Sarvadaman Chowla (1907–1995)

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Sarvadaman Chowla

Sarvadaman Chowla was an extremely talented mathematician who was internationally known for his research in number theory and related topics. His name is associated with several important theorems. His mathematical output was impressive and reflected his special gift for expressing complex ideas simply. One of the best-known number theorists from India who followed in the tradition of Ramanujan, Chowla's fertile and creative imagination justified the title "poet of mathematics" given to him by his associates.

Chowla was born in London, England, on October 22, 1907, the son of Gopal and Shankuntala Chowla. His father, Gopal, was a professor of mathematics from Lahore in India who had gone to Cambridge University in England for further studies. Chowla was reared in India and showed an early aptitude for mathematics. His first published papers appeared when he was only eighteen. In 1928 he received a master's degree from Government College in Lahore. Between 1929 and 1931 Chowla studied at Trinity College, Cambridge, where he received his doctorate under the guidance of J. E. Littlewood. Following his return to India, Chowla held professorships at St. Stephen's College in Delhi, at Benares Hindu University in Benares, at Andhra University in Waltair, and lastly at Government College of Punjab University in Lahore, where he was head of the Department of Mathematics from 1936 to 1947. While at St. Stephen's College, he met and married a student, Himani Mozoomdar. Their only child, Paromita, also became a number theorist and is now a retired professor from Pennsylvania State University.

Upon the partition of India in 1947 Chowla fled from Lahore with his family first to Delhi and then to the United States, where he visited the Institute for Advanced Study in Princeton until the fall of 1949, when he assumed a professorship at the University of Kansas in Lawrence. In 1952 he moved to the University of Colorado in Boulder. Then in 1963 he accepted the offer of a position as research professor at Pennsylvania State University, where he remained until his retirement in 1976. After his retirement Chowla spent several years at the Institute for Advanced Study. From there Chowla eventually moved to Laramie, Wyoming, to work with his former doctoral student Mary Jane Cowles. Chowla died in Laramie on December 10, 1995.

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Those who knew Chowla were always struck by his enthusiasm for and love of his subject. One of Chowla's doctoral students referred to him as a "perpetual ambassador for number theory," and this reflects accurately Chowla's great capacity to stimulate and excite others about the theory of numbers. Chowla's lectures were both interesting and inspiring. He introduced students to the main ideas of the subject by means of illuminating examples and by giving proofs of important special cases of more general theorems. He taught by suggestion and encouragement, confident that his doctoral students would find their own interesting problems to work on. This approach brought out the best in his students, who often made progress with minimal assistance.

Chowla was a lively, friendly, and good-humored person who was extremely modest about his accomplishments. He was engaged in mathematics to such an extent that he had few outside interests. Mrs. Chowla was a constant source of support and freed him from the more mundane matters of everyday life. Though the death of his wife in 1970 was a tremendous loss to Chowla, after her death he managed reasonably well on his own with the assistance of friends.

Chowla was a member of the Indian National Academy of Sciences (from which he received a Padmabhushan Award) and an honorary foreign member of the Royal Norwegian Society of Sciences and Letters.

Chowla's first paper appeared in 1925 and his last in 1986. During this period of sixty-two years he wrote about 350 papers. His papers encompassed a wide variety of interests. He wrote on additive number theory (lattice points, partitions, Waring's problem), analysis (Bernoulli numbers, class invariants, definite integrals, elliptic integrals, infinite series, the Weierstrass approximation theorem), analytic number theory (Dirichlet L-functions, primes, Riemann and Epstein zeta functions), binary quadratic forms and class numbers, combinatorial problems (block designs, difference sets, Latin squares), diophantine equations and diophantine approximation, elementary number theory (arithmetic functions, continued fractions, and Ramanujan's tau function), and exponential and character sums (Gauss sums, Kloosterman sums, trigonometric sums). This list is not intended to be exhaustive. Among his coauthors are such well-known mathematicians as N. C. Ankeny, T. M. Apostol, E. Artin, R. P. Bambah, P. T. Bateman, B. C. Berndt, B. J. Birch, R. C. Bose, R. Brauer, H. Davenport, B. Dwork, P. Erdős, R. J. Evans, J. B. Friedlander, D. M. Goldfeld, H. Gupta, M. Hall Jr., H. Hasse, I. N. Herstein, B. W. Jones, D. J. Lewis, H. B. Mann, W. E. Mientka, L. J. Mordell, M. B. Nathanson, S. S. Pillai, K. Ramanchandra, C. R. Rao, H. J. Ryser, A. Schinzel, A. Selberg, G. Shimura, Th. Skolem, E. G. Straus, J. Todd, A. Walfisz, A. L. Whiteman, and H. Zassenhaus. Here we can do no more than touch upon some of Chowla's results. The selection will of course reflect the tastes of the authors.

