Singular Integrals: The Roles of Calderón and Zygmund

Elias M. Stein

Editor's Note: Alberto Pedro Calderón died on April 16, 1998. A memorial article appears elsewhere in this issue.

The subject matter of this essay is Alberto Calderón's pivotal role in the creation of the modern theory of singular integrals. In that great enterprise Calderón had the good fortune of working with Antoni Zygmund, who was at first his teacher and mentor and later his collaborator. For that reason any account of that theory has to be in part the story of the efforts of both Zygmund and Calderón. With this in mind, I shall explain the various goals that motivated them, describe some of their shared accomplishments and later work of Calderón, and discuss briefly the wide influence of their achievements.

Zygmund's Vision: 1927–1949

In the first period of his scientific work, from 1923 to the middle 1930s, Zygmund devoted himself to what is now called "classical" harmonic analysis: that is, Fourier and trigonometric series of the circle, related power series of the unit disc, conjugate functions, Riemannian theory connected to uniqueness, lacunary series, etc. An account of much of what he did, as well as the work of his contemporaries and predecessors, is contained in his famous treatise *Trigonometrical Series*, published in 1935. The time in which this took place may be viewed as the concluding decade of the brilliant century of classical harmonic analysis: the approximately one hundred-year span which began with Dirichlet and Riemann, continued with Cantor and Lebesgue among others, and culminated with the achievements of Kolmgorov, M. Riesz, and Hardy and Littlewood.

It was during that last decade that Zygmund began to turn his attention from the one-dimensional situation to problems in higher dimensions. At first this represented merely an incidental interest, but then later he followed it with increasing dedication, and eventually it was to become the main focus of his scientific work. I want now to describe how this point of view developed with Zygmund.

In outline, the subject of one-dimensional harmonic analysis as it existed in that period can be understood in terms of what were then three closely interrelated areas of study and which in many ways represented the central achievements of the theory: real-variable theory, complex analysis, and the behavior of Fourier series. Zygmund's first excursion into questions of higher dimensions dealt with the key issue of real-variable theory, the averaging of functions. The question was as follows. The classical theorem of Lebesgue guaranteed that for almost every x

$$\lim_{\substack{x\in I \\ \operatorname{diam}(I)\to 0}} \frac{1}{|I|}\int_I f(y)\,dy = f(x), \tag{1.1}$$

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This essay, published in the Notices *by arrangement with the University of Chicago Press, is a modified version of the article entitled "Calderón and Zygmund's theory of singular integrals", to appear in the proceedings of the 1996 conference in honor of Calderón's seventy-fifth birthday held at the University of Chicago. That article also contains an extended list of references and further bibliographical notes which have been omitted from the present version.*

where I ranges over intervals and when f is an integrable function on the line $\mathbb{R}^1$.

In higher dimensions it is natural to ask whether a similar result holds when the intervals I are replaced by appropriate generalizations in $\mathbb{R}^n$. The fact that this is the case when the I's are replaced by balls (or more general sets with bounded "eccentricity") was well known at that time. What must have piqued Zygmund's interest in the subject was his realization (in 1927) that a paradoxical set constructed by Nikodym showed that the answer is irretrievably false when the I's are taken to be rectangles (each containing the point in question) but with arbitrary orientation. To this must be added the counterexample found by Saks several years later, which showed that the desired analogue of (1.1) still failed even if we now restricted the rectangles to have a fixed orientation (e.g., with sides parallel to the axes) as long as one allowed f to be a general function in L^1.

It was at this stage that Zygmund effectively transformed the subject at hand by an important advance: he proved that the wished-for conclusion (when the sides are parallel to the axes) held if f was assumed to belong to L^p, with $p > 1$. He accomplished this by proving an inequality for what is now known as the "strong" maximal function. The original Hardy-Littlewood maximal function involved a supremum over the averages in (1.1); in the definition of the strong maximal function the intervals containing x are replaced by rectangles with sides parallel to the axes. Shortly afterwards in Jessen, Marcinkiewicz, and Zygmund (1935) this was refined to the requirement that f belong to $L(\log L)^{n-1}$ locally.

This study of the extension of (1.1) to $\mathbb{R}^n$ was the first step taken by Zygmund. It is reasonable to guess that it reinforced his fascination with what was then developing as a long-term goal of his scientific efforts: the extension of the central results of harmonic analysis to higher dimensions. But a great obstacle stood in the way: it was the crucial role played by complex function theory in the whole of one-dimensional Fourier analysis, and for this there was no ready substitute.

In describing this special role of complex methods we shall content ourselves with highlighting some of the main points. The theory can be formulated equally in the unit disc or the upper half-plane, and we shall freely pass back and forth between these settings.

(i) *The conjugate function and its basic properties*

As is well known, the Hilbert transform comes directly from the Cauchy integral formula. Closely connected with this is the fact that the Hilbert transform of a function f is obtained by passing to the Poisson integral of f in the upper half-plane, taking the conjugate harmonic function and passing back to boundary values. We also recall the fact that M. Riesz proved the L^p boundedness properties of the Hilbert transform

$$f \mapsto H(f) = \frac{p.v.}{\pi} \int_{-\infty}^{\infty} f(x-y)\frac{dy}{y}$$

by applying a contour integral to $(F)^p$, where F is the analytic function whose boundary limit has f as its real part. It should be noted that the Hilbert transform has a simple expression as a Fourier multiplier, that is,

$$H(f)\hat{}(\xi) = \frac{sign(\xi)}{i}\hat{f}(\xi),$$

where $\hat{}$ denotes the Fourier transform; from this the L^2 boundedness is an immediate consequence via Plancherel's theorem.

(ii) *The theory of the Hardy spaces H^p*

These arose in part as substitutes for L^p, when $p \le 1$, and were by their very nature complex-function-theory constructs. (It should be noted, however, that for $1 < p < \infty$ they were essentially equivalent with L^p by Riesz's theorem.) The classical space H^p consists of analytic functions F in the unit disc for which

$$\sup_{r<1} \int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta < \infty.$$

The main tool used in their study was the Blaschke product of their zeroes in the unit disc. Using it, one could reduce matters to elements $F \in H^p$ with no zeroes, and from these one could pass to $G = F^{p/2}$; the latter was in H^2 and hence could be treated by more standard (L^2) methods.

