$m\times n$ **MATRIX** $\mathbf{A}$ has m rows and n columns	$\mathbf{A}=\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}$ is 2×3
$i,j-$entry is in the $i^{th}-$row and $j^{th}-$column	the $2,3-$entry of $\mathbf{A}$ is 6, $a_{23}=6$
row vector or **row**, matrix with one row	$(1\quad 2\quad 3)$ a row of size 3
column vector or **column**, matrix with one column	$\begin{pmatrix}a\\b\\c\end{pmatrix}$ is a column of size 3
Square matrix, $m=n$, number of rows is the same as the number of columns	$\begin{pmatrix}1&2\\3&4\end{pmatrix}$ is a square matrix.
1×1 is a **scalar**, a	4 is a scalar
$\mathbf{A}^{\mathrm{T}}$ is the **transpose** where columns and rows are switched	$\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}1&4\\2&5\\3&6\end{pmatrix}$
sum, $\mathbf{A}+\mathbf{B}$ only possible if of the same size, and then add entry-per-entry	if $\mathbf{B}=\begin{pmatrix}a&b&c\\d&e&f\end{pmatrix}$ then $\mathbf{A}+\mathbf{B}=\begin{pmatrix}1+a&2+b&3+c\\4+d&5+e&6+f\end{pmatrix}$
scalar product, $a\mathbf{A}$ always possible	$4\mathbf{B}=\begin{pmatrix}4a&4b&4c\\4d&4e&4f\end{pmatrix}$
multiplication of $(1\times n)(n\times 1)$, $(a_1\quad a_2\quad a_3\quad\cdots\quad a_n)\begin{pmatrix}b_1\\b_2\\b_3\\\vdots\\b_n\end{pmatrix}=$ $a_1b_1+a_2b_2+a_3b_3+\cdots+a_nb_n$	$(1\quad 2\quad 3)\begin{pmatrix}a\\b\\c\end{pmatrix}=1\times a+2\times b+3\times c=a+2b+3c$
multiplication $\mathbf{AB}$ only possible if the number of columns of $\mathbf{A}$ is the number of rows of $\mathbf{B}$: then **each row times each column.**	$\mathbf{AB}$ is not possible, but $\mathbf{AB}^{\mathrm{T}}=\begin{pmatrix}a+2b+3c&d+2e+3f\\4a+5b+6c&4d+5e+6f\end{pmatrix}$
stacking, making matrices out of smaller matrices	$(\mathbf{A}\quad\mathbf{B})=\begin{pmatrix}1&2&3&a&b&c\\4&5&6&d&e&f\end{pmatrix}$
blocking, thinking of a matrix as made up of smaller matrices	$\begin{pmatrix}1&2&3&a&b&c\\4&5&6&d&e&f\end{pmatrix}=(\mathbf{A}\quad\mathbf{B})$
adjacency matrix, $(0,1)-$matrix	$i\to j$ if and only if $a_{ij}=1$
main diagonal: the $1,1-$, $2,2-,\ldots,$ $n,n-$positions	$\begin{pmatrix}1&2\\3&4\end{pmatrix}$ has 1 and 4 on its main diagonal.

Diagonal Matrix—square one which has only zeroes off the main diagonal	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is diagonal
$\mathbf{I}$, the **identity matrix**: diagonal matrix with 1's along the diagonal of any square size	$\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\mathbf{IM} = \mathbf{MI} = \mathbf{M}$	$(a \quad b)\mathbf{I}_2 = (a \quad b) = \mathbf{I}_1(a \quad b)$
$\mathbf{0}$ and $\mathbf{J}$, zero and jay, of any size, having all zeroes and all ones respectively	$\mathbf{0}_{2\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathbf{J}_{2\times 3} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
Upper triangular—square with zeroes below diagonal. **Lower triangular** similarly.	$\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ is upper, $\mathbf{B}^{\mathrm{T}} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ is lower
Product of **upper triangulars** is also upper triangular with diagonal just the product	$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & b+2c \\ 0 & 3c \end{pmatrix}$
Block multiplication: If $\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{pmatrix}$ and $\mathbf{MN}$ exists and so does $\mathbf{AX}$, then $\mathbf{MN} = \begin{pmatrix} \mathbf{AX}+\mathbf{BZ} & \mathbf{AY}+\mathbf{BW} \\ \mathbf{CX}+\mathbf{DZ} & \mathbf{CY}+\mathbf{DW} \end{pmatrix}$	Let $\mathbf{M} = \begin{pmatrix} \mathbf{J}_{2\times 3} & \mathbf{I} \\ \mathbf{0}_{4\times 3} & \mathbf{D} \end{pmatrix}$ and let $\mathbf{N} = \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{pmatrix}$ be $5\times ?$. Then block multiplication can be used when and only when $\mathbf{X}$ is $3\times ?$ and then $\mathbf{MN} = \begin{pmatrix} \mathbf{J}_{2\times 3}\mathbf{X}+\mathbf{Z} & \mathbf{J}_{2\times 3}\mathbf{Y}+\mathbf{W} \\ \mathbf{DZ} & \mathbf{DW} \end{pmatrix}$
$\mathbf{A}$ is invertible if $\mathbf{AX} = \mathbf{XA} = \mathbf{I}$ for some $\mathbf{X}$. In particular $\mathbf{A}$ is then square.	$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$ is invertible since $\mathbf{A}\begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}\mathbf{A} = \mathbf{I}_2$
Inverse is unique, $\mathbf{A}^{-1}$ and $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$	$\begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$
