Louis Nirenberg Receives National Medal of Science

On October 18, 1995, Louis Nirenberg, former director and professor of mathematics at New York University's Courant Institute of Mathematical Sciences received the National Medal of Science, the nation's highest scientific honor.

Nirenberg was selected for his fundamental contributions to mathematical analysis. His work has had a major influence in the development of different areas of mathematics and in their applications. It has focused on partial differential equations and related parts of analysis, the basic mathematical tools of modern science. He developed intricate interactions between mathematical analysis, differential geometry, and complex analysis and applied them to the theory of fluid flow and other physical phenomena.

"Nirenberg's influence is not limited to the many original and fundamental contributions he has made to the subject," said David W. McLaughlin, professor of mathematics and director of the Courant Institute. "He continues to lead research at the Institute, of which he has been a member since 1949.

"He has not only played a major role in the development of mathematical analysis worldwide but has had significant influence on the development of young mathematicians, as indicated by his direction of over forty Ph.D. students. As a past director of Courant, he has demonstrated leadership and vision for the mathematical sciences community. The clarity of his writing, his lectures, and numerous expository articles continue to inspire generations of mathematicians," McLaughlin concluded.

Nirenberg has received many honors and awards over the course of his career. He is a member of the National Academy of Sciences, the American Philosophical Society, and the American Academy of Arts and Sciences. In 1982 he was named the first winner of the Crafoord Prize, created by the Royal Swedish Academy as an equivalent to the Nobel Prize for scientific research in areas outside existing Nobel categories.

He also received the AMS Bôcher Memorial Prize for "outstanding contributions to mathematical analysis." Nirenberg has been a Sloan Fellow and a Guggenheim Fellow. Most recently, he received the AMS Leroy P. Steele Prize for Lifetime Achievement and the New York City Mayor's Award for Excellence in Science and Technology.

Born in Hamilton, Canada, Nirenberg received his undergraduate training at McGill University and his M.S. and Ph.D. at NYU. He served as director of the Courant Institute from 1970–1972. He is a foreign member of Accademia Nazionale dei Lincei, Académie des Sciences de Paris, Accademia Mediterreanea delle Scienze, Istituto Lombardo Accademia Scienze a Lettere, and Academy of Sciences of Ukraine.

The editors of the *Notices* asked Luis Caffarelli of the Institute for Advanced Study and Joseph J. Kohn of Princeton University to write about the work of Louis Nirenberg. Their articles follow.

Nirenberg's Work in Partial Differential Equations

Luis Caffarelli

The work of Louis Nirenberg has enormously influenced all areas of mathematics linked one way or another with partial differential equations: real and complex analysis, calculus of variations, differential geometry, continuum and fluid mechanics. From this work I have chosen to mention two papers and four areas in which Nirenberg had a long-standing interest.

The first is his paper on functions of bounded mean oscillation (in collaboration with Fritz John [8]). Probably motivated by the work of John in elasticity, they consider the space of functions of bounded mean oscillation (BMO): those functions u on R^n for which there exists a constant K such that

$$u : \frac{1}{|Q|}\int |u - \bar{u}_Q|dx \leq K$$

holds for any cube Q, where $\bar{u}_Q$ is the average of u on Q. They prove the cornerstone theorem for BMO: If u is in BMO, then u is in an exponential class; that is,

$$\int e^{\lambda|u-\bar{u}_Q|} < \infty$$

for some λ. Since that paper, BMO spaces have become a central element of analysis because of their invariance under dilations, because of their duality to H^1, and as a natural replacement for L^∞ because many important operators which fail to be continuous on L^∞ are so on BMO.

The second paper, with R. Kohn and me [20], is on the possible singularities to solutions of incompressible Navier-Stokes equations. In that paper it is proven that for a singularity to occur the energy density must concentrate around the point at a given rate. From this, in particular, it follows that the set of singularities has one-dimensional Hausdorff measure zero (are "less than a curve") in 3-space-time.

The four long-standing areas of interest are:

1. Regularity and solvability of elliptic equations of order $2n$. Of the many contributions in this area, it is necessary to mention the papers with Agmon and Douglis [5, 9], where they set up the basic modern theory of regularity up to the boundary for solutions to elliptic partial differential equations and systems with coercive (i.e., elliptic) boundary conditions. This is accomplished in both the Hölder and L^p function spaces.

