Victor L. Klee 1925–2007

Peter Gritzmann and Bernd Sturmfels

Victor L. Klee passed away on August 17, 2007, in Lakewood, Ohio. Born in San Francisco in 1925, he received his Ph.D. in mathematics from the University of Virginia in 1949. In 1953 he moved to the University of Washington in Seattle, where he was a faculty member for 54 years. Klee specialized in convex sets, functional analysis, analysis of algorithms, optimization, and combinatorics, writing more than 240 research papers. He received many honors, including a Guggenheim Fellowship; the Ford Award (1972), the Allendoerfer Award (1980 and 1999), and the Award for Distinguished Service (1977) from the Mathematical Association of America; and the Humboldt Research Award (1980); as well as honorary doctorates from Pomona College (1965) and the Universities of Liège (1984) and Trier (1995). For collaborations with the first listed editor he received the Max Planck Research Award (1992). Klee served as president of the Mathematical Association of America from 1971 to 1973, was a fellow of the American Academy of Arts and Sciences, and was a fellow of the American Association for the Advancement of Science.

In 1990, in honor of Klee's 65th birthday and the broad range of his mathematical interests, the two of us (long-time co-worker and former Ph.D. student, respectively) edited the volume *Applied Geometry and Discrete Mathematics*, which was published by the American Mathematical Society.

For this obituary, we invited a group of former colleagues and mentees to contribute short pieces on Klee's mathematical life. This resulted in ten individual spotlights, followed by some personal remarks by the editors. The emphasis lies on Klee's work in the more recent decades of his rich scientific life, and hence they focus on finite-dimensional convexity, discrete mathematics, and optimization. His bibliography, however, makes it clear that by the late 1960s he already had more than a career's worth of papers in continuous and infinite dimensional convexity.

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Louis J. Billera
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Algebraic Combinatorics and the g-Theorem

Victor Klee was a pioneer in two closely related aspects of convex polytopes that have subsequently played an important role in algebraic combinatorics, namely, f-vectors and shellings. The f-vector of a polytope (or of more general geometric complexes) encodes the number of faces of each dimension [33].

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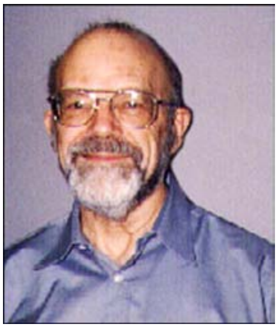

Victor L. Klee

Klee had the key insight of proving results concerning f-vectors in their greatest possible generality. Thus he proved the Dehn-Sommerville equation for Eulerian manifolds (a vast generalization of polytopes), generalized the Upper Bound Conjecture for simplicial polytopes to triangulations of spheres, and proved a special case for Eulerian manifolds [29]. He also proved the Lower Bound Conjecture for polytopes in the general setting of pseudomanifolds [34].

Klee was the first person to deal with shellings of simplicial complexes in a systematic way [11]. His papers in this area paved the way for the use of shellings as a major tool in proving combinatorial and topological properties of wide classes of complexes. His work on shellings and f-vectors had a big influence on our own research and led to some of our best papers.

Much of Klee's interest in polytopes related to questions originally arising in optimization theory. In the early 1960s he began to write and lecture on f-vectors and diameters of polytopes. Interest in both of these topics was spurred by their relevance to computational techniques for linear programming problems. The d-step conjecture poses a linear bound on the diameter of the graph of a polytope in terms of its dimension and number of facets. (In spite of all the progress on the combinatorics of polytopes since that time, this question remains unsettled.) Klee's paper [32] was seminal to part of the proof of the g-theorem, which characterizes f-vectors of simplicial polytopes [3, 47]. It suggested a way to construct extremal examples of polytopes by placing points over cyclic polytopes and led to the essential geometric step in the construction of simplicial polytopes having predetermined f-vectors.

Richard A. Brualdi

Sign Patterns of Matrices

The idea that the signs (some or all) of the solution vector of some linear systems of equations could be determined knowing only the signs of the system parameters originated in the economics literature [46] in 1947. The subject, including the consideration of other matrix properties, e.g., stability, that could be determined solely on the basis of the signs of entries, caught the attention of a few economists, mathematicians, and computer scientists (including Klee) in the 1960s and 1970s. But it was the 1984 paper by Klee et al. [36] that was a catalyst for its further substantial development and its further extension to many matrix properties. The authors of the book [9] were enticed into the subject by this paper.