Upper triangular is invertible if and only if every diagonal entry is nonzero. Inverse is also upper triangular.	$\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ is invertible, $\mathbf{B}^{-1} = \frac{1}{3}\begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}$
Diagonal matrix is invertible if and only if diagonal entries are not zero. Inverse is the diagonal of the reciprocals.	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$
If $\mathbf{A}$ and $\mathbf{B}$ are invertible, then so is $\mathbf{AB}$ and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$	$(\mathbf{AB})^{-1} = \begin{pmatrix} 2 & 13 \\ 3 & 21 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 21 & -13 \\ -3 & 2 \end{pmatrix} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
$(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$	$(\mathbf{AB})^{\mathrm{T}} = \begin{pmatrix} 2 & 13 \\ 3 & 21 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 2 & 3 \\ 13 & 21 \end{pmatrix} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$
Permutation if every entry is either a 0 or a 1, and only one 1 in each row. Permutation matrix $\mathbf{P}$ is invertible, $\mathbf{P}^{-1} = \mathbf{P}^{\mathrm{T}}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$\mathbf{P}$ invertible, then $\mathbf{Ax}=\mathbf{b}$ and $\mathbf{PAx}=\mathbf{Pb}$ have the same solutions	Let $\mathbf{A}=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\\10&11&12\end{pmatrix}$, $\mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}6\\15\\24\\33\end{pmatrix}$. and let $\mathbf{P}=\begin{pmatrix}2&5&2&1\\3&1&4&7\\1&2&1&1\\1&1&1&1\end{pmatrix}$. Then $\mathbf{Ax}=\mathbf{b}$ and $\begin{pmatrix}46&56&66\\105&120&135\\26&31&36\\22&26&30\end{pmatrix}\mathbf{x}=\begin{pmatrix}168\\360\\93\\78\end{pmatrix}$ have the same solutions.
A matrix $\mathbf{A}$ is said to be in (**row**) **reduced form** (or **row echelon form**, or **row reduced echelon form**) if the following 4 conditions are satisfied: ① The first nonzero term of any row is a 1. This entry is called a **pivot**, and thus every row is either all zeros or it has a unique pivot. Hence a nonzero row will be called **pivotal**, and such a row will have exactly one pivot. Conversely, every pivot is in a pivotal row. ② A column that contains a pivot, a **pivotal column**, has all zeroes except for the pivot. Hence any pivotal column is identical to a column of the identity matrix, and a **pivotal column** has exactly one pivot, and every pivot is in a pivotal column. ③ Any zero row is below any pivotal row. ④ The pivots in the matrix lie from upper left to lower right.	$\begin{pmatrix}1&?&0&?&0\\0&0&1&?&0\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix}$ is reduced $\begin{pmatrix}1&2&0\\0&-1&0\\0&0&0\\0&0&0\end{pmatrix}$, $\begin{pmatrix}1&2&0\\0&1&0\\0&0&0\\0&0&0\end{pmatrix}$, $\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\\0&0&1\end{pmatrix}$ and $\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\\0&0&0\end{pmatrix}$ are not reduced
For any matrix $\mathbf{M}$ there exists an invertible matrix $\mathbf{P}$ such that $\mathbf{PM}$ is reduced	$\mathbf{M}=\begin{pmatrix}3&2&1&39\\2&3&1&34\\1&2&3&26\end{pmatrix}$, then $\begin{pmatrix}1&0&0&\frac{37}{4}\\0&1&0&\frac{17}{4}\\0&0&1&\frac{11}{4}\end{pmatrix}=\mathbf{PM}$ where $\mathbf{P}=\frac{1}{12}\begin{pmatrix}7&-4&-1\\-5&8&-1\\1&-4&5\end{pmatrix}$.
The reduced form of $\mathbf{A}$ is unique, $rref(\mathbf{A})$ is a common designation	No matter what invertible matrix $\mathbf{Q}$ we use, if $\mathbf{QM}$ is reduced, then $\mathbf{QM}$ is as above

$\mathbf{A}$ is invertible if and only if $rref(\mathbf{A}) = \mathbf{I}$	$\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$ reduces to $\mathbf{I}_2$
If $\mathbf{A}$ is invertible, then $(\mathbf{A} \quad \mathbf{I})$ reduces to $(\mathbf{I} \quad \mathbf{A}^{-1})$	$rref\begin{pmatrix} 2 & 3 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 5 & -3 \\ 0 & 1 & -3 & 2 \end{pmatrix}$
To solve, $\mathbf{Ax} = \mathbf{b}$ reduce augmented matrix $(\mathbf{A} \quad \mathbf{b})$ and proceed.	$(1 \quad 2)\mathbf{x} = 7$ has all solutions where $x_1 = 7 - 2x_2$ so x_2 is a **free** variable
$r(\mathbf{A})$, the **rank** of $\mathbf{A}$, is the number of pivots in the reduced form of $\mathbf{A}$. The **nullity** is the number of free variables.	$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ has rank 3 and nullity 2.
