Fields Medals and Nevanlinna Prize Presented at ICM-94 in Zürich

On August 3, 1994, during the Opening Ceremonies for ICM-94 in Zürich, four Fields Medals and the Nevanlinna Prize were presented. David Mumford of Harvard University, chair of the Fields Medal Committee and vice-president of the International Mathematical Union (IMU), called the medalists to the stage. ICM Honorary President Beno Eckmann, founder of the Mathematical Research Institute at the ETH and former IMU secretary presented the medals. The Nevanlinna Prize Committee Chair Jacques-Louis Lions of Collège de France presented that prize.

Fields Medals were presented to:

JEAN BOURGAIN of the Institut des Hautes Études Scientifiques and University of Illinois at Urbana–Champaign (now at the Institute for Advanced Study in Princeton);

PIERRE-LOUIS LIONS of CEREMADE, Université de Paris–Dauphine;

JEAN-CHRISTOPHE YOCCOZ of Université de Paris–Sud (Orsay); and

EFIM I. ZELMANOV of the University of Wisconsin at Madison (now at the University of Chicago).

The Nevanlinna Prize was presented to:

AVI WIGDERSON of the Hebrew University in Jerusalem.

The members of the Fields Medal Committee were: Luis Caffarelli, Masaki Kashiwara, Barry Mazur, David Mumford (chair), Alexander Schrijver, Dennis Sullivan, Jacques Tits, and S. R. S. Varadhan. The members of the Nevanlinna Prize Committee were: Hendrik Lenstra, Jacques-Louis Lions (chair), Yuri Matiyasevich, Robert Tarjan, and K. Yamaguti.

During the afternoon following the Opening Ceremonies, lectures about the contributions of the awardees were presented: Luis Caffarelli of the Institute for Advanced Study spoke on the work of Bourgain; S. R. S. Varadhan of the Courant Institute of Mathematical Sciences at New York University spoke on the work of Lions; Adrien Douady of Université de Paris–Sud (Orsay) spoke on the work of Yoccoz; Walter Feit of Yale University spoke on the work of Zelmanov; and Yuri Matiyasevich of the Steklov Institute of Mathematics in St. Petersburg spoke on the work of Wigderson.

The *Notices* solicited the following five articles describing the work of the Fields Medalists and Nevanlinna Prize winner.

Allyn Jackson

Fields Medals

At the 1924 International Congress of Mathematicians in Toronto, a resolution was adopted that at each ICM, two gold medals should be awarded to recognize outstanding mathematical achievement. Professor J. D. Fields, a Canadian mathematician who was secretary of the 1924 Congress, later donated funds establishing the medals which were named in his honor. Consistent with Fields's wish that the awards recognize both existing work and the promise of future achievement, it was agreed to restrict the medals to mathematicians not over forty years of age. In 1966 it was agreed that, in light of the great expansion of mathematical research, up to four medals could be awarded at each Congress.

