André Weil and the Foundations of Algebraic Geometry

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The goal of algebraic geometry is the study of solutions of polynomial equations with coefficients in a commutative field k, or more generally a commutative ring. An "affine variety" is defined by equations $f_i = 0$ with i in some set I and f_i in $k[X_1, \ldots, X_n]$; a "projective variety" is defined similarly by homogeneous equations $f_i = 0$, for $i \in I$, in the projective space P^n with homogeneous coordinates $X_0, \ldots, X_n$. According to a theorem of Hilbert one can limit oneself to finite families I. When k is the field of real numbers $\mathbb{R}$ or that of the complex numbers $\mathbb{C}$, the set of solutions inherits a topology from $\mathbb{R}$ or $\mathbb{C}$; this is compact in the projective case. For the needs of arithmetic, it is appropriate to consider the case where $k = \mathbb{Q}$ or $\mathbb{Z}$. By reduction modulo a prime number p, one then obtains a variety over the finite field $\mathbb{F}_p$. If X is an algebraic variety defined over a finite field k with q elements, X possesses a finite number of points ν_n in the field k_n, the extension of k of degree n. The knowledge of ν_n is equivalent with the knowledge of the "zeta function" of X, introduced by E. Artin [1]. In 1940, in a note in the *Comptes Rendus de l'Académie des Sciences* [11], André Weil announced the proof of an analog of the Riemann hypothesis for the zeta function of curves over finite fields. In a long letter to E. Artin dated July 1942 [15], Weil explained the principle of the argument.

It rested on the establishment of an intersection theory for surfaces. In the simplest case, intersection theory permits one, starting from two plane curves C and D without common components, to attach to the points of $C \cap D$ certain multiplicities in such a way as to define the intersection cycle $C \cdot D$. Thus when C and D pass through the origin of the plane with coordinates x and y and have $f = 0$ and $g = 0$ as respective equations, the intersection multiplicity at the origin is the vector-space dimension over k of $k[[x, y]]/(f, g)$. This is 1 if and only if C and D meet transversally. The case that interested Weil is the one where the surface is $S = X \times X$. Weil needed to extend a result of G. Castelnuovo established over the complex numbers. He had to take into account the need to deal with an intersection theory valid over an arbitrary field k and for varieties in any number of dimensions. To make this theory rigorous was going to require some years and would lead to works to which we are briefly going to refer.

André Weil's book ***Foundations of Algebraic Geometry*** was published by the AMS in 1946. The goal of this book is to establish algebraic geometry rigorously over an arbitrary commutative field and to insist quite particularly on an intersection theory. It marks a break with respect to the works of his predecessors—B. L. Van der Waerden [10] and the German school, O. Zariski and the Italian geometers. To signal this clearly, the book contains no bibliography. The emphasis is systematically put on fields: fields of definition of varieties, fields of rational functions. Fifteen years later, Grothendieck [3], in developing the language of schemes, would bring out the role of rings.

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Weil fixes a *universal domain K*, an algebraically closed field of infinite transcendence degree over the prime field. Let $p \geq 0$ be its characteristic. The points of algebraic varieties are going to have coordinates in K. In what follows, k is a subfield of K such that K has infinite transcendence degree over k.

For an integer $n \geq 0$, a point of the affine space S^n is an n-tuple $x = (x_1, \ldots, x_n)$ of elements of K. Let $k(x)$ be the subfield of K generated by $k, x_1, \ldots, x_n$. Let I be the prime ideal in the ring of polynomials $k[X_1, \ldots, X_n]$ formed from all F such that $F(x_1, \ldots, x_n) = 0$, so that $k(x)$ is the field of fractions of the quotient integral domain $R = k[X_1, \ldots, X_n]/I$. A point $y = (y_1, \ldots, y_n)$ of S^n is a k specialization of x if $F(y_1, \ldots, y_n) = 0$ for all F in I. For example, if $k = \mathbb{Q}$ and $K = \mathbb{C}$ and $x = (i, \pi)$, then I is the principal ideal $(X_1^2 + 1)$ in $\mathbb{Q}[X_1, X_2]$, and the specializations of x are all points $(\pm i, y_2)$.