$r(\mathbf{A})$ is the number of **leading** variables of $\mathbf{Ax} = \mathbf{b}$	To solve $\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\mathbf{x} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$ we have $x_1 + x_2 + x_4 = b_1$, $x_3 + x_4 = b_2$, $x_5 = b_3$ and $0 = b_4$ and the leading variables are x_1, x_3 and x_5
$\mathbf{Ax} = \mathbf{b}$ will have a solution if and only if $r(\mathbf{A}) = r(\mathbf{A} \quad \mathbf{b})$	As above, $r(\mathbf{A}) = r(\mathbf{A} \quad \mathbf{b})$ if and only if $b_4 = 0$ which is equivalent to the system having a solution (infinitely many in this case)
If $\mathbf{Mx} = \mathbf{b}$ has a solution, then that solution is unique if and only if $r(\mathbf{M}) = n$ where n is the number of columns of $\mathbf{M}$ — otherwise it has infinitely many solutions	Let $\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & ? \\ 0 & 0 & 0 \end{pmatrix}$ be reduced. Then $\mathbf{Mx} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$ is equivalent to $x_1 = b_1$, $x_2 = b_2$, and $r(\mathbf{M}) = 3$ if and only if $? = 1$. And then $x_3 = b_3$, otherwise x_3 is free and can be anything. Of course, $b_4 = 0$ if there is to be a solution
If $\mathbf{Mx} = \mathbf{b}$ has infinitely many solutions, then all solutions can be written in the form $\mathbf{u} + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k$ where $\mathbf{u}$ is an arbitrary solution to $\mathbf{Mx} = \mathbf{b}$, $\mathbf{Mv}_i = 0$ for each $i = 1, \ldots, k$, k is the number of free variables, and the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ when viewed in the positions corresponding to the free variables constitute $\mathbf{I}_k$	The system $\begin{pmatrix} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 7 \\ 8 \\ 9 \\ ? \end{pmatrix}$ when written out becomes $x_1 - 2x_2 + 3x_4 = 7$ so $x_1 = 7 + 2x_2 - 3x_4$, $x_3 + 4x_4 = 8$ so $x_3 = 8 - 4x_4$, and $x_5 = 9$, so its solution set is given by $\begin{pmatrix} 7 \\ 0 \\ 8 \\ 0 \\ 9 \end{pmatrix} + t_1\begin{pmatrix} 2 \\ \mathbf{1} \\ 0 \\ \mathbf{0} \\ 0 \end{pmatrix} + t_2\begin{pmatrix} -3 \\ \mathbf{0} \\ -4 \\ \mathbf{1} \\ 0 \end{pmatrix}$

An expression of the form $a_1\mathbf{A}_1 + a_2\mathbf{A}_2 + \cdots + a_t\mathbf{A}_t$ is called a **linear combination** of $\mathbf{A}_1, \ldots, \mathbf{A}_t$	$\begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix} = 2\mathbf{J}_2 + 3\mathbf{I}_2$
To write a linear combination is equivalent to matrix multiplication by a vector	$\begin{pmatrix} 5 \\ 2 \\ 2 \\ 5 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ because $\begin{pmatrix} 5 \\ 2 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$
the **span** of $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_t$, $[\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_t]$, is the collection of all linear combinations of them	the span of $\mathbf{I}_2, \mathbf{J}_2$, $[\mathbf{I}_2, \mathbf{J}_2] = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right\}$
A collection V is called a **space** if the following occur: ❶ $\mathbf{0} \in V$ ❷ If $\mathbf{u}, \mathbf{v} \in V$ then so does $\mathbf{u} + \mathbf{v}$ ❸ If $\mathbf{u} \in V$ then so does $a\mathbf{u}$ for any scalar a	The set $\{t\mathbf{J}_2\}$ is a space since it satisfies the three conditions: $0\mathbf{J}_2$, $t\mathbf{J}_2 + s\mathbf{J}_2 = (t+s)\mathbf{J}_2$ and $s(t\mathbf{J}_2) = (st)\mathbf{J}_2$. On the other hand: $\{(t \quad 0), (0 \quad t)\}$ is **NOT** a space although it satisfies the first and third condition, it does not satisfy the second one: $(1 \quad 0) + (0 \quad 1) = (1 \quad 1) \notin V$
the **column space** of a matrix, $C(\mathbf{A})$, is the span of its columns	$C\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \left[\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \right] = \left\{ a\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + b\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + c\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \right\}$
For any $\mathbf{b}$, $\mathbf{Ax} = \mathbf{b}$ has a solution when and only when $\mathbf{b} \in C(\mathbf{A})$	$\begin{pmatrix} 6 \\ 15 \\ 24 \end{pmatrix} \in C\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ since $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 15 \\ 24 \end{pmatrix}$
the **null space** of a matrix, $N(\mathbf{A})$, is the set of solutions of the homogeneous system $\mathbf{Ax} = \mathbf{0}$	$N\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \left[\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right] = \left\{ a\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$
the **row space** of a matrix, $R(\mathbf{A})$, is the span of its rows	$R\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \left[\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}^T, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}^T, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}^T \right] = \left\{ a\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}^T + b\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}^T + c\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}^T \right\}$
If $\mathbf{P}$ is invertible, then $R(\mathbf{PA}) = R(\mathbf{A})$ and $N(\mathbf{PA}) = N(\mathbf{A})$ but not necessarily true that $C(\mathbf{PA}) = C(\mathbf{A})$	$\begin{pmatrix} -5 & 2 & 0 \\ 4 & -1 & 0 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{pmatrix}$ so $\left[(1 \quad 2 \quad 3), (4 \quad 5 \quad 6), (7 \quad 8 \quad 9) \right] = \left[(1 \quad 0 \quad -1), (0 \quad 1 \quad 2) \right]$ $N\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \left[\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right] = N\begin{pmatrix} 3 & 0 & -3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{pmatrix}$, but $\left[\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \right] \neq \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]$

let $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_t \end{pmatrix}$, then the following statements are equivalent: ❶ Every column of $\mathbf{A}$ is pivotal ❷ $\mathbf{A}$ is of full column rank, $r(\mathbf{A}) = n$ ❸ If $\mathbf{Au} = \mathbf{0}$, then $\mathbf{u} = \mathbf{0}$, *i.e.*, $\mathrm{N}(\mathbf{A}) = \mathbf{0}$ ❹ If $\mathbf{Au} = \mathbf{Av}$, then $\mathbf{u} = \mathbf{v}$ ❺ There exists a matrix **B** such that $\begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix}$ is invertible	Let $\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$ Then $rref(\mathbf{M}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ Also $r(\mathbf{M}) = 2$. The system $\mathbf{Mx} = \mathbf{0}$ reduces to $x_1 = 0$ and $x_2 = 0$. The matrix $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 3 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{pmatrix}$ is invertible.