(iii) *The Littlewood-Paley theory*

This proceeded by studying the dyadic decomposition in frequency space and had many applications; among them was the Marcinkiewicz multiplier theorem. This gave conditions on a Fourier multiplier, in terms of certain differential inequalities, that were sufficient to guarantee that it defined a bounded operator on L^p. The theory initiated and exploited certain basic "square functions", and these we originally studied by complex-variable techniques closely related to what were used in H^p spaces.

(iv) *The boundary behavior of harmonic functions*

The main result obtained here (Privalov (1923), Marcinkiewicz and Zygmund (1938), and Spencer (1943)) stated that for any harmonic function $u(re^{i\theta})$ in the unit disc the following three properties are equivalent for almost all boundary points $e^{i\theta}$:

(1.2) $\quad u$ has a nontangential limit at $e^{i\theta}$,

(1.3) $\quad u$ is nontangentially bounded at $e^{i\theta}$, the "area integral" $(S(u)(\theta))^2 =$

$$\iint_{\Gamma(e^{i\theta})} |\nabla u(z)|^2 \, dx\, dy \text{ is finite,} \tag{1.4}$$

where $\Gamma(e^{i\theta})$ is a nontangential approach region with vertex $e^{i\theta}$.

The crucial first step in the proof was the application of the conformal map (to the unit disc) of the famous saw-tooth domain in Figure 1.[1] This mapping allowed one to reduce the implication (1.3) $\Rightarrow$ (1.2) to the special case of bounded harmonic functions in the unit disc (Fatou's theorem) and also played a corresponding role in the other parts of the proof.

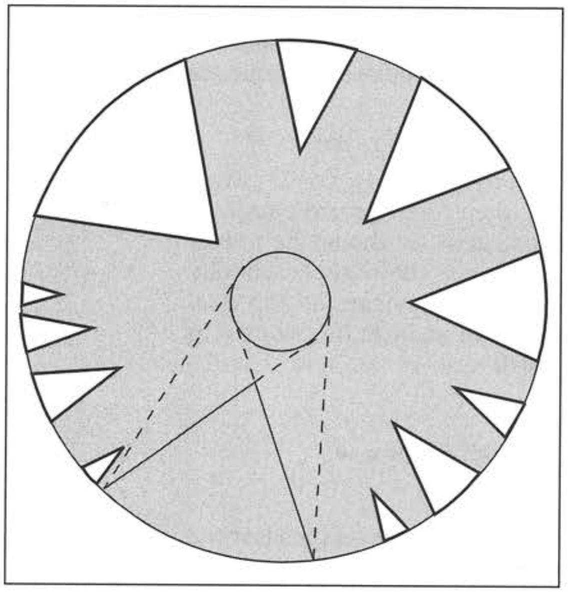


Figure 1. Saw-tooth domain.

It is ironic that complex methods with their great power and success in the one-dimensional theory actually stood in the way of progress to higher dimensions and appeared to block further progress. The only way past, as Zygmund foresaw, required a further development of "real" methods. Achievement of this objective was to take more than one generation and in some ways is not yet complete. The mathematician with whom he was to initiate the effort to realize much of this goal was Alberto Calderón.

Calderón and Zygmund: 1950–1957

Zygmund spent the academic year 1948–49 in Argentina, and there he met Calderón. Zygmund brought him back to the University of Chicago, and soon thereafter (in 1950), under his direction, Calderón obtained his doctoral thesis. The dissertation contained three parts, the first about ergodic theory, which will not concern us here. It is the second and third parts that interest us, and these represented breakthroughs in the problem of freeing oneself from complex methods and, in particular, in extending to higher dimensions some of the results described in (iv) above. In a general way we can say that his efforts here already typified the style of much of his later work: he begins by conceiving some simple but fundamental ideas that go to the heart of the matter and then develops and exploits these insights with great power.

In proving (1.3) $\Rightarrow$ (1.2) we may assume that u is bounded inside the saw-tooth domain Ω that arose in (iv) above: this region is the union of approach regions $\Gamma(e^{i\theta})$, ("cones") with vertex $e^{i\theta}$, for points $e^{i\theta} \in E$, and E a closed set. Calderón introduced the auxiliary harmonic function U, with U the Poisson integral of χ_{cE}, and observed that all the desired facts flowed from the dominating properties of U: namely, u could be split as $u = u_1 + u_2$, where u_1 is the Poisson integral of a bounded function (and hence has nontangential limits a.e.), while by the maximum principle $|u_2| \le c\,U$, and therefore u_2 has (nontangential) limits $= 0$ at a.e. point of E.

The second idea (used to prove the implication (1.2) $\Rightarrow$ (1.4)) has as its starting point the simple identity

$$\Delta u^2 = 2|\nabla u|^2 \tag{2.1}$$

valid for any harmonic function. This can be combined with Green's theorem

$$\iint_{\Omega} (B\Delta A - A\Delta B)\, dx\, dy = \int_{\partial\Omega} \left(B\frac{\partial A}{\partial n} - A\frac{\partial B}{\partial n}\right) d\sigma,$$

where $A = u^2$ and B is another ingeniously chosen auxiliary function depending on the domain Ω only. This allowed him to show that $\iint_{\Omega} y|\nabla u|^2\, dx\, dy < \infty$, which is an integrated revision of (1.4).

It may be noted that the above methods and the conclusions they imply make no use of complex analysis and are very general in nature. It is also a fact that these ideas played a significant role in the later real-variable extension of the H^p theory.

Starting in the year 1950, a close collaboration developed between Calderón and Zygmund which lasted almost thirty years. While their joint research dealt with a number of different subjects, their preoccupying interest and most fundamental contributions were in the area of singular integrals. In this connection the first issue they addressed was—to put the matter simply—the extension to higher dimensions of the theory of the Hilbert transform. A real-variable analysis of the Hilbert transform had been carried out by Besicovitch, Titchmarsh, and Marcinkiewicz, and this is what needed to be extended in the $\mathbb{R}^n$ setting.