The study of sign solvability can be broken down into the study of L-matrices (linear independence of the rows can be determined solely from the sign pattern) and S-matrices (the sign pattern implies that the null space is spanned by one positive vector). In [36] it was shown that recognizing L-matrices is NP-complete, even when the matrix is "almost square". The recognition problem for square L-matrices (also called sign-nonsingular matrices), which can be formulated as a pure graph-theoretic problem, was later shown to be of polynomial complexity [45].

In the 1980s, Klee, with various coauthors, continued his work on the class of S-matrices, including recursive structure and recognition algorithms. He also investigated linear systems of differential equations from the sign pattern point of view, constructing and classifying such systems [5]. In what I believe to be his last paper in this area, Klee investigated in 2000 the idea of conditional sign solvability where the sign pattern determines the sign pattern of a solution *when a solution exists.* Klee's papers in this area were full of original ideas and clever combinatorial, geometrical, and analytic arguments. He, more than anyone, is responsible for the explosion of interest in sign patterns in the last twenty years.

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Jacob E. Goodman
Richard Pollack

Geometric Transversal Theory

Vic Klee's interest in the combinatorial and topological properties of convex sets, manifested in his early papers [26, 27, 28], led him to help found the subfield of discrete geometry that has recently been recognized with its own AMS classification, geometric transversal theory. After Helly's theorem was published [22, 44], there were some scattered papers written by Santaló, Vincensini, Horn, and others; but it was not until after Vic published [27] his first paper on transversals, that new people, such as Grünbaum, Hadwiger, and

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Danzer, began publishing results relating to common transversals of families of convex sets. In the past few decades the subject has blossomed, and many long-outstanding problems have been resolved and new problems have taken their place. A major inspiration to the field was the very comprehensive survey [12] that Vic wrote with Danzer and Grünbaum in 1963, which summarized virtually all the work that had been done in the field up to that time. Some recent surveys of GTT can be found in [13, 14, 48, 49]. Vic's interest in geometric transversal theory resurfaced in recent years, as evidenced in the papers [37, 42].

In addition to his foundational work in a subject close to our hearts, we can attest to Vic's personal generosity and his encouragement. His interests, perhaps more than anyone's, spanned the complete scope of the journal *Discrete & Computational Geometry* in its early years, on whose editorial board he served from its beginning in 1986. He once commented that DCG was his favorite journal, the one whose new issues he looked forward to reading the most. That offhand remark has meant a great deal to us over the years and continues to inspire us as editors.

Peter Gritzmann
Bernd Sturmfels

From the Klee-Minty Cube to Computational Convexity

Vic Klee has always applied his strong geometric insight to problems in mathematical programming. Arguably his most famous contribution to this field was his paper with Minty [38] on the worst-case behavior of the simplex algorithm. Since the simplex method worked so well in practice, there was a long-standing conjecture that the number of required arithmetic operations (in particular, the number of pivots) is bounded by some polynomial in the dimension n and the number m of inequalities. However, [38] showed that the worst-case behavior of Dantzig's pivot rule is exponentially bad. The offending polytope is combinatorially equivalent to an n-cube; in particular, it is defined by $2n$ linear inequalities in n variables. For a bad choice of starting vertex, the resulting path to the maximizer involves all 2^n vertices of P. Similarly bad behavior was later established by other authors for other pivot rules. However, it is still unknown whether there exists a pivot rule under which the worst-case behavior of the simplex method is polynomially bounded, though certain pivot rules have been shown to have good average-case behavior. The Klee-Minty example was the starting point for the quest for less combinatorial paradigms leading to polynomial-time algorithms for linear programming like the ellipsoid and the interior point methods; see [18].

In retrospect, his work on the simplex method is at the heart of the more recent field of *computational convexity*, the name having first appeared in print in 1988 in [15]. The subject of computational convexity draws its methods from discrete mathematics and convex geometry, and many of its problems from operations research, computer science, and other applied areas. In essence, it is the study of the computational and algorithmic aspects of convex bodies in normed vector spaces of finite but generally not restricted dimension, especially polytopes, with a view to applying the knowledge gained to bodies that arise in a wide range of disciplines in the mathematical sciences.