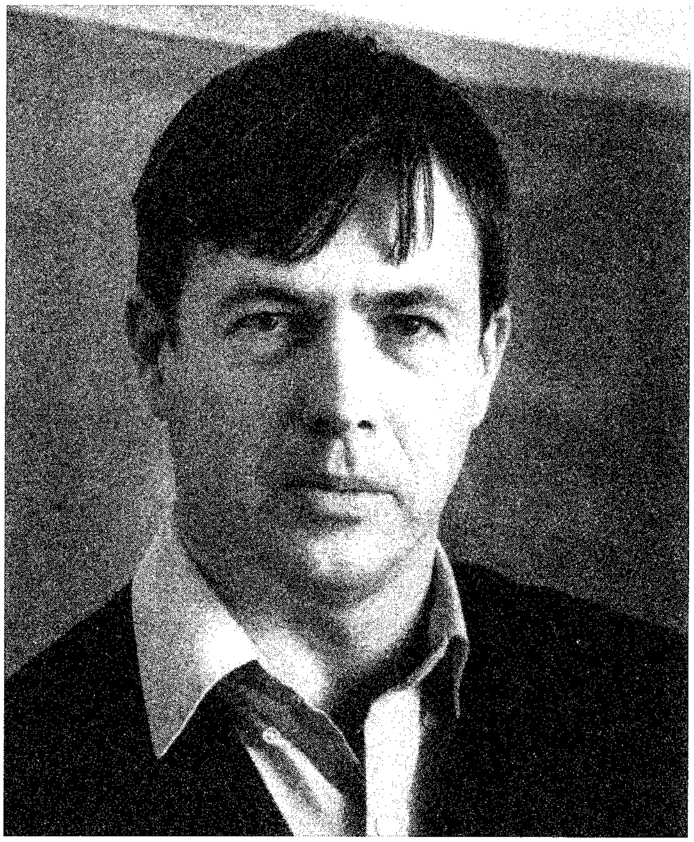
Jean Bourgain

The area of research of Jean Bourgain is analysis in almost all its aspects. He has an extremely strong analytical power which he often combined with ideas and methods of "soft" analysis to solve a very long list of well-known hard problems from many different areas. In his work one also finds many surprising and fascinating connections between areas which were, prior to his work, quite unrelated.

Among the areas of analysis to which Bourgain made very substantial contributions are: Banach space theory, commutative harmonic analysis, complex analysis, ergodic theory, finite dimensional convexity, geometric measure theory, analytic number theory, and, most recently, nonlinear partial differential equations and mathematical physics.

It is impossible even to list a substantial part of Bourgain's achievements in this short essay. What I do here is mention just a few of his significant results (mainly among those which can be stated without elaborate definitions). In the references below I refer also to several papers which contain significant additional results.

1. Using tools from analytic number theory and harmonic analysis, Bourgain proved the following. Let T be an ergodic measure preserving transformation on a probability space Ω. Let $P(k)$, $k = 1, 2, \ldots$ be a "nice" arithmetic sequence (e.g.



Jean Bourgain

$P(k)$ the values taken by a polynomial P with positive integer coefficients on the positive integers or the kth prime). Then for every $f \in L_p(\Omega)$, $p > 1$, the sequence

$$n^{-1} \sum_{k=1}^{n} f(T^{P(k)}x) \ , \ n = 1, 2, \ldots$$

converges almost everywhere on Ω [11].

2. Let G be a compact commutative group (e.g. the circle). Let Γ be the character group of G, and let $2 < p < \infty$. A subset A of Γ is called a $\Lambda(p)$ set if there is a constant C such that

$$\left\| \sum_{\gamma \in A} \alpha_\gamma \gamma(x) \right\|_p \leq C \left(\sum_{\gamma \in A} | \alpha_\gamma |^2 \right)^{1/2}$$

for every choice of scalars α_γ. Combinatorial methods were known for constructing "large" $\Lambda(p)$ sets only for p an even integer. A famous problem in Banach space theory and harmonic analysis was, for example, whether there are $\Lambda(3)$ sets which were not already $\Lambda(4)$ sets. By using hard analytic and probabilistic arguments, Bourgain [7] proved that for every $p > q > 2$ there are $\Lambda(q)$ sets which are not $\Lambda(p)$ sets. This proof works in a more general setting of bounded orthonormal sequences in $L_2(\Omega)$ and leads to the solution of further open questions on orthonormal systems.

3. Let f be a continuous function on the plane, and let $F(x)$ be the maximum of the averages of the values of f on circles centered at x (the maximum is with respect to the radius). Is there a way to estimate the "size" of F by that of f? For $n \geq 3$ estimates of this type were known by using L_2 estimates. It was known that L_2 estimates fail in the plane. Bourgain [3] solved this problem by proving L_p estimates for $p > 2$ using delicate geometric arguments. This result also solved a long-standing open problem concerning the existence of certain fractal sets in the plane.

4. A well-known problem in complex analysis was the following. Let $f(z)$ be a bounded analytic function on the open unit disc in the plane. Are there directions θ , $0 \leq \theta < 2\pi$, so that the restriction of f to the radius vector with direction θ is of bounded variation? It was known that these "good" θ are in general not of positive measure or of the second category. Bourgain [13] proved that such θ always exist and even form a set of Hausdorff dimension 1.

5. Central problems in harmonic analysis in R^n (and also partial differential equations) are questions concerning the boundedness of Bochner-Riesz multipliers (in suitable L_p norms), the boundedness of restrictions of the Fourier transform to some submanifolds (like spheres), and in general estimates concerning oscillatory integrals. This area still has many difficult open problems. A major breakthrough of Bourgain [14, 15] in this direction is that he obtained for the first time results on some of these problems (including some definitive results) in dimension $n \geq 3$ which go beyond results which follow from known L_2 estimates by interpolation. His work contains some new structural results concerning Kakeya type sets in R^n, $n \geq 3$.

