Niels Vigand Pedersen (1949–1996)

V. S. Varadarajan

Niels Vigand Pedersen, one of the leading mathematicians in the representation theory of solvable and nilpotent Lie groups, passed away on November 24, 1996, at the age of forty-seven. During his relatively short career he discovered many new themes and results in this area. His untimely death is a real tragedy for his mathematical colleagues and for mathematics in Denmark in general.

Let me begin with a few biographical details.[1] Niels was born on March 12, 1949, and grew up near the village of Algestrup, not very far from Copenhagen. He started studying mathematics and physics at the University of Copenhagen when he was nineteen. His brilliance showed up almost at once, and he obtained his master's degree in 1975 with a thesis written under the guidance of Esben Kehlet. He became interested in the theory of representations of nilpotent and solvable Lie groups when he came across the remarkable papers of Dixmier, Kirillov, Moore, Auslander, Kostant, and Pukanszky in this subject, and this interest became deeper through visits to the Universities of Pennsylvania and Berkeley and contacts with Pukanszky and Michele Vergne. These experiences were critical in his maturation and development. His interaction with Pukanszky was especially deep and fruitful. They became good friends and remained so throughout their lives.

Niels joined the University of Copenhagen as a lektor (associate professor) in 1986 and remained there till the end. He was a very inspiring lecturer, and his interest in physics and other sciences allowed him to reach out to a very wide circle of students. In recognition of his ability as well as his dedication to teaching, he was given the teaching award of the science faculty in 1994.

I shall discuss his scientific work a little later in more detail, but let me mention here that he had great interest in computer algebra, especially in the development of algorithms at a very sophisticated level. He really understood the intimate relationship between proofs and their implementability by algorithms. Actually, in a case that I knew personally, the computerized calculations led him to a beautiful conjecture that he then was able to prove, to his great satisfaction. He represented Denmark in Euromath for some time, and I am sure his knowledge and enthusiasm for the role of computers in modern mathematics was a source of great help and inspiration to his colleagues in that organization.

He became the chair of the mathematics institute at the University of Copenhagen in 1993. It was not a particularly easy time to be at the helm, since the institutes of mathematics, mathematical statistics, and actuarial mathematics were struggling to come to terms with a number of difficult issues. To try to resolve them, an advisory group was created to review the work of these institutes. I was invited to be a member of this group. I visited

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[1] *For these I am indebted to an obituary written by Professor Henrik Schlichtkrull of the University of Copenhagen.*

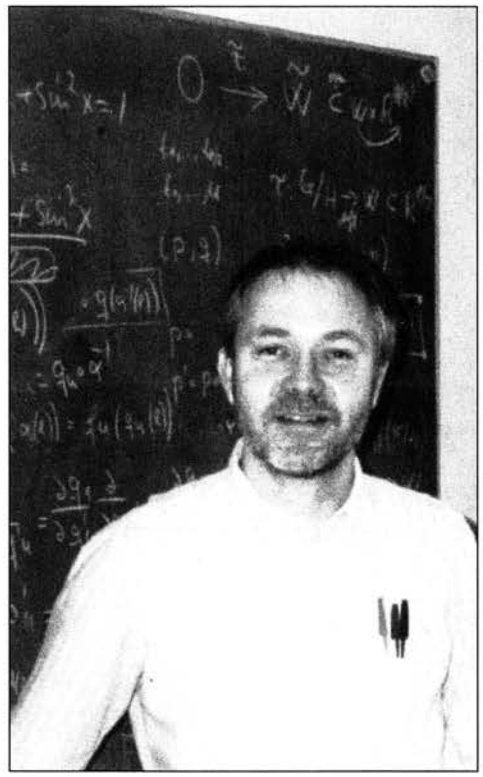
Copenhagen several times during the next two years. It was during these visits that I came to know Niels really well and to admire his science and personality. The task before our group was a complicated one, and the principal reason why we were able to come up with a unanimous report that was substantially acceptable to the institutes concerned was the collaboration of Niels. He was the epitome of courtesy and consideration for us, but at the same time his decisiveness and integrity were very instrumental in communicating to the group the views and interests of the institutes. His charm, sense of humor, and total lack of pretension made it very easy for us to do our job well.

