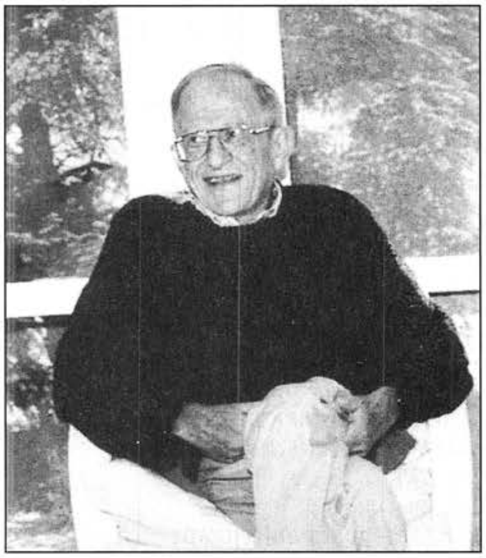
Bernard Dwork (1923–1998)

Nicholas M. Katz and John Tate



All photographs courtesy of Shirley Dwork.

Bernard Dwork, May 27, 1996.

Bernard Morris Dwork, "Bernie" to those who had the privilege of knowing him, died on May 9, 1998, just weeks short of his seventy-fifth birthday. He is survived by his wife of fifty years, Shirley; his three children, Andrew, Deborah, and Cynthia; his four granddaughters; his brothers Julius and Leo; and his sister, Elaine Chanley.

We mention family early in this article, both because it was such a fundamental anchor of Bernie's life and because so many of his mathematical associates found themselves to be part of Bernie's extended family. The authors of this article, although not family by blood, felt themselves to be son and older brother to him.

Bernie was perhaps the world's greatest p-adic analyst. His proof of the rationality of the zeta function of varieties over finite fields, for which he was awarded the AMS Cole Prize in Number Theory, is one of the most unexpected combinations of ideas we know of. In this article we will try to describe that proof and sketch Bernie's other main contributions to mathematics.

But let us start at the beginning. Bernie was born on May 27, 1923, in the Bronx. In 1943 he graduated from the City College of New York with a bachelor's degree in electrical engineering. He served in the United States Army from March 30, 1944, to April 14, 1946. After eight months of training as repeaterman at the Central Signal Corps School, he served in the Asiatic Pacific campaign with the Headquarters Army Service Command. He was stationed in Seoul, Korea, which, according to reliable sources, he once deprived of electricity for twenty-four hours by "getting his wires crossed". This is among the first of many "Bernie stories", some of which are so well known that they are referred to with warm affection in shorthand or code. For instance, "Wrong Plane" refers to the time Bernie put his ninety-year-old mother on the wrong airplane. "Wrong Year" refers to the time Bernie was prevented from flying to Bombay in January 1967 by the fact that his last-minute request for a visa to attend a conference at the Tata Institute was denied on the grounds that the conference was to be held the following year, in 1968.

After his discharge from the army, Bernie worked as an electrical engineer by day and went to school by night, getting his master's degree in electrical engineering from Brooklyn Polytechnic Institute in 1948. Bernie and Shirley Kessler were married on October 26, 1947. He worked successively for I.T.T. (1943–48, minus his army years), the Atomic Energy Commission (1948–50), and the Radiological Research Laboratory of Columbia Medical Center (1950–52). During these years he

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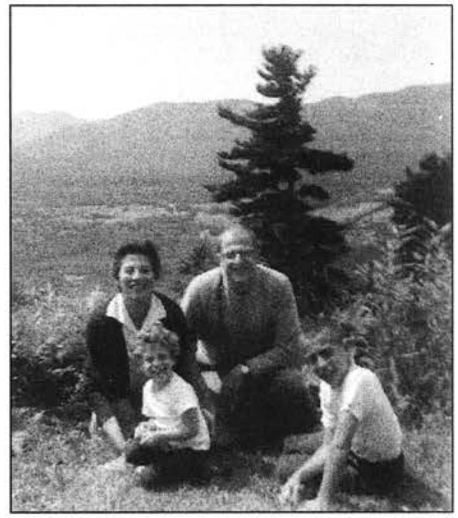
wrote a number of technical reports and published a few papers, including the earliest paper of his listed by *MathSciNet*, "Detection of a pulse superimposed on fluctuation noise", dating from 1950. In view of Bernie's later interest in differential equations (albeit from a p-adic point of view), it is interesting to note that this paper was reviewed by Norman Levinson.