6. Bourgain did some seminal work [16] on the well posedness in L_2 of the initial value problem for such equations as the Schrödinger or KdV equations in case the data are periodic in the space variables. Results of this type were known in the nonperiodic case. The treatment of the periodic case required several new tools, including estimates on the $\Lambda(p)$ constant (see 2 above) of certain arithmetic subsets of characters in R^2. The results of Bourgain also give precise information on the required minimal amount of smoothness of the given data. Moreover, he obtained also new well posedness results for nonsmooth initial data in the nonperiodic case. In his proof he used new space time norms originating mainly from the investigation of the periodic case. As an application of his approach he was able to construct for the nonlinear Schrödinger equation a flow on the support of the Gibbs measure and to prove the invariance of this flow [17, 18].

Many of the results and methods of Bourgain have already had significant impact on the research work of other mathematicians. The many gates which he opened also led naturally to a better understanding of his own results (including some simplifications of his original proofs). On the other hand, many aspects of his genial analytical methods have not yet been understood to the fullest by the mathematical community. His work is bound to influence strongly the research in analysis in the coming decades.

Some selected papers of Bourgain

[1] *A class of special $\mathcal{L}^\infty$ spaces* (with F. Delbaen), Acta Math. **145** (1980), 155–176 [Banach space theory].

[2] *New Banach space properties of the disc algebra and H^∞*, Acta Math. **152** (1984), 1–48 [Banach spaces of analytic functions].

[3] *Averages in the plane over convex curves and maximal operators*, J. Analyse Math. **47** (1986), 69–85 [harmonic analysis].

[4] *On high dimensional maximal functions in convex bodies*, Amer. J. Math. **108** (1986), 1467–1476 [harmonic analysis and high dimensional convexity].

[5] *New volume ratio properties for convex symmetric bodies in R^n* (with V. Milman), Invent. Math. **88** (1987), 319–340 [high dimensional convexity].

[6] *On the Hausdorff dimension of harmonic measure in higher dimensions*, Invent. Math. **87** (1987), 477–483 [geometric measure theory].

[7] *Bounded orthogonal systems and the $\Lambda(p)$ set problem*, Acta Math. **162** (1989), 227–245 [harmonic analysis and Banach space theory].

[8] *Restricted invertibility of matrices and applications* (with L. Tzafriri), Cambridge Univ. Press. Lecture Notes **137** (1989), 61–108 [operator theory, Banach space theory and harmonic analysis].

[9] *Approximation of zonoids by zonotopes* (with J. Lindenstrauss and V. Milman), Acta Math. **162** (1989), 73–141 [high dimensional convexity and Banach space theory].

[10] *Distribution of points on spheres and approximation by zonotopes* (with J. Lindenstrauss), Israel J. Math. **64** (1988), 25–31 [convexity].

[11] *Pointwise ergodic theorems for arithmetic sets*, IHES Publ. Math. **69** (1989), 5–45 [ergodic theory].

[12] *Double recurrence and almost sure convergence*, Crelle's J. **404** (1990), 140–161 [ergodic theory].

[13] *On the radial variation of bounded analytic functions on the disc* Duke Math. J. **69** (1993), 671–682 [complex analysis].

[14] *Besicovitch type maximal operators and applications to Fourier analysis*, Geom. Funct. Anal. **1** (1991), 144–187 [geometric measure theory and harmonic analysis].

[15] *L^p estimates for oscillatory integrals in several variables*, Geom. Funct. Anal. **1** (1991), 321–374 [geometric measure theory and harmonic analysis].

[16] *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, Part I: Schrödinger equations; Part II: The KdV equation*, Geom. Funct. Anal. **3** (1993), 107–156, 209–261 [nonlinear PDE and harmonic analysis].

[17] *Periodic nonlinear Schrödinger equations and invariant measures*, Comm. Math. Phys. (to appear) [nonlinear PDE and mathematical physics].

[18] *Invariant measures for the 2 dimensional defocusing nonlinear Schrödinger equation*, Comm. Math. Phys. (to appear) [nonlinear PDE and mathematical physics].

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Pierre-Louis Lions

Pierre-Louis Lions, professor of mathematics at the University of Paris–Dauphine, is one of the world's greatest researchers in nonlinear partial differential equations and related subjects. He has made truly fundamental discoveries cutting across many disciplines, pure and applied, and his publications are so numerous and varied as to defy easy classification.

Keep in mind that there is in truth no central core theory of nonlinear PDE, nor can there be. The sources of PDE are so many—physical, probabilistic, geometric, etc.—that the subject is a confederation of diverse subareas, each studying different phenomena for different nonlinear PDE by utterly different methods. Pierre-Louis Lions is unique in his unbelievable ability to transcend these boundaries and to solve pressing problems throughout the field.

