The Mathematics of Lars Valerian Ahlfors

Frederick Gehring; Irwin Kra; Steven G. Krantz, Editor; and Robert Osserman

Lars Valerian Ahlfors died October 11, 1996, and a memorial article appears elsewhere in this issue. This feature article contains three essays about different aspects of his mathematics. Ahlfors's contributions were so substantial and so diverse that it would not be possible to do all of them justice in an article of this scope (the reference [36] serves as a detailed map of Ahlfors's contributions to the subject). These essays give the flavor of some of the ideas that Ahlfors studied.

—Steven G. Krantz, Editor

Conformal Geometry

Robert Osserman

There are two directions in which one can pursue the relations between Riemann surfaces and Riemannian manifolds. First, if a two-dimensional Riemannian manifold is given, then not only lengths but also angles are well defined, so that it inherits a conformal structure. Furthermore, there always exist local isothermal coordinates, which are local conformal maps from the plane into the surface. (That was proved by Gauss for real analytic surfaces and early in the twentieth century for the general case by Korn and Lichtenstein.) The set of all such local maps forms a complex structure for the manifold, which can then be thought of as a Riemann surface. One then has all of complex function theory to bring to bear in studying the geometry of the surface. The most notable successes of this approach have been in the study of minimal surfaces, as exemplified in the contributions to that subject made by some of the leading function theorists of the nineteenth century: Riemann, Weierstrass, and Schwarz.

In the other direction, given a Riemann surface, one can consider those metrics on the surface that induce the given conformal structure. By the Koebe uniformization theorem, such metrics always exist. In fact, for "classical Riemann surfaces" of the sort originally considered by Riemann, which are branched covering surfaces of the plane, there is the natural euclidean metric obtained by pulling back the standard metric on the plane under the projection map. One can also consider the Riemann surface to lie over the Riemann sphere model of the extended complex plane and to lift the spherical metric to the surface. Both of those metrics prove very useful for obtaining information about the complex structure of the surface.

For simply connected Riemann surfaces the Koebe uniformization theorem tells us that they are all conformally equivalent to the sphere, the plane, or the unit disk. Since the first case is distinguished from the other two by the topological property of compactness, the interesting question concerning complex structure is deciding in the noncompact case whether a given surface is conformally the plane or the disk, which became known as the parabolic and hyperbolic cases, respectively. In 1932 Andreas Speiser formulated the "problem of type", which was to find criteria that could be applied to various classes of Riemann surfaces to decide whether a given one was parabolic or hyperbolic. That problem and variants of it became a central focus of Ahlfors's work for several decades. He started by obtaining conditions

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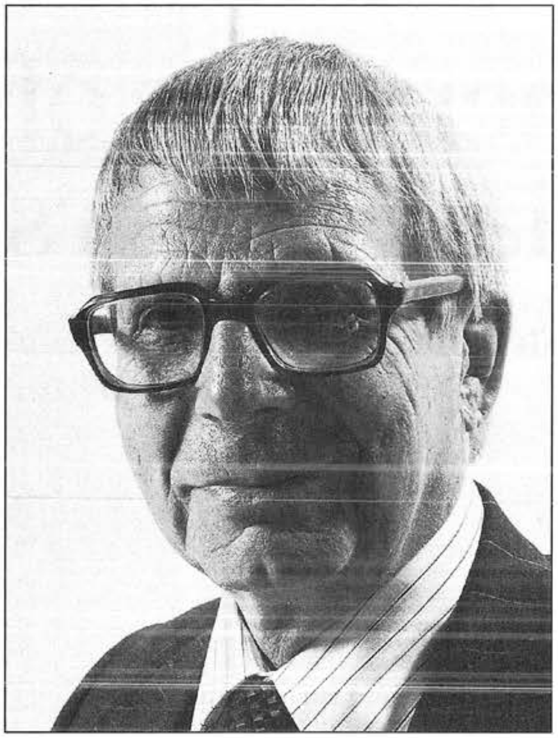

Lars Valerian Ahlfors

for a branched surface to be of parabolic type in terms of the number of branch points within a given distance of a fixed point on the surface, first using the euclidean metric and later realizing that a much better result could be obtained from the spherical metric. But perhaps his main insight was that one could give a necessary and sufficient condition by looking at the totality of all conformal metrics on the surface.

The problem of type may be viewed as a special case of the general problem of finding conformal invariants. There one has some class of topologically defined objects, such as a simply or double-connected domain or a simply connected domain with boundary and four distinguished points on the boundary, and one seeks to define quantities that determine when two topologically equivalent configurations are conformally equivalent. One example is the "extremal length" of a family of curves in a domain, which is defined by a minimax expression in terms of all conformal metrics on the domain and is thereby automatically a conformal invariant. Ahlfors and Beurling, first independently and then jointly, developed the idea into a very useful tool that has since found many further applications.