Basic and typical problems deal with the efficient computation or approximation of geometric functionals such as the volume or the diameter of a polytope, or with the algorithmic reconstruction of a polytope from data concerning it, or with algorithmic versions of geometric theorems; see [19, 20] for surveys. One emphasis in Vic's work on computational convexity was the computation of radii of convex polytopes and more general bodies, leading to far reaching theoretical and algorithmic results that have turned out to be of great relevance in applications ranging from data analysis to medical surgery planning; see [6, 7, 16, 17].

Branko Grünbaum

Convexity

Convexity is a topic that has been studied since the late nineteenth century. A "final report" of sorts was the survey *Theorie der konvexen Körper* by Bonnesen and Fenchel, published in 1934. A new direction in convexity research arose mid-century, combining aspects of convexity and discrete (or combinatorial) mathematics. This was a topic to which Vic was attracted all his life; in later years he also dealt with computational aspects of convexity.

In June 1961 Vic organized the first ever symposium entitled *Convexity*. This served, in many ways, as a starting point of widespread interest in questions of combinatorial convexity. One of the papers included in the proceedings of that symposium was the 80-page survey "Helly's Theorem and its Relatives" [12]. Its genesis occurred through Vic's diplomacy: Ludwig Danzer, I, and Vic were all three interested in writing a survey about Helly's theorem. As the organizer of the symposium and the editor of the proceedings (not

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to mention his senior status relative to Danzer and myself) Vic could have decided that only his paper was to be published. Instead, he proposed to have a joint paper coauthored by all three of us (there were very few three-authors papers at that time). Although Danzer and I did contribute to it, the overwhelming majority of the work on the survey paper was done by Vic. The paper was immediately hailed as a landmark; even more than forty years after it was written, this survey is the most quoted paper of Vic's: According to the MR Citation Database it was referenced in 72 works reviewed since 2000.

Another notable activity of Vic's was the Unsolved Problems column in the *American Mathematical Monthly*, which Vic started in 1969 and to which he contributed many items. This was an outgrowth of an earlier endeavor: Vic compiled in the early 1960s a collection of unsolved problems, meant to be part of a joint effort with Paul Erdős, Laszlo Fejes Tóth, and Hugo Hadwiger; however, this collaboration never materialized.

The level of Vic's activity during the 1960s can also be appreciated by recalling that it was during this time that he wrote his well-known papers on convex polytopes, which ushered in the still-continuing flourishing of that field. Indirectly, these papers are responsible for my book *Convex Polytopes*: In 1963 I conducted a seminar on convex polytopes at the Hebrew University, based on preprints of Vic's papers. The students had difficulties understanding the material, so I started writing explanatory notes; ultimately, these notes became the book [21].

Robert Jamison

The Shift from Continuous to Discrete

Klee's early work was largely in the area of the topology of normed spaces and the geometry of convex bodies. But right from the start there was an indication of the flexibility of his interest. Among his earliest papers are several on the Euler totient [25]. When I arrived in Seattle to study with Vic in 1970, a major shift from the continuous and infinite to the discrete and finite was taking place. It is only fair to say that Vic did not drop one subject for another, rather he expanded his field of interest.

One of Vic's major discrete papers, co-authored with George Minty [38], showed the simplex algorithm could be exponentially bad. He also did work on the greedy algorithm in infinite matroids [31]. In addition to his own work, Klee was a popularizer of ideas and problems. In public lectures he liked to link Hamiltonian cycles with "life on Mars", the idea being that a Hamiltonian cycle in a chemical structure would make the encoding and transmission of its structure easier. Vic was also fond of talking about the Lekkerkerker-Boland characterization of interval graphs. In a research problem in the *American Mathematical Monthly*, he promoted research on a variant, the circular arc graphs, a class that is now widely studied and known to have many interesting properties.

In addition to being an outstanding mathematician, Vic was an outstanding person. Once he agreed to hear a graduate student present a "proof" of Fermat's Last Theorem. I asked Vic why he was willing to invest his time in a project that almost certainly would end in failure. He said, "There was a small chance he was right. Then he would need someone to vouch for him." Vic was generous with his time, with his encouragement, and with his friendship.

