Sum of Squares (SOS) program for 3-variable symmetric polynomials of degree 6

github.com/duong-db/paramSOS

Inequalites and Solutions

Problem 1. Prove that for all real numbers a, b, c, we have:

$$\frac{a\cdot b}{a^2+b^2+3\,c^2}+\frac{b\cdot c}{b^2+c^2+3\,a^2}+\frac{c\cdot a}{c^2+a^2+3\,b^2}\le\frac{3}{5}$$

Equality holds at $a=b=c$, or $a=b=\frac{2}{3}c$, or any cyclic permutations.

(Pham Kim Hung)

Proof. After clearing denominators, the inequality is equivalent with:

> $numer\left(\frac{3}{5}-\frac{a\cdot b}{a^2+b^2+3\,c^2}-\frac{b\cdot c}{b^2+c^2+3\,a^2}-\frac{c\cdot a}{c^2+a^2+3\,b^2}\right)\ge 0$

$$0\le 9\,a^6-15\,a^5\,b-15\,a^5\,c+39\,a^4\,b^2-5\,a^4\,b\,c+39\,a^4\,c^2-50\,a^3\,b^3-20\,a^3\,b^2\,c-20\,a^3\,b\,c^2-50\,a^3\,c^3+39\,a^2\,b^4-20\,a^2\,b^3\,c+114\,a^2\,b^2\,c^2-20\,a^2\,b\,c^3+39\,a^2\,c^4-15\,a\,b^5-5\,a\,b^4\,c-20\,a\,b^3\,c^2-20\,a\,b^2\,c^3-5\,a\,b\,c^4-15\,a\,c^5+9\,b^6-15\,b^5\,c+39\,b^4\,c^2-50\,b^3\,c^3+39\,b^2\,c^4-15\,b\,c^5+9\,c^6 \quad (1)$$

We used the built-in Maple function **`numer()`** to get the numerator of a fractional expression. Define the polynomial $f(a,b,c)$ as follows:

> $f:=numer\left(\frac{3}{5}-\frac{a\cdot b}{a^2+b^2+3\,c^2}-\frac{b\cdot c}{b^2+c^2+3\,a^2}-\frac{c\cdot a}{c^2+a^2+3\,b^2}\right)$

$$f:=9\,a^6-15\,a^5\,b-15\,a^5\,c+39\,a^4\,b^2-5\,a^4\,b\,c+39\,a^4\,c^2-50\,a^3\,b^3-20\,a^3\,b^2\,c-20\,a^3\,b\,c^2-50\,a^3\,c^3+39\,a^2\,b^4-20\,a^2\,b^3\,c+114\,a^2\,b^2\,c^2-20\,a^2\,b\,c^3+39\,a^2\,c^4-15\,a\,b^5-5\,a\,b^4\,c-20\,a\,b^3\,c^2-20\,a\,b^2\,c^3-5\,a\,b\,c^4-15\,a\,c^5+9\,b^6-15\,b^5\,c+39\,b^4\,c^2-50\,b^3\,c^3+39\,b^2\,c^4-15\,b\,c^5+9\,c^6 \quad (2)$$

To initialize the program, read the script file **`paramSOS`** with your saved path in computer:

```
> read "F:/Maple/scripts/paramSOS.txt"
          "Sum of Squares (SOS) program for symmetric polynomials (a, b, c) of degree 6"
                        "[sigma, prod, getParam, getSOS]"
                                  "Example"
           "[Step 1]>  getParam(a^6 + b^6 + c^6 - a*b*c* (a^3 + b^3 + c^3))"
          "[Step 2]>  getSOS(a^6 + b^6 + c^6 - a*b*c* (a^3 + b^3 + c^3), -1) "       (3)
```

The program offers four functions as displayed, where ours two main ones are **`getParam()`** and **`getSOS()`**. First, you need to get the parameter for the use of **`getSOS()`**, which is nothing just the way the program works. Use **`getParam()`**:

> $getParam(f)$

$$\left\{x = \frac{4}{9}\right\}$$

$$\{x = 0.4444444444\} \tag{4}$$

The output returns 4/9, take this number and put it in **getSOS()** function:

> $getSOS\left(f, \frac{4}{9}\right)$

$$\frac{1}{90}(a-b)^2\left(9a^2+34ab-4ac+9b^2-4bc-29c^2\right)^2+\frac{1}{90}(b-c)^2\left(-29a^2-4ab-4ac+9b^2+34bc+9c^2\right)^2+\frac{1}{90}(c-a)^2\left(9a^2-4ab+34ac-29b^2-4bc+9c^2\right)^2+\frac{4}{5}\left(3a^3-4a^2b-4a^2c-4ab^2+15abc-4ac^2+3b^3-4b^2c-4bc^2+3c^3\right)^2+\frac{596}{15}(a-b)^2(b-c)^2(-c+a)^2 \tag{5}$$

This is our result. Let's check if this expression is true. Use the Maple's built-in function **factor()** for factorizing, note that the notation % reffers to the previous output.

