Michio Suzuki (1926–1998)

Michael Aschbacher, Helmut Bender, Walter Feit, and Ronald Solomon

Editor's Note. Michio Suzuki, an early leader in the effort to classify finite simple groups, died May 31, 1998, in Tokyo at the age of seventy-one. Born October 2, 1926, in Japan, he obtained his Ph.D. from the University of Tokyo in 1952, with Shoukichi Iyanaga as official advisor. Suzuki's teachers included also Yasuo Akizuki and Kenkichi Iwasawa. Suzuki assumed a faculty position at the University of Illinois, Urbana-Champaign, beginning the next year. In 1956–57 he took a leave of absence to work at Harvard University as research associate with Richard Brauer, with support from the National Science Foundation. He was a professor in the Center for Advanced Study at the University of Illinois from 1968 until his death.

Suzuki held a postdoctoral fellowship in 1952–53 and a Guggenheim Fellowship in 1962–63, received the Academy Prize from the Japan Academy in 1974 for his work in group theory, and was awarded an honorary doctoral degree from the University of Kiel, Germany, in 1991. He had visiting appointments at the University of Chicago (1960–61); the Institute for Advanced Study in Princeton (1962–63, 1968–69, and spring 1981); the University of Tokyo (spring 1971); the Universities of Hokkaido, Osaka, and Tokyo (1981 and 1985); and the University of Padua, Italy (1994).

Walter Feit

Michio Suzuki was one of the group of brilliant young Japanese mathematicians who entered college after World War II. He received his Ph.D. in 1952 from the University of Tokyo in absentia. Prior to that he came to the University of Illinois in 1952 as a research fellow. He joined the faculty of the University in 1953, a position he held until his sudden death.

His thesis was in the theory of finite groups, and this subject was to occupy him for his whole career. His early work included a study of the lattice of all subgroups $L(G)$ of a group G. He proved that if G is a noncyclic simple finite group and H is a finite group with $L(G \times G) = L(H \times H)$, then G is isomorphic to H. At the time it was not known whether $L(G)$ determines G up to isomorphism. However, by using the classification of the finite simple groups, it is possible to prove the more natural result that if G and H are noncyclic finite simple groups with $L(G) = L(H)$, then G is isomorphic to H. (Consideration of a cyclic group G of prime order indicates why $L(G \times G)$ should always be much richer than $L(G)$.)

During the summer of 1952 he came to Ann Arbor, attracted by the presence of Richard Brauer, who was on the faculty there. Brauer was one of the very few senior mathematicians in the USA who worked on questions concerning the structure of finite simple groups. He and Brauer ran a seminar that summer, which John Walter and I and others attended. I met him in that seminar while I was a graduate student at the University of Michigan.

The theory of finite groups became a subject of intensive research during the next few years. One reason was John Thompson's thesis, which introduced new methods and ideas to the subject; another was the progress in character theory sparked by Brauer and Suzuki.

It is necessary here to make some definitions. By way of background, an important theorem due to Frobenius says that if H is a finite transitive permutation group such that the subgroup fixing a letter is nontrivial and no nonidentity element fixes two or more letters, then H contains a proper nontrivial normal subgroup M such that every nonidentity element x in M has centralizer $C_H(x)$ contained in M. All known proofs of this theorem use character theory. We use this theorem to make a definition: A finite group H is a *Frobenius group*

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Michio Suzuki, December 1994, Manila, Phillipines.

with *Frobenius kernel M* if M is a proper nontrivial normal subgroup of H and if every nonidentity element x in M has centralizer $C_H(x)$ contained in M.

A subset S of a group G is a *trivial intersection set* in G (briefly, a *T.I. set in G*) if $x^{-1}Sx \cap S \subseteq \{1\}$ for every x in G with $x^{-1}Sx \neq S$.

Suppose that G contains a subgroup H that is a Frobenius group with an abelian Frobenius kernel M. Assume that H is the normalizer of M and M is a T.I. set in G. Let $|M| = m$, and $|H| = me$. Then $e \mid (m-1)$ and H has exactly $(m-1)/e$ irreducible characters that do not contain M in their kernel. Furthermore, they all have degree e.

Brauer used his characterization of characters to observe the following.

Let χ_1 be an irreducible character of G, and let ζ_1 and ζ_2 be irreducible characters of H which do not have M in their kernel such that ζ_1 and χ_1 agree on $M - \{1\}$. Let χ_2 be derived from χ_1 by replacing ζ_1 by ζ_2 on the conjugates of $M - \{1\}$. Then χ_2 is an irreducible character of G.