From the first, Ahlfors viewed classical results like Picard's theorem and Bloch's theorem as special cases of the problem of type, in which conditions such as the projection of a Riemann surface omitting a certain number of points would imply that the surface was hyperbolic and hence could not be the image of a function defined in the whole plane. He felt that Nevanlinna theory should also be fit into that framework. Finally, in 1935 he produced one of his most important papers, in which he used the idea of specially constructed conformal metrics (or "mass distributions" in the terminology of that paper) to give his own geometric version of Nevanlinna theory. When he received his Fields Medal the following year, Carathéodory remarked that it was hard to say which was more surprising: that Nevanlinna could develop his entire theory without the geometric picture to go with it or that Ahlfors could condense the whole theory into fourteen pages.

Not satisfied that he had yet got to the heart of Nevanlinna theory from a geometric point of view, Ahlfors went on to present two further versions of the theory. The first, also from 1935, was one of his masterpieces: the theory of covering surfaces. The guiding intuition of the paper is this: If a meromorphic function is given, then the fundamental quantities studied in Nevanlinna theory, such as the counting function, determined by the number of points inside a disk of given radius where the function assumes a given value, and the Nevanlinna characteristic function, measuring the growth of the function, can be reinterpreted as properties of the Riemann surface of the image of the function, viewed as a covering surface of the Riemann sphere. The counting function, for example, just tells how many points in the part of the image surface given by the image of the disk lie over the given point on the sphere. The exhaustion of the plane by disks of increasing radii is replaced by an exhaustion of the image surface. Ahlfors succeeds in showing that by using a combination of metric and topological arguments (the metric being that of the sphere and its lift to the covering surface), one can not only recover basically all of standard Nevanlinna theory but that—quite astonishingly—the essential parts of the theory all extend to a far wider class of functions than the very rigid special case of meromorphic functions, namely, to functions that Ahlfors calls "quasiconformal"; in this theory the smoothness requirements may be almost entirely dropped, and, asymptotically, images of small circles—rather than having to be circles—can be arbitrary ellipses as long as the ratio of the radii remains uniformly bounded.

Of his three geometric versions of Nevanlinna theory, Ahlfors has described the one on covering surfaces as a "much more radical departure from Nevanlinna's own methods" and as "the most original of the three papers," which is certainly the case. (According to Carathéodory that paper was singled out in the decision of the selection committee to award the Fields Medal to Ahlfors.) Nevertheless, the last of the three, published two years later in 1937, was destined to be probably at least as influential. Here the goal was to apply the methods of differential geometry to the study of covering surfaces. The paper is basically a symphony on the theme of Gauss-Bonnet. The explicit relation between topology and total curvature of a surface, now called the "Gauss-Bonnet theorem", had not been around all that long at the time, perhaps first appearing in Blaschke's 1921 *Vorlesungen über Differentialgeometrie* [15].

It occurred to Ahlfors that if one hoped to develop a higher-dimensional version of Nevanlinna theory, it might be useful to have a higher-dimensional Gauss-Bonnet formula, a fact that he mentioned to André Weil in 1939, as Weil recounts in his collected works. (A letter from Weil says, "I learnt from Ahlfors, in 1939, all the little I ever knew about Gauss-Bonnet (in dimension 2).") When Weil spent the year 1941–42 at Haverford, where Allendoerfer was teaching, he heard of Allendoerfer's proof of the higher-dimensional Gauss-Bonnet theorem; and remembering Ahlfors's suggestion, he worked with Allendoerfer on their joint paper, proving the generalized Gauss-Bonnet theorem for a general class of manifolds that need not be embedded. That in turn led to Chern's famous intrinsic proof of the general Gauss-Bonnet theorem. As for Ahlfors's idea of adapting the method to obtain a higher-dimensional Nevanlinna theory, that had to wait until the paper by Bott and Chern in 1965.

The year following his Gauss-Bonnet Nevanlinna theory paper, there appeared a deceptively short and unassuming paper called "An extension of Schwarz's lemma" [8, v. 1, p. 350]. The main theorem and its proof take up less than a page. That is followed by two brief statements of more general versions of the theorem and then four pages of applications. Initially, it was the applications that received the most attention and that Ahlfors was most pleased with, since they were anything but a straightforward consequence of the main theorem. That is particularly true of the second application, which gives a new proof of Bloch's theorem in a remarkably precise form. Bloch's theorem states that there is a uniform constant B such that every function analytic in the unit disk, and normalized so that its derivative at the origin has modulus 1, must map some subdomain of the unit disk one-to-one conformally onto a disk of radius B. Said differently, the image Riemann surface contains an unbranched disk of radius B. The largest such B is known as Bloch's constant. In 1937 Ahlfors and Grunsky [8, v. 1, p. 279] published a paper giving an upper bound for B that they conjectured to be the exact value. The conjectured extremal function maps the unit disk onto a Riemann surface with simple branch points in every sheet over the lattice formed by the vertices obtained by repeated reflection over the sides of an equilateral triangle, where the center of the unit disk maps onto the center of one of the triangles. One obtains the map by taking three circles orthogonal to the boundary of the unit disk that form an equilateral triangle centered at the origin with 30° angles and mapping the interior of that triangle onto the interior of a euclidean equilateral triangle. Under repeated reflections one gets a map of the entire unit disk onto the surface described. Ahlfors and Grunsky write down an explicit expression for the function of that description with the right normalization at the origin and thereby get the size of the largest circular disk in the image, which is just the circumscribed circle of one of the equilateral triangles whose vertices are the branch points of the image surface. The size of that circle turns out to be a bit under 1/2, or approximately .472, which is therefore an upper bound for Bloch's constant B.

