CS-E4850 COMPUTER VISION

Exercise round 1

Exercise 1:

a) Line $l: ax + by + c = 0$
$\rightarrow$ In homogenous coordinates, $l = (a, b, c)^T$

Define a point x that lies on l as $(x, y)^T$ in $\mathbb{R}^2$
$\rightarrow$ In homogenous coordinates, $x = (x, y, 1)^T$

$$\Rightarrow \quad x^T l = (x, y, 1)(a, b, c)^T = ax + by + c \quad (= l^T x)$$
$$= 0 \text{ (since } x \text{ lies on } l)$$

b) Let $l = (a, b, c)^T$
$l' = (a', b', c')^T$
$x = l \times l'$

We have the triple scalar product identity $l.(l \times l') = l'.(l \times l') = 0$
$\rightarrow$ we can see that $l^T x = l'^T x = 0$ (according to textbook)
As previous proved, $l^T x = 0 \rightarrow x$ lies on l
$l'^T x = 0 \rightarrow x$ lies on l'
$\rightarrow$ **x** is the intersection of l and l'

c) Let $x = (x, y, 1)^T$
$x' = (x', y', 1)^T$
$l = x \times x'$

We have from the triple scalar product rule:
$$(\mathbf{x} \times x')^T x = (x \times x')^T x' = 0$$
$$\Rightarrow \quad l^T x = l^T x' = 0$$
$$\Rightarrow \quad \begin{cases} x \text{ lies on } l \\ x' \text{ lies on } l \end{cases}$$
$\Rightarrow$ l is the line passing through x and x'

d)

$x = (a, b, 1)^T$

$x' = (a', b', 1)^T$

$\Rightarrow$ Line passing through x and x': $l = x \times x'$

$= (a, b, 1)^T \times (a', b', 1)^T$

$= (b - b', a' - a, ab' - a'b)^T$

$$
\begin{aligned}
y &= \alpha x + (1-\alpha) x' \\
&= \alpha (a, b, 1)^T + (1-\alpha)(a', b', 1)^T \\
&= (\alpha a + (1-\alpha) a', \alpha b + (1-\alpha) b', \alpha + (1-\alpha))^T \\
&= (\alpha a + a' - \alpha a', \alpha b + b' - \alpha b', 1)^T
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad y^T l &= (\alpha a + a' - \alpha a')(b - b') + (\alpha b + b' - \alpha b')(a' - a) \\
&\quad + (ab' - a'b) \\
&= \alpha ab - \alpha ab' + a'b - a'b' - \alpha a'b + \alpha a'b' \\
&\quad + \alpha a'b - \alpha ab + a'b' - ab' - \alpha a'b' + \alpha ab' \\
&\quad + ab' - a'b \\
&= 0
\end{aligned}
$$

Therefore, as previously proved, y lies on l.

Exercise 2:

a) Let $x = (x, y, 1)^T$ in homogenous coordinates

+) Translation matrix: $T = \begin{bmatrix} \varepsilon \cos 0^\circ & -\sin 0^\circ & t_x \\ \varepsilon \sin 0^\circ & \cos 0^\circ & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$

+) Euclidean transformation matrix: (rotation + translation) $(\varepsilon = \pm 1)$

$$E = \begin{bmatrix} \varepsilon \cos\theta & -\sin\theta & t_x \\ \varepsilon \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \quad (\varepsilon = \pm 1)$$

+) Similarity transformation matrix: (scaling + rotation + translation)

$$S = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \quad (s: \text{isotropic scaling})$$

+) Affine transformation matrix: (non-singular linear + translation)

$$A = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

+) Projective transformation matrix: (all of the above)

$$P = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

b) The degree of freedom in each transformation:

+) Translation: 2 dof

+) Euclidean: 3 dof

+) Similarity 4 dof

+) Affine: 6 dof

+) Projective 8 dof

c)

A projective transformation matrix H is represented by a non-singular 3×3 matrix, which means there are 9 elements. As H is a transformation matrix in homogenous coordinates, the exact value of each element is not important, but rather only the ratios between elements are significant. Therefore, there are 8 independent ratios amongst the 9 elements and the degree of freedom of H is 8.

Exercise 3:

a) We have:

$$l = (a, b, c)^T$$

$$x' = Hx \Rightarrow x = H^{-1}x'$$

Let l' be the planar projective transformation representation of l

Then, $l'^T x' = 0$ (since x' lies on l')

$$\Rightarrow l'^T x' = l^T x = l^T H^{-1} x'$$

$$\Rightarrow l'^T = l^T H^{-1}$$

$$\Rightarrow l' = H^{-T} l$$

Therefore, the line transformation $l \rightarrow l'$ is H^{-T}

b) We have:

$$x' = Hx \Leftrightarrow x = H^{-1}x'$$

$$l^T x = l'^T x' = l'^T H x \Rightarrow l^T = l'^T H$$

$$l' = H^{-T} l$$

$$I = \frac{(l_1^T x_1)(l_2^T x_2)}{(l_1^T x_2)(l_2^T x_1)}$$

Let l_1', l_2', x_1', x_2' be the projective transformation of l_1, l_2, x_1, x_2 respectively.

$$\rightarrow I = \frac{(l_1'^T H H^{-1} x_1')(l_2'^T H H^{-1} x_2')}{(l_1'^T H H^{-1} x_2')(l_2'^T H H^{-1} x_1')}$$

$$= \frac{(l_1'^T x_1')(l_2'^T x_2')}{(l_1'^T x_2')(l_2'^T x_1')}$$

+) Providing l_1 & l_2 being the projective transformation of each other, and x_1 & x_2 also being the projective transformation of each other:

$$I = \frac{(l_1'^T x_1')(l_2'^T x_2')}{(l_1'^T x_2')(l_2'^T x_1')} = \frac{(l_2^T x_2)(l_1^T x_1)}{(l_2^T x_1)(l_1^T x_2)} = I$$

→ This is an invariant under projective transformation

It is clear that the correlation between elements of I before and after the projective transformation is the same, which means I is an invariant under projective transformation.

+) Given an arbitrary scaling of the homogenous coordinate vectors with a non-zero scaling factor, we have:

$$I_s = \frac{(a l_1^T c x_1)(b l_2^T d x_2)}{(a l_1^T d x_2)(b l_2^T c x_1)}$$ (where a,b,c,d are scaling factor for l_1, l_2, x_1, x_2

→ Projective transformation:

$$\rightarrow I_s = \frac{(a l_1'^T H H^{-1} c x_1')(b l_2'^T H^{-1} H d x_2')}{(a l_1'^T H H^{-1} d x_2')(b l_2'^T H^{-1} H c x_1')}$$

$$= \frac{ac(l_1'^T x_1')\, bd(l_2'^T x_2')}{ad(l_1'^T x_2')\, bc(l_2'^T x_1')} = \frac{(l_1'^T x_1')(l_2'^T x_2')}{(l_1'^T x_2')(l_2'^T x_1')}$$

Providing (l_1, l_2) and (x_1, x_2) satisfy that each value is the projective transformation of the other value,

→ $I_s = I$

This means the scaling factors are cancelled out during the projective transformation, assuming 1 important property is assured, that the number of terms in the invariant is unchanged. Any reconstruction with smaller number of terms (fewer points and lines) would result in different number of scaling factors that cannot be cancelled out.