He told me that he wrote to Suzuki about this and immediately received by return mail Suzuki's alternative proof. (Presumably Suzuki had independently been thinking along these lines.)

In the above notation suppose that $e \neq m-1$. Then there are at least two faithful irreducible characters ζ_1, ζ_2 of H that do not have M in their kernel. It is easily seen that

$$\|\zeta_1^G - \zeta_2^G\|_G^2 = \|\zeta_1 - \zeta_2\|_H^2 = 2.$$

This argument can be used to prove the following result due to Brauer and Suzuki.

Theorem. Let H, M, m, e be as above. Assume that

(i) M is a T.I. set.

(ii) $m - 1 \neq e$.

(iii) M is abelian.

Let $\{\zeta_i\}$ be the set of irreducible characters of H that do not have M in their kernel. Then there exist a sign $\varepsilon = \pm 1$ and a set $\{\chi_i\}$ of irreducible characters of G such that

$$\zeta_i^G - \zeta_j^G = \varepsilon(\chi_i - \chi_j)$$

for all i, j.

The characters χ_i constructed in the theorem are called *exceptional characters*. In effect, the theorem defines a one-to-one correspondence between the set of irreducible characters of H that do not have M in their kernel and a set of irreducible characters of G, the exceptional characters of G. The construction of the exceptional characters makes it possible to derive information concerning their values on certain elements of G. In this way it is possible to construct a fragment of the character table of G. In some circumstances this fragment of the character table is enough to yield significant information about the group G.

This formulation has one fundamental advantage over Brauer's original observation, in that it is not necessary to begin with an irreducible character of G. Rather, it implies the existence of irreducible characters of G with certain properties. That is precisely the essence of this approach.

In 1957 Suzuki [Su3] showed that a group of odd order in which the centralizer of every nonidentity element is abelian is solvable. If it is assumed that G is a counterexample of minimum order, then it is not difficult to show that G is simple and every maximal subgroup of G is a Frobenius group H and the hypotheses of the theorem are satisfied for H. Hypothesis (ii) follows from the fact that $|G|$ is odd. Thus the theorem shows the existence of exceptional characters corresponding to H. Suzuki showed that every irreducible nonprincipal character of G is exceptional for some maximal subgroup of G. He was then able to reach a contradiction to the assumed existence of G by using various properties of exceptional characters.

This paper is a gem! At the time its importance was not fully grasped, either by him or by others, as it seemed to be simply an elegant exercise in character theory. However, the result and the methods used had a profound impact on much succeeding work, and in particular it is an essential ingredient of the work by John Thompson and me on groups of odd order. In fact it was the first significant result on groups of odd order since Burnside's work. Much of his later work had a more immediate impact and so overshadowed this paper, which is perhaps even today still not fully appreciated.

By Thompson's thesis M is always nilpotent if H is a Frobenius group as above. Soon thereafter Marshall Hall, John Thompson, and I were able to show that if G is a group of odd order in which the centralizer of every nonidentity element is nilpotent, then G is solvable. The proof consisted of two parts: The first part was an application of Thompson's methods to reduce the problem to the case that every proper maximal subgroup of G is a Frobenius group whose kernel is a T.I. set in G. Unlike the abelian case, this was not easy to prove. The second part is a modification of Suzuki's argument. This case is an essential ingredient in the proof that groups of odd order are solvable.

Before this I was attempting to generalize the construction of exceptional characters by dropping assumption (iii) of the theorem above. I had succeeded in doing this in case M is nilpotent unless M is a p-group and H satisfies certain stringent conditions.

This work of mine was partially motivated by the following question. Let G be a doubly transitive permutation group on a set Ω. Suppose that no element of $G - \{1\}$ fixes three or more letters. What can be said about G? It is clear that either G is a doubly transitive Frobenius group or that the subgroup G_a leaving a fixed is a Frobenius group for a in Ω. It suffices to consider the case that G is simple. Thus $H = G_a$ is a Frobenius group. Let M be the Frobenius kernel of H. Let $|M| = m$ and let $|H| = me$. If $e \geq \frac{1}{2}(m-1)$, then a theorem of Zassenhaus implies that $G \simeq PSL_2(q)$ and $|M| = q$ for some prime power q. If $e < \frac{1}{2}(m-1)$, then hypotheses (i) and (ii) of the theorem above are satisfied. I showed that if one could generalize the concept of exceptional characters to this situation, then $e \geq \frac{1}{2}(m-1)$ and so Zassenhaus's theorem applied. Thus by using Thompson's result that M is nilpotent, it followed that m is a prime power. Groups satisfying these conditions were later called *Zassenhaus groups,* as Zassenhaus, and later Suzuki, in unpublished work had independently classified these groups in case M is abelian.

