Non-stationary Bandits and Meta-Learning with a Small Set of Optimal Arms

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Abstract

We study a sequential decision problem where the learner faces a sequence of K-armed bandit tasks. The task boundaries might be known (the bandit meta-learning setting), or unknown (the non-stationary bandit setting). For a given integer $M \leq K$, the learner aims to compete with the best subset of arms of size M. We design an algorithm based on a reduction to bandit submodular maximization, and show that, for T rounds comprised of N tasks, in the regime of large number of tasks and small number of optimal arms M, its regret in both settings is smaller than the simple baseline of $\tilde{O}(\sqrt{KNT})$ that can be obtained by using standard algorithms designed for non-stationary bandit problems. For the bandit meta-learning problem with fixed task length τ, we show that the regret of the algorithm is bounded as $\tilde{O}(NM\sqrt{M\tau} + N^{2/3}M\tau)$. Under additional assumptions on the identifiability of the optimal arms in each task, we show a bandit meta-learning algorithm with an improved $\tilde{O}(N\sqrt{M\tau} + N^{1/2}\sqrt{MK\tau})$ regret.

1 Introduction

Consider a recommendation platform with a large catalog of items. Every day, the system gets to interact with different customers and must sequentially discover what is the preferred item of the day. But if we suppose that days correspond to recurring groups of customers, it naturally implies that only a small subset of the catalog is eventually preferred. In this problem, like in many personalization or recommendation scenarios, the learner needs to solve a sequence of tasks. When the tasks share some similarities, the learner may be able to utilize this by sequentially learning the underlying common structure in the tasks and hence perform better on the sequence than solving each task in isolation. In this paper we study a special class of these *meta-learning* problems, where each task is an instance of a multi-armed bandit problem.

Formally, we consider a problem where a learner faces N instances of K-armed bandit tasks sequentially. At the beginning of task $n \in [N]$,[1] the adversary chooses the mean reward function $r_n \in \mathcal{R} = [0, 1]^K$. Then, the learner interacts with the K-armed bandit task specified by the reward function r_n for τ_n time steps: for each $n \in [N]$, when action $a \in [K]$ is played, it receives reward $r_n(a) + \eta_{n,t}(a)$ where $\eta_{n,t}(a)$ are independent zero-mean and $[-1/2, 1/2]$-valued noise variables for all n, t and a, so that the expected reward is $r_n(a)$.

We consider two settings: the *meta-learning setting* where the learner knows the length τ_n of each task n when it starts, and the more general *non-stationary setting* where the task boundaries are unknown (Wei and Luo, 2021; Hong et al., 2020b; Russac et al., 2019; Auer et al., 2019a).

We use T to denote the total number of rounds, $T = \sum_{n=1}^{N} \tau_n$. Let $\mathcal{H}_{n,t}$ be the history of the learner's actions and the corresponding observed rewards up to, but not including, time step t in task n. For this time step, the learner chooses a distribution $\pi_{n,t}$ over the actions as a function of $\mathcal{H}_{n,t}$ and other problem

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[1] For any integer k, we let $[k] = \{1, 2, \ldots, k\}$, and for any (multi-)set S, $|S|$ denotes the number of distinct elements in S.

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data (such as T, and τ_n in bandit meta-learning), and samples an action $A_{n,t}$ from $\pi_{n,t}$ to be used in this time step. The learner's policy π is a collection of the mappings $\pi_{n,t}$ (formally, π is a mapping from the set of possible histories to distributions over the action set $[K]$). Its regret on a sequence of reward functions $(r_n)_{n=1}^{N}$ with respect to a sequence of actions $(a_n)_{n=1}^{N}$ is defined as

$$R(\pi, T, (r_n)_{n=1}^{N}, (a_n)_{n=1}^{N}) = \mathbb{E}\left[\sum_{n=1}^{N} \sum_{t=1}^{\tau_n} r_n(a_n) - \sum_{n=1}^{N} \sum_{t=1}^{\tau_n} r_n(A_{n,t}) \right],$$

where the expectation is over the random actions of the learner, which may depend on the realization of the noise in the rewards as well as some internal randomness used by the learner. Note that, in the notation, we suppressed the dependence of the regret on the segment lengths $(\tau_n)_{n=1}^{N}$ and only indicated their sum $T = \sum_{n=1}^{N} \tau_n$. The learner's worst-case expected regret relative to the best set of M arms (in the sense of achieving the highest expected reward) is defined as

$$R_T \doteq R(\pi, T, N, M) \doteq \sup_{(r_n)_{n=1}^{N}} \max_{(a_n)_{n=1}^{N}: |\{a_n\}_{n=1}^{N}| \leq M} R(\pi, T, (r_n)_{n=1}^{N}, (a_n)_{n=1}^{N}).$$

Note that, as usual in bandit problems, we use the mean reward functions r_n in the regret definition rather than the noisy rewards. Importantly, the learner competes with a sequence of arms that has at most M distinct elements. We call such problems *sparse* non-stationary bandit problems (Kwon et al., 2017). We say a sequence of reward functions $r_1, \ldots, r_N$ is *realizable* if there exists a set of arms of size at most $M \leq K$ which contains an optimal arm (achieving reward $\max_a r_n(a)$) for every task n. Otherwise, the problem is called *agnostic*. All our results, apart from those in Section 4, hold in the agnostic setting. In the meta-learning problems, our regret guarantees in fact hold even in the more general adversarial setting where the reward vector can change within each task and regret is defined as

$$R_T = \sup_{(r_{n,t})_{n=1,t=1}^{N,\tau_n}} \max_{(a_n)_{n=1}^{N}: |\{a_n\}_{n=1}^{N}| \leq M} \mathbb{E}\left[\sum_{n=1}^{N} \sum_{t=1}^{\tau_n} r_{n,t}(a_n) - \sum_{n=1}^{N} \sum_{t=1}^{\tau_n} r_{n,t}(A_{n,t}) \right].$$

Our goal in this paper is to provide algorithms achieving small worst-case expected regret for both the above problems. In the bandit meta-learning problem, the task lengths (and therefore, the change points) are known at the beginning of each task, while in the sparse non-stationary bandit problems, the start and duration of tasks are unknown, but the number of tasks N is known at the beginning of the game.

1.1 Approach: reduction to bandit subset selection

Our approach is based on a reduction to the bandit subset selection problem. We describe it first in the simpler bandit meta-learning problem. A natural solution to this problem is to use an appropriate base bandit algorithm restricted to the optimal subset of M arms at the beginning of each task. Unfortunately, this set is unknown. Nevertheless, it can be learned, and if the learner identifies the correct subset, a restricted version of the base bandit algorithm can be applied to the next task. For any K, let $B_{\tau,K}$ be a worst-case regret upper bound in τ steps for the *base* K-armed bandit algorithm. Note that $B_{\tau_n,K} = \widetilde{O}(\sqrt{K\tau_n})$ for the standard bandit algorithms (such as `UCB` and Thompson sampling in stochastic tasks, or EXP3 in adversarial tasks, see, e.g., Lattimore and Szepesvari, 2020);[2] accordingly, we assume throughout that $B_{\tau,K}$ is a non-decreasing concave function of both τ and K. When restricted to the correct M-subset (i.e, a subset of size M containing the best arm) of the K arms, the regret of the base algorithm is bounded by $B_{\tau_n,M} \leq B_{\tau_n,K}$. Otherwise, if the chosen subset does not contain the best arm, the regret on this task can be linear in τ_n. Therefore, it is important to identify the correct subset quickly.

This way the bandit meta-learning problem can be reduced to a bandit subset-selection problem, where the decision space is the set of M-subsets of $[K]$ and the reward in round n is the maximum cumulative reward

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[2]The notation $\widetilde{O}$ hides polylogarithmic terms.

in task n over the chosen subset. For example, when all tasks have length τ, and $\tau \ll N$, an $o(N)$-regret for the bandit subset-selection problem translates to an $O(NB_{\tau,M}) + o(N)$ bound on $R(\pi, T, N, M)$ for the bandit meta-learning problem. Notice that the leading term in the regret is $O(NB_{\tau,M})$, which is an upper bound on the regret of a bandit strategy that runs the base algorithm on the correct subset in each task. The subset-selection problem is a bandit submodular maximization problem, for which the online greedy approximation (OG°) algorithm[3] of Streeter and Golovin (2007) achieves regret $O(N^{2/3})$.

The approach to the sparse non-stationary bandit problem is similar to the one described above, with the difference that pre-determined segments are used in place of task boundaries, and we require the base algorithm to have a guarantee for a notion of *dynamic regret*: for any $1 \leq i \leq j \leq N$, and letting $L = \sum_{n=i}^{j} \tau_n$,

$$\sup_{(r_n)_{n=i}^{j}} \mathbb{E}\left[\sum_{n=i}^{j}\sum_{t=1}^{\tau_n}(\max_{a \in [K]} r_n(a) - r_n(A_{n,t}))\right] \leq B_{L,j-i,K}, \tag{1}$$

where we naturally extend the upper-bound notation to indicate the number of task changes in the segment of length L, and typically $B_{L,j-i,K} = \widetilde{O}(\sqrt{L(j-i+1)K})$. An example of such a base algorithm, which does not need to know the task boundaries and lengths, is AdSwitch (Auer et al., 2019b).

1.2 Contributions

In Section 3, we present an algorithm for our sparse non-stationary bandit problem, and show that its dynamic regret scales as $\widetilde{O}(MN^{-1/3}T(\log K)^{1/3} + M\sqrt{MNT})$. This algorithm is based on the aforementioned reduction to bandit subset selection problem and uses a bandit submodular optimization method, and needs to know the number of change points in advance. Without any restrictions on the set of optimal arms, the optimal rate for the non-stationary problem is $\widetilde{O}(\sqrt{KNT})$ (Auer et al., 2019b; Chen et al., 2019; Wei and Luo, 2021).[4] For small N, this baseline rate cannot be improved. Therefore, we are mainly interested in the regime of a large number of tasks and a small number of optimal arms, for which our regret upper bound is smaller than the baseline regret.

The same reduction to submodular optimization also applies to the bandit meta-learning problem, and leads to a regret bound that is better than the simple baseline of $O(\sum_{n=1}^{N} B_{\tau_n,K})$; this is shown in Section 3.2. Importantly, in this setting, our regret guarantees hold even in the more general adversarial problems where the reward vector can change within each task.

In Section 4, we analyze the bandit meta-learning problem in the realizable setting with stochastic tasks, and provide two algorithms which provide sample-efficient solutions to the problem under some assumptions on the identifiability of the optimal arms in each task. When all tasks have the same length τ, our computationally efficient algorithm G-BASS achieves an $O(NB_{\tau,M(1+\log N)}) + \widetilde{O}(N^{1/2})$ regret, paying a small $\log N$ factor in the regret for computational efficiency. To remove this extra factor we design E-BASS, which achieves the desired $O(NB_{\tau,M}) + \widetilde{O}(N^{1/2})$ regret, at the cost of a computational complexity that is potentially exponential in M. Although G-BASS is similar to the greedy solution in offline submodular maximization, the proof technique is quite different and exploits the special structure of the problem and results in an improved regret. A naive reduction to bandit submodular maximization would yield only an $\widetilde{O}(NB_{\tau,M}) + \widetilde{O}(N^{2/3})$ regret.

2 Reduction to subset-selection and bandit submodular maximization

As explained in Section 1.1, we reduce our problems to a bandit subset-selection problem. However, instead of learning the subset of the best arms task by task (whose boundaries are unknown in non-stationary problems), it will be convenient to divide the time horizon T into N' segments of equal length $\tau' = T/N'$ (in meta-learning problems when tasks are of equal length, we can choose $N' = N$ making segments and tasks

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[3] The superscript ^{o} stands for "opaque" feedback model in the cited paper.
[4] This rate is achieved by algorithms such as AdSwitch of Auer et al. (2019b) without any prior knowledge of T or N.

coincide), and learn the optimal arms from segment to segment. Consider segment n, $[(n-1)\tau'+1, n\tau')$. Without loss of generality, we assume that a new task starts at the beginning of each segment (we can redefine the task boundaries accordingly), so we have at most $N + N' - 1$ tasks. That is, denoting by N_n the number of tasks in segment n, we have

$$\sum_{n=1}^{N'} N_n \leq N + N' - 1 . \tag{2}$$

We also denote $\tau_{n,u}$ and $r_{n,u}$ respectively the length and mean reward of u'th task in segment n.

In segment n, the learner chooses a subset of arms $S_n \in \mathcal{S} = \{S : S \in 2^{[K]}, |S| \leq M\}$ and runs a base bandit algorithm on that subset for τ' steps and receives a pseudo-reward[5]

$$\sum_{u=1}^{N_n} \sum_{t=1}^{\tau_{n,u}} r_{n,u}(A_{n,u,t}) = \sum_{u=1}^{N_n} \left(\tau_{n,u} \max_{a \in S_n} r_{n,u}(a) - \sum_{t=1}^{\tau_{n,u}} \left(\max_{a \in S_n} r_{n,u}(a) - r_{n,u}(A_{n,u,t}) \right) \right)$$
$$\doteq f_n(S_n) - \tau' \varepsilon_n ,$$

where $A_{n,u,t}$ is the learner's action in time step t of task u of segment n, $f_n(S) \doteq \sum_{u=1}^{N_n} \tau_{n,u} f_{n,u}(S)$ and $f_{n,u}(S) \doteq \max_{a \in S} r_{n,u}(a)$ is the max-reward function for a set of arms S and reward function $r_{n,u}$, and ε_n is the average "noise" per time step observed by the agent. We assume the base algorithm satisfies (1), $\tau' \mathbb{E}[\varepsilon_n] \leq B_{\tau', N_n, M}$. We prove in Appendix A that $f_n \in \mathcal{F}$, the family of submodular functions, since it is an affine combination of finitely many submodular functions. Now we can bound the regret in the sparse non-stationary bandit problem as follows (proof in Appendix B):

Lemma 2.1. *For any policy running a base algorithm satisfying* (1) *in each segment n on a selected subset of arms S_n, we have* $R_T \leq \sup_{f_1,\dots,f_{N'} \in \mathcal{F}} \max_{S \in \mathcal{S}} \mathbb{E}\left[\sum_{n=1}^{N'} \left(f_n(S) - f_n(S_n) + B_{\tau', N_n, M} \right) \right].$

Thus, the sparse non-stationary bandit problem can be reduced to minimizing a notion of regret where in segment n, the learner chooses a subset $S_n \in \mathcal{S}$ and observes $f_n(S_n) - \tau' \varepsilon_n$. Since f_n is submodular, this reduction allows us to leverage the literature on bandit submodular optimization to get a bound on R_T.

