Karl Menger

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Karl Menger died on October 5, 1985, in Chicago. Except in his native Austria [5], no obituary notice seems to have appeared. This note marks ten years since his passing.

Menger's career spanned sixty years, during which he published 234 papers, 65 of them before the age of thirty. A partial bibliography appears in [15]. His early work in geometry and topology secured him an enduring place in mathematics, but his reach extended to logic, set theory, differential geometry, calculus of variations, graph theory, complex functions, algebra of functions, economics, philosophy, and pedagogy. Characteristic of Menger's work in geometry and topology is the reworking of fundamental concepts from intrinsic points of view (curve, dimension, curvature, statistical metric spaces, hazy sets). A few of these and some of his other accomplishments will be mentioned here.

Menger was born in Vienna on January 13, 1902, into a distinguished family. His mother, Hermione, was an author and musician; his father, Carl Menger, well known as a founder of the Austrian School of Economics, was tutor in economics to Crown Prince Rudolph (the ill-starred Hapsburg heir-apparent, played by Charles Boyer in "Mayerling").

From 1913 to 1920 Menger attended the Doblinger Gymnasium in Vienna, where he was recognized as a prodigy. Two of his fellow students were Nobel Laureates Richard Kuhn and Wolfgang Pauli. He entered the University of Vienna in 1920 to study physics, attending the lectures of physicist Hans Thirring. Hans Hahn joined the mathematics faculty in March 1921, and Menger attended his seminar "News about the Concept of Curves". In the first lecture Hahn formulated the problem of making precise the idea of a curve, which no one had been able to articulate, mentioning the unsuccessful attempts of Cantor, Jordan, and Peano. The topology used in the lecture was new to Menger, but he "was completely enthralled and left the lecture room in a daze" [10, p. 41]. After a week of complete engrossment, he produced a definition of a curve and confided it to fellow student Otto Schreier, who could find no flaw but alerted Menger to recent commentary by Hausdorff and Bieberbach as to the problem's intractability, which Hahn hadn't mentioned. Before the seminar's second meeting Menger met with Hahn, who, unaccustomed to giving first-year students a serious hearing, nevertheless *listened* and after some thought agreed that Menger's was a promising attack on the problem.

Inspired, Menger went to work energetically, but having been diagnosed with tuberculosis, he left Vienna to recuperate in a sanatorium in the mountains of Styria, an Austrian counterpart of "The Magic Mountain". He wrote later that at the same time Kafka was dying in another sanatorium. In Styria he elaborated his theory of curves and dimension, submitting a paper to *Monatshefte fur Mathematik und Physik* in 1922 which contained a recursive definition of dimension in

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a separable metric space. P. S. Urysohn simultaneously and independently of Menger developed an equivalent definition. The Menger/Urysohn definition has become the cornerstone of the theory [6].

After receiving a Ph.D. in 1924, Menger, interested in L. E. J. Brouwer's fundamental work in topology and wanting to clarify his thought about intuitionism, which he saw as the counterpart in mathematics to Ernst Mach's positivism in science, accepted Brouwer's invitation to Amsterdam, where he remained two years as docent and Brouwer's assistant. Though he found Brouwer testy, he retained warm feelings about his stay in Amsterdam and cited the good Brouwer did for some young mathematicians and "the beautiful experience of watching him listen to reports of new discoveries" [15].

In 1927, at age 25, Menger accepted Hahn's invitation to fill Kurt Reidemeister's chair in geometry at the University of Vienna. During the year 1927–28, in preparing a course on the foundations of projective geometry, Menger set out to construct an axiomatic foundation in terms of the principal features of the theory: the operations of joining and intersecting, which Garrett Birkhoff later called lattice operations. Menger was critical of Hilbert's and Veblen's formulations: Hilbert's requiring a different primitive for each dimension; Veblen's giving points a distinguished role not played in the theory itself, since hyperplanes play the same role as points. Von Neumann in [17] took a line parallel to Menger's, seeking "to complete the elimination of the notion of point (and line and plane) from geometry". He refers to Menger as "the first to replace distinct classes of 'undefined entities' by a unique class which consists of all linear subspaces of the given space, an essential part of his system being the axiomatic requirement of a linear dimensionality function." Menger subsequently gave a self-dual set of axioms for the system.