Photograph of Lars Ahlfors by Bill Graustein, the nephew of Harvard mathematician William Caspar Graustein. Ahlfors was at the time the William Caspar Graustein Professor of Mathematics.

As one application of his generalized Schwarz lemma, Ahlfors proves Bloch's theorem with a lower bound for B of $\sqrt{3}/4 \approx .433$. The method is worth describing, since it so typifies Ahlfors's approach to many problems in function theory.

First we give the statement of his generalized Schwarz lemma. The original Schwarz lemma said in particular that for an analytic function mapping the unit disk into the disk, with $f(0) = 0$, one has $|f'(0)| < 1$, unless f is a rotation, in which case $|f'(0)| = 1$. Pick observed in 1916 that since every one-to-one conformal map of the unit disk onto itself is an isometry of the hyperbolic metric of the disk, one could drop the assumption that the origin maps into the origin and conclude that for any analytic map of the unit disk into itself, the hyperbolic length of the image of any curve is at most equal to the hyperbolic length of the original curve and, in fact, those lengths are strictly decreased unless the map is one-to-one onto or an isometry preserving all hyperbolic lengths. Ahlfors's great insight came from viewing the Schwarz-Pick lemma as a statement about two conformal metrics on the unit disk: the original hyperbolic metric and the pullback of the hyperbolic metric on the image. That led him to a truly far-reaching generalization, in which he replaces the second conformal metric by any one whose curvature is bounded above by the (constant) negative curvature of the original hyperbolic metric. The conclusion is that, again, the lengths of curves in the second metric are at most equal to the original hyperbolic lengths. That is the main result of his paper and is the content of the Ahlfors-Schwarz lemma. The proof requires the metrics to be smooth, but in his generalized versions he shows that the conclusion continues to hold in particular for various piecewise smooth metrics. In order to apply the result, Ahlfors constructs specific conformal metrics adapted to specific problems. As already mentioned, that is anything but straightforward and may take considerable ingenuity in each case. For Bloch's theorem he constructs a conformal metric based on the conjectured optimal surface described above and uses it to obtain the lower bound of $\sqrt{3}/4$ for Bloch's constant.[1]

When his collected papers [8] were published in 1982, Ahlfors commented that this particular paper "has more substance than I was aware of," but he also said: "Without applications my lemma would have been too lightweight for publication." It is a lucky thing for posterity that he found applications that he considered up to his standard, since it would have been a major loss for us not to have the published version of the Ahlfors-Schwarz lemma. As elegant and important as his applications were, I believe that they have long ago been dwarfed by the impact of the lemma itself, which has proved its value in countless other applications and has served as the underlying insight and model for vast parts of modern complex manifold theory, including Kobayashi's introduction of the metric that now bears his name and Griffiths's geometric approach to higher-dimensional Nevanlinna theory. It demonstrates perhaps more strikingly than anywhere else the power that Ahlfors was able to derive from his unique skill in melding the complex analysis of Riemann surfaces with the metric approach of Riemannian geometry.

Kleinian Groups

Irwin Kra

Of the many significant contributions of Lars Ahlfors to the modern theory of Kleinian groups, I will discuss only two closely related contributions: the finiteness theorem (AFT) and the use of Eichler cohomology as a tool for proving this and related results. Both originated in the seminal paper [2].

[1] *It may be noted that Ahlfors's lower bound for the Bloch constant stood for many years. The bound was shown to be strict by M. Heins [21] in 1962; C. Pommerenke [30] contributed some refinements in 1970; the lower estimate was improved by 10^{-14} by M. Bonk [16] in 1990; the related Landau constant was improved by 10^{-335} by H. Yanagihara [35] in 1995; and finally the lower bound for the Bloch constant was improved by 10^{-4} by H. Chen and P. Gauthier [17] in 1996.*

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For the purposes of this note a *Kleinian group* G will always be finitely generated, nonelementary, and of the second kind. Thus it consists of Möbius transformations (a subgroup of $\mathrm{PSL}(2,\mathbb{C})$) and acts discontinuously on a nonempty maximal open set $\Omega \subset \mathbb{C} \cup \{\infty\}$, the *region of discontinuity* of G, whose complement Λ in $\mathbb{C} \cup \{\infty\}$, the *limit set* of G, is an uncountable perfect nowhere dense subset of the Riemann sphere.