2.1 Bandit submodular maximization

As mentioned above, the problem of subset selection is an instance of online submodular maximization. Streeter and Golovin (2007) studied the online submodular maximization problem in four different settings: the full-information setting where the function f_n is fully observed at the end of round n; the priced feedback model where the learner can observe f_n by paying a price and the price is added to the total regret; a partially transparent model where values of f_n for some subsets are revealed, and the bandit setting where only $f_n(S_n)$ is observed. For the full-information and partially transparent models, an $O(\sqrt{N})$ regret is shown, while for the priced feedback and bandit model, an $O(N^{2/3})$ regret is shown. Radlinski et al. (2008) provide a similar algorithm to that of Streeter and Golovin (2007) for a particular ranking problem, with the more informative partially transparent feedback model, and also obtain an $O(N^{1/2})$ regret bound. The priced feedback model is similar to problems where the best arms can be identified in every task, which we study in Section 4.

3 A general solution based on submodular maximization

In this section we present our first algorithm, `OS-BASS`, for "online submodular bandit subset selection", based on the above reduction and the general submodular maximization algorithm `OG°` of Streeter and Golovin (2007). The algorithm is applicable in the agnostic setting and its pseudo-code is given in Algorithm 1.

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[5]We call it pseudo-reward since it considers the mean reward functions not the actual random actions.

Algorithm 1 OS-BASS: Online Submodular Bandit Subset Selection for Non-Stationary Bandits and Meta-Learning

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1: **Input**: Subset size M, segment length τ', Expert algorithms $\mathcal{E}_1, \cdots, \mathcal{E}_{\tilde{M}}$;
2: **for** $n \in [N']$ **do**
3:      For each $i \in [\tilde{M}]$, use $\mathcal{E}_i$ to choose an action a_i
4:      Set $S_n = \{a_1, \cdots, a_{\tilde{M}}\}$
5:      With prob. γ_n, $E_n = \text{EXR}$, otherwise $E_n = \text{EXT}$
6:      **if** $E_n = \text{EXT}$ **then**
7:          Run `Base` on S_n for τ' steps
8:      **else**
9:          Choose $i \in [\tilde{M}]$ uniformly at random
10:         Choose a new action a_i' uniformly at random
11:         Replace i'th element of S_n with a_i':
12:            $S_{n:i} \leftarrow \{a_1, \cdots, a_{i-1}, a_i'\}$
13:         Run `Base` on $S_{n:i}$ for τ' steps
14:         Feed back the average reward over the segment as a reward to $\mathcal{E}_i$ for a_i', and zero reward for all other actions and experts
15:      **end if**
16: **end for**

`OS-BASS` applies $\tilde{M} = \lceil M \log N' \rceil$ expert algorithms (i.e, regret minimization algorithms in the full information setting) over the K arms, denoted by $\mathcal{E}_1, \ldots, \mathcal{E}_{\tilde{M}}$, where the role of $\mathcal{E}_i$ is to learn the ith "best" action.

In segment n, each expert picks an action. The algorithm *exploits* (EXT) the set of $\tilde{M}$ actions picked by the experts with probability $1 - \gamma_n$, i.e, a base algorithm `Base` is executed on this set for τ' steps. With probability γ_n, the algorithm *explores* (EXR), meaning that first a random index $i \in [\tilde{M}]$ is chosen uniformly at random, and then the set of $i - 1$ actions picked by the experts $\{\mathcal{E}_1, \cdots, \mathcal{E}_{i-1}\}$ is chosen, and together with a random action a_i', `Base` is executed on this set of i actions, $S_{n:i}$, for τ' steps. When exploring, the expert whose action is replaced with a random action ($\mathcal{E}_i$) is updated with a reward function where the reward for the chosen action a_i' is the average reward of `Base` in the segment (approximately $f_n(S_{n:i})/\tau'$ up to error ε_n/τ'), while for all other actions the reward is zero, and all other experts are updated with an all-zero reward function.[6] In this way, $\mathcal{E}_1, \ldots, \mathcal{E}_i$ jointly learn the identity of the best i-set given the (approximate) reward function $\sum_{n=1}^{N} f_n$, hence approximating a greedy solution for finding the top M arms (essentially, the "real" reward of expert $\mathcal{E}_i$ for action a is $f_n(S_{n:i-1} \cup \{a\}) - f_n(S_{n:i-1})$). `OS-BASS` should be viewed as a simulation of the off-line greedy procedure, which constructs its solution incrementally (Streeter and Golovin, 2007). When an index i is chosen, the algorithm is learning the ith choice of the off-line greedy procedure and therefore only a subset of size i is played.

Our requirement about the expert algorithm is that it achieves an $O(\sqrt{v \log(K)})$ regret over v time steps relative to the best action selected in hindsight; this is achieved by all standard expert algorithms, such as exponential weights (Cesa-Bianchi and Lugosi, 2006). The choice of `Base` depends on which setting we solve.

3.1 The sparse non-stationary bandit setting

To choose `Base`, notice that in the sparse non-stationary bandit problem, every segment may contain multiple tasks. Therefore, we need to choose an algorithm that is able to solve non-stationary bandit problems, without the knowledge change points. Hence, we use the `AdSwitch` algorithm of Auer et al. (2019b) as `Base` in Algorithm 1, giving $B_{\tau', N_n, \tilde{M}} = \sqrt{\tilde{M} N_n \tau'}$ for segment n in Lemma 2.1. The performance of the algorithm is analyzed in the next theorem (proved in Appendix C.3). Note that the algorithm does not need to know

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[6]This is simply to construct an unbiased estimator and use importance sampling. We provide no feedback to the experts during the exploitation phase. It is reasonable to expect that by using such feedback we can improve the empirical performance of the algorithm, but using the feedback is not going to improve the theoretical guarantees.

when the tasks change and the tasks can be of different lengths.

Theorem 3.1. The regret of `OS-BASS` using `AdSwitch` as `Base`, with N' segments each of size $\tau' = T/N'$, and using exploration probability $\gamma_n = \left(\frac{\tilde{M}K\log K}{N'}\right)^{1/3}$ is $R_T = \tilde{O}(TN'^{-1/3}(\tilde{M}^4 K \log K)^{1/3} + \tilde{M}\sqrt{\tilde{M}N'T})$.

We are mainly interested in the regime of large number of switches and small number of optimal arms. If $N \geq \frac{T^{3/5}(\log K)^{2/5}}{\tilde{M}^{3/5}}$ and $M \leq K^{1/3}$, with the choice of $N' = N$, we have $\tilde{M}(\log K)^{1/3}N^{-1/3}T \leq \tilde{M}^{3/2}\sqrt{NT}$, and the regret of `OS-BASS` improves upon the $\tilde{O}(\sqrt{KTN})$ bound of standard non-stationary bandit algorithms (such as `AdSwitch`). In the case of a small number of switches, however, the simple baseline of $\tilde{O}(\sqrt{KTN})$ can be smaller than our bound, and the learner should simply play a standard non-stationary bandit algorithm. At the extreme that $N = O(1)$, i.e. only a few changes in the environment, the regret bound $\tilde{O}(\sqrt{KNT})$ cannot be improved. On the other hand, when N is large compared to T, it is easy to establish a $O(\sqrt{MNT})$ lower bound, and therefore our $\tilde{O}(M\sqrt{MNT})$ bound is optimal up to a factor of M. Closing this gap remains an open question. We can improve the bounds in certain regimes by further tuning of the parameters, which is discussed in more details in Appendix C.3.

The algorithm needs to know the number of changes N in advance: the segment length τ' depends on N', and as the proof of Theorem 3.1 reveals, we need $N' \geq N$ to have the above regret upper bound. In the next section, we consider a meta-learning setting where segment lengths are fixed in advance. In that case, we can have a version of the algorithm that does not need to have N as input.

3.2 Bandit meta-learning

In this section we consider the bandit meta-learning problem with tasks of equal and known length τ. Again, we apply `OS-BASS`, instantiated now with different parameters and a different `Base` algorithm. Since we know τ, we can always introduce a segmentation such that each segment contains a single task only (possibly only a part of a task), and choose `Base` to be an algorithm designed for a multi-armed bandit problem such as the well-known `UCB` algorithm (Auer et al., 2002) when tasks are stochastic bandits, or EXP3 algorithm (Auer et al., 1995) when tasks are adversarial bandits.

Corollary 3.2. Consider `OS-BASS` with `UCB` as `Base` in a meta-learning problem with N stochastic bandit tasks of equal length τ. By the choice of $\tau' = \tau$ and $\gamma_n = \left(\frac{\tilde{M}K\log K}{n}\right)^{1/3}$, the regret is $R_T = \tilde{O}(\tau N^{2/3}(\tilde{M}^4 K \log K)^{1/3} + \tilde{M}N\sqrt{\tilde{M}\tau})$.

The proof is omitted as it follows the proof argument of Theorem 3.1. Again, in the regime of large number of tasks and small number of optimal arms, our bound improves upon the $\tilde{O}(N\sqrt{K\tau})$ bound of the trivial solution of running an independent UCB algorithm in each task.

We can extend the above result to an adversarial setting, where the reward vector can change in every time step, by using an EXP3 algorithm instead of UCB. Such an extension is not possible in the sparse non-stationary setting as currently we are not aware of a base algorithm with *adaptive* dynamic regret guarantee in the adversarial setting.

Remark 3.3 (Variable task lengths). Consider problems with non-equal task lengths where the learner only gets to know the length of each task when they begin. To deal with this situation, we construct an exponential grid for the task lengths with $b := \log(\max_n \tau_n) \leq \log T$ buckets where bucket i is $[\underline{\tau}_i := 2^{i-1}, \overline{\tau}_i := 2^i]$. Now we run a copy of `OS-BASS` on the tasks falling in each bucket as they arrive. Let $N^{(i)}$ denote the number of tasks that fall in bucket i. Then by Corollary 3.2, the total regret satisfies $R_T \leq \sum_{i=1}^{b} \tilde{O}(\tilde{M}^{4/3}(N^{(i)})^{2/3}K^{1/3}\overline{\tau}_i + N^{(i)}\tilde{M}^{3/2}\sqrt{\overline{\tau}_i})$.

4 Bandit meta-learning under an identifiability condition

In this section, we study the bandit meta-learning problem in the *realizable* setting under the assumption that the learner has access to an exploration method that reveals optimal actions. We further assume that the tasks are of equal length τ.

Assumption 4.1 (Efficient Identification). There exists a set of M arms that has a non-empty intersection with the set of optimal arms in each task. Also, the learner has access to a best-arm-identification (BAI) procedure that for some $\delta \in [0,1]$, with probability at least $1 - \delta/N$, identifies the set of optimal arms if executed in a task (for at most τ steps)[7].

The Efficient Identification assumption is a special case of the priced feedback model of Streeter and Golovin (2007). If for any task n with optimal arms $S_n^* \subset [K]$, we have $r_n(a_n^*) - \max_{a \notin S_n^*} r_n(a) \geq \Delta$ for all $a_n^* \in S_n^*$ (note that $r_n(a_n^*)$ is the same for all $a_n^* \in S_n^*$) for some $\Delta = \Theta(\sqrt{K \log(N/\delta)/\tau})$, a properly tuned *phased elimination* (PE)[8] procedure (Auer and Ortner, 2010) returns the set of optimal arms with probability at least $1 - \delta/N$. The cumulative worst-case regret of PE in a task with K arms is $B'_{\tau,K} = \Theta(B_{\tau,K})$ (Auer and Ortner, 2010), see Lattimore and Szepesvári (2020) for details. With a slight abuse of notation, in this section we use $B_{\tau,K}$ to denote $\max\{B_{\tau,K}, B'_{\tau,K}\}$.

Akin to `OS-BASS`, we disentangle exploration (ExR) and exploitation (ExT) at a meta-level. In ExR mode, the learner executes a BAI on all arms, and (by Assumption 4.1) observes, with high probability, the set of optimal actions, S_n^*. The price of this information is a large regret denoted by $\mathbf{C}_{\texttt{info}}$, which for a properly tuned PE, we know $\mathbf{C}_{\texttt{info}} = B_{\tau,K} > B'_{\tau,M}$. So, since we aim for $R_T \leq \widetilde{O}(NB_{\tau,M}) + o(N)$, we should keep the number of ExR calls small. In ExT mode, the learner executes a base bandit algorithm on a chosen subset S_n, constructed using the previously identified optimal actions $\mathcal{I}_n = \bigcup_{j<n:E_j=\text{ExR}}\{S_j^*\}$.

Let s_n be the size of S_n. If $S_n \cap S_n^* \neq \emptyset$, the regret of the base algorithm is bounded by $\mathbf{C}_{\texttt{hit}} = B_{\tau,s_n}$. Otherwise, since the performance gap between the optimal arms and the arms in S_n can be arbitrary, the regret in the task can be as large as $\mathbf{C}_{\texttt{miss}} = \tau$. Note that to keep $\mathbf{C}_{\texttt{hit}}$ small, the subset S_n should be as small as possible. Ideally, S_n should be a subset of size M that has non-empty overlap with all members of $\mathcal{I}_n$. However, the problem of finding such S_n is the so-called *hitting set* problem, which is known to be NP-Complete (Feige et al., 2004).

A simple greedy algorithm can be used to get an approximate solution efficiently (see, e.g, Streeter and Golovin, 2007): it has polynomial computation complexity and it finds a subset of size at most $M(1+\log N)$ that contains an optimal action for each task. We say an action $a \in [K]$ covers task j if $a \in S_j^*$. The greedy method, denoted by GREEDY, starts with an empty set and at each stage, it adds the action that covers the largest number of uncovered tasks in $\mathcal{I}_n$, until all tasks are covered.

We also propose `E-BASS`, that is based on an elimination procedure; the learner maintains an active set of possible M-subsets compatible with the ExR history, and all subsets that are inconsistent with $\mathcal{I}_n$ are eliminated. In ExT mode, a subset is selected uniformly at random from the set of active subsets. As we will show, this algorithm improves the regret by a factor of $\log N$, although it is not computationally efficient.