Chowla's name is identified with a number of mathematical results, including the Bruck-Chowla-Ryser theorem that gives a criterion for the nonexistence of certain block designs, the Ankeny-Artin-Chowla congruence for the class number of a real quadratic field, the Chowla-Mordell theorem on Gauss sums, and the Chowla-Selberg formula relating special values of the Dedekind eta function. We now describe each of these briefly.

Let v, k, and λ be positive integers. A (v, k, λ)-*design* is a set of v elements arranged into v sets such that every set contains exactly k distinct elements and every pair of sets has exactly $\lambda = k(k-1)/(v-1)$ elements in common.

Bruck-Chowla-Ryser Theorem [1]. If a (v, k, λ)-design exists, then $k - \lambda$ is a square when v is even, and the diophantine equation $x^2 - (k-\lambda) y^2 - (-1)^{(v-1)/2}\lambda z^2 = 0$ has a nontrivial solution when v is odd.

Let $h(d)$ denote the class number of the real quadratic field K of discriminant d, and let $(t + u\sqrt{d})/2$ be the fundamental unit (> 1) of K. Let p be an odd prime divisor of d.

Ankeny-Artin-Chowla Theorem [2]. If $3 < p < d$, then

$$-2\frac{u}{t}h(d) \equiv \sum_{0<n<d} \frac{p}{dn}\left(\frac{d}{n}\right)\left[\frac{n}{p}\right] \pmod p,$$

where (d/n) is the Kronecker symbol for discriminant d and $[x]$ denotes the greatest integer less than or equal to x.

Let p be an odd prime, and let χ be a nonprincipal character (mod p). The Gauss sum $G(\chi)$ is defined by

$$G(\chi) = \sum_{n=1}^{p-1} \chi(n) e^{2\pi i n/p}.$$

We set $\varepsilon(\chi) = G(\chi)/\sqrt{p}$. It is well known that $|\varepsilon(\chi)| = 1$. Chowla and Mordell proved the following result independently in 1962.

Chowla-Mordell Theorem [3]. $\varepsilon(\chi)$ is a root of unity if and only if $\chi(n)$ is the Legendre symbol (n/p).

Let d be a negative integer with $d \equiv 0$ or 1 (mod 4). The equivalence classes of positive-definite, primitive, integral binary quadratic forms $(a, b, c) = ax^2 + bxy + cy^2$ of discriminant $d = b^2 - 4ac$ form a finite abelian group under Gaussian composition denoted by $H(d)$. Denote the class containing the form (a, b, c) by $[a, b, c]$ and the number of classes by $h(d)$.

In 1967 Selberg and Chowla discovered a beautiful formula, now known as the Chowla-Selberg formula, giving explicitly the value of the product of $|\eta((b+\sqrt{d})/2a)|$ as $[a,b,c]$ runs through the classes of $H(d)$ where d is a fundamental discriminant and $\eta(z)$ is the Dedekind eta function defined by

$$\eta(z) = e^{2\pi i z/24} \prod_{m=1}^{\infty} (1 - e^{2\pi i m z})$$

for $z = x + iy$, $y > 0$.