The case that M is a 2-group was studied by Suzuki. In a major piece of work he succeeded in constructing a new class of simple groups that are Zassenhaus groups. These are now known as Suzuki groups.

They were the first (and are now known to be the only) finite simple groups whose order is not divisible by 3. Their discovery, and especially the latter fact about their order, was a great surprise to everyone.

For further progress it was necessary to weaken the hypotheses of the above theorem considerably. An early example of this occurred in part of the proof by Brauer and Suzuki of the following result [BS].

Theorem. If the finite group G has a Sylow 2-group that is a generalized quaternion group, then G/K has a center of order 2, where K is the maximal normal subgroup of G of odd order. In particular, G is not simple.

In case the Sylow 2-group has order 8, modular representation theory was used. Exceptional characters were adequate to settle the remaining cases.

This result is unlike any that had been proved previously, as the usual way of showing that a group is not simple was either to show that it was not equal to its commutator group or was a Frobenius group or to reach a contradiction from the assumptions.

During this period I learned a lot from Suzuki, both personally and also through his papers. I consider him to be one of my teachers.

In 1960–61 A. A. Albert arranged the group theory year at the University of Chicago. During that year at Chicago, Suzuki proved a deep result that gives a characterization of his groups. This is of great importance in the classification of the finite simple groups, since, aside from the actual result, the method is a model for the characterization of the Ree groups achieved much later by Thompson and Bombieri.

Over the years he made many other significant contributions to the classification of the finite simple groups. Some of these will be discussed in the remaining segments of this article, but I must at least mention that he discovered a new sporadic group that bears his name.

Many of us who attended the group theory year at Chicago lived in the same apartment house and so got to know each other better. In later years Michio and I frequently met at group theory conferences and occasionally visited each other's University.

When I heard about his illness, I was shocked. It was so totally unexpected. Almost a year ago there was a meeting in Tokyo to celebrate his seventieth birthday. It was a happy event, and I am very glad that I was able to attend. It was the last time I saw him. No one then could foresee that this sad occasion would follow so soon.

He will be missed by his family, his friends, and the mathematical community. However, his name will forever be remembered in mathematics for his pioneering work, especially for the groups named after him.

Michael Aschbacher

I first met Michio Suzuki in September of 1969, when I began my tenure as a postdoc at the University of Illinois under Suzuki's sponsorship. During that year I became interested in finite groups "disconnected" at some prime. The study of such groups constitutes one of the most important chapters in the theory of finite groups. Moreover, it was Suzuki who initiated work on this subject, and, as we will see, Suzuki, his students, and his postdocs played the leading role in this work.

Let G be a finite group and p a prime. The *commuting graph* Γ_p for G is the graph whose vertices are the subgroups of G of order p and whose edges are pairs of commuting subgroups. Define

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G to be *connected at p* if Γ_p is connected. It is an elementary exercise to see that there are the following equivalent formulations of this condition:

Lemma. Let G be a finite group, p a prime divisor of the order of G, Δ a connected component of Γ_p, and $H = N_G(\Delta)$. Then the following are equivalent:

(i) Γ_p is disconnected.

(ii) H is proper of order divisible by p, and $|H \cap gHg^{-1}|$ is prime to p for all $g \in G - H$.

(iii) Each nontrivial p-element of H fixes a unique point of the coset space G/H.

We say that H is a *strongly p-embedded subgroup* of G if the condition in (ii) holds.

The theorem determining the groups disconnected at the prime 2 was one of the major steps in the classification of the finite simple groups. This theorem was proved by Helmut Bender, who reduced the problem to an earlier theorem of Suzuki. Groups disconnected at odd primes are known only as a corollary to the classification of the finite simple groups, although treating a very special case of disconnected groups at odd primes was one of the hard steps in the classification. The concept of disconnectedness is of interest in contexts other than the classification. It arises, for example, in modular representation theory and in the study of subgroup complexes of finite groups.