The analysis of `G-BASS` depends on the *cost-to-go* function of the following game between the learner and environment. At round n, the learner may choose $E_n \in \{\text{ExR}, \text{ExT}\}$ with distribution $p_n = P(E_n = \text{ExR})$ and cost $\mathbf{C}_{\texttt{info}}$. The environment may choose a best arm a_n^* that the learner already knows about which costs $\mathbf{C}_{\texttt{hit}}$ (i.e, $a_n^* \in S_n$) or choose an optimal arm set S_n^* so that $S_n^* \cap S_n = \emptyset$, with cost $\mathbf{C}_{\texttt{miss}}$. Let $q_n = P(S_n^* \cap S_n = \emptyset)$. The regret of `G-BASS` is bounded by the cost of the learner in this simple game, if we assume $\delta = 0$ in Assumption 4.1. The learner is a (randomized) function of $\mathcal{I}$, hence we can easily define and write the minimax cost-to-go function as

$$
\begin{aligned}
V_N(\mathcal{I}) = 0 \ &\text{ and for } n < N, \\
V_n(\mathcal{I}) = \min_p \max_q \{ &p\mathbf{C}_{\texttt{info}} + q(1-p)\mathbf{C}_{\texttt{miss}} + (1-q)(1-p)\mathbf{C}_{\texttt{hit}} \\
&+ (1-pq)V_{n+1}(\mathcal{I}) + pqV_{n+1}(\mathcal{I} \cup \{S_n^*\})\}.
\end{aligned}
\tag{3}
$$

For the last equality note that when the environment reveals a new action (happens with probability q) and the learner explores (with probability p), its current knowledge set $\mathcal{I}$ is incremented. The optimal cost-to-go

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[7]Note that the Efficient Identification assumption requires the BAI procedure to return only optimal arms. This choice is for simplicity and could easily be relaxed to allow the returned set to be all arms with sub-optimality gap smaller than $\Theta(\sqrt{M \log(N/\delta)/\tau})$; we discuss this in more details in the analysis.

[8]Improved `UCB` of Auer and Ortner (2010) runs exponentially growing phases and maintains an active set of arms with gaps twice smaller in every step.

Algorithm 2 BASS: Bandit Subset Selection for Meta-Learning

1: **Option:** Greedy G-BASS (G), Elimination-based E-BASS (E)
2: **Input:** Base (efficient K-armed bandit algorithm), best arm identification algorithm BAI, EXR probabili-
    ties p_n, (E) subset size M
3: **Initialize:** Let (G) $\mathcal{I}_0 = \emptyset$; (E) $\mathcal{X}_0$ be the set of all M-subsets of $[K]$.
4: **for** $n = 1, 2, \ldots, N$ **do**
5:     With prob. p_n, let $E_n = \text{EXR}$; otherwise $E_n = \text{EXT}$
6:     **if** $E_n = \text{EXR}$ or $n = 1$ **then**
7:         Run BAI on all arms and observe the best arms S_n^* of this task
8:         (G) Update $\mathcal{I}_n = \mathcal{I}_{n-1} \cup \{S_n^*\}$
9:         (E) Let $\mathcal{X}_n = \{S \in \mathcal{X}_{n-1} : S \cap S_n^* \neq \emptyset\}$ be elements of $\mathcal{X}_{n-1}$ with non-empty overlap with S_n^*;
10:    **else**
11:        (G) Find S_n by GREEDY s.t. $\forall S \in \mathcal{I}_n, S_n \cap S \neq \emptyset$
12:        (E) Sample S_n uniformly at random from $\mathcal{X}_n$
13:        Run Base algorithm on S_n
14:    **end if**
15: **end for**

---

function V_n above corresponds to the case of $\delta = 0$ in Assumption 4.1, and $V_0(\emptyset)$ gives the minimax regret for the family of algorithms with the limited choice described. Therefore, when the BAI algorithm is successful, almost surely, $R_T \leq V_0(\emptyset)$. For $\delta > 0$, using a union bound, we can show $R_T \leq V_0(\emptyset) + \delta N\tau$. Setting $\delta = 1/(N\tau)$ ensures that $\delta N\tau$ is negligible. Finally, if the BAI algorithm only returns a set of approximately optimal arms satisfying $r_n(a) \geq r_n(a_n^*) - \Delta$ for all arms a selected, the meta-regret can be bounded trivially as $R_T \leq V_0(\emptyset) + (\delta + \Delta)N\tau$.

Before deriving $V_0(\emptyset)$ in the general case, we consider the more restricted setting where each task has a unique optimum.

Characterizing an optimal policy in the case of unique optimal arms: Assume there is a unique and identifiable optimal arm in each task.

Assumption 4.2 (Unique Identification). Assumption 4.1 holds, and each task has a unique best arm.

Theorem 4.3. In the simple game,

$$V_0(\emptyset) \leq NB_{\tau,M} + M\sqrt{2(\mathbf{C}_{\texttt{info}} - \mathbf{C}_{\texttt{hit}})(\mathbf{C}_{\texttt{miss}} - \mathbf{C}_{\texttt{hit}})N} \ .$$

Therefore, under Assumption 4.2, the regret of G-BASS can be bounded, for an appropriate selection of the parameters $p_n = \Theta(1/\sqrt{N-n})$, as

$$R_T \leq NB_{\tau,M} + M\sqrt{B_{\tau,K}N\tau} + \delta N\tau \ .$$

We prove this in Appendix D.2 by solving the min-max problem in (3) for V_n. Interestingly, the exploration probability p_n increases as $\Theta(1/\sqrt{N-n})$. This might seem counter-intuitive at first as typically the exploration rate decreases in most online learning algorithms. The intuition is that as n gets closer to N, if $s < M$, the adversary does not have much time left to use the remaining budget to make the learner suffer a big cost. Therefore, the adversary needs to increase its probability of choosing a new optimal arm.

The more general case: Now we consider the more general case with potentially multiple optimal arms in each task.

Theorem 4.4. Let $M' = M(1 + \log N)$. Under Assumption 4.1, the regret of the G-BASS algorithm with exploration probability $p_n = \sqrt{\frac{|S_n|K\tau}{NB_{\tau,K}}}$ for all n is bounded as

$$R_T \leq NB_{\tau,M'} + MB_{\tau,K} + \sqrt{MKB_{\tau,K}N\tau} + \delta N\tau \ .$$

The proof is similar in spirit to that of Theorem 4.3 and is deferred to Appendix D.3. The regret guarantee holds in the realizable setting, but the `G-BASS` algorithm does not need M as input. The next theorem shows that the regret of `E-BASS` is bounded as $NB_{\tau,M} + o(N)$, which is smaller than the regret of the `G-BASS` algorithm by a factor of $\log N$; note, however, that `E-BASS` is not computationally efficient and also requires M as input. The proof of the theorem is in Appendix E.

Theorem 4.5. Under Assumption 4.1, with constant exploration probability $p_n = \left(\frac{\tau}{K}\right)^{1/4} \sqrt{\frac{\log K}{N}}$, the regret of the `E-BASS` algorithm is bounded as $R_T \leq NB_{\tau,M} + O(\tau^{3/4}K^{1/4}\sqrt{NM\log K})$.

Remark 4.6 (Connections with partial monitoring games). The setting of this section can be viewed more generally as a partial monitoring game. Partial monitoring is a general framework in online learning that disentangles rewards and observations (information). In our bandit meta-learning problem, different actions of the meta-learner (EXR and EXT) provide different levels of information and have different costs, and the problem can be reduced to a partial monitoring game on $\mathcal{X}$, the set of M-subsets of $[K]$. More details are in Appendix E.

5   Related work

Our bandit solution is based on subset selection, which connects it to many branches of online learning and bandit literature.

Non-stationary experts/bandits with long-term memory. The sparse non-stationary bandit problem is the bandit variant of experts problem with small set of optimal arms whose study goes back to Bousquet and Warmuth (2002). The only result in the bandit setting that we are aware of is the work of Zheng et al. (2019), who show an algorithm for competing against a small set of optimal arms in an adversarial setting with sparse reward vectors. The solution of Zheng et al. (2019) has similarities with our approach, as both solutions are reduction-based and employ a meta-learner that plays with base algorithms. Similar to our approach, the meta-learner of Zheng et al. (2019) needs to satisfy a static regret guarantee, while the base algorithm needs to satisfy a dynamic regret guarantee. The algorithm of Zheng et al. (2019) also requires the number of change points as input and its dynamic regret is $\tilde{O}((sN)^{1/3}(MT)^{2/3} + M\sqrt{sT\log K} + MK^3\log T)$, where $s = \max_n \|r_n\|_0$ is the number of non-zero elements in reward vectors. Notice that without this sparsity condition, the above dynamic regret is worse than $\tilde{O}(\sqrt{KNT})$ regret of EXP3.S. When s is a small constant, the above bound improves upon the regret of EXP3.S when $M^{4/3}(T/N)^{1/3} < K < (TN/M^2)^{1/5}$.

Meta-learning in adversarial bandits. Balcan et al. (2022) studies bandit meta-learning problems with adversarial bandit tasks. They introduce a meta-learning algorithm that tunes the initialization and step-size of the online mirror decent base algorithm. Balcan et al. (2022) show that their algorithm achieves $\min_{\beta \in (0,1]} \tilde{O}(N\sqrt{H_\beta K^\beta \tau/\beta} + N^{1-\beta/6})$ total regret, where H_β is a notion of entropy. When a subset of size M contains the optimal arms of most tasks, then by the choice of $\beta = 1/\log(K)$ the regret bound simplifies to $\tilde{O}(N\sqrt{M\tau} + N^{1-1/(6\log K)})$. Compared to our results in Section 3.2, their convergence rate of $\tilde{O}(1/N^{1/(6\log K)})$ is much slower than our convergence rate of $\tilde{O}(1/N^{1/3})$. However, their task-averaged regret is $O(\sqrt{M\tau})$, whereas it is $O(M\sqrt{M\tau})$ in our case. Finally, the regret bound of Balcan et al. (2022) adaptively holds for the best value of M, while M is an input to our algorithm. Osadchiy et al. (2022) further improves the results of Balcan et al. (2022) under extra identifiability assumptions.

Slate bandits. The reduction in Section 2 is an instance of slate bandit problems with a non-separable cost function (Dimakopoulou et al., 2019; Rhuggenaath et al., 2020; Kale et al., 2010). Rhuggenaath et al. (2020) study the problem in the stochastic setting, where the reward parameter is fixed throughout the game. Merlis and Mannor (2019) study a problem that includes the probabilistic maximum coverage as a special case. They obtain problem-dependent logarithmic and problem-independent $O(\sqrt{N})$ regret bounds. However, the feedback structure in this work is richer than our setting. Applied to our problem, they assume that in each round, for each item and task pair, a random variable is observed whose expected value is the probability that the item is the optimal arm in that task.

Meta-learning and bandit meta-learning. Meta, Multi-Task, and Transfer Learning (Baxter, 2000; Caruana, 1997; Thrun, 1996) are related machine learning problems concerned with learning a lower dimen-

sional subspace across tasks. In that sense, our work is connected to other theoretical studies (Franceschi et al., 2018; Denevi et al., 2018b,a, 2019; Kong et al., 2020; Khodak et al., 2019; Tripuraneni et al., 2021) though indeed we focus on the bandit learning setting. Various other ways of modelling structure have been proposed and studied in bandit meta-learning. A special case of our problem was studied by Azar et al. (2013) where K-armed bandit problems are sampled from a prior over a finite set of tasks. Park et al. (2021) consider a continual learning setting where the bandit environment changes under a Lipschitz condition. Kveton et al. (2020) observe that the hyperparameters of bandit algorithms can be learned by gradient descent across tasks. Learning regularization for bandit algorithms (Kveton et al., 2021; Cella et al., 2020) are also proposed, building on the biased regularization ideas from Baxter (2000). Interestingly, these contextual problems are also connected with latent and clustering of bandit models (Maillard and Mannor, 2014; Gentile et al., 2014; Hong et al., 2020a,b).

Bandits with very large action spaces. As K grows very large, our bandit meta-learning problem is akin to infinitely many armed bandits (Berry et al., 1997; Wang et al., 2008; Bonald and Proutiere, 2013; Carpentier and Valko, 2015; Chan and Hu, 2020) and countable-armed bandits (Kalvit and Zeevi, 2020) though these settings do not have a meta-learning aspect.

6   Experiments

In this section, we study the performance of our algorithms on synthetic environments. The experiments include: 1) G-BASS [9] from Section 4, 2) Algorithm 1, OS-BASS, 3) MOSS which is agnostic MOSS (Audibert and Bubeck, 2009) running independently on the tasks without any knowledge of the optimal M-subset, 4) Opt-MOSS, an oracle MOSS that plays only the arms in the optimal M-subset, and its performance constitutes an empirical lower bound on the achievable regret, and 5) OG°, which is OS-BASS with the choice of $\tau' = \tau$ and optimized γ. Error bars are ± 1 standard deviation computed over 5 independent runs.

We study the impact of four variables on the regret: number of tasks N, length of each task τ, number of arms in each task K, and the optimal subset size M. To do so, we fix a default setting of (N, τ, K, M) and for each experiment we let one of these parameters vary. The problems are generated by an *oblivious* adversary (see Appendix F for further details).

Fig. 1 demonstrates the impact of N and M. Further experiments in Appendix F illustrate the effect of all four variables including τ and K. Under Assumption 4.1 (left two plots), G-BASS outperforms all methods with a regret close to that of the oracle Opt-MOSS. When this assumption does not hold (right two plots), OS-BASS outperforms the other algorithms, while G-BASS naturally has high variance.
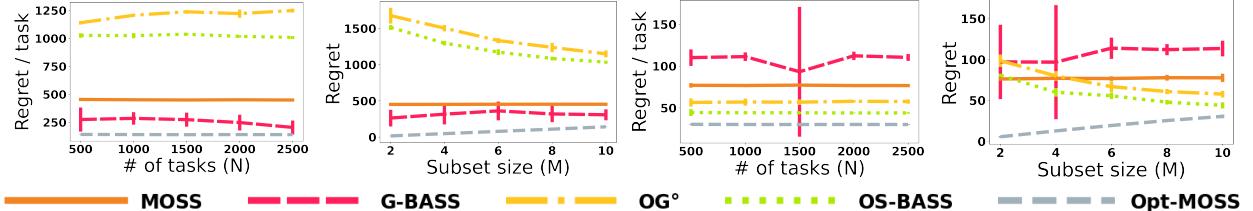

Figure 1:   Default setting: $(N, \tau, K, M) = (500, 4500, 30, 10)$. In the right two plots $\tau = 450$. Left to Right: Regret as a function of N and M under Assumption 4.1. Regret as a function of N and M without Assumption 4.1.

7   Discussion and Future Work

We analyze a problem of N tasks of K-armed bandits arriving sequentially, where the adversary is constrained to choose the optimal arm of each task in a smaller subset of M arms. We consider two settings: bandit

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[9] E-BASS is computationally too expensive so we only run it on smaller settings in Appendix F.

meta-learning and non-stationary bandits. We design an algorithm based on a reduction to bandit sub-modular optimization, and prove that its regret is $\tilde{O}(MN^{-1/3}T(\log K)^{1/3} + M^{3/2}\sqrt{NT})$. Under additional identifiability assumptions, we develop a meta-learning algorithm with an improved regret.

