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CS-E4850 COMPUTER VISION

Exercise Round 6

Exercise 1:

a) $$E = \sum_{i=1}^{n} \| x_i' - M x_i - t \|^2 = \sum_{i=1}^{n} \left\| \begin{bmatrix} x_i' \\ y_i' \end{bmatrix} - \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} - \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \right\|^2$$

$$= \begin{bmatrix} \sum_{i=1}^{n} (x_i' - m_1 x_i - m_2 y_i - t_1)^2 \\ \sum_{i=1}^{n} (y_i' - m_3 x_i - m_4 y_i - t_2)^2 \end{bmatrix}$$

We have the gradient of E with respect to each parameter:

+) $$\frac{\partial E}{\partial m_1} = \begin{bmatrix} \sum_{i=1}^{n} -2x_i (x_i' - m_1 x_i - m_2 y_i - t_1) \\ 0 \end{bmatrix}$$

+) $$\frac{\partial E}{\partial m_2} = \begin{bmatrix} \sum_{i=1}^{n} -2y_i (x_i' - m_1 x_i - m_2 y_i - t_1) \\ 0 \end{bmatrix}$$

+) $$\frac{\partial E}{\partial m_3} = \begin{bmatrix} 0 \\ \sum_{i=1}^{n} -2x_i (y_i' - m_3 x_i - m_4 y_i - t_2) \end{bmatrix}$$

+) $$\frac{\partial E}{\partial m_4} = \begin{bmatrix} 0 \\ \sum_{i=1}^{n} -2y_i (y_i' - m_3 x_i - m_4 y_i - t_2) \end{bmatrix}$$

+) $$\frac{\partial E}{\partial t_1} = \begin{bmatrix} \sum_{i=1}^{n} -2(x_i' - m_1 x_i - m_2 y_i - t_1) \\ 0 \end{bmatrix}$$

+) $$\frac{\partial E}{\partial t_2} = \begin{bmatrix} 0 \\ \sum_{i=1}^{n} -2(y_i' - m_3 x_i - m_4 y_i - t_2) \end{bmatrix}$$

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b) Setting all partial derivatives to 0, we have:

+) $\sum_{i=1}^{n} -2x_i(x_i' - m_1 x_i - m_2 y_i - t_1) = 0$

$\Rightarrow \sum_{i=1}^{n} x_i x_i' = m_1 \sum_{i=1}^{n} x_i^2 + m_2 \sum_{i=1}^{n} x_i y_i + t_1 \sum_{i=1}^{n} x_i$

+) $\sum_{i=1}^{n} -2y_i(x_i' - m_1 x_i - m_2 y_i - t_1) = 0$

$\Rightarrow \sum_{i=1}^{n} y_i x_i' = m_1 \sum_{i=1}^{n} x_i y_i + m_2 \sum_{i=1}^{n} y_i^2 + t_1 \sum_{i=1}^{n} y_i$

+) $\sum_{i=1}^{n} -2x_i(y_i' - m_3 x_i - m_4 y_i - t_2) = 0$

$\Rightarrow \sum_{i=1}^{n} x_i y_i' = m_3 \sum_{i=1}^{n} x_i^2 + m_4 \sum_{i=1}^{n} x_i y_i + t_2 \sum_{i=1}^{n} x_i$

+) $\sum_{i=1}^{n} -2y_i(y_i' - m_3 x_i - m_4 y_i - t_2) = 0$

$\Rightarrow \sum_{i=1}^{n} y_i y_i' = m_3 \sum_{i=1}^{n} x_i y_i + m_4 \sum_{i=1}^{n} y_i^2 + t_2 \sum_{i=1}^{n} y_i$

+) $\sum_{i=1}^{n} -2(x_i' - m_1 x_i - m_2 y_i - t_1) = 0 \Rightarrow \sum_{i=1}^{n} x_i' = m_1 \sum_{i=1}^{n} x_i + m_2 \sum_{i=1}^{n} y_i + nt_1$

+) $\sum_{i=1}^{n} -2(y_i' - m_3 x_i - m_4 y_i - t_2) = 0 \Rightarrow \sum_{i=1}^{n} y_i' = m_3 \sum_{i=1}^{n} x_i + m_4 \sum_{i=1}^{n} y_i + nt_2$