In the early 1960s Suzuki proved:

Theorem (Suzuki, [Su4]). Let G be a transitive group of permutations on a set X of odd order for which the stabilizer H of $x \in X$ contains a normal subgroup Q regular on $X - \{x\}$ with H/Q of odd order. Then either

(i) G is solvable and known, or

(ii) G is an extension of a rank 1 group L of Lie type and even characteristic and the permutation action is on the Borel subgroups of L.

The "rank 1 groups" of Lie type and "even characteristic" are $L_2(q)$, $Sz(q)$, and $U_3(q)$, q even. The groups $Sz(q)$ are the "Suzuki groups" and were discovered and constructed by Suzuki in the process of proving this theorem. Only later was it discovered that the Suzuki groups are of Lie type.

In the late 1960s Bender extended Suzuki's result to a classification of groups disconnected at the prime 2 by showing:

Theorem (Bender, [B]). Let G be a group with a strongly embedded subgroup H. Then either

(i) G has cyclic or quaternion Sylow 2-subgroups, or

(ii) the representation of G on the cosets of H satisfies the hypotheses of Suzuki's theorem.

In particular, G is a simple group with a strongly 2-embedded subgroup if and only if G is a rank 1 group of Lie type and even characteristic. It is worth noting that Brauer and Suzuki [BS] showed that a group with quaternion Sylow 2-groups is not simple.

In 1969, shortly after Bender proved his theorem, I received my Ph.D. with a thesis in combinatorics. But already in my last year as a graduate student I had become more interested in permutation groups than in combinatorics. That year I also took a course from Steven Bauman (a former student of Suzuki's) based on galleys of Gorenstein's book on finite groups that was published later in the year. This got me interested in finite simple groups. Thus it was logical that one of the places to which I would apply for my first job was the University of Illinois, since both Michio Suzuki and John Walter were then on the Illinois faculty. I spent the year 1969–70 as a postdoc at Illinois. At that time Suzuki was in the process of writing his excellent two-volume text on finite groups, and I can recall sitting in on a class where he lectured from a draft of that book.

It was in the Suzuki-Walter seminar that I first was introduced to disconnected groups. It was also at this seminar that I heard Ernie Shult (another of Suzuki's former students) discuss the new fusion theorem he had just proved. Given subgroups $V \subseteq H \subseteq G$, define V to be *strongly closed in H with respect to G* if $v^G \cap H \subseteq V$ for all $v \in V$.

Theorem (Shult's Fusion Theorem, [Sh]). Let G be a finite group, and let V be an abelian subgroup of G such that $V = \langle t^G \cap C_G(t)\rangle$ for some involution t and V is strongly closed in $H = N_G(V)$ with respect to G. Then $G = L_0 \cdots L_m$, where $[L_i, L_j] = 1$ for $i \neq j$, $L_0/O(L_0)$ is an elementary abelian 2-group, and for $i > 0$, $L_i/O(L_i)$ is $L_2(2^n)$, $Sz(2^n)$, $U_3(2^n)$, or a covering of $Sz(8)$.

At first glance this may not look like a connectedness result, but the hypotheses are equivalent to:

> G is a finite group, H is a subgroup of G, t is an involution in H fixing a unique point of the coset space G/H, and $V = \langle t^H\rangle$ is abelian and strongly closed in H with respect to G.

Thus the commuting graph on t^G (rather than the set of *all* involutions of G) is disconnected, and a connected component of the graph is complete. If t^H is of odd order, one can omit the condition that V is strongly closed in H with respect to G.

Shult's Fusion Theorem was an important tool in the classification, but even more, it inspired at least three other important tools. First, in [G] David Goldschmidt classified all groups with a strongly closed abelian 2-subgroup. Thus he weakened Shult's hypotheses, but he also appealed to Shult's result in his proof. Goldschmidt's theorem is not strictly speaking a connectedness result.

Shult never published his Fusion Theorem, but one section of his proof is reproduced with slight variations in the next paper. To simplify the statement of the theorem, I assume G is simple, but this is not really necessary if one slightly extends the class of examples.

Theorem (Aschbacher [A1], [A2]). Let G be a finite simple group, H a proper subgroup of G, and $z \in H$ an involution in the center of a Sylow 2-subgroup of G. Assume

(i) z fixes a unique point of the coset space G/H, and

(ii) if $z \neq t \in z^G \cap C_G(z)$, then $C_G(tz)$ is contained in H.

Then H is strongly embedded in G, so that $G \cong L_2(2^n)$, $Sz(2^n)$, or $U_3(2^n)$.