Pierre-Louis Lions

Here are some highlights.

A really great breakthrough in the study of nonlinear first-order PDE was Lions's 1983 paper with M. G. Crandall, which introduced "viscosity solutions" for Hamilton-Jacobi equations. This work resolved a fundamental difficulty in the subject, namely, that such nonlinear PDE simply do not have smooth or even C^1 solutions existing after short times. Classical characteristics (= curves along which a solution of the PDE can be computed) typically collide, and the resulting focusing effects so destroy regularity that the PDE itself no longer makes sense.

The only option is therefore to search for some kind of appropriate "weak" solution. This undertaking is in effect to figure out how to allow for certain kinds of "physically correct" singularities and how to forbid others. But there seemed no way even to begin, especially for nonconvex Hamiltonians. Lions and Crandall at last broke open the problem by focusing attention on viscosity solutions, which are defined in terms of certain inequalities holding wherever the graph of the solution is touched on one side or the other by a smooth test function. Otherwise said, the idea is to require by definition that a viscosity solution satisfy a comparison principle with respect to smooth sub- or supersolutions. Lions and Crandall's amazing insight is that then such viscosity solutions actually satisfy comparison principles with respect to other viscosity solutions. Uniqueness and an arsenal of existence, convergence, stability, and numerical methods follow. The entire theory can be regarded as a profoundly unconventional application of the maximum principle in highly nonlinear settings.

Viscosity solution methods, and their subsequent extension to fully nonlinear second-order elliptic equations by R. Jensen, have totally revolutionized much of PDE, resolving basic questions left open by Hamilton and Jacobi themselves, Carathéodory, and many others. In addition, Lions has made the decisive observation that optimality conditions in differential control and game problems imply the defining inequalities for viscosity solutions of appropriate Bellman or Issacs PDE. This insight has opened essentially the entirety of dynamic programming methods to viscosity solution interpretation.

Completely different ideas lie at the heart of Pierre-Louis Lions's studies of kinetic equations. His extraordinary 1989 paper with R. J. DiPerna is the first mathematical work to show rigorously the existence of solutions to Boltzmann's equation for the density of colliding hard spheres, with general initial data. To construct this "renormalized" solution, Lions and DiPerna introduce a sequence of solutions to easier, approximating problems had by simplifying the collision operator. The trick is then to show that these approximations in fact converge, and in a good enough sense to justify interpreting the limit as actually solving Boltzmann's equation. Since only certain physically natural, but analytically quite weak, moment and entropy estimates are available, this is a seemingly impossible prospect. However, Lions and DiPerna manage just this, mostly by building upon an important earlier observation of Golse, Lions, Perthame, and Sentis that velocity space averaging forces a slight gain of smoothness and therefore compactness. Lions and DiPerna introduce as well several other innovations, notably cutoff renormalization and generalized integration along particle paths.

In later papers they similarly treat an array of related Vlasov, Maxwell-Vlasov, and other equations. More recently, and often in collaboration with B. Perthame and E. Tadmor, Lions has exploited the ideas of kinetic formulations for other types of nonlinear PDE, primarily to uncover new and unexpected compactness phenomena for hyperbolic conservation laws and the Navier-Stokes equations. This entire body of work is a deep and rigorous contribution to the great theoretical problems in science concerning microscopic modeling and corresponding macroscopic behavior.

Yet another ongoing theme in Pierre-Louis Lions's vast research program concerns "concentrated compactness" techniques, which study energy concentration effects. In a long series of papers he has characterized the failure of compactness for minimizing sequences for various functionals in mathematical physics and has in addition devised ways to restore compactness using group invariants of the functionals. Primary tools here are certain measures which record concentrations, the structure of which Lions then restricts by subadditivity inequalities and the nonlinear features of the particular problems. With Coifman, Meyer, and Semmes, he has discovered as well a simple criterion for various natural expressions from PDE theory to lie in Hardy spaces. This insight has led for instance to the important discoveries of Bethuel on harmonic maps.

Let me mention also Lions's extensive work in applied mathematics. His early research on stochastic control theory has led by now to a fairly complete understanding of dynamic programming PDE for the control of differential equations disturbed by noise. The significant technical accomplishment, as in related discoveries of N. V. Krylov, is handling the case that the controls appear in the coefficients of the noise terms: in this setting the PDE are fully nonlinear. Lions has written extensively as well on the rigorous analysis of all sorts of numerical algorithms for PDE. He has in particular (with many recent collaborators) employed PDE techniques to introduce and study various "shape from shading" and modified curvature motion algorithms in image processing.