A reasonable candidate for consideration presented itself. It was the operator $f \mapsto Tf$, with

$$T(f)(x) = p.v. \int_{\mathbb{R}^n} K(y)f(x-y)\, dy, \tag{2.2}$$

[1] *Figure 1 is based on a diagram in Zygmund [1959], Vol. II, p. 200.*

when K was homogeneous of degree $-n$, satisfied some regularity, and in addition satisfied the cancellation condition $\int_{|x|=1} K(x)\, d\sigma(x) = 0$.

Besides the Hilbert transform (which is the only real example when $n = 1$), higher-dimensional examples include the operators that arise as second derivatives of the fundamental solution operator for the Laplacian (which can be written as $\frac{\partial^2}{\partial x_2 \partial x_j}(\Delta)^{-1}$), as well as the related Riesz transforms $\frac{\partial}{\partial x_j}(-\Delta)^{-1/2}$. For the Hilbert transform, n is equal to 1 and $K(x)$ is equal to $\frac{1}{\pi x}$; the Riesz transforms are given (up to a constant multiple) by $K(x) = x_j/|x|^{n+1}$, $j = 1, \dots, n$.

All of this is the subject matter of their historic memoir "On the existence of singular integrals", which appeared in *Acta Mathematica* in 1952. There is probably no paper in the last fifty years which had such widespread influence in analysis. The ideas in this work are now so well known that I will only outline its contents. It can be viewed as having three parts.

First, there is the Calderón-Zygmund lemma and the corresponding Calderón-Zygmund decomposition. The main thrust of the former is as a substitute for F. Riesz's "rising sun" lemma, which F. Riesz used in re-proving Lebesgue's theorem about the almost everywhere differentiability of monotone functions and which had implicitly played a key role in the earlier treatment of the Hilbert transform. A schematic representation of their decomposition is given in Figure 2.[2]

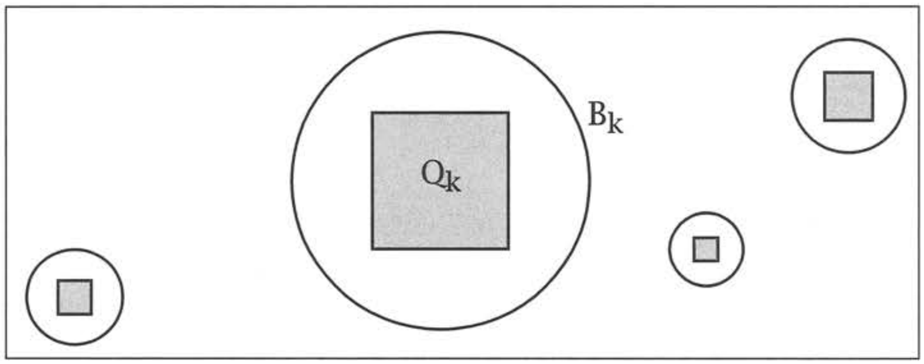


Figure 2. Calderón-Zygmund decomposition of a nonnegative function f at altitude α. The cubes $Q = Q_k$ have the property that $\alpha < |Q|^{-1} \int_Q f\, dx \le 2^n \alpha$. In the complement of the balls B_k, a favorable L^1 estimate holds.

Second, using their decomposition, they then proved the weak-type L^1 and L^p, $1 < p < \infty$, estimates for the operator T in (2.2). As a preliminary step they disposed of the L^2 theory of T using Plancherel's theorem.

Third, they applied these results to the examples mentioned above, and in addition they proved a.e. convergence for the singular integrals in question.

It should not detract from one's great admiration for this work to note two historical anomalies contained in it. The first is the fact that there is no mention of Marcinkiewicz's interpolation theorem or of the paper in which it appeared (Marcinkiewicz [1939a]), even though its ideas play a significant role. In the Calderón-Zygmund paper the special case that is needed is in effect reproved. The explanation for this omission is that Zygmund had simply forgotten about the existence of Marcinkiewicz's note. To make amends, he published (in 1956) an account of Marcinkiewicz's theorem and various generalizations and extensions he had since found. In it he conceded that the paper of Marcinkiewicz "... seems to have escaped attention and does not find allusion to it in the existing literature."

The second point, like the first, also involves important work of Marcinkiewicz. He had been Zygmund's brilliant student and collaborator until his death at the beginning of World War II. It is a mystery why no reference was made to the paper by Marcinkiewicz [1939b] and the multiplier theorem in it. This theorem had been proved by Marcinkiewicz in an n-dimensional form (as a product "consequence" of the one-dimensional form). As an application the L^p inequalities for the operators $\frac{\partial^2}{\partial x_i \partial x_j}(\Delta^{-1})$ were obtained[3]; these he had proved at the behest of Schauder.

As has already been indicated, the n-dimensional singular integrals had their main motivation in the theory of partial differential equations. In their further work, Calderón and Zygmund pursued this connection, following the trail that had been explored earlier by Giraud, Tricomi, and Mihlin. Starting from those ideas (in particular the notion of "symbol"), they developed their version of the symbolic calculus of "variable-coefficient" analogues of the singular integral operators. To describe these results, one considers an extension of the class of operators arising in (2.2), namely, of the form

$$(2.3) \quad T(f)(x) = a_0(x)f(x) + p.v. \int_{\mathbb{R}^n} K(x,y) f(x-y)\, dy,$$

where $K(x,y)$ is for each x a singular integral kernel of the type (2.2) in y, which depends smoothly and boundedly on x; also $a_0(x)$ is a smooth and bounded function.

[2] *I still have a graphic recollection of a similar picture shown me by Harold Widom in 1952–53, when we were both graduate students at the University of Chicago.*

[3] *In truth, he had done this for their periodic analogues, but this is a technical distinction.*

To each operator T of this kind there corresponds its symbol $a(x,\xi)$, defined by

$$a(x,\xi) = a_0(x) + \widehat{K}(x,\xi), \tag{2.4}$$

where $\widehat{K}(x,\xi)$ denotes the Fourier transform of $K(x,y)$ in the y variable. Thus $a(x,\xi)$ is homogeneous of degree 0 in the ξ variable (reflecting the homogeneity of $K(x,y)$ of degree $-n$ in y), and it depends smoothly and boundedly on x. Conversely, to each function $a(x,\xi)$ of this kind there exists a (unique) operator (2.3) for which (2.4) holds. One says that a is the symbol of T and also writes $T = T_a$.