Reading Shult's paper and Bender's paper on groups with a strongly embedded subgroup helped me to prove the result above, and that theorem is used in turn to prove the final connectedness result I will mention. Define Γ_2^2 to be the commuting graph on elementary abelian 2-subgroups of G of rank at least 2 and $\Gamma_2^{2,\circ}$ the subgraph of nonisolated vertices in Γ_2^2.

Theorem (Aschbacher [A1]). Let G be a finite simple group of 2-rank at least 3 such that $\Gamma_2^{2,\circ}$ is disconnected. Then G is $L_2(2^n)$, $Sz(2^n)$, $U_3(2^n)$, or J_1.

This last result is used in conjunction with signalizer functor theory to control the groups $O(C_G(t))$, t an involution in G. The problem of determining all such groups (phrased somewhat differently) was posed by Gorenstein and Walter. I first learned of this problem in the Suzuki-Walter seminar and treated a very special case of the problem during my year at Illinois.

Since space in this tribute is limited, I have chosen to concentrate on Suzuki's work on disconnected groups, pointing out also how he inspired his students and postdocs to take that work to its logical conclusion. Unfortunately this only hints at the many important contributions Suzuki made to the theory of finite groups. During the 1950s and 1960s Suzuki was one of the masters of the theory of group characters and one of the initiators of the local theory of finite groups. He also discovered his infinite family of simple groups, which were only later recognized to be groups of Lie type, and he discovered the sporadic Suzuki group.

I last saw Michio Suzuki in July 1997 at the conference in honor of his seventieth birthday in Tokyo. While his hair was a little grey, he was as active as ever, and it was hard to believe that he was almost thirty years older than when we first met. Unfortunately, later that year it was discovered he was in the last stages of terminal cancer, and he died a few months after that diagnosis. His death is a great loss to mathematics and to those of us who knew him.

Helmut Bender

The name of Michio Suzuki was forever engraved in my mind when in 1964 Bernd Fischer, who had just become an assistant of Reinhold Baer at Frankfurt, handed me a paper by Suzuki to be studied and presented in Baer's seminar. That paper [Su5] lies at the intersection of two main streams of Suzuki's work:

(1) Characterize the known simple groups by the centralizer of an involution.

(2) Determine doubly transitive permutation groups with a regular fixed point behavior, especially *Zassenhaus groups* (a nonidentity element does not fix three points) and *Suzuki-transitive groups* (the stabilizer of a point has a normal subgroup regular on the remaining points).

Michio Suzuki and daughter Kazuko at the Institute for Advanced Study, Princeton, fall of 1968.

The stream (1) was initiated by Richard Brauer in his 1954 ICM lecture and on the basis of philosophical as well as practical considerations: A given group can occur as the centralizer of an involution in only finitely many finite simple groups, up to isomorphism, and involutions allow certain arguments, elementary as well as character-theoretic ones, that do not work for elements of higher order.

Brauer's dream was that in a hypothetical future classification of the finite simple groups one would reach a point where such characterizations could be used. This dream indeed became true by Aschbacher's later work, but the relevance of Suzuki's main contributions in that area was obvious much earlier. His most general result is the determination of all finite groups in which the centralizer of any involution has a normal Sylow 2-subgroup [Su6]. The only such simple groups are $PSL_2(2^n)$, $Sz(2^n)$, and $PSL_3(2^n)$.

A further discussion of that story will carry us quickly to the main topic, 2-connectedness, of Aschbacher's segment of the present article. So, as a bridge in some other direction, let me recall a re-

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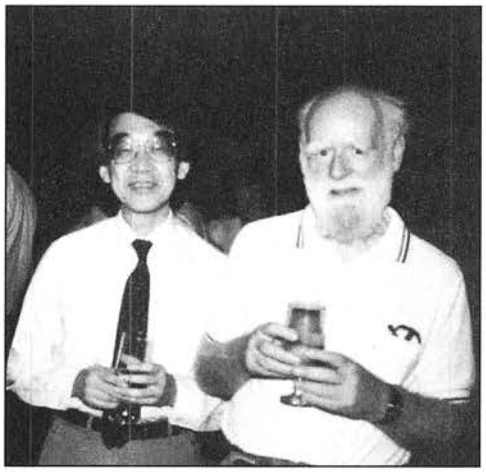
Ichiro Satake (left) and Suzuki, Japan, 1981.



Suzuki (left) with Graham Higman, July 1987.

mark Fischer once made to me in the old days, namely, "Suzuki now works with BN-pairs rather than characters." This referred to [Su6] and reflects, as I understood only much later, a twofold deep inner harmony in the world of finite groups.