> $factor(f - \%)$

$$0 \tag{6}$$

Which is indeed an indentity.

Thus f can be expressed as sum of squares, which means $f \geq 0$ for all real numbers a, b, c. The proof is completed.

Problem 2. Let a, b, c be real numbers such that $a+b+c=3$, prove that:

$$\frac{1}{a^2+7b^2+7c^2}+\frac{1}{b^2+7c^2+7a^2}+\frac{1}{c^2+7a^2+7b^2} \leq \frac{1}{5}$$

Equality holds when $a=b=c=1$, or $a=b=\frac{1}{2}$, $c=2$, or any cyclic permutations.

(Vasile Cirtoaje)

Proof. After homologizing, we need to prove:

$$\sum_{cyc}\frac{1}{a^2+7b^2+7c^2} \leq \frac{9}{5(a+b+c)^2}$$

Let's define the polynomial $f(a, b, c)$, we may use **sigma()** as the built-in function of the **paramSOS** script for the simplicity in writing.

> $\frac{9}{5(a+b+c)^2} - \mathrm{sigma}\left(\frac{1}{a^2+7b^2+7c^2}\right)$

$$\frac{9}{5(a+b+c)^2}-\frac{1}{a^2+7b^2+7c^2}-\frac{1}{7a^2+b^2+7c^2}-\frac{1}{7a^2+7b^2+c^2} \tag{7}$$

> $f := numer(\%)$

$$f := 126a^6-630a^5b-630a^5c+2466a^4b^2-630a^4bc+2466a^4c^2-1620a^3b^3-1620a^3b^2c-1620a^3bc^2-1620a^3c^3+2466a^2b^4-1620a^2b^3c+5076a^2b^2c^2 \tag{8}$$

$-1620\,a^2\,b\,c^3+2466\,a^2\,c^4-630\,a\,b^5-630\,a\,b^4\,c-1620\,a\,b^3\,c^2-1620\,a\,b^2\,c^3$
$-630\,a\,b\,c^4-630\,a\,c^5+126\,b^6-630\,b^5\,c+2466\,b^4\,c^2-1620\,b^3\,c^3+2466\,b^2\,c^4$
$-630\,b\,c^5+126\,c^6$

As decribed in the previous problem, a SOS expression of f can be obtained as follows:

> $getParam(f)$

$$\left\{x=\frac{22}{7}\right\}$$
$$\{x=3.142857143\} \qquad \textbf{(9)}$$

> $getSOS\left(f, \frac{22}{7}\right)$

$$\frac{3}{5}\,(a-b)^2\,(7\,a^2+2\,a\,b-22\,a\,c+7\,b^2-22\,b\,c-17\,c^2)^2+\frac{3}{5}\,(b-c)^2\,(-17\,a^2-22\,a\,b-22\,a\,c+7\,b^2+2\,b\,c+7\,c^2)^2+\frac{3}{5}\,(c-a)^2\,(7\,a^2-22\,a\,b+2\,a\,c-17\,b^2-22\,b\,c+7\,c^2)^2+\frac{336}{5}\,(a^3-3\,a^2\,b-3\,a^2\,c-3\,a\,b^2+15\,a\,b\,c-3\,a\,c^2+b^3-3\,b^2\,c-3\,b\,c^2+c^3)^2+\frac{9432}{5}\,(a-b)^2\,(b-c)^2\,(-c+a)^2 \qquad \textbf{(10)}$$

Thus f is non-negative for all real numbers a, b, c. We are done.

Problem 3. Let a, b, c be reals numbers, prove that:

$$a^6+b^6+c^6\geq a\cdot b\cdot c\cdot(a^3+b^3+c^3)$$

> $f:=a^6+b^6+c^6-a\cdot b\cdot c\cdot(a^3+b^3+c^3)$

$$f:=a^6+b^6+c^6-a\,b\,c\,(a^3+b^3+c^3) \qquad \textbf{(11)}$$

> $getParam(f)$

$$\{x\leq RootOf(19\,\_Z^3+33\,\_Z^2+3\,\_Z-1,\ -0.25\,..\,-0.1875),\ RootOf(19\,\_Z^3+33\,\_Z^2+3\,\_Z-1,\ -1.625\,..\,-1.562)\leq x\}$$
$$\{-1.619258506\leq x,\ x\leq -0.2484229730\} \qquad \textbf{(12)}$$

Note that sometimes **`getParam()`** function would give a range where our parameter could belong to. Just choose one satisfied. It's quite often that polynomial f may have more than one SOS expression.

