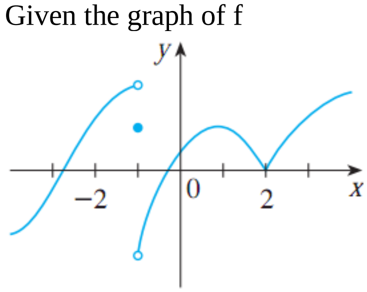
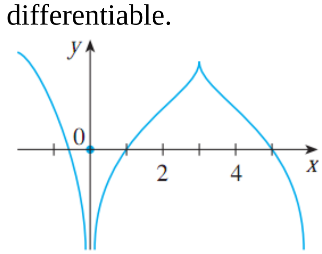
CALCULUS – KEY TERMS & MAIN RESULTS

<table>
<tr><th>Key terms</th><th>Problems with solutions</th><th>Exercises - Do yourself</th></tr>
<tr><td colspan="3">Chapter 1. Functions and limits:</td></tr>
<tr><td>Functions

domain and range</td><td>Find the domain and range of y = $\sqrt{4-x^2}$
Solution.
Domain: the set of all x-values such that f(x) is defined
$4 - x^2 \geq 0 \Leftrightarrow 4 \geq x^2 \Leftrightarrow 2 \geq x \geq -2$
Range (the set of all y-values): y = $\sqrt{4-x^2}$ ➔ $\sqrt{4} \geq y \geq 0$</td><td>1/ Find the domain and range of the functions:
a/ y = $\sqrt{16-x^2}$
b/ y = $\frac{1}{x^4+1} + 3$</td></tr>
<tr><td>Odd functions
Even functions</td><td>Ex1. A function f is called odd if f(-x) = -f(x) for all x in domain. For example, f(x) = sinx. (f(-x) = sin(-x) = -sin(x) = -f(x)).
Suppose f is an odd function and (-3, 2) is a point in the graph of y = f(x). Show that (3, -2) is also a point of the graph of f.
Solution.
(-3, 2) is a point of the graph of f ➔ f(-3) = 2
f is odd ➔ f(-x) = -f(x) ➔ f(-3) = -f(3) = -2.
So, (3, -2) is also a point in the graph of f.

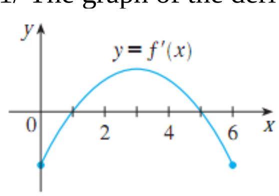
Ex2. Even function. A function f is called even if f(-x) = f(x) for all I in domain. For example, f(x) = x^2. (f(-x) = $(-x)^2 = x^2$ = f(x)).
Suppose f is an even function and (-3, 2) is a point in the graph of y = f(x). Show that (3, 2) is also a point of the graph of f.
Solution.
(-3, 2) is a point of the graph of f ➔ f(-3) = 2
f is odd ➔ f(-x) = f(x) ➔ f(-3) = f(3) = 2.
So, (3, 2) is also a point in the graph of f.</td><td>2/
a/ Suppose f, g are odd functions. Show that h(x) = f(x) + g(x) is also an odd function.
b/ Suppose f, g are even functions. Show that h(x) = f(x) + g(x) is also an even function.</td></tr>
<tr><td>composite function
(fog)(x) = f(g(x))</td><td>Given f(x) = x^3 and g(x) = x + 3, find f(g(x)) and g(f(x)).
Solution.
f(g(x)) = f(x+3) = $(x+3)^3$
g(f(x)) = g(x^3) = x^3 + 3</td><td>3/ Find f(g(x)) and g(f(x)) if f(x) = 1/(x+1) and g(x) = x^2.
4/ Find (fog)(3), g(f(4)) from the table<table><tr><td>x</td><td>1</td><td>2</td><td>3</td><td>4</td></tr><tr><td>f</td><td>7</td><td>1</td><td>5</td><td>2</td></tr><tr><td>g</td><td>2</td><td>8</td><td>1</td><td>4</td></tr></table></td></tr>
<tr><td>from y = f(x) to y = f(x+c) and y = f(x) + c</td><td>Suppose the graph of y = f(x) is given. Say how the graphs of a) y=f(x) + 3 b) y = f(x+3) c) y = f(x-3) + 2 are obtained.
Solution.<ol type="a">
The graph of y = f(x) + 3 is obtained by shifting the given graph 3 units UP.
The graph of y = f(x+3) is obtained by shifting the given graph 3 units to the LEFT.</td><td>5/How to obtain the graph of y = f(x - 3) from the graph of y = f(x)?
6/ How to obtain the graph of y = $(x-3)^2$ + 1 from the graph of y = x^2?</td></tr>
</table>

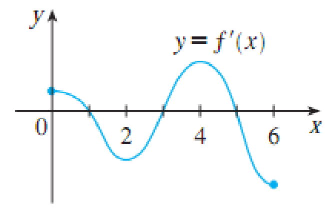
	c. The graph of y = f(x-3) + 2 is obtained by shifting the given graph 3 units to the RIGHT, then 2 units UP.	
find **limits** of **functions**	Find $\lim\limits_{x\to 3^-}\frac{x^2-9}{\lvert x-3\rvert}$ *and* $\lim\limits_{x\to 3}\frac{x^2-9}{\lvert x-3\rvert}$ (if any) Solution. • $x \to 3^-$ means is near 3 and x < 3 ➔ |x – 3| = -(x-3) ➔ $\lim\limits_{x\to 3^-}\frac{x^2-9}{\lvert x-3\rvert} = \lim\limits_{x\to 3^-}\frac{(x-3)(x+3)}{-(x-3)}$ $= -\lim(x+3) = -6$ • $x \to 3^+$: |x – 3| = x-3 and $\lim\limits_{x\to 3^+}\frac{x^2-9}{\lvert x-3\rvert} = \lim\limits_{x\to 3^+}\frac{(x-3)(x+3)}{(x-3)}$ $= \lim(x+3) = 6$ • $\lim\limits_{x\to 3^-}\frac{x^2-9}{\lvert x-3\rvert} \neq \lim\limits_{x\to 3^+}\frac{x^2-9}{\lvert x-3\rvert}$ ➔ $\lim\limits_{x\to 3}\frac{x^2-9}{\lvert x-3\rvert}$ *does not exist.* [Trick: try with x near 3, for example, x = 3.01, x = 2.99 and consider the results.]	**7/** Find $\lim\limits_{x\to 2}\frac{x^2-4}{\lvert x-2\rvert}$ (if any) **8/** Find $\lim\limits_{x\to\infty}(\sqrt{x^2+2x} - x)$
test for **continuity** (at x = a)	Given the function $f(x) = \begin{cases} x^2 - x \text{ } if \text{ } x \leq 3 \\ x - m \text{ } if \text{ } x > 3 \end{cases}$. Find m such that f is **continuous** at x = 3. Solution. • f(3) = 6 • $\lim\limits_{x\to 3^-}$ f(x) = 3^2 – 3 = 6 • $\lim\limits_{x\to 3^+}$ f(x) = 3 – m • f is **continuous** at x = 3 ⇔ $\lim_{x\to 3}$f(x) = f(3) ⇔ $\lim\limits_{x\to 3^-}$ f(x) = $\lim\limits_{x\to 3^+}$ f(x) = f(3) ⇔ 3 – m = 6 ⇔ m = -3	**9/** Find all values of a, b such that $f(x) = \begin{cases} \frac{x^2-1}{x-1} \text{ } if \text{ } x < 1 \\ x^2 + a \text{ } if \text{ } 1 \leq x < 2 \\ x + b \text{ } if \text{ } x \geq 2 \end{cases}$ is **continuous** at x = 1 and x = 2.
Chapter 2. Derivatives		
differentiable	Example 1. Given $f(x) = \begin{cases} x^2 \text{ } if \text{ } x \leq 2 \\ 4x - 4 \text{ } if \text{ } x > 2 \end{cases}$ Find f'(2) or say it is not **differentiable** at x = 2. Solution. • $\lim\limits_{x\to 2^-}\frac{f(x)-f(2)}{x-2} = \lim\limits_{x\to 2^-}\frac{x^2-4}{x-2} = 4$ • $\lim\limits_{x\to 2^+}\frac{f(x)-f(2)}{x-2} = \lim\limits_{x\to 2^+}\frac{4x-4-4}{x-2} = 4$ • f'(2) = $\lim\limits_{x\to 2}\frac{f(x)-f(2)}{x-2} = 4$ Example 2. Given the graph of f  Find x in which f is not differentiable.	**10/** Compute dy/dx or y': a/ y = $\frac{3}{1-\sqrt{x}}$ b/ y = ln(3x) – e^{-2x} **11/** Compute $\frac{d^2y}{dx^2}$ or y'' a/ y = $\frac{3}{1-2x}$ b/ y = e^{-2x} -1/x 12/ Given the graph of the function f. Find the numbers at which f is not differentiable. 

