Alberto P. Calderón Receives National Medal of Science

Alberto P. Calderón, University Professor Emeritus of Mathematics at the University of Chicago, received the National Medal of Science on September 16, 1991. The medal is the nation's highest award for scientific achievements. One of twenty medalists, Professor Calderón was cited "for his ground-breaking work on singular integral operators leading to their application to important problems in partial differential equations, including his proof of uniqueness in the Cauchy problem, the Atiyah-Singer index theorem, and the propagation of singularities in non-linear equations."

Alberto P. Calderón

Commentary on Calderón's Research

The Managing Editor of the *Notices* solicited the following piece describing Calderón's mathematical achievements. The piece was written by Richard W. Beals, Ronald R. Coifman, and Peter W. Jones, all of Yale University.

The last forty years have seen remarkable progress in analysis, and much of this is a result of Calderón's seminal work and ideas. In long term collaboration with Antoni Zygmund, he established the so-called Calderón-Zygmund school of analysis. (Further background may be found in the article "The School of Antoni Zygmund," by Ronald R. Coifman and Robert S. Strichartz, which appears in *A Century of Mathematics in America* (AMS, 1989), vol. 3, pages 343–368.) His fundamental contributions to partial differential equations and concrete operator theory have profoundly affected modern mathematics.

The Zygmund program bucked the trend in the fifties toward abstract mathematics by concentrating on basic questions of real and complex analysis. It had as a goal the development of methods for understanding the structure of natural operations on functions, culminating in "Calderón-Zygmund Theory." Calderón, Zygmund, and their students developed tools for understanding the relations between differentiability properties of functions and the properties of their harmonic or holomorphic extensions (Hardy spaces and boundary value problems). For example, holomorphic functions $f = u + iv$ on the upper half-plane $\{x + iy : y > 0\}$ have the property that, on the boundary, $v = Hu$, where H is the Hilbert transform:

$$Hu(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|y-x|>\varepsilon} \frac{f(y)}{x-y}\,dy.$$

The Hilbert transform is now understood to be one of the most important operators in analysis. Prior to the work of Calderón and Zygmund, this operator was primarily studied via methods from complex analysis. Calderón and Zygmund, following previous work of Marcinkiewicz, Riesz, and Zygmund, provided the modern, real variable tools for understanding this operator and explained why the limit in the definition of Hu exists almost everywhere and is a bounded operator when u is in a suitable space (e.g. L^2).

These questions tied in naturally to the study of various operators generalizing the Hilbert transform (Calderón-Zygmund operators) which permitted a detailed understanding of the size relations between partial derivatives of functions on $\mathbf{R}^n$. In $\mathbf{R}^n$ one natural generalization is the Riesz transform $R_j f(x) = \lim_{\varepsilon\to 0} c_n \int_{|x-y|>\varepsilon} \frac{(y_j-x_j)f(y)}{|x-y|^{n+1}} dy$. By taking Fourier transforms one can derive many identities such as $\frac{\partial^2}{\partial x_j \partial x_k} f = R_j R_k \Delta f, f \in C^2$, so the Calderón-Zygmund results imply L^p-boundedness of mixed partials when $\Delta f \in L^p$.

An example of the power of this point of view is given by considering the existence problem for solutions in $\mathbf{R}^2$ of the Beltrami equation:

$$\bar{\partial} F = \mu \partial F$$

where $\mu \in L^\infty(\mathbf{R}^2)$, $\|\mu\|_\infty < 1$. The Calderón-Zygmund theory shows that solutions may be written in a Neumann series and thus vary analytically in μ. The solutions F are quasi-conformal mappings with a given dilatation μ. This fact is one of the central results in modern complex analysis.