> $getSOS(f,-1)$

$$\frac{1}{72}\,(a-b)^2\,(3\,a^2+7\,a\,b+3\,a\,c+3\,b^2+3\,b\,c-c^2)^2+\frac{1}{72}\,(b-c)^2\,(-a^2+3\,a\,b+3\,a\,c+3\,b^2+7\,b\,c+3\,c^2)^2+\frac{1}{72}\,(c-a)^2\,(3\,a^2+3\,a\,b+7\,a\,c-b^2+3\,b\,c+3\,c^2)^2+\frac{1}{432}\,(18\,a^3-7\,a^2\,b-7\,a^2\,c-7\,a\,b^2-12\,a\,b\,c-7\,a\,c^2+18\,b^3-7\,b^2\,c-7\,b\,c^2+18\,c^3)^2+\frac{5}{108}\,(a^2\,b+a^2\,c+a\,b^2-6\,a\,b\,c+a\,c^2+b^2\,c+b\,c^2)^2+\frac{23}{48}\,(a-b)^2\,(b-c)^2\,(-c+a)^2 \qquad \textbf{(13)}$$

> $getSOS\left(f, -\frac{1}{2}\right)$

$$\frac{1}{18}(a-b)^2\left(2a^2+3ab+ac+2b^2+bc\right)^2+\frac{1}{18}(b-c)^2\left(ab+ac+2b^2+3bc+2c^2\right)^2+\frac{1}{18}(c-a)^2\left(2a^2+ab+3ac+bc+2c^2\right)^2+\frac{1}{45}\left(5a^3-2a^2b-2a^2c-2ab^2-3abc-2ac^2+5b^3-2b^2c-2bc^2+5c^3\right)^2+\frac{1}{20}\left(a^2b+a^2c+ab^2-6abc+ac^2+b^2c+bc^2\right)^2+\frac{5}{12}(a-b)^2(b-c)^2(-c+a)^2 \quad \textbf{(14)}$$

Problem 4. Represent the following polynomial as sum of squares:

$$f(a,b,c)=\prod_{cyc}\left(a^2+b^2\right)-8\cdot a^2\cdot b^2\cdot c^2$$

Use the built-in function **`prod()`** of the **`paramSOS`** script to present the cyclic product:

> $f := prod\left(a^2+b^2\right)-8\cdot a^2\cdot b^2\cdot c^2$

$$f := \left(b^2+c^2\right)\left(a^2+c^2\right)\left(a^2+b^2\right)-8a^2b^2c^2 \quad \textbf{(15)}$$

> $getParam(f)$

$$\{x \in R\} \quad \textbf{(16)}$$

In this case, the value of the parameter x does not matter. Let's take $x=1$, for example:

> $getSOS(f, 1)$

$$\frac{1}{9}(a-b)^2\left(ab+2ac+2bc+c^2\right)^2+\frac{1}{9}(b-c)^2\left(a^2+2ab+2ac+bc\right)^2+\frac{1}{9}(c-a)^2\left(2ab+ac+b^2+2bc\right)^2+\frac{1}{3}(a-b)^2(b-c)^2(-c+a)^2 \quad \textbf{(17)}$$

Problem 5. Let k be a real numbers such that $k>-2$ and $k^4+12k^3-8k^2+48k-80\le 0$. Prove that for all three real numbers a, b, c, we have:

$$\sum_{cyc}\frac{8a^2+k\cdot(4-k)\cdot b\cdot c}{b^2+k\cdot b\cdot c+c^2}\ge\frac{3\cdot\left(8+4k-k^2\right)}{k+2}$$

(Vasile Cirtoaje)

