2a+1$, gossiping problem in multidimensional grids. We showed that by restricting to the case where the data distribution paths are translationally and rotationally symmetric, the problem corresponds to finding Hamiltonian paths in orbit graphs. Solutions were found for certain dimensions and side lengths by finding solutions in the case $a=1$, and then giving a method of extending those solutions to the cases $a > 1$.

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The Wave Equation Revisited

Cathleen S. Morawetz



Cathleen Morawetz

It is a surprising fact that there is still much to learn about the wave equation. Beginning with representing the solution of the initial value problem in terms of a fundamental solution, it is an easy step to see that in 3D+T Huyghens's principle holds, i.e., if the surface of the light cone from a point backwards in time does not cut the support of the initial values, then the solution at that point is identically zero. This will be true even if there is a perturbation of the wave equation confined to the inside of the cone.

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This has led to many investigations, including scattering theory for perturbations, and with Huyghens's principle gives neater results including the exponential decay of the local energy if the wave equation is perturbed locally in certain ways, e.g., the presence of a star-shaped reflecting obstacle. See (LPM) and (M).

More recent questions are about finding "transparency conditions". These are conditions on a boundary which guarantee that the solution could be continued outside the boundary as a solution of the wave equation. See (EM), (BT), and (KM) for approximate transparency conditions. H. Warchall (W) has shown under more general but analytic conditions that if, and we think here only of the wave equation, the initial data is supported in a ball of radius b , then the solution inside that ball for $t > t_1 > 0$ depends only on the data inside the ball at $t = t_1$.

In fact, for the simple case of the wave equation the transparency condition can be written down explicitly by giving the outward characteristic derivative in terms of an integral over the ball of support at the same time. Conversely, if the solution satisfies this integral condition, then it can be continued outside as a solution of the wave equation. Cauchy integrals are involved, and thus the solution must have three derivatives.

Finally, this circle of ideas leads one to a way of avoiding transparency conditions but still solving in a finite domain by repeated use of Huyghens's principle.

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Surface Motion Due to Surface Energy Reduction

Jean E. Taylor



Suppose one has an irregular blob of something, without any external forces such as gravity acting on it. Because it is not a round ball, it does not have the smallest surface area for its volume. If it evolves so as to reduce its surface area, how does it evolve? What different evolution laws are reasonable, and how do the shapes differ under these different types of motion? Can you tell, by some not-too-delicate features of the shapes, which law is, in fact, governing the motion?

For a crystalline material, such as a single crystal of ice, the shape of least surface energy for a given volume is not a round ball, but something else, like a hexagonal prism. How does a snowflake evolve towards this?

The claim is that physically interesting motion problems can be formulated in a variational setting, as *gradient flows* under various natural inner products. This enables existence proofs and a unifying framework for different motion laws and different surface free energy functions. It also underlies the whole *crystalline* approach, which is interesting both as a model in its own right and as a possible means of approximation for other surface free energy functions, and which has been implemented by computer programs in many cases.

Surface Energy and Weighted Mean Curvature

Why does mean curvature H arise for variational problems involving surface area? Because it is the rate of decrease of surface area with volume under deformations. More precisely, the first variation of area of a smooth oriented surface S is the bounded linear operator on continuous vector fields g defined by

$$\delta S(g) = \frac{d}{dt} \int_{x \in h_t S} 1 d\mathcal{H}^2 x,$$

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where $h_t S = \{y : y = x + t g(x) \text{ for some } x \in S\}$.¹ One can also compute that with $n(x)$ denoting the oriented normal at x and H_n the mean curvature vector,

$$\delta S(g) = \int_{x \in S} g \cdot (-H_n(x)) d\mathcal{H}^2 x.$$

For solids, with crystal lattice orientations fixed in space, the surface free energy per unit area is a function γ from the unit sphere to the real numbers, and it is convenient to extend the domain of γ to all vectors by $\gamma(p) = |p| \gamma(p/|p|)$. If γ is C^2 and convex and S smooth,

$$\begin{aligned} \delta_\gamma S(g) &= \frac{d}{dt} \int_{x \in h_t S} \gamma(n(x)) d\mathcal{H}^2 x \\ &= \int_{x \in S} g \cdot (-\kappa_\gamma n) d\mathcal{H}^2 x \end{aligned}$$

where $\kappa_\gamma = \frac{\partial^2 \gamma}{\partial p_1^2} \kappa_1 + \frac{\partial^2 \gamma}{\partial p_2^2} \kappa_2$. Here κ_i is the i th principal curvature and p_i the corresponding principal direction. We call κ_γ the *weighted mean curvature*. Note that if $\Delta E = \int_{h_t S} \gamma - \int_S \gamma$ and ΔV denotes the signed volume from S to $h_t S$, then for g with small support around x_0 , $\frac{E}{t} \approx \delta S(g) \approx -\kappa_\gamma(x_0) \frac{V}{t}$, so $\kappa_\gamma(x_0) \approx -\frac{E}{V}$.