In the summer of 1947 Bernie, encouraged by his brother, took an evening math course at New York University (NYU) with Emil Artin. The course was Higher Modern Algebra Part I: Galois Theory. The course notes, taken by Albert A. Blank, are still one of the best "textbooks" on Galois theory available. The following summer Bernie was back, to take Higher Modern Algebra Part II: Algebraic Number Theory, again with Artin. In the summer of 1949 Bernie took a course from Harold Shapiro on Selected Topics in Additive Number Theory, and in the summer of 1950 he took a course from Artin which was the precursor of Artin's 1950–51 Princeton course Algebraic Numbers and Algebraic Functions.[1]

Bernie was hooked.[2] He continued taking evening courses, both at NYU and at Columbia. He tried to enter NYU as a graduate student in mathematics, but NYU found his educational background wanting. Fortunately for us, Columbia was less picky: in February 1952 Columbia admitted him as a math graduate student, with a scholarship for the year 1952–53. He quit his job and took up full-time study in September of 1952. This was hardly a light decision. He was walking away from a secure career as an engineer, and he had a wife and son to support. Years later he wrote, "Imagine my horror when I found that the scholarship was only for tuition." The family savings soon dwindled, and to make ends meet, Bernie taught night courses at Brooklyn Polytechnic from February 1953 to June 1954.

Bernie had intended, once at Columbia, to study under Chevalley. But Chevalley returned to France that year, so he turned to Artin for advice.[3] Artin gave Bernie a thesis problem and introduced him to the second author of this article, then a young instructor at Princeton who later became Bernie's formal thesis adviser (despite being two years Bernie's junior) when he spent 1953–54 visiting Columbia.



In Europe, en route to 1962 ICM in Stockholm. Shirley and Bernie Dwork with children Cynthia (left) and Andrew.

Bernie received his Ph.D. from Columbia in 1954, with a thesis entitled "On the root number in the functional equation of the Artin-Weil L-series", about the possibility of defining "local root numbers" for nonabelian Artin L-functions, whose product would be the global root number (:= constant in the functional equation connecting $L(s,\chi)$ and $L(1-s,\bar{\chi})$). Bernie solved this problem up to sign; Langlands (unpublished) solved it definitively in 1968, and soon after, Deligne, inspired by the work of Langlands, found a more conceptual solution.

Before we discuss Bernie's later work, let us record the key dates of his professional career as a mathematician. He spent 1954–57 at Harvard as a Peirce Instructor, then 1957–64 at Johns Hopkins (1957–60 as assistant professor, 1960–61 as associate professor, 1961–64 as professor). In 1964 he moved to Princeton, where in 1978 he was named Eugene Higgins Professor of Mathematics. During this time, Bernie spent numerous sabbatical years in France and Italy. In 1992 he was named Professore di Chiara Fama by the Italian government and was awarded a special chair at the University of Padua, which he occupied until his death.