A k *affine variety* V is defined by what is called a *generic* point x, one such that $k(x)$ is a "regular" extension of k; the *points* of V are the specializations y of x.[1] The condition of regularity is a technical condition that assures that for every extension k' of k, $k(x) \otimes_k k'$ is still free of zero divisors. In the example above, $x = (i, \pi)$ is not generic since $i \otimes 1 + 1 \otimes i$ is a zero divisor in $\mathbb{Q}(i, \pi) \otimes_{\mathbb{Q}} \mathbb{Q}(i)$; this property corresponds to the fact that the set of specializations of x is reducible, being the union of all (i, y_2) and all $(-i, y_2)$. If we had chosen $k = \mathbb{Q}(i)$, however, then the ideal I would have been $(X_1 - i)$ and x would have been generic, defining a variety.

This said, the scene is set: Weil is occupied first with generic points, and then he specializes.

If U and V are k varieties, one can choose generic points x and y such that $k(x) \otimes_k k(y)$ is an integral domain, and then $k(x, y)$ is a generic point of the product variety $U \times V$. The fact that the transcendence degree of K over k is infinite thus permits the consideration of products of an arbitrary finite number of varieties over k.

In algebraic geometry, correspondences and rational mappings between varieties emerged before the notion of everywhere-defined mappings or "morphisms". Weil does not depart from this presentation. A *correspondence* between two varieties U and V is given by its graph W, which is a particular kind of subvariety of $U \times V$. If the projection of W on U is *regular*, that is to say, sends a generic point w of W to a generic point u of U

in such a way that $k(w)$ is algebraic over $k(u)$ of degree 1, then the correspondence furnishes a *rational mapping* of U into V. The domain of definition of this mapping is formed from points of U above which the projection of W into U is itself regular in a suitable sense. This indirect approach to domains of definition of mappings is rather hard to use. Later Grothendieck would insist, on the contrary, on morphisms and a functorial point of view and would minimize the role of rational mappings. Weil introduced *abstract varieties*, defined by an atlas of affine charts, on the model of differentiable manifolds. He distinguished those that are "complete" (these correspond to compact varieties in the case where $k = \mathbb{C}$) by means of specializations and freed himself thereby from projective geometry. But it is only with M. Nagata [7] in 1958 and H. Hironaka [5] in 1962 that one would know that there exist complete varieties that are not projective.

We come to the heart of the book—intersection theory. Given two subvarieties A and B of a variety V, one wants, under certain conditions, to define the intersection $A \cdot B$, which will be a cycle on the variety V with integer coefficients ≥ 0. The difficulty consists in defining the coefficients of this cycle, that is to say, the *intersection multiplicities*.

Weil begins by defining a "simple point" x of a variety V. We note first that for polynomials, partial derivatives exist, given by the usual formulas, without reference to a topology on k and the notion of limit. Consequently, over an arbitrary field k, one has available a purely algebraic differential calculus. One says that a point x of a variety V is *simple* (nowadays one says *smooth*) if locally V can be defined by m equations $f_1 = \cdots = f_m = 0$ that satisfy a Jacobian condition: the matrix $((\partial f_i / \partial X_j))$ is of rank m at x. When V is smooth of dimension r, embedded in an affine space S^n, one constructs affine linear mappings of S^n into S^r such that the composition $V \to S^n \to S^r$ is "étale"[2] at x. These Weil calls *linearizations* of V at x. If A and B are subvarieties of V, of respective dimensions a and b, there is no difficulty in defining the irreducible components of $A \cap B$. Let C be one of them. One says that C is a *proper component* of $A \cap B$ if a generic point of C is smooth in V and if C is of dimension $a + b - r$. The goal is to define the multiplicity in the cycle $A \cdot B$ of a proper component C of the intersection of A and B.