	Solution. See the graph and find the points that the graph is not "smooth", or discontinuous. ⇨ x = -2, x = 2.	
Slope, Tangent line and **linearization** of y= f(x) at a: y = f'(a)(x-a) + (a)	Find an equation of the **tangent line** to the curve $y = \sqrt{x^2+3}$ at the point (1, 2). Solution. $$y' = \frac{(x^2+3)'}{2\sqrt{x^2+3}} = \frac{x}{\sqrt{x^2+3}}$$ • $y'(1) = \frac{1}{2}$ // ***slope*** of the tangent line • *An equation of the* ***tangent line***: $$y = f'(x_0)(x - x_0) + f(x_0)$$ $$y = \frac{1}{2}(x-1) + 2$$ $$y = \frac{1}{2}x + \frac{3}{2}$$ -The tangent line $y = \frac{1}{2}x + \frac{3}{2}$ *is also* called the **linearization** of $y = \sqrt{x^2+3}$ at x = 1 and we can use this line to **approximate** the value of f(x) for x near 1. -For example, to **approximate** $\sqrt{x^2+3}$ with x = 0.98 (near 1), we can use $y = \frac{1}{2}x + \frac{3}{2}$ *and the result is* $\frac{1}{2}0.98 + \frac{3}{2}$ = 1.99.	**13/** Given the curve y = x^3 – 2x a/ Find the **tangent line** of the curve at the point (2, 4). b/ Find the point on the graph of the curve at which the tangent line has ***slope*** 1. **14/** Find the **linearization** of the function f(x) = $\frac{1}{4}x^4$ – 5x + 3 at x = 2.
Find (fog)'(x) By chain rule: (fog)'(x) = f'(g(x)).g'(x)	Given f(u) = $\sqrt{u}$, g(x) = 1 + 3x^2, find (fog)'(1). Solution. • Let u = g(x), then u'(x) = 6x • f'(u) = $\frac{1}{2}\sqrt{u}$ = $\frac{1}{2}\sqrt{1+3x^2}$ • (fog)'(x) = f'(g(x)).g'(x) = f'(u).u'(x) = 6x$\frac{1}{2}\sqrt{1+3x^2} = 3\sqrt{1+3x^2}$ ⇨ (fog)'(1) = 6	**15/** a/ Given f(u) = u^2, g(x) = 1 + 2x. Find (fog)'(2). b/ Given F(x) = f(g(x)), and f(-2) = 8, f'(-2) = 4, f'(5) = 3, g(5) = -2, g'(5) = 6. Find F'(5). **16/** Suppose H(x) = $(2x + 1)^3$ – 5 can be expressed as (fogoh)(x), and f(x) = x – 5, h(x) = 2x + 1, what is g(x)?
find **dy/dt** (rate of y) when given **dx/dt** (rate of x), x and y.	Example 1. Given $x^2 + y^3$ = 12 and dx/dt = -3, find dy/dt when x = 2. Solution. $x^2 + y^3$ = 12 ➔ y = 2 if x = 2 ⇨ $\frac{d}{dt}(x^2 + y^3) = \frac{d}{dt}(12)$ ⇨ $2x\frac{dx}{dt} + 3y^2\frac{dy}{dt} = 0$ ⇨ 2.2.(-3) + 3.(2).$\frac{dy}{dt}$ = 0 ⇨ $\frac{dy}{dt}$ = 2 Example 2. Each side of a square is increasing at a ***rate*** of 6cm/s . At what ***rate*** is the area of the square increasing when the area of the 16cm^2?	**17/** Given $x^3 + y^3$ = 9 and dx/dt = -3, find dy/dt when x = 2. **18/** A ladder 5m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of ½ m/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3 m from the

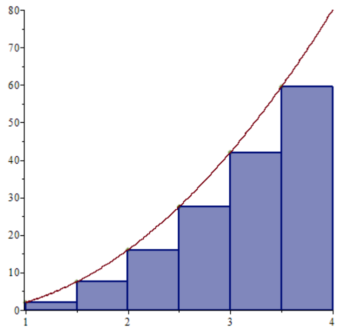
	Solution. • A: area of square, x : length of a side • Rate of side: x'(t), rate of area A'(t) • $A = x^2$ And $A'(t) = 2x.x'(t) = 2.4.6 = 48\ cm^2/s$	wall?
differential **dy** = f'(x)dx and approximation ςy $\approx$ f'(x)dx	The **radius** of a circular disk is given as 24 cm with a **maximum error** in measurement of 0.2 cm. Use **differentials** to estimate the **maximum error** in the calculated **area** of the disk. Solution. A: area $= \pi r^2$ r: radius = 24cm **maximum error of r:** $\varsigma r = 0.2$ cm maximum error of area = $\varsigma A \approx A'(r)\varsigma r = 2\pi r \varsigma r \approx$ 30.15929 cm^2	**19/** The **edge** of a **cube** was found to be 30 cm with a **possible error** in measurement of 0.1 cm. Use **differentials** to estimate the **maximum possible error**.
Find dy/dx by **implicit differentiation**.	Use **implicit differentiation** to find an equation of the **tangent line** to the curve $x^2 + xy + y^2 = 3$ at the given point (1, 1). Solution. $\frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}(3)$ ⇨ $(x^2)' + (xy)' + (y^2)' = 0$ ⇨ $2x + x'y + xy' + 2y.y' = 0$ ⇨ $2x + y + (x+2y).y' = 0$ ⇨ $y' = -(2x+y)/(x+2y)$ ⇨ $y'(1) = -3/3 = -1$ Equation of the **tangent line**: $y = y'(1)(x - x_0) + f(x_0)$ ⇨ $y = -(x - 1) + 1$ ⇨ $y = -x$	**20/** Find dy/dx by **implicit differentiation**. $x^2 + xy - y^2 + x = 2$. **21/** Use **implicit differentiation** to find an equation of the **tangent line** to the curve at the given point. $x^2 + 2xy - y^2 + x = 2$, (1, 2)