In the early sixties not much was known about Kleinian groups. Around the beginning of this century Poincaré suggested a program for studying discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$; Poincaré's program was based on the fact that $\mathrm{PSL}(2,\mathbb{R})$ acts on the upper half plane H^2, a model for hyperbolic 2-space. The quotient of H^2 by a discrete subgroup (a *Fuchsian* group) of $\mathrm{PSL}(2,\mathbb{R})$ is a 2-dimensional orbifold (a Riemann surface with some "marked" points). By analogy $\mathrm{PSL}(2,\mathbb{C})$ acts on H^3, hyperbolic 3-space, and the quotient of H^3 by a torsion free discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ is a 3-dimensional hyperbolic manifold. The study of subgroups of $\mathrm{PSL}(2,\mathbb{R})$ was successful because of its connection to classical function theory and to 2-dimensional topology and geometry, about which a lot was known, including the uniformization theorem classifying all simply connected Riemann surfaces. Poincaré's program was to take advantage of the connection of $\mathrm{PSL}(2,\mathbb{C})$ to 3-dimensional topology and geometry to study groups of Möbius transformations. However, in 1965 very little was known about hyperbolic 3-manifolds. Research in the field seemed to be stuck and going nowhere. Ahlfors completely ignored Poincaré's program and took a different route to prove the finiteness theorem. He used complex analytic methods, and his result described the Riemann surfaces that can be represented by a Kleinian group. About fifteen years later in the mid-seventies, as a result of the fundamental contributions of W. Thurston [34], 3-dimensional topology came to the forefront in the study of Kleinian groups. However, in the interest of keeping this presentation to reasonable length, I ignore this subject.

The history of Ahlfors's work on Kleinian groups is also part of the history of a remarkable collaboration between Lars Ahlfors and Lipman Bers.[2] Ahlfors's finiteness theorem says that the ordinary set Ω of a (finitely generated) Kleinian group G factored by the action of the group is an orbifold Ω/G of finite type (finitely many "marked" points *and* compactifiable as an orbifold by adding a finite number of points).

Bers [11], in approximately 1965, reproved an equivalent known result in the Fuchsian case, a much simpler case to handle. The finiteness the-

[2] *Although they co-authored only one paper [9], their work and the work of many of their students was intertwined. See [27].*

orem for PSL$(2,\mathbb{R})$ had been known for a long time, and Bers reproved it using modern methods, Eichler cohomology. Bers constructs Eichler cohomology classes from analytic potentials (by integrating cusp forms sufficiently many times, using methods developed by Eichler [18] for number theory). To be specific, let G be a Fuchsian group (finitely generated) operating on the upper half plane H^2. Fix an integer $q \geq 2$. Let φ be a holomorphic q-form for G on H^2. Choose a holomorphic function F on H^2 whose $(2q-1)^{\mathrm{st}}$ derivative is φ. Then, for every element $g \in G$,

$$\chi_g = (F \circ g)(g')^{1-q} - F \tag{1}$$

is the restriction to H^2 of a polynomial of degree at most $2q-2$; we denote the vector space of such polynomials by Π_{2q-2}. The polynomials constructed satisfy the *cocycle condition*

$$\chi_{g_1 \circ g_2} = (\chi_{g_1} \circ g_2)(g_2{}')^{1-q} + \chi_{g_2}, \quad \text{all } g_1 \text{ and } g_2 \in G. \tag{2}$$

One obtains in this way a holomorphic potential F for the automorphic form φ and a cohomology class $[\chi] = \beta(\varphi)$ in $H^1(G, \Pi_{2q-2})$. If G is finitely generated, then $H^1(G, \Pi_{2q-2})$ is a finite-dimensional vector space. Now if G is of the first kind and φ is a cusp form, then, as a consequence of the Riemann-Roch theorem, $\beta(\varphi) = 0$ if and only if $\varphi = 0$. The space of cusp forms for G on H^2 is finite dimensional if and only if H^2/G is a finitely punctured compact surface. From the injectivity of the linear map β, Bers concludes the finiteness result in this case. Bers's argument for groups of the second kind is more complicated but proceeds along similar lines.

Ahlfors generalized Bers's methods to a much wider class of subgroups of PSL$(2,\mathbb{C})$. This generalization was completely nontrivial. It required the passage from holomorphic potentials to smooth potentials. This involved a conceptual jump forward—a construction of Eichler cohomology classes via an integral operator, producing a conjugate linear map α that assigns an Eichler cohomology class $\alpha(\varphi)$ to a bounded holomorphic q-form φ for the group G. In addition, there appeared a very difficult technical obstacle that Ahlfors had to surmount to prove the injectivity of α. To surmount this obstacle, Ahlfors introduces a "mollifier", a function used to construct an approximate identity. Ahlfors works only with the case $q = 2$. Using a modified Cauchy kernel, he constructs a potential for $\lambda^{2-2q}\bar{\varphi}$, a continuous function F on $\mathbb{C}$ whose $\bar{z}$-derivative is $\lambda^{2-2q}\bar{\varphi}$. Here λ is a weight function whose exact form need not concern us. To be more specific, without loss of generality, we can assume that $\infty \in \Lambda$. Choose $2q-2$ distinct finite points $a_1, \dots, a_{2q-2} \in \Lambda$. Form the potential $F_{\lambda^{2-2q}\bar{\varphi}}$ for $\lambda^{2-2q}\bar{\varphi}$,

$$F(z) = F_{\lambda^{2-2q}\bar{\varphi}}(z) = \frac{(z-a_1)\dots(z-a_{2q-2})}{2\pi i} \times \int\int_{\Omega} \frac{\lambda^{2-2q}(\zeta)\overline{\varphi}(\zeta)\, d\zeta \wedge \overline{d\zeta}}{(\zeta - z)(\zeta - a_1)\dots(\zeta - a_{2q-2})}.$$