Weil examines first the case where V is the affine space S^r and where B is a linear affine subvariety L of S^r, of dimension $r - a$. When L is generic, the intersection $L \cap A$ is situated in the smooth locus of A and the intersection multiplicity is elementary to define as the length of a suitable Artinian ring. When L is no longer generic, Weil

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[1] *In terms of the integral domain $R = k[X_1, \ldots, X_n]/I$, one is at the same time very close to the language of schemes, since each point y of V corresponds to a prime ideal of R, and also very far, partly because the language of specializations masks the role of the ring R and partly because a prime ideal Q of R corresponds to several points—the different ways of embedding the zero-divisor-free k algebra R/Q into K—therefore to infinitely many points as soon as R/Q is not algebraic over k.*

[2] *An étale mapping is the algebraic geometer's version of a local isomorphism in the complex analytic case.*

Castelnuovo

Severi

Zariski

Artin Hasse Schmidt Weil

reduces to the previous case by an argument of specialization of multiplicities. He treats in an analogous way the intersection of A with a linear subvariety of codimension $\leq a$. Finally, the general case of a proper component C of $A \cap B$ reduces to the previous case by using linearizations and an argument of reduction to the diagonal, due to Van der Waerden. The latter consists in realizing $A \cdot B$ as the intersection of $A \times B$ with the diagonal Δ of $V \times V$.

Let us not lose sight of the fact that these works are contemporaneous with those of C. Chevalley [2] developing the notion of multiplicity of a local Noetherian ring. Then would come the contributions of P. Samuel [8]. It is only in 1957 that J.-P. Serre [9] would give a local definition of intersection multiplicity as an alternating sum of lengths of certain "Tor" modules, making the argument of specialization unnecessary. Afterwards Weil presents the operations that one can make on cycles, in particular in the case of complete varieties.

In 1948 André Weil published **Courbes Algébriques et Variétés Abéliennes**. This magnificent book, clearly less austere than *Foundations*, rich in geometry and topological motivation, is a masterful illustration of the use of the theory of intersections.

The first part treats curves. Suppose that an algebraic curve X over k, complete and smooth, has genus g. As in the complex case, g is the dimension over k of the vector space of global algebraic differential 1-forms. After a study of divisors on X leading to the Riemann-Roch theorem comes the study of correspondences, already considered by the Italian geometers, in particular F. Severi. If X is a complete smooth curve of genus g, a correspondence on X is given by a divisor D on $X \times X$, modulo the equivalence relation generated by linear equivalence of divisors and by the divisors that come from X by one or the other of the two projections. These correspondences can be added and composed to form a ring with an involution $D \mapsto D'$ that exchanges the two factors of $X \times X$. Intuitively a correspondence corresponds to a multivalued algebraic map of X into X: if p and q denote the projections of D on the first and second factors X, the correspondence associates to a point x of X the cycle on X equal to $q(p^{-1}(x))$. Moreover, in the second part, Weil will show that this ring of correspondences is naturally identified with the ring of endomorphisms of the "Jacobian variety" of X, the involution providing duality.

To a correspondence ξ represented by a divisor D, Weil associates an integer $\sigma(\xi)$ equal to the degree of the intersection $D \cdot (X \times a) + D \cdot (b \times X) - D \cdot \Delta$ (where a and b are points of X and Δ is the diagonal of $X \times X$). The integer $\sigma(\xi)$ plays the role of a *trace*. More precisely, when X is defined over the complex numbers, the Lefschetz fixed-point formula shows that $\sigma(\xi)$ is interpreted as the trace of the correspondence ξ operating on the cohomology in degree 1, say rational cohomology, of the topological variety associated to X. The essential point is that $\sigma(\xi\xi') \geq 0$ and that $\sigma(\xi\xi') = 0$ if and only if $\xi = 0$. Castelnuovo had proved this in the complex case. The reader can see in [6] a short proof, due to Grothendieck, tying Castelnuovo's result to the index theorem.

The preceding result is going to allow Weil to establish the "Riemann hypothesis" for the zeta function of curves over finite fields. Suppose that the curve X is defined over a finite field k with q elements, and let k_n be the unique extension of k of degree n. Let ν_n be the number of points of X with values in k_n. The *zeta function* of X,

$$Z(T) = \prod_{\substack{x \in X, \\ \deg x < \infty}} (1 - T^{\deg(x)})^{-1},$$

is tied to ν_n by the formula

$$TZ'/Z = \sum_{n \geq 1} \nu_n T^n.$$

This zeta function for curves over a finite field was introduced by E. Artin in his thesis [1] by analogy with the classical Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}.$$

Suppose that X is smooth and complete. Use of the Riemann-Roch theorem permits one to rewrite $Z(T)$, as Hasse and Schmidt [4] already observed, in the form

$$Z(T) = P(T)/(1 - T)(1 - qT),$$

where

$$P(T) = \prod_{i=1}^{2g}(1 - \alpha_i T)$$

is a polynomial of degree $2g$ with integer coefficients such that if α is the reciprocal of a root of P, so is q/α.