Lawrence C. Evans
University of California, Berkeley

Jean-Christophe Yoccoz

Jean-Christophe Yoccoz was a student at L'Ecole Normale Supérierre, where he received first place on the entrance exam in 1975 (as well as first place for Ecole Polytechnique in the same year) and shared first place for the *agrégation* for mathematics in 1977. Today, at 38 years of age, Yoccoz is a professor at Université de Paris–Sud (Orsay), a member of Institut Universitaire de France, and a member of Unité Recherche Associé "Topology and Dynamics" of the Centre National de la Recherche Scientifique at Orsay.

Yoccoz can be considered to be the most brilliant specialist in the theory of dynamical systems. This theory consists of the study of the long-term evolution of a system for which one had a law of momentary evolution describing how the state of the system changes from one moment to the next. It is therefore a matter of repeating many times the operation that describes this evolution.

It was for the study of the stability of the solar system that Henri Poincaré founded this theory at the turn of the century. The laws of Kepler, which predict that the planets follow elliptical orbits around the sun, neglect the perturbations caused by the attraction of the planets to each other. These perturbations slowly change the parameters of the orbits, and the question was—and still is—to understand whether this change has a limited effect or whether over several hundreds of millions of years it could cause a planet to be ejected from the solar system or to fall into the sun.

Poincaré discovered very complicated phenomena that arise out of even very simple evolution laws. This has very general consequences, and the theory can be applied to a great variety of fields: mechanics, chemistry, biology, ecology—all fields which consider evolving systems.

Since Poincaré, many mathematicians have developed the theory, describing long-term properties in very general situations. They have also done detailed studies of examples which are both simple and typical where the general phenomena occur but where the explicit descriptions can be pooled quite far. This is the case with the iteration of complex polynomials, which produces fractal objects, Julia and Mandelbrot sets—beautiful pictures which one sees everywhere. Fatou, Julia, Kolmogorov, Siegel, Arnold, Smale, Herman, Palis, Yoccoz, and many others have contributed to this gigantic effort, spending, as always, a great deal of time pursuing dead ends before producing remarkable results. For example, they have proved stability properties—dynamical stability, such as that sought for the solar system, or structural stability, meaning persistence under parameter changes of the global properties of the system.

Jean-Christophe Yoccoz

Yoccoz, who was a student of Michael Herman, quickly established himself as a leader in this area, establishing, among other results, precise limits on the validity of certain stability theorems. He combines an extremely acute geometric intuition, an impressive command of analysis, and a penetrating combinatorial sense to play the chess game at which he excels. He occasionally spends half a day on mathematical "experiments", by hand or by computer. "When I make such an experiment" he says, "it is not just the result that interests me, but the manner in which it unfolds, which sheds light on what is really going on." Yoccoz has developed a method of combinatorial study of Julia sets and Mandelbrot sets—called "Yoccoz puzzles"—which permit deep insight. Yoccoz works regularly in Brazil, especially at IMPA in Rio de Janeiro. He developed an attachment to Brazil after doing his military service there during 1981–1983—he married a Brazilian woman, and they have one child. Many young researchers have written their theses under his direction, including Ricardo Pérez-Marco, who pushed further some of Yoccoz's results.

Adrien Douady
Université de Paris–Sud (Orsay)

Efim I. Zelmanov

Efim Zelmanov* was born in 1955 in the former USSR and spent most of his life in Novosibirsk. In 1980 he completed his Ph.D. thesis, under the supervision of Shirshov and Bokut. Nine years later he completed the solution to one of the most difficult and long-standing problems in group theory—the so-called Restricted Burnside Problem.

Curiously enough, Zelmanov is not a group theorist by training. Most of his mathematical work falls under the category of nonassociative algebras. In his thesis, as well as in his early papers (see, for instance, [10]), Zelmanov revolutionized the theory of Jordan algebras. He was able to extend classical results in finite dimensional Jordan algebras to the infinite dimensional case, where structure theorems are significantly harder. This was to become a leitmotiv in several of his works. For example, it is a basic result in finite dimensional Lie algebras that the Engel identity

$$ad(y)^n = 0$$

implies nilpotency. However, the analogous problem for infinite dimensional Lie algebras remained open for many years. In [11] Zelmanov shows that the Engel identity implies nilpotency in arbitrary Lie algebras of characteristic zero. His beautiful proof makes use of Lie super-algebras and representation theory for symmetric groups.