Then as before, (1) defines an element of Π_{2q-2}, and these polynomials satisfy (2). There is no analogue of the Riemann-Roch theorem. Let $\mathcal{R}$ be the span of the rational functions R^b with $b \in \Lambda \setminus \{a_1, a_2, \dots, a_{2q-2}, \infty\}$, where $R^b(\zeta) = \frac{b-a_1)\cdots(b-a_{2q-2})}{(\zeta-b)(\zeta-a_1)\cdots(\zeta-a_{2q-2})}$. Ahlfors needs to establish the density of $\mathcal{R}$ in the Banach space $\mathbb{A}$ of integrable holomorphic functions on Ω. He needs to use Stokes's theorem. However, the functions confronting him are not smooth at the boundary, *and* the boundary is not even rectifiable. The outline of Ahlfors's argument follows. Assume that $q = 2$. It is easily seen that $\alpha(\varphi) = 0$ if and only if $F_{\lambda^{2-2q}\bar{\varphi}}$ vanishes on Λ. A bounded measurable function[3] μ induces a bounded linear functional l_μ on $\mathbb{A}$ by the formula

$$l_\mu(\psi) = \int\int_{\Omega} \psi(z)\mu(z) dz \wedge \overline{dz}, \ \psi \in \mathbb{A}.$$

Hence the injectivity of α is equivalent to the density of $\mathcal{R}$ in $\mathbb{A}$. It suffices by Hahn-Banach and the Riesz representation theorem to prove that for every bounded measurable function[3] μ on Ω the condition

$$\int\int_{\Omega} \mu(z) r(z) dz \wedge \overline{dz} = 0, \quad \text{all } r \in \mathcal{R},$$

is enough to guarantee that $\mu = 0$, a.e.[4] The hypothesis on μ tells us that F_μ vanishes on $\Lambda = \partial\Omega$. A fake "proof" of the density, using divergent integrals, is provided by

$$\begin{aligned} l_\mu(\psi) &= \int\int_{\Omega} \mu\psi dz \wedge \overline{dz} \\ &= \int\int_{\Omega} \left(\frac{\partial F_\mu}{\partial \bar{z}}\right) \psi dz \wedge \overline{dz} \\ &= \int\int_{\Omega} \frac{\partial}{\partial \bar{z}} \left(F_\mu \psi\right) dz \wedge \overline{dz} \\ &= -\int\int_{\Omega} \bar{\partial} \left(F_\mu \psi\, dz\right) \\ &= -\int_{\partial\Omega} F_\mu \psi\, dz = 0, \quad \text{all } \psi \in \mathbb{A}. \end{aligned}$$

To convert the above fake proof into a real one, choose a smooth function j on $\mathbb{R}$ with values in the closed interval $[0, 1]$ with the properties that

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[3] *In particular,* $\lambda^{-2}\bar{\varphi}$ *with* φ *a bounded holomorphic 2-form on* Ω *for* G.

[4] *Bers's paper [11] shows that every bounded linear functional on* $\mathbb{A}$ *is of this form with* $\mu = \lambda^{-q}\bar{\varphi}$ *and* φ *a holomorphic bounded 2-form. But this observation does* not *simplify the argument.*

it vanishes on $(-\infty, 1]$ and takes on the value 1 on $[2, \infty)$. Then for $n \in \mathbb{Z}^+$, define

$$\omega_n(z) = j\left(\frac{n}{\log\log \frac{1}{\delta(z)}}\right), \quad z \in \Omega,$$

where $\delta(z)$ is the distance from z to $\partial\Omega$. Let $R > 0$. We now let Ω_R and Λ_R be the intersection of Ω with the disc and circle centered at the origin and radius R, respectively. Because ω_n vanishes in a neighborhood of Λ, we can use integration by parts to conclude that

$$\int\int_{\Omega_R} \omega_n \psi \mu \, dz \wedge \overline{dz}$$
$$= -\int_{\Lambda_R} \omega_n \psi F_\mu - \int\int_{\Omega_R} \psi F_\mu \frac{\partial \omega_n}{\partial \bar{z}} \, dz \wedge \overline{dz},$$

and from here conclude that

$$\left| \int\int_{\Omega_R} \psi \mu \, dz \wedge \overline{dz} \right|$$
$$\leq \left| \int_{\Lambda_R} \psi F_\mu \, dz \right| \leq \text{ const. } R \log R \int_{\Lambda_R} |\psi \, dz|.$$

Standard potential-theoretic estimates for the growth of $|F_\mu|$ are used to obtain the last inequality. The finiteness of $\int\int_\Omega |\psi \, dz \wedge \overline{dz}|$ now implies that the last integral in the series of inequalities tends to zero as R becomes large. This completes the proof of the injectivity of α for $q = 2$. It is important to observe that the argument did *not* require G to be finitely generated. Ahlfors's "mollifier" is so delicate that it has not found any other uses even though it has been around for more than thirty-three years. In particular, it is an open problem to determine necessary and sufficient conditions on G for α to be injective for a fixed $q > 2$. For finitely generated G, once one has the injectivity for $q = 2$, it is easy to obtain the same conclusion for bigger q. Alternative approaches to the finiteness theorem are found in [33, 14, 28].[5]