Given any γ there is associated the Wulff shape

$$W_\gamma = \{x : x \cdot p \leq \gamma(p) \forall p\}.$$

If W_γ is polyhedral, γ is *crystalline*, and the first variation is neither bounded nor linear. Instead, the crystalline weighted mean curvature (abbreviated crystalline curvature) is nonlocal: one uses the rate of decrease of surface free energy with volume, with the deformations moving entire segments. Edges and corners must be energy-minimizing to avoid infinite contributions to the analog of first variation. For polygonal curves, the formula reduces to something quite simple: $\kappa_\gamma(\text{segment } i) = -\sigma_i \Lambda_i / \ell_i$, where ℓ_i is the length of the segment, Λ_i is the length of the segment of the Wulff shape boundary with the same normal direction, and σ_i is 1 if the curve makes left turns at each end (when traveling in the direction of its orientation) is -1 if both ends make right turns, and is 0 otherwise [T].

Inner Products and Gradient Flows

Observe that the action of the linear operator δS on a function g is given by the L^2 inner product of g with $-H_n$. Thus, flow by mean curvature is gradient flow for surface area in the L^2 inner product. It is an unusual type of gradient

flow, in that the inner product space keeps changing, as S itself keeps changing.

One can use this fastest decrease property to construct a variational approach to motion by weighted mean curvature, by selecting a time step Δt and minimizing $\int_{h_t S} \gamma + \frac{1}{2t} h \cdot h$ starting with S_0 to get S_t , then with S_t to get S_{2t} , etc. Convergence of these piecewise constant flows to a continuous flow as Δt goes to zero has been shown [ATW] (with an approximately equal volume integral replacing the L^2 inner product). The resulting flow is the PDE solution to motion by weighted mean curvature where that exists, and is motion by crystalline curvature for curves and crystalline γ [AT].

Are there other reasonable inner products than the L^2 inner product? Yes. In particular, motion by the negative Laplacian of weighted mean curvature arises if one does gradient flow in the H^{-1} inner product. Mullins (who also introduced motion by curvature) derived that motion by the negative Laplacian of curvature models surface diffusion: H is a potential, $D\nabla H$ gives the flux of atoms down the gradient of the potential, and $-D\Delta H$ is then the accumulation of atoms, which translates into a velocity with the appropriate constants. For a survey of this field, see [CT].

This type of motion is difficult to model, however. There is no maximal principle, and no level-set (viscosity solution) formulation. However, a crystalline formulation of this motion for curves has been devised and implemented in a computer program [CRCT].

We have also derived from physical principles a whole family of motion laws [CT]. At one extreme, where attachment kinetics are rapid and surface diffusion is slow, there is motion by $-\Delta \kappa_\gamma$. At the other, where attachment kinetics are slow and surface diffusion is rapid, there is motion by $\kappa_\gamma - \kappa_{av}$ where κ_{av} is an overall average; this is L^2 gradient flow restricted to deformations with integral 0. In between is a whole family of motion laws, which turn out to be flow by the inner product, which is the appropriate linear combination of the L^2 and H^{-1} inner products!

Comparing Shapes

We return to the original question of whether there are gross features of shapes moving under various laws that can be used to distinguish one law from another. We have made videos of computer simulations of several initial shapes moving under each of the two extreme-case motion laws discussed above. These videos can be viewed using Mosaic [M]. We comment on the following observable features:

- (1) In motion by $\kappa_\gamma - \kappa_{av}$, zigzag portions of the curve translate as a whole, with the same ve-

¹There are better definitions that allow for singularities in $S[A]$.

locity, whereas in surface diffusion, there is a proximity effect, and each segment can have a different velocity.

(2) Curves that are convex (or, more generally, which have κ_y nonpositive) remain so under motion by $\kappa_y - \kappa_{av}$, whereas long skinny portions tend to develop bulbs at the end under motion by surface diffusion.

(3) Topology changes can happen under either type of motion, but tend to be more frequent and dramatic under motion by surface diffusion. In particular, a simple closed curve with κ_y nonpositive everywhere can break into two connected components under motion by surface diffusion.