Calderón and Zygmund, in a series of seminal papers, developed fundamental real variable tools and established a calculus of singular integral operators, currently known as pseudodifferential calculus, profoundly affecting partial differential equations and creating interactions between geometry and analysis as in the Atiyah-Singer Index Theorem. The main impetus for acceptance of the singular integral calculus as a principal tool in partial differential equations came through Calderón's elegant treatment of uniqueness of solutions to the Cauchy problem for hyperbolic equations, and more spectacularly through his very general existence and uniqueness results for linear equations and systems. This groundbreaking result illustrated the power and flexibility of the new methodology and quickly became part of modern "calculus."

Calderón continued this work (again bucking the trend), pushing for a calculus with minimum regularity assumptions on coefficients of differential operators. He felt that the development of such a calculus would enable a better analysis of nonlinear partial differential operators. This work led to deep estimates on commutator integrals and the establishment of Calderón-Zygmund analysis in a nonlinear context. In particular, Calderón succeeded by ingenious analytic methods in proving the boundedness on L^2 of the Cauchy integral for Lipschitz curves (with small constant), which implied the so-called Denjoy conjecture in complex analysis. The Cauchy integral is defined on a rectifiable curve Γ by

$$\mathcal{C}f(z) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int\limits_{\substack{\Gamma \\ |\zeta - z| > \varepsilon}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

It is not even clear that the limit exists for continuous f. A remarkable aspect of this work is that, after developing the (real variable) Calderón-Zygmund theory, Calderón was not reluctant to return to the methods of complex analysis. This view had fallen out of favor, mostly due to the power of Calderón-Zygmund theory! His philosophy that one should use the Cauchy integral formula ($\mathcal{C}(1) \equiv 0$) has led to our present understanding of necessary and sufficient conditions for L^2 boundedness of general singular integrals.

Calderón, like his teacher and collaborator Zygmund, has focused his energy on the development of analytical tools permitting a blend of the "miracles" of complex analysis with a deep understanding of real variable inequalities. His paper developing the complex method of interpolation is a beautiful illustration of these ideas. Here the main point is to prove hard inequalities by embedding them in a one complex parameter family of inequalities where the extremities are easier to prove, thereby enabling the use of the maximum principle to prove the desired intermediate results. (Formally, this is known as interpolation of Banach spaces with norms depending on the complex parameter.) The paper on interpolation also introduced (as a synthesis of Littlewood-Paley theory) the so-called Calderón identities (currently known as the continuous wavelet transform) as a main tool for describing function spaces and their approximation properties. The Calderón identity states that for a large class of functions ψ, one has for any F which is continuous and of compact support,

$$F(x) = \int_0^\infty F * \psi_t * \psi_t \frac{dt}{t},$$

where $\psi_t(x) = t^{-n} \psi\left(\frac{x}{t}\right)$, $x \in \mathbf{R}^n$, $t > 0$. This identity allows one to write many natural operators T as

$$T = \int_0^\infty T_t \frac{dt}{t},$$

where the operators T_t are "simple" pieces of T which are more amenable to analysis. Clever choices of T_t allow one to write $T = \sum_{n=-\infty}^{\infty} T_n$, $T_n = \int_{2^n}^{2^{n+1}} T_t \frac{dt}{t}$, where the operators T_n are essentially spectral projections which are almost diagonal. This has proved to be of great significance in numerical analysis and signal processing.

Another example of interaction between complex and real analysis is provided by the method introduced by Calderón to reduce boundary value problems to the solution of boundary pseudodifferential equations through the Calderón operator. He discovered this by recognizing the "real variable" role played by the Cauchy projection in solving Dirichlet and Neumann problems in the unit disk.

Calderón's influence on analysis and related areas is due in large part to the many methods that he invented and perfected. In modern Fourier analysis, theorems are usually much less important than the techniques developed to prove them. Calderón's techniques have been absorbed as standard tools of harmonic analysis and are now propagating into nonlinear analysis, partial differential equations, complex analysis, and even signal processing and numerical analysis.