> $f := numer\left(\text{sigma}\left(\frac{8a^2+k\cdot(4-k)\cdot b\cdot c}{b^2+k\cdot b\cdot c+c^2}\right)-\frac{3\cdot\left(8+4k-k^2\right)}{k+2}\right)$

$$f := a^3b^2ck^4+a^3bc^2k^4+a^2b^3ck^4-6a^2b^2c^2k^4+a^2bc^3k^4+ab^3c^2k^4+ab^2c^3k^4+10a^4bck^3+2a^3b^3k^3-6a^3b^2ck^3-6a^3bc^2k^3+2a^3c^3k^3-6a^2b^3ck^3-6a^2bc^3k^3+10ab^4ck^3-6ab^3c^2k^3-6ab^2c^3k^3+10abc^4k^3+2b^3c^3k^3+8a^5bk^2+8a^5ck^2+3a^4b^2k^2+6a^4bck^2+3a^4c^2k^2-10a^3b^3k^2-10a^3b^2ck^2-10a^3bc^2k^2-10a^3c^3k^2+3a^2b^4k^2-10a^2b^3ck^2+6a^2b^2c^2k^2-10a^2bc^3k^2+3a^2c^4k^2+8ab^5k^2+6ab^4ck^2-10ab^3c^2k^2-10ab^2c^3k^2+6abc^4k^2+8ac^5k^2+8b^5ck^2+3b^4c^2k^2-10b^3c^3k^2+3b^2c^4k^2+8bc^5k^2+8a^6k+16a^5bk+16a^5ck \quad \textbf{(18)}$$

$$- 4 a^4 b^2 k - 16 a^4 b c k - 4 a^4 c^2 k - 16 a^3 b^3 k - 16 a^3 c^3 k - 4 a^2 b^4 k - 4 a^2 c^4 k + 16 a b^5 k - 16 a b^4 c k - 16 a b c^4 k + 16 a c^5 k + 8 b^6 k + 16 b^5 c k - 4 b^4 c^2 k - 16 b^3 c^3 k - 4 b^2 c^4 k + 16 b c^5 k + 8 c^6 k + 16 a^6 - 8 a^4 b^2 - 8 a^4 c^2 - 8 a^2 b^4 - 8 a^2 c^4 + 16 b^6 - 8 b^4 c^2 - 8 b^2 c^4 + 16 c^6$$

> $getParam(f)$

(19)

The output returns nothing, since **`getParam()`** can only deal with a speacific value of k. Let's check the output of **`getSOS()`** function with a parameter x in general:

> $getSOS(f, x)$

$$\frac{1}{36} \frac{1}{(x-1)^2} \Big((k+2)^2 (a-b)^2 \big(-a b k x - 2 a c k x - 2 b c k x - c^2 k x + 2 a^2 k - a b k - 16 a b x - 8 a c x + 2 b^2 k - 8 b c x + 3 c^2 k + 8 c^2 x + 8 a^2 + 8 a b + 8 b^2 \big)^2 \Big) + \frac{1}{36} \frac{1}{(x-1)^2} \Big((k+2)^2 (b-c)^2 \big(-a^2 k x - 2 a b k x - 2 a c k x - b c k x + 3 a^2 k + 8 a^2 x - 8 a b x - 8 a c x + 2 b^2 k - b c k - 16 b c x + 2 c^2 k + 8 b^2 + 8 b c + 8 c^2 \big)^2 \Big) + \frac{1}{36} \frac{1}{(x-1)^2} \Big((k+2)^2 (c-a)^2 \big(-2 a b k x - a c k x - b^2 k x - 2 b c k x + 2 a^2 k - 8 a b x - a c k - 16 a c x + 3 b^2 k + 8 b^2 x - 8 b c x + 2 c^2 k + 8 a^2 + 8 a c + 8 c^2 \big)^2 \Big) - \frac{1}{72} \Big(\big(-3 a^2 b k^3 x - 3 a^2 c k^3 x - 3 a b^2 k^3 x + 18 a b c k^3 x - 3 a c^2 k^3 x - 3 b^2 c k^3 x - 3 b c^2 k^3 x + 4 a^3 k^3 - 3 a^2 b k^3 - 42 a^2 b k^2 x - 72 a^2 b k x^2 - 3 a^2 c k^3 - 42 a^2 c k^2 x - 72 a^2 c k x^2 - 3 a b^2 k^3 - 42 a b^2 k^2 x - 72 a b^2 k x^2 + 6 a b c k^3 + 252 a b c k^2 x + 432 a b c k x^2 - 3 a c^2 k^3 - 42 a c^2 k^2 x - 72 a c^2 k x^2 + 4 b^3 k^3 - 3 b^2 c k^3 - 42 b^2 c k^2 x - 72 b^2 c k x^2 - 3 b c^2 k^3 - 42 b c^2 k^2 x - 72 b c^2 k x^2 + 4 c^3 k^3 + 40 a^3 k^2 - 144 a^3 x^2 - 18 a^2 b k^2 - 24 a^2 b k x - 18 a^2 c k^2 - 24 a^2 c k x - 18 a b^2 k^2 - 24 a b^2 k x - 12 a b c k^2 + 144 a b c k x + 432 a b c x^2 - 18 a c^2 k^2 - 24 a c^2 k x + 40 b^3 k^2 - 144 b^3 x^2 - 18 b^2 c k^2 - 24 b^2 c k x - 18 b c^2 k^2 - 24 b c^2 k x + 40 c^3 k^2 - 144 c^3 x^2 + 128 a^3 k + 288 a^3 x - 96 a^2 b k - 192 a^2 b x - 96 a^2 c k - 192 a^2 c x - 96 a b^2 k - 192 a b^2 x + 192 a b c k + 288 a b c x - 96 a c^2 k - 192 a c^2 x + 128 b^3 k + 288 b^3 x - 96 b^2 c k - 192 b^2 c x - 96 b c^2 k - 192 b c^2 x + 128 c^3 k + 288 c^3 x - 16 a^3 + 48 a b c - 16 b^3 - 16 c^3 \big)^2 (k+2) \Big) \Big/ \Big(\big(k^3 + 10 k^2 - 36 x^2 + 32 k + 72 x - 4 \big) (x-1)^2 \Big) + \frac{1}{6} \frac{1}{\big(k^3 + 10 k^2 - 36 x^2 + 32 k + 72 x - 4 \big) (x-1)} \Big((k+2)^2 \big(2 k^3 + 3 k^2 x + 17 k^2 + 36 k x + 28 k + 72 x - 8 \big) (1 + k + 3 x)^2 \big(a^2 b + a^2 c + a b^2 - 6 a b c + a c^2 + b^2 c \tag{20}$$