The Riemann hypothesis in the case of curves over finite fields says that the complex numbers α_i have modulus $\sqrt{q}$. It was first proved by Hasse [4] in the case of elliptic curves ($g = 1$).

To treat the general case, Weil introduces the graph I_n of the n-th iterate of the Frobenius endomorphism[3] of X and notes that it cuts Δ transversally, from which he deduces that $\sigma(I_n) = 1 + q^n - \nu_n$. Upon writing, for x and y integers, that the correspondence ξ defined by $x\Delta + yI_n$ satisfies $\sigma(\xi\xi') \geq 0$, one finds that $|\sigma(I_n)| = |1 + q^n - \nu_n| \leq 2gq^{1/2}$. Since $d \log P(T)/dT = -\sum_n \sigma(I_n)T^n dT/T$, Weil

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[3]*In coordinates, the Frobenius endomorphism takes each coordinate to its q-th power.*

concludes that the α_i are of modulus $\leq q^{1/2}$ and therefore, by symmetry, are of modulus $q^{1/2}$.

The second part of the book is devoted to Jacobian varieties of curves and to abelian varieties. This algebraic study, valid over an arbitrary field, requires approaches quite different from the transcendental methods used over the complex numbers.

Weil begins by defining a "group variety", and *abelian varieties* are complete group varieties.[4] To construct group varieties, Weil introduces the notion of *normal law* on a variety V: it is a partially defined composition law satisfying certain conditions, a sort of "kernel of group variety". He shows that there exists a group variety G, unique up to isomorphism, and a birational mapping between V and G, compatible with the composition laws. If now X is a curve of genus g with base point, the Riemann-Roch theorem furnishes a normal law on the g-fold symmetric product of X. The associated group variety is the *Jacobian variety*. Later, M. Rosenlicht would construct in the same way generalized Jacobian varieties, and A. Néron would use a relative version of normal laws to obtain his models of abelian varieties over a discrete valuation ring.

Every endomorphism u of an abelian variety A has a degree $\nu(u)$. If one takes u to be multiplication in A by an integer $n \geq 0$, one finds that the degree is n^{2g}. If n is prime to the characteristic p, the kernel A_n of multiplication by n is a finite group isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$. If one then chooses a prime number $\ell \neq p$, the projective system of the A_{ℓ^m}, as m varies, furnishes a free $\mathbb{Z}_\ell$ module $T_\ell(A)$, of rank $2g$. These ℓ-adic modules, constructed starting from the torsion points of A, have become one of the basic tools in the arithmetic study of abelian varieties. They furnish ℓ-adic representations of the Galois group $\mathrm{Gal}(\bar{k}/k)$, and one has a faithful representation of the ring $\mathrm{End}\, A$ on the ring of endomorphisms of the $\mathbb{Z}_\ell$ module $T_\ell(A)$. Finally, starting from an endomorphism u of A, Weil defines its characteristic polynomial. Over the complex numbers, one can take the characteristic polynomial of the endomorphism induced by u on the lattice of periods. In the general case, Weil shows, by intersection theory, that for x and y integers, $\nu(x\,\mathrm{id} + yu)$ is a homogeneous polynomial P of degree $2g$ with integer coefficients. He establishes that P is equal also to the characteristic polynomial of u in its ℓ-adic representation; in particular, this last polynomial has integer coefficients, independent of the prime number $\ell \neq p$.

This study of curves and abelian varieties convinced Weil that algebraic varieties over finite fields could behave like topological varieties and that "Betti cohomology" was still playing a role. In the case of curves the interesting degree of cohomology is degree 1; it corresponds to the numerator $P(T)$ of the zeta function. After having studied the zeta functions of monomial hypersurfaces of arbitrary dimension d, for which the interesting degree of cohomology is d, Weil was ready to formulate in 1949 his conjectures concerning smooth complete varieties over finite fields.

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[4] *The name incorporates the theorem that the group structure of a complete group variety is necessarily abelian. Abelian varieties over the complex numbers correspond to algebraic complex tori, each such torus being the quotient of a complex vector space by a maximal lattice.*

Weil Chevalley Samuel Serre Zariski Grothendieck