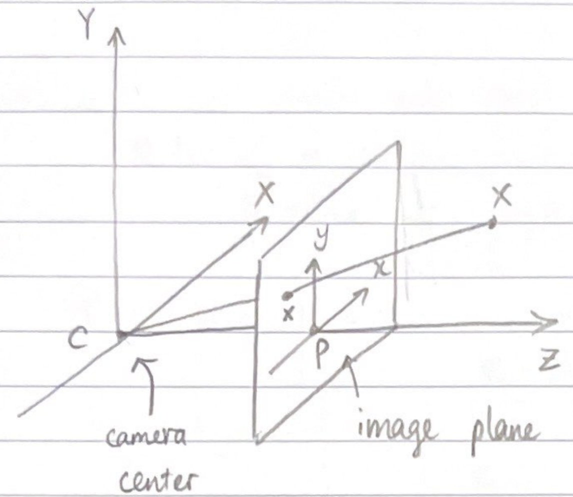
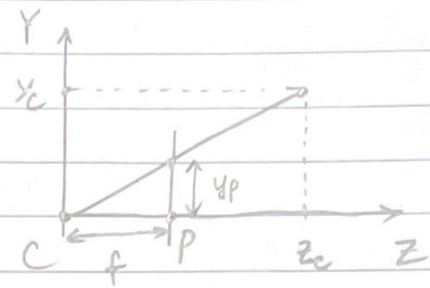
CS-E4850 COMPUTER VISION

Exercise Round 2

Exercise 1:





⇒ Consider this graph, we can see that by using similar triangles,

$$\frac{f}{z_c} = \frac{y_p}{y_c} \Rightarrow y_p = f\frac{y_c}{z_c}$$

+) Similarly, $x_p = f\frac{x_c}{z_c}$

Therefore, the projected point's coordinates are:

$$(x_p, y_p) = \left(f\frac{x_c}{z_c}, f\frac{y_c}{z_c}\right)$$

Exercise 2:

a) $x_p = (x_p, y_p)^T \longmapsto p = [u, v]^T$

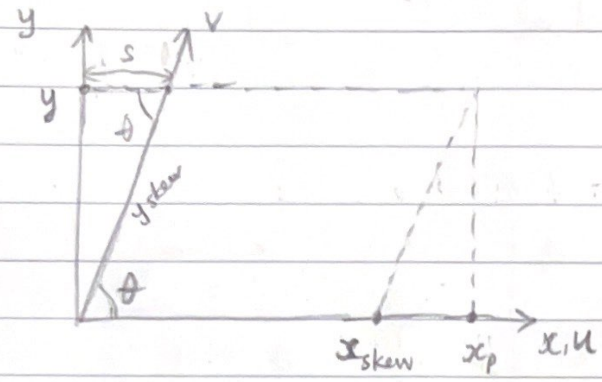
+) Since x_p and y_p are not in pixel coordinates, they need to be converted into pixels by the pixels-per-unit coefficient m_u and m_v (with $u \parallel x$, $v \parallel y$)

$$\Rightarrow (x_p, y_p) \longrightarrow (m_u x_p, m_v y_p)$$

+) Since the coordinates of the principal point is not (0,0) but rather (u_0, v_0), we need to shift the coordinates accordingly.

Therefore, $\begin{cases} u = m_u . x_p + u_0 \\ v = m_v . y_p + v_0 \end{cases} \rightarrow \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} m_u x_p + u_0 \\ m_v y_p + v_0 \end{bmatrix}$

b) $u \parallel x$; $(u,v) = \theta$



$s = y.\cot\theta$

We have:

$$p = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} m_u x_p + v_0 \\ m_v y_p + v_0 \end{bmatrix}$$

From this graph, we have:

$$\begin{cases} x_{p\,skew} = x_p - y_p.\cot\theta \\ y_{p\,skew} = y_p / \sin\theta \end{cases}$$

$$\rightarrow \quad p_{skew} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} m_u (x_p - y_p.\cot\theta) + u_0 \\ m_v\, y_p/\sin\theta \quad + v_0 \end{bmatrix}$$

Exercise 3:

We have:

$$p = \begin{bmatrix} u \\ v \end{bmatrix} \rightarrow \tilde{p} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} z_c u \\ z_c v \\ z_c \end{bmatrix} = \begin{bmatrix} z_c\left(f_x \frac{x_c}{z_c} + u_0\right) \\ z_c\left(f_y \frac{y_c}{z_c} + v_0\right) \\ z_c \end{bmatrix}$$

$$= \begin{bmatrix} f_x x_c + z_c u_0 \\ f_y y_c + z_c v_0 \\ z_c \end{bmatrix} \quad \left(\text{where } \begin{matrix} f_x = m_u \cdot f \\ f_y = m_v \cdot f \end{matrix}\right)$$

$$\tilde{p} = K x_c$$

$$\rightarrow \quad \underset{3\times 3}{K} = \begin{bmatrix} f_x & & u_0 \\ & f_y & v_0 \\ & & 1 \end{bmatrix}$$