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The fundamental interpolation inequalities in [6] establish how to control intermediate Sobolev norms by lower and higher norms. This is a crucial technique for semilinear equations, where certain combinations of zero- and first-order derivatives must be controlled by second-order operations. His works on degenerate equations with J. J. Kohn [12, 13] are a masterful and beautiful blend of geometry, barriers, and functional analytical a priori estimates.

2. The next important area involves the Minkowski problem and fully nonlinear equations. Here we mention his thesis [1], where he solves the Weyl and Minkowski problems by the method of continuity, and a series of papers written between 1984 and 1988 with J. J. Kohn, Spruck, and me, where (simultaneously with Krylov, Ivochkina, and others) a complete up-to-the-boundary regularity theory is developed for solutions of fully nonlinear equations that are uniformly elliptic or have special symmetries (Monge-Ampère and similar types).

3. An area basically due to Nirenberg (in collaboration with Kinderlehrer and Spruck) is the theory of higher regularity for free boundary problems (compare with contemporary work of Isaacov). This theory applies to a wide range of problems, from two-phase elliptic and parabolic problems to the singular configuration of three minimal surfaces coming together in an edge. More recently, in the context of flame propagation [22], the convergence of singular perturbation of free boundary problems is studied, and there is ongoing research on the geometry of the free boundary.

4. The final area we could call symmetry properties of solutions to invariant equations: Given equations and domains that have some symmetry or monotonicity properties, when do solutions inherit this property? Starting with Nirenberg's celebrated paper [18] with Gidas and Ni, where the Alexandrov method was applied to solve some long-standing questions in conformal deformations, this has become a beautiful theory thanks to the many methods and ideas contributed by Nirenberg. Let me just mention its application to combustion [23] and the small domain eigenfunction theory developed in [24].

Nirenberg's Work in Complex Analysis

Joseph J. Kohn

Introduction

Louis Nirenberg is recognized throughout the world for his many important research contributions to mathematics, his marvelous lectures, and his lucid expository writing. His range of in-

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terest is very broad: differential equations, harmonic analysis, differential geometry, functional analysis, complex analysis, etc. Here I will concentrate on his work in complex analysis and closely related areas even though only a small proportion of Nirenberg's publications are in this field. To justify this selection, I offer the following anecdote ("*se non e vero e ben trovato*"). In a lecture, Professor Oka drew a large circle on the blackboard and said: "This closed disc represents all of mathematics: the interior represents the theory of several complex variables and each of the boundary points represents other mathematical fields such as topology, algebra, probability, etc."

Perhaps a more convincing justification of my selection is that this is the area of Nirenberg's work that I know best, and I have had the privilege and pleasure to participate in some of it.

I first met Louis Nirenberg in the early 1950s when I was a graduate student at Princeton. My thesis advisor, D. C. Spencer, would meet frequently with Nirenberg either in Fine Hall or in the Courant Institute, and I was often included in these sessions. Those meetings were a real education for me. They usually started with a discussion of some general question concerning analysis on complex manifolds and/or pseudogroups. Nirenberg would then elicit clarifications, specific examples, and motivation; then he would proceed to explain where the "heart of the matter" is—it was usually an a priori estimate for a partial differential equation. Then we would grapple with the estimate or with a special case. It was in these discussions that Nirenberg's technical virtuosity shone through. The problem was most often attacked by "calculus tricks", and invariably integration by parts played a crucial role. In those days, the Courant Institute had a certain "mystique": there was a tremendous sense of loyalty and unity personified by the patriarch Courant himself surrounded by a galaxy of shining stars and dedicated staff members. There was an enthusiastic spirit of common undertaking. Around noon a group would gather, and we would go to lunch at one of the local restaurants. The conversation was always animated: the topics ranged from mathematics to politics, and the talk was liberally sprinkled with the latest jokes. Visiting the Courant felt like being a part of an extremely close-knit family.

At this time Nirenberg started getting involved in the area between the theory of several complex variables and the theory of partial differential equations, more precisely problems in several complex variables which can be solved by the techniques of partial differential equations and problems in partial differential equations which are motivated by several complex variables. From the late 1950s to the early 1970s Nirenberg was very active in this area. Nirenberg's contributions to other areas (i.e., harmonic analysis, nonlinear partial differential equations, differential geometry, etc.) are described in the accompanying article by Luis Caffarelli.

Complex Vector Fields

A fundamental question that arises in the study of complex manifolds is: When is an almost-complex structure given by a complex structure? In terms of vector fields, the problem can be formulated as follows: Let U be a neighborhood of the origin in R^{2n}, and let $L_1, \dots, L_n$ be n complex vector fields on U such that $L_1, \dots, L_n, \bar{L}_1, \dots, \bar{L}_n$ are linearly independent over $\mathbb{C}$ at each point, where:

$$(1) \qquad L_j = \sum_k a_j^k \, \partial/\partial x_k \text{ and } \bar{L_j} = \sum_k \bar{a}_j^k \, \partial/\partial x_k.$$

These vector fields define an *almost complex structure on U*. The problem is to find complex coordinates $\{z_1, z_2, \dots, z_n\}$ on a neighborhood V of the origin such that at each point of V the space spanned by $L_1, \dots, L_n$ equals the space spanned by $\partial/\partial z_1, \dots, \partial/\partial z_n$. It is clear that a necessary condition for this is:

$$(2) \qquad [L_i, L_j] = L_i L_j - L_j L_i = \sum_k a_{ij}^k L_k.$$

An almost complex structure satisfying (2) is called *integrable*. In case the real and imaginary parts of the $\{a_{ij}^k\}$ are real analytic functions, then the problem can be solved by a modification of the Cauchy-Kowalevsky theorem. In 1957 Newlander and Nirenberg (see [1]) proved this when the $\{a_{ij}^k\}$ are in C^∞. This is a fundamental result; the problem had been open for years, and its resolution is used in the study of many aspects of complex manifolds, particularly deformation theory.

Although the problem is linear, Newlander and Nirenberg's proof is via nonlinear partial differential equations. The idea of the proof is to reduce the problem to a system of nonlinear partial differential equations such that each equation involves derivatives with respect to only one (complex) variable. This is done as follows: first, introduce the "approximate" complex coordinates $\{z_1, z_2, \dots, z_n\}$ in a neighborhood of the origin in $\mathbb{R}^{2n}$ such that

$$(3) \qquad L'_j = \partial/\partial z_j + \sum_k b_j^k \, \partial/\partial \bar{z}_k, \; j = 1, \dots, n$$

where the L' are linear combinations of the L and $b_j^k(0) = 0$. Next, assuming that the required complex coordinates $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$ exist and are differentiable functions of the coordinates $\{z_1, z_2, \dots, z_n\}$, the z, as functions of the ζ, satisfy the equations

$$\partial z_k/\partial\bar\zeta_j - \sum_m b_m^k\,\partial\bar z_m/\partial\zeta_j = 0, \quad j = 1, \dots, n. \tag{4}$$

Here each j^{th} equation involves only differentiations with respect to ζ_j and $\bar\zeta_j$. The system is then solved using an inverse of $\bar\partial_{\bar\zeta}$.

A more general problem is the problem of local embedding of CR manifolds. This problem can be described in terms of complex vector fields as follows. Let $L_1, \dots, L_k$ be complex vector fields in a neighborhood of the origin in $\mathbb{R}^m$ such that $L_1, \dots, L_k, \bar L_1, \dots, \bar L_k$ are linearly independent and such that the $[L_i, L_j]$ are linear combinations of the $L_1, \dots, L_k$. The problem is to find an embedding F of a neighborhood U of the origin in $\mathbb{R}^m$ into $\mathbb{C}^N$ such that the space of vector fields tangent to $F(U)$ spanned by the images of the $L_1, \dots, L_k$ is equal to the space of vector fields spanned by the fields of the form

$$\sum_1^N a_i\partial/\partial z_i$$

which are tangent to $F(U)$; here $z_1, \dots, z_N$ denote the coordinates on $\mathbb{C}^N$. In case $m = 2k$ this problem is just the integrability problem solved by the Newlander-Nirenberg theorem. Again, in the analytic case, the problem can be analyzed by a modification of the Cauchy-Kowalevsky method. In the differentiable case Nirenberg solved the problem when the structure is "flat", that is, if

$$[L_i, \bar L_j] = \sum_k a_{ij}^k L_k + \sum_k b_{ij}^k \bar L_k. \tag{5}$$

This result is known as the Nirenberg-Frobenius theorem (see [2]).

In case condition (5) does not hold, there are some results on the local problem only under special circumstances. The main result, due to Kuranishi, is concerned with CR manifolds of co-dimension one, that is, $m - 2k = 1$, which are strongly pseudo-convex. To define strongly pseudo-convex, let T denote a vector field such that $\{L_1, \dots, L_k, \bar L_1, \dots, \bar L_k, T\}$ are linearly independent and such that $\bar T = -T$. Then we have

$$[L_i, \bar L_j] = c_{ij}T + \sum_s a_{ij}^s L_s + \sum_s b_{ij}^s \bar L_s. \tag{6}$$

The matrix (c_{ij}) is the *Levi form* of the CR manifold, and the manifold is *strongly pseudo-convex* if this form is positive definite. Kuranishi's result is that the CR manifold of co-dimension one is locally embeddable if it is strongly pseudo-convex and has dimension $m > 7$. Akahori proved this result for dimension $m = 7$. The problem for $m = 5$ is still open. Nirenberg in [12] gave a remarkable example of a three-dimensional strongly pseudo-convex CR manifold which is not embeddable. To do this, he constructed a vector field L in a neighborhood of the origin in $\mathbb{R}^3$ such that L, $\bar L$, and $[L, \bar L]$ are linearly independent and such that if u satisfies the equation $\bar L u = 0$ in any open set, then u is a constant. Hence, there does not exist an embedding, since, if there was an embedding, the pullbacks of holomorphic functions would be nonconstant solutions of the equation $\bar L u = 0$.

Deformations of Complex Structures

One of the first applications of the Newlander-Nirenberg theorem was in the study of deformations of complex structures (see [3] and [4]). The problem concerns the existence of deformations. Suppose that for each t in a neighborhood U of the origin in $\mathbb{C}^m$ there exists a compact complex manifold V_t such that any two of these are diffeomorphic and such that the complex structure depends differentiably on t. The family $\{V_t\}$ is called a deformation of V_0. Then any tangent vector at the origin gives rise to an infinitesimal deformation. There is a natural mapping of the tangent space at the origin into $H^1(V_0, \Theta_0)$, the cohomology of the sheaf of germs of holomorphic tangent vectors of V_0. An important problem is to find sufficient conditions for a nontrivial family of deformations of a compact complex manifold V_0. Kodaira, Nirenberg, and Spencer (in [3]) proved the following result. If $H^2(V_0, \Theta_0) = 0$, then there exists a family of deformations $\{V_t\}$ such that the tangent space of U at 0 is mapped isomorphically onto $H^1(V_0, \Theta_0)$. This result was the first such existence theorem. Its proof is based on the theory of elliptic partial differential equations and depends on the Newlander-Nirenberg theorem.

Regularity and Pseudo-differential Operators

In 1957 Alberto P. Calderón proved a remarkable theorem on the Cauchy problem. A crucial step of his proof is to put a matrix of differential operators into Jordan canonical form by using compositions of differential operators and singular integral operators. This represents a striking application of the theory of pseudo-differential operators even before these were formally defined. In the early 1960s Nirenberg and I (and independently Bobkobza and Unterberger) developed a calculus of pseudo-differential operators. Nirenberg and I were motivated by regularity problems that arise in the theory of several complex variables (see [6] and [7]) in connection with boundary regularity of the $\bar\partial$-Neumann problem and related questions. Roughly speaking, we deal with integro-differential forms Q that can be expressed as:

$$Q(u, v) = \sum_{|\alpha|,|\beta|\le 1} \int_\Omega a_{\alpha\beta}^{ij} D^\alpha u_i \overline{D^\beta v_j}\, dV, \tag{7}$$

where Ω is a domain with a smooth boundary, and that satisfy the sub-elliptic estimate

$$\|u\|_\epsilon^2 \le CQ(u,u) \tag{8}$$

for all u satisfying certain boundary conditions. Here $\|u\|_\epsilon$ denotes the Sobolev norm, that is, the L_2-norm of "ϵ derivatives". Given $f = (f_1, f_2, \dots, f_N)$, we wish to establish local regularity for $u = (u_1, u_2, \dots, u_n)$ that satisfies

$$Q(u,v) = \int_\Omega f\bar{v}dV \tag{9}$$

for all v satisfying the boundary conditions. In the case of the $\bar{\partial}$-Neumann problem, the Q is defined on forms on a domain in $\mathbb{C}^n$, and it is a kind of "energy" associated with the operator $\bar{\partial}$. The estimate (8) holds with $\epsilon = 1$ if and only if the problem is elliptic. In that situation one estimates derivatives of order m by substituting $D^{m-1}u$ for u in (9), and then, using integration by parts, one obtains

$$\begin{aligned} Q(D^{m-1}u, D^{m-1}u) &= Q(u, D^{2m-2}u) \\ &+ \text{error terms.} \end{aligned} \tag{10}$$

The first term on the right is under control by (9), and the second term can be estimated by norms of derivatives of orders less than or equal to $m-1$. Thus, in the elliptic case, estimates for derivatives of order m can be obtained by induction. Here, for the sake of brevity, we have oversimplified—the derivatives have to be localized, and near the boundary they must be tangential, preserving the boundary conditions. When $\epsilon < 1$ in (8), the substitution of $D^{m-1}u$ for u no longer works; instead we substitute $\Lambda^{(m-1)\epsilon}u$ for u, where Λ is the square root of the Laplacian (again properly localized and tangential near the boundary). Then, in order to estimate the error terms in the analogue of (11) in this situation, we must deal with operators that arise from the operators Λ^s and $a^{ij}_{\alpha\beta}D^\alpha$ under repeated commutation and taking adjoints. It is this process that led us to study an algebra of pseudo-differential operators which contains differential operators, the operators Λ^s and their adjoints.

The Influence of PDE on SCV

In many instances problems which arose from complex analysis have had an influence on partial differential equations. Aside from the work described above, there are numerous examples, such as conformal mappings in one complex variable and Weyl's lemma, which was motivated by the study of harmonic integrals in Hodge theory. Regularity questions of the type discussed above have led to very interesting discoveries about the geometry of domains. In particular, a large body of research, culminating in the work of J. P. D'Angelo and D. Catlin, has led to the discovery of deep geometrical properties of pseudo-convex domains of finite type. These are the domains on which (8) holds for the $\bar{\partial}$-Neumann problem.

On strongly pseudo-convex domains a solution of the $\bar{\partial}$-equation

$$\bar{\partial}u = f \tag{11}$$

can be obtained by means of integral operators (early work on this was done by Grauert, Ramirez, and Henkin). These operators have the feature that their kernel is built up from holomorphic separating functions. If P is on the boundary of a domain Ω, then a holomorphic function h on a neighborhood U of P is a separating function if $h(P) = 0$ and if $h(P') \ne 0$ for all $P' \in U \cap \bar{\Omega} - \{P\}$. Searching for such a kernel led Nirenberg and me (see [10]) to the domain $\Omega \subset \mathbb{C}^n$ defined by

$$\begin{aligned} &Re(w) + |zw|^2 + |z|^8 + \\ &\frac{15}{7}|z|^2 Re(z^6) < 0. \end{aligned} \tag{12}$$

This domain is pseudo-convex of finite type and has the property that if U is a neighborhood of the origin and if h is a function defined on U with $h(0) = 0$, then there exist $P \in U \cap \Omega$ and $P' \in U \cap (\bar{\Omega})^c$ such that $h(P) = h(P') = 0$. Thus Ω does not have a separating function at P. This situation contrasts totally with the case of strongly pseudo-convex domains. Another remarkable weakly pseudo-convex domain W, the "worm domain", was constructed by Diederich and Fornaess. This domain is pseudo-convex (so there exist holomorphic functions on it which cannot be extended to a larger domain), and it has the property that there exists a domain $\widetilde{W} \supset \overline{W}$ such that any holomorphic function defined in a neighborhood of $\overline{W}$ has a holomorphic extension on $\widetilde{W}$. Through the work of D. Barrett and M. Christ, the worm domain also provides some important insights on problems of global regularity of the $\bar{\partial}$-Neumann problem and suggests some fundamental questions in the theory of partial differential equations.

Conclusion

Before concluding, I call attention to the papers [8, 9, 13, 16]. These are important contributions to: partial differential equations whose degeneracies can be treated analogously as those arising in SCV (see [8]), geometry of complex manifolds in [9], boundary regularity of holomorphic mappings in [13], and complex Monge-Ampère equations in [14].

I also want to call attention to Nirenberg as an expositor and teacher. Nirenberg is a wonderful expositor. His expository papers and notes are models of clarity and are sources for learning and enjoying important mathematical concepts. As examples of this, I refer to [5, 11, 15, 16]. Nirenberg's career has been an inspiration; his numerous students, collaborators, and colleagues have learned a great deal from him. Aside from mathematics, Nirenberg has taught all of us the enjoyment of travel, movies, and gastronomy. An appreciation of Nirenberg also must include his ever-present sense of humor. His humor is irrepressible, so that on occasion it makes its way to the printed page. To illustrate, I quote from [8], in which, after pages of intricate calculations, Nirenberg inserted on page 844: "With these tedious estimates, as well as some readers, behind us, we may proceed to deal with ..."

In conclusion, looking back at Nirenberg's work in complex analysis (and more generally in mathematics) forces one to appreciate the subject and his many contributions to it.

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