The central themes of the theory of unitary representations of nilpotent and solvable Lie groups were well understood by the time Niels started his career, and it is a tribute to his penetration and persistence that he was able to obtain many beautiful results in this mature topic. Starting from around 1988, he discovered a more direct approach to many aspects of this theory that did not rely on induction but was based on ideas from quantization and symplectic geometry. In what follows I shall attempt in a limited way to give the reader a feeling for this aspect of his work. He carried out his program in the papers [1]–[5] and had obtained results for many additional papers when his life was cut short. I think these papers reveal him at his characteristic best. One of the guiding principles in the representation theory of Lie groups is that there is—or should be—a correspondence between the irreducible unitary representations of a Lie group and the orbits of the action of the Lie group on the dual of its Lie algebra. Actually this is one of the most fertile and profound principles that governs all of representation theory, not only of solvable groups, but also of semisimple groups. It relates representation theory to symplectic geometry, dynamical systems, and quantization, which are themes of unsurpassed beauty and scope. To mention only one aspect of this correspondence, the orbits, being symplectic manifolds, allow one to do Hamiltonian mechanics on them, and the unitary representation associated to an orbit may therefore be regarded as the quantization of the kinematics of this Hamiltonian manifold. Thus, in some way, the structure and geometry of the orbits give one a brief glimpse into the much richer world of irreducible unitary representations of the group.

Let G be a Lie group, connected and simply connected, and $\mathfrak{g}$ its Lie algebra. Then G has a natural action on $\mathfrak{g}$ through its adjoint representation $x \mapsto Ad(x)$, and hence a natural action on $\mathfrak{g}^*$, the dual of $\mathfrak{g}$. We write $(x, X) \mapsto xX$ and $(x, \ell) \mapsto x\ell$ for these actions. For any $\ell \in \mathfrak{g}^*$ let G_ℓ be its stabilizer in G, and let $\mathfrak{g}_\ell$ be the Lie algebra of G_ℓ. The orbit $O_\ell \simeq G/G_\ell$ of ℓ is then a regularly embedded submanifold and carries a natural G-invariant symplectic structure. In fact, the bilinear form

$$(X, Y) \mapsto \ell([X, Y]) \quad (X, Y \in \mathfrak{g})$$

has $\mathfrak{g}_\ell$ for its radical and so induces a symplectic form on $\mathfrak{g}/\mathfrak{g}_\ell \times \mathfrak{g}/\mathfrak{g}_\ell$, which can be identified with $T_\ell(O_\ell) \times T_\ell(O_\ell)$; here $T_\ell(O_\ell)$ is the tangent space to O_ℓ at ℓ. This form can then be transported to all the points of the orbit by the action of G, and the resulting 2-form ω_ℓ is smooth and closed. One can then in various ways construct a line bundle on a suitable quotient space of O_ℓ on which G acts, and the resulting action of G on the space of sections of this line bundle is a representation of G that is associated to the orbit O_ℓ. The theories of Kirillov, Moore-Auslander-Kostant, Pukanszky, and others all have the goal of describing these line bundles precisely and showing that all irreducible unitary representations of the group G may be obtained by this procedure when G is nilpotent or, more generally, solvable. This procedure, with its many generalizations and variants, is known as *geometric quantization*. In fact, if G is the three-dimensional Heisenberg group, it leads to the representation that is central to (one-dimensional) quantum mechanics, involving the famous operators q, p and the commutation rule