We now return to Bernie's work. In 1959 he electrified the mathematical community when he proved the first part of the Weil conjecture in a strong form, namely, that the zeta function of *any* algebraic variety ("separated scheme of finite type" in the modern terminology) over a finite field was a rational function. What's more, his proof did not at all conform to the then widespread idea that the Weil conjectures would, and should, be solved by the construction of a suitable cohomology theory for varieties over finite fields (a "Weil cohomology" in later terminology) with a plethora of marvelous properties.[4]

Bernie's proof of rationality was an incredible tour de force, making use of a number of new and

[1] *Artin was in those years professor of mathematics at Princeton University, but he regularly taught evening courses at NYU in algebra and number theory.*

[2] *Strictly speaking, he was "rehooked", because he had been very drawn to mathematics in high school and college, but his parents convinced him that he would never earn a living as a mathematician and that engineering would be an adequate outlet for his mathematical enthusiasm. Parents are not always right.*

[3] *The (alphabetically) first author of this article, who was Bernie's thesis student in 1964–66, has fond memories of Bernie's describing his anxieties and his car troubles, which included once breaking down on the George Washington Bridge, on his early trips to Princeton to see Artin.*

[4] *We should point out, however, that by 1964 the ℓ-adic étale cohomology with compact supports of M. Artin and Grothendieck had caught up with Bernie and provided an alternate proof not only of Bernie's rationality result but also of most of the rest of the Weil conjecture for projective smooth varieties over finite fields, all but the "Riemann Hypothesis", which Deligne proved in 1973.*

Shirley and Bernie Dwork with grandchildren at Bernie's retirement party, October 1993.

unexpected ideas. We will describe it in some detail, because even after nearly forty years it remains strikingly fresh and original and gives a good idea of the way Bernie thought. But before we say more about Bernie's proof, we must digress to say a few words about zeta functions of varieties over finite fields.

Thus, we begin with a finite field k—e.g., k might be $\mathbf{Z}/p\mathbf{Z}$ for p a prime number—and an algebraic variety X/k—e.g., X might be an "affine variety", namely, the common zeroes of a finite collection of polynomials $f_i(X_1, \dots, X_n)$ in some finite number n of variables with coefficients in the field k. In this affine example, by a k-valued point of X we mean an n-tuple $(a_1, \dots, a_n)$ in k^n such that $f_i(a_1, \dots, a_n) = 0$ in k for each defining equation f_i. In any case, for any algebraic variety X/k we have the notion of a k-valued point of X, and an easy but essential observation is that the set $X(k)$ of k-valued points of X/k is a finite set. It is known that inside a given algebraic closure $\bar{k}$ of k there is one and only one field extension k_d/k of each degree $d \geq 1$. Each field k_d is itself finite, so the sets $X(k_d)$ are finite for all $d \geq 1$. We denote by $N_d \geq 0$ the integer

$$N_d := \text{Card}\,(X(k_d)).$$

The integers N_d are fundamental diophantine invariants of X/k. The zeta function of X/k, $\text{Zeta}(X/k, T)$, is simply a convenient packaging of these integers: it is defined as the formal series in T with coefficients in $\mathbf{Q}$ and constant term 1 by

$$\text{Zeta}(X/k, T) := \exp\left(\sum_{d\geq 1} N_d T^d/d\right).$$

An important fact is that $\text{Zeta}(X/k, T)$ has integer coefficients. To see this, let us define a *closed point* $\wp$ of X/k to be an orbit of the Galois group $\text{Gal}(\bar{k}/k)$ on the set $X(\bar{k})$, and let us define the degree of $\wp$ to be the cardinality of the orbit which $\wp$ "is". If we denote by B_d the number of closed points of degree d, we have the relation

$$N_d = \sum_{r|d} rB_r,$$

hence the Euler product expansion

$$\text{Zeta}(X/k, T) = \prod_{r\geq 1}(1 - T^r)^{-B_r} = \prod_{\wp}(1 - T^{\deg(\wp)})^{-1}.$$

This last formula makes clear that $\text{Zeta}(X/k, T)$ as power series has integer coefficients.

