Olga Arsen′evna Oleĭnik (1925–2001)

Willi Jäger, Peter Lax, and Cathleen Synge Morawetz

Her Life[1]

On October 13, 2001, Olga Arsen′evna Oleĭnik, one of a handful of truly exceptional women mathematicians of the twentieth century, died in Moscow at the age of seventy-six, succumbing, after a long struggle, to cancer.

Olga Oleĭnik was born in the Ukraine on July 2, 1925. In 1941 the Ukraine was invaded by Germany, and the machine factory where Olga's father was bookkeeper was evacuated to Perm in the Urals. The sixteen-year-old Olga accompanied him and finished high school there. Her mother, her sister, and her nephew remained in the Ukraine. Olga then attended the University of Perm, to which the mathematics and mechanics faculty of the Moscow State University had also been evacuated. There her talents came to the attention of professors Sof′ya Yanovskaya and Dynnikov, who arranged in 1944 for her to become a student in Moscow at the university. She was married briefly and had one son, to whom she was devoted and who predeceased her.

Oleĭnik continued at the university, receiving her doctor's degree in 1954 with a thesis in partial differential equations (PDEs) under the guidance of I. G. Petrovskiĭ. The topic, partial differential equations with small coefficients multiplying the highest-order terms, heralded some of the underlying approaches to much of her future work. Although she also made contributions to algebraic geometry, to Hilbert's sixteenth problem, and to other topics, her forte and lifetime contributions were focused on PDEs that arise in very important applications such as boundary-layer theory and elasticity. In fact, she constantly emphasized the role of PDEs in applications [O1].

She remained a devoted student and disciple of Petrovskiĭ and his memory for her whole life. She succeeded him to the chair of differential equations, and despite the political difficulties of the seventies and eighties, she steered the department forward successfully throughout her tenure. Oleĭnik received many prizes and awards in her lifetime; she was named a member of the Russian Academy of Sciences in 1990 but had earlier been made an honorary member of too many foreign academies and societies to list here. She was also awarded many honorary degrees as well as prizes both in her native land and abroad. In all, she published more than three hundred papers and eight monographs.

She was deeply involved in improving and extending contacts between Russian and Western scientists. She was particularly eager to bring the Western world to Russian mathematicians and was therefore anxious to see Western mathematical literature translated into Russian. An early beneficiary of such a translation was the new edition of Courant-Hilbert, Vol. 2, which appeared in Russian in 1962, the same year it appeared in English. When Courant died in 1972, Oleĭnik and Paul Aleksandrov wrote a long obituary in *Uspekhi*[2] emphasizing

Willi Jäger (WJ) is a professor in the Institute for Applied Mathematics at the University of Heidelberg, Germany. His email address is jaeger@iwr.uni-heidelberg.de.

Peter Lax (PL) is professor emeritus at the Courant Institute, New York University. His email address is lax@cims.nyu.edu.

Cathleen Synge Morawetz (CSM) is professor emeritus at the Courant Institute, New York University. Her email address is morawetz@cims.nyu.edu.

[1] *CSM. The author (CSM) is indebted to Tatiana A. Shaposhnikova of Moscow University for much of this information.*

[2] *P. S. Aleksandrov and O. A. Oleĭnik,* Uspekhi Mat. Nauk ***30** (1975), no. 4 (184), 205–26; MR **52** #7786.*

Courant's contributions to American mathematics through his institute at New York University.

Oleĭnik loved to travel. In 1960 she met Courant and Lax in Moscow on their visit to Alexandrov, an old friend of Courant's. Thus when she was invited to a women's congress in California shortly afterwards, she was able to accept Courant's invitation to visit New York. It is hard now to remember how unusual such a visit was in the decade following Stalin's death. She retained an abiding affection and interest in those working in Courant's group: K. O. Friedrichs, F. John, P. D. Lax, and L. Nirenberg. I met her then for the first time, but I became close to her only when, at G. Fichera's invitation, she and I spent a month in 1965 together at the University of Rome. The university was "under siege" by the student body, and it was only Fichera's political skill that made it possible for Oleĭnik to give her talks. We were excluded from the campus much of the time, so together we happily toured the wonderful sights of Rome instead.

Oleĭnik was a very private person. In the latter part of her life she suffered badly from knee trouble, which prevented her from walking properly and sometimes kept her hospitalized. During one such long stay she kept occupied by writing another book. For a woman who had been so active as a mathematician and with her students (she supervised fifty-six dissertations), these spells of inactivity were trying. She told me once that as a young student she had worked in what we would call a lumber camp and how much she, in contrast to the other girls, had enjoyed doing the hard physical work.

The life span of Olga Oleĭnik was from the early days of the sovietization of Russia to the complete collapse of communism. I never heard her express a political opinion nor make a complaint about how things were going. She clearly tried to live within the system she had been raised in, and her model was always her teacher, Petrovskiĭ. Uprooted from the Ukraine by the war, she identified herself as a Russian, not as a Ukrainian. Of one thing I am sure: she believed a larger picture of her country was the right one.