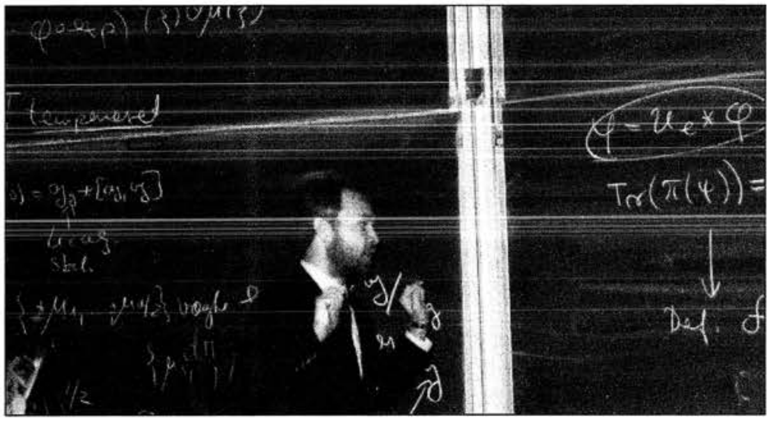
Niels Vigand Pedersen

$$pq - qp = \frac{\hbar}{i} I.$$

The Kirillov theory, which dealt with nilpotent groups and which served as the model for all subsequent work along this path, is especially elegant and beautiful. Given ℓ, one chooses a maximal isotropic subspace of $\mathfrak{g}$ for the form ω_ℓ that is also a subalgebra, say $\mathfrak{h}_\ell$; if H_ℓ is the analytic subgroup of G defined by $\mathfrak{h}_\ell$, then H_ℓ is closed and carries a unitary character (= one-dimensional representation) χ_ℓ such that

$$\chi_\ell(\exp X) = e^{i\ell(X)} \quad (X \in \mathfrak{h}_\ell).$$

The unitary representation of G obtained by inducing χ_ℓ from H_ℓ is then irreducible, and the class



of this representation depends only on the orbit O_ℓ and not on the choice of ℓ within the orbit or the choice of $\mathfrak{h}_\ell$ ($\mathfrak{h}_\ell$ is called a *real polarization at* ℓ). All irreducible unitary representations of G are obtained this way. With suitable modifications, some of which require new ideas going beyond the Kirillov theory, this scheme can be extended to all solvable G; the extension is particularly nice when G is in addition *exponential*, namely, when the exponential map is a diffeomorphism from $\mathfrak{g}$ to G. Then G_ℓ is connected and the above description extends with few changes (for an excellent description of these basic facts, see [7]). Now $O = O_\ell$ is foliated by the orbits $H_\ell \cdot \ell$, and the quotient space of the orbit by this foliation is the space $M = G/H_f$ where f is a fixed point of O_ℓ. M is the *configuration space*, the functions on M define the *position observables*, and the functions that are linear on the fibres of the map

$$O_\ell \longrightarrow M, \qquad sf \longrightarrow sH_f$$

are the *momenta*. In this context it is of course natural to ask whether one can construct *global coordinates*

$$q_1, q_2, \ldots, q_{d/2}, p_1, p_2, \ldots, p_{d/2} \qquad (d = \dim(O))$$

on O so that the symplectic form reduces to

$$dp_1 \wedge dq_1 + \cdots + dp_{d/2} \wedge dq_{d/2}.$$

Such coordinates are called *canonical*, and they always exist locally; this is the statement of the classical theorem of Darboux. But enlarging a canonical chart globally or even semiglobally is another matter. That this can be done for nilpotent groups was known to be true [6, 8] when Niels got interested in this problem. He wanted to develop a computer algorithm for writing down the canonical coordinates for all nilpotent groups and all coadjoint orbits. But, as he told me when I was visiting Copenhagen in 1988, he noticed that the algorithm also applied to many solvable groups and supplied *global canonical coordinates* on the coadjoint orbits [1]. He was then able to prove a very general theorem from which the existence of the canonical coordinates followed for all coadjoint orbits of all exponential solvable groups [2]. This is a perfect illustration of how his mind operated; there was total harmony between the computational and conceptual aspects of mathematics in his mind. A recent calculation of mine (with a student) in the Lie theoretic aspects of the dynamics of Riemann (= self-gravitating) ellipsoids and ellipses has shown that for the case of the rotating ellipses (which, for instance, model rotating galaxies), although the Lie group in question is not solvable, nevertheless there is a reduction to the solvable case, and by the theorem of Niels one can explicitly construct the canonical coordinates.