Let $\mathbf{w}_1$, $\mathbf{w}_2$, …, $\mathbf{w}_n$ be vectors in $\mathbb{R}^m$. We will refer to them as **transparent** if there is a set of n positions among the rows of the **w**'s where we can find $\mathbf{I}_n$.	$\begin{pmatrix} 3 \\ 1 \\ 7 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 4 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ is transparent since we can see the identity matrix $\mathbf{I}_3$ in rows 2, 4 and 5
Consider $\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_n$. Then the following are equivalent: ❶ $\mathbf{u}_1 \neq \mathbf{0}$ and for each $i = 2, \ldots, n$, $\mathbf{u}_i \notin [\mathbf{u}_1, \ldots, \mathbf{u}_{i-1}]$ ❷ There exists an invertible matrix **P** such that $\mathbf{Pu}_1$, $\mathbf{Pu}_2, \ldots, \mathbf{Pu}_n$ is transparent ❸ Whenever $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n = \mathbf{0}$, we must have $a_1 = a_2 = \cdots = a_n = 0$ ❹ For any scalars, if $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n = b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \cdots + b_n\mathbf{u}_n$ we must have $a_1 = b_1$, $a_2 = b_2$, …, $a_n = b_n$ ❺ There exists an invertible matrix **P** whose first n columns are $\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_n$ Such a collection of vectors is known as **linearly independent**	Take $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$. Then $\mathbf{u}_2 \notin [\mathbf{u}_1]$, and $\mathbf{P} = \begin{pmatrix} 0 & 0 & 4 & -3 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{pmatrix}$ satisfies $\begin{pmatrix} \mathbf{Pu}_1 & \mathbf{Pu}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{I}_2 \\ \mathbf{0} \end{pmatrix}$. If $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 = \mathbf{0}$ then $a_1 + a_2 = 0$ and also $a_1 + 2a_2 = 0$ and so we get $a_2 = 0$ and then $a_1 = 0$ also.
Let V be a space and $\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_n$ be in V. We say they **span** V if $[\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n] = V$. They are also called a **spanning set**.	$\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$ is a spanning set for $\mathcal{C}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$
Let V be a space and $\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_n$ be a spanning set for V. Then we say they form a **basis** if they are also linearly independent.	$\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$ is **NOT** a basis for $\mathcal{C}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$
A space has many bases but all of them have the same size. That common size is called the **dimension** of V	$\dim \mathbb{R}^5 = 5$

Every column of $\mathbf{A}$ is a linear combination of pivotal columns of $\mathbf{A}$	$\mathbf{A}=\begin{pmatrix}1&2&3&4\\5&6&7&8\\9&10&11&12\end{pmatrix}$, $rref(\mathbf{A})=\begin{pmatrix}1&0&-1&-2\\0&1&2&3\\0&0&0&0\end{pmatrix}$, and $\mathbf{C}_3=-\mathbf{C}_1+2\mathbf{C}_2$ in both matrices and also $\mathbf{C}_4=-2\mathbf{C}_1+3\mathbf{C}_2$ where $\mathbf{C}$ is for column
$\dim\mathcal{C}(\mathbf{A})=r(\mathbf{A})$ $\dim\mathcal{R}(\mathbf{A})=r(\mathbf{A})$ $\dim\mathcal{N}(\mathbf{A})=n-r(\mathbf{A})$ where n is the number of columns of $\mathbf{A}$	$\begin{pmatrix}1\\5\\9\end{pmatrix}$ and $\begin{pmatrix}2\\6\\10\end{pmatrix}$ is a basis for $\mathcal{C}(\mathbf{A})$; $(1\quad 0\quad -1\quad -2)$, $(0\quad 1\quad 2\quad 3)$ is a basis for $\mathcal{R}(\mathbf{A})$, and $\begin{pmatrix}1\\-2\\1\\0\end{pmatrix},\begin{pmatrix}2\\-3\\0\\1\end{pmatrix}$ is a basis for $\mathcal{N}(\mathbf{A})$
$r(\mathbf{A})=r(\mathbf{A}^{\mathrm{T}})$	$rref(\mathbf{A}^{\mathrm{T}})=\begin{pmatrix}1&0&0&0\\0&1&0&0\\-1&2&0&0\end{pmatrix}^{\mathrm{T}}$