Chapter 3. App. Of differentiation

critical numbers	Find the ***critical numbers*** of the function. $f(x) = 2x^3 + 3x^2 - 36x$ Solution. $f'(x) = 6x^2 + 6x - 36$ $f'(x) = 0 \Leftrightarrow x = 2, x = -3$ critical numbers: 2 and -3	**22/** Find the ***critical numbers*** of the function. $f(x) = f(x) = x^4 - 2x^2 + 3$
increasing/decreasing **local (relative) min/max**: 1st derivative test and 2nd derivative test **concave upward/downward**	1/ The graph of the derivative of a function is shown.  a/ On what intervals is *f* increasing or decreasing? b/ At what values of x does *f* have a local maximum or minimum? Solution. a/ Based on the graph above, f'(x) < 0 on the intervals (0, 1) and (5, 6) ➔ f is decreasing on (0, 1) and (5, 6); f	**23/** The graph of the derivative of a function is shown. a/ On what intervals is increasing or decreasing? b/ At what values of x does have a local maximum or minimum?

inflection points	is increasing on (1, 5) because f'(x) > 0 on (1, 5). b/ f' changes sign from (-) to (+) at x = 1 ➔ f has local minimum at x = 1. f' changes sign from (+) to (-) at x = 5 ➔ f has local maximum at x = 5. 2/ Given $f(x) = 2x^3 + 3x^2 - 36x$ a/ Find the intervals on which is *f* increasing or decreasing. b/ Find the local maximum and minimum values of *f* . c/ On what intervals is *f* concave upward or concave downward? d/ Find all inflection points of *f*. Solution. • $f'(x) = 6x^2 + 6x - 36$ $f'(x) = 0 \Leftrightarrow x = 2, x= - 3$ sign of f' x -∞ -3 2 ∞ f' + 0 - 0 + a/ f is increasing on (-∞, -3), and increasing on (2, ∞) f is decreasing on (-3, 2). b/ local max: f(-3) = 81, local min: f(2) = -44 • $f''(x) = 12x + 6$ $f''(x) = 0 \Leftrightarrow x = - ½$ sign of f'' x -∞ - ½ ∞ f'' - 0 + c/ f is concave downward on (-∞, - ½) and concave upward on (- ½, ∞) d/ at x = - ½, f changes from concave downward to concave upward ➔ inflection point is (-1/2, f(-1/2)) or (-1/2, 20)	 **24/** Given $f(x) = x^4 - 2x^2 + 3$ a/ Find the intervals on which is *f* increasing or decreasing. b/ Find the local maximum and minimum values of *f* . c/ On what intervals is *f* concave upward or concave downward? d/ Find all inflection points of *f*.
abs. max/min and **Optimization problems**	Find two numbers whose difference is 40 and product is minimum. Solution. We find x and y such that x – y = 40 and x.y is minimum. Let $f(x) = x.y = x.(x-40) = x^2 - 40x$ $f'(x) = 2x - 40$ $f'(x) = 0 \Leftrightarrow x = 20$ and $f''(20) = 2 > 0$ ⇨ f(20) = -400 is minimum value of f. So, x = 20 and y = -20	**25/** a/ Find two numbers whose difference is 20 and product is minimum. b/ Find the absolute maximum and minimum of the function $f(x) = \frac{1}{3}x^3 - 2x^2 + 5x - 1$ on [0, 3].
Rolle's theorem	Rolle's Theorem. If a function f satisfies the following: • f is continuous on [a, b] • f is differentiable on (a, b) • f(a) = f(b) Then, there exists some numbers c in (a, b)	**26/** Find all numbers c satisfying the Rolle's theorem if $f(x) = x^3 - x^2 - 5x - 11$.

	such that f'(c) = 0. **Ex.** Find all numbers c satisfying the Rolle's theorem if $f(x) = x^3 – 2x^2 – 7x – 15$. Solution. Based on the theorem, f'(c) = 0 ⟷ $3c^2 – 4c – 7 = 0$ ⟷ c = -1, c = 7/3.	
$\frac{f(b)-f(a)}{b-a}$ **= f'(c)** (Mean value theorem)	If f(1) = 10 and f'(x) ≥ 5 for all x, how small can f(4) possibly be? Solution. Based on MVT, there exists c in (1, 4) such that $f'(c) = \frac{f(4)-f(1)}{4-1}$ ➔ f(4) – f(1) = 3.f'(c) ≥ 3.5 ⇨ f(4) ≥ 15 + f(1) = 25 ⇨ smallest value of f(4) is 25.	**27/** If f(3) = 7 and f'(x) ≤ 4 for all x, how large can f(8) possibly be?
Newton's method: find **n**$^{\text{th}}$ **approximation** to the solution of an **equation f(x) = 0**.	Use Newton's method to find x_3 to approximate the solution of the equation $x^3 – x = 7$. Choose $x_1 = 2$ and round the result to 2 decimal places. Solution. • $x^3 – x = 7 \Leftrightarrow x^3 – x – 7 = 0$ • Let $f(x) = x^3 – x – 7$ ⇨ $f'(x) = 3x^2 - 1$ • Use the formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $x_1 = 2$ $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 2.090909091$ $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 2.086754310 \approx 2.09$	**28/** Use Newton's method to find x_3, the 3rd approximation to the solution of the equation $x^3 + 2x = 5$. Choose $x_1 = 1$ and round the result to 2 decimal places.
position function s(t) **velocity v(t)** **acceleration a(t)**	A particle is moving on a straight line with acceleration a(t) = 12t + 4 (cm/s^2). a/ Find velocity v(t) if v(0) = 0. b/ Find position of the particle after 5 seconds if s(0) = 3. c/ Find the total distance traveled by the particle after 5 seconds. Solution. a/ velocity $v(t) = \int a(t)dt = 6t^2 + 4t + C$ $v(0) = 0 \Leftrightarrow C = 0$ So, $v(t) = 6t^2 + 4t$ b/ position = $s(t) = \int v(t)dt = 2t^3 + 2t^2 + C$ s(0) = 3 ➔ $2.0^3 + 2.0^2 + C = 3$ ➔ C = 3 ⇨ $s(t) = 2t^3 + 2t^2 + 3$ ⇨ position after 5 seconds is: s(5) = 303 (cm) c/ total distance = $\int_0^5 \lvert v(t)\rvert dt = 300$ (cm)	**29/** A particle is moving on a straight line with acceleration a(t) = 12t + 4 (cm/s^2). a/ Find velocity v(t) if v(0) = 0. b/ Find position of the particle after 5 seconds if s(0) = 3. c/ Find the total distance traveled by the particle after 5 seconds.
Find **antiderivatives**	Find f(x) if $f''(x) = 6x^2 – 4x$ and f(0) = f(1) = 3. Solution. $f''(x) = 6x^2 – 4x$ ➔ $f'(x) = 2x^3 – 2x^2 + C$ ⇨ $f(x) = \frac{1}{2}x^4 - \frac{2}{3}x^3 + Cx + D$ So, f(0) = D = 3 and f(1) = -1/6 + C + 3 = 3 ➔ C = 1/6	**30/** Find f(x) if $f''(x) = 12x^2 – 2x + 3$ and f(0) = f'(0) = 2.