For a brief description of this, let me relate Suzuki's procedure in [Su6] for the classification of all finite simple groups. In either case, after some reduction of a minimal counterexample G,

(a) the normalizer of any nonidentity 2-subgroup (in short, any 2-*local subgroup*) has a normal 2-subgroup containing its centralizer, and then

(b) one either has non-2-connectedness in Aschbacher's sense or a geometry (building or BN-pair) that allows one to identify G.

In [Su6] that geometry is a projective plane with the cosets Lx ($x \in G$) of some 2-local L as lines, the cosets Py of some other 2-local as points, incidence defined by $Lx \cap Py \neq \emptyset$, and G acting on the plane by multiplication from the right. In either case the reduction to (a) is a big story in itself; in the general situation it involves the study of (a) also for odd primes p in place of 2, and again (b) is the goal to be reached. The way to a geometry is by the study of the p-locals and their interaction. The most exciting recent developments in finite group theory are in this field.

The main obstacle to any kind of local analysis as just indicated is non-p-connectedness; that is, some proper subgroup H of G contains a Sylow p-subgroup of G and the normalizers of all (or nearly all) nonidentity p-subgroups of H. So from the p-local point of view, H is a kind of black hole and hard to distinguish from G. As Solomon has pointed out, character theory may come to the rescue when things get very tight. Indeed, local analysis and character theory are somehow complementary methods, and it was the work of Brauer, Feit, and Suzuki on so-called exceptional characters that made this clear.

The rule of the prime 2 in the realm of finite groups is via the Sylow 2-subgroups. I like to compare them with fortresses. In order to exert an effective control, they need good lines of communication, that is, good connectedness, and most of all strength. For every type of small Sylow 2-subgroup, Suzuki has obtained substantial results. Most impressive are his CA Theorem [Su3], discussed in the segment below by Solomon,[1] and the Brauer-Suzuki-Burnside Theorem [Su5] on groups G of even order with no elementary 2-subgroup of order 4.[2]

After a long period of relatively little global progress, much light has been shed on finite groups in the second half of this century, not least by Suzuki and by others using his work. In particular, the finite simple groups have been determined. The earlier state of affairs was described very nicely by Brauer at the 1970 ICM: "Up to the early 1960s, really nothing of real interest was known about general simple groups of finite order." On the solvability of groups of odd order, the main cornerstone of the classification, Brauer said, "Nobody ever did anything about it, simply because nobody had any idea how to get even started."

Unfortunately, public statements about the classification are sometimes more governed by the bureaucratic mind than by mathematical interest and insight. The most exciting observation, however, is open to every mathematician, not only to experts: On the one hand, apart from sets, mappings, and integers, there is no other mathematical concept of such a general nature as the concept of a group: any mathematical structure has an automorphism group, which a priori has a good chance of being more or less simple. On the other hand, not only is there some order among the finite simple groups, but essentially all of them can be derived in a certain way from certain very special types of objects, namely, finite-dimensional Lie algebras over the complex numbers, which in their own right are of central importance for mathematics as a whole.

In today's light we see that Suzuki's work centers around the "rank 1 groups" of Lie type. Their

[1] *A* CA group *is a group in which the centralizer of every nonidentity element is abelian. The CA Theorem says that CA groups of odd order are solvable.*

[2] *The theorem says that G is the product of the centralizer of an involution and a normal subgroup of odd order.*

Ph.D. Students of Michio Suzuki

Steven F. Bauman (1962)
Ernest E. Shult (1964)
Anne P. Street (1966)
Jon M. Laible (1967)
John S. Montague (1967)
Robert E. Lewis (1968)
C. Gomer Thomas (1968)
Mark P. Hale (1969)
Wen-Jin Woan (1970)
Zon-I Chang (1974)
George M. Whitson (1974)
Arthur A. Yanushka (1974)
David Chillag (1975)
Seung-Ahn Park (1975)
David Redmond (1977)
Mark R. Hopkins (1978)
Robert F. Mortenson (1978)
Michael D. Fry (1980)
Philip Abram Cobb (1987)
Randall Reed Holmes (1987)
Harald Erich Ellers (1989)
Tsung-Luen Sheu (1989)
Jose Maria Balmaceda (1991)
Tuval Foguel (1992)
Abdellatif Laradji (1993)

unifying group-theoretical property is "Suzuki-transitivity", essentially that the underlying geometry degenerates. This explains why Suzuki's main papers on permutation groups contain such long and difficult (though ingenious) calculations to identify the group under consideration. Is this effort really worthwhile? Why do group theorists always want to establish isomorphisms rather than being satisfied with the relevant group-theoretic information? This question was once raised by a mathematician in an interview. The answer is that information on some aspect of a group may suffice for an isomorphism with a known group but is usually not strong enough to clarify any other important aspect, e.g., representations. This can be illustrated by the (slightly generalized) Brauer-Suzuki-Wall Theorem [BSW]: A certain abelian subgroup of the given group G will be known to be cyclic only after an isomorphism of G with some $PSL_2(q)$ has been established.