The basic properties that were proved were, first, the regularity properties

$$T_a : L_k^p \to L_k^p, \tag{2.5}$$

where L_k^p are the usual Sobolev spaces, involving L^p norms of the function and its partial derivatives through order k, with $1 < p < \infty$.

Also proved were the basic facts of symbolic manipulations

$$T_{a_1} \cdot T_{a_2} = T_{a_1 \cdot a_2} + Error \tag{2.6}$$

$$(T_a)^* = T_{\overline{a}} + Error \tag{2.7}$$

where the *Error* operators are smoothing of order 1, in the sense that $Error : L_k^p \to L_{k+1}^p$.

A consequence of the symbolic calculus is the factorizability of any linear partial differential operator L of order m,

$$L = \sum_{|\alpha| \le m} a_\alpha(x) \left(\frac{\partial}{\partial x}\right)^\alpha,$$

where the coefficients a_α are assumed to be smooth and bounded. On can write

$$L = T_a(-\Delta)^{m/2} + (Error)' \tag{2.8}$$

for an appropriate symbol a, where the operator $(Error)'$ refers to an operator that maps $L_k^p \to L_{k-m+1}^p$, for $k \ge m-1$. It seemed clear that this symbolic calculus should have wide applications to the theory of partial differential operators and to other parts of analysis. This was soon to be borne out.

Acceptance: 1957–1965

At this stage of my narrative I would like to share some personal reminiscences. I had been a student of Zygmund at the University of Chicago, and in 1956 at his suggestion I took my first teaching position at MIT, where Calderón was at that time. I had met Calderón several years earlier when he came to Chicago to speak about the "method of rotations" in Zygmund's seminar. I still remember my feelings when I saw him there; these first impressions have not changed much over the years: I was struck by the sense of his understated elegance, reserve, and quiet charisma.

At MIT we would meet quite often, and over time an easy conversational relationship developed between us. I do recall that we, in the small group who were interested in singular integrals then, felt a certain separateness from the larger community of analysts—not that this isolation was self-imposed, but more because our subject matter was seen by our colleagues as somewhat arcane, rarefied, and possibly not very relevant. However, this did change, and a fuller acceptance eventually came. I want to relate now how this occurred.

Starting from the calculus of singular integral operators that he had worked out with Zygmund, Calderón obtained a number of important applications to hyperbolic and elliptic equations. His most dramatic achievement was in the uniqueness of the Cauchy problem (Calderón [1958]). There he succeeded in a broad and decisive extension of the results of Holmgren (for the case of analytic coefficients) and Carleman (in the case of two dimensions). Calderón's theorem can be formulated as follows.

Suppose u is a function which in the neighborhood of the origin in $\mathbb{R}^n$ satisfies the equation of m^{th} order:

$$\frac{\partial^m u}{\partial x_n^m} = \sum_\alpha a_\alpha(x) \frac{\partial^\alpha u}{\partial x^\alpha}, \tag{3.1}$$

where the summation is taken over all indices $\alpha = (\alpha_1, \dots, \alpha_n)$, with $|\alpha| \le n$ and $\alpha_n < m$. We also assume that u satisfies the null initial Cauchy conditions

$$\left.\frac{\partial^j u(x)}{\partial x_n^j}\right|_{x_n=0} = 0, \qquad j = 0, \dots, m-1. \tag{3.1$'$}$$

Besides (3.1) and (3.1$'$) it suffices that the coefficients a_α belong to $C^{1+\epsilon}$, that the characteristics are simple, and that $n \neq 3$ or $m \le 3$. Under these hypotheses u vanishes identically in a neighborhood of the origin.

Calderón's approach was to reduce matters to a key "pseudo-differential inequality" (in a terminology that was used later). This inequality is complicated but somewhat reminiscent of a differential inequality that Carleman had used in two dimensions. The essence of it is that

$$\begin{aligned} &\int_0^a \phi_k \left\| \frac{\partial u}{\partial t} + (P + iQ)(-\Delta)^{1/2} u \right\|^2 dt \\ &\qquad \le c \int_0^a \phi_k \|u\|^2 \, dt, \end{aligned} \tag{3.2}$$

where $u(0) = 0$ implies $u \equiv 0$ if (3.2) holds for $k \to \infty$.

Here P and Q are singular integral operators of the type (2.3), with real symbols and P invertible; we have written $t = x_n$, and the norms are L^2

norms taken with respect to the variables $x_1, \ldots, x_{n-1}$. The function ϕ_k is meant to behave like t^{-k}, which, when $k \to \infty$, emphasizes the effect taking place near $t = 0$. In fact, in (3.2) we can take $\phi_k(t) = (t + 1/k)^{-k}$.

The proof of assertions like (3.2) is easier in the special case when all the operators commute; their general form is established by using the basic facts (2.6) and (2.7) of the calculus.

The paper by Calderón was at first not well received. In fact, I learned from him that it was rejected when submitted to what was then the leading journal in partial differential equations, *Communications of Pure and Applied Mathematics.*

At about that time, because of the applicability of singular integrals to partial differential equations, Calderón became interested in formulating the facts about singular integrals in the setting of manifolds. This required the analysis of the effect coordinate changes had on such operators. A hint that the problem was tractable came from the observation that the class of kernels, $K(y)$, of the type arising in (2.2) was invariant under linear (invertible) changes of variables $y \mapsto L(y)$. (The fact that $K(L(y))$ satisfied the same regularity and homogeneity that $K(y)$ did was immediate; that the cancellation property also holds for $K(L(y))$ is a little less obvious.)