<table>
<tr><td></td><td>Hence, f(x) = = $\frac{1}{2}x^4 - \frac{2}{3}x^3$ + x/6 + 3</td><td></td></tr>
<tr><th colspan="3">Chapter 4-6: Integrals</th></tr>
<tr><td>Integrals and areas, Riemann sum, left endpoint, right endpoint, midpoint</td><td>Given f(x) = $6x^2 - 4x$
a/ Approximate the area under f(x) from x = 1 to x = 4 using Riemann sum with n = 6 and left endpoints.
b/ Find the area under f(x) from x =1 to x = 4 by computing the integral $\int_1^4 f(x)$dx.
Solution.
a/ area ≈ $\frac{4-1}{6}$(f(1) + f(1.5) + f(2) + f(2.5) + f(3) + f(3.5)) = 77.25000000


b/Actual area = $\int_1^4 f(x)$dx = $\int_1^4 (6x^2 - 4x)$dx = 96.</td><td>31/ Given f(x) = $3x^2 - 2x$
a/ Approximate the $\int_0^8 f(x)$dx by computing the area under f(x) using Riemann sum with n = 4 and right endpoints.
b/ Find the area under f(x) from x = 0 to x = 8.</td></tr>
<tr><td>$\int_a^b f(x)\, dx$ = F(b) – F(a)</td><td>1/ Given f(1) = 3, f’ is continuous and $\int_1^4 f'(x)$dx = 7. Find f(4).
Solution.
f is an antiderivative of f’ ➔ $\int_1^4 f'(x)$dx = f(4) – f(1)
⇨ f(4) – f(1) = 7 ➔ f(4) = 7 + 3 = 10
2/ Suppose h is a function such that h(1) = -2, h’(1) = 2, h’’(1) = 3, h(2) = 6, h’(2) = 5, h’’(2) = 13 and h’’ is continuous everywhere.
Evaluate $\int_1^2 h''(x)$dx.
Solution.
h’ is an antiderivative of h’’ ➔
$\int_1^2 h''(x)$dx = h’(2) – h’(1) = 5 – 2 = 3.</td><td>32/ Given $\int_0^b \sqrt{x}$ dx = $\frac{\sqrt{2}}{6}$
Find b.

33/ Compute $\int_0^3 f(x)$ dx, where
$f(x) = \begin{cases} 5 & if\ x < 2 \\ x^2 + 1 & if\ x \geq 2 \end{cases}$
Hint:
$\int_0^3 f(x)$ dx = $\int_0^2 f(x)$ dx + $\int_2^3 f(x)$ dx</td></tr>
<tr><td>Trapezoidal rule and Simpson’s rule</td><td>Given the table of values of f(x)
<table><tr><td>x</td><td>0</td><td>2</td><td>4</td><td>6</td><td>8</td><td>10</td></tr><tr><td>f(x)</td><td>0</td><td>5</td><td>4</td><td>2</td><td>-3</td><td>2</td></tr></table>Approximate $\int_0^{10} f(x)$ dx using trapezoidal rule with n = 5 and the given data.
Solution.
$\int_0^{10} f(x)$ dx ≈ $\frac{1}{2}$(f(0) + 2f(2) + 2f(4) + 2f(6) + 2f(8) + f(10)) = $\frac{1}{2}$(0 + 10 + 8 + 4 + (-6) + 2) = 9.</td><td>34/ Use trapezoidal rule with n = 4 to approximate the integral $\int_0^4 f(x)$dx if:
a/
<table><tr><td>x</td><td>f(x)</td></tr><tr><td>0</td><td>2</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>-1</td></tr><tr><td>3</td><td>3</td></tr><tr><td>4</td><td>5</td></tr></table>b/ f(x) = $\sqrt{x^4 + 1}$</td></tr>
<tr><td>Average value of</td><td>Find the average value of the function f(x) = $3x^2$ –</td><td>35/ Suppose the average</td></tr>
</table>