For the general non-stationary setting, we assume N is known. Relaxing this assumption adaptively is left for further work.

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A   Submodular functions

In this section, we verify that $f(S) \doteq f(r, S) = \max_{a \in S} r(a)$ is a submodular function. It is obviously monotone. We have

$$f(S_1 \cup S_2 \cup \{a\}) - f(S_1 \cup S_2) \leq f(S_1 \cup \{a\}) - f(S_1)$$

This is because (1) if $r(a) \leq f(S_1)$ then the inequality holds as $f(S_1 \cup \{a\}) = f(S_1)$ and $f(S_1 \cup S_2 \cup \{a\}) = f(S_1 \cup S_2)$. (2) if $r(a) > f(S_1)$ then (2.i) if $r(a) > f(S_1 \cup S_2)$ the inequality holds as $r(a) - f(S_1 \cup S_2) \leq r(a) - f(S_1) \iff f(S_1 \cup S_2) \geq f(S_1)$, which holds by monotonicity of f, (2.ii) if $r(a) \leq f(S_1 \cup S_2)$ then $f(S_1 \cup S_2 \cup \{a\}) = f(S_1 \cup S_2)$ and the inequality simplifies to $0 \leq r(a) - f(S_1)$ which holds as we assumed $r(a) > f(S_1)$ in (2).

B   Proofs of Section 2

Proof of Lemma 2.1. We have that

$$
\begin{aligned}
R_T &= \sup_{\substack{r_1,\ldots,r_N \\ \in [0,1]^K}} \max_{\substack{a_1,\ldots,a_N \in [K], \\ |\{a_1,\ldots,a_N\}| \leq M}} \mathbb{E}\left[\sum_{n=1}^{N} \sum_{t=1}^{\tau_n} (r_n(a_n) - r_n(A_{n,t}))\right] \\
&= \sup_{\substack{r_1,\ldots,r_N \\ \in [0,1]^K}} \max_{S \in \mathcal{S}} \mathbb{E}\left[\sum_{n=1}^{N'} \sum_{u=1}^{N_n} \sum_{t=1}^{\tau_{n,u}} (\max_{a \in S} (r_{n,u}(a) - r_{n,u}(A_{n,u,t}))\right] \\
&= \sup_{\substack{r_1,\ldots,r_N \\ \in [0,1]^K}} \max_{S \in \mathcal{S}} \mathbb{E}\left[\sum_{n=1}^{N'} \sum_{u=1}^{N'_n} \left(\sum_{t=1}^{\tau_{n,u}} (\max_{a \in S} r_{n,u}(a) - \max_{a \in S_n} r_{n,u}(a)) + \sum_{t=1}^{\tau_{n,u}} (\max_{a \in S_n} r_{n,u}(a) - r_{n,u}(A_{n,u,t}))\right)\right] \\
&\leq \sup_{(f_{n,u} \in \mathcal{F})_{n \in [N'], u \in [N_n]}} \max_{S \in \mathcal{S}} \mathbb{E}\left[\sum_{n=1}^{N'} \left(\sum_{u=1}^{N_n} \tau_{n,u}(f_{n,u}(S) - f_{n,u}(S_n)) + \tau' \varepsilon_n\right)\right] \\
&= \sup_{f_1,\cdots,f_{N'} \in \mathcal{F}} \max_{S \in \mathcal{S}} \mathbb{E}\left[\sum_{n=1}^{N'} (f_n(S) - f_n(S_n) + B_{\tau',N_n,M})\right] .
\end{aligned}
$$

$\square$

C   Proofs of Section 3

First, we present the relevant results from Streeter and Golovin (2007) with appropriate modifications. We start with the regret analysis of the `OG` algorithm, which is designed to solve the online submodular maximization in the full feedback model.

C.1   The `OG` algorithm

For a submodular function g, consider an ordered set of actions $\bar{S} = \langle \bar{a}_1, \bar{a}_2, \cdots \rangle$ that satisfies the following *greedy* condition for any j:

$$g(\bar{S}_j \cup \bar{a}_j) - g(\bar{S}_j) \geq \max_{a \in [K]} \{g(\bar{S}_j \cup a) - g(\bar{S}_j)\} - \alpha_j, \tag{4}$$

where α_j are some positive error terms. Let $\bar{S}_0 = \emptyset$, $\bar{S}_j = \langle \bar{a}_1, \bar{a}_2, \cdots, \bar{a}_{j-1} \rangle$, and for a sequence of actions $S \subset [K]$, let $S_{\langle M \rangle}$ be the set of actions in S truncated at the M'th action. The following result shows near-optimality of $\bar{S}$ as constructed above. Recall that $\tilde{M} = \lceil M \log N' \rceil$.

Algorithm 3 OG algorithm

1: **Input**: Subset size M, Expert algorithms $\mathcal{E}_1, \cdots, \mathcal{E}_{\tilde{M}}$;
2: **for** $n \in [N']$ **do**
3:     Let $S_{n,0} = \emptyset$
4:     **for** $j \in \{1, \cdots, \tilde{M}\}$ **do**
5:         Let action $a_j^n \in [K]$ be the choice of expert $\mathcal{E}_j$
6:         Let $S_{n,j} \leftarrow S_{n,j-1} \cup \{a_j^n\}$
7:     **end for**
8:     Play subset S_n and observe function g_n.
9:     **for** $j \in \{1, \cdots, \tilde{M}\}$ and any $a \in [K]$ **do**
10:        Let $x_{j,a}^n \leftarrow g_n(S_{n,j-1} \cup \{a\}) - g_n(S_{n,j-1})$
11:        Expert $\mathcal{E}_j$ receives payoff vector $(x_{j,a}^n)_{a \in [K]}$
12:     **end for**
13: **end for**

Theorem C.1 (Based on Streeter and Golovin (2007), Theorem 6). Consider the greedy solution in Eq. (4). Then

$$g(\bar{S}_{\langle \tilde{M} \rangle}) > \left(1 - \frac{1}{N'}\right) \max_{S \in \mathcal{S}} g(S_{\langle M \rangle}) - \sum_{j=1}^{\tilde{M}} \alpha_j \ .$$

Proof. Let $C^* = \max_{S \in \mathcal{S}} g(S_{\langle M \rangle})$ and for any j let $\Delta_j = C^* - g(\bar{S}_j)$. Then, by Fact C.1 below, we have $C^* \leq g(\bar{S}_j) + M(s_j + \alpha_j)$. Therefore, $\Delta_j \leq M(s_j + \alpha_j) = M(\Delta_j - \Delta_{j+1} + \alpha_j)$ which means $\Delta_{j+1} \leq \Delta_j(1 - \frac{1}{M}) + \alpha_j$. Unrolling this $\tilde{M}$ times (and noting $1 - \frac{1}{N'} < 1$) gives

$$\Delta_{\tilde{M}+1} \leq \Delta_1 \left(1 - \frac{1}{M}\right)^{\tilde{M}} + \sum_{j=1}^{\tilde{M}} \alpha_j$$

$$< \Delta_1 \frac{1}{N'} + \sum_{j=1}^{\tilde{M}} \alpha_j \leq C^* \frac{1}{N'} + \sum_{j=1}^{\tilde{M}} \alpha_j.$$

This concludes the proof since $C^* - g(\bar{S}_{\tilde{M}+1}) = \Delta_{\tilde{M}+1}$ and $g(\bar{S}_{\tilde{M}+1}) = g(\bar{S}_{\langle \tilde{M} \rangle})$. $\qquad \square$

Fact C.1 (Streeter and Golovin (2007), Fact 1). For any subset of arms S, and any positive integer j, and any $t > 0$, we have $g(S_{\langle t \rangle}) \leq g(\bar{S}_j) + t(s_j + \alpha_j)$ where $s_j = g(\bar{S}_{j+1}) - g(\bar{S}_j)$.

Proof. The proof is akin to Fact 1 of Streeter and Golovin (2007) and it goes $g(S_{\langle t \rangle}) \leq g(\bar{S}_j \cup S_{\langle t \rangle}) \leq g(\bar{S}_j) + t(s_j + \alpha_j)$. The first inequality holds because g is a monotone function. The second inequality is by definition of s_j and Condition 1, (Streeter and Golovin, 2007, Lemma 1) – for any submodular function g, and any $S_1, S \in \mathcal{S}$, $\frac{g(S_1 \cup S) - g(S_1)}{|S|} \leq \max_{a \in \mathcal{A}} g(S_1 \cup \{a\}) - g(S_1)$ and we replace $S_1 \leftarrow \bar{S}_j$, $S \leftarrow S_{\langle t \rangle}$, so $|S| = t$. $\qquad \square$

Consider a sequence of submodular functions $g_1, \cdots, g_{N'}$ for a fixed $N' \in \mathbb{N}$. Define the coverage regret of a submodular maximization policy by

$$R_{\text{coverage}}(N') := \left(1 - \frac{1}{N'}\right) \max_{S \in \mathcal{S}} \sum_{n=1}^{N'} g_n(S_{\langle M \rangle}) - \sum_{n=1}^{N'} g_n(S_n) \ .$$

Algorithm 3 is the OG algorithm of Streeter and Golovin (2007) for the full feedback model modified for our setting with $\tilde{M}$ experts. In this algorithm, N' is the number of rounds (analogous to segments/tasks). The algorithm uses a set of experts and each expert is a *randomized weighted majority* (RWM) algorithm (Littlestone

Algorithm 4 `OG°` algorithm
***
1: **Input**: Subset size M, `Expert` algorithms $\mathcal{E}_1, \cdots, \mathcal{E}_{\tilde{M}}$, Probabilities of exploration $\{\gamma_n\}_{n=1}^{N'}$;
2: **for** $n \in [N']$ **do**
3:     Observe g_n
4:     For $i \in [\tilde{M}]$, let a_i be the choice of $\mathcal{E}_i$
5:     Set $S_n = \{a_1, \cdots, a_{\tilde{M}}\}$
6:     With prob. γ_n, $E_n = \text{EXR}$, otherwise $E_n = \text{EXT}$
7:     **if** $E_n = \text{EXT}$ **then**
8:         All experts receive the zero vector as the payoff vector
9:     **else**
10:         Choose $i \in [\tilde{M}]$ uniformly at random
11:         Choose a new action a_i' uniformly at random
12:         Replace i'th element of S_n with a_i': $S_{n:i} \leftarrow \{a_1, \cdots, a_{i-1}, a_i'\}$
13:         Expert $\mathcal{E}_i$ receives a payoff vector that is zero everywhere except at position a_i' that has the value of $g_n(S_{n:i})$
14:         All other experts receive the zero vector as the payoff vector
15:     **end if**
16: **end for**
***

and Warmuth, 1994). See Chapter 4.2 of Cesa-Bianchi and Lugosi (2006) for more information. The following lemma connects the coverage regret of the `OG` algorithm and the regret of the experts.

Lemma C.2 (Lemma 3 of Streeter and Golovin (2007)). Let $G_j(N')$ be the cumulative regret of expert $\mathcal{E}_j$ in `OG` algorithm, and let $G(N') = \sum_{j=1}^{\tilde{M}} G_j(N')$. Then, $R_{\text{coverage}}(N') \leq G(N')$.

Proof. As we will show, the `OG` algorithm is an approximate version of the offline greedy subset selection, defined by Eq. (4), for function $g = \frac{1}{N'} \sum_{n=1}^{N'} g_n$. First, let's view the sequence of actions selected by $\mathcal{E}_j$ as a single "batch-action" $\tilde{a}_j$, and extend the domain of each g_n to include the batch-actions by defining $g_n(S \cup \{\tilde{a}_j\}) = g_n(S \cup \{a_j^n\})$ for all $S \in \mathcal{S}$. Thus, the online algorithm produces a single set $\tilde{S} = \{\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_{\tilde{M}}\}$. By definition we have

$$\frac{G_j(N')}{N'} = \max_{a \in [K]} \left(g(\tilde{S}_{\langle j-1 \rangle} \cup \{a\}) - g(\tilde{S}_{\langle j-1 \rangle}) \right) - \left(g(\tilde{S}_{\langle j-1 \rangle} \cup \{\tilde{a}_j\}) - g(\tilde{S}_{\langle j-1 \rangle}) \right),$$

where $\tilde{S}_{\langle j \rangle}$ is $\tilde{S}$ truncated at j'th action. Thus the `OG` algorithm simulates the greedy schedule (4) for function g, where the j'th decision is made with additive error $\alpha_j = \frac{G_j(N')}{N'}$. By Theorem C.1 and the fact that function g is submodular, we get that $R_{\text{coverage}}(N') \leq \sum_{j=1}^{\tilde{M}} G_j(N') = G(N')$. $\qquad\square$

By Lemma 4 of Streeter and Golovin (2007), $\mathbb{E}[G(N')] = O(\sqrt{\tilde{M} N' \log(K)})$.

C.2   The `OG°` algorithm

Algorithm 4 is based on the `OG°` algorithm of Streeter and Golovin (2007) for the bandit (opaque) feedback model. This algorithm is very similar to `OS-BASS` algorithm so we omit the description. The difference is that in the `OS-BASS` algorithm, the meta-learner observes the value of the submodular function up to a noise term $\varepsilon_n = (1/\tau') B_{\tau', N_n, \tilde{M}}$. So we extend the analysis of Streeter and Golovin (2007) to the case that the observation of the submodular function is corrupted by a noise term.

Lemma C.3. Consider an expert prediction problem with K actions, and let x_a^n be the payoff for action $a \in [K]$ in round n. Let $\mathcal{E}$ be a an expert algorithm that gets payoff vector $(x_a^n)_{a \in [K]}$ in round n, let e_n be its action in round n, and let $R(N')$ be its worst-case expected regret over N' rounds: $R(N') = \max_a \sum_{n=1}^{N'} (x_a^n - x_{e_n}^n)$.