The analysis of the Engel identity in Lie algebras forms part of a profound PI theory for nonassociative algebras (PI stands for polynomial identities) which Zelmanov and others have shaped over the years. The theory of associative PI rings, developed by Amitsur, Shirshov, and others, served here as an inspiring model. But establishing the nonassociative analogues is by no means straightforward, and an arsenal of powerful new tools and ideas had to be developed.

It is perhaps his interest in the Engel identity which sparked Zelmanov's work on the Restricted Burnside Problem. In 1902 William Burnside [1] wrote: "A still undecided point in the theory of discontinuous groups is whether the order of a group may not be finite, while the order of every operation it contains is finite." This led to the investigation of the following fundamental problems:

1. The General Burnside Problem: is every finitely generated torsion group finite?

2. The Burnside Problem: is every finitely generated group of bounded exponent finite?

3. The Restricted Burnside Problem: is there a bound on the order of d-generated finite groups of exponent n?

*Zelmanov will be addressing the AMS at the Central Section Meeting in Chicago in spring 1995.

Efim I. Zelmanov

Recall that a torsion group is a group in which every element has finite order. The exponent of a group G is the minimal n (possibly infinity) such that $x^n = 1$ for all $x \in G$.

A positive answer to Problem 2 for groups of exponent 2 is an easy exercise for undergraduate students. Less easy, but still not hard, is the case of exponent 3, which was solved by Burnside in [1]. In view of the seemingly elementary nature of the problems, and the positive answers for small exponents, it was hoped at the beginning of the century that Problems 1 and 2 would have affirmative solutions. This was not to be. In the late 1930s, after many years of unsuccessful attempts to solve these questions, the study of Problem 3 was initiated; it was Magnus [6] who gave the problem its current name. To clarify the relation between Problems 2 and 3, let F_d be the free group on d generators, and let F_d^n be the subgroup generated by all nth powers in F_d. Then $F_d^n \triangleleft F_d$ and we can form the quotient group $B(d,n) = F_d/F_d^n$. The question whether $B(d,n)$ is finite is the content of the Burnside Problem. The Restricted Burnside Problem asks whether the (possibly infinite) group $B(d,n)$ has a maximal finite quotient. It is now clear that (for given d and n) a positive solution to Problem 2 implies a positive solution to Problem 3. In fact the Restricted Burnside Problem may be regarded as the Burnside Problem for residually finite groups (i.e., groups whose finite index subgroups intersect at the identity).

The Burnside problems have attracted wide attention throughout this century. Gradually it became clear (as often happens) that the simple-sounding formulations are rather misleading: naive methods did not yield fruits, and highly subtle mathematical theories had to be developed. These include important aspects of combinatorial group theory, geometry, combinatorics, finite group theory, and the theory of infinite dimensional Lie algebras. Needless to say, the tools developed in the context of the Burnside problems had numerous additional applications.

Early work by Sanov and by M. Hall provided positive solutions to Problem 2 for exponents 4 and 6 respectively. In 1964 Golod provided a negative answer to the General Burnside Problem by constructing an infinite finitely generated p-group. Four years later Novikov and Adian showed that, if $n \geq 4381$ is an odd integer, then there are infinite 2-generated groups of exponent n. The restriction on n has subsequently been weakened, and results for (large) even exponents have recently been published by Ivanov. Geometric methods developed by Ol'shanski and by Rips gave rise to new constructions and demonstrated how wild groups of finite exponent can be. However, The Burnside Problem is still very much open for small exponents, most notably $n = 5$.

The Restricted Burnside Problem went down a different route. A reduction theorem proved by P. Hall and G. Higman in 1956 shows that, assuming there are only finitely many finite simple groups of any given exponent and that the Schreier conjecture on the solubility of the outer automorphism groups of finite simple groups holds, it suffices to provide a positive answer to Problem 3 for prime power exponents $n = p^k$ [2]. In this case the groups G in question are finite p-groups, and their lower central series give rise to Lie algebras L over the field $\mathbb{F}_p$ with p elements . It was shown by Magnus [6] that, if $n = p$, then these Lie algebras satisfy the Engel identity

$$ad(y)^{p-1} = 0.$$

The Restricted Burnside Problem for prime exponent p was thus reduced to the following Lie-theoretic question: Are Lie algebras over $\mathbb{F}_p$ satisfying the $(p-1)$ Engel identity locally nilpotent? A theorem of Kostrikin [3, 4] provides an affirmative answer to this question, and thus solves the Restricted Burnside Problem in the case of prime exponents.