f(x) over [a, b]	2x over [1, 3]. Solution. $f_{ave} = \frac{\int_1^3 f(x)dx}{3-1} = \frac{\int_1^3 (3x^2-2x)dx}{3-1}$ = 18/2 = 9	value of f over [1, 5] is 7/2. Find $\int_1^5 f(x)\,dx$
$\int udv = uv - \int vdu$	1/ Find $\int 4xe^{-2x}dx$ Solution. Let u = 4x, dv = $e^{-2x}dx$ ➔ du = 4dx, v = $\frac{-1}{2}e^{-2x}$ So, $\int 4xe^{-2x}dx = \int udv = uv - \int vdu$ $= -2xe^{-2x} + 2\int e^{-2x}dx = -2xe^{-2x} - e^{-2x} + C$ 2/ Find $\int_1^e 2x\ln x dx$ Solution. Let u = lnx, dv = 2xdx ➔ du = dx/x, v = x^2 $\int_1^e 2x\ln x dx = \int_1^e udv = uv\Big\|_1^e - \int_1^e vdu$ $= x^2\ln x\Big\|_1^e - \int_1^e xdx = x^2\ln x\Big\|_1^e - \frac{1}{2}x^2\Big\|_1^e$ $= e^2 - \frac{1}{2}(e^2 - 1) = \frac{1}{2}e^2 + ½$	**36/** Find the integrals: a/ $\int 2xe^{-x}dx$ b/ $\int 4x\ln(2x)dx$ Hint: u = ln(2x), dv = 4xdx ⇨ du = u'dx $= \frac{(2x)'}{2x}$ dx = dx/x c/ $\int \frac{2x}{\sqrt{x+2}}dx$ Hint: u = 2x, dv = $\frac{1}{\sqrt{x+2}}dx$ ⇨ v = $2\sqrt{x+2}$
$\int f(x)dx = \int g(t)dt$ by substitution t = u(x)	1/ Find $\int 2x(x^2+3)^9dx$ Solution. Let t = $x^2 + 3$ ➔ dt = 2xdx So, $\int 2x(x^2+3)^9dx = \int t^9dt = \frac{t^{10}}{10} + C$ $= \frac{(x^2+3)^{10}}{10} + C$ 2/ Find $\int \frac{\ln x}{x}dx$ Solution. Let t = lnx ➔ dt = dx/x So, $\int \frac{\ln x}{x}dx = \int t\,dt = \frac{1}{2}t^2 + C = \frac{1}{2}(\ln x)^2 + C$	**37/** Evaluate the integrals: a/ $\int_0^2 3x^2\sqrt{x^3+1}\,dx$ b/ $\int \frac{2}{\sqrt{x}}(1+\sqrt{x})^3\,dx$ Hint: t = $1+\sqrt{x}$
$\frac{d}{dx}(\int_{v(x)}^{u(x)} f(t)dt)$ = u'(x).f(u(x)) – v'(x).f(v(x))	1/ Find $$\frac{d}{dx}(\int_x^{x^2} \sqrt{1+t^2}\,dt)$$ Solution. $\frac{d}{dx}(\int_x^{x^2} \sqrt{1+t^2}\,dt) = (x^2)'\sqrt{1+x^4} - (x)'\sqrt{1+x^2}$ $= 2x\sqrt{1+x^4} - \sqrt{1+x^2}$ 2/ Find g'(x) and g'(2) if g(x) = $\int_x^{3x}(1+t)\,dt$ Solution. g(x) = $\int_x^{3x}(1+t)\,dt$ ⇨ g'(x) = $\frac{d}{dx}[\int_{x^2}^{3x}(1+t)\,dt]$ = 3(1+ 3x) – 2x(1+ x^2) = $-2x^3 + 7x + 3$ Hence, g'(2) = -1	**38/** Find the following derivatives: a/ $$\frac{d}{dx}(\int_{\sin x}^{x}(1+t)\,dt)$$ b/ Suppose $\int_1^x f(t)dt = x\sqrt{x} - 3$, find f(x). Hint: If g(x) = $\int_1^x f(t)dt$, then g'(x) = f(x)
improper integral: Test for **convergence** or **divergence**	1/ Which of the following integrals are convergent? a/ $\int_1^\infty \frac{1}{x\sqrt{x}}dx$ b/ $\int_3^\infty \frac{x+3}{x^2+x\sqrt{x}}dx$	**39/** Which of the following integrals are convergent? a/ $\int_1^\infty \frac{1}{\sqrt{x}}dx$

$\int_a^{\infty} \frac{1}{x^p}dx$	c/ $\int_0^{\infty} e^{\frac{-x}{2}}dx$ Solution. a/ $\int_1^{\infty} \frac{1}{x\sqrt{x}}dx = \int_1^{\infty} \frac{1}{x^{3/2}}dx$ converges (p = 3/2 > 1) b/ $\int_3^{\infty} \frac{x+3}{x^2+x\sqrt{x}}dx \sim \int_3^{\infty} \frac{x}{x^2}dx = \int_3^{\infty} \frac{1}{x}dx$ diverges (p = 1) c/ $\int_0^{\infty} e^{\frac{-x}{2}}dx = -2e^{\frac{-x}{2}} \Big\|_0^{\infty} = -2(e^{-\infty} - e^0) = -2(0-1) = 2$	b/ $\int_2^{\infty} \frac{x+x\sqrt{x}}{x^3+1}dx$ c/ $\int_0^{\infty} e^{-2x}dx$
improper integral (type 2)	Evaluate the improper integral or say it diverges $$\int_1^5 \frac{1}{\sqrt{5-x}}dx$$ Solution. • $\int_1^{t} \frac{1}{\sqrt{5-x}}dx = \int_1^{t} \frac{1}{(5-x)^{0.5}}dx = \int_1^{t}(5-x)^{-0.5}dx$ $= -\frac{(5-x)^{-0.5+1}}{-0.5+1}\Big\|_1^t = -2\sqrt{5-x}\,\Big\|_1^t$ $= -2\sqrt{5-t} + 2\sqrt{5-1} \to 4$ when $t \to 5$ • $\int_1^{5} \frac{1}{\sqrt{5-x}}dx = 4$	**40/** Evaluate each of these improper integrals or say it diverges a/ $\int_1^{5} \frac{1}{\sqrt{x-1}}dx$ b/ $\int_0^{4} \frac{1}{x^{0.8}}dx$

END OF PART I – CALCULUS

LINEAR ALGEBRA – KEY TERMS & MAIN RESULTS

Key terms	Problems with solutions	Exercises - Do yourself
Chapter 1. Systems of Linear Equations		
Reduced row-echelon form	**Ex.** Find x and y such that the matrix $\begin{pmatrix} 1 & 1 & -1 & 3 & 5 \\ 0 & x & 1 & 0 & -2 \\ 0 & y & x & 2 & 1 \end{pmatrix}$ is a ***reduced row-echelon matrix***. **Solution.** Consider row 2, two possible cases for x's value: 0 or 1 • x = 0 ➔ y = 0, so the 3rd row becomes [0 0 0 2 1], which is impossible. • x = 1 ➔ y = 0 and row 3 is [0 0 1 2 1], which is possible. Conclusion: x = 1 and y = 0.	**1/** Find x and y such that the matrix $\begin{pmatrix} 1 & 1 & -1 & 3 & 5 \\ 0 & y & x & 0 & -2 \\ 0 & 0 & x & 1 & -1 \end{pmatrix}$ is a ***reduced row-echelon matrix***.
Consistent and inconsistent system	**Ex1.** ***Solve*** the system x + 2y + 3z = 0 2x + 4y – z = 0 x + 2y – z = 0 **Solution.** • Step 1. Carry augmented matrix to reduced row-echelon form:	**2/** a/ ***Solve*** the system x - y + 2z = 0 -x + y – z = 0 b/ Solve the system corresponding to the augmented matrix