Let $\mathcal{E}'$ and $\tilde{\mathcal{E}}$ be the same algorithm but with payoff vectors $(\hat{x}_a^n)_{a \in [K]}$ and $(\tilde{x}_a^n)_{a \in [K]}$ instead of $(x_a^n)_{a \in [K]}$. These feedbacks are such that $\mathbb{E}[\hat{x}_a^n] = \gamma_n x_a^n + \delta_n$ for some constant $\gamma_n \in [0,1]$ and δ_n, and

$$\mathbb{E}[\tilde{x}_a^n] \in [\mathbb{E}[\hat{x}_a^n - \gamma_n \varepsilon_n'], \mathbb{E}[\hat{x}_a^n]]$$

for some $\varepsilon_n' \geq 0$. Let u_n be the action of algorithm $\tilde{\mathcal{E}}$ in round n. Then the worst-case expected regret of $\tilde{\mathcal{E}}$ is bounded as

$$\max_a \sum_{n=1}^{N'} (x_a^n - x_{u_n}^n) \leq \frac{1}{\min_n \gamma_n} R(N') + \sum_{n=1}^{N'} \mathbb{E}[\varepsilon_n'] \,.$$

Proof. By the regret guarantee of the expert algorithm,

$$R(N') \geq \max_{a \in [K]} \sum_{n=1}^{N'} (\tilde{x}_a^n - \tilde{x}_{u_n}^n) \,.$$

Thus, for any a,

$$R(N') \geq \mathbb{E}\left[\sum_{n=1}^{N'} \tilde{x}_a^n - \tilde{x}_{u_n}^n \,\middle|\, \tilde{x}\right]$$

$$\geq \sum_{n=1}^{N'} \mathbb{E}[\hat{x}_a^n - \gamma_n \varepsilon_n' - \hat{x}_{u_n}^n]$$

$$= \sum_{n=1}^{N'} \mathbb{E}[\gamma_n x_a^n + \delta_n - \gamma_n \varepsilon_n' - \gamma_n x_{u_n}^n - \delta_n]$$

$$= \sum_{n=1}^{N'} \gamma_n \mathbb{E}[x_a^n - \varepsilon_n' - x_{u_n}^n]$$

$$\geq \min_n \gamma_n \sum_{n=1}^{N'} \mathbb{E}[x_a^n - \varepsilon_n' - x_{u_n}^n] \,.$$

Therefore,

$$\frac{1}{\min_n \gamma_n} R(N') + \sum_{n=1}^{N'} \mathbb{E}[\varepsilon_n'] \geq \sum_{n=1}^{N'} \mathbb{E}[x_a^n - x_{u_n}^n] \,,$$

and the result follows as the above inequality holds for all a. $\qquad\square$

The next lemma bounds the coverage regret of the `OG°` algorithm when N' (i.e. N) is known.

Lemma C.4 (Coverage Regret for known N'). Let $\gamma_n = \left(\frac{\tilde{M}K \log K}{N'}\right)^{1/3}$ for all n. Assume the jth expert $\tilde{\mathcal{E}}_j$ in the `OG°` algorithm gets a payoff vector $(\tilde{x}_a^n)_{a \in [K]}$ in round n such that the following holds:

$$\gamma' \left(g_n(S_{n:j-1} \cup \{a\}) - \varepsilon_n'\right) \leq \mathbb{E}[\tilde{x}_a^n] \leq \gamma g_n(S_{n:j-1} \cup \{a\}) \,.$$

where $\gamma = \left(\frac{\tilde{M}K \log K}{N'}\right)^{1/3}$ and $\gamma' = \frac{\gamma}{\tilde{M}K}$. Then for the sequence of subsets $(S_n)_{n=1}^{N'}$ chosen by the `OG°` algorithm,

$$R_{\text{coverage}}(N') \leq (\tilde{M}^4 N'^2 K \log k)^{1/3} + \tilde{M} \sum_{n=1}^{N'} \mathbb{E}[\varepsilon_n'] \,.$$

Proof. We start with another expert $\mathcal{E}'$ that gets payoff vector $\hat{x}^n$ such that $\mathbb{E}[\hat{x}_a^n] = \gamma' g_n(S_{n-1:j} \cup \{a\})$ for any action a. Then we can write

$$\mathbb{E}[\hat{x}_a^n] = \gamma' x_a^n + \delta_n$$

for $x_a^n = (g_n(S_{n-1:j} \cup \{a\}) - g_n(S_{n-1:j}))$ and $\delta_n = \gamma' g_n(S_{n-1:j})$, where $\gamma' = \frac{\gamma}{\tilde{M}K}$. Let N'_{EXR} be the number of exploration rounds. Let $G'_j(N')$ be the total regret of expert $\mathcal{E}'_j$. By Lemma C.3 and the regret guarantee of the expert algorithm, the total regret of expert $\mathcal{E}'_j$ is bounded as

$$\mathbb{E}[G'_j(N')] \leq \frac{1}{\gamma'}\mathbb{E}\sqrt{\left(\max_a \sum_{n=1}^{N'} \hat{x}_a^n\right) \log K}$$

$$\leq \frac{1}{\gamma'}\mathbb{E}\sqrt{N'_{\text{EXR}} \log K}$$

$$\leq \sqrt{\frac{N'}{\gamma'} \log K},$$

where we used Jensen's inequality and $\mathbb{E}[N'_{\text{EXR}}] = \gamma' N'$ in the last step. Let $\tilde{G}_j(N')$ be regret of expert $\tilde{\mathcal{E}}_j$. We observe that $\mathbb{E}[\tilde{x}_a^n] \in [\mathbb{E}[\hat{x}_a^n - \gamma'\varepsilon'_n], \mathbb{E}[\hat{x}_a^n]]$. Given that the $\texttt{OG}^\circ$ algorithm takes random actions in the exploration rounds, it incurs an extra $\gamma' N'$ regret, and therefore together with Lemma C.3, we have $\mathbb{E}[\tilde{G}_j(N')] \leq \mathbb{E}[G'_j(N')] + \sum_{n=1}^{N'} \mathbb{E}[\varepsilon'_n] + \gamma N'$. By summing over $j \in [\tilde{M}]$,

$$\mathbb{E}\left[\sum_{j=1}^{\tilde{M}} \tilde{G}_j(N')\right] \leq \tilde{M}\sqrt{\frac{N'}{\gamma'} \log K} + \tilde{M}\sum_{n=1}^{N'} \mathbb{E}[\varepsilon'_n] + \gamma\tilde{M}N' .$$

By Lemma C.2 we get

$$\mathbb{E}[R_{\text{coverage}}(N')] \leq \tilde{M}\sqrt{\frac{N'}{\gamma'} \log K} + \tilde{M}\sum_{n=1}^{N'} \mathbb{E}[\varepsilon'_n] + \gamma\tilde{M}N'$$

$$= \tilde{M}\sqrt{\frac{N'}{\gamma}\tilde{M}K \log K} + \tilde{M}\sum_{n=1}^{N'} \mathbb{E}[\varepsilon'_n] + \gamma\tilde{M}N' .$$

Finally, choosing $\gamma = \left(\frac{\tilde{M}K \log K}{N'}\right)^{1/3}$[10] yields

$$\tilde{M}\sqrt{\frac{N'}{\gamma}\tilde{M}K \log K} + \gamma\tilde{M}N' = (\tilde{M}^4 N'^2 K \log k)^{1/3} .$$

Therefore

$$\mathbb{E}[R_{\text{coverage}}(N')] \leq (\tilde{M}^4 N'^2 K \log k)^{1/3} + \tilde{M}\sum_{n=1}^{N'} \mathbb{E}[\varepsilon'_n] .$$

$\square$

A similar result holds for the case of unknown N by using a time-varying exploration probability.

---

[10]To be more precise, $\gamma = (3/2)\left(\frac{\tilde{M}K \log K}{N'}\right)^{1/3}$.

Lemma C.5 (Coverage regret for unknown N). Let $\gamma_n = \left(\frac{\tilde{M}K\log K}{n}\right)^{1/3}$ and assume the jth expert $\tilde{\mathcal{E}}_j$ in the OG° algorithm gets a payoff vector $(\tilde{x}_a^n)_{a\in[K]}$ in round n such that the following holds:

$$\gamma_n'\left(g_n(S_{n:j-1}\cup\{a\})-\varepsilon_n'\right) \le \mathbb{E}[\tilde{x}_a^n] \le \gamma_n'g_n(S_{n:j-1}\cup\{a\}),$$

where $\gamma_n = \left(\frac{\tilde{M}K\log K}{n}\right)^{1/3}$ and $\gamma_n' = \frac{\gamma_n}{\tilde{M}K}$. Then for the sequence of subsets $(S_n)_{n=1}^{N'}$ chosen by the OG° algorithm,

$$R_{\text{coverage}}(N') \le \tilde{M}(\log K)^{1/3}N'^{2/3} + \tilde{M}\sum_{n=1}^{N'}\mathbb{E}[\varepsilon_n'].$$

Proof. The proof is analogous to Lemma C.4. Let expert $\mathcal{E}'$ be an expert that gets payoff vector $\hat{x}^n$ such that $\mathbb{E}[\hat{x}_a^n] = \gamma_n'g_n(S_{n-1:j}\cup\{a\})$ for any action a, then $\mathbb{E}[\hat{x}_a^n] = \gamma_n'x_a^n + \delta_n$ for $x_a^n = (g_n(S_{n-1:j}\cup\{a\}) - g_n(S_{n-1:j}))$ and $\delta_n = \gamma_n'g_n(S_{n-1:j})$, where $\gamma_n' = \frac{\gamma_n}{\tilde{M}K}$. Let N'_{EXR} and $G'_j(N')$ be the same as in Lemma C.4. Akin to Lemma C.4, by Lemma C.3 and the regret guarantee of the expert algorithm, we know

$$\mathbb{E}[G'_j(N')] \le \frac{1}{\min_n \gamma_n'}\mathbb{E}\sqrt{N'_{\text{EXR}}\log K}$$

$$\le \frac{1}{\min_n \gamma_n'}\sqrt{\sum_{n=1}^{N'}\gamma_n'\log K},$$

where we used Jensen's inequality and $\mathbb{E}[N'_{\text{EXR}}] = \sum_{n=1}^{N'}\gamma_n$ in the last step. Again $\tilde{G}_j(N')$ is the regret of expert $\tilde{\mathcal{E}}_j$. It is easy to see that $\mathbb{E}[\tilde{x}_a^n] \in [\mathbb{E}[\hat{x}_a^n - \gamma'\varepsilon_n'], \mathbb{E}[\hat{x}_a^n]]$. The exploration rounds regret is $\le \sum_{n=1}^{N'}\gamma_n'$. Therefore, by Lemma C.3 it holds that $\mathbb{E}[\tilde{G}_j(N')] \le \mathbb{E}[G'_j(N')] + \sum_{n=1}^{N'}\gamma_n + \sum_{n=1}^{N'}\mathbb{E}[\varepsilon_n']$. Summing over $j \in [\tilde{M}]$ gives

$$\mathbb{E}\left[\sum_{j=1}^{\tilde{M}}\tilde{G}_j(N')\right] \le \frac{\tilde{M}}{\min_n \gamma_n'}\sqrt{\sum_{n=1}^{N'}\gamma_n'\log K} + \tilde{M}\sum_{n=1}^{N'}\mathbb{E}[\varepsilon_n'] + \tilde{M}\sum_{n=1}^{N'}\gamma_n.$$

We know $\mathbb{E}[R_{\text{coverage}}(N')] \le \mathbb{E}\left[\sum_{j=1}^{\tilde{M}}\tilde{G}_j(N')\right]$ by Lemma C.2, i.e., using the definition of γ_n' gives

$$\mathbb{E}[R_{\text{coverage}}(N')] \le \frac{\tilde{M}^2 K}{\min_n \gamma_n}\sqrt{\frac{\sum_{n=1}^{N'}\gamma_n}{\tilde{M}K}\log K} + \tilde{M}\sum_{n=1}^{N'}\mathbb{E}[\varepsilon_n'] + \tilde{M}\sum_{n=1}^{N'}\gamma_n.$$

Now, $\gamma_n = \left(\frac{\tilde{M}K\log K}{n}\right)^{1/3}$ gives the same result as in Lemma C.4 noting that $\sum_{n=1}^{N'}(1/n)^{1/3} \le \int_{x=1}^{N'}(1/x)^{1/3} = (N')^{2/3}$. Therefore

$$\mathbb{E}[R_{\text{coverage}}(N')] \le (\tilde{M}^4 N'^2 K \log k)^{1/3} + \tilde{M}\sum_{n=1}^{N'}\mathbb{E}[\varepsilon_n'].$$

$\square$

C.3 The OS-BASS algorithm for non-stationary bandits

Now we are ready to bound the regret of the OS-BASS algorithm (shown in Algorithm 1).

Theorem 3.1. The regret of OS-BASS using AdSwitch as Base, with N' segments each of size $\tau' = T/N'$, and using exploration probability $\gamma_n = \left(\frac{\tilde{M}K\log K}{N'}\right)^{1/3}$ is $R_T = \tilde{O}(TN'^{-1/3}(\tilde{M}^4 K \log K)^{1/3} + \tilde{M}\sqrt{\tilde{M}N'T})$.

Proof. Fix a sequence of N tasks with unknown and potentially variable task lengths $\{\tau_n\}_{n\in[N]}$. Let π_{OS} be the policy used by Algorithm 1. By the decomposition in Lemma 2.1 the regret of Algorithm 1 when updated every τ' steps (with $N' = T/\tau'$ updates) satisfies the following,

$$
R(\pi_{\mathrm{OS}}, T, N', \tilde{M}) = \sup_{f_n \in \mathcal{F}} \max_{S \in \mathcal{S}} \mathbb{E}\left[\sum_{n=1}^{N'} \left(f_n(S) - f_n(S_n) \right) + B_{\tau', N_n, \tilde{M}} \right]
$$

$$
\leq \sup_{f_n \in \mathcal{F}} \max_{S \in \mathcal{S}} \mathbb{E}\bigg[\sum_{n=1}^{N'} \tfrac{1}{N'} f_n(S)
$$

$$
+ \sum_{n=1}^{N'} (1 - \tfrac{1}{N'}) f_n(S) - \sum_{n=1}^{N'} f_n(S_n) + B_{\tau', N_n, \tilde{M}} \bigg]
$$

$$
\leq \frac{N'\tau'}{N'} + \tau' \mathbb{E}[R_{\mathrm{coverage}}] + \sum_{n=1}^{N'} \sqrt{\tilde{M} \tau' N_n}
$$

$$
\leq \tau' + \tau' \mathbb{E}[R_{\mathrm{coverage}}] + \sqrt{N' \tilde{M} \tau' \sum_{n=1}^{N'} N_n} \tag{5}
$$

$$
\leq \tau' + \tau' \mathbb{E}[R_{\mathrm{coverage}}] + \sqrt{N' \tilde{M} \tau' \max\{2T/\tau', 2N\}} \,, \tag{6}
$$

where in Eq. (5) we use the Cauchy-Schwarz inequality. For Eq. (6) we used the fact that if $N' \geq N$ then $\sum_{n=1}^{N'} N_n \leq 2N' = 2T/\tau'$, otherwise $\sum_{n=1}^{N'} N_n \leq 2N$ by Eq. (2). Let $\varepsilon_n = B_{\tau', N_n, \tilde{M}}/\tau'$. By Lemma C.4 we can set $\gamma = \left(\frac{\tilde{M} K \log K}{N'} \right)^{1/3}$ and bound $\mathbb{E}[R_{\mathrm{coverage}}]$ to get

$$
R(\pi_{\mathrm{OS}}, T, N', \tilde{M}) \leq \tau' + \tau' \left((\tilde{M}^4 N'^2 K \log k)^{1/3} + \tilde{M} \sum_{n=1}^{N'} \varepsilon_n \right) + \sqrt{N' \tilde{M} \tau' \max\{2T/\tau', 2N\}}
$$

$$
\leq \tau' + \tau' (\tilde{M}^4 N'^2 K \log k)^{1/3} + \tilde{M} \sqrt{N' \tilde{M} \tau' \max\{2T/\tau', 2N\}}
$$

$$
= TN'^{-1} + TN'^{-1/3}(\tilde{M}^4 K \log K)^{1/3} + \tilde{M}\sqrt{2\tilde{M}N'T} \,.
$$

Here, the second inequality follows from Cauchy-Schwarz inequality and the same argument as above for bounding $\sum_{n=1}^{N'} N_n$, and the last step holds as the bound is minimized with $N' \geq N$. $\qquad\square$

If $N \geq N_1 \doteq \left(\frac{T^3 (K \log K)^2}{\tilde{M}} \right)^{1/5}$ and $M \leq (K \log K)^{1/3}$ (large number of changes and small number of optimal arms), then our regret upper bound is $\tilde{O}(\tilde{M}^{3/2}\sqrt{NT})$, and the regret of OS-BASS improves upon the $\tilde{O}(\sqrt{KTN})$ bound of standard non-stationary bandit algorithms (such as AdSwitch).