To follow those calculations in my seminar paper was the hardest job I ever did in group theory. Fortunately, there was enough other motivation to go on and study further papers of Suzuki and related work of Feit. This became my main activity for about two years as a student. Shortly thereafter I spent a year with Suzuki at Urbana, the most productive and pleasant time in my academic life.

Once Suzuki mentioned that he came to Urbana at exactly my age. So it must have been 1952. He came on the initiative of Baer, then at Urbana, who was impressed by his work [Su1, Su2] on subgroup lattices, recently described to me by experts as revolutionary.

In 1991 Suzuki received a honorary degree from the University of Kiel for his pioneering work, more precisely "für seine Verdienste auf dem Gebiet der Gruppentheorie, vor allem in Würdigung seiner wegweisenden Arbeiten zur Klassifikation der endlichen einfachen Gruppen wie auch für sein grundlegendes Werk über Untergruppenverbände und seine Beiträge zur Theorie der Permutationsgruppen."[3] The next time I saw him was in July 1997 at a meeting in Tokyo connected with his seventieth birthday. Those who know him and met him there will have no doubt that his passion for finite groups and his warm interest in the life and work of his colleagues lasted until the end of his days.

Ronald Solomon

In preparing a talk for the July 1997 conference in Tokyo in honor of Michio Suzuki, I took a moment to check the bibliography of my battered copy of Danny Gorenstein's book, *Finite Groups*, (the *Book of Common Prayer* for students of my generation interested in finite simple groups). I found that the author (or coauthor) with the most citations (19) was Michio Suzuki, a crude numerical indication of the importance of Suzuki to the rapidly developing theory of finite simple groups.

A far more eloquent synopsis of the influence of Michio Suzuki on the project of classifying the finite simple groups can be found in the following remarks of John Thompson:[4]

> A third strategy (or was it a tactic?) in OOP [the Odd Order Paper] attempted to build a bridge from Sylow theory to character theory. The far shore was marked by the granite of Suzuki's theorem on CA-groups....
>
> Suzuki's CA-theorem is a marvel of cunning. In order to have a genuinely satisfying proof of the odd order theo-

[3] *"for his achievements in the domain of group theory, above all in recognition of his path-breaking works on the classification of simple finite groups as well as for his fundamental work on lattices of subgroups and his contributions to the theory of permutation groups."*

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[4] *For CA groups and Suzuki's theorem that are mentioned in these remarks, see footnote 1 within Bender's segment of this article.*

Photographs courtesy of Naoko Suzuki, photographer Masahiko Miyamoto.

At the conference in honor of his seventieth birthday, Tokyo, 1997. Top: Suzuki with wife Naoko. Bottom: left to right, Suzuki, Donald G. Higman, and Walter Feit.

> rem, it is necessary, it seems to me, not to assume this theorem. Once one accepts this theorem as a step in a general proof, one seems irresistibly drawn along the path which was followed. To my colleagues who have grumbled about the tortuous proofs in the classification of simple groups, I have a ready answer: find another proof of Suzuki's theorem.

Thus Suzuki was either Moses leading his people to the Promised Land of the classification or the Pied Piper leading a generation of thoughtless children down a tortuous path of no return.

Burnside was intrigued by the differences between groups of even and odd order and was fascinated by the thought that all groups of odd order might be solvable, but for all his brilliance he never established a major subcase of the problem. Brauer in the early 1950s focused attention on centralizers as a key to unlocking the mysteries of simple groups, but his ideas were directed primarily at centralizers of involutions and had little applicability to groups of odd order. Suzuki took up the idea of centralizer conditions (CA means that the centralizers of nonidentity elements are abelian) and devised and implemented a strategy applicable to groups of both odd and even order:

(1) Determine the structure of all maximal local subgroups (normalizers of nonidentity p-subgroups) via Sylow theory.