	$\left(\begin{array}{ccc\|c}1&2&3&0\\2&4&-1&0\\1&2&-1&0\end{array}\right)\xrightarrow{\substack{-2r_1+r_2\\-r_1+r_3}}\left(\begin{array}{ccc\|c}1&2&3&0\\0&0&-7&0\\0&0&-4&0\end{array}\right)$ $\xrightarrow{-\frac{1}{7}r_2}\left(\begin{array}{ccc\|c}1&2&3&0\\0&0&1&0\\0&0&-4&0\end{array}\right)\xrightarrow{4r_2+r_3}\left(\begin{array}{ccc\|c}1&2&3&0\\0&0&1&0\\0&0&0&0\end{array}\right)$ $\xrightarrow{-3r_2+r_1}\left(\begin{array}{ccc\|c}1&2&0&0\\0&0&1&0\\0&0&0&0\end{array}\right)$ • Step 2. From the last matrix, the system has infinitely many solution described as below: y = t (parameter = any number) // no leading one with respect to y z = 0 x = -2t • Step 3. Conclusion: solution set is {(-2t, t, 0) where t is arbitrary} **Ex2.** Find all values of m such that the system $\begin{cases}x-y+2z=2\\-2x+y-z=-1\\x+y+mz=0\end{cases}$ has ***unique solution***. **Solution.** $\left[\begin{array}{ccc\|c}1&-1&2&2\\-2&1&-1&1\\1&1&m&0\end{array}\right]\to\left[\begin{array}{ccc\|c}1&-1&2&2\\0&-1&3&5\\0&2&m-2&-2\end{array}\right]$ $\to\left[\begin{array}{ccc\|c}1&-1&2&2\\0&1&-3&-5\\0&2&m-2&-2\end{array}\right]\to\left[\begin{array}{ccc\|c}1&-1&2&2\\0&1&-3&-5\\0&0&m+4&8\end{array}\right]$ From the last matrix, the system has unique solution when m + 4 ≠ 0 Conclusion: m ≠ -4.	$\begin{bmatrix}1&0&-1&2\\0&1&-1&0\\0&0&-2&10\\0&0&0&0\end{bmatrix}$ **3/** Find all values of m such that the system $\begin{cases}x-y+2z=-1\\-y+z=1\\x-y+mz=0\end{cases}$ has ***unique solution***.
Rank of a matrix r(A)	**Ex.** Find the ***rank*** of the matrix. $A=\begin{bmatrix}0&-2&1&3\\1&-2&-1&1\\-2&2&3&1\end{bmatrix}.$ **Solution.** In general, carry A to a row echelon matrix, and	**4/** Find the ***rank*** of the matrix. $A=\begin{bmatrix}1&-2&1&-3\\-2&0&-1&1\\2&2&-2&3\end{bmatrix}.$

	rank(A) = number of leading ones. $\begin{bmatrix} 0 & -2 & 1 & 3 \\ 1 & -2 & -1 & 1 \\ -2 & 2 & 3 & 1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & -2 & -1 & 1 \\ 0 & -2 & 1 & 3 \\ -2 & 2 & 3 & 1 \end{bmatrix}$ $\xrightarrow{2r_1+r_3} \begin{bmatrix} 1 & -2 & -1 & 1 \\ 0 & -2 & 1 & 3 \\ 0 & -2 & 1 & 3 \end{bmatrix} \xrightarrow{-r_2+r_3} \begin{bmatrix} 1 & -2 & -1 & 1 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ We can see the latest matrix can be carried to a row echelon matrix with ***2 leading ones***. So, rank(A) = 2.	
the number of free parameters p = n – r of a homogeneous system	**Ex.** A homogeneous system has the coefficient matrix of **rank 8**. If there are 11 linear equations involving **13 variables (or unknowns)** in the system, then how many ***free parameters*** in the solution set of the system? **Solution.** p: number of parameters n: number of variables r = rank of the coefficient matrix p = n – r = 13 – 8 = 5.	5/ A homogeneous system has the coefficient matrix of **rank 7**. If there are 13 linear equations involving **15 variables (or unknowns)** in the system, then how many ***free parameters*** in the solution set of the system?
Chapter 2-3. Matrix Algebra		
Matrix addition A + B, scalar multiplication (k.A) and transpose A^T **Matrix multiplication A·B** **Matrix inverse A^{-1}**	**Ex1.** Given A = $A = \begin{bmatrix} -2 & 1/2 & 3 \\ 3/2 & -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ -3 & 2 \\ 1 & 5 \end{bmatrix}$ Find 2A – B^T. **Solution.** $2A = \begin{bmatrix} -4 & 1 & 6 \\ 3 & -2 & 0 \end{bmatrix}$ $B^T = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 2 & 5 \end{bmatrix}$ So, $2A - B^T = \begin{bmatrix} -4 & 1 & 6 \\ 3 & -2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -3 & 1 \\ 0 & 2 & 5 \end{bmatrix}$ $= \begin{bmatrix} -5 & 4 & 5 \\ 3 & -4 & -5 \end{bmatrix}$ **Ex2.** Find $(2A)^{-1}$ if $A = \begin{bmatrix} -1 & 3 \\ 2 & 3 \end{bmatrix}$ **Solution.**	6/ Find A^{-1} if $A = \begin{bmatrix} -1 & -2 \\ 3 & 3 \end{bmatrix}$ 7/ Find A if $\left(A^T - 2I\right)^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$.

	$A=\begin{bmatrix}-1 & 3\\ 2 & 3\end{bmatrix} \Rightarrow 2A=\begin{bmatrix}-2 & 6\\ 4 & 6\end{bmatrix}$ $\Rightarrow (2A)^{-1}=\frac{1}{-36}\begin{bmatrix}6 & -6\\ -4 & -2\end{bmatrix}=\begin{bmatrix}-1/6 & 1/6\\ 1/9 & 1/18\end{bmatrix}$ **Another way**. $A=\begin{bmatrix}-1 & 3\\ 2 & 3\end{bmatrix} \Rightarrow A^{-1}=\frac{1}{-9}\begin{bmatrix}3 & -3\\ -2 & -1\end{bmatrix}$ $\Rightarrow (2A)^{-1}=\frac{1}{2}A^{-1}=\begin{bmatrix}-1/6 & 1/6\\ 1/9 & 1/18\end{bmatrix}$ **Ex3.** Find A if $\left(A^T-2I\right)^{-1}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$. **Solution.** $\left(A^T-2I\right)^{-1}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix} \Leftrightarrow A^T-2I=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$ $A^T=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}+2I=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}+2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ $A^T=\begin{bmatrix}6 & -3\\ -1 & 3\end{bmatrix}$ $\Rightarrow A=\begin{bmatrix}6 & -1\\ -3 & 3\end{bmatrix}$	
Invertible and determinant	**Ex.** Find all values of x such that the matrix $\begin{bmatrix}-1 & 2 & 1\\ 2 & 0 & -3\\ 1 & x & 1\end{bmatrix}$ has **an inverse**. **Solution.** $\det\begin{bmatrix}-1 & 2 & 1\\ 2 & 0 & -3\\ 1 & x & 1\end{bmatrix}=-x-10$ A has an inverse iff det(A) ≠ 0 ⇔ x ≠ -10.	**8/** Find all values of x such that the matrix $\begin{bmatrix}1 & -2 & 1\\ -2 & 0 & 3\\ -1 & 2 & x\end{bmatrix}$
Linear transformations $T\left(a\vec{u}+b\vec{v}\right)$ $=aT\left(\vec{u}\right)+bT\left(\vec{v}\right)$	**Ex.** Let T: R^2 → R^2 be a **linear transformation** such that T(u) = (-1, 2) and T(v) = (-1, 1) Find T(2u – 3v). **Solution.** T(2u – 3v) = 2T(u) – 3T(v) = 2(-1, 2) -3(-1, 1) = (1, 1).	**9/** Let T: R^2 → R^2 be a **linear transformation** such that T(u) = (1, -2) and T(v) = (1, 2) Find T(3u – 2v).
Determinants of 2x2, 3x3, 4x4 matrices det(A)	**Ex.**	**10/** Given