If $\mathbf{M}$ has independent columns, then $\mathbf{Q}=\mathbf{M}(\mathbf{M}^{\mathrm{T}}\mathbf{M})^{-1}\mathbf{M}^{\mathrm{T}}$ is the projection onto $\mathcal{C}(\mathbf{M})$	Let $\mathbf{M}=\begin{pmatrix}1&2\\5&6\\9&10\end{pmatrix}$ so $\mathbf{Q}=\frac{1}{6}\begin{pmatrix}5&2&-1\\2&2&2\\-1&2&5\end{pmatrix}$
Solve $\mathbf{Mx}=\mathbf{b}$ by solving $\mathbf{Mx}=\mathbf{Qb}$	To solve $\mathbf{Mx}=\begin{pmatrix}0\\1\\1\end{pmatrix}$, one solves instead for $\frac{1}{6}\begin{pmatrix}1\\4\\7\end{pmatrix}$
If $\mathbf{B}$ is a square matrix, then there is always a unique monic polynomial $m_{\mathbf{B}}(x)$ of least degree such that $m_{\mathbf{B}}(\mathbf{B})=\mathbf{0}$, the **minimum polynomial** of $\mathbf{B}$	If $\mathbf{B}=\mathbf{I}$, then $m_{\mathbf{B}}(x)=x-1$ If $\mathbf{B}=\mathbf{0}$, then $m_{\mathbf{B}}(x)=x$ If $\mathbf{B}=\mathbf{J}_n$, then $m_{\mathbf{B}}(x)=x^2-nx$
$\langle\mathbf{B}\rangle=\left[\mathbf{I},\mathbf{B},\mathbf{B}^2,\mathbf{B}^3,\ldots\right]$ is the **algebra** generated by $\mathbf{B}$, which is a space closed under multiplication	If $\mathbf{B}=\mathbf{I}$, then $\langle\mathbf{B}\rangle=[\mathbf{I}]=\{a\mathbf{I}\}$ If $\mathbf{B}=\mathbf{J}_n$, then $\langle\mathbf{B}\rangle=[\mathbf{I},\mathbf{J}]=\{a\mathbf{I}+b\mathbf{J}\}$ If $\mathbf{B}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, then $\langle\mathbf{B}\rangle=\left\{\begin{pmatrix}a&b\\0&a\end{pmatrix}\right\}$
$\dim\langle\mathbf{B}\rangle=\deg m_{\mathbf{B}}(x)$	Consider $\mathbf{B}=\begin{pmatrix}2&1&0\\0&2&1\\0&0&2\end{pmatrix}$, then $m_{\mathbf{B}}(x)=(x-2)^3=x^3-6x^2+12x-8$, so $\mathbf{B}^3=6\mathbf{B}^2-12\mathbf{B}+8\mathbf{I}$ and $\mathbf{I},\mathbf{B},\mathbf{B}^2$ constitute a basis for $\langle\mathbf{B}\rangle$ and $\dim\langle\mathbf{B}\rangle=3$

the **determinant** of a matrix is a scalar	$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$
$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - ceg - afh - bdi$	$\det\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} = 45 + 96 + 84 - 105 - 48 - 72 = 0$
If $\mathbf{A}$ is a square, then a new matrix $\tilde{\mathbf{A}}$ (the **adjoint**) can be built whose $i, j-$entry is $(-1)^{i+j} \det \mathbf{A}_{ji}$ where $\mathbf{A}_{ji}$ is obtained from $\mathbf{A}$ by deleting the $j^{th}-$row and the $i^{th}-$column (note the switch)	If $\mathbf{A} = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix}$, then $\tilde{\mathbf{A}} = \begin{pmatrix} 2 & 2 & -3 \\ 4 & -11 & 6 \\ -3 & 6 & -3 \end{pmatrix}$
$\mathbf{A}\tilde{\mathbf{A}} = (\det \mathbf{A})\mathbf{I} = \tilde{\mathbf{A}}\mathbf{A}$ $\mathbf{A}$ is invertible when and only when $\det \mathbf{A} \neq 0$ and then $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \tilde{\mathbf{A}}$	$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix}\begin{pmatrix} 2 & 2 & -3 \\ 4 & -11 & 6 \\ -3 & 6 & -3 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ $\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix}^{-1} = \frac{-1}{3}\begin{pmatrix} 2 & 2 & -3 \\ 4 & -11 & 6 \\ -3 & 6 & -3 \end{pmatrix}$
For any i, $\det \mathbf{A} =$ $(-1)^{i+1} a_{i1} \det \mathbf{A}_{1i} + (-1)^{i+2} a_{i2} \det \mathbf{A}_{2i} + \cdots + (-1)^{i+n} a_{in} \det \mathbf{A}_{ni}$	$\det\begin{pmatrix} 2 & 0 & 0 \\ ? & a & b \\ ? & c & d \end{pmatrix} = 2(ad - bc)$
$\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$	$522 = \det\begin{pmatrix} 46 & 26 \\ 79 & 56 \end{pmatrix} =$ $\det\begin{pmatrix} 2 & 5 \\ 5 & 8 \end{pmatrix} \det\begin{pmatrix} 3 & 8 \\ 8 & 2 \end{pmatrix} = -9 \times -58$