Oleĭnik was devoted to mathematics. She drove her theorems to their absolute limits. She never seemed happier than when she was doing mathematics or working with her students, the most important elements in her life. She left a great deal of unfinished work, although she had published so much. We may hope that this work will be completed by her students and colleagues.

Oleĭnik was one of the major figures in the study, during the fifties and sixties, of elliptic and parabolic equations. This study was the major field of partial differential equations at the time, and many mathematicians such as Agmon, John, Ladyzhenskaya, Morrey, Nirenberg, and Vishik were involved in it. It would make this article too technical to describe Oleĭnik's most significant achievements in this area, and we will confine the scientific description to some of the many other areas to which she made even more significant contributions.

Olga Oleĭnik.

Nonlinear Hyperbolic Equations[3]

During the years 1954–61, Olga Oleĭnik studied the theory of nonlinear hyperbolic conservation laws and the propagation of shock waves. Her contributions were basic and extremely original. Her 1957 paper in the *Uspekhi* [O2] was particularly influential. The starting point of her work, like much in this field, was Eberhard Hopf's fundamental paper of 1950 [H].

Oleĭnik's results deal mainly with the existence, uniqueness, and properties of solutions of the initial-value problem for single conservation laws. She showed that solutions satisfy one-sided Lipschitz conditions and formulated for flux functions that have points of inflection what today is called the Oleĭnik entropy condition.

Her principal tool was the parabolic perturbation of a conservation law. She proved that as the coefficient of viscosity tends to zero, the solution of the initial-value problem for the parabolic equation tends to a solution, in the sense of distributions, of the conservation law, and that this limit satisfies the Oleĭnik entropy condition.

In 1957 Oleĭnik and Vvedenskaya considered a discretized form of a conservation law and proved the convergence of their solutions to solutions of the conservation law, as the discretization parameter tends to zero [OV]. This provided another approach to the problem.

Oleĭnik also studied special systems of pairs of conservation laws, which are derivable from second-order equations; in 1966 she proved uniqueness of the initial-value problem in the class of solutions that satisfy a one-sided Lipschitz condition [O4].

Boundary Layer Theory[4]

In their simplest form, Prandtl's steady 2D boundary-layer equations are obtained from the Navier-Stokes equations by stretching the variables appropriately. If x is distance along the boundary,

[3] *PL.*

[4] *CSM.*

and y, a depth variable, is distance from the boundary, the equations become after rescaling:

$$u_x + v_y = 0 \qquad \text{(mass)},$$

$$uu_x + vu_y = -p_x + \mu u_{xx} \qquad (x\text{-momentum}),$$

$$p_y = 0 \qquad (y\text{-momentum}).$$

The standard boundary conditions are $u = v = 0$ on $y = 0$ and $u \to U(x)$ as $y \to \infty$. The function U is the speed of the flow "along the body" given by inviscid incompressible flow and is related to the pressure by Bernoulli's law,

$$p + \tfrac{1}{2}U^2 = \text{constant}.$$

Upstream, say $x = 0$, the incoming flow is prescribed. Experimental observations confirm the scaling.

Thus we have a nonlinear parabolic equation for u with the role of time derivative played by the Lagrangian along the particle path. The coefficient v is to be found from the mass equation.

The crucial question of how the separation or breaking away of the boundary layer from the boundary takes place is still not resolved. Oleĭnik's work concentrated on where and why separation does NOT take place by proving that there exists a unique boundary flow and hence one without separation, provided in our simple case that $p_x \neq 0$ [O3]. In real life on an airplane wing this is achieved by artificial suction, and Oleĭnik examined this question in detail (see [O4]). In 1997, after many papers with others from 1963 on concerning many different kinds of flow from magnetohydrodynamics to non-Newtonian fluids, she wrote a book on the subject summarizing and proving many background results and her own key theorems [O5].

Singular Elliptic and Parabolic Theory[5]

Equations of elliptic or parabolic type enjoyed a great deal of attention in the decades following World War II, and Oleĭnik's thesis was in this area. She followed it with many other results, but she may well be best remembered for her work on degenerate problems where an elliptic equation becomes parabolic at points or segments on the boundary or even in whole patches of a domain. One of the simplest examples is for the solutions u of the wave equation depending only on x/t. Thus with $r = |x|$, $\rho = r/t$, the equation for $w = ru$ is $(\rho^2 - 1)w_{\rho\rho} + \rho w_\rho - \Lambda w = 0$, where Λ is the angular part of the Laplacian, and the degeneracy, not surprisingly, occurs on the light cone $\rho = 1$.