Let us interpret what it means for $\text{Zeta}(X/k, T)$ to be a rational function of T, say

$$\text{Zeta}(X/k, T) = \prod_i (1 - \alpha_i T) \Big/ \prod_j (1 - \beta_j T)\,.$$

Taking logarithms of both sides and equating coefficients of like powers of T, we see that rationality means precisely that all the integers N_d are determined by the finitely many numbers α_i and β_j by the rule

$$N_d = \sum_j \beta_j^d - \sum_i \alpha_i^d\,.$$

This has the striking consequence that all the N_d are determined by the first few of them. More precisely, once we know upper bounds, say A and B, for the degrees of the numerator and denominator of Zeta, then *all* the integers N_d are determined by the N_d for $d \leq A + B$.

Let us now describe Bernie's proof of the rationality of $\text{Zeta}(X/k, T)$. By an elementary inclusion-exclusion argument, he reduces first to the case when X/k is affine, then to the case when X/k is defined by one equation $f = 0$ for some polynomial f in $k[X_1, \dots, X_n]$, and finally to the case when X/k is the open subset of $f = 0$ where all the coordinates x_i are nonzero. To count points over k, he then uses a nontrivial additive character ψ of k with values in a field K of characteristic zero. For a while in the argument, K could be $\mathbf{C}$, but a different choice of K will be handy in a moment. Bernie exploits the classical orthogonality relation: for $a \in k$, we have, in K,

$$\sum_{y\in k} \psi(ya) = \begin{cases} 0\,, & \text{if } a \neq 0 \\ q := \text{Card}\,(k)\,, & \text{if } a = 0\,. \end{cases}$$

Taking for a the value $f(x)$ at a point x in k^n, we get

$$\sum_{y\in k} \psi(yf(x)) = \begin{cases} 0\,, & \text{if } f(x) \neq 0 \\ q\,, & \text{if } f(x) = 0\,. \end{cases}$$

Papers and Books by Bernie

Under Dwork, B., *MathSciNet* lists over seventy items, including three books. Bernie also wrote two papers as Maurizio Boyarsky. Rather than recopy his complete list of publications, we give a complete alphabetical list of Bernie's coauthors, from which the reader can correctly infer Bernie's delight and enthusiasm in sharing ideas with colleagues around the world.

A. Adolphson	F. Loeser
F. Baldassarri	A. Ogus
S. Bosch	P. Robba
S. Chowla	S. Sperber
G. Christol	F. J. Sullivan
R. Evans	F. Tovena
G. Gerotto	A. J. van der Poorten

Ph.D. Students of Bernard Dwork

Kenneth Ireland (1964)
Alvin Thaler (1966)
Nicholas Katz (1966)
Daniel Reich (1967)
Stefan Burr (1968)
Philippe Robba* (c. 1972)
Steven Sperber* (1975)
Edith Stevenson (1975)
Jack Diamond (1975)
Alan Adolphson (1976)
Mark Heiligman (1982)

*Unofficial students of Bernie

Letting $k^* = k \setminus \{0\}$ we get

$$\sum_{\substack{y\in k\\ x\in k^{*n}}} \psi(yf(x)) = qN_1 \ .$$

The terms with $y=0$ in the above sum are each 1, so we get

$$\sum_{\substack{y\in k^*\\ x\in k^{*n}}} \psi(yf(x)) = qN_1 - (q-1)^n \ .$$

So far, nothing too exciting. But now come the fireworks, in three new ideas. The first is to express analytically a nontrivial additive character of k in a way well adapted to the passage from k to k_d. For this, Bernie now takes for K a large p-adic field (large enough to contain the p-th roots of unity and all roots of unity of order prime to p) and introduces the concept of a *splitting function*, i.e., a power series in one variable over K, say $\Theta(T)$, that converges in a disc $|T| < r$ for some $r > 1$ such that the following two conditions hold:

1) For a in k $(= \mathbf{F}_q)$, denote by Teich(a) its *Teichmuller representative*, the unique solution of $X^q = X$ in the ring O_K of integers in K that reduces to a in the residue field. Then $a \mapsto \Theta(\mathrm{Teich}(a))$ is a nontrivial additive character ψ of k, with values in K.