If $N \leq N_1$ and $M \leq (K \log K)^{1/3}$, and we can obtain an improved bound by using a larger number of segments. By choosing $N' = \left(\frac{T^3 (K \log K)^2}{\tilde{M}} \right)^{1/5}$ and $M \leq (K \log K)^{1/3}$ segments, each of size $\tau' = T/N'$, the bound improves to $\tilde{O}(\tilde{M}^{7/5}(K \log K)^{1/5} T^{4/5})$.

If $N \leq N_2 \doteq \tilde{M}^{14/5}(T/K)^{3/5}(\log K)^{2/5}$ (even small number of changes), then $\sqrt{KNT} \leq \tilde{M}^{7/5}(K \log K)^{1/5} T^{4/5}$. In this case, the simple baseline of $\tilde{O}(\sqrt{KTN})$ is smaller than our bound, and the learner should simply play a standard non-stationary bandit algorithm. Notice that $N_2 \leq N_1$ as long as $M \leq K^{1/3}$.

D   Proofs of Section 4

D.1   Complement to the proof of Theorem 4.3

We are left to prove that

$$G_n(s) \leq G_{n+1}(s) + \frac{(\mathbf{C}_{\texttt{info}} - \mathbf{C}_{\texttt{hit}})(\mathbf{C}_{\texttt{miss}} - \mathbf{C}_{\texttt{hit}})}{\mathbf{C}_{\texttt{miss}} - \mathbf{C}_{\texttt{hit}} + G_{n+1}(s)}, \tag{7}$$

given in (11) implies that for any $n \leq N$,

$$G_n(s) \leq \sqrt{2(\mathbf{C}_{\texttt{info}} - \mathbf{C}_{\texttt{hit}})(\mathbf{C}_{\texttt{miss}} - \mathbf{C}_{\texttt{hit}})(N - n)}. \tag{8}$$

Proof. We proceed by (backward) induction. First, by definition, $G_N(s) = V_N(s) - V_N(s+1) = 0$ for all s, thus (8) holds for $n = N$. Next, assume that (8) holds for $\{N, N - 1, \ldots, n + 1\}$, and we show that it also holds for n.

Consider positive constants $b \geq a$ and consider the function $h(z) = z + \frac{ab}{b+z}$ defined on $[0, c]$ for some $c > 0$. Then $h'(z) = 1 - ab/(b + z)^2 \geq 0$. Therefore, h is maximized at $z = c$. Since the right-hand side of (7) is of the form $h(G_{n+1}(s))$ with $a = \mathbf{C}_{\texttt{info}} - \mathbf{C}_{\texttt{hit}}$ and $b = \mathbf{C}_{\texttt{miss}} - \mathbf{C}_{\texttt{hit}}$, which indeed satisfy $b \geq a$. By this argument, the induction assumption, and $0 \leq G_{n+1}(s) \leq \sqrt{ab(N - n - 1)}$ by the induction hypothesis, we obtain that

$$\begin{aligned} G_n(s) &\leq \sqrt{2(\mathbf{C}_{\texttt{info}} - \mathbf{C}_{\texttt{hit}})(\mathbf{C}_{\texttt{miss}} - \mathbf{C}_{\texttt{hit}})(N - n - 1)} \\ &\quad + \frac{(\mathbf{C}_{\texttt{info}} - \mathbf{C}_{\texttt{hit}})(\mathbf{C}_{\texttt{miss}} - \mathbf{C}_{\texttt{hit}})}{\mathbf{C}_{\texttt{miss}} - \mathbf{C}_{\texttt{hit}} + \sqrt{2(\mathbf{C}_{\texttt{info}} - \mathbf{C}_{\texttt{hit}})(\mathbf{C}_{\texttt{miss}} - \mathbf{C}_{\texttt{hit}})(N - n - 1)}} \\ &= \sqrt{2ab(N - n - 1)} + \frac{ab}{b + \sqrt{2ab(N - n - 1)}} \end{aligned} \tag{9}$$

It remains to show that the right-hand side above is bounded from above by $\sqrt{2ab(N - n)}$. This follows since

$$\begin{aligned} \sqrt{2ab(N - n)} - \sqrt{2ab(N - n - 1)} &= \frac{\sqrt{2ab}}{\sqrt{N - n} + \sqrt{N - n - 1}} \\ &= \frac{ab}{\sqrt{ab(N - n)/2} + \sqrt{ab(N - n - 1)/2}} \geq \frac{ab}{b + \sqrt{2ab(N - n - 1)}} \end{aligned}$$

where the last inequality holds because

$$\begin{aligned} b + \sqrt{2ab(N - n - 1)} &\geq \left[\sqrt{ab} + \sqrt{ab(N - n - 1)/2}\right] + \sqrt{ab(N - n - 1)/2} \\ &\geq \sqrt{ab(N - n)/2} + \sqrt{ab(N - n - 1)/2} \end{aligned}$$

(where we used that $1 + \sqrt{z} \geq \sqrt{z + 1}$ for $z \geq 0$). Thus, $G_n(s) \leq \sqrt{2ab(N - n)}$, proving the induction hypothesis (8) for n.

$\square$

D.2   Proof of Theorem 4.3

The proof relies on solving the min-max problem in (3). First, we consider the case that the best-arm-identification can be performed with probability 1 (i.e., $\delta = 0$ in the efficient identification assumption). From symmetry, it is easy to see that $V_n(\mathcal{I}_n)$ only depends on the size of $\mathcal{I}_n$, and not the actual arms in $\mathcal{I}_n$. Therefore, to simplify notation and emphasize the dependence on the number of discovered optimal arms, we use below $V_n(|\mathcal{I}_n|) := V_n(\mathcal{I}_n)$. Let $n_M = \text{argmin}_n\{|\mathcal{I}_n| = M\}$ be the first round when all optimal arms

have been discovered. Then from any $n > n_M$, the adversary no longer can reveal new arms ($q = 0$), and the learner should no longer explore ($p = 0$), and so

$$\forall n \geq n_M, \ V_n(M) = (N - n)\mathbf{C}_{\mathtt{hit}}.$$

Denoting $s = |\mathcal{I}_n|$, the min-max optimization objective in (3) can be written as

$$
\begin{aligned}
L(q, p) = \mathbf{C}_{\mathtt{hit}} + p(\mathbf{C}_{\mathtt{info}} - \mathbf{C}_{\mathtt{hit}}) + V_{n+1}(s) + \\
q(1 - p)(\mathbf{C}_{\mathtt{miss}} - \mathbf{C}_{\mathtt{hit}}) - p\left[q^1(V_{n+1}(s)\right. \\
\left. -V_{n+1}(s + 1)) + \ldots \right. \\
\left. +q^{M-s}(V_{n+1}(s) - V_{n+1}(M))\right] ,
\end{aligned}
$$

where q^i denotes the probability that the environment reveals i optimal arms in the round, and $q = \sum_{i=1}^{M-s} q^i$. Given that $V_{n+1}(s) - V_{n+1}(s + 1) < \cdots < V_{n+1}(s) - V_{n+1}(M)$, the maximizing q is such that $q^i = 0$ for $i > 1$ and $q = q^1$. Using this, the saddle point can be obtained by solving $\partial L(q, p)/\partial q = 0$ and $\partial L(q, p)/\partial p = 0$:

$$
\begin{aligned}
p = p_n &= \frac{\mathbf{C}_{\mathtt{miss}} - \mathbf{C}_{\mathtt{hit}}}{\mathbf{C}_{\mathtt{miss}} - \mathbf{C}_{\mathtt{hit}} + V_{n+1}(s) - V_{n+1}(s + 1)} \\
q_n &= \frac{\mathbf{C}_{\mathtt{info}} - \mathbf{C}_{\mathtt{hit}}}{\mathbf{C}_{\mathtt{miss}} - \mathbf{C}_{\mathtt{hit}} + V_{n+1}(s) - V_{n+1}(s + 1)} .
\end{aligned}
\tag{10}
$$

Plugging these values in (3), we get

$$V_n(s) = V_{n+1}(s) + \mathbf{C}_{\mathtt{hit}} + \frac{(\mathbf{C}_{\mathtt{info}} - \mathbf{C}_{\mathtt{hit}})(\mathbf{C}_{\mathtt{miss}} - \mathbf{C}_{\mathtt{hit}})}{\mathbf{C}_{\mathtt{miss}} - \mathbf{C}_{\mathtt{hit}} + V_{n+1}(s) - V_{n+1}(s + 1)} .$$

Given N and M, the policy of the learner and the adversary can be computed by solving the above recursive equation. Given that for any $s < M$, $V_n(s + 1) \geq V_{n+1}(s + 1) + \mathbf{C}_{\mathtt{hit}}$,

$$V_n(s) - V_n(s + 1) \leq V_{n+1}(s) - V_{n+1}(s + 1) + \frac{(\mathbf{C}_{\mathtt{info}} - \mathbf{C}_{\mathtt{hit}})(\mathbf{C}_{\mathtt{miss}} - \mathbf{C}_{\mathtt{hit}})}{\mathbf{C}_{\mathtt{miss}} - \mathbf{C}_{\mathtt{hit}} + V_{n+1}(s) - V_{n+1}(s + 1)} .$$

Let $G_n(s) = V_n(s) - V_n(s + 1) \geq 0$ be the cost difference in state s relative to state $s + 1$. We have

$$G_n(s) \leq G_{n+1}(s) + \frac{(\mathbf{C}_{\mathtt{info}} - \mathbf{C}_{\mathtt{hit}})(\mathbf{C}_{\mathtt{miss}} - \mathbf{C}_{\mathtt{hit}})}{\mathbf{C}_{\mathtt{miss}} - \mathbf{C}_{\mathtt{hit}} + G_{n+1}(s)} , \tag{11}$$

and indeed by a telescopic argument,

$$R_T - NB_{\tau, M} = V_0(0) - V_0(M) = \sum_{s=0}^{M-1} (V_0(s) - V_0(s + 1)) = \sum_{s=0}^{M-1} G_0(s) .$$

The proof is completed by bounding $G_0(s)$ by backward induction on $n \leq N$:

$$G_{N-n}(s) \leq \sqrt{2(\mathbf{C}_{\mathtt{info}} - \mathbf{C}_{\mathtt{hit}})(\mathbf{C}_{\mathtt{miss}} - \mathbf{C}_{\mathtt{hit}})n} .$$

The proof of this inequality relies on standard algebraic manipulations that can be found in Appendix D.1. When the BAI routine returns the best arm with probability at least $1 - \delta/N$, with a simple union bound argument, the probability that $\mathcal{I}_n$ ever contains wrong elements is bounded by δ and the above derivations again hold.

D.3 Proof of Theorem 4.4

Proof. The proof relies on the analysis of the optimization problem defined as in Eq. (3) with a fixed $p_n = \hat{p}_{|S_n|} = \sqrt{|S_n|K\tau/(NB_{\tau,K})}$ (no minimization over the exploration probability of the learner in task n). As in the proof of Theorem 4.3, we assume that the best arm identification is successful, and the extension to $\delta \neq 0$ can be done the same way. After some algebraic manipulation, similarly to the proof of Theorem 4.3, the optimization objective can be written as

$$L(q) = V_{n+1}(S_n) + B_{\tau,|S_n|} + \hat{p}_{|S_n|}(B_{\tau,K} - B_{\tau,|S_n|})$$
$$+ q\left\{(1 - \hat{p}_{|S_n|})(\tau - B_{\tau,|S_n|}) - \hat{p}_{|S_n|}(V_{n+1}(S_n) - V_{n+1}(S'_n))\right\}$$

where S'_n is the new greedy subset selected by the learner in time step $n+1$ if $S^*_n \cap S_n = \emptyset$ and the learner chooses to explore at time n (we use the notation S'_n instead of S_{n+1} to emphasize that this corresponds to the aforementioned choices of the learner and the adversary). Given that L is linear in q, the optimal adversary choice is either $q = 0$ or $q = 1$ (similarly as in Theorem 4.3, it is suboptimal for the adversary to reveal multiple optimal arms). We have

$$q = \begin{cases} 0 & \text{if } (1 - \hat{p}_{|S_n|})(\tau - B_{\tau,|S_n|}) - \hat{p}_{|S_n|}(V_{n+1}(S_n) - V_{n+1}(S'_n)) \leq 0, \\ 1 & \text{otherwise} \end{cases}$$