Chowla-Selberg Formula [4].

$$\prod_{[a,b,c]\epsilon H(d)} a^{-1/4} \left| \eta\left((b+\sqrt{d})/2a\right) \right| =$$

$$(2\pi|d|)^{-h(d)/4} \left\{ \prod_{m=1}^{|d|} (\Gamma(m/|d|))^{(d/m)} \right\}^{w(d)/8},$$

where $\Gamma(z)$ is the gamma function, (d/m) is the Kronecker symbol for discriminant d, and $w(d)$ is the number of roots of unity in the ring of integers of the quadratic field $\mathbf{Q}(\sqrt{d})$.

The Chowla-Selberg formula has been extended recently to nonfundamental discriminants as well as to genera of $H(d)$.

The study of class numbers from an analytic point of view is a recurring theme throughout Chowla's work. We describe briefly another of Chowla's results in this area.

Heilbronn showed in 1934 that $h(d) \to \infty$ as $d \to -\infty$. Upon learning of this result, Chowla [8] showed the stronger result that $h(d)/2^{t(d)} \to \infty$ as $d \to -\infty$, where $t(d)$ denotes the number of distinct prime divisors of d.

A number of Chowla's papers show that he was a master of the methods of analytic number theory. We mention briefly two examples. The first concerns the representation of a positive integer as the sum of four squares and a prime; the second, the behavior of the error term in the asymptotic formula for the summatory function of Euler's phi function.

Let $N_{r,s}(n)$ denote the number of representations of the positive integer n as the sum of r squares and s primes. By making use of the Brun-Titchmarsh Theorem, Chowla [5] showed in 1935 that

$$N_{4,1}(n) \sim \frac{\pi^2 n^2}{2\log n} \prod_{\substack{p|n \\ p>2}} \frac{(p-1)^2(p+1)}{(p^3-p^2+1)} \times \prod_{p>2} \left(1 + \frac{1}{p^2(p-1)}\right),$$

as $n \to \infty$, where p denotes a prime, thereby establishing a conjecture made by Hardy and Littlewood in 1922 in their famous work on *Partitio Numerorum.*

Next let $\phi(n)$ denote Euler's phi function; that is, $\phi(n)$ counts the number of positive integers not exceeding n that are coprime to n. It has long been known that

$$\sum_{n \le x} \phi(n) \sim \frac{3}{\pi^2} x^2$$

as $x \to \infty$. It was shown by Mertens in 1874 that the error term

$$E(x) = \sum_{n \le x} \phi(n) - \frac{3}{\pi^2} x^2$$

satisfies

$$E(x) = O(x \log x).$$

In 1930 in joint work with Pillai, Chowla [6] considered the sum

$$S(x) = \sum_{n \le x} E(n)$$

and showed that

$$S(x) \sim \frac{3}{2\pi^2} x^2$$

as $x \to \infty$. He also established that

$$E(x) \neq o(x \log\log\log x).$$

In 1932 Chowla returned to the subject of the behavior of $E(x)$ and proved in a long paper [7] that the integral

$$I(x) = \int_1^x E^2(u)du$$

satisfies

$$I(x) \sim \frac{1}{6\pi^2} x^3,$$

as $x \to \infty$.

Chowla's work was also characterized by a large number of short papers, many of them mathematical gems, others interesting conjectures. We give six examples.

1. Let p be a prime with $p > 3$. In 1862 Wolstenholme showed that

$$1 + \frac{1}{2} + \cdots + \frac{1}{p-1} \equiv 0 (\text{mod } p^2),$$

where $1/i$ denotes the inverse of i modulo p^2. There are many generalizations of Wolstenholme's theorem. In 1934 Chowla [9] gave a very short elegant proof of one of these due to Leudesdorf; namely, if n is a positive integer coprime with 6, then

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{n} \frac{1}{m} \equiv 0 (\text{mod } n^2).$$

2. Quintic equations that are solvable by radicals have attracted a great deal of attention in recent years since the publication of Dummit's important paper (*Solving solvable quintics*, Math.