2) For each $d \geq 1$, the nontrivial additive character ψ_d of k_d obtained by composing ψ with the trace from k_d to k is given as follows. For a in k_d, denote by Teich(a) the unique solution of $X^{q^d} = X$ in O_K which reduces to a in the residue field. Then we have

$$\psi_d(a) = \prod_{i=0}^{d-1} \Theta(\mathrm{Teich}(a)^{q^i}).$$

The simplest example of a splitting function $\Theta(T)$ relative to $\mathbf{F}_q$ is $\exp(\pi(T - T^q))$, where $\pi^{p-1} = -p$, although the fact that this is a splitting function is nontrivial.

To see what this has bought us, lift the polynomial $f(X)$ over k to a polynomial over O_K by lifting each coefficient to its Teichmuller lifting, say to $\sum A_w X^w$, with each A_w its own q-th power. The series

$$F(Y,X) := \prod_w \Theta(A_w Y X^w)$$

has the property that for any point (y,x) in k^{n+1}, we have

$$\psi(yf(x)) = F(\mathrm{Teich}(y,x)).$$

More generally, for any $d \geq 1$, if we define

$$F_d(Y,X) := \prod_{i=0}^{d-1} F(Y^{q^i}, X^{q^i}),$$

then for any point (y,x) in $(k_d)^{n+1}$, we have

$$\psi_d(yf(x)) = F_d(\mathrm{Teich}(y,x)).$$

Let us recall how this relates to counting points. We have

$$\sum_{\substack{y\in k^*\\ x\in k^{*n}}} F(\mathrm{Teich}(y,x)) = qN_1 - (q-1)^n,$$

and, more generally, for each $d \geq 1$ we have

$$\sum_{\substack{y\in k_d^*\\ x\in k_d^{*n}}} F_d(\mathrm{Teich}(y,x)) = q^d N_d - (q^d-1)^n.$$

The second new idea is to express a sum of that form as the trace of a completely continuous operator on a p-adic Banach space. On the space of formal series over K in $n+1$ variables, Bernie defines an operator Ψ_q by

Prizes and Fellowships Awarded to Bernie

Sloan Fellowship, 1961
Cole Prize in Number Theory, 1962
Guggenheim Fellowship, 1964
Townsend Harris Medal, 1969 (from the Alumni Association of the City College of New York)
Guggenheim Fellowship, 1975

$$\Psi_q\left(\sum B_w X^w\right) := \sum B_{qw} X^w.$$

Arguing heuristically, one sees that for any formal series $F = \sum C_w X^w$ in the $n+1$ variables $X_0 := Y$ and X_i for $i = 1$ to n, the composition of Ψ_q with multiplication by F has a trace, given by

$$\begin{aligned}\mathrm{Trace}(\Psi_q \circ F) &= \mathrm{Trace}\left(\Psi_q \circ \sum C_w X^w\right)\\ &= \sum C_w \ \mathrm{Trace}(\Psi_q \circ X^w)\\ &= \sum C_{(q-1)w}\ .\end{aligned}$$

The last equality is valid because in the "basis" given by all monomials, the "matrix" of $\Psi_q \circ X^w$ has no nonzero terms on the diagonal unless w is $(q-1)v$ for some v, in which case the (v, v) entry is 1 and all other diagonal entries vanish. Still arguing heuristically, we interpret $\sum C_{(q-1)w}$ in terms of values of $F = \sum C_w X^w$ at $(n+1)$-tuples of $(q-1)$-th roots of unity in K by the identity

$$\begin{aligned}(q-1)^{n+1} \sum C_{(q-1)w} &= \sum_{a \in (\mu_{q-1}(K))^{n+1}} F(a)\\ &= \sum_{\substack{y \in k^* \\ x \in k^{*n}}} F(\mathrm{Teich}(y, x))\ ,\end{aligned}$$

an identity that does hold, in fact, when F is a single monomial X^w or when F has coefficients tending to zero. Thus we get a heuristic identity, using now the particular choice of F

$$F(Y, X) := \prod_w \Theta(A_w Y X^w),$$

namely,

$$\begin{aligned}(q-1)^{n+1}\mathrm{Trace}(\Psi_q \circ F) &= \sum_{\substack{y \in k^* \\ x \in k^{*n}}} F(\mathrm{Teich}(y, x))\\ &= qN_1 - (q-1)^n.\end{aligned}$$