In 1928 he published *Dimensiontheorie* (Teubner). Fifty years later, J. Keesling wrote: "This book has historical value. It reveals at one and the same time the naiveté of the early investigators by modern standards and yet their remarkable perception of what the important results were and the future direction of the theory" [7]. The authors of [10] illustrate the remark with Menger's theorem that every n-dimensional separable metric space is homeomorphic to part of a certain "universal" n-dimensional space, which can in turn be realized as a compact set in $(2n+1)$-dimensional Euclidean space. The universal 1-dimensional space, (the "Menger universal curve" or "Menger sponge"), appears in Mandelbrot's *The Fractal Geometry* [9] and in [2]. Menger's *Kurventheorie* appeared in 1932 (Teubner; Chelsea reprint, 1967). It contains Menger's n-Arc Theorem: Let G be a graph with A and B two disjoint n-tuples of vertices. Then either G contains n pairwise disjoint AB-paths (each connecting a point of A and a point of B), or there exists a set of fewer than n vertices that separates A and B. Menger's account of the theorem's origins appears in [11], an issue of the *Journal of Graph Theory* dedicated to his work. In an introductory note Frank Harary calls it "the fundamental theorem on connectivity of graphs" and "one of the most important results in graph theory". Variations of Menger's theorem and some of its applications are given in Harary's *Graph Theory* (Addison-Wesley, 1969).

In that same year Menger joined philosopher Moritz Schlick's fortnightly discussion group, which became known as the Vienna Circle [10], [18]. In 1931 in tandem with the Circle, Menger ran a mathematical colloquium in which Rudolph Carnap, Kurt Gödel, Alfred Tarski, Olga Taussky, and others took part. Gödel (also a Hahn student) first announced his epoch-making incompleteness results at the colloquium. Menger edited the series *Ergebnisse eines Mathematischen Kolloquiums* in the years 1931–37. Additionally, during this period a series of public lectures in science was held. Einstein and Schroedinger were among the speakers. Admission was charged, and the money generated was used to support students.

In the posthumously published *Reminiscences of the Vienna Circle and the Mathematical Colloquium* [10] Menger gives perceptive and lucid accounts of the philosophical and cultural atmosphere in Vienna, the personalities that streamed through the Circle and those who participated in the colloquium, and the topics discussed and issues that engrossed the participants. There are chapters on Wittgenstein and a moving account of Menger's long association with Gödel. There is also an account of Menger's time at Harvard and The Rice Institute (1930–31), where he met most of the leading U.S. mathematicians of the day. Though he recognized E. H. Moore as the father of American mathematics, Percy Bridgman and Emil Post made the strongest impression on him. Menger regarded Bridgman as the successor to Mach. And he had great admiration for Post, who (with Haskell Curry) is a source for Menger's later work in the algebra of functions.

In the early 1930s Menger developed a notion of general curvature of an arc A in a compact convex metric space. Consider a triple of points of A, where A is an ordered continuum, not necessarily described by equations or functions. The triangle inequality implies the existence of three points in the Euclidean plane isometric to the given triple, and their Menger curvature is the reciprocal of the radius of the circumscribing circle. This curvature is zero if and only if one of the points is between the other two. Menger defined the curvature at a point of A to be the number (if it exists) from which the curvature of any three sufficiently close points in the Euclidean plane differs arbitrarily little. Numerous results and modifications of Menger's concept were obtained by his student Franz Alt and by Gödel [4]. The extension to higher-dimensional manifolds was achieved by Menger's student Abraham Wald (later a distinguished statistician), who obtained a fundamentally new way of introducing Gaussian curvature. Menger's comment: "This result should make geometers realize that (contrary to the traditional view) the fundamental notion of curvature does not depend on coordinates, equations, parametrizations, or differentiability assumptions. The essence of curvature lies in the general notion of convex metric space and a quadruple of points in such a space" [12].