In his proof of the finiteness theorem [2], Ahlfors made a small mistake: he left out the possibility of infinitely many thrice-punctured spheres appearing in Ω/G. (Such surfaces admit no moduli (deformations) and alternatively carry no nontrivial integrable quadratic differentials.) That deficiency was remedied in subsequent papers by work of Bers [13], Greenberg [20], and Ahlfors himself [3]. Ahlfors initially limited his work to quadratic differentials (which include squares of "classical" abelian differentials), in part because this case and the abelian case are the only ones with geometric significance. Perhaps more significantly, it was Ahlfors's style to make the pioneering contributions to a field and leave plenty of room for others to continue in the same area. In this particular case much remained to be done.

Bers [12] saw that if he studied the more general case of q-differentials, he would be able to improve on the results of Ahlfors and get quantitative versions of the finiteness theorem that have become known as the Bers area theorems. The first of these theorems [12] states: *If G is generated by N-motions, then* $\text{Area}(\Omega/G) \leq 4\pi(N-1)$. This paper by Bers led, in turn, to investigations of the structure of (the Eichler cohomology groups of) Kleinian groups by Ahlfors [4], based on his earlier paper [3], and this author [23, 24]. Whereas Ahlfors used meromorphic Eichler integrals (in a sense going back to Bers's studies [11]) to describe the structure of the cohomology groups, I relied on smooth potentials (in a sense combining methods of Ahlfors [2] and Bers [12]). Ahlfors's generosity, as evidenced by the footnote in [4] regarding the relation between his and my approach to this problem, was very much appreciated; his remarks were most encouraging to a young mathematician.

Since the minimal area of a hyperbolic orbifold is $\pi/21$, Bers's area theorem gives an upper bound on the number of (connected) Riemann surfaces represented by a nonelementary Kleinian group as $84(N-1)$. Ahlfors [3] lowered that bound to $18(N-1)$. Even after some important work of Abikoff [1], there is still no satisfactory bound on the number of surfaces that a Kleinian group represents, especially if one insists on using only 2-dimensional methods. Bers's paper [12] also showed that the thrice-punctured spheres issue can be resolved "without new ideas." It, together with Ahlfors's discoveries on Kleinian groups, led fifteen years later to work on the vanishing[6] of Poincaré and relative Poincaré series [25, 26].

The so-called measure zero problem first surfaced during the 1965 Tulane conference.[7] In his 1964 paper Ahlfors remarked that perhaps of greater interest than the theorems (AFT) that he has been able to prove were the ones he was *not* able to prove. First of these was the assertion that the limit set of a finitely generated Kleinian group has two-dimensional Lebesgue measure zero.[8] This

[5] *This paper deals with the topological aspects of AFT.*

[6] *A different, more classical, approach to the problem of deciding when a relative Poincaré series vanishes identically is found in the earlier work of Hejhal [22].*

[7] *The first of the periodic meetings, roughly every four years, of researchers in fields related to the mathematical interests of Lars Ahlfors and Lipman Bers. The tradition continues; the next meeting of the Ahlfors-Bers Colloquium will take place in November 1998.*

[8] *Although quasiconformal mappings do not appear in the proof of AFT, Ahlfors's motivation and ideas came from*

has become known as the Ahlfors measure zero conjecture. It is still unsolved, although important work on it has been done by Ahlfors, Maskit, Thurston, Sullivan, and Bonahan. In some sense the problem has been solved for analysts by Sullivan [31], who showed that a nontrivial deformation of a finitely generated Kleinian group cannot be supported entirely on its limit set; topologists are still interested in the measure zero problem. Formulae in Ahlfors's nonsuccessful attempt [5] to prove the measure zero conjecture led Sullivan [32] to a finiteness theorem on the number of maximal conjugacy classes of purely parabolic subgroups of a Kleinian group. The measure zero problem not only opened up a new industry in the Kleinian groups "industrial park", it also revived the connection with 3-dimensional topology following the fundamental work of Marden [29] and Thurston [34]. It showed that Poincaré was not at all wrong when he thought that we could study Kleinian groups by 3-dimensional methods. The second "theorem" that Ahlfors "wanted" to establish for his 1964 paper can be rephrased in today's language to say that *a finitely generated Kleinian group is geometrically finite*; a counterexample was produced shortly thereafter [19].

Quasiconformal Mappings

Frederick Gehring

In 1982 Birkhäuser published two fine volumes of Lars Ahlfors's collected papers [8] and his fascinating commentaries on them. Volume 2 contains forty-three articles: twenty-one of these are directly concerned with quasiconformal mappings and Teichmüller spaces, twelve with Kleinian groups, and ten with topics in geometric function theory. This distribution shows clearly the dominant role that quasiconformal mappings played in this part of Ahlfors's work. Moreover, quasiconformal mappings play a key role in several other papers, for example, the important finiteness theorem for Kleinian groups. For this reason I have chosen quasiconformal mappings as the subject of this survey. In particular, I will consider the four papers [6, 7, 9, 10], which have had great impact on contemporary analysis.