$+ b c^2)^2\Big) - \frac{1}{24} \frac{1}{(x-1)^2}\Big((-c+a)^2 (b-c)^2 (a-b)^2 \big(k^4 x^2 + 10 k^4 x + 48 k^3 x^2 + 25 k^4 + 192 k^3 x + 388 k^2 x^2 + 192 k^3 + 1000 k^2 x + 1200 k x^2 + 484 k^2 + 1632 k x + 928 x^2 + 624 k + 1216 x + 160\big)\Big)$

One may notice that if we take $x = -\frac{k+1}{3}$, the coefficent of term $\left(a^2 b + a^2 c + a b^2 - 6 a b c + a c^2 + b^2 c + b c^2\right)^2$ will be zero.

> $getSOS\left(f, -\frac{k+1}{3}\right)$

$$\frac{1}{36} (k+2)^2 (a-b)^2 \left(a b k + 2 a c k + 2 b c k + c^2 k + 6 a^2 + 10 a b + 2 a c + 6 b^2 + 2 b c - 2 c^2\right)^2 + \frac{1}{36} (k+2)^2 (b-c)^2 \left(a^2 k + 2 a b k + 2 a c k + b c k - 2 a^2 + 2 a b + 2 a c + 6 b^2 + 10 b c + 6 c^2\right)^2 + \frac{1}{36} (k+2)^2 (c-a)^2 \left(2 a b k + a c k + b^2 k + 2 b c k + 6 a^2 + 2 a b + 10 a c - 2 b^2 + 2 b c + 6 c^2\right)^2 - \frac{1}{8} (k-2) (k+2) \left(a^2 b k + a^2 c k + a b^2 k - 6 a b c k + a c^2 k + b^2 c k + b c^2 k + 4 a^3 - 2 a^2 b - 2 a^2 c - 2 a b^2 - 2 a c^2 + 4 b^3 - 2 b^2 c - 2 b c^2 + 4 c^3\right)^2 - \frac{1}{24} \left(k^4 + 12 k^3 - 8 k^2 + 48 k - 80\right) (a-b)^2 (b-c)^2 (-c+a)^2 \quad \textbf{(21)}$$

Clearly, the above expression is non-negative under the conditions $k > -2$ and $k^4 + 12 k^3 - 8 k^2 + 48 k - 80 \leq 0$. Note that if $k \geq 2$ then $k^4 + 12 k^3 - 8 k^2 + 48 k - 80 = (k-2)\left(k^3 + 14 k^2 + 20 k + 88\right) + 96 > 0$, which leads to a contradiction. The proof is completed.

Problem 6. Let a, b, c be reals numbers, prove that:

$$\sum_{cyc} \frac{(a+b)\cdot(a+c)}{a^2 + 4 b^2 + 4 c^2} \leq \frac{4}{3}$$

Equality holds at $a = b = c$, or $a = b = \frac{7}{2} c$, or any cyclic permutations.