Niels deduced the existence of global canonical coordinates from a more general theorem that he was able to establish. He considers a connected G not necessarily solvable and a coadjoint orbit O for G. To simplify matters, I shall assume that G is exponential solvable. Thus for any point $\ell \in O$, G_ℓ, the stabilizer of ℓ in G, is connected with Lie algebra $\mathfrak{g}_\ell$, and $\ell \mapsto \mathfrak{h}_\ell$ is a covariant real polarization ($\ell \in O$). Clearly $\mathfrak{g}_\ell \subset \mathfrak{h}_\ell$. The analytic subgroup H_ℓ defined by $\mathfrak{h}_\ell$ is known to be always closed, the leaves of the polarization $\ell \mapsto \mathfrak{h}_\ell$ are the orbits $H_\ell \cdot \ell$, and the quotient of O by the foliation defined by the polarization is a manifold M identified with G/H where $H = H_f$ for a fixed $f \in O$. Under the condition

$$Hf = f + \mathfrak{h}^{\text{ann}}$$

where $\mathfrak{h}^{\text{ann}}$ is the annihilator of $\mathfrak{h}$, known as the *Pukanszky condition*, the orbits $H_\ell \cdot \ell$ are flat affine spaces. It is known that if G is exponential solvable and $f \in \mathfrak{g}^*$ is arbitrary, there is always a real polarization $\mathfrak{h}$ that satisfies the Pukanszky condition. The smooth locally defined functions on M, also viewed as smooth locally defined functions on O via the natural map $\pi : O \longrightarrow M$, form a sheaf $\mathcal{Q}$ on M, the *sheaf of the position observables*; for any two local sections q_1, q_2 of this sheaf we have

$$\{q_1 \circ \pi, q_2 \circ \pi\} = 0.$$

To introduce the momenta, we have to consider the larger sheaf $\mathcal{P}$ on M with the property that, for any open set $U \subset M$, $\mathcal{P}(U)$ is the space of smooth functions p on $\pi^{-1}(U)$ that normalize the space of position observables on $\pi^{-1}(U)$ (with respect to $\{\ ,\ \}$; i.e., for each $q \in \mathcal{Q}(U)$

$$\{p, q \circ \pi\} = q' \circ \pi \text{ for some } q' \in \mathcal{Q}(U)$$
$$(U \text{ open in } M).$$

Then

$$U \longmapsto \mathcal{P}(U)$$

is a sheaf of Lie algebras and at the same time a sheaf of modules for the sheaf of functions $\mathcal{Q}$. On the other hand, each $X \in \mathfrak{g}$ may be viewed as a linear function on $\mathfrak{g}^*$, and the restriction of this function to O is denoted by ψ^X; thus $\psi^X(\ell) = \langle X, \ell \rangle$. It is an easy exercise to show that these functions are global sections of the sheaf $\mathcal{P}$. Niels shows that

$\mathcal{P}$ is generated as a module by the $\psi^X (X \in \mathfrak{g})$ over $\mathcal{Q}$ and that $\mathcal{P}$, as a sheaf of modules over $\mathcal{Q}$, is *locally free*. There is another sheaf of Lie modules over M—in fact, an infinity of them—defined as follows. Consider the character (not necessarily unitary)