When $q = 0$, $V_n(S_n) = V_{n+1}(S_n) + B_{\tau,|S_n|} + \hat{p}_{|S_n|}(B_{\tau,K} - B_{\tau,|S_n|})$, and given that $|S_n| \leq M'$, the total contribution of these rounds to the regret is bounded by

$$NB_{\tau,M'} + \hat{p}_{M'}NB_{\tau,K} .$$

Next, consider the rounds where $q = 1$. Among these rounds, consider rounds where the adversary chooses a particular arm $a \in S^*$ and the learner chooses to explore (EXR). This arm is not added to the future EXT subset of the learner if instead another arm is used to cover this round. This means that after at most K such rounds, the learner adds a to the EXT subset. Since the learner's regret in the exploration rounds is $B_{\tau,K}$, in these rounds the cumulative regret is bounded by $MKB_{\tau,K}$. Since the random choices made by the learner and the adversary are independent in the same round, we discover the first arm in $K/\hat{p}_1$ tasks in expectation, the second in $K/\hat{p}_2$ tasks in expectation, and so on. Thus, since the size of our cover is at most M', we get

$$K\sum_{s=1}^{M'} \frac{1}{\hat{p}_s} = \sqrt{\frac{KNB_{\tau,K}}{\tau}} \sum_{s=1}^{M'} \frac{1}{\sqrt{s}} \leq 2\sqrt{\frac{M'KNB_{\tau,K}}{\tau}} .$$

The adversary reveals all positions after $2\sqrt{\frac{M'KNB_{\tau,K}}{\tau}}$ such tasks in expectation where the adversary's choice is $q = 1$. If in these tasks the learner chooses to exploit, it can suffer a regret τ, leading to a total expected regret of at most $2\sqrt{M'KNB_{\tau,K}\tau}$. Thus, the total regret of rounds with $q = 1$ is bounded by

$$2\sqrt{M'KNB_{\tau,K}\tau} + MKB_{\tau,K} .$$

Therefore,

$$R_T = V_0(\emptyset) \leq NB_{\tau,M'} + MKB_{\tau,K} + 3\sqrt{MK\tau B_{\tau,K}N} .$$

$\square$

E Partial monitoring and bandit meta-learning

Partial monitoring is a general framework in online learning that disentangles rewards and observations (information). It is a game where the learner has Z actions and the adversary has D actions, and it is characterized by two $Z \times D$ matrices (not observed): matrix C maps the learner's action to its cost given the

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Algorithm 5 The partial monitoring algorithm

1: Exploration probability $p \in (0,1)$, learning rate $\eta > 0$, base costs $\mathbf{C}_{\texttt{info}}, \mathbf{C}_{\texttt{hit}}, \mathbf{C}_{\texttt{miss}}$
2: **for** $n = 1, 2, \ldots, N$ **do**
3:    With probability p, let $E_n = \text{EXR}$ and otherwise $E_n = \text{EXT}$
4:    **if** $E_n = \text{EXR}$ **then**
5:       Observe the best arms S_n^* of this round and for all $i \in \text{EXT}$ experts, observe cost C_{i,S_n^*} and let
      $\widehat{C}_n(i) = (C_{i,S_n^*} - \mathbf{C}_{\texttt{hit}})/p$
6:       Update exponential weights $Q_{n,i} \propto \exp(-\eta \sum_{\tau=1}^{n} \widehat{C}_n(i))$ Suffer cost $\mathbf{C}_{\texttt{info}}$
7:    **else**
8:       Sample $S_n \sim Q_{n-1}$
9:       Suffer (but do not observe) cost $\mathbf{C}_{\texttt{hit}}$ if $S_n^* \cap S_n \neq \emptyset$ and suffer cost $\mathbf{C}_{\texttt{miss}}$ otherwise
10:    **end if**
11: **end for**

---

adversary's choice, and matrix X maps the learner's action to its observation given the adversary's choice. In all generality, we consider bandit meta-learning problems with $Z + 1$ learner actions: an EXR action that provides information for a cost $\mathbf{C}_{\texttt{info}}$, and Z other actions that do not provide information but have a hidden cost $\mathbf{C}_{\texttt{hit}}$ or $\mathbf{C}_{\texttt{miss}}$ depending on whether the chosen action had low or high cost respectively.

As defined in the introduction, a bandit subset selection problem is realizable when there is a subset of size M that contains an optimal arm in all rounds. Otherwise, the problem is called agnostic.

In our bandit subset selection problem, $Z = \binom{K}{M} \leq K^M$ and the adversary can have up to 2^K choices depending on the realizable or agnostic nature of the problem. We have $D = M$ if the problem is realizable and if the adversary is constrained to picking a unique optimal arm in each round. For example, let $M = 2$ and $K = 4$. There are $Z + 1 = \binom{4}{2} + 1 = 7$ learner actions and only $D = 2$ possible choices for the adversary

$$
\begin{pmatrix} \text{EXR} \\ \{1,2\} = x^* \\ \{1,3\} \\ \{1,4\} \\ \{2,3\} \\ \{2,4\} \\ \{3,4\} \end{pmatrix} \rightarrow C = \begin{pmatrix} \mathbf{C}_{\texttt{info}} & \mathbf{C}_{\texttt{info}} \\ \mathbf{C}_{\texttt{hit}} & \mathbf{C}_{\texttt{hit}} \\ \mathbf{C}_{\texttt{hit}} & \mathbf{C}_{\texttt{miss}} \\ \mathbf{C}_{\texttt{hit}} & \mathbf{C}_{\texttt{miss}} \\ \mathbf{C}_{\texttt{miss}} & \mathbf{C}_{\texttt{hit}} \\ \mathbf{C}_{\texttt{miss}} & \mathbf{C}_{\texttt{hit}} \\ \mathbf{C}_{\texttt{miss}} & \mathbf{C}_{\texttt{miss}} \end{pmatrix}, \qquad X = \begin{pmatrix} 1 & 2 \\ \bot & \bot \\ \vdots & \vdots \\ \bot & \bot \end{pmatrix}.
$$

The symbol $\bot$ is used to denote no observations. We use $C_{i,y}$ to denote the cost of action $i \in \{\text{EXR}, x_1, \ldots, x_Z\}$ when adversary chooses $a \in [D]$. Thanks to this reduction, we can leverage the partial monitoring literature to obtain an algorithm and the corresponding bounds for our problem as well. We detail this process below. Note that using the vocabulary of online learning, the learner's actions are referred to as "experts".

Next, we describe an algorithm based on the Exponentially Weighted Average (EWA) forecaster. The learner estimates the cost matrix by importance sampling when action EXR is chosen. When EXT is chosen, the learner samples an expert according to EWA weights that depend on the estimated cost matrix. The pseudo-code of the method is shown in Algorithm 5.

To analyze the algorithm, we consider the realizable and agnostic cases. In the realizable case, there a subset of size M that contains an optimal arm in all rounds. In this case, the exponential weights distribution reduces to a uniform distribution over the subsets that satisfy this condition.

Theorem E.1. Consider the partial monitoring algorithm shown in Algorithm 5. In the agnostic case, with the choice of $p = O\left(\left(\frac{\mathbf{C}_{\texttt{miss}}^2 \log Z}{\mathbf{C}_{\texttt{info}}^2 N} \right)^{1/3} \right)$ and $\eta = O\left(\left(\frac{\log^2 Z}{\mathbf{C}_{\texttt{info}} \mathbf{C}_{\texttt{miss}}^2 N^2} \right)^{1/3} \right)$, the regret of the algorithm is bounded as $O((\mathbf{C}_{\texttt{info}} \mathbf{C}_{\texttt{miss}}^2 N^2 \log Z)^{1/3})$. In the realizable case, with the choice of $p = \sqrt{\frac{\mathbf{C}_{\texttt{miss}} \log Z}{\mathbf{C}_{\texttt{info}} N}}$ and $\eta = 1$, the regret of the algorithm is bounded as $O(\sqrt{\mathbf{C}_{\texttt{info}} \mathbf{C}_{\texttt{miss}} N \log Z})$.

Proof. Let function $f_n : [Z+1] \times [D] \to \mathbb{R}^{Z+1}$ be defined by

$$f_n(k, X_{k,y})_i = \mathbf{1}\{k = \text{EXR}\}(C_{i,y} - \mathbf{C}_{\text{hit}}) .$$

Therefore, $\sum_{k=1}^{Z+1} f_n(k, X_{k,y})_i = C_{i,y} - \mathbf{C}_{\text{hit}}$. With probability p, let $E_n = \text{EXR}$ and otherwise $E_n = \text{EXT}$. Let $C_n(i) = C_{i,Y_n}$. Define cost estimator

$$\widehat{C}_{n,i} = \frac{f_n(E_n, X_{E_n,Y_n})_i}{p} = \frac{\mathbf{1}\{E_n = \text{EXR}\}(C_n(i) - \mathbf{C}_{\text{hit}})}{p} .$$

Let Q_n be the weights of the EWA forecaster defined using the above costs. For any i, we have $\mathbb{E}(\widehat{C}_n(i)) = C_n(i) - \mathbf{C}_{\text{hit}}$. Let E_n be the learner's decision in round n, that is either EXR or a subset chosen by EWA, in which case it is denoted by x_n. We have

$$\mathbb{E}(C_n(E_n)) = p\mathbf{C}_{\text{info}} + (1-p)\mathbb{E}(C_n(S_n)) .$$

Let S^* be the optimal subset. By the regret bound of EWA (Cesa-Bianchi and Lugosi, 2006),

$$\sum_{n=1}^{N} \widehat{C}_n(S_n) - \sum_{n=1}^{N} \widehat{C}_n(S^*) \le \frac{\log Z}{\eta} + \frac{\eta}{2} \sum_{n=1}^{N} \|\widehat{C}_n\|_\infty^2 .$$

Thus,

$$\sum_{n=1}^{N} \mathbb{E}(C_n(E_n)) - \sum_{n=1}^{N} \mathbb{E}(C_n(S^*)) \le \mathbf{C}_{\text{info}} \sum_{n=1}^{N} p + \frac{\log Z}{\eta} + \frac{\eta}{2} \sum_{n=1}^{N} \mathbb{E}(\|\widehat{C}_n\|_\infty^2)$$

$$\le \mathbf{C}_{\text{info}} \sum_{n=1}^{N} p + \frac{\log Z}{\eta} + \frac{\eta \mathbf{C}_{\text{miss}}^2}{2} \sum_{n=1}^{N} \frac{1}{p} .$$

With the choice of $p = O((\mathbf{C}_{\text{miss}}/\mathbf{C}_{\text{info}})^{2/3}(\log^{1/3} Z)/N^{1/3})$ and $\eta = O((\log^{2/3} Z)/(\mathbf{C}_{\text{miss}}^{2/3}\mathbf{C}_{\text{info}}^{1/3}N^{2/3}))$, the regret of the partial monitoring game is bounded as $O(\mathbf{C}_{\text{miss}}^{2/3}\mathbf{C}_{\text{info}}^{1/3}N^{2/3}\log^{1/3} Z)$. The regret scales logarithmically with the number of experts, and is independent of the number of adversary choices.

Next, we show a fast $O(\sqrt{N})$ rate when the optimal expert always has small cost. More specifically, we assume that $C_n(S^*) = \mathbf{C}_{\text{hit}}$ for the optimal expert S^*. The fast rate holds independently of the relative values of $\mathbf{C}_{\text{hit}}, \mathbf{C}_{\text{info}}$, and $\mathbf{C}_{\text{miss}}$. The algorithm can also be implemented efficiently.

Let $\widehat{\ell}_n = p\widehat{C}_n/\mathbf{C}_{\text{miss}}$, which is guaranteed to be in $[0, 1]$. Notice that $\sum_{n=1}^{N} \widehat{\ell}_n(S^*) = 0$ as $C_n(S^*) = \mathbf{C}_{\text{hit}}$ by assumption. In this case, the regret of EWA is known to be logarithmic:

$$\sum_{n=1}^{N} \widehat{\ell}_n(S_n) - \sum_{n=1}^{N} \widehat{\ell}_n(S^*) = O(\log Z) .$$

Thus,

$$\sum_{n=1}^{N} \mathbb{E}(C_n(E_n)) - \sum_{n=1}^{N} \mathbb{E}(C_n(S^*)) \le \mathbf{C}_{\text{info}} \sum_{n=1}^{N} p + \frac{\mathbf{C}_{\text{miss}} \log Z}{p} .$$

Therefore, with the choice of $p = \sqrt{\frac{\mathbf{C}_{\text{miss}} \log Z}{\mathbf{C}_{\text{info}} N}}$,

$$\sum_{n=1}^{N} \mathbb{E}(C_n(E_n)) - \sum_{n=1}^{N} \mathbb{E}(C_n(S^*)) \le O(\sqrt{\mathbf{C}_{\text{info}}\mathbf{C}_{\text{miss}}N \log Z}) .$$

The meta-regret scales logarithmically with the number of experts, and is independent of the number of adversary choices. Given that the optimal expert is known to have small loss in all rounds, the learner can eliminate all other experts. Therefore, the EWA strategy reduces to a uniform distribution over the surviving experts.

$\square$

E.1   Proof of Theorem 4.5

`E-BASS` is constructed as a special case of the EWA algorithm above, where the sampling distribution at each EXT round is simply the uniform distribution over the surviving experts. The proof of Theorem 4.5 is therefore a direct consequence of the more general analysis done for the EWA forecaster in Theorem E.1 above.

Proof. The BAI algorithm might return a number of extra arms in addition to the optimal arm. However, since with high probability the optimal arm is always in the surviving set, the cost estimate for the optimal subset is always zero, and costs of all other subsets are under-estimated. Therefore, if S_n is the expert (subset) selected in task n and S^* is the optimal subset, by fast rates of the previous section,

$$\sum_{n=1}^{N} \mathbb{E}(C_n(S_n)) - \sum_{n=1}^{N} \mathbb{E}(C_n(S^*)) \leq O(\sqrt{\mathbf{C}_{\texttt{info}}\mathbf{C}_{\texttt{miss}}N \log Z}) \,.$$

Given that with high probability the optimal arm is always in the surviving set and therefore $C_n(S^*) = \mathbf{C}_{\texttt{hit}}$,

$$\begin{aligned}
R_T &= \sum_{n=1}^{N} \mathbb{E}\left(\tau r_n(a_n^*) - \sum_{t=1}^{\tau} r_n(A_{n,t}) \right) \leq \sum_{n=1}^{N} \mathbb{E}(C_n(S_n)) \leq N\mathbf{C}_{\texttt{hit}} + O(\sqrt{\mathbf{C}_{\texttt{info}}\mathbf{C}_{\texttt{miss}}N \log Z}) \\
&= N\sqrt{M\tau} + O(\sqrt{\mathbf{C}_{\texttt{info}}\mathbf{C}_{\texttt{miss}}N \log Z}) \\
&= N\sqrt{M\tau} + O(\tau^{3/4}K^{1/4}\sqrt{NM\log(K)}\,,
\end{aligned}$$

where the first inequality holds by the fact that $\mathbb{E}(C_n(S_n))$ is an upper bound on the regret for task n. $\square$

F   Further experimental details and results

This section consists of further experimental details and results. We use the code in the following repository: https://anonymous.4open.science/r/meta-bandit-760E/README.md. We used a server machine with the following configuration: OS: Ubuntu 18.04 bionic, Kernel: x86_64 Linux 4.15.0-176-generic, CPU: Intel Core i9-10900K @ 20x 5.3GHz, GPU: GeForce RTX 2080 Ti, RAM: 128825 MiB, DISK: 500 GB SSD.