On Quasiconformal Mappings [6]

In his commentary to this paper Ahlfors wrote that "It had become increasingly evident that Teichmüller's ideas would profoundly influence analysis and especially the theory of functions of one complex variable.…The foundations of the theory were not commensurate with the loftiness of Teichmüller's vision, and I thought it was time to re-examine the basic concepts."

The quasiconformal mappings considered by Grötzsch and Teichmüller were assumed to be continuously differentiable except for isolated points or small exceptional sets. Teichmüller's theorem concerned the nature of the quasiconformal mappings between two Riemann surfaces S and S' that have minimum maximal dilatation. This and the fact that any useful theory that generalizes conformal mappings should have compactness and reflection properties led Ahlfors to formulate a geometric definition that was free of all a priori smoothness hypotheses.

A *quadrilateral* Q is a Jordan domain Q with four distinguished boundary points. The *conformal modulus* of Q, denoted $\mathrm{mod}(Q)$, is defined as the side ratio of any conformally equivalent rectangle R. Grötzsch showed that if $f: D \to D'$ is K-quasiconformal in the classical sense, then

$$(1) \qquad \frac{1}{K}\mathrm{mod}(Q) \le \mathrm{mod}(f(Q)) \le K\mathrm{mod}(Q)$$

for each quadrilateral $Q \subset D$. Ahlfors used this inequality to define his new class of quasiconformal mappings: a homeomorphism $f: D \to D'$ is K-quasiconformal if (1) holds for each quadrilateral $Q \subset D$. Ahlfors then established all of the basic properties of conformal mappings for this general class of homeomorphisms, including a uniform Hölder estimate, a reflection principle, a compactness principle, and an analogue of the Hurwitz theorem. He did all of this in nine pages.

The major part of this article was, of course, concerned with a statement, several interpretations, and the first complete proof of Teichmüller's theorem.

In his commentary on this paper Ahlfors modestly wrote that "My paper has serious shortcomings, but it has nevertheless been very influential and has led to a resurgence of interest in quasiconformal mappings and Teichmüller theory."

This is an understatement! Ahlfors's exposition made Teichmüller's ideas accessible to the mathematical public and resulted in a flurry of activity and research in the area by scientists from many different fields, including analysis, topology, algebraic geometry, and even physics.

Next, Ahlfors's geometric approach to quasiconformal mappings stimulated analysts to study

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his work on quasiconformality. Ahlfors knew that a finitely generated Kleinian group had a finite-dimensional deformation space and for a family of such groups a deformation should be induced by a quasiconformal map whose Beltrami coefficient is supported on the ordinary set. These assertions would follow if the limit set had measure zero. They follow from Sullivan's theorem [31].

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Ahlfors in the classroom.

this class of mappings in the plane, in higher-dimensional euclidean spaces, and now in arbitrary metric spaces. His inspired idea to drop all analytic hypotheses eventually led to striking applications of these mappings in other parts of complex analysis, such as discontinuous groups, classical function theory, complex iteration, and in other fields of mathematics, including harmonic analysis, partial differential equations, differential geometry, and topology.

The Boundary Correspondence under Quasiconformal Mappings [10]

In the previous paper Ahlfors proved that a quasiconformal mapping $f : D \to D'$ between Jordan domains has a homeomorphic extension to their closures. A classical theorem due to F. and M. Riesz implied that the induced boundary correspondence ϕ is absolutely continuous with respect to linear measure whenever ∂D and $\partial D'$ are rectifiable and f is conformal. Mathematicians asked whether this conclusion holds when f is K-quasiconformal.

By composing f with a pair of conformal mappings, one can reduce the problem to the case where $D = D' = H$, where H is the upper half plane and $\phi(\infty) = \infty$. Next, if x and t are real with $t > 0$ and if Q is the quadrilateral with vertices at $x - t$, x, $x + t$, ∞, then $\mathrm{mod}(Q) = 1$ and inequality (1) implies that

$$\frac{1}{\lambda} \le \frac{\phi(x+t) - \phi(x)}{\phi(x) - \phi(x-t)} \le \lambda, \tag{2}$$

where $\lambda = \lambda(K)$. Inequality (2) is a quasisymmetry condition that it was thought would imply that ϕ is absolutely continuous.

In 1956 Ahlfors and Beurling published a paper in which they exhibited for each $K > 1$ a K-quasiconformal mapping $f : H \to H$ for which the boundary correspondence $\phi : \partial H \to \partial H$ is completely singular. The importance of this example was, however, overshadowed by the authors' main theorem, which stated that inequality (2) characterizes the boundary correspondences induced by quasiconformal self-mappings of H. The sufficiency part consisted in showing that the remarkable formula

$$\begin{aligned} f(z) = \frac{1}{2y} \int_0^y \left[\phi(x+t) + \phi(x-t) \right] dt \\ + \frac{i}{2y} \int_0^y \left[\phi(x+t) - \phi(x-t) \right] dt \end{aligned} \tag{3}$$

yields a K-quasiconformal self-mapping of H with $K = K(\lambda)$ whenever ϕ satisfies (2). Moreover, f is a *hyperbolic quasi-isometry* of H, a fact that turns out to have many important consequences.