R. Seeley was Calderón's student at that time, and he dealt with this problem in his thesis (1959). Suppose $x \mapsto \psi(x)$ is a local diffeomorphism. Then the result is that modulo error terms (which are "smoothing" of one degree) the operator (2.3) is transformed into another operator of the same kind,

$$T'(f)(x) \cdot a_0'(x) = f(x) + p.v. \int K'(x, y) f(x - y)\, dy,$$

but now

$$a_0'(x) = a_0(\psi(x))$$

and

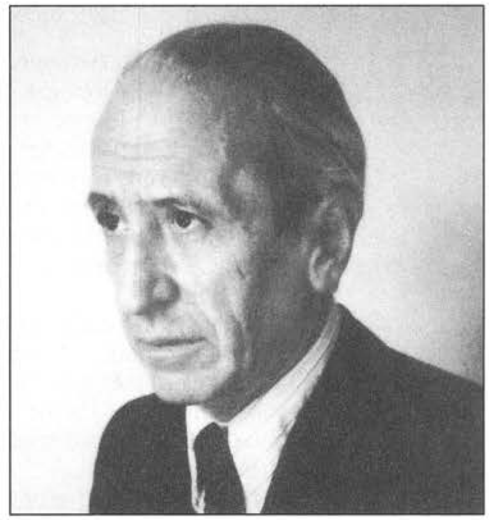
$$K'(x, y) \; = \; K'(\psi(x), L_x(y)),$$

where L_x is the linear transformation given by the Jacobian matrix $\frac{\partial \psi(x)}{\partial x}$. On the level of symbols this meant that the new symbol a' was determined by the old symbol according to the formula

$$a'(x, \xi) = a(\psi(x), L_x'(\xi)),$$

with L_x' the transpose inverse of L_x. Hence the symbol is actually a function on the cotangent space of the manifold.

The result of Seeley was not only highly satisfactory as to its conclusions, it was also very timely in terms of events that were about to take place. Following an intervention by Gelfand (1960), interest grew in calculating the "index" of an elliptic operator on a manifold. This index is the difference of the dimension of the null space and the codimension of the range of the operator and is an invariant under deformations. The problem of determining it was connected with a number of interesting issues in geometry and topology. The result of the "Seeley calculus" proved quite useful in this context: the proofs proceeded by appropriate deformations, and matters were facilitated if these could be carried out in the more flexible context of "general" symbols instead of restricting attention to the polynomial symbols coming from differential operators. A contemporaneous account of this development (during the period 1961–64) may be found in the notes of the seminar on the Atiyah-Singer index theorem (see Palais [1965]); for an historical survey of some of the background, see also Seeley [1967].



Alberto Pedro Calderón, circa 1979

Antoni Zygmund, circa 1979.

With the activity surrounding the index theorem, it suddenly seemed as if everyone was interested in the algebra of singular integral operators. However, one further step was needed to make this a household tool for analysts: it required a change of point of view. Even though this change of perspective was not major, it was significant psychologically and methodologically, since it allowed one to think more simply about certain aspects of the subject and because it suggested various extensions.

The idea was merely to change the role of the definitions of the operators, from (2.3) for singular integrals to pseudo-differential operators

$$T_a(f)(x) = \int_{\mathbb{R}^n} a(x, \xi) \hat{f}(\xi) e^{2\pi i x \cdot \xi}\, d\xi, \tag{3.3}$$

with symbol a. (Here $\hat{f}$ is the Fourier transform $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x)\, dx$.)

Although the two operators are identical (when $a(x,\xi) = a_0(x) + \widehat{K}(x,\xi)$), the advantage lies in the emphasis in (3.3) on the L^2 theory and Fourier transform and the wider class of operators that can be considered, in particular, differential operators. The formulation (3.3) allows one to deal more systematically with the composition of such operators and incorporate the lower-order terms in the calculus.

One way of doing this was to adopt a wider class of symbols of "homogeneous type": roughly speaking, $a(x,\xi)$ belongs to this class (and is of *order m*) if $a(x,\xi)$ is for large ξ asymptotically the sum of terms homogeneous in ξ of degrees $m-j$, with $j = 0, 1, 2, \ldots$.

The change in point of view described above came into its full flowering with the papers of Kohn and Nirenberg (1965) and Hörmander (1965), (after some work by Unterberger and Bokobza (1964) and Seeley (1965)). It is in this way that singular integrals were subsumed by pseudo-differential operators. Despite this, singular integrals, with their formulation in terms of kernels, still retained their primacy when treating real-variable issues, issues such as L^p or L^1 estimates (and even for some of the more intricate parts of the L^2 theory). The central role of the kernel representation of these operators became, if anything, more pronounced in the next twenty years.

Calderón's New Theory of Singular Integrals: 1965–

In the years 1957–58 there appeared the fundamental work of DeGiorgi and Nash dealing with smoothness of solutions of partial differential equations, with minimal assumptions of regularity of the coefficients. One of the most striking results, for elliptic equations, was that any solution u of the equation

$$L(u) \equiv \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = 0 \tag{4.1}$$

in an open ball satisfies an a priori interior regularity as long as the coefficients are uniformly elliptic, i.e.,

$$c_1|\xi|^2 \le \sum_{i,j} a_{ij}(x)\xi_i\xi_j \le c_2|\xi|^2. \tag{4.2}$$

In fact, no regularity is assumed about the a_{ij} except for the boundedness implicit in (4.2), and the result is that u is Hölder continuous with an exponent depending only on the constants c_1 and c_2.

Calderón was intrigued by this result. He initially expected, as he told me, that one could obtain such conclusions and others by refining the calculus of singular integral operators (3.2), making minimal assumptions of smoothness on $a_0(x)$ and $K(x,y)$. While this was plausible—and indeed in his work with Zygmund they had already derived properties of the operators (2.3) and their calculus when the dependence on x was e.g. of class $C^{(1+\epsilon)}$—this hope was not to be realized. Further understanding about these things could be achieved only if one were ready to look in a somewhat different direction. I want to relate now how this came about.

The first major insight arose in answer to the following:

Question: Suppose M_A is the operator of multiplication (by the function A),

$$M_A : f \mapsto A \cdot f.$$

What are the least regularity assumptions on A needed to guarantee that the commutator $[T, M_A]$ is bounded on L^2 whenever T is of order 1?