Summary

The importance of regarding flows as being gradient flows under different inner products has several advantages. First, it shows there is nothing magic about “decreasing energy fastest”—each decreases energy fastest in its own metric. Second, it provides a unifying approach, putting different surface energies, different motion laws, and even sharp versus diffuse interface approaches under one umbrella [TC]. Third, it allows variational techniques to be used for motion problems. Often both elliptic and highly nonelliptic problems can be handled in the same proofs. And, in particular, in the crystalline context the gradient flow approach provides a good criterion for when a facet must be stepped [CRCT].

There are also advantages to considering the crystalline approach. Some physical materials are faceted. One can see what is special about area, by comparing area-reducing motions to corresponding surface-energy reducing motions. It is flexible; in both theory and computation, one can alter the motion law without affecting the basic underlying structure. It is computable: it uses a natural parametrization, by the Gauss map. It is a good way to measure curvature numerically, and one can fairly easily detect and make topological changes. One can use it for approximations (convergence has been proved in some cases [G][S]). And it has a nice theory in its own right, which is sometimes similar to that for area-related problems and sometimes different.

A longer version of this lecture summary is expected to appear [T2].

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Soliton Equations and Differential Geometry

Chuu-Lian Terng



In this talk, I explained some symplectic geometric properties of harmonic maps from the Lorentz space $R^{1,1}$ to a symmetric space: the existence of a sequence of compatible symplectic structures that make the harmonic map equation Hamiltonian, a hierarchy of commuting flows, and an action of the affine

Kac-Moody group on the space of harmonic maps.

It is well known that most finite dimensional, completely integrable, Hamiltonian systems can be obtained by applying the Adler-Kostant-Symes (AKS) theorem to some Lie algebra \mathcal{G} equipped with an ad-invariant, nondegenerate bilinear form, and a decomposition $\mathcal{G} = \mathcal{K} + \mathcal{N}$. The symplectic manifold is some coadjoint N -orbit $M \subset \mathcal{K}^\perp \simeq \mathcal{N}^*$, and the equation is the Hamiltonian equation of the restriction of some $\text{Ad}(G)$ -

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invariant function f on G to M . Applying the AKS theorem to the loop algebra of an affine Kac-Moody algebra, we obtain many well-known soliton equations and equations from differential geometry. In fact, let G/K be a symmetric space, $G = \mathcal{K} + \mathcal{P}$ the Cartan decomposition, \mathcal{A} a maximal abelian subalgebra in \mathcal{P} , $\mathcal{P}_1 = \mathcal{A}^\perp \cap \mathcal{P}$, $a \in \mathcal{A}$ a fixed regular element, and $b \in \mathcal{A}$. We obtain a sequence of symplectic structures $\{\omega_r \mid r \in \mathbb{Z}\}$ on the space of Schwartz functions $\mathcal{M} = S(R, \mathcal{P}_1)$, functionals $\{F_{b,n} : \mathcal{M} \rightarrow R \mid n \geq 1\}$, and evolution equations, the n -th flow associated to G/K with respect to a, b , $u_t = [Q_{b,n}(u), a]$ with $n = -1$ or $n \geq 2$ such that

- (1) all ω_r are compatible. That is, $\{\cdot, \cdot\}_r + \mu \{\cdot, \cdot\}_s$ is again a Poisson structure on \mathcal{M} for any $\mu \in R$ and $r, s \in \mathbb{Z}$, where $\{\cdot, \cdot\}_r$ is the Poisson structure corresponding to ω_r ,
- (2) the Hamiltonian equation for $F_{b,n}$ with respect to ω_r is $u_t = [Q_{b,n+r+1}(u), a]$,
- (3) $F_{b,n}$ are commuting Hamiltonians with respect to ω_r for all n, r .

These flows contain many interesting equations. For example, the equation for harmonic maps from $R^{1,1}$ to G/K is the -1 -flow associated to G/K , the nonlinear Schrödinger equation is the third flow associated to $SU(2)$, the Modified KdV equation is the fourth flow associated to $SU(2)/SO(2)$, and the Beals-Coifman evolution equations are the n -th flows associated to a compact Lie group G with $n \geq 2$.

There is also a system of partial differential equations of n variables naturally arising from the second flow associated to a rank n symmetric space. For example, the natural PDE system associated to

$$\begin{aligned} SO(2n+1)/SO(n) \times SO(n+1), \\ SO(2n)/SO(n) \times SO(n) \\ \text{and } SO(2n, 1)/SO(n) \times SO(n, 1) \end{aligned}$$

is the equation for isometric immersions from $N^n(c)$ into $N^{2n}(c)$ with flat normal bundle and independent curvature normals with $c = 1, 0, -1$ respectively, where $N^n(c)$ is an n -dimensional space form with sectional curvature c .

For these sequence of flows we also obtain analogues of K. Uhlenbeck's loop group action on the space of harmonic maps from $R^{1,1}$ into Lie group and the classical Bäcklund transformations for the Sine-Gordon equation.