(Vasile Cirtoaje)

> $f := \frac{4}{3} - \text{sigma}\left(\frac{(a+b)\cdot(a+c)}{a^2 + 4 b^2 + 4 c^2}\right)$

$$f := \frac{4}{3} - \frac{(a+b)(c+a)}{a^2 + 4 b^2 + 4 c^2} - \frac{(b+c)(a+b)}{4 a^2 + b^2 + 4 c^2} - \frac{(c+a)(b+c)}{4 a^2 + 4 b^2 + c^2} \quad \textbf{(22)}$$

> $getParam(numer(f))$

$$\left\{x = \frac{23}{8}\right\}$$

$$\{x = 2.875000000\} \quad \textbf{(23)}$$

> $getSOS\left(numer(f), \frac{23}{8}\right)$

$$\frac{1}{18}(a-b)^2(8a^2+5ab-23ac+8b^2-23bc-20c^2)^2+\frac{1}{18}(b-c)^2(-20a^2-23ab-23ac+8b^2+5bc+8c^2)^2+\frac{1}{18}(c-a)^2(8a^2-23ab+5ac-20b^2-23bc+8c^2)^2+\frac{5}{9}(4a^3-11a^2b-11a^2c-11ab^2+54abc-11ac^2+4b^3-11b^2c-11bc^2+4c^3)^2+\frac{626}{3}(a-b)^2(b-c)^2(-c+a)^2 \quad \textbf{(24)}$$

Problem 7. Let a, b, c be reals numbers, prove that:

$$\sum_{cyc}\frac{(a+b)\cdot(a+c)}{7a^2+b^2+c^2} \le \frac{4}{3}$$

Equality holds at $a=b=c$, or $a=b=\frac{4}{5}c$, or any cyclic permutations.

(Vasile Cirtoaje)

> $f := \frac{4}{3} - \text{sigma}\left(\frac{(a+b)\cdot(a+c)}{7a^2+b^2+c^2}\right)$

$$f := \frac{4}{3}-\frac{(a+b)(c+a)}{7a^2+b^2+c^2}-\frac{(b+c)(a+b)}{a^2+7b^2+c^2}-\frac{(c+a)(b+c)}{a^2+b^2+7c^2} \quad \textbf{(25)}$$

> $getParam(numer(f))$

$$\left\{x=\frac{2}{5}\right\}$$
$$\{x=0.4000000000\} \quad \textbf{(26)}$$

> $getSOS\left(numer(f), \frac{2}{5}\right)$

$$\frac{1}{18}(a-b)^2(5a^2+38ab-2ac+5b^2-2bc-35c^2)^2+\frac{1}{18}(b-c)^2(-35a^2-2ab-2ac+5b^2+38bc+5c^2)^2+\frac{1}{18}(c-a)^2(5a^2-2ab+38ac-35b^2-2bc+5c^2)^2+\frac{8}{9}(5a^3-7a^2b-7a^2c-7ab^2+27abc-7ac^2+5b^3-7b^2c-7bc^2+5c^3)^2+\frac{332}{3}(a-b)^2(b-c)^2(-c+a)^2 \quad \textbf{(27)}$$

Problem 8. Let a, b, c be reals numbers such that $a+b+c=3$, prove that:

$$\sum_{cyc}\frac{1}{8+5(b^2+c^2)} \le \frac{1}{6}$$

Equality holds at $a=b=c=1$, or $a=b=\frac{1}{5}, c=\frac{13}{5}$, or any cyclic permutations.

(Vasile Cirtoaje)

> $f := \frac{1}{6}$ - sigma$\left(\frac{1}{8+\frac{45(b^2+c^2)}{(a+b+c)^2}}\right)$

$$f := \frac{1}{6} - \frac{1}{8+\frac{45\,(b^2+c^2)}{(a+b+c)^2}} - \frac{1}{8+\frac{45\,(a^2+c^2)}{(a+b+c)^2}} - \frac{1}{8+\frac{45\,(a^2+b^2)}{(a+b+c)^2}} \quad \textbf{(28)}$$