(2) Count elements in G via character theory.

Suzuki's paper was seductively short and elegant, only ten pages long. Little did anyone know that it was the seed that would germinate into the 255-page Odd Order Paper of Feit and Thompson.

After the Odd Order Paper, the role of character theory in the proof of the classification declined precipitously and disappeared completely in the final decade. One heuristic explanation for this is that the primary role of character theory in the classification is to obtain group order formulas. But Thompson's Order Formula shows that the group order can be obtained in an elementary character-free way whenever the group has at least two conjugacy classes of involutions (elements of order 2). Empirically (I do not know of a proof without quoting the classification), simple groups with only one class of involutions have Sylow 2-subgroups either of very small (sectional) 2-rank or of very small nilpotence class. As groups of these types were among the first handled in the classification project, it is not surprising that the need for character theory was exhausted early.

A somewhat deeper reason is the following. Character theory comes to the rescue when local group theory finds itself trapped in a cul-de-sac, a so-called strongly embedded subgroup or something very close to one. If the local structure of the group G in the neighborhood of the prime 2 is sufficiently robust to provide a family of local subgroups with large intersections which generate G, then it is possible to avoid character theory and identify G by either Lie theoretic or geometric methods. A single subgroup (and its cosets) provides a permutation action. A web of intersecting subgroups (and their cosets) provides a geometry. But the absence of rich intersections can also be exploited, as Suzuki taught us in his deepest papers, "On a class of doubly transitive groups I, II". Here Suzuki wrote the final chapter in the long saga of Zassenhaus groups and also the first chapter of the classification of groups with a strongly embedded subgroup, which was completed by Helmut Bender. This marvelous work complements the Odd Order Paper to give the final assurance that not only do all nonabelian finite simple groups have even order but, with the exception of the groups of Lie rank 1 over fields of even order, they are rich in 2-local subgroups. (Technically speaking, for any fixed Sylow 2-subgroup S, G is generated by the normalizers of nonidentity subgroups of S.)

This does not exhaust Suzuki's contributions to the classification. He was a mentor, official or unofficial, of Walter Feit, Helmut Bender, Koichiro Harada (whose collaboration with Gorenstein yielded a large portion of the classification proof), and Ernie Shult (whose work had a profound influence on Goldschmidt and Aschbacher). He also discovered one of the twenty-six sporadic simple groups (thereby robbing Conway of a fourth Conway group). Finally, he pioneered in the late 1960s the program of characterizing the finite groups of Lie type over fields of even order by the centralizers of a central involution. (Brauer and his school had focused on the analogous problem for groups of Lie type over fields of odd order, where involutions are semisimple elements.) Historically this program was stopped in its tracks after a few years by the express train of Gorenstein, Lyons, and Aschbacher (plus Gilman and Griess), who championed a different philosophy: When G is "of characteristic 2 type", switch attention to "semisimple elements" of odd prime order. (This may be viewed as a revival of the philosophy of Killing.) The semisimple strategy carried the day in the 1970s and 1980s, yielding a proof of the Classification Theorem.

But Suzuki's influence is mighty still. His principal disciple in Japan was Kensaku Gomi, who did some beautiful work in the 1970s and 1980s in the spirit of the "unipotent approach" of Suzuki. Some of Gomi's key ideas, combined with methods of David Goldschmidt, were crafted by Bernd Stellmacher into a unipotent strategy for the classification of finite groups of characteristic p type. This approach is being vigorously pursued by Ulrich Meierfrankenfeld, Gernot Stroth, and others. It may well yield an alternate proof of a major portion of the Classification Theorem. So even today we may not yet have felt the full glow of the light that Michio Suzuki's work shines upon the study of finite simple groups.

I would like to close with some personal remarks. I first saw Suzuki at the Summer Institute on Finite Groups in 1970. I saw him last at the Tokyo Conference in 1997. He never seemed to change. The twinkle in his eyes, the vitality in his step, and the enthusiasm for his subject were constants. Also constant were his graciousness and generosity of spirit. I have treasured a letter that I received from him dated February 27, 1996, from which I quote the first paragraph:

> Dear Ron,
>
> I like to congratulate the publication of the second volume of the classification series which I have just glanced through. It is very well organized and readable. I have an elated feeling that I may be able to understand the proof of the classification in my life time. Keep up the good work. I also want to thank you for your kind words on my book.

Professor Suzuki, I am sorry we were too slow. But I suppose you know a better proof by now anyway.

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