One of the difficulties in extending Kostrikin's approach to prime power exponents $n = p^k$ was that in this case it is not known whether the associated Lie rings L satisfy any fixed Engel identity. Consequently, there was no reduction to a natural Lie-theoretic question. The first significant step in Zelmanov's work on the Restricted Burnside Problem was to establish such a reduction. He proved in [12] that in order to provide an affirmative solution, it suffices to show that a Lie algebra over an infinite field of characteristic p which satisfies an Engel identity is locally nilpotent. The reduction is nontrivial and uses combinatorics of words, in the spirit of Shirshov.

Having completed the reduction, Zelmanov set out to prove the local nilpotency of Engel Lie algebras of characteristic p. His stunning proof, published in [13] for $p > 2$ and in [14] for $p = 2$, combines an amazing technical capability with highly original ideas from various disciplines. The proof uses a deep structure theory for (quadratic) Jordan algebras, previously developed by McCrimmon and Zelmanov [7], as well as divided powers and other tools; it also relies on the joint work of Kostrikin and Zelmanov [5], which establishes the local nilpotency of so-called sandwich algebras.

While Lie algebras have long been considered a natural playground in the context of the Restricted Burnside Problem, the appearance of Jordan algebras in this context is unprecedented and quite surprising. The role of Jordan algebras is crucial for $p = 2$, but they are also used for odd primes. It is perhaps no coincidence that the Restricted Burnside Problem was solved by a mathematician whose skills and interests are so broad: a proper group theorist (or Lie-theorist) could never have discovered such a proof.

A more elementary presentation of Zelmanov's proof can be found in Vaughan-Lee's recent monograph [8].

Zelmanov's Theorem, combined with the Classification of Finite Simple Groups, shows that there exists a function f such that, for any n and d, the order of a finite group of exponent n with d generators is at most $f(d, n)$. Can one give explicit bounds on the function f? A recent work of Vaughan-Lee and Zelmanov [9] shows that, for a prime power n we have

$$f(d,n) \leq d^{d^{\cdot^{\cdot^{d}}}}, \text{where } d \text{ occurs } n^{n} \text{ times.}$$

The real order of magnitude of f still remains a mystery.

Zelmanov's work on the Restricted Burnside Problem has had profound impact on both ring theory and group theory; within group theory proper it has applications to compact (profinite) groups, pro-p groups, the theory of p-adic Lie groups, the study of various group-identities, etc. It is impossible to describe all these applications here, so let me mention just one. A classical result of P. Hall and C.R. Kulatilaka shows that every infinite locally finite group has an infinite abelian subgroup. The analogous problem for compact groups has been open for quite some time. By slightly adapting the techniques of [13, 14], Zelmanov has recently solved this problem [15]. The main result of [15] asserts that torsion compact groups are locally finite. This result, which was conjectured by Platonov, may be regarded as a positive solution to the General Burnside Problem for compact groups; it readily implies that every infinite compact group has an infinite abelian subgroup.

Since 1989 Zelmanov has held positions at various universities in the West, including Yale, Oxford, and the University of Wisconsin at Madison. He is currently a professor of mathematics at the University of Chicago.

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The University of Helsinki has granted funds to award the Rolf Nevanlinna Prize in the mathematical aspects of information science to a young mathematician, to be given at the International Congress of Mathematicians. The Nevanlinna Prize was first awarded in 1982.

Avi Wigderson

Avi Wigderson has been one of the most prolific of theoretical computer scientists, and even an unannotated bibliography of his publications would exceeded the length appropriate for this article. Attention will be confined, therefore, to two particular aspects of his work, which have been chosen because they illustrate central themes of interest in theoretical computer science.