$\widetilde{\mathbf{A}^{\mathrm{T}}} = \tilde{\mathbf{A}}^{\mathrm{T}}$ and $\det\left(\mathbf{A}^{\mathrm{T}}\right) = \det \mathbf{A}$	$\widetilde{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}} = \begin{pmatrix} 2 & 4 & -3 \\ 2 & -11 & 6 \\ -3 & 6 & -3 \end{pmatrix}$
Let $\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{pmatrix}$ where $\mathbf{A}$ and $\mathbf{C}$ are square (not necessarily of the same size). Then $\det \mathbf{M} = \det \mathbf{A} \det \mathbf{C}$.	Let $\mathbf{M} = \begin{pmatrix} 2 & 1 & ? & ? & ? \\ 1 & 2 & ? & ? & ? \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 \end{pmatrix}$, then $\det \mathbf{M} = \det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \det\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} = 3 \times 20 = 60$
Let $\mathbf{A}$ be upper triangular, then $\det \mathbf{A} = a_{11} a_{22} \cdots a_{nn} = \prod_{i=1}^{n} a_{ii}$	$\det\begin{pmatrix} 1 & 0 & 0 \\ ? & 2 & 0 \\ ? & ? & 3 \end{pmatrix} = 6$
If $\mathbf{P}$ is a permutation, then $\det \mathbf{P} = \pm 1$	$\det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1,\ \det\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = 1$

Throughout the remaining of the course, our matrices will be square	
The polynomial $c_{\mathbf{A}}(x)=\det(\mathbf{A}-x\mathbf{I})$ has leading coefficient ±1 and has degree the size of $\mathbf{A}$. It is called the **characteristic polynomial** of $\mathbf{A}$	$c_{\mathbf{J}_2}(x)=-2x+x^2$ $c_{\mathbf{J}_3}(x)=3x^2-x^3$
Let λ be a scalar. Then the following are equivalent: ❶ $\det(\mathbf{A}-\lambda\mathbf{I})=0$ ❷ $\mathbf{Au}=\lambda\mathbf{u}$ for some nonzero vector $\mathbf{u}$ ❸ λ is a root of $c_{\mathbf{A}}(x)$	❶ $\det(\mathbf{J}_3-3\mathbf{I})=0$ ❷ $\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}3\\3\\3\end{pmatrix}=3\begin{pmatrix}1\\1\\1\end{pmatrix}$ ❸ $c_{\mathbf{J}_3}(3)=(3x^2-x^3)(3)=3(3)^2-(3)^3=0$
Such a λ is called an **eigenvalue** of $\mathbf{A}$ and such a $\mathbf{u}$ is an **eigenvector** of $\mathbf{A}$	3 is an eigenvalue of $\mathbf{J}_3$ and $\begin{pmatrix}1\\1\\1\end{pmatrix}$ is a corresponding eigenvector
The complete list of roots of $c_{\mathbf{A}}(x)$ (that is the list eigenvalues) is known as the **spectrum** of $\mathbf{A}$	The spectrum of $\mathbf{J}_3$ is $3,0,0$ The spectrum of $\mathbf{J}_2$ is $2,0$
The spectrum of an upper triangular matrix consists of its diagonal entries	The spectrum of $\mathbf{A}=\begin{pmatrix}1&?&?\\0&2&?\\0&0&a\end{pmatrix}$ is $1,2,a$
$\mathbf{A}$ and $\mathbf{B}$ are **similar** if $\mathbf{P}^{-1}\mathbf{AP}=\mathbf{B}$ for some invertible matrix $\mathbf{P}$	Since $\begin{pmatrix}5&-3\\-3&2\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}2&3\\3&5\end{pmatrix}=\begin{pmatrix}-14&-22\\12&19\end{pmatrix}$, $\begin{pmatrix}1&2\\3&4\end{pmatrix}$ and $\begin{pmatrix}-14&-22\\12&19\end{pmatrix}$ are similar
Two similar matrices have the same characteristic polynomial	x^2-5x-2 is the common characteristic polynomial of the example above
$\mathbf{A}$ is **diagonalizable** if $\mathbf{A}$ is similar to a diagonal matrix.	$\mathbf{J}_3$ is diagonalizable since $\frac{1}{3}\begin{pmatrix}1&1&1\\1&-2&1\\1&1&-2\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&-1&0\\1&0&-1\end{pmatrix}=\begin{pmatrix}3&0&0\\0&0&0\\0&0&0\end{pmatrix}$
If $\mathbf{P}$ diagonalizes $\mathbf{A}$, then its columns are eigenvectors of $\mathbf{A}$	Indeed $\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\-1\\0\end{pmatrix},\begin{pmatrix}1\\0\\-1\end{pmatrix}$ are eigenvectors of $\mathbf{J}_3$
Not every matrix is diagonalizable	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$ has spectrum $1,1$ so if it were diagonalizable it would be similar to $\mathbf{I}$ which is impossible