> $getParam(numer(f))$

$$\left\{x = \frac{239}{23}\right\}$$

$$\{x = 10.39130435\} \quad \textbf{(29)}$$

> $getSOS\left(numer(f), \frac{239}{23}\right)$

$$\frac{5}{18}\,(a-b)^2\,(23\,a^2-29\,a\,b-239\,a\,c+23\,b^2-239\,b\,c-187\,c^2)^2+\frac{5}{18}\,(b-c)^2\,(-187\,a^2 -239\,a\,b-239\,a\,c+23\,b^2-29\,b\,c+23\,c^2)^2+\frac{5}{18}\,(c-a)^2\,(23\,a^2-239\,a\,b -29\,a\,c-187\,b^2-239\,b\,c+23\,c^2)^2+\frac{2125}{36}\,(2\,a^3-15\,a^2\,b-15\,a^2\,c-15\,a\,b^2 +84\,a\,b\,c-15\,a\,c^2+2\,b^3-15\,b^2\,c-15\,b\,c^2+2\,c^3)^2+\frac{172875}{4}\,(a-b)^2\,(b -c)^2\,(-c+a)^2 \quad \textbf{(30)}$$

Problem 9. Let a, b, c be reals numbers, prove that:

$$\sum_{cyc} \frac{a\cdot b+a\cdot c-b\cdot c}{a^2+3\,b^2+3\,c^2} \leq \frac{3}{7}$$

Equality holds at $a=b=c$, or $a=b=\frac{4}{3}c$, or any cyclic permutations.

(Vasile Cirtoaje)

> $f := \frac{3}{7}$ - sigma$\left(\frac{a\cdot b+a\cdot c-b\cdot c}{a^2+3\,b^2+3\,c^2}\right)$

$$f := \frac{3}{7} - \frac{a\,b+a\,c-b\,c}{a^2+3\,b^2+3\,c^2} - \frac{a\,b-a\,c+b\,c}{3\,a^2+b^2+3\,c^2} - \frac{-a\,b+a\,c+b\,c}{3\,a^2+3\,b^2+c^2} \quad \textbf{(31)}$$

> $getParam(numer(f))$

$$\left\{x = \frac{34}{27}\right\}$$

$$\{x = 1.259259259\} \quad \textbf{(32)}$$

> $getSOS\left(numer(f), \frac{34}{27}\right)$

$$\frac{1}{126}\,(a-b)^2\,(27\,a^2+34\,a\,b-34\,a\,c+27\,b^2-34\,b\,c-41\,c^2)^2+\frac{1}{126}\,(b-c)^2\,(-41\,a^2 \quad \textbf{(33)}$$

$$-34\,a\,b-34\,a\,c+27\,b^2+34\,b\,c+27\,c^2\big)^2+\frac{1}{126}\,(c-a)^2\,\big(27\,a^2-34\,a\,b$$
$$+34\,a\,c-41\,b^2-34\,b\,c+27\,c^2\big)^2+\frac{12}{7}\,\big(3\,a^3-5\,a^2\,b-5\,a^2\,c-5\,a\,b^2+21\,a\,b\,c$$
$$-5\,a\,c^2+3\,b^3-5\,b^2\,c-5\,b\,c^2+3\,c^3\big)^2+\frac{2588}{21}\,(a-b)^2\,(b-c)^2\,(-c+a)^2$$

Problem 10. Let a, b, c be reals numbers, prove that:

$$\sum_{cyc}\frac{a\cdot(a-4\,b-4\,c)}{b^2+c^2}\geq -\frac{21}{2}$$

Equality holds at $a=b=c$, or $a=b=3\,c$, or any cyclic permutations.

(Grotex@AoPS)

> $f:=\mathrm{sigma}\left(\frac{a\cdot(a-4\,b-4\,c)}{b^2+c^2}\right)+\frac{21}{2}$

$$f:=\frac{a\,(a-4\,b-4\,c)}{b^2+c^2}+\frac{b\,(b-4\,c-4\,a)}{a^2+c^2}+\frac{c\,(c-4\,a-4\,b)}{a^2+b^2}+\frac{21}{2} \quad \textbf{(34)}$$

> $getParam(numer(f))$

$$\left\{x=\frac{7}{3}\right\}$$
$$\{x=2.333333333\} \quad \textbf{(35)}$$

> $getSOS\left(numer(f),\frac{7}{3}\right)$

$$\frac{1}{18}\,(a-b)^2\,\big(3\,a^2+a\,b-7\,a\,c+3\,b^2-7\,b\,c-5\,c^2\big)^2+\frac{1}{18}\,(b-c)^2\,\big(-5\,a^2-7\,a\,b \quad \textbf{(36)}$$
$$-7\,a\,c+3\,b^2+b\,c+3\,c^2\big)^2+\frac{1}{18}\,(c-a)^2\,\big(3\,a^2-7\,a\,b+a\,c-5\,b^2-7\,b\,c+3\,c^2\big)^2$$
$$+\frac{1}{4}\,\big(2\,a^3-5\,a^2\,b-5\,a^2\,c-5\,a\,b^2+24\,a\,b\,c-5\,a\,c^2+2\,b^3-5\,b^2\,c-5\,b\,c^2+2\,c^3\big)^2$$
$$+\frac{221}{12}\,(a-b)^2\,(b-c)^2\,(-c+a)^2$$