In other words,

$$\tilde{p} = K x_c \iff \begin{bmatrix} z_c u \\ z_c v \\ z_c \end{bmatrix} = \begin{bmatrix} f_x & 0 & u_0 \\ 0 & f_y & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$$

Exercise 4:

$$x_c = R x_w + t$$

+) We have an extrinsic calibration matrix: world $\rightarrow$ camera

$$[R \mid t] = \left[\begin{array}{ccc|c} \cos\theta & -\sin\theta & 0 & t_x \\ \sin\theta & \cos\theta & 0 & t_y \\ 0 & 0 & 1 & 1 \end{array}\right]$$

+) We have an intrinsic calibration matrix: camera $\rightarrow$ pixels

$$K = \begin{bmatrix} f_x & 0 & u_0 \\ 0 & f_y & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \underset{3\times 4}{P} : x_w \longmapsto x_{pixels} = K\,[R \mid t] = \begin{bmatrix} f_x\cos\theta & -f_x\sin\theta & u_0 & f_x t_x \\ f_y\sin\theta & f_y\cos\theta & v_0 & f_y t_y \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Exercise 5:

a) Rodrigues Formula:

$$Rx = \cos\theta\, x + \sin\theta\, u \times x + (1-\cos\theta)(u\cdot x)\, u$$

⊛ Geometric justification for Rodrigues Formula:

+) Vector x decomposed into components parallel & perpendicular to axis u:

$$x = x_{\parallel} + x_{\perp}$$

+) $x_{\parallel} = (x\cdot u)\, u$

+) $x_{\perp} = x - x_{\parallel} = x - (x\cdot u)u = -u \times (u \times x)$

+) Component parallel to u will not change magnitude nor direction under the rotation: $x_{\parallel\, rot} = x_{\parallel}$ ①

+) Perpendicular component changes direction but keeps magnitude:

$$|x_{\perp\, rot}| = |x_{\perp}|$$

$$x_{\perp\, rot} = \cos\theta\, x_{\perp} + \sin\theta\, u \times x_{\perp}$$

+) $u \parallel x_{\parallel} \Rightarrow u \times x_{\parallel} = 0$

$$\Rightarrow u \times x_{\perp} = u \times (x - x_{\parallel}) = u \times x - u \times x_{\parallel} = u \times x$$

$$\Rightarrow x_{\perp\, rot} = \cos\theta\, x_{\perp} + \sin\theta\, u \times x \quad ②$$

+) Full rotated vector:

$$\begin{aligned} x_{rot} &= x_{\parallel\, rot} + x_{\perp\, rot} \\ &= x_{\parallel} + \cos\theta\, x_{\perp} + \sin\theta\, u \times x \qquad (\text{from ① \& ②}) \\ &= x_{\parallel} + \cos\theta\,(x - x_{\parallel}) + \sin\theta\, u \times x \\ &= \cos\theta\, x + (1-\cos\theta)\, x_{\parallel} + \sin\theta\, u \times x \\ &= \cos\theta\, x + \sin\theta\, u \times x + (1-\cos\theta)(x\cdot u)\, u \end{aligned}$$

b) Derive the expressions for elements of R as function of θ and u.

$$\begin{cases} u = (u_1, u_2, u_3)^T \\ x = (x_1, x_2, x_3)^T \end{cases}$$

We have: $Rx = x\cos\theta + \sin\theta\, u \times x + (1-\cos\theta)(u\cdot x)u$

$$\Rightarrow R\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \cos\theta \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \sin\theta\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + (1-\cos\theta)\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1\cos\theta + \sin\theta(x_3u_2 - x_2u_3) + (1-\cos\theta)(x_1u_1^2 + x_2u_1u_2 + x_3u_1u_3) \\ x_2\cos\theta + \sin\theta(x_1u_3 - x_3u_1) + (1-\cos\theta)(x_1u_1u_2 + x_2u_2^2 + x_3u_2u_3) \\ x_3\cos\theta + \sin\theta(x_2u_1 - x_1u_2) + (1-\cos\theta)(x_1u_1u_3 + x_2u_2u_3 + x_3u_3^2) \end{bmatrix}$$

$$\Rightarrow R = \begin{bmatrix} \cos\theta + (1-\cos\theta)u_1^2 & (1-\cos\theta)u_1u_2 - \sin\theta\, u_3 & (1-\cos\theta)u_1u_3 + \sin\theta\, u_2 \\ (1-\cos\theta)u_1u_2 + \sin\theta\, u_3 & \cos\theta + (1-\cos\theta)u_2^2 & (1-\cos\theta)u_2u_3 - \sin\theta\, u_1 \\ (1-\cos\theta)u_1u_3 - \sin\theta\, u_2 & (1-\cos\theta)u_2u_3 + \sin\theta\, u_1 & \cos\theta + (1-\cos\theta)u_3^2 \end{bmatrix}$$