In $\mathbb{R}^1$ if T happens to be $\frac{d}{dx}$, then $[T, M_A] = M_{A'}$, and so the condition is exactly

$$A' \in L^\infty(\mathbb{R}^1). \tag{4.3}$$

In a remarkable paper Calderón [1965] showed that this is also the case more generally. The key case, containing the essence of the result he proved, arose when $T = H\frac{d}{dx}$, with H the Hilbert transform. Then T is actually $\left|\frac{d}{dx}\right|$, its symbol is $2\pi|\xi|$, and $[T, M_A]$ is the "commutator" C_1,

$$C_1(f)(x) = \frac{1}{\pi}\, p.v. \int_{-\infty}^{\infty} \frac{A(x) - A(y)}{(x-y)^2} f(y)\, dy. \tag{4.4}$$

Calderón proved that $f \mapsto C_1(f)$ is bounded on $L^2(\mathbb{R})$ if (4.3) holds.

There are two crucial points that I want to emphasize about the proof of this theorem. The first is the reduction of the boundedness of the bilinear term $(f,g) \to \langle C_1(f), g\rangle$ to a corresponding property of a particular bilinear mapping, $(F,G) \to B(F,G)$, defined for (appropriate) holomorphic functions in the upper half-plane $\{z = x + iy, y > 0\}$ by

$$B(F,G)(x) = i\int_0^\infty F'(x+iy)G(x+iy)\, dy. \tag{4.5}$$

This B is a primitive version of a "para-product" (in this context, the justification for this terminology is the observation that $F(x)\cdot G(x) = B(F,G)(x) + B(G,F)(x)$). It is, in fact, not too difficult to see that $f \mapsto C_1(f)$ is bounded on $L^2(\mathbb{R}^1)$ if B satisfies the Hardy-space estimate

$$\|B(F,G)\|_{H^1} \le c\|F\|_{H^2}\|G\|_{H^2}. \tag{4.6}$$

The second major point in the proof is the assertion needed to establish (4.6). It is the converse part of the equivalence

$$\|S(F)\|_{L^1} \sim \|F\|_{H^1} \tag{4.7}$$

for the area integral S (which appeared in (1.4)).

The theorem of Calderón, and in particular the methods he used, inspired a number of significant developments in analysis. The first came because of the enigmatic nature of the proof: a deep L^2 theorem had been established by methods (using complex function theory) that did not seem susceptible to a general framework. In addition, the non-translation-invariance character of the operator C_1 made Plancherel's theorem of no use here. It seemed likely that a method of "almost-orthogonal" decomposition—pioneered by Cotlar for the classical Hilbert transform—might well succeed in this case also. This led to a reexamination of Cotlar's lemma (which had originally applied to the case of commuting self-adjoint operators). A general formulation was obtained as follows: Suppose that on a Hilbert space $T = \sum T_j$. Then

$$\|T\|^2 \leq \sum_k \sup_j \{\|T_j T_{j+k}^*\| + \|T_j^* T_{j+k}\|\}. \tag{4.8}$$

Despite the success in proving (4.8) this alone was not enough to re-prove Calderón's theorem. As understood later, the missing element was a certain cancellation property. Nevertheless, the general form of Cotlar's lemma, (4.8), quickly led to a number of highly useful applications, such as singular integrals on nilpotent groups (intertwining operators), pseudo-differential operators, etc.

Calderón's theorem also gave added impetus to the further evolution of the real-variable H^p theory. This came about because the equivalence (4.7) and its generalizations allowed one to show that the usual singular integrals (2.2) were also bounded on the Hardy space H^1 (and in fact on all H^p, $0 < p < \infty$). Taken together with earlier developments and some later ideas, the real-variable H^p theory reached its full flowering a few years later. One owes this long-term achievement to the work of G. Weiss, C. Fefferman, Burkholder, Gundy, and Coifman, among others.

It became clear after a time that understanding the commutator C_1 (and its "higher" analogues) was in fact connected with an old problem that had been an ultimate but unreached goal of the classical theory of singular integrals: the boundedness behavior of the Cauchy integral taken over curves with minimal regularity. The question involved can be formulated as follows: in the complex plane, for a contour γ and a function f defined on it form the Cauchy integral

$$F(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta,$$

with F holomorphic outside γ. Define the mapping $f \to C(f)$ by $C(f) = F_+ + F_-$, where $F_\pm$ are the limits of F on γ approached from either side. When γ is the unit circle, or real axis, then $f \to C(f)$ is essentially the Hilbert transform. Also, when γ has some regularity (e.g., γ is in $C^{(1+\epsilon)}$), the expected properties of C (i.e., L^2, L^p boundedness, etc.) are easily obtained from the Hilbert transform. The problem was what happened when, say, γ was less regular, and here the main issue that presented itself was the behavior of the Cauchy integral when γ was a Lipschitz curve.

If γ is a Lipschitz graph in the plane, $\gamma = \{x + iA(x), x \in R\}$, with $A' \in L^\infty$, then up to a multiplicative constant,

$$C(f)(x) = p.v. \int_{-\infty}^{\infty} \frac{1}{x - y + i(A(x) - A(y))} f(y) b(y)\, dy \tag{4.9}$$

where $b = 1 + iA'$. The formal expansion

$$\begin{aligned} &\frac{1}{x - y + i(A(x) - A(y))} \\ &\quad = \frac{1}{x-y} \cdot \sum_{k=0}^{\infty} (-i)^k \left(\frac{A(x) - A(y)}{x - y} \right)^k \end{aligned} \tag{4.10}$$

then makes clear that the fate of Cauchy integral C is inextricably bound up with that of the commutator C_1 and its higher analogues C_k given by

$$C_k f(x) = \frac{p.v.}{\pi} \int_{-\infty}^{\infty} \left(\frac{A(x) - A(y)}{x - y} \right)^k \frac{f(y)}{x - y} \, dy.$$

The further study of this problem was begun by Coifman and Meyer in the context of the commutators C_k, but the first breakthrough for the Cauchy integral was obtained by Calderón [1977] (using different methods) in the case where the norm $\|A'\|_{L^\infty}$ was small. His proof made decisive use of the complex-analytic setting of the problem. It proceeded by an ingenious deformation argument, leading to a nonlinear differential inequality; this nonlinearity accounted for the limitation of small norm for A' in the conclusion. But even with this limitation the conclusion obtained was stunning.