Not every matrix has real eigenvalues but when it does, it is similar to an upper triangular matrix	If $\mathbf{A}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, then $c_{\mathbf{A}}(x)=x^2+1$, but also $\frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}3&-2\\2&-1\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix}=\begin{pmatrix}1&4\\0&1\end{pmatrix}$

The **trace** of a matrix is the sum of its diagonal elements. Abbreviated, $\operatorname{tr}\mathbf{A}$.	$\operatorname{tr}\mathbf{I}_n = n = \operatorname{tr}\mathbf{J}_n$
If $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{B}$, then $c_{\mathbf{A}}(x) = c_{\mathbf{B}}(x)$, $m_{\mathbf{A}}(x) = m_{\mathbf{B}}(x)$, $\det\mathbf{A} = \det\mathbf{B}$ and $\operatorname{tr}\mathbf{A} = \operatorname{tr}\mathbf{B}$	$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} -14 & -22 \\ 12 & 19 \end{pmatrix}$ both have determinant -2 and trace 5
If $\lambda_1, \lambda_2, \ldots, \lambda_n$ is the spectrum of $\mathbf{A}$ then $\lambda_1 \times \lambda_2 \times \cdots \times \lambda_n = \det\mathbf{A}$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = \operatorname{tr}\mathbf{A}$	Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 0 & 1 \\ 5 & 0 & 1 \end{pmatrix}$ then $\mathbf{A}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{A}\begin{pmatrix} 1 \\ 7 \\ -5 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 7 \\ -5 \end{pmatrix}$ and $\mathbf{A}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = -4\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ so its spectrum is $0, 6, -4$ and $\operatorname{tr}\mathbf{A} = 2 = 0 + 6 - 4$ while $\det\mathbf{A} = 0 = 0 \times 6 \times 4$
If $p(x)$ is a polynomial, and $\mathbf{A}$ is similar to $\mathbf{B}$, then $p(\mathbf{A})$ is similar to $p(\mathbf{B})$	$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^2 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 10 \\ 15 & 22 \end{pmatrix}$ is similar to $\begin{pmatrix} -14 & -22 \\ 12 & 19 \end{pmatrix}^2 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -67 & -110 \\ 60 & 98 \end{pmatrix}$
If $\lambda_1, \lambda_2, \ldots, \lambda_n$ is the spectrum of $\mathbf{A}$ and $p(x)$ is a polynomial, then $p(\lambda_1), p(\lambda_2), \ldots, p(\lambda_n)$ is the spectrum of $p(\mathbf{A})$	Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 0 & 1 \\ 5 & 0 & 1 \end{pmatrix}$, then $\mathbf{A}^2 + \mathbf{I} = \begin{pmatrix} 27 & 2 & 8 \\ 10 & 11 & 16 \\ 10 & 10 & 17 \end{pmatrix}$ has spectrum $1, 37, 17$
$c_{\mathbf{A}}(\mathbf{A}) = \mathbf{0}$	If $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $c_{\mathbf{M}}(x) = x^2 - (a+d)x + (ad - bc)$ and $\mathbf{M}^2 - (a+d)\mathbf{M} + (ad - bc)\mathbf{I} = \mathbf{0}$
$m_{\mathbf{A}}(x)$ is a factor of $c_{\mathbf{A}}(x)$ and every eigenvalue is a root of $m_{\mathbf{A}}(x)$	Consider $\mathbf{M} = \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then $c_{\mathbf{M}}(x) = x^2(x-1)^2$ and $m_{\mathbf{M}}(x) = x^2(x-1)^2$ if $z \neq 0$ and $x \neq 0$, $m_{\mathbf{M}}(x) = x(x-1)^2$ if $x \neq 0$ and $z = 0$, $m_{\mathbf{M}}(x) = x^2(x-1)$ if $z \neq 0$ and $x = 0$, $m_{\mathbf{M}}(x) = x(x-1)$ if $z = 0$ and $x = 0$
If λ is an eigenvalue of $\mathbf{A}$ then its **algebraic multiplicity** is the number of times λ occurs as a root of $c_{\mathbf{A}}(x)$	In the previous example, the algebraic multiplicity of 1 is 2 and so is the algebraic multiplicity of 0
If λ is an eigenvalue of $\mathbf{A}$ then its **geometric multiplicity** is the number of independent eigenvectors for λ	In the example above, if $x = 0$, then $r(\mathbf{A} - \mathbf{I}) = 2$ so its nullity is 2, and so the geometric multiplicity of 1 is also 2. If $x \neq 0$, then $r(\mathbf{A} - \mathbf{I}) = 3$ and then the geometric multiplicity is only 1.