Menger had a lifelong interest in economics [15]. Oscar Morgenstern reports that Menger's paper "Das Unsicherherheitsmoment in der Wertlehre" (1934) played a primary role in persuading von Neumann to undertake a formal treatment of utility [8]. Two essays of Menger's appear in *Economic activity analysis*, the 1954 Princeton Economics Research Project's collection of essays edited by Morgenstern.

With Hitler's coming to power in 1933, Austrian agitation for unification with Germany intensified. With the resulting turmoil and street violence and Chancellor Dolfuss taking dictatorial powers in 1934, the vigorous intellectual life in Vienna atrophied. Extreme pro-German "nationalists" ruled the faculty and the student body of the university, and the Circle was disparaged and maligned. Hahn, a progressive force, had died in 1934, and in June 1936 Schlick, founder of the Circle, was shot dead by a deranged student. Still stunned by the tragedy, Menger attended the 1936 International Congress of Mathematicians in Oslo, becoming one of its vice-presidents. He described the deteriorating situation to friends and associates. Shortly thereafter he received a cable offering him a professorship at the University of Notre Dame. He and his family arrived in South Bend in 1937. In March 1938, the month of the *Anschluss*, Menger resigned his professorship in Vienna. He did not return until the 1960s.

European intellectuals who fled Europe for refuge in America were sometimes uncomfortable with American ways and unaccustomed to teaching elementary courses. Though central European in dress, manner, and style, Menger felt at home in America and enjoyed teaching undergraduates, believing that when properly done, it stimulated research. During the 1960s he lectured to high school students on the subject "What is x?".

His fourth child was born in 1942, and the others were then reaching school age. Menger was drawn to Chicago, where Carnap and others were developing a sort of "Chicago Circle" and where his children's education would benefit from a cosmopolitan environment. L. R. Ford, whom he had met at Rice in 1931, had become IIT mathematics department chairman. Thus in 1948 Menger came to IIT and remained in Chicago the rest of his life.

In the note "Statistical Metrics" [14] Menger made a contribution to resolving the interpretative issue of quantum mechanics. He proposed transferring the probabilistic notions of quantum mechanics from the physics to the underlying geometry. He showed how one could replace a numerical distance between points p and q by a distribution function Fpq whose value $Fpq(x)$ at the real number x is interpreted as the probability that the distance between p and q is less than x. Studies of such spaces by numerous authors followed, as did the book *Probabilistic metric spaces* (North Holland, 1983), by Berthold Schweizer and Abe Sklar.

In 1951 Menger introduced the concept of a "hazy set" [16], in which the element-set relation is replaced by the probability of an element belonging to a set. Hazy sets were rediscovered and renamed "fuzzy sets" in [1] and have become the subject of a broad research area.

In the war years Menger taught calculus to future navy officers in the V-12 program, which prompted him to rethink the foundations of the subject. It led to the monograph *Algebra of analysis* [13] and some papers which aimed to systematize and clarify the foundations of analysis, and it rekindled his interest in the algebra of functions. It also acquainted him with the calculus textbooks of the time, which he found made scant distinction between f and $f(x)$, and in which specific functions, used everywhere in calculus—like the identity function and selector functions for ordered n-tuples—were not identified. But Menger was operating in the framework of the Vienna Circle and the emphasis there on clear thought and expression. One might guess that he found an assault on the traditional calculus irresistible. Thus Menger's attempt to reform calculus: *Calculus, a modern approach* (Ginn, 1955). Written in characteristically vigorous style, it was a radical revision of textbooks of the period, scrapping some traditional notation. It received a lengthy, thoughtful, and cautiously favorable review in the *Monthly* by H. E. Bray [3] but was never accorded serious attention. He sent a copy to Einstein, who replied that he liked it and recognized the need for some clarity in notation, but advised against attempting too much "housecleaning".