The first of these is the use of communication complexity for the derivation of lower bounds in circuit complexity. Circuit complexity deals with the question of how complex a circuit must be to perform a given computational task. A typical computational task is: given a graph on n vertices, determine whether there exists a path connecting two distinguished vertices. Such a problem might be presented to a circuit through $\binom{n}{2}$ Boolean inputs (one to indicate the presence or absence of each possible edge), and the circuit could produce its answer as a single Boolean output. A circuit in this context is a sequence of Boolean functions of the inputs, with each function in the sequence being either one of the inputs or a Boolean combination of one or two previous functions in the sequence (the complement of a previous function, or the "and" or "or" of two previous functions), and with the last function in the sequence being the desired output. The complexity of such a circuit can be measured in many ways; one common measure is the "depth", the length of the longest chain of dependence between an input and the output. For most questions of circuit complexity, upper bounds on complexity are given by explicitly describing a circuit that solves the problem; lower bounds are much harder to obtain, and there are very few problems for which nontrivial lower bounds (depth growing faster than logarithmically with the number of inputs) have been obtained. A frequent practice is to restrict attention to "monotone" circuits (disallowing complementation, and allowing only "and" and "or" operations); this of course is only possible when the computational task is itself monotone. (The graph-connection problem is monotone: adding an edge to a graph can connect two vertices, but it cannot disconnect them.) For the graph-connection problem, it has long been known that monotone circuits of depth $O\big((\log n)^2\big)$ suffice, and Wigderson's work with Mauricio Karchmer has shown that depth at least $c(\log n)^2$ (for some constant $c > 0$) is necessary as well.

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The proof of this result involves an unexpected transformation to a seemingly unrelated problem of "communication complexity". Suppose that an agent is given a "path" (a minimal set of edges connecting two distinguished vertices), while another agent is given a "cut" (a minimal set of edges separating these vertices). The two agents are to communicate with each other by transmitting Boolean values over a two-way communication channel according to some agreed-upon protocol until they have exchanged enough information to identify an edge common to their sets (of course, any path and cut must have at least one edge in common). Miraculously, the minimum number of Boolean values they must exchange is essentially the minimum possible depth of a monotone circuit for the graph-connection problem. The translation into communication-theoretic terms, however, greatly increases the scope for probabilistic and information-theoretic arguments, which played a key role in Karchmer and Wigderson's proof.

Another transformation between circuits and communication was discovered by Wigderson and Ran Raz for another classical problem of complexity theory, the "perfect matching" or "marriage" problem. In this problem, there are n^2 Boolean inputs, one telling whether or not each of n boys knows each of n girls. The circuit is to determine whether or not there is a one-to-one correspondence pairing each boy to a girl he knows. For this problem, circuits of depth $O\big((\log n)^2\big)$ are known to exist (though the proof of this does not give a completely explicit description), but if the circuits must be monotone only the much weaker bound $O(n)$ is known. Raz and Wigderson showed that for monotone circuits, depth at least cn (for some constant $c > 0$) is necessary, illustrating dramatically the importance of allowing complementation even when it is not obviously called for by the statement of the problem. For this problem, the communication problem is particularly simple: two agents are each given a subset of an n-element set, and they are to determine by exchanging Boolean values whether or not their subsets have a nonempty intersection.

The second central theme of Wigderson's work concerns the replacement of random numbers by deterministically generated "pseudorandom" numbers. This enterprise has its origin in a famous remark of John von Neumann following the birth of Monte Carlo methods: "Anyone who considers arithmetical methods of producing random digits is, of course, in a state of sin... (It is true that a problem that we suspect of being solvable by random methods may be solvable by some rigorously defined sequence, but this is a deeper mathematical question than we can now go into.)" Going into this question has appeared prominently on the agenda of theoretical computer science during the last decade or so, and it has been a frequent theme in Wigderson's work. A beautiful illustration of this theme concerns the graph-connection problem considered above, but with the complexity being measured not by the depth of a circuit but rather by the "space" (the working storage, measured in bits) taken by a computing machine. (Many models for such a machine are possible, but almost any responsible attempt to define the number of bits of space used will result in the same measure of complexity, to within constant factors.) Around 1970, Walter Savitch showed that the graph-connection problem can be solved by a machine using space $O\big((\log n)^2\big)$. The first improvement of this bound came in 1992, when Wigderson, together with Noam Nisan and Endre Szemerédi, reduced it to $O\big((\log n)^{3/2}\big)$. Their method exhibits particularly clearly the randomization-derandomization paradigm that is becoming increasingly successful. First one finds a randomized solution to the problem that uses a very restricted amount X of resources (for the graph-connection problem there is one using space $O(\log n)$). Then one devises a pseudorandom sequence that can be produced deterministically with the restricted amount $Y > X$ of resources, but that "looks random" to any computational process using only X resources. Composition then yields a deterministic solution using $X + Y$ resources. In the result of Nisan, Szemerédi, and Wigderson, as is often the case, there is some extra ingenuity required going beyond the paradigm described, making this decisive result an appealing combination of old and new ideas.

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