<table>
<tr>
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<td>Find $\begin{vmatrix} a & -2 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & -2 \end{vmatrix}$
Solution.
$\begin{vmatrix} a & -2 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & -2 \end{vmatrix} = -2(-1)^{3+3}\det\begin{bmatrix} a & -2 \\ 1 & 1 \end{bmatrix} = -2(a+2)$</td>
<td>$A = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & k \\ 0 & 1 & -3 \end{bmatrix}$
a/ Find det(A).
b/ Find k such that A has an inverse.</td>
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<tr>
<td>Properties of determinants</td>
<td>Ex. Suppose A and B are 3x3 matrices such that |A| = 3, |B| = -6.
a/ Find $\lvert 2AB^{-1}\rvert$
b/ Find $\lvert 3A^TBA^{-2}\rvert$
Solution.
a/ $\lvert 2AB^{-1}\rvert = 2^3\lvert A\rvert\frac{1}{\lvert B\rvert} = \frac{8\cdot 3}{-6} = -4$
b/ $\lvert 3A^TBA^{-2}\rvert =$
$3^3\lvert A^T\rvert\lvert B\rvert\frac{1}{\lvert A\rvert^2} = 3^3\lvert A\rvert\lvert B\rvert\frac{1}{\lvert A\rvert^2} = \frac{3^3\cdot(-6)}{3} = -54$</td>
<td>11/ Suppose A and B are 4x4 matrices such that |A| = -2, |B| = 3.
a/ Find $\lvert 2AB^T\rvert$
b/ Find $\lvert A^2B^{-1}A^{-1}\rvert$</td>
</tr>
<tr>
<td>(i, j)-cofactor and A^{-1}.

$(-1)^{i+j}$ det(delete row i, delete column j)</td>
<td>Ex. Find (2, 3)-cofactor and (3, 1)-cofactor of A if
$A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix}$
Solution.
(2, 3)-cofactor = $c_{23} = (-1)^{2+3}\det\begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix} = 4$
(3, 1)-cofactor = $c_{31} = (-1)^{3+1}\det\begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix} = -6$</td>
<td>12/ Find (2, 3)-cofactor and (3, 1)-cofactor of A if
$A = \begin{bmatrix} -1 & 2 & 5 \\ 2 & 3 & -1 \\ -1 & 0 & 2 \end{bmatrix}$</td>
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<tr>
<td>Adjugate matrix</td>
<td>Ex.
Find the first row of the adjugate of $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 5 \\ 0 & 1 & -1 \end{bmatrix}$
Solution.</td>
<td>13/ Find the second row of the adjugate matrix of
$A = \begin{bmatrix} -1 & 2 & 5 \\ 2 & 3 & -1 \\ -1 & 0 & 2 \end{bmatrix}$</td>
</tr>
</table>

	Find the first row of the adjugate of $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 5 \\ 0 & 1 & -1 \end{bmatrix}$ Solution. The first row of **adjugate matrix** of A is **cofactors** c_{11}, c_{21}, c_{31}: $c_{11} = (-1)^{1+1} \det\begin{pmatrix} 0 & 5 \\ 1 & -1 \end{pmatrix} = -5$ $c_{21} = (-1)^{2+1} \det\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = 1$ $c_{31} = (-1)^{3+1} \det\begin{pmatrix} -1 & 2 \\ 0 & 5 \end{pmatrix} = -5$ The first row of adj(A) is: [-5 1 -5].	
eigenvalues	**Ex.** Find all **eigenvalues** of the matrix of the matrix $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix}$ **Solution.** • det(xI – A) = $\begin{vmatrix} x-1 & -1 & 1 \\ 0 & x & 1 \\ 0 & -2 & x+3 \end{vmatrix} = (x-1)\left[x(x+3)+2\right]$ $= (x-1)(x^2+3x+2)$ • det(xI – A) = 0 ⇔ x = 1, x = -1, x = -2. • Eigenvalues: 1, -1, -2 * Note that for a multiple choice question: first we can find det(A) = 2. Then, choose options which the product of values is 2 a) $2, 3, 4$ b) $-3, 3, 4$ d) $-3, 0, 4$ e) $-1, -2, 1$ e) is the possible option because det(A) = 2 = (-1)(-2).1	**14/** Find all **eigenvalues** of the matrix of the matrix $\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$ Choose one from options below (i) -2, 1, 3 (ii) 2, -1, -3 (iii) -1, -2, 3 (iv) 2, 1, -1
Chapter 5. The Vector Space R^n		
Linear independence, Linear dependence	**Ex1.** Find all values of x such that the set {(1, 0, -2); (-2, 1, 1); (1, -3, x)} is linearly **independent**. **Solution.** • We solve the system for a, b, c a(1, 0, -2) + b(-2, 1, 1) + c(1, -3, x) = (0, 0, 0) Equivalently, in augmented matrix	**15/** Find all values of x such that the set {(1, -1, 2); (-2, 0, 1); (-1, x, 3)} is linearly **independent**. **16/** Find all values of *a* such that the set {(1, 1, 0); (2, 1, 3); (-1, 0, *a*)} is linearly **dependent**.

	$\left[\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ -2 & 1 & x & 0 \end{array}\right]$ • Carry the matrix to row-echelon form $\left[\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ -2 & 1 & x & 0 \end{array}\right] \to \left[\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -3 & x-2 & 0 \end{array}\right]$ $\to \left[\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & x-11 & 0 \end{array}\right]$ We want the set is linearly independent, so the system must have solution a = 0, b = 0, c = 0. ⇨ x – 10 ≠ 0. **Ex2.** Find all values of a such that the set {(1, -1, 1); (2, 1, 3); (-1, a, 2)} is linearly **dependent**. **Solution.** Similar to the previous example, solve the system $\left[\begin{array}{ccc\|c} 1 & 2 & -1 & 0 \\ -1 & 1 & 2 & 0 \\ 1 & 3 & a & 0 \end{array}\right] \to \left[\begin{array}{ccc\|c} 1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & a+1 & 0 \end{array}\right]$ $\to \left[\begin{array}{ccc\|c} 1 & 2 & -1 & 0 \\ 0 & 1 & a+1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc\|c} 1 & 2 & -1 & 0 \\ 0 & 1 & a+1 & 0 \\ 0 & 0 & 1-3(a+1) & 0 \end{array}\right]$ $\to \left[\begin{array}{ccc\|c} 1 & 2 & -1 & 0 \\ 0 & 1 & a+1 & 0 \\ 0 & 0 & -3a-2 & 0 \end{array}\right]$ We need values of a such that the set is linearly **dependent** ➔ -3a – 2 = 0 ⇔ a = -2/3.	
Spanning sets, span	**Ex.** Given U = span{(-1, 0, 1); (2, -1, 1)}. a/ Does the vector (1, -2, 3) belong to U? b/ Find all values of m such that (-2, 2, m)∈ U. **Solution.** a/ We want to find a, b such that (1, -2, 3) = a(-1, 0, 1) + b(2, -1, 1) Or equivalent, 1 = -a + 2b (1) -2 = 0a – b (2) 3 = a + b (3) Solve for a, b from (1), (2) ➔ a = 3, b = 2 ⇨ (3) becomes: 3 = 5 (!)	17/ Given U = span{(1, -1, 0); (-2, 1, 1)}. Find all values of m such that (0, -1, m)∈ U.