That his book was ignored saddened Menger's later years. When Menger addressed the foundations of dimension theory, topology, projective n-space, or differential geometry, attention was paid by the best mathematical minds of his generation: Hahn, Brouwer, von Neumann, Gödel. The failure of the calculus endeavor strained his relations with the mathematical community.

Menger has been described as a fiery personality. As a junior faculty member at IIT in the 1960s, I found him gracious, charming, and vivacious. Menger was solicitous of students. From his early days in Vienna onward he invited students and faculty to his home. In Chicago it included a tour of his decorative tile collection, which lined the walls of his living room. And he sometimes invited doctoral students for early morning mathematical walks along Lake Michigan.

His office was a showplace of chaos, the desktop covered with a turbulent sea of papers. He knew the exact position of each scrap. On the telephone he could instruct a secretary exactly how to locate what he needed. Once, in his absence, a new secretary undertook to "make order", making little stacks on his desk. Upon his return, discovering the disaster, he nearly wept, because "Now I don't know where anything is."

Menger liked America. He even liked the Marx Brothers. I once met him emerging from the Clark Theater in Chicago, where "A Night at the Opera" was playing. Still suffused with laughter, all he could say was, "Funny! Funny!"

Though born into a family with ties to the Austrian crown, Menger did not like establishments. His work shows that he could shed traditional ways as called for. He was a peerless mathematician and an independent and original spirit.

References

[1] R. Bellman, R. Kalabe, and L. Zadeh, *Abstraction and pattern design*, J. Math. Anal. Appl. **13** (1966), 1–7.

[2] L. Blumental and K. Menger, *Studies in geometry*, W. H. Freeman, 1970.

[3] H. E. Bray, Math. Monthly of America, vol. 61 (1954), 483–493.

[4] K. Gödel, *Collected works*, vol. 1, Oxford Univ. Press, 1986, pp. 272–281.

[5] E. Hlawka and K. Menger, *Sonderabdruck aus dem Almanach der Osterreichischen Akademie der Wissenschaften*, 136 Jahrang, 1986.

[6] W. Hurewicz, *Dimension theory*, Princeton Univ. Press, 1941.

[7] J. Keesling, Bull. Amer. Math. Soc. **83** (1958), 203.

[8] H. W. Kuhn and A. W. Tucker, *John von Neumann's work on the theory of games*, Bull. Amer. Math. Soc. **64** (1958).

[9] B. B. Mandelbrot, *The fractal geometry of nature*, W. H. Freeman, 1982.

[10] K. Menger, *Reminiscences of the Vienna Circle and the Mathematical Colloquium*, (L. Golland, B. McGuinness, and A. Sklar, eds.), Kluwer Academic Publishers, Dortmund, 1994.

[11] ——, *On the origin of the n-arc theorem*, J. Graph Theory **5** (1981), 341–350.

[12] ——, *The formative years of Abraham Wald and his work in geometry*, Ann. of Math. Statistics **23** (1952), 14–20.

[13] ——, *Algebra of analysis*, Notre Dame Math. Lectures, vol. **3**, Univ. Notre Dame, IN, 1944.

[14] ——, *Statistical metrics*, Proc. Nat. Acad. Sci. U.S.A. **28** (1942).

[15] ——, *Selected papers in logic and foundations, didactics, economics*, Reidel, 1979.

[16] ——, *Ensembles flous et fonctions aleatoires*, CRP **232** (1951), 2001–2003.

[17] J. von Neumann, *Continuous geometry*, Proc. Nat. Acad. Sci. U.S.A. **22** (1936), 92–100.

[18] K. Sigmund, *A philosopher's mathematician: Hans Hahn and the Vienna Circle*, The Mathematical Intelligencer **17** (1995), 6–29.