The crowning result came in 1982 when Coifman and Meyer, having enlisted the help of McIntosh and relying on some of their earlier ideas, together proved the desired result without limitation on the size of $\|A'\|_{L^\infty}$. The method they used was operator-theoretic, emphasizing the multilinear aspects of the C_k, and in distinction to Calderón's approach was not based on complex-analytic techniques.

The major achievement represented by the theory of the Cauchy integral led to a host of other results, either by a rather direct exploitation of the conclusions involved or by extensions of the techniques that were used. I will briefly discuss two of these developments.

The first was a complete analysis of the L^2 theory of "Calderón-Zygmund operators". By this terminology is meant operators of the form

$$T(f)(x) = \int_{\mathbb{R}^n} K(x, y) f(y)\, dy, \tag{4.11}$$

initially defined for test functions $f \in S$, with the kernel K a distribution. It is assumed that away from the diagonal, K agrees with a function that satisfies familiar estimates such as

$$\begin{aligned} |K(x, y)| &\leq A|x - y|^{-n}, \\ |\nabla_{x,y} K(x, y)| &\leq A|x - y|^{-n-1}. \end{aligned} \tag{4.12}$$

The main question that arises (and is suggested by the commutators C_k) is, what are the additional conditions that guarantee that T is a bounded operator on $L^2(\mathbb{R}^n)$ to itself? The answer, found by David and Journé (1984), is highly satisfying: a certain "weak boundedness" property, namely, $|(Tf, g)| \leq Ar^n$ wherever f and g are suitably normalized bump functions, supported in a ball of radius r; also, that both $T(1)$ and $T^*(1)$ belong to BMO. (Here BMO denotes the space of functions of bounded mean oscillation. This space first arose in partial differential equations as a useful substitute for the space L^∞ and later played a key role in H^p theory.) These conditions are easily seen to also be necessary.

The argument giving the sufficiency proceeded in decomposing the operator into a sum $T = T_1 + T_2$, where for T_1 the additional cancellation condition $T_1(1) = T_1^*(1) = 0$ held. As a consequence the method of almost-orthogonal decomposition, (4.8), could be successfully applied to T_1. The operator T_2 (for which L^2 boundedness was proved differently) was of para-product type, chosen so as to guarantee the needed cancellation property.

The conditions of the David-Journé theorem, while applying in principle to the Cauchy integral, are not easily verified in that case. However, a refinement (the "$T(b)$ theorem"), with $b = 1 + iA'$, was found by David, Journé, and Semmes, and this does the job needed.

A second area that was substantially influenced by the work of the Cauchy integral was that of second-order elliptic equations in the context of minimal regularity. Side by side with the consideration of the divergence-form operator L in (4.1) (where the emphasis is on the minimal smoothness of the coefficients), one was led to study also the potential theory of the Laplacian (where the emphasis was now on the minimal smoothness of the boundary). In the latter setting a natural assumption to make was that the boundary is Lipschitzian. In fact, by an appropriate Lipschitz mapping of domains, the situation of the Laplacian in a Lipschitz domain could be realized as a special case of the divergence-form operator (4.1) where the domain was smooth, say, a half-space.

The decisive application of the Cauchy integral to the potential theory of the Laplacian in a Lipschitz domain was in the study of the boundedness of the double-layer potential (and the normal derivative of the single-layer potential). These are $n-1$ dimensional operators, and they can be realized by applying the "method of rotations" to the one-dimensional operator (4.9). One should mention that another significant aspect of Laplacians on Lipschitz domains was the understanding brought to light by Dahlberg of the nature of harmonic measure and its relation to A_p weights. These two strands, initially independent, have been linked together, and with the aid of further ideas a rich theory has developed, owing to the added contributions of Jerison, Kenig, and others.

Finally, we return to the point where much of this began—the divergence-form equation (4.1). Here the analysis growing out of the Cauchy integral also had its effect. Here I will mention only the usefulness of multilinear analysis in the study of the case of "radially-independent" coefficients, also in the work on the Kato problem: the determination of the domain of $\sqrt{L}$ in the case where the coefficients can be complex-valued.

Some Perspectives on Singular Integrals: Past, Present, and Future

The modern theory of singular integrals, developed and nurtured by Calderón and Zygmund, has proved to be a very fruitful part of analysis. Beyond the achievements described above, a number of other directions have been cultivated with great success, with work being vigorously pursued up to this time. In addition, here several interesting open questions present themselves. I want to allude briefly to three of these directions and mention some of the problems that arise.

1. *Method of the Calderón-Zygmund lemma.* As is well known this method consists of decomposing an integrable function into its "good" and "bad" parts, the latter being supported on a disjoint union of cubes and having mean value zero on each cube. Together with an L^2 bound and estimates of the type (4.12), this leads ultimately to the weak-type (1,1) results, etc.

It was recognized quite early that this method allowed substantial extension. The generalizations that were undertaken were not so much pursued for their own sake, but rather were motivated in each case by the interest of the applications. Roughly, in order of appearance, here were some of the main instances:

(i) *The heat equation and other parabolic equations.* This began with the work of F. Jones (1964) for the heat equation, with the Calderón-Zygmund cubes replaced by rectangles whose dimensions reflected the homogeneity of the heat operator. The theory was extended by Fabes, Rivière, and Sadosky to encompass more general singular integrals respecting "nonisotropic" homogeneity in Euclidean spaces.

(ii) *Symmetric spaces and semisimple Lie groups.* To be succinct, the crucial point was the extension to the setting of nilpotent Lie groups with dilations ("homogeneous groups"), motivated by problems connected with Poisson integrals on symmetric spaces, and construction of intertwining operators.