$$\chi : \exp X \longmapsto e^{\kappa f(X)} \qquad (\kappa \in \mathbf{C}).$$

Then χ gives rise to a homogeneous line bundle L_χ on M, and one has the sheaf of differential operators $\mathcal{D}_\chi$ of order ≤ 1 operating on the sections of L_χ, which is also a sheaf of Lie modules over $\mathcal{Q}$. Niels's fundamental theorem [2] asserts that the two sheaves $\mathcal{P}$ and $\mathcal{D}_\chi$ are naturally isomorphic as Lie modules; i.e., the isomorphism preserves the Lie algebra structure as well as the module structure of the sheaves. If we now assume that M has a global chart consisting of the coordinates t_u (this is the case when G is exponential solvable), Niels's theorem shows that there are functions q_u, p_u on O that correspond to $t_u, \partial/\partial t_u$ under his isomorphism. It is then immediate that the q_u, p_u are canonical coordinates for O. The isomorphism he constructs is sufficiently transparent that an algorithm can be developed for getting the q_u, p_u, which is what he did in [1] and [3].

Niels did not stop with this result. He wanted to know how the canonical coordinates he constructed changed when the orbit was changed; he also wanted to explore the relationship between his isomorphism and the unitary representation of G that one can construct on the space of sections of the line bundle L_χ when χ is unitary, i.e., κ is pure imaginary. The first question is answered in [3] in the case when the ambient group G is a connected, simply connected nilpotent Lie group. Then the group and all its coadjoint orbits have *polynomial structures* derived ultimately from the polynomial structure on the Lie algebra $\mathfrak{g}$ that results from identifying G with $\mathfrak{g}$ and viewing multiplication $G \times G \longrightarrow G$ as a map $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$. It then makes sense to ask whether a polynomial version of the main theorem of [2] can be proved. The paper [3] carries out this program. A consequence of this is the derivation of an earlier result of Vergne [8] that one can find rational functions $q_1, q_2, \ldots, q_e, p_1, p_2, \ldots, p_e$ that restrict on each orbit belonging to a specified stratum to canonical coordinates on that orbit. Needless to say, Niels's methods are algorithmic, and he indicates in [3] how these algorithms can be made very explicit.

The second question mentioned above is even more interesting. Its study led Niels to a direct construction of the irreducible representation associated to the orbit. We choose $\kappa = i$ and let the canonical coordinates on M be $q_i, p_i (1 \leq i \leq d/2)$. The functions in $\mathcal{P}(U)$ are precisely those that can be expressed (uniquely) in the form

$$\psi = \sum_{u=1}^{d/2} a_u p_u + a_0$$

where $a_0, a_1, \ldots, a_{d/2} \in C^\infty(U)$. In particular, for any $X \in \mathfrak{g}$ we can write the global expression

$$\psi^X = \sum_{u=1}^{d/2} a_{X,u} p_u + a_{X,0} \qquad (a_{X,u}, a_{X,0} \in C^\infty(M)).$$

Identifying M with $\mathbf{R}^{d/2}$ via the coordinates q_u, Niels obtains the following theorem: There exists a unique unitary representation π of G on $\mathcal{H} = L^2(\mathbf{R}^{d/2})$ such that $C_c^\infty(\mathbf{R}^{d/2}) \subset \mathcal{H}_\infty$ (the subscript ∞ refers to the subspace of smooth vectors) and such that for all $X \in \mathfrak{g}$ and all $f \in C_c^\infty(\mathbf{R}^{d/2})$,

$$d\pi(X)f = \sum_{u=1}^{d/2} a_{X,u} \frac{\partial f}{\partial t_u} + i a_{X,0} f + \frac{1}{2} \sum_{u=1}^{d/2} \frac{\partial a_{X,u}}{\partial t_u} f.$$

Niels was able to show, *without using induction*, that in the nilpotent case the image of the universal enveloping algebra under $d\pi$ is precisely the *Weyl algebra* of polynomial coefficient differential operators on $\mathbf{R}^{d/2}$.