The geometric multiplicity is always less than or equal to the algebraic multiplicity	Let $\mathbf{M}=\begin{pmatrix}1&x&0\\0&1&y\\0&0&1\end{pmatrix}$. Then the algebraic multiplicity of 1 is3. But if $x\neq 0$ and $y\neq 0$, then $r(\mathbf{M}-\mathbf{I})=2$, so the geometric multiplicity of 1 is just 1. If $x\neq 0$ and $y=0$, then geometric multiplicity is 2, and if $x=y=0$, then it is of course 3
Eigenvectors corresponding to different eigenvalues are linearly independent	Let $\mathbf{A}=\begin{pmatrix}1&2&3\\5&0&1\\5&0&1\end{pmatrix}$, then $\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}1\\7\\-5\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ are independent
The following are equivalent for a matrix $\mathbf{A}$: ❶ $\mathbf{A}$ is diagonalizable ❷ There is a basis of $\mathbb{R}^n$ consisting of eigenvectors of $\mathbf{A}$ ❸ The geometric multiplicity equals the algebraic multiplicity for every eigenvalue	$\mathbf{M}=\begin{pmatrix}1&1&2\\1&1&2\\1&1&2\end{pmatrix}$. Then ❶ holds since $\mathbf{P}^{-1}\mathbf{M}\mathbf{P}=\mathbf{D}$ where $\mathbf{P}=\begin{pmatrix}1&1&1\\1&1&-1\\1&-1&0\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}4&0&0\\0&0&0\\0&0&0\end{pmatrix}$. ❷ holds since $\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}1\\1\\-1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ form a basis of eigenvectors. ❸holds since the multiplicities of 4 are both 1 while those of 0 are both 2
A matrix with different eigenvalues is diagonalizable	$\mathbf{A}=\begin{pmatrix}1&2&3\\5&0&1\\5&0&1\end{pmatrix}$ is diagonalizable since its eigenvalues are $0,6,-4$
A matrix is diagonalizable if and only if its minimum polynomial has different roots	The minimum polynomial of $\mathbf{M}=\begin{pmatrix}1&1&2\\1&1&2\\1&1&2\end{pmatrix}$ is x^2-4x while its characteristic polynomial is $x(x^2-4x)$
$\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_t$ are **mutually orthogonal** if $\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_j=\mathbf{u}_i\cdot\mathbf{u}_j=0$ for any $i\neq j$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}1\\1\\-2\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ are mutually orthogonal
If $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_t$ are **mutually orthogonal** and not zero, then they are linearly independent	same as above
If $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_t$ are independent, and $\mathbf{u}_1=\mathbf{v}_1$, $\mathbf{u}_2=\mathbf{v}_2-\frac{\mathbf{v}_2\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1$, $\mathbf{u}_3=\mathbf{v}_3-\frac{\mathbf{v}_3\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1-\frac{\mathbf{v}_3\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2$, then $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_t$ are **mutually orthogonal**	$\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\\0\end{pmatrix}$ become $\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}1\\1\\-2\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\0\end{pmatrix}$

A symmetric matrix has real eigenvalues	If $\mathbf{A}=\begin{pmatrix} a & b \\ b & c \end{pmatrix}$, then its spectrum is given by $\dfrac{a+c\pm\sqrt{(a-c)^2+4b^2}}{2}$
Eigenvectors corresponding to different eigenvalues of a symmetric matrix are orthogonal	Let $\mathbf{A}=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$ then $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ are eigenvectors for 0 and 14 respectively and they are orthogonal
A square matrix $\mathbf{P}$ is **orthogonal** if $\mathbf{P}\mathbf{P}^{\mathrm{T}}=\mathbf{I}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and $\frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & \sqrt{2} & 1 \\ -\sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \end{pmatrix}$ are
The following are equivalent: ❶ $\mathbf{P}$ is orthogonal ❷ $\mathbf{P}^{\mathrm{T}}$ is orthogonal ❸ The columns of $\mathbf{P}$ form an **orthonormal** basis (mutually orthogonal of length 1) ❹ The rows of $\mathbf{P}$ form an orthonormal basis (mutually orthogonal of length 1)	The examples just above illustrate the theorem. Peculiarly, not enough for the columns to be mutually orthogonal to obtain the rows to be orthogonal. $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix}$ has orthogonal columns but not rows
Let $\mathbf{A}$ have real eigenvalues. Then there exists an orthogonal matrix $\mathbf{P}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}=\mathbf{T}$ is upper triangular.	$\frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & \sqrt{2} & 1 \\ -\sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{pmatrix}\frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & \sqrt{2} & 1 \\ -\sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 3\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$
Let $\mathbf{A}$ be symmetric, then there exists an orthogonal matrix $\mathbf{P}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}=\mathbf{D}$ is diagonal $\mathbf{A}$ is **orthogonally diagonalizable**	$\frac{1}{\sqrt{42}}\begin{pmatrix} \sqrt{3} & \sqrt{7} & 4\sqrt{2} \\ 2\sqrt{3} & -2\sqrt{7} & \sqrt{2} \\ 3\sqrt{3} & \sqrt{7} & -2\sqrt{2} \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}\frac{1}{\sqrt{42}}\begin{pmatrix} \sqrt{3} & \sqrt{7} & 4\sqrt{2} \\ 2\sqrt{3} & -2\sqrt{7} & \sqrt{2} \\ 3\sqrt{3} & \sqrt{7} & -2\sqrt{2} \end{pmatrix}=\begin{pmatrix} 14 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
An equation of the form $ax+bxy+cy+dx+ey+f=0$ is a **general quadratic equation**	$4x^2+4xy+y^2-24x+38y-139=0$
A general quadratic equation always represents a **conic** (although some are so called simplistic conic)	$x^2+y^2=1$ is the unit circle $x^2-y^2=1$ is a hyperbola $x^2-y^2=0$ represents two lines
One associates the symmetric matrix $\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$ with the equation	$4x^2+4xy+y^2-24x+38y-139=0$ has matrix $\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$
the shape of the conic is determined by the determinant of the matrix: 0, **parabola**, positive, **ellipse** and negative **hyperbola**	$4x^2+4xy+y^2-24x+38y-139=0$ is a parabola