Biographical Sketch

Alberto P. Calderón was born on September 14, 1920, in Mendoza, Argentina. After completing his secondary education at the state high school of his home town, he enrolled in the engineering school of the University of Buenos Aires, from which he graduated as a civil engineer in 1947. He had always been interested in mathematics and soon became a student of Alberto González Domínguez and Antoni Zygmund, who was a visiting professor at the University of Buenos Aires in 1948. He finally received his doctorate in Mathematics from the University of Chicago in 1950, which he attended as a fellow of the Rockefeller Foundation.

Professor Calderón began his academic career as an assistant to the chair of electric circuit theory at the engineering school of the University of Buenos Aires (1948). After graduation from the University of Chicago, he became visiting associate professor at the Ohio State University (1950–1953). He also was a member of the Institute for Advanced Study in Princeton (1954-1955) and served as an associate professor at the Massachusetts Institute of Technology (1955–1959). He then moved to the University of Chicago, where he served as professor of mathematics (1959–1968), Louis Block Professor (1968–1972), and chairman of the mathematics department (1970–1972). In 1972, he returned to MIT as a professor of mathematics, and, in 1975, he became University Professor of Mathematics at the University of Chicago until his retirement in 1985. Currently, he is professor emeritus with a post-retirement appointment at Chicago and an honorary professor at the University of Buenos Aires; he had held the latter position since 1975. He also was, for short periods of time, a visiting professor at several American and European universities such as Cornell University, Stanford University, National University of Bogotá (Colombia), Collège de France, University of Paris (Sorbonne), University of Madrid, University of Rome, and University of Göttingen.

Professor Calderón has received the following awards and honors (in chronological order): "Provincia de Sante Fe" prize (I.P.C.L.A.R.), Argentina (1969); Honorary Doctorate, University of Buenos Aires (1969); AMS Bôcher Memorial Prize (1979); Konex Prize, Argentina (1983); Union Carbide Prize, Argentina (1984); "Consagración Nacional" Prize, Argentina (1989); Wolf Prize in Mathematics, Jerusalem, Israel (1989); AMS Steele Prize (fundamental research work category) (1989); and Honorary Doctorate, Technion, Israel (1989).

Professor Calderón has been elected a member of the following academies: American Academy of Arts and Sciences, U.S.A. (1957); National Academy of Exact, Physical and Natural Sciences, Buenos Aires, Argentina (1959); National Academy of Sciences, U.S.A. (1968); Royal Academy of Sciences, Madrid, Spain (1970); Latin American Academy of Sciences, Caracas, Venezuela (1983); French Academy of Sciences, Paris, France (1984); and Third World Academy of Sciences, Trieste, Italy (1984).

Professor Calderón presented an AMS Invited Address in University Park in 1957 and delivered the American Mathematical Society Colloquium Lectures on Singular Integrals in Ithaca in 1965. He also gave an Invited Address at the International Congress of Mathematicians in Moscow (1966).

Professor Calderón has been a member of the AMS for forty years. He was a Member-at-Large of its Council (1965–1967) and served on several of its committees, such as the *Transactions* and *Memoirs* Editorial Committee, the Nominating Committee, the Colloquium Editorial Committee, etc. He has also served as an associate editor of the *Duke Mathematical Journal*, *Illinois Mathematical Journal*, *Journal of Functional Analysis*, *the Journal of Differential Equations*, and *Advances in Mathematics*.

Professor Calderón has published some seventy-six scientific papers on various topics such as: Real Variables, Harmonic Analysis, Functional Analysis, Singular Integrals, and Partial Differential Equations. A number of these papers, mostly on Singular Integrals, were written in collaboration with his teacher, Professor Antoni Zygmund, and made his initial reputation as a mathematician. He also has had some twenty-seven doctoral students. Some of them in turn became reputed mathematicians as, for example, Robert T. Seeley, whose extension of the work of Calderón and Zygmund to singular integral operators on manifolds became the foundation of the famous Atiyah-Singer index theorem.