Q4.10

Find the SVD of $B = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

Solutions:

First, find the transpose of B, $B^T = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$

Multiply the transposed matrix with initial:

$$Y = B^T \cdot B = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

Now, find the eigenvalues and eigenvectors of Y:

$$det(Y - \lambda I) = \begin{vmatrix} 13-\lambda & 12 & 2 \\ 12 & 13-\lambda & -2 \\ 2 & -2 & 8-\lambda \end{vmatrix} = 0$$

$$\iff (-1)^{1+1} \times (13-\lambda) \times \begin{vmatrix} 13-\lambda & -2 \\ -2 & 8-\lambda \end{vmatrix} + (-1)^{1+2} \times 12 \times \begin{vmatrix} 12 & -2 \\ 2 & 8-\lambda \end{vmatrix} + (-1)^{1+3} \times 2 \times \begin{vmatrix} 12 & 13-\lambda \\ 2 & -2 \end{vmatrix} = 0$$

$$\iff -\lambda^3 + 34\lambda^2 - 225\lambda = 0$$

$$\implies \lambda = \{25, 9, 0\}$$

With $\lambda_1 = 25$, we have eigenvector:

$$\begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, t \neq 0$$

With $\lambda_2 = 9$, we have eigenvector:

$$\begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1/4 \\ -1/4 \\ 1 \end{bmatrix}, t \neq 0$$

With $\lambda_3 = 0$, we have eigenvector:

$$\begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, t \neq 0$$

Find the square roots of nonzero eigenvalues (denoted σ_i) :

$\sigma_1 = \sqrt{\lambda_1} = 5$

$\sigma_2 = \sqrt{\lambda_2} = 3$

The Σ matrix is a zero matrix with σ_i on its diagonal: $\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

The columns of the matrix V are the normalized (unit) vectors:

$$V = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/6 & -2/3 \\ \sqrt{2}/2 & -\sqrt{2}/6 & 2/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{bmatrix}$$

Find $u_i = \frac{1}{\sigma_i} \cdot \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \cdot v_i$:

$$u_1 = \frac{1}{\sigma_1} \cdot \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \cdot v_1 = \frac{1}{5} \cdot \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} \cdot \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \cdot v_2 = \frac{1}{3} \cdot \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2}/6 \\ -\sqrt{2}/6 \\ 2\sqrt{2}/3 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

Therefore, $U = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}$

The matrices U, Σ, and V are such that the initial matrix $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = U \Sigma V^T$

with:

$$U = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/6 & -2/3 \\ \sqrt{2}/2 & -\sqrt{2}/6 & 2/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{bmatrix}$$