(iii) *Several complex variables and subelliptic equations.* Here we return again to the source of singular integrals, complex analysis, but now in the setting of several variables. An important conclusion obtained was that for a broad class of domains in $\mathbb{C}^n$ the Cauchy-Szegö projection is a singular integral susceptible to the above methods. This was realized first for strongly pseudo-convex domains, next weakly pseudo-convex domains of finite type in $\mathbb{C}^2$, and more recently convex domains of finite type in $\mathbb{C}^n$.

Connected with this is the application of the above ideas to the $\bar{\partial}$-Neumann problem, and its boundary analogue for certain domains in $\mathbb{C}^n$, as well as the study solving operators for subelliptic problems, such as Kohn's Laplacian, Hörmander's sum of squares, etc. These matters also involved using ideas originating in the study of nilpotent groups as in (ii).

The three kinds of extensions mentioned above are prime examples of what one may call "one-parameter" analysis. This terminology refers to the fact that the cubes (or their containing balls) which occur in the standard $\mathbb{R}^n$ setup have been replaced by a suitable one-parameter family of generalized "balls" associated to each point. While the general one-parameter method clearly has wide applicability, it is not sufficient to resolve the following important question:

Problem. Describe the nature of the singular integral operators that are given by Cauchy-Szegö projection, as well as those that arise in connection with the solving operators for the $\bar{\partial}$ and $\bar{\partial}_b$ complexes for general smooth finite-type pseudo-convex domains in $\mathbb{C}^n$.

Some speculation about what may be involved in resolving this question can be found below.

2. *The method of rotations.* The method of rotations is both simple in its conception and far-reaching in its consequences. The initial idea was to take the one-dimensional Hilbert transform, induce it on a fixed (subgroup) $\mathbb{R}^1$ of $\mathbb{R}^n$, rotate this $\mathbb{R}^1$, and integrate in all directions, obtaining in this way the singular integral (2.2) with odd kernel, which can be written as

$$T_\Omega(f)(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^n} f(x-y)\,dy \tag{5.1}$$

where Ω is homogeneous of degree 0, integrable in the unit sphere, and odd.

In much the same way the general maximal operator

$$M(f)(x) = \sup_{r>0} \frac{1}{r^n} \left| \int_{|y|\le r} \Omega(y) f(x-y)\,dy \right| \tag{5.2}$$

arises from the one-dimensional Hardy-Littlewood maximal function.

This method worked very well for L^p estimates for $p > 1$, but not for L^1 (since the weak-type L^1 "norm" is not subadditive). The question of what happens for L^1 was left unresolved by Calderón and Zygmund. It is now to a large extent answered: we know that both (5.1) and (5.2) are indeed of weak-type (1,1) if Ω is in $L(\log L)$. This is the achievement of a number of mathematicians, in particular Christ and Rubio de Francia.

When the method of rotations is combined with the singular integrals for the heat equation (as in 1(i) above), one arrives at the "Hilbert transform on the parabola". Consideration of the Poisson integral on symmetric spaces leads one also to inquire about some analogous maximal functions associated to homogeneous curves. The initial major breakthroughs in this area of research were obtained by Nagel, Rivière, and Wainger. The subject has since developed into a rich and varied theory, beginning with its translation invariant setting on $\mathbb{R}^n$ (and its reliance on the Fourier transform), and then prompted by several complex variables, to a more general context connected with oscillatory integrals and nilpotent Lie groups, where it was rechristened as the theory of "singular Radon transforms".

A common unresolved enigma remains about these two areas that have sprung out of the method of rotations. This is a question that has intrigued workers in the field and whose solution, if positive, would be of great interest.

Problem

(a) Is there an L^1 theory for (5.1) and (5.2) if Ω is merely integrable?[4]

(b) Are the singular Radon transforms, and their corresponding maximal functions, of weak-type (1,1)?

3. *Product theory and multiparameter analysis.* To oversimplify matters, one can say that "product theory" is that part of harmonic analysis in $\mathbb{R}^n$ which is invariant with respect to the n-fold dilations: $x = (x_1, x_2, \dots, x_n) \to (\delta_1 x_1, \delta_2 x_2, \dots, \delta_n x_n)$, $\delta_j > 0$. Another way of putting it is that its initial concern is with operators that are essentially products of operators acting on each variable separately and then more generally with operators (and associated function spaces) that retain some of these characteristics. Related to this is the multiparameter theory, standing partway between the one-parameter theory discussed above and product theory: here the emphasis is on operators

[4] *For (5.1) we assume also that Ω is odd.*

which are "invariant" (or compatible with) specified subgroups of the group of n-parameter dilations.

The product theory of $\mathbb{R}^n$ began with Zygmund's study of the strong maximal function, continued with Marcinkiewicz's proof of his multiplier theorem, and has since branched out in a variety of directions where much interesting work has been done. Among the things achieved are an appropriate H^p and BMO theory and the many properties of product (and multiparameter) singular integrals that have came to light. This is due to the work of S. Y. Chang, R. Fefferman, and Journé, to mention only a few of the names.

Finally, I want to come to an extension of the product theory (more precisely, the induced "multiparameter analysis") in a direction that has particularly interested me recently. Here the point is that the underlying space is no longer Euclidean $\mathbb{R}^n$, but rather a nilpotent group or another appropriate generalization. On the basis of recent but limited experience, I would hazard the guess that multiparameter analysis in this setting could well turn out to be of great interest in questions related to several complex variables. A first vague hint that this may be so came with the realization that certain boundary operators arising from the $\bar{\partial}$-Neumann problem (in the model case corresponding to the Heisenberg group) are excellent examples of multiple-parameter singular integrals (in the work of Müller, Ricci, and the author (1995)). A second indication is the description of Cauchy-Szegö projections and solving operators for $\bar{\partial}_b$ for a wide class of quadratic surfaces of higher codimension in $\mathbb{C}^n$ in terms of appropriate quotients of products of Heisenberg groups (in yet unpublished joint work with Nagel and Ricci). And even more suggestive are recent calculations (made jointly with A. Nagel) for such operators in a number of pseudo-convex domains of finite type. All this leads one to hope that a suitable version of multiparameter analysis will provide the missing theory of singular integrals needed for a variety of questions in several complex variables. This is indeed an exciting prospect.

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