	Conclusion: vector (1, -2, 3) does not belong to U. b/ (-2, 2, m)$\in$ U if and only if the system (-2, 2, m) = a(-1, 0, 1) + b(2, -1, 1) has solution a, b. Or equivalent, -2 = -a + 2b (1) 2 = 0a – b (2) m = a + b (3) Solve for a, b from (1), (2) ➔ a = -2, b = -2 ⇨ (3) becomes: m = -4 Conclusion: m = -4	
Basis of a vector space, Dimension	**Ex1.** Given U = span{(1, 2, 1); (3, 2, 0); (-1, 2, 2)}. Find the **dimension** of U (find dim(U)). **Solution.** First, check for independence of the set {(1, 2, 1); (3, 2, 0); (-1, 2, 2)} $\left[\begin{array}{ccc\|c} 1 & 3 & -1 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc\|c} 1 & 3 & -1 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$ $\to \left[\begin{array}{ccc\|c} 1 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc\|c} 1 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ From the last matrix, the set is NOT INDEPENDENT. Only **two vectors** make an independent set ➔ Two vectors are chosen to form a basis of U ➔ dim(U) = 2. **Ex2.** Find all values of x such that dim(V) = 2 where V = span{(1, -1, 2); (-1, 0, 3); (2, -3, x)}. **Solution.** $\left[\begin{array}{ccc\|c} 1 & -1 & 2 & 0 \\ -1 & 0 & -3 & 0 \\ 2 & 3 & x & 0 \end{array}\right] \to \left[\begin{array}{ccc\|c} 1 & -1 & 2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 5 & x-4 & 0 \end{array}\right]$ $\left[\begin{array}{ccc\|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & x-9 & 0 \end{array}\right]$ dim(V) = 2 if and only if x = 9.	**18/** Given U = span{(1, 2, 0); (-3, 1, 1); (1, 3, -1)}. Find the **dimension** of U (find dim(U)). **19/** Given U = span{(1, 2, 0, 1); (-3, 0, 1, -2); (1, 1, -1, 3)}. Find the **dimension** of U (find dim(U)).
Column space Col(A) and row space row(A)	**Ex.** Find **dim(col(A))** if A = $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & -2 & 3 \\ -2 & 6 & -4 & 0 \end{bmatrix}$. **Solution.**	**20/** Find **dim(col(A))** if A = $\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -2 & 1 \\ 2 & -3 & 4 & 2 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & -2 & 3 \\ -2 & 6 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 2 & -4 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

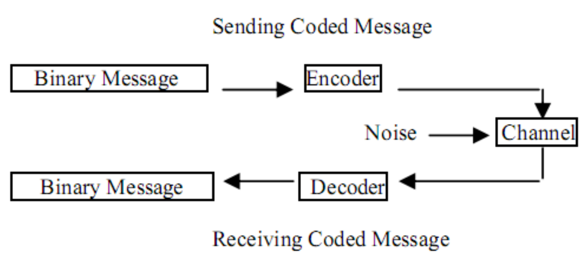
⇨ Dim(col(A)) = rank(A) = 3.

Applications.

Error correction. Hamming code.

1/ First, let consider the **addition mod 2**.
Addition mod 2 is described by the addition rules:
0 + 0 = 0
0 + 1 = 1
1 + 0 = 1
1 + 1 = 0 (because 2 mod 2 = 0),
which you can think of as “even plus even is even", “even plus odd is odd," etc.

2/ Hamming codes.



In 1950, Richard Hamming provided a method to send messages with error-detecting and –correcting. We focus on what is known as the “(7; 4) Hamming code", which takes each group of four bits of the sender's message and encodes it as seven bits.
Suppose the message we wish to send consists of 4 bits $x_1; x_2; x_3; x_4$, denoted by a column vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

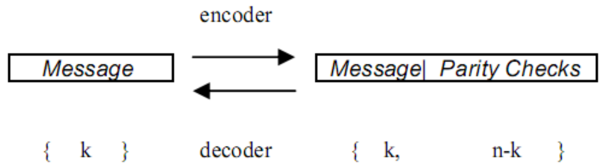
(we will write [x_1 x_2 x_3 x_4] to reduce space)

Three extra bits, called parity bits, are added to these four bits to obtain a 7-bit string. A code generator matrix G, called the parity matrix, is used to construct this 7-bit string.

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

The user sends $G\vec{x}$ (a 7x1 vector) in place of the original message $\vec{x}$.
The matrix G is chosen precisely so that the first four entries of $G\vec{x}$ are x_1; x_2; x_3; x_4, and the last three entries are the parity bits $x_1 + x_3 + x_4$; $x_1 + x_2 + x_4$; and $x_2 + x_3 + x_4$ (mod 2).
For example, if the message is [0 1 0 1], then the parity bits are $x_1 + x_3 + x_4 = 0+0+1=1$, $x_1 + x_2 + x_4 = 0+1+1 = 0$ (mod 2), $x_2 + x_3 + x_4 = 1+0+1 = 0$.

So in this case, the encoded message $G\vec{x}$ would be [0 1 0 1 1 0 0] and it will be sent instead of [0 1 0 1].



Next, if the recipient receives some message of seven bits, call it $\vec{c} = (c_1, c_2, c_3, c_4, c_5, c_6, c_7)$. They expect the original message will just be the first four entries of $\vec{c}$, but they must check for errors.
The recipient is checking to see if the following three equations are satisfied
$c_1 + c_3 + c_4 + c_5 = 0$ (in fact, $c_1 + c_3 + c_4 = c_5$ but these two equations are the same in mod 2)
$c_1 + c_2 + c_4 + c_6 = 0$
$c_2 + c_3 + c_4 + c_7 = 0$.
The coefficient matrix of the above linear system is

$$P = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix},$$

which we will again call the parity check matrix.
Thus, the recipient computes $P\vec{c}$ and obtains one of eight possible outcomes. Each outcome tells the recipient about the correctness of bits received.
For example, if $\vec{c}$ = [0 1 0 1 1 0 0], then $P\vec{c}$ = [0 0 0], it follows that the first four bit of $\vec{c}$ is correct.
In general, if the outcome of $P\vec{c}$ is the i^{th} column of P, then the i^{th} bit in $\vec{c}$ is not correct, it must be changed from 0 to 1 or from 1 to 0.

Exercises:
1/
Suppose the original message is [1 0 1 1 0 1 1 0]. What is the encoded message after using the (7,4) Hamming code? (Hint: first divide the message into two parts, each contains four bits).
2/ Suppose you receive a message [0 1 1 0 1 0 0]. Check for the correctness of the first four bits. If there is an error, say which is the incorrect bit?

END OF PART II – LINEAR ALGEBRA