Problem 11. Let a, b, c be reals numbers, prove that:

$$\sum_{cyc}\frac{a\cdot(a-4\,b-4\,c)}{9\,a^2+b^2+c^2}\geq -\frac{21}{11}$$

Equality holds at $a=b=c$, or $a=b=\frac{6}{5}\,c$, or any cyclic permutations.

(Grotex@AoPS)

> $f:=\mathrm{sigma}\left(\frac{a\cdot(a-4\,b-4\,c)}{9\,a^2+b^2+c^2}\right)+\frac{21}{11}$

$$f:=\frac{a\,(a-4\,b-4\,c)}{9\,a^2+b^2+c^2}+\frac{b\,(b-4\,c-4\,a)}{a^2+9\,b^2+c^2}+\frac{c\,(c-4\,a-4\,b)}{a^2+b^2+9\,c^2}+\frac{21}{11} \quad \textbf{(37)}$$

> $getParam(numer(f))$

$$\left\{x = \frac{26}{15}\right\}$$

$$\{x = 1.733333333\} \tag{38}$$

> $getSOS\left(numer(f), \frac{26}{15}\right)$

$$\frac{4}{99}\,(a-b)^2\,(15\,a^2+158\,a\,b-26\,a\,c+15\,b^2-26\,b\,c-169\,c^2)^2+\frac{4}{99}\,(b-c)^2\,(-169\,a^2-26\,a\,b-26\,a\,c+15\,b^2+158\,b\,c+15\,c^2)^2+\frac{4}{99}\,(c-a)^2\,(15\,a^2-26\,a\,b+158\,a\,c-169\,b^2-26\,b\,c+15\,c^2)^2+\frac{80}{11}\,(5\,a^3-8\,a^2\,b-8\,a^2\,c-8\,a\,b^2+33\,a\,b\,c-8\,a\,c^2+5\,b^3-8\,b^2\,c-8\,b\,c^2+5\,c^3)^2+\frac{20032}{33}\,(a-b)^2\,(b-c)^2\,(-c+a)^2 \tag{39}$$

Problem 12. Let a, b, c be reals numbers, prove that:

$$\sum_{cyc}\frac{(3\,a-4\,b)\cdot(3\,a-4\,c)}{b^2+c^2}\geq\frac{3}{2}$$

Equality holds at $a=b=c$, or $a=b=\frac{c}{3}$, or any cyclic permutations.

(Grotex@AoPS)

> $f := \text{sigma}\left(\frac{(3\,a-4\,b)\cdot(3\,a-4\,c)}{b^2+c^2}\right)-\frac{3}{2}$

$$f := \frac{(3\,a-4\,b)\,(3\,a-4\,c)}{b^2+c^2}+\frac{(3\,b-4\,c)\,(3\,b-4\,a)}{a^2+c^2}+\frac{(3\,c-4\,a)\,(3\,c-4\,b)}{a^2+b^2}-\frac{3}{2} \tag{40}$$

> $getParam(numer(f))$

$$\left\{x = \frac{5}{9}\right\}$$

$$\{x = 0.5555555556\} \tag{41}$$

> $getSOS\left(numer(f), \frac{5}{9}\right)$

$$\frac{1}{18}\,(a-b)^2\,(9\,a^2+11\,a\,b-5\,a\,c+9\,b^2-5\,b\,c-7\,c^2)^2+\frac{1}{18}\,(b-c)^2\,(-7\,a^2-5\,a\,b-5\,a\,c+9\,b^2+11\,b\,c+9\,c^2)^2+\frac{1}{18}\,(c-a)^2\,(9\,a^2-5\,a\,b+11\,a\,c-7\,b^2-5\,b\,c+9\,c^2)^2+\frac{1}{4}\,(6\,a^3-7\,a^2\,b-7\,a^2\,c-7\,a\,b^2+24\,a\,b\,c-7\,a\,c^2+6\,b^3-7\,b^2\,c-7\,b\,c^2+6\,c^3)^2+\frac{341}{12}\,(a-b)^2\,(b-c)^2\,(-c+a)^2 \tag{42}$$