F.1   Setup

In each experiment, the adversary first samples the size M set of optimal arms, $S^* := \cup_n S_n^*$, uniformly at random (without replacement) from $[K]$. The mean reward of task n, $r_n \in \mathcal{R} = [0,1]^K$, is then generated according to the experiment setup as described in the following.

The optimal arm: We categorize the experiments into three settings based on how the optimal arm is generated: i) *non-oblivious adversarial*, ii) *oblivious adversarial*, and iii) *stochastic*.

i) In the adversarial setting with a *non-oblivious* adversary, the adversary peeks into the learner's set of discovered arms, S_n, at the end of each task. With probability q_n (see Eq. (10)), the adversary chooses a new optimal arm uniformly at random from $[K]\backslash S_n$. Otherwise, the next optimal arm is chosen uniformly at random from S_n.

ii) The *oblivious* adversary is applicable against any learner even if the learner does not maintain a set of discovered arms. Here the adversary simulates an imaginary `G-BASS` algorithm with a minimax optimal p_n (see Eq. (10)). Then it samples new optimal arms and generates the reward sequence with respect to this imaginary learner. This is the same as the non-oblivious adversary except here the adversary plays against an imaginary learner.

iii) In the *stochastic* setting, for each task n, the environment samples the optimal arm uniformly at random from the optimal set, i.e., $a_n^* \sim \text{Uniform}(S^*)$.

Note that in the *non-oblivious* setting, the rewards are generated at the start of each task, according to the learner's discovered arms. In the other settings, however, rewards of all the tasks could be generated at the very beginning, independently of the learner.

The sub-optimal arms (min gap): Based on the discussion after Assumption 4.1, the minimum gap for the assumption to hold is

$$r_n(a^*) - \max_{a \neq a^*} r_n(a) \geq \Delta \;,$$

where $\Delta = \Theta(\sqrt{K \log(N/\delta)/\tau})$. After generating the optimal arm, depending on the setting, the rewards of other arms are generated in two ways: 1) with a minimum gap condition uniformly at random in $[0, r_n(a_n^*) - \Delta)$ and 2) without a minimum gap condition uniformly at random in $[0, r_n(a_n^*)]$. In the second case, the mean reward is generated such that the gap condition is violated by at least 1 sub-optimal arm.

Task length and PE: As we know, task length plays an important role in regard to PE performance. In the case where Assumption 4.1 holds, we set the phase length based on Δ and make sure τ is longer than the length of the first phase of PE. For more details, see the analysis of PE (Auer and Ortner, 2010) in exercise 6.8 (elimination algorithm) of Lattimore and Szepesvári (2020).

Assumption 4.1: We have two types of experiments considering Assumption 4.1: i) In the experiments where Assumption 4.1 is supposed to hold, we make sure the task length is longer than the first phase of PE and the minimum gap condition holds (case 1 in the discussion on the min gap). ii) In the tasks where Assumption 4.1 is supposed to be violated, we use case 2 in the discussion on the min gap above with a small τ so that PE fails.

F.2 Further experiments

Next, we report the experimental results under different conditions. Error bars are ± 1 standard deviation, computed over 5 independent runs.

Fig. 2 shows the result when Assumption 4.1 holds, where `G-BASS` almost matches the `Opt-MOSS`, outperforming the other algorithms. Fig. 3 shows the results when Assumption 4.1 does not hold. In this case, we observe that `OS-BASS` outperforms the other algorithms and is close to `Opt-MOSS`. Here `G-BASS` is less effective and sometimes has large variance due to the failure of PE.
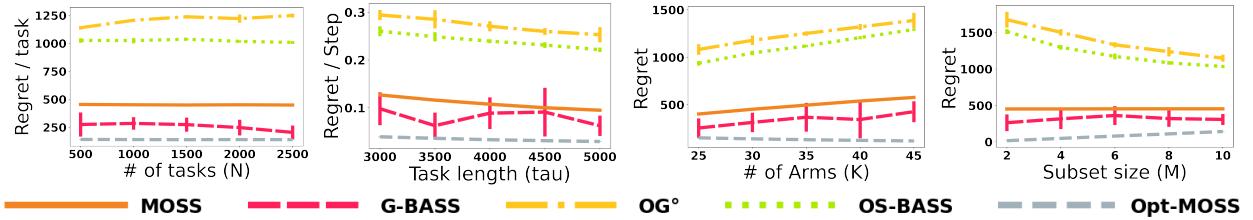

Figure 2: Oblivious adversarial setting with Assumption 4.1. Default setting: $(N, \tau, K, M) = (500, 4500, 30, 10)$. `G-BASS` is near-optimal on all tasks. Left to Right: Regret as a function of N, τ, K, and M.

Fig. 4 demonstrates the performance of `E-BASS` when Assumption 4.1 holds. We can see that `E-BASS` outperforms all other baselines. For large M, `G-BASS` seems to be more effective than the others. Fig. 5 compares `E-BASS` to the other algorithms when Assumption 4.1 does not hold. `OS-BASS` is competitive with `E-BASS` and outperforms it for larger M. Comparing Fig. 4 and Fig. 5, we can see that `G-BASS` and `E-BASS` perform better if Assumption 4.1 holds.

Figs. 6 and 7 show the experimental results with a non-oblivious adversary. We observe similar trends as in the previous experiments.
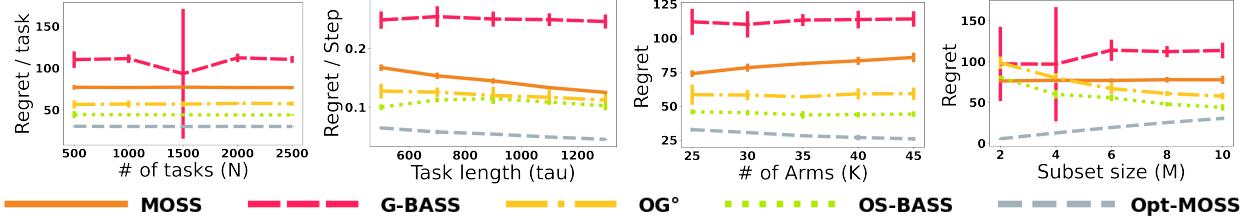
Figure 3: Oblivious adversarial setting without Assumption 4.1 (no minimum gap and small task length). Default setting: $(N, \tau, K, M) = (500, 450, 30, 10)$. OS-BASS is near-optimal on all tasks and outperforms OG°. Left to Right: Regret as a function of N, τ, K, M.
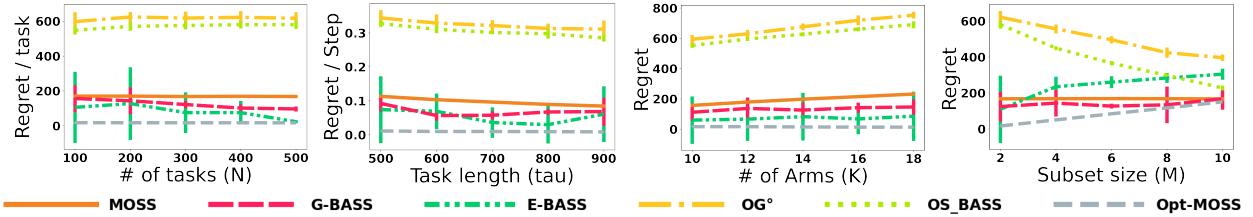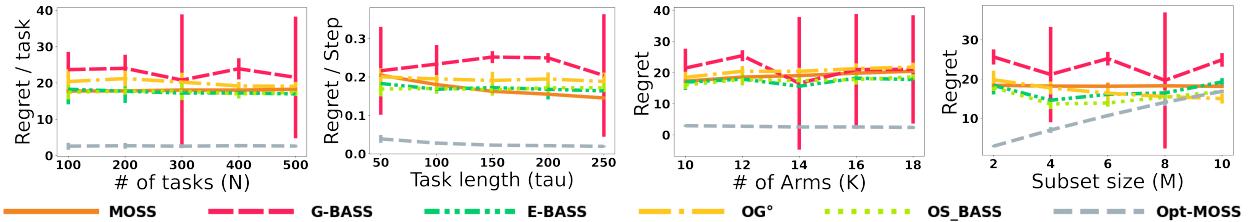

Figure 4: E-BASS's performance in the oblivious adversarial setting with Assumption 4.1. Default setting: $(N, \tau, K, M) = (400, 2000, 11, 2)$. E-BASS outperforms other algorithms. Left to Right: Regret as a function of N, τ, K, M.

Figure 5: E-BASS's performance in the oblivious adversarial setting without Assumption 4.1 (no minimum gap and small task length). Default setting: $(N, \tau, K, M) = (400, 100, 11, 2)$. E-BASS and OS-BASS win all the settings, while MOSS is competitive. Left to Right: Regret as a function of N, τ, K, M.

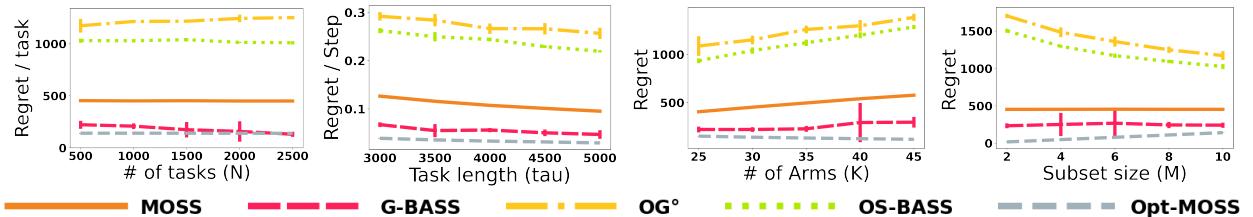

Figure 6: The non-oblivious adversarial setting, where Assumption 4.1 holds. Default setting: $(N, \tau, K, M) = (500, 4500, 30, 10)$. G-BASS is near-optimal on all tasks. Left to Right: Regret as a function of N, τ, K, M.

The results for the stochastic setting are shown in Figs. 8 and 9. In Fig. 8 it seems that G-BASS performs the best while MOSS is closer to the oracle Opt-MOSS. However, in Fig. 9 OS-BASS outperforms G-BASS and the other algorithms and gets closer to the oracle baseline. We can see in the stochastic setting the variance is higher than the adversarial setting.

In all the experiments, OS-BASS outperforms OG° which confirms the choice of γ and τ in our analysis for OS-BASS.
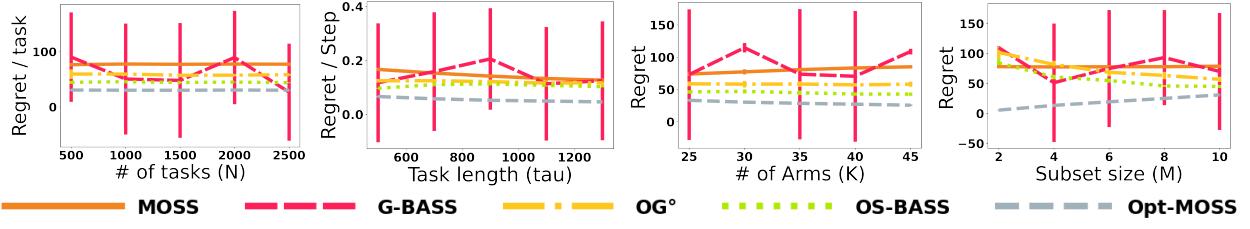
Figure 7: Non-oblivious adversarial setting without Assumption 4.1 (no minimum gap and small task length). Default setting: $(N, \tau, K, M) = (500, 450, 30, 10)$. `OS-BASS` mostly outperforms the other method. `G-BASS` has a high variance as PE fails in this experiment. Left to Right: Regret as a function of N, τ, K, M.
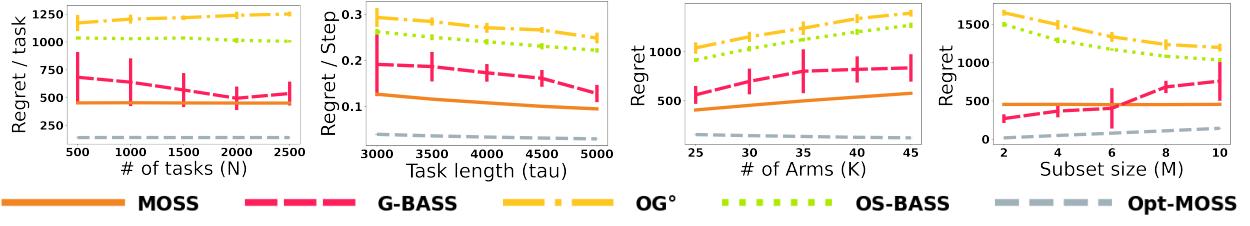

Figure 8: Stochastic setting, where Assumption 4.1 holds. Default setting: $(N, \tau, K, M) = (500, 4500, 30, 10)$. `G-BASS` and `MOSS` have the best performance in all the experiments. Left to Right: Regret as a function of N, τ, K, M.
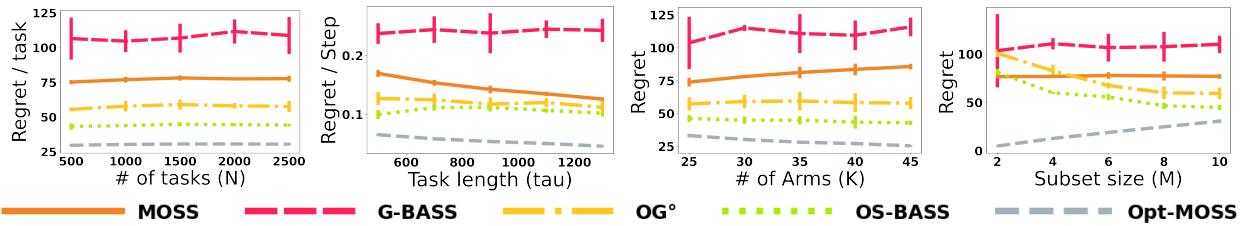

Figure 9: Stochastic setting without Assumption 4.1 (no minimum gap and small task length). Default setting: $(N, \tau, K, M) = (500, 450, 30, 10)$. `OS-BASS` has the best performance in all experiments. Left to Right: Regret as a function of N, τ, K, M.