The d-th iterate of $\Psi_q \circ F$ is easily checked to be $\Psi_{q^d} \circ F_d$, so for each $d \geq 1$ we have a heuristic identity

$$(q^d - 1)^{n+1}\mathrm{Trace}((\Psi_q \circ F)^d) = q^d N_d - (q^d - 1)^n.$$

Because $\Theta(T)$ converges in a disc strictly bigger than the unit disc, F has good convergence properties. By restricting the action of $\Psi_q \circ F$ to a space of series with suitable growth conditions, one can make sense of $\Psi_q \circ F$ as a completely continuous endomorphism of a p-adic Banach space. The Fredholm characteristic series $\det(1 - T\Psi_q \circ F)$ is an entire function of T, which as a formal power series is given by

$$\begin{aligned}&\det(1 - T\Psi_q \circ F)\\ &\quad = \exp\left(-\sum_{d \geq 1} \mathrm{Trace}((\Psi_q \circ F)^d)T^d/d\right).\end{aligned}$$

Denote by $\Delta(T)$ the entire function $\det(1 - T\Psi_q \circ F)$. Then from the identities

$$(q^d - 1)^{n+1}\mathrm{Trace}((\Psi_q \circ F)^d) = q^d N_d - (q^d - 1)^n$$

for $d \geq 1$ we get the identity of series

$$\begin{aligned}&\prod_{i=0}^{n+1} \Delta(q^i T)^{(-1)^{n-i}\binom{n+1}{i}}\\ &\quad = \mathrm{Zeta}(X/k, qT)\prod_{i=0}^{n}(1 - q^i T)^{(-1)^{n-i}\binom{n}{i}}.\end{aligned}$$

Since $\Delta(T)$ is p-adically entire, we see that the zeta function is the ratio of two p-adically entire functions.

To recapitulate, we now know that the zeta function as power series has integer coefficients and that it is the ratio of two p-adically entire functions. We also know the zeta function has a nonzero radius of archimedean convergence (since we have the trivial archimedean bound $N_d \leq (q^d - 1)^n$). Bernie's third new idea is to generalize a classical but largely forgotten result of E. Borel to show that *any* power series with these three properties is a rational function. Thus he proves the rationality of the zeta function.

Bernie then further developed his p-adic approach and applied it to study in detail the zeta function in the special case of a projective smooth hypersurface X/k, say of dimension n and degree d. The Weil conjectures predicted that its zeta function should look like

$$P(T)^{(-1)^{n+1}} \Big/ \prod_{i=0}^{n}(1 - q^i T),$$

with P a polynomial of known degree (namely, the middle "primitive" Betti number of any smooth projective hypersurface $H_{n,d}$ over the complex numbers, of the same degree d and dimension n as X) and that $P(T)$ should satisfy a certain functional equation. Bernie's theory allowed him to confirm these predictions and to study the p-adic valuations of the reciprocal zeros of $P(T)$. He proved, for instance, that, provided p does not divide d, the Newton polygon of $P(T)$ always lies above the middle dimensional primitive "Hodge polygon" of $H_{n,d}$; cf. his Stockholm ICM talk, p. 259. This is the first instance we know of a nontrivial relation between Newton and Hodge polygons. Such relations were later established in great generality by Mazur.