Comp. **57** (1991), 387–401). Chowla in conjunction with Bhalotra wrote on this subject in 1942 [10]. Let $x^5+ax^3+bx^2+cx+d$ be irreducible in $\mathbf{Z}[x]$. They showed that if a and b are even and c and d are odd, then the quintic equation

$$x^5+ax^3+bx^2+cx+d=0$$

is not solvable by means of radicals. In particular, the quintic trinomial equation

$$x^5+cx+d=0$$

is not solvable by radicals when c and d are both odd. They also showed that this last equation is not solvable by radicals if c is odd and not divisible by any prime p with $p\equiv 3 \pmod 4$.

3. Let $p(n)$ denote the number of partitions of the positive integer n into integer parts. In joint work with A. M. Mian, Chowla [11] showed that the function

$$y(x)=\sum_{n=1}^{\infty} p(n)x^n \quad (|x|<1)$$

satisfies an algebraic differential equation.

4. The Bernoulli numbers $B_n (n=1,2,\dots)$ can be defined by

$$\frac{x}{e^x-1}=1-\frac{x}{2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}B_n x^{2n}}{(2n)!}.$$

There is a vast literature devoted to the properties of these numbers. In 1972 Chowla together with P. Hartung [12] considered the problem of determining an exact closed-form formula for B_n. They proved that

$$B_n = \frac{1}{2(2^{2n}-1)}\left(1+\left[\frac{2(2^{2n}-1)(2n)!}{2^{2n-1}\pi^{2n}}\sum_{m=1}^{3n}\frac{1}{m^{2n}}\right]\right).$$

5. Let p be a prime with $p\equiv 1 \pmod 4$. It is well known that there are integers a and b such that $p=a^2+b^2$, where a is odd and b is even. Indeed a is uniquely determined by this equation and the condition that $a\equiv 1 \pmod 4$. Gauss showed that the binomial coefficient

$$A=\binom{(p-1)/2}{(p-1)/4}$$

satisfies the congruence $A\equiv 2a \pmod p$. In one of his last papers Chowla, in joint work with Dwork and Evans [13], proved, using the Gross-Koblitz formula, the congruence

$$A\equiv\left(1+\frac{2^{p-1}-1}{2}\right)\left(2a-\frac{p}{2a}\right) \pmod{p^2},$$

which had been conjectured by Beukers.

6. For a positive integer N the continued fraction expansion of $\sqrt{N}$ is of the form

$$\sqrt{N}=[a_0;\overline{a_1,a_2,\cdots,a_\ell}],$$

where $a_0=[\sqrt{N}]$ and $a_l=2a_0$. It was shown by T. Vijayaraghavan (Proc. London Math. Soc. **26** (1927), 403–414) that the length l of the period $a_1a_2\dots a_l$ satisfies

$$l=O(N^{1/2}\log N).$$

He also showed that if $\delta>0$, then there are infinitely many values of the positive integer N for which

$$l>N^{1/2-\delta}.$$

The estimation of l is an important problem in number theory, and it was a problem that interested Chowla. In 1929 he [14] showed that if N is squarefree and exceeds a certain bound depending on $\delta>0$, then

$$l<\left(\frac{6}{\pi^2}+\delta\right)N^{1/2}\log N.$$

This inequality should be compared with that of R. G. Stanton, C. Sudler Jr., and H. C. Williams (*An upper bound for the period of the simple continued fraction for $\sqrt{D}$*, Pacific J. Math. **67** (1976), 525–536), who showed that

$$l<0.72\,N^{1/2}\log N$$

for all squarefree $N>7$. In joint work with Pillai in 1931 Chowla [15] showed that under a certain condition

$$l=O(N^{1/2}\log\log N)$$

and without the condition that there are positive constants C_1 and C_2 such that the inequalities

$$C_1N^{1/2}<l<C_2N^{1/2}$$

hold for infinitely many positive integers N. They showed further that l is "on the average" of order $N^{1/2}$.

The first author wrote an article on Chowla and his work on the occasion of Chowla's seventieth birthday. This article, which contained a list of Chowla's publications up to that point and a list of Chowla's twenty-three doctoral students in the United States, appeared in a special issue of the *Journal of Number Theory* dedicated to Professor Chowla [J. Number Theory **11** (1979), 286–301; erratum, ibid. **12** (1980), 139.] The other two authors are in the process of preparing Chowla's collected papers for publication.

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