Moduli Spaces and Adiabatic Limits

Karen K. Uhlenbeck



The last ten years of differential geometry have been dominated by the study of very special geometric partial differential equations. The equations I have in mind include the minimal and constant mean curvature equations, the Yamabe equation, the Kähler-Einstein equation, harmonic maps, and Yang-Mills with all its variations.

The origins of the equations vary tremendously, as they come from both very classical geometry and modern high-energy physics, but the equations themselves are clearly very special. One of the interesting and useful features of most of the equations is the "large", by which we mean multi-dimensional, solution spaces which are regularly found for these equations in special circumstances.

By and large the origin of the equations of geometry is intimately connected with variational principles. The classical equations are usually posed as minimization problems. It is a hard and fast rule of physics that interesting equations are the "equations of motion" for some integral with a classical Lagrangian. In these cases, the Euler-Lagrange equation, which is generated by the variational principle, is nearly always second order. However, for technical reasons, the second variation is self-adjoint, and hence has zero index if it can be represented as a Fredholm operator between Banach spaces. This means that formally speaking, the dimension of a critical set, or the solution space of a geometric equation, tends to be zero dimensional.

There are a multitude of origins for the phenomena of "nonaccidental" solution manifolds of large dimension. By "nonaccidental" we mean that a small perturbation which does not change the nature of the problem will not destroy them. The most common cause is the existence of a continuous family of symmetries which leaves the equation invariant. This is closely allied to the phenomena of "integrals" which is important

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for mean curvature surfaces. A second philosophically separate cause is the reduction of the original variational problem, with its second-order geometric equation, to a first-order equation for a topological minima. This mechanism reduces the second-order harmonic map equation from a Riemann surface into an almost complex symplectic manifold to a pseudo-holomorphic curve equation, and Yang-Mills to self-dual Yang-Mills. The index of the first-order equation obtained in this fashion is topologically determined and typically not necessarily zero. A final and not-so-well-known origin for multi-dimensional parameter spaces is the introduction of point singularities, which changes the index. The Yamabe equation, which has zero index on a compact manifold, has index k on a manifold with k punctures.

An optimum strategy for progress includes the search for new equations, as well as technical progress on the well-studied equations. Another angle is to search for new uses of these already-studied moduli spaces of solutions to geometric partial differential equations. We describe how a reduction technique can result in the moduli space of solutions to one problem as the target manifold for more geometry. Credit for this idea, at least in its modern form in an application to monopoles, is due to the physicist Nick Manton. Here is a simplified version.

Consider the question of locating long, slow, closed orbits of the classical mechanics problem of a point mass moving in \mathbf{R}^n with potential energy $V(x)$ and kinetic energy $\frac{1}{2}|\dot{x}|^2$. To make the problem interesting we must have

$$\mathcal{M} = \{x \in \mathbf{R}^n : dV(x) = 0\}$$

a nontrivial manifold, and technically speaking we assume that \mathcal{M} is a nondegenerate critical manifold in the sense of Bott. The action for the problem is

$$J(x) = \int_0^T \left(\frac{1}{2} |\dot{x}|^2 - V(x) \right) dt$$

with $x(0) = x(T)$. In our search for long, slow, closed orbits we renormalize $t = T\tau$ to get

$$J(x) = \int_0^1 \frac{1}{2} T^{-2} \left| \frac{dx}{d\tau} \right|^2 - V(x) \right) T d\tau.$$

Let $\varepsilon = T^{-2}$. In renormalizing time τ , the Euler-Lagrange equations are

$$(*) \quad \varepsilon \left(\frac{d}{d\tau} \right)^2 x(\tau) + dV(x(\tau)) = 0.$$

As $T \rightarrow \infty$, $\varepsilon \rightarrow 0$ and we hypothesize that there is a smooth family of solutions $x(\varepsilon, \tau)$ which for each ε solves (*), and which has a limit as $\varepsilon \rightarrow 0$. We call this orbit at $\varepsilon = 0$ and *adiabatic limit*. The following theorem is cute, if nonobvious, and the proof can be done with advanced calculus.

Theorem. *A necessary condition for $x(\tau)$ to be an adiabatic limit for the original mechanics problem is that x is a closed geodesic on \mathcal{M} .*

By replacing \mathbf{R}^n by a function manifold and $V : \mathbf{R}^n \rightarrow \mathbf{R}$ by one of the geometric integrals of the type with a moduli space of critical points, one finds adiabatic approximations for hyperbolic equations. It is also possible to replace t by static parameters to obtain "adiabatic" limits of static problems. Proving that the adiabatic limit has some physical or geometric reality is much more difficult.

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