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Peter Kleinschmidt

The d-Step Conjecture

The d-step conjecture, first formulated by Warren Hirsch in 1957 and published in 1963 in George Dantzig's classical book on linear programming, arose from an attempt to understand the computational complexity of edge-following algorithms for linear programming as exemplified by the simplex algorithm. It can be stated in terms of diameters of graphs of convex polytopes, in terms of the existence of nonrevisiting paths in such graphs, in terms of an exchange process for simplicial bases of a vector space, and in terms of matrix pivot operations. This variety of equivalent formulations of the conjecture—largely due to Klee himself and reported in Klee-Walkup [40], Klee-Kleinschmidt [35] and Grünbaum [21]—made it a typical field of his research areas: geometry and combinatorics of polytopes, linear programming, and complexity theory. Warren Hirsch died about a month prior to Victor on July 9, 2007. He spent most of his career at NYU were he worked mainly as a probabilist and statistician. He is best known for his work in mathematical biology, particularly on the transmission of parasitic diseases, but several of his earlier papers concerned optimization.

The conjecture states that the maximum diameter of (the graphs of) d-polytopes with $2d$ facets is d. It is equivalent—though not necessarily on a dimension-for-dimension basis—to the Hirsch conjecture, which states that $\Delta(d, n)$, the maximum diameter of the graph of a d-polytope with

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n facets, is not greater than $n - d$. Klee believed the conjecture to be false. However, the fact that it obviously reflects the worst-case behavior of a best edge-following LP-algorithm makes the study of the behavior of $\Delta(d,n)$ very important. The various edge-following LP algorithms, apart from numerical and implementational issues, differ principally in the pivot rule by which the sequence of edges is chosen. A pivot rule that generates a polynomial (in n and d) number of edges would imply a polynomial bound for $\Delta(d,n)$. However, to this day no such rule is known. The famous Klee-Minty example [38] shows that the original rule of Dantzig that maximizes the gradient in the space of nonbasic variables is exponential. For the most common pivot rules exponential behavior was proved subsequently by various authors. The best currently known general upper bound for $\Delta(d,n)$ is quasi-polynomial (Kalai-Kleitman [24]):

$$\Delta(d,n) \quad \leq \quad 2n^{log(d)+1}.$$

This result contains the unbounded case. Results of Barnette [2] and Larman [41] provide an upper bound that is linear in the number of facets but exponential in the dimension. For the important class of 0-1 polytopes Naddef [43] has proved the validity of the Hirsch conjecture. For other classes of polyhedra arising in applications it is also known to be correct. For a survey see [35]. Slightly stronger versions of the Hirsch conjecture have been proved to be false (e.g., the case of unbounded polyhedra and the monotone version). Holt-Klee [23] showed that $n - d$ is the best possible lower bound for $\Delta(n,d)$ in the bounded case for $n > d \geq 8$.

Jim Lawrence

Unfinished Business

Not long ago I received email from Vic in which he mentioned health problems and described actions that he had taken relating to the end of his career; but he said that he was so encumbered by unfinished business that his wife Jodey had made a plaque for him reading "I was put on this earth to accomplish certain things. Right now, I'm so far behind that I'm sure I'll never die." Perhaps one of those pieces of unfinished business related to some hand-written, mimeographed notes from a class he taught when I was a graduate student. They seem to be part of Chapter 2 of a book in preparation, on the subject of convexity. (Some years later, I overheard his answer when asked why he hadn't finished that book; after hesitating a moment, he said, "I think it's because I don't know enough about the subject yet." This engendered much laughter, although he seemed serious.)

Items in *Mathematical Reviews* concerning Vic's papers often use modifiers such as "commendably clear", "precise", "elegant", occasionally preceded by the phrase "as always". I'd like to add two others: "fun" and "exciting". Chapter 2 certainly deserved all of these. It was an energetic account of such topics as Helly's Theorem, Radon's Theorem, and the notion of a positive basis.