Bernie also studied the way his p-adic construction varied when the projective smooth hypersurface varied in a family. The rich structure he discovered was the first nontrivial instance of what later came to be called an F-crystal. Roughly speaking, an F-crystal is a differential equation upon which a "Frobenius" operates. Bernie correctly conjectured that the underlying differential equation to his F-crystal was the relative (primitive, middle dimensional) de Rham cohomology of the family, endowed with its Gauss-Manin connection, i.e., the classical Picard-Fuchs equations attached to the family. He also studied the variation in a family of the Newton polygon attached to an F-crystal, which led him to discover the "slope filtration" of an F-crystal. It seems fair to say that a desire to understand Bernie's results in a more cohomological context was one of the main motivations for the development, by Grothendieck and Berthelot in the late 1960s, of crystalline cohomology.

One of Bernie's key discoveries was that those differential equations that "admit a Frobenius", i.e., that underlie an F-crystal, have very special properties as p-adic differential equations (for instance, solutions have p-adic radius of convergence 1 in generic discs). By crystalline theory, any Picard-Fuchs equation underlies an F-crystal and hence has these special properties for almost all primes p. One enduring fascination of Bernie's was the still open problem of characterizing, by p-adic conditions for almost all p, those differential equations over, say, number fields, that "come from geometry" (or "are motivic", in the new terminology) in the sense, say, that they are successive extensions of pieces, each of which is a subquotient of a Picard-Fuchs equation.

Another of Bernie's fundamental and iconoclastic discoveries concerns the arithmetic significance of equations with irregular singular points. Picard-Fuchs equations are known to have regular singular points, and for a long time it was generally believed that the only differential equations relevant to algebraic geometry were those with regular singular points. But in the early 1970s Bernie achieved the remarkable insight that equations with irregular singular points (e.g., those for the hypergeometric function ${}_pF_q$ for arbitrary p and q, $q \neq p-1$) were not only not to be regarded as pathological, but they were in fact a fundamental feature of the p-adic algebro-geometric landscape, playing the same role for exponential sums in families as regular singular equations play for counting points in families.

Pursuing these ideas led Bernie to a long and deep study of the p-adic and arithmetic properties of differential equations, both for their own sake and for their interaction with the arithmetic of varieties over finite fields and with the algebraic geometry of families over $\mathbf{C}$. He remained actively engaged in this study right up to his death.

This is perhaps an appropriate point to comment on three early mathematical influences on Bernie.

1. We have already explained how it was an NYU evening course taught by Emil Artin in 1947 which hooked Bernie on mathematics.
2. His interest in mod p and p-adic properties of Picard-Fuchs equations probably dates from the late 1950s at Johns Hopkins, when he learned from Igusa that the Hasse invariant for the Legendre family of elliptic curves

$$y^2 = x(x-1)(x-\lambda)$$

   in any odd characteristic p is a mod p solution of the Picard-Fuchs equation for that family (explicitly, the differential equation for the hypergeometric function $F(1/2, 1/2, 1; \lambda)$).
3. Where did Bernie get the idea that there could be a connection between p-adic analysis and zeta functions? It grew out of a letter from the second author of this article to Bernie, dated February 13, 1958, an extract of which is quoted on page 257 of Bernie's ICM Stockholm talk. The letter contained a result on the "unit root" of an ordinary elliptic curve, which could be proved by using work of Michel Lazard on formal groups to show that certain p-adic power series have integral coefficients. The letter writer, considering Bernie to be the world's leading expert in such matters, challenged him to prove those results by p-adic analysis. Bernie met the challenge almost by return mail and, going further, discovered the "close connection" that he mentions in the following quote from loc. cit., p. 250, "...but using unpublished results of Tate and Lazard, we give indications of the existence of a deformation theory, involving a close connection between hypergeometric series and the zeros of the zeta functions of elliptic curves. We became aware of this connection in 1958; it was the first suggestion of a connection between p-adic analysis and the theory of zeta functions."

Nurtured in Dwork's amazingly original mind, what marvelous fruit these three seeds bore.