Oleĭnik determined the conditions for well-posedness in many cases and in 1964 generalized and completed the problems posed by G. Fichera [F] (see [O6]). The technique was to add a term so that the equation remains elliptic and to obtain estimates for the limiting case when this term vanishes. By 1977, however, Oleĭnik was directly using a priori energy estimates [O7].

Homogenization of Differential Equations[6]

Oleĭnik's contributions to homogenization and her impact on this field were very broad. Mathematical modelling leads naturally to systems with multiple scales, for example in electromagnetism, mechanics, material science, and flows through porous media or biological tissues. The resulting underlying structures may be highly oscillatory in space and time. To analyze the transition between the different scales and to derive effective model equations are great challenges to mathematical research and imply many practical consequences. The main problem lies in identifying the proper scaling and, if possible, deriving effective equations for limits when a "scale" goes to 0 or ∞. The random situation especially, the most important in real life, poses a lot of difficult questions.

The challenge is to derive quantitative results, including estimates for the approximations. Homogenization, a special kind of averaging, started as a mathematical discipline only about thirty years ago, although in physics and engineering such averaging methods had been used for many years to determine effective properties and effective laws for heterogeneous media.

The theory of homogenization is strongly linked to Oleĭnik's name. An impressive group of mathematicians from the Department of Differential Equations at Moscow State University formed a leading team, and with them Oleĭnik developed a fruitful cooperation with France, Germany, and Italy, especially after the fall of the iron curtain. The long list of contributions of Oleĭnik and her coworkers, covering all of the main areas of homogenization, can be found in the monographs by Oleĭnik, Shamaev, and Yosifian [OSY] and by Jikov, Kozlov, and Oleĭnik [JKO]. These books are important sources of information and original ideas, and cover important topics in theory and applications.

Oleĭnik's contributions were mainly in developing the necessary tools to control the scale limit for initial and boundary value problems for systems with oscillatory coefficients or in domains with complex structures, holes, or oscillatory boundaries. Essentially two methods are available: energy methods, based on proper estimates of the solutions and compactness results; and multiscale expansions.

Introducing fast and slow (microscopic and macroscopic) variables, one starts from a formal expansion with respect to the scale parameter, obtaining a recursive system for the coefficients

[5] *CSM.*

[6] *WJ.*

of the expansion. Oleĭnik and her coworkers developed a systematic technique for determining approximations and validating the expansion. In the case of periodic structures, the approximation problem in the interior of the domain is reduced to solving a system with respect to fast variables in the scaled periodicity cell coupled with slow equations in the whole domain. Effective equations can often be obtained by such averaging. The cited monographs contain many examples.

The analysis at the boundary or at interfaces is more complicated, and here the contributions of Oleĭnik have been especially important. She and her coworkers did pioneering work on the approximation near the boundary. The analysis of boundary layers is a nontrivial problem in the case of periodic structures. Oleĭnik and her coworkers obtained optimal results when the boundary is flat and in rational position with respect to the periodicity lattice. In this situation an unbounded boundary-layer cell has to be considered. Ideas developed in the papers of Oleĭnik and Yosifian can be used to study problems in domains where the scale changes across an interface. Typical examples are processes in partially porous or perforated domains where the derivation of effective transmission conditions and the estimate of the errors at the interface are the main aims. Flow and transport through filters are practical, important examples. During her last years, despite her serious health problems, Olga Oleĭnik was also involved with such multiscale problems, strongly motivated by the many possible applications. Oleĭnik also developed a spectral theory adapted to homogenization.

Stochastic homogenization, mainly in the case of random coefficients, is the main topic in [JKO]. Several important contributions were obtained in Oleĭnik's group. Homogenization of stochastic processes, also in random geometry, is a crucial topic of ongoing research, and here Oleĭnik's influence is clear.

Problems in Elasticity[7]

Traditionally the links between mathematics and mechanics are very strong in Russia, and Oleĭnik's mathematical research reflected this. Prominent examples are Korn-type inequalities, basic in proving existence and estimating the solutions of the main boundary-value problems in elasticity. Consider the tensor $e(u)$ defined by

$$e_{ij}(u) = \partial_j u_i + \partial_i u_j.$$

Korn inequalities relate the L^2-norm of $e(u)$ with the L^2-norm of ∇u, for instance in the form of the inequality

$$\|u\|_{H^1(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + \|e(u)\|_{L^2(\Omega)}).$$

[7] *WJ.*

In general, assumptions on the domain are necessary: for example, the assumption that Ω is bounded and a Lipschitz domain. Also, an optimal constant C is wanted, with the constant depending on Ω. Kondratev and Oleĭnik obtained the estimate by a rather simple proof, with asymptotically sharp constants in the case of star-shaped domains. As Oleĭnik liked to say, the original proof was nineteen pages, Friedrichs's proof was nine pages, and they had reduced it to four pages [OSY, Chapter I, §2]. They also proved further Korn-type inequalities that are even more useful.

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