The paper [4] is a change of direction in his work. In it Niels studies matrix coefficients and quantization for nilpotent groups. I would like to venture the suggestion that he wanted to develop a complete *operator Fourier transform theory* on nilpotent groups, and the results of [4] were to be the foundation for such a theory. If π is an irreducible unitary representation of a connected simply connected nilpotent group G, then for any function φ in the Schwartz space of G (defined by carrying over the Schwartz space of $\mathfrak{g}$ via the exponential map), $\pi(\varphi)$ is a smooth operator[2] and the map $\varphi \mapsto \pi(\varphi)$ is surjective. But this map is far from injective, and for an operator transform theory one has to somehow "invert" this map. By deep Fourier analysis Niels achieves this. He constructs a subspace $\mathfrak{g}_e$ of $\mathfrak{g}$ where e is a set of discrete parameters of the orbit corresponding to π and shows that the map $\varphi \mapsto \pi(\varphi)$ factors through the restriction

$$\varphi \circ \exp \Big|_{\mathfrak{g}_e},$$

inducing an isomorphism of the Schwartz space $S(\mathfrak{g}_e)$ with the nuclear space of smooth operators associated to π, and gets an explicit formula for the inverse of this map. He then sets up an isomorphism between the Schwartz space $S(O)$ of the orbit O associated to π and $S(\mathfrak{g}_e)$. In fact, for $a \in S(O)$ define $\check{a}$ as a function on $\mathfrak{g}$ by

[2] *A bounded operator A in the space of π is smooth if $d\pi(u)A$ and $Ad\pi(u)$ are bounded operators for all elements u of the universal enveloping algebra of $\mathfrak{g}$.*

$$\check{a}(X) = \int_O a(\ell) e^{-i\langle X,\ell\rangle} d\beta_O(\ell) \qquad (X \in \mathfrak{g})$$

where β_O is the canonical (suitably normalized) invariant measure on O. Then $\check{a}$ is in $C^\infty(\mathfrak{g})$, and its restriction to $\mathfrak{g}_e$ is in $S(\mathfrak{g}_e)$. The map

$$a \longmapsto \check{a}\Big|_{\mathfrak{g}_e}$$

is the topological linear isomorphism between $S(O)$ and $S(\mathfrak{g}_e)$ described above. Niels uses this isomorphism to set up a far-reaching generalization of the Weyl correspondence in quantum mechanics taking functions in $S(O)$ to smooth operators in the space of π. This correspondence extends to the duals of these spaces and reduces to the Weyl correspondence when the group is the Heisenberg group. The assignment

$$a \longmapsto Op_\pi(a)$$

that he constructs is a topological linear $*$-isomorphism of $S(O)$ with the nuclear space of smooth operators in the space of π that extends to an isomorphism Op_π between the dual spaces and has many elegant properties. Among these are

$$\mathrm{Tr}\,(Op_\pi(a)) = \int_O a(\ell) d\beta_O(\ell)$$

$$\mathrm{Tr}\,(Op_\pi(a) Op_\pi(b)) = \int_O a(\ell) b(\ell) d\beta_O(\ell)$$

$$Op_\pi(\psi^X) = d\pi(X).$$

It follows from these that the image of the polynomial functions on O is the image of the enveloping algebra under $d\pi$.

He mentions in the introduction to this paper that he had obtained extensions of many of these results to the case when G is solvable. In addition to these there were a number of additional results and computer algorithms that he had mentioned from time to time, both in his papers and to his friends in private conversation. At the time of this writing, attempts were being made to recover what he did and to put it in some definitive form.[3]

Niels was diagnosed in the fall of 1995 as having a malignant tumor in the brain. Surgery was performed, and he was able to spend a few weeks with his family, but he was stricken again and never recovered. In the year between the discovery of his illness and his passing away, he kept his composure and continued some teaching and thinking about mathematics almost to the very end. His courage in the face of such adversity was truly inspirational and revealed what a special person he was and how irreplaceable his loss was going to be. The end came on November 24, 1996, and left his family and friends in a state of utter helplessness and profound grief. But viewed from an ancient perspective of how a person's life ought to be spent in pursuit of deeper knowledge and harmony, the journey of Niels Vigand Pedersen was an exemplary one.

[3] *Information on downloading the programs and examples of solvable Lie groups will be available at* `http://www.math.purdue.edu/~rcp/pedersen.html`.

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