Therefore, we can rewrite $Sh = u$ as:

$$\begin{bmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i y_i & 0 & 0 & \sum_{i=1}^{n} x_i & 0 \\ \sum_{i=1}^{n} x_i y_i & \sum_{i=1}^{n} y_i^2 & 0 & 0 & \sum_{i=1}^{n} y_i & 0 \\ 0 & 0 & \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i y_i & 0 & \sum_{i=1}^{n} x_i \\ 0 & 0 & \sum_{i=1}^{n} x_i y_i & \sum_{i=1}^{n} y_i^2 & 0 & \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} y_i & 0 & 0 & n & 0 \\ 0 & 0 & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} y_i & 0 & n \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} x_i x_i' \\ \sum_{i=1}^{n} y_i x_i' \\ \sum_{i=1}^{n} x_i y_i' \\ \sum_{i=1}^{n} y_i y_i' \\ \sum_{i=1}^{n} x_i' \\ \sum_{i=1}^{n} y_i' \end{bmatrix}$$

c) We have: $\begin{cases} (x_i, y_i) \in \{(0,0), (1,0), (0,1)\} \\ (x_i', y_i') \in \{(1,2), (3,2), (1,4)\} \end{cases} \Rightarrow n = 3$

+) The components of S could be calculated by:

$$\begin{bmatrix} \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i y_i & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i y_i & \sum_{i=1}^n y_i^2 & \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n y_i & n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

+) The components of u could be calculated by:

$$\begin{bmatrix} \sum_{i=1}^n x_i x_i' & \sum_{i=1}^n x_i y_i' & \sum_{i=1}^n x_i \\ \sum_{i=1}^n y_i x_i' & \sum_{i=1}^n y_i y_i' & \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i' & \sum_{i=1}^n y_i' & n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 1 \\ 5 & 8 & 3 \end{bmatrix}$$

Therefore, we have $Sh = u \Leftrightarrow h = S^{-1}u$

$$\Leftrightarrow \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

Exercise 2:

$$x' = sRx + t \iff \begin{pmatrix} x' \\ y' \end{pmatrix} = s\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

$$\{\mathbf{x}_1 \to x_1'\}, \{x_2' \to x_2'\}$$

a) $$v = x_2 - x_1 = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}, \quad v' = x_2' - x_1' = \begin{bmatrix} x_2' - x_1' \\ y_2' - y_1' \end{bmatrix}$$

+) $$x_2' - x_1' = s(\cos\theta\, x_2 - \sin\theta\, y_2) + t_x - s(\cos\theta\, x_1 - \sin\theta\, y_1) - t_x$$
$$= s(\cos\theta(x_2 - x_1) - \sin\theta(y_2 - y_1))$$

+) $$y_2' - y_1' = s(\sin\theta\, x_2 + \cos\theta\, y_2) + t_y - s(\sin\theta\, x_1 + \cos\theta\, y_1) - t_y$$
$$= s(\sin\theta(x_2 - x_1) + \cos\theta(y_2 - y_1))$$

$$\Rightarrow v' = x_2' - x_1' = \begin{bmatrix} x_2' - x_1' \\ y_2' - y_1' \end{bmatrix} = s\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$

+) We have, from the dot product:

$$\cos\theta = \frac{v' \cdot v}{\|v'\| \|v\|} = \frac{(x_2 - x_1)(x_2' - x_1') + (y_2 - y_1)(y_2' - y_1')}{\sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

$$\Rightarrow \theta = \arccos\left(\frac{(x_2 - x_1)(x_2' - x_1') + (y_2 - y_1)(y_2' - y_1')}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2}}\right)$$

b) $$s = \frac{\|v'\|}{\|v\|} = \sqrt{\frac{(x_2' - x_1')^2 + (y_2' - y_1')^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

c) We have:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = s\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} sx\cos\theta - sy\sin\theta + t_x \\ sx\sin\theta + sy\cos\theta + t_y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} x' - sx\cos\theta + sy\sin\theta \\ y' - sx\sin\theta - sy\cos\theta \end{bmatrix}$$

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d) $\{(\frac{1}{2},0) \to (0,0)\}$, $\{(0,\frac{1}{2}) \to (-1,-1)\}$

$$v = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}, \quad v' = \begin{bmatrix} x_2' - x_1' \\ y_2' - y_1' \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \theta = \arccos\left(\frac{(x_2-x_1)(x_2'-x_1') + (y_2-y_1)(y_2'-y_1')}{\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \cdot \sqrt{(x_2'-x_1')^2 + (y_2'-y_1')^2}} \right) = \arccos(0)$$

$$\Rightarrow \theta = \frac{\pi}{2} \; (+ k\pi)$$

Also, we have $s = \sqrt{\frac{(x_2'-x_1')^2 + (y_2'-y_1')^2}{(x_2-x_1)^2 + (y_2-y_1)^2}} = 2$

$$\begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} x_1' - s x_1 \cos\theta + s y_1 \sin\theta \\ y_1' - s x_1 \sin\theta - s y_1 \cos\theta \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Therefore, the complete transformation in this case:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = 2 \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$