Training Problems 6 - Solutions

The below exercises are based on Chapter 3 from the *Coding the Matrix* book by Philip Klein.

1. Multiply matrix X and Y:

$$X = \begin{bmatrix} 5 & 3 \\ -2 & 4 \\ 8 & 2 \\ 11 & 5 \end{bmatrix} \qquad Y = \begin{bmatrix} 5 & -8 & 3 & -1 & -3 \\ 11 & 13 & 5 & 2 & 4 \end{bmatrix}$$

Solution:

We see that matrix X has dimensions 4x2 and matrix Y has dimensions 2x5. Thus, the matrices are compatible to be multiplied and the resulting matrix will be 4x5, with positions listed below:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix}$$

$a_{11} = 5 \times 5 + 3 \times 11 = 58$

$a_{12} = 5 \times -8 + 3 \times 13 = -1$

$a_{13} = 5 \times 3 + 3 \times 5 = 30$

$a_{14} = 5 \times -1 + 3 \times 2 = 1$

$a_{15} = 5 \times -3 + 3 \times 4 = -3$

$a_{21} = -2 \times 5 + 4 \times 11 = 34$

$a_{22} = -2 \times -8 + 4 \times 13 = 68$

$a_{23} = -2 \times 3 + 4 \times 5 = 14$

$a_{24} = -2 \times -1 + 4 \times 2 = 10$

$a_{25} = -2 \times -3 + 4 \times 4 = 22$

$a_{31} = 8 \times 5 + 2 \times 11 = 62$

$a_{32} = 8 \times -8 + 2 \times 13 = -38$

$a_{33} = 8 \times 3 + 2 \times 5 = 34$

$a_{34} = 8 \times -1 + 2 \times 2 = -4$

$a_{35} = 8 \times -3 + 2 \times 4 = -16$

$a_{41} = 11 \times 5 + 5 \times 11 = 110$

$a_{42} = 11 \times -8 + 5 \times 13 = -23$

$a_{43} = 11 \times 3 + 5 \times 5 = 58$

$a_{44} = 11 \times -1 + 5 \times 2 = -1$

$a_{45} = 11 \times -3 + 5 \times 4 = -13$

The resulting matrix is:

$$\begin{bmatrix} 58 & -1 & 30 & 1 & -3 \\ 34 & 68 & 14 & 10 & 22 \\ 62 & -38 & 34 & -4 & -16 \\ 110 & -23 & 58 & -1 & -13 \end{bmatrix}$$

2. Find the determinant of the following matrix:

$$\begin{bmatrix} 3 & 15 & 72 \\ 35 & 17 & -28 \end{bmatrix}$$

Solution:

Only square matrices (matrices with dimension n x n) can have a determinant. The above matrix is 2x3, and thus isn't a square matrix, and because of that, doesn't have a determinant.

3. Find the determinant of the following matrix:

$$\begin{bmatrix} 22 & 44 & 0 & -1 \\ 15 & 3 & 0 & 72 \\ 19 & 35 & 0 & -28 \\ -32 & 55 & 0 & 7 \end{bmatrix}$$

Solution:

If a square matrix has 0's fill up all the values in a row and/or column, then its determinant is zero. The above matrix's determinant is zero because column 3 has all of its values as zero.

4. Find the determinant of the following matrix:

$$\begin{bmatrix} 20 & 5 & 0 & 0 \\ 6 & 2 & 0 & 4 \\ 2 & 3 & 2 & 0 \\ 2 & 2 & 2 & 2 \end{bmatrix}$$

Solution:

We see the above matrix is square (n x n) and doesn't have any columns or rows have all of its values as zero. Thus, we need to calculate and find the determinant:

We need to expand out the matrix, by using a Laplace expansion.

$$\begin{bmatrix} 20 & 5 & 0 & 0 \\ 6 & 2 & 0 & 4 \\ 2 & 3 & 2 & 0 \\ 2 & 2 & 2 & 2 \end{bmatrix} = 20\begin{bmatrix} 2 & 0 & 4 \\ 3 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix} - 5\begin{bmatrix} 6 & 0 & 4 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix} + 0\begin{bmatrix} 6 & 2 & 4 \\ 2 & 3 & 0 \\ 2 & 2 & 2 \end{bmatrix} - 0\begin{bmatrix} 6 & 2 & 0 \\ 2 & 3 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

In the above calculation, we know whether to add or subtract by using the following checkerboard pattern:

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Here is the pattern generalized:

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Sub-matrices with a zero coefficient will equal zero, so those will be skipped in calculations going forward:

$$= 20\begin{bmatrix} 2 & 0 & 4 \\ 3 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix} - 5\begin{bmatrix} 6 & 0 & 4 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix} = 20\left(2\begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} + 4\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}\right) - 5\left(6\begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} + 4\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\right)$$

$$=20\big(2(2 \times 2 - 2 \times 0) + 4(3 \times 2 - 2 \times 2)\big) - 5(6(2 \times 2 - 2 \times 0) + 4(2 \times 2 - 2 \times 2))$$

$$=20\big(2(4-0) + 4(6-4)\big) - 5\big(6(4-0) + 4(4-4)\big)$$

$$=20(2 \times 4 + 4 \times 2) - 5(6 \times 4 + 4 \times 0) = 20(8+8) - 5(24) = 20(16) - 5(24) = 200$$

The determinant of the original matrix is 200.

5. Let v_1 = [3, 5, 2] and v_2 = [8, 6, 7]. Use matrices to find coefficients k_1 and k_2 that would produce the linear combination vector of [42, 48, 33].

Solution:

We can represent the linear combination as:

$k_1[3, 5, 2] + k_2[8, 6, 7] = [42, 48, 33]$

Which is the same as:

$3k_1 + 8k_2 = 42$

$5k_1 + 6k_2 = 48$

$2k_1 + 7k_2 = 33$

Since we have three equations and two unknowns, we actually only need any two of the equations to find what k_1 and k_2 are. Let's use the top two equations to produce:

$$\begin{bmatrix} 3 & 8 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 42 \\ 48 \end{bmatrix}$$

In general, if we have matrices of the form AC = B, then C = A^{-1}B. Where A^{-1} means the inverse of matrix A.

$$A^{-1} = \frac{1}{18 - 40} \begin{bmatrix} 6 & -8 \\ -5 & 3 \end{bmatrix} = -\frac{1}{22} \begin{bmatrix} 6 & -8 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} \frac{-6}{22} & \frac{8}{22} \\ \frac{5}{22} & \frac{-3}{22} \end{bmatrix}$$

$$C = A^{-1}B = \begin{bmatrix} \frac{-6}{22} & \frac{8}{22} \\ \frac{5}{22} & \frac{-3}{22} \end{bmatrix} \begin{bmatrix} 42 \\ 48 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

Thus, $k_1 = 6$ and $k_2 = 3$.

6. Let $v_1 = [1, 2, 0]$, $v_2 = [3, 1, 1]$, and $w = [4, -7, 3]$. Determine whether w belongs to Span(v_1, v_2).

Solution:

We have to check if there exists $k_1, k_2 \in \mathbb{R}$ such that $w = k_1 v_1 + k_2 v_2$. If k_1 and k_2 values fit in all three equations, then the answer is yes, otherwise no.

$k_1+3 k_2 = 4$

$2 k_1+ k_2=-7$

$0 k_1+ k_2=3$

We see that $k_2=3$. So then we plug the k_2 in the first and second equations above and see if we can the same value for k_1:

First equation:

$k_1+3(3)=4$

$k_1+9=4$

$k_1=-5$

Second equation:

$2 k_1+3=-7$

$2 k_1=-10$

$k_1=-5$

As you can see, plugging in k_1 and k_2 to all three equations will yield the same result. Thus, w belongs to Span(v_1, v_2).

7. Let $v_1 = [1, 2, 0]$, $v_2 = [3, 1, 1]$, and $w = [4, -7, 4]$. Determine whether w belongs to Span(v_1, v_2).

Solution:

We have to check if there exists $k_1, k_2 \in \mathbb{R}$ such that $w = k_1 v_1 + k_2 v_2$. If k_1 and k_2 values fit in all three equations, then the answer is yes, otherwise no.

$k_1+3 k_2 = 4$

$2 k_1+ k_2=-7$

$0 k_1+ k_2=4$

We see that $k_2=4$. So then we plug the k_2 in the first and second equations above and see if we can the same value for k_1:

First equation:

$k_1+3(4)=4$

$k_1+12=4$

$k_1=-8$

Second equation:

$2 k_1+4=-7$

$2 k_1=-11$

$k_1=-5.5$

As you can see, plugging in k_2 to the first and second equations will yield different values for k_1. Thus, w does not belong to Span(v_1, v_2).