In 1962 it was observed that a self-quasi-conformal mapping of the n-dimensional upper half space H^n induces an $(n-1)$-dimensional self-mapping ϕ of the $(n-1)$-dimensional boundary plane ∂H^n; this fact, for the case $n = 3$, was an important step in the original proof of Mostow's rigidity theorem. It was then natural to ask if every quasiconformal self-mapping ϕ of ∂H^n admits a quasiconformal extension to H^n. This question was eventually answered in the affirmative by Ahlfors in 1963 for $n = 3$, by Carleson in 1972 for $n = 4$, and by Tukia-Väisälä in 1982 for all $n \ge 3$.

Riemann's Mapping Theorem for Variable Metrics [9]

If $f : D \to D'$ is K-quasiconformal according to (1), then f is differentiable with $f_z \ne 0$ a.e. in D, and

$$\mu_f = \frac{f_{\bar z}}{f_z} \tag{4}$$

is measurable with

$$|\mu_f| \le k = \frac{K-1}{K+1} \tag{5}$$

a.e. in D. The *complex dilatation* μ_f determines f uniquely up to postcomposition with a conformal mapping.

The main result of this article states that for any μ that is measurable with $|\mu| \le k$ a.e. in D, there exists a K-quasiconformal mapping f that has μ as its complex dilatation. Moreover, if f is suitably normalized, then f depends holomorphically on μ.

The above result, known by many as the "measurable Riemann mapping theorem", has proved to be an enormously influential and effective tool in analysis. It is a cornerstone for the study of Teichmüller space; it was the key for settling outstanding questions of classical function theory, including Sullivan's solution of the Fatou-Julia problem on wandering domains; and it currently plays a major role in the study of iteration of rational functions. Indeed, application of this theorem has become a verb in this area. In a lecture at the 1986 International Congress of Mathematicians a distinguished French mathematician was heard to explain that "before mating two polynomials, one must first Ahlfors-Bers the structure."

Quasiconformal Reflections [7]

A Jordan curve C is said to be a *quasicircle* if it is the image of a circle or line under a quasiconformal self-mapping of the extended plane; a domain D is a *quasidisk* if ∂D is a quasicircle. Quasicircles can be very wild curves. Indeed, for $0 < a < 2$ there exists a quasicircle C with Hausdorff dimension at least a.

Nevertheless, the first theorem of this elegant paper contains the following remarkable characterization for this class of curves. A Jordan curve C is a quasicircle if and only if there exists a constant b such that

$$|z_1 - z_2| \le b|z_1 - z_3| \tag{6}$$

for each ordered triple of points $z_1, z_2, z_3 \in C$. The proof for the sufficiency of (6) depends on the fact that the function in (3) is a hyperbolic quasi-isometry.

The fact that quasicircles admit such a simple geometric description is the reason why these curves play such an important role in many different areas of analysis. Inequality (6) is universally known as the "Ahlfors condition", and many regard it as the best way to define the notion of a quasicircle. This is an example of a beautiful theorem that is in danger of becoming a definition!

The second main result of this paper asserts that the set S of Schwarzian derivatives

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 \tag{7}$$

of conformal mappings f that map the upper half plane H onto a quasidisk D is an open subset of the Banach space of holomorphic functions ϕ with norm

$$\|\phi\|_H = \sup_H |\phi(z)|^2 y^2. \tag{8}$$

This fact is the key step in proving that the Bers universal Teichmüller space $T(1)$ is the interior of S with the topology induced by the norm (8). It also led to another surprising connection between quasiconformal mappings and classical function theory, namely, that a simply connected domain D is a quasidisk if and only if each function f, analytic with small Schwarzian S_f in D, is injective. Here the size of S_f is measured by

$$\|S_f\| = \sup_D |S_f(z)|^2 \rho_D(z)^{-2}, \tag{9}$$

where ρ_D is the hyperbolic metric in D.

Lars Ahlfors lectured on this beautiful paper at the Oberwolfach Tagung in 1963. He, Olli Lehto, and I were scheduled to speak the same morning. After several hours of socializing and wine the evening before, Olli and I tried to excuse ourselves so that we could get some sleep before our talks. We were told by Lars that that was "a very silly idea indeed" and that it would be far better to relax and drink with the pleasant company. The next day Lars's talk went extremely well, and he was subsequently asked if he believed that staying up late always improved his lectures. "I am not sure," he replied, "but at least they always sound better to me!"

Conclusion

Quasiconformal mappings first appeared under this name in Ahlfors's paper *Zur theorie der Überlagerungsflächen* in 1935. In his commentary for this article he wrote, "Little did I know at the time what an important role quasiconformal mappings would come to play in my own work." This class of mappings offers a stripped-down picture of the geometric essentials of complex function theory and, as such, admits applications of these ideas to many other parts of analysis and geometry. They constitute just one illustration of the profound and lasting effect that the deep, central, and seminal character of Lars Ahlfors's research has had on the face of modern mathematics.

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