Training Problems 7 – Solutions

1. $B = [5, 7]$

   $B^T = ?$

   Solution:
   We need to find the transpose of B (written as B^T), which is turning all the rows to columns and all the columns to rows: $B^T = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$
   A transpose of an n x m matrix will have dimensions m x n. Transposing a matrix twice will return the original matrix.

2. $A = \begin{bmatrix} 5 & 2 & 4 & 12 \\ 3 & 7 & -1 & 8 \\ -2 & 2 & 5 & 3 \end{bmatrix}$

   $A^T = ?$

   Solution:
   A is 3 x 4, so its transpose A^T will be 4 x 3.

   $$A^T = \begin{bmatrix} 5 & 3 & -2 \\ 2 & 7 & 2 \\ 4 & -1 & 5 \\ 12 & 8 & 3 \end{bmatrix}$$

3. What does it mean for a matrix to be in row echelon form? Also give an example.

   Solution:

   A matrix is in row echelon form if it meets all of the following conditions:

   a. All nonzero rows (rows with at least one nonzero element) are above any rows of all zeros.
   b. Each leading entry of a row is in a column to the right of the leading entry of the row above it.

   These two conditions imply that all entries in a column below a leading coefficient are zeros.

   Below are a couple examples of matrices in row echelon form:

   $$\begin{bmatrix} -2 & 5 & 0 & 3 \\ 0 & 11 & 0 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 12 & -1 & 3 \\ 0 & 6 & 2 & 8 \\ 0 & 0 & 4 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4. What does it mean for a matrix to be in reduced row echelon form? Also give an example.

Solution:

A matrix is in reduced row echelon form if it meets all of the following conditions:

a. It is in row echelon form.
b. The leading entry in each nonzero row is a 1 (called a leading 1).
c. Each column containing a leading 1 has zeros everywhere else in that column.

Below are some examples of matrices in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 6 & 10 & 35 & 11 & 7 \\ 0 & 1 & 22 & -1 & 100 & 1 & 9 \end{bmatrix}$$

5. Use Gaussian Elimination to solve the following system of linear equations:
   x + 5y = 7
   -2x - 7y = -5

   Solution:

   Gaussian Elimination is an algorithm for solving systems of linear equations. It will result in a matrix that's in reduced row echelon form.

   You can only use a sequence of elementary row operations. The row operations are:
   - Swapping two rows,
   - Multiplying a row by a nonzero number,
   - Adding a multiple of one row to another row.

   Augmenting (combining the left and right side of the equal sign) the above linear equations in a matrix will be:

$$\begin{bmatrix} 1 & 5 & 7 \\ -2 & -7 & -5 \end{bmatrix}$$

   We need to get the above matrix in row echelon form. To start, we need to get the -2 to be a 0. How do we do that? Well, we see that we can multiply the first row by 2 and then adding that row to the 2nd row. We get:

$$\begin{bmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{bmatrix}$$

   The above matrix is in row echelon form. We now need to get it in reduced row echelon form. To do that, we first need to get the 3 to be a 1. We can do that by multiplying the 2nd row by 1/3:

$$\begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 3 \end{bmatrix}$$

   We now need to get the 5 to be a 0. That can be done by multiplying the 2nd row by -5 and then adding that to the first row:

$$\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \end{bmatrix}$$

   The above matrix is in reduced row echelon form and we're now done:

   x = -8

   y = 3

6. Use Gaussian Elimination to get the below system of linear equations to row echelon form:

$$x + y - z = 9$$

$$y + 3z = 3$$

$$-x - 2z = 2$$

You should then use back-substitution to solve.

Solution:

The augmented matrix of the above linear equations will look like the following:

$$\begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 1 & 3 & 3 \\ -1 & 0 & -2 & 2 \end{bmatrix}$$

To get it in row echelon form, we first need to get the -1 in the bottom row to be a 0. To do that, we can add the first row to the bottom row. The result is:

$$\begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 1 & 3 & 3 \\ 0 & 1 & -3 & 11 \end{bmatrix}$$

We then need the 1 in the bottom row to be a 0. We can multiply the second row with -1, then add it to the first row. The result is a row echelon matrix:

$$\begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & -6 & 8 \end{bmatrix}$$

Generally, we can either proceed with the Gaussian Elimination to get it in reduced row echelon form (which will therefore solve the linear equation), or we can solve it by back-substitution. In this case, per the problem requirements, we should use back-substitution. Based on the row echelon form matrix above, the system of linear equations is now:

$$x + y - z = 9$$

$$y + 3z = 3$$

$$-6z = 8$$

We can easily get z:

$z = -8/6 = -4/3$

Then plug z into the middle equation to get y:

$y + 3(-4/3) = 3$

$y - 4 = 3$

$y = 7$

So we have y and z, now all we have to do is plug them into the first equation to get x:

$x + 7 - (-4/3) = 9$

$x + (25/3) = 9$

$x = 2/3$

So using back-substitution:

$x = 2/3$

$y = 7$

$z = -4/3$