Chapter 7. Continuous optimization

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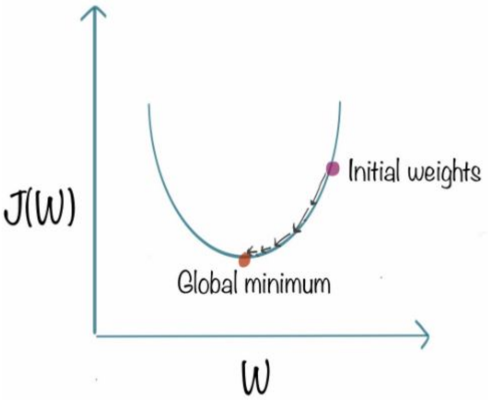
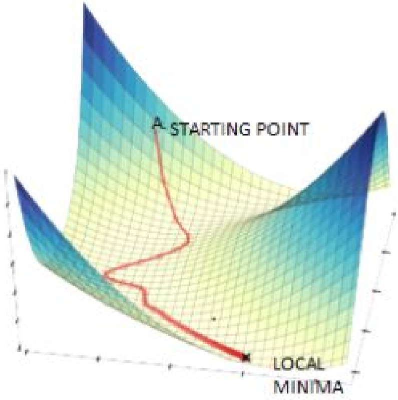
7.1 Optimization Using Gradient Descent





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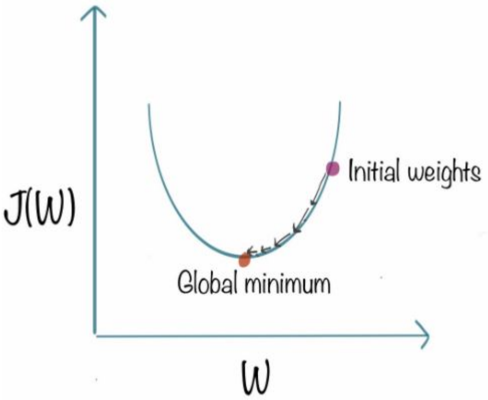

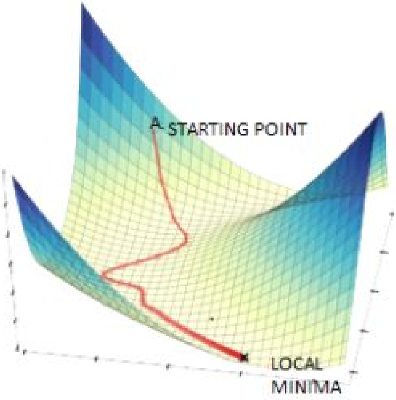

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$$\min_x f(x),$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable.

Gradient descent is a first-order optimization algorithm. To find a local minimum of a function using gradient descent, one takes steps proportional to the negative of the gradient of the function at the current point.

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We can see that, if

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for a small step size (learning rate) $\gamma > 0$ then $f(x_1) \leq f(x_0)$.

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$$x_{i+1} = x_i - \gamma_i \left((\nabla f)(x_i) \right)^T.$$

For suitable learning rate γ_i, the sequence $f(x_0) \geq f(x_1