Vic greatly liked David Barnette's work in settling what (before Barnette) was called the "lower bound conjecture" for convex polytopes [1], and his paper [34] related to that work provides another example in which a complicated topic is treated with utter simplicity. In this paper, by using the method of Barnette in a graph-theoretical setting, he obtained a generalization for connected pseudomanifolds. The mathematics was quite technically involved, but the paper was written in such a way that one could read it in an almost leisurely manner, and it imparted the feeling that he had actually had a lot of fun in writing it! (I once mentioned to him that I liked the paper. He minimized his contribution, saying that he was just making use of David's methods.)

I suppose that he didn't finish his book on convexity. We would certainly have liked to see the rest of it.

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Joseph Zaks

Shapes of the Future

I first met Victor Klee when I came to the University of Washington as a graduate student in 1966; I was fascinated by his talks in the Geometry Seminar and later on by his Open Problems section in the *Monthly*. In the early 1970s I had enjoyed his two films "Shapes of the Future". (See also [30].) I worked on many problems that Vic raised in his papers and films, in particular on the illumination of planar polygonal simply-connected regions. I gave an example of a non-spherical, non-convex body in 3-space that has constant HA-Measurements, and I have shown, with the aid of a computer (and my son Ayal) that there exist no nine neighborly tetrahedra in 3-space—these are two of the many open problems that Vic mentioned in his articles. I frequently use Vic's part in his book [39] with S. Wagon, and quite often I refer to Chapter 8, concerning the colorings of the rational spaces $\mathbf{Q}^d$; this is related to the Beckman-Quarles Theorem, concerning one-distance preserving mappings from $\mathbf{Q}^d$ to itself.

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I have enjoyed Vic's company during my recent visits to UW, and during many conferences all over. I last saw Vic when he came to my talk at UW in October 2006. In a small tribute to Branko and to Vic, I had organized "The Klee-Grünbaum Festival of Geometry" in Ein Gev, Israel, in 2000, in which we celebrated the 70th birthday of Branko and the 75th birthday of Vic. My wife Sara and I had the pleasure of having Jodey and Vic spend an overnight at our house in Israel, a few days before the festival. Vic was a great teacher, a devoted inspirer, and an extremely warm colleague. He will be missed by all of his students, colleagues, and friends.

Günter M. Ziegler

Generosity

Vic Klee was a wonderful poser of problems, who guided *others* to great success: For example, problems asked by Klee and Erdős in the early 1960s (before I was born) led to the 1962 Danzer–Grünbaum paper about point sets without obtuse angles—which suddenly in 2006 was the key to breakthroughs in Barvinok–Novik's work on centrally symmetric polytopes. But I also remember him asking about cube tilings in Oberwolfach, which stimulated Lagarias and Shor in their stunning disproof of Keller's cube tiling conjecture in 1992.

From our work on the second edition of Branko Grünbaum's *Convex Polytopes* [21], I remember most vividly Vic's generosity: Indeed, the first edition from 1967 appeared "with the cooperation of Victor Klee, Micha Perles, and Geoffrey C. Shephard"; Vic wrote two influential chapters for the book; the re-edition of course would have never happened without his sense of duty, taking responsibility for the project. He provided a wealth of ideas, references, and suggestions from his decades of work on convexity and polytopes, which reflected richly his stunning influence on work of others; this wealth shows in the final product. In the end, the book won the 2005 Steele Prize. I hope that our lasting image of Vic Klee reflects some of the splendor and success that he helped others achieve.

Editors' Epilogue

Looking at the photographs we have of Vic and his family, we know that Vic will live on in our personal memories. He remains with us through all the things we learned from him and admired in him, through his strong direct influence on our paths of life, but also through his modesty, his warmth, and his humor that enriched us, and the thousand little things that left their traces.

Of course, Vic found immortal results in many different branches of mathematics and neighboring fields. What was even more important to us was his wonderful personality. Given his scientific achievements, his numerous awards, and his exceptional standing in the scientific community, it is by no means self-evident how modest, friendly, open-minded, and encouraging he was, how fun to be with, great to talk to, hike with, or play table tennis or billiards with.

Vic always created an atmosphere that made it easy for others to grow despite their own imperfection. He always encouraged others to explain their ideas, no matter how vague they were. It has been incredibly wonderful to be with him and to enjoy mathematics together. **Thank you, Vic!**

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