Addresses on Works of Fields Medalists and Nevanlinna Prize Recipient

The following are summaries of the lectures on the work of the 1986 Fields Medal and Nevanlinna Prize recipients. These talks were presented following the Opening Ceremonies at the International Congress of Mathematicians in Berkeley. Michael Atiyah spoke on the work of Simon Donaldson, Barry Mazur spoke on the work of Gerd Faltings, John Milnor spoke on the work of Michael Freedman, and Volker Strassen spoke on the work of Leslie Valiant. It should be noted that in the October 1986 issue of Notices, the names of Mazur and Strassen were interchanged in the list of who presented these addresses. The complete manuscripts of these talks are expected to appear in the Proceedings of the Congress. These summaries are being published in Notices with the permission of ICM-86.

Fields Medals

The Work of Simon Donaldson

Michael Atiyah
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In 1982, when he was a second year graduate student, Simon Donaldson proved a result which stunned the mathematical world. Together with the important work of Michael Freedman, Donaldson's result implied that there exist "exotic" 4-spaces–4-dimensional manifolds which are topologically but not differentially equivalent to the standard Euclidean 4-space $\mathbf{R}^4$. This result is especially surprising because $n = 4$ is the only value for which such exotic n-spaces exist. These spaces, unlike $\mathbf{R}^4$, contain compact sets that cannot even be contained inside any differentially embedded 3-sphere.

To a closed, oriented 4-manifold one can associate a topological invariant, an integer matrix of determinant ± 1 defined by the intersection properties of the 2-cycles, and depending on the choice of basis. Freedman showed that all such matrices can occur for topological manifolds, while Donaldson showed that only those equivalent to the unit matrix can occur for differentiable manifolds. This shows an inherent difference between the topological and differentiable cases.

Donaldson's results are derived from the Yang-Mills equations of theoretical physics, which are the nonlinear generalization of Maxwell's equations for electromagnetism. In the Euclidean case, the solutions to the Yang-Mills equations giving the absolute minimum (for given boundary conditions at infinity) are of special interest and are called instantons. Mathematicians have worked with instantons before, but it was Donaldson who boldly used instantons (or, more precisely, the nonlinear space of instanton parameters) as a new geometrical tool. This approach revealed completely new phenomena and showed that the Yang-Mills equations are beautifully suited to studying this new field.

When Donaldson proved his first results it was unclear that instantons could be used more generally. Since then, Donaldson has developed and exploited instantons with great insight and skill. He has proved further constraints on the topology of differentiable 4-manifolds, and has produced new invariants. He proved an existence theorem which implied that on an algebraic surface, instanton parameter spaces have a purely algebraic description coinciding with stable vector bundles. Donaldson used this result to calculate algebraically his new invariants, and to exhibit two algebraic surfaces which are homeomorphic but not diffeomorphic.

Donaldson linked the problem of explicitly finding all instantons in Euclidean 4-space to algebraic vector bundles on the complex projective plane (viewed as a compactification of $\mathbf{R}^4 = \mathbf{C}^2$). Applying similar ideas, he proved the remarkably simple result that the parameter space of monopoles of magnetic charge k can be identified with the space of rational functions of a complex variable of degree k.

Donaldson's methods are very subtle and difficult in their use of nonlinear partial differential equations, and required a thorough understanding of the complex and delicate theory of the Yang-Mills equations. The mastery of such a wide range of ideas and techniques by such a young mathematician indicates that mathematics has lost neither its unity, nor its vitality.

The Work of Gerd Faltings

Barry Mazur
Harvard University

One of the recent great moments in Mathematics was when Gerd Faltings revealed the circle of ideas which led him to a proof of the conjecture of Mordell. The conjecture, marvelous in the simplicity of its statement, had stood as a goad and an elusive temptation for over half a century: it is even older than the Fields Medal! In modern language it takes the following form:

K is any number field and X is any curve of genus > 1 defined over K, then X has only a finite number of K-rational points.

To get a feeling for our level of ignorance in the face of such questions, consider that, before Faltings, there was not a single curve X (of genus > 1) for which we knew this statement to be true for all number fields K over which X is defined!

Already in the 20s, Weil and Siegel made serious attempts to attack the problem. Siegel, influenced by Weil's thesis, used methods of diophantine approximation, to prove that the number of *integral* solutions to a polynomial equation $f(X < Y) = 0$ (i.e., solutions in the ring of integers of a number field K) is finite, provided that f defines a curve over K of genus > 0, or a curve of genus 0 with at least three points at infinity.

In his thesis, Weil generalized Mordell's theorem on the finite generation of the group of rational points on an elliptic curve, to abelian varieties of any dimension. Weil then hoped to use this finite generation result for the rational points on the jacobian of a curve to go on to show that when a curve of genus > 1 is imbedded in its jacobian, only a finite number of the rational points of the jacobian can lie on the curve. Not finding a way to do this, he decided to call his proof of finite generation (the "theorem of Mordell-Weil") a thesis, despite Hadamard's advice not to be satisfied with half a result!

After this work of Weil and Seigel there was little progress for 30 years. It was in the 60s and early 70s that several new developments occurred in algebraic geometry and number theory which were to influence Faltings (work of Grothendieck, Serre, Mumford, Lang, Néron, Tate, Manin, Shafarevich, Parsin, Arakelov, Zarhin, Raynaud, and others).

These developments enter, and come together, in an essential way in the work of Faltings who proved the conjecture of Mordell, by first establishing the truth of a number of other outstanding conjectures–fundamental to arithmetic, and to arithmetic algebraic geometry.

Gerd Faltings's approach to the conjecture of Mordell, as well as his other mathematical contributions, immediately impress one as the work of a marvelously original mind from which we may expect similarly wonderful things in the future.

The Work of Michael Freedman

John Milnor
Institute for Advanced Study

Michael Freedman has not only proved the celebrated Poincaré hypothesis for 4-dimensional topological manifolds, but has also given us classification theorems for important classes of topological 4-manifolds. These theorems are simple to state and use, and are in marked contrast to the extreme complications that are now known to occur in the study of differentiable and piecewise linear 4-manifolds.

The n-dimensional Poincaré hypothesis is the conjecture that every topological n-manifold which has the same homology and the same fundamental group as the n-sphere is actually homeomorphic to the n-sphere. The cases $n = 1, 2$ and $n \geq 5$ had been proved, but the 3- and 4-dimensional cases were found to be much more difficult.

Freedman's 1982 proof of the 4-dimensional Poincaré hypothesis was an extraordinary tour de force. He characterized, up to homeomorphism, all compact, simply connected topological 4-manifolds by two simple invariants, thereby providing a complete classification. This work unearthed many previously unknown examples of such manifolds, and many previously unknown homeomorphisms between known manifolds.

Freedman was able to extend his methods to noncompact manifolds. For example, he showed that $S^3 \times \mathbf{R}$ can be given an exotic differentiable structure containing a smoothly embedded Poincaré homology sphere, which prevents a smooth embedding in Euclidean 4-space. He also addressed the case of nonsimply connected 4-manifolds, by showing that, for example, a "flat" 2-sphere in 4-space is unknotted if and only if its complement has free cyclic fundamental group, and that a flat 1-sphere in S^3 has trivial Alexander polynomial if and only if it bounds a flat 2-disk in the unit 4-disk whose complement has free cyclic fundamental group.

In the difficult proofs of these results, Freedman used a variation of the methods used in low dimensions by Moebius and Poincaré and in high dimensions by Smale and Wallace. The basic idea is to start with a 4-dimensional disk and build up the 4-manifold by successively adding handles. To get around the essential difficulties that arise in four dimensions, Freedman's major technical tool is a theorem stating that every Casson handle is actually homeomorphic to the standard open handle, (closed 2-disk) $\times$ (open 2-disk). The proof involves a delicately controlled infinite repetition argument in the spirit of the Bing school of topology, and is nondifferentiable in a crucial way.

Nevanlinna Prize

The Work of Leslie Valiant

Volker Strassen
University of Zürich

Valiant has contributed in a decisive way to the growth of almost every branch of theoretical computer science. In order to convey some impression of the scope of his work and the impetuous pace of its creation, this address first discusses three early papers, published within a single year, which contain spectacular advances in three widely different areas. Then it turns to Valiant's most important and mature work, centering around his theory of counting problems.

Context-free grammars are being used extensively for describing the structure of programming languages. The central algorithmic problem is to recognize sentences of the language defined by such a grammar. For a number of years, recognition algorithms which run in a time proportional to n^3 on sentences of length n had been known, but despite great efforts no significant improvement had been obtained until 1975 when Valiant showed that the problem can be solved in less than cubic time by reducing it to integer matrix multiplication. He has never returned to this subject, but ingenious algorithmic reductions from combinatorial to algebraic problems have become one of the main themes of his work.

Let m be a positive integer. An m-superconcentrator is a directed graph with m input and m output nodes, such that for every $r \leq m$ any r input nodes may be connected to any r output nodes by r disjoint directed paths. By the size of an m-superconcentrator one means its number of edges. Superconcentrators first appeared in algebraic complexity theory: any straightline algorithm for computing the discrete Fourier transform of order m yields an m-superconcentrator of a size proportional to the length of the algorithm. This observation led to the intriguing idea of proving the optimality of the fast Fourier transform by purely graph theoretical means. Building on previous work of Pinsker, Valiant dashed such hopes by showing that there exist m-superconcentrators of size only linear in m. This is just one example of Valiant's systematic and penetrating study of efficient imbedding and routing properties of graphs, leading in recent years to a theory of the general purpose parallel computer (the so-called supercomputer).

Turing machines are the principal theoretical model on which notions of computability and computational complexity are being based. Given a function: $t : \mathbf{N} \to \mathbf{N}$, let $\mathrm{TIME}(t)$ be the class of all decision problems (coded as sets of binary strings) which can be decided by a multitape Turing machine using $0(t(n))$ steps on inputs of length n. Define $\mathrm{SPACE}(t)$ similarly in terms of the number of tape squares visited. $\mathrm{TIME}(t) \subset \mathrm{SPACE}(t)$, since in one step a Turing machine can reach at most a constant number of new tape squares. Hopcroft, Paul, and Valiant caused a scientific sensation by showing that the above inclusion is strict. In fact they proved

$$\mathrm{TIME}(t) \subset \mathrm{SPACE}(t/\log t),$$

which implies $\mathrm{TIME}(t) \subsetneqq \mathrm{SPACE}(t)$ by a classical diagonalization argument. The class $P := \bigcup_k \mathrm{TIME}(n^k)$ of problems decidable in *polynomial time* has gained a central position in complexity theory, since it appears to be best suited for distinguishing between what can be and what cannot be computed in practice. For brevity, problems in P are called easy, those not in P hard. Now consider a map $f : \mathbf{N} \to 2^{\mathbf{N}}$ such that

$$(1) \qquad \{(x,y) : y \in f(x)\} \text{ is easy,}$$

$$(2) \qquad y \in f(x) \Rightarrow |y| \leq |x|^k$$

for a suitable constant k. ($|x|$ denotes the binary length of x). The set of all numbers x such that $f(x)$ is nonempty is called a *search problem.* The set of composite numbers is an example: $f(x)$ may be taken to consist of all proper divisors of x. The class NP of all search problems contains P, but also includes numerous decision problems occurring in mathematics and its applications that apparently are not easy. Cook in 1971 introduced the fundamental notion of NP-completeness. Roughly speaking, NP-complete problems have maximal degree of difficulty among all search problems. *Cook's hypothesis* $N \neq NP$ therefore implies that NP-complete problems are hard. Cook, Karp and others have succeeded in classifying most of the naturally occurring search problems as either easy or NP-complete.

In 1979, Valiant drew attention to what he called *counting problems*. These are the numerical functions $x \mapsto \#f(x)$ associated with maps f satisfying (1) and (2). Clearly, a counting problem is at least as difficult as the corresponding search problem. Hence complete counting problems are hard under Cook's hypothesis. Most exciting is Valiant's discovery of various complete counting problems that correspond to easy search problems, thereby considerably enlarging the theory of NP-completeness. Examples are: counting subtrees of a directed graph, evaluating the probability of failure of an unreliable connecting network, counting perfect matchings of a bipartite graph. The proofs of completeness of these problems intricately combine ideas belonging to mathematical logic, graph theory, and algebra.

The number of perfect matchings of a bipartite graph is equal to the value of the permanent function at the zero-one matrix representing the graph over the ring of integers. Replacing $\mathbf{Z}$ by $\mathbf{Z}/m\mathbf{Z}$, Valiant showed that the permanent becomes easy when m is a fixed power of two, while the existence of a fast algorithm for the permanent modulo m for some m that is not a power of two implies that any polynomially bounded number theoretical function, whose graph is easy to decide, is itself easy to compute. It can be deduced that if the permanent modulo 3 (say) is easy, then so is prime factorization of integers. The permanent is a polynomial function of the entries in the matrix. Thus Valiant was naturally led into algebraic complexity theory. Here the basic model is that of a straightline algorithm, i.e., a finite sequence of arithmetical instructions, to be executed over a suitable algebraic structure. For a number of years it had seemed that this subject would remain unaffected by the notion of NP-completeness. Motivated by his work on the permanent, however, Valiant developed a convincing analogue of the theory of search and counting problems entirely in the algebraic framework.

Theoretical computer science is in the stage of formulating its central problems and devising the proof techniques for their solution. Valiant has been eminently involved in this process, not only by answering a number of recalitrant open questions, but above all by developing important new concepts, which have led him to discover deep and beautiful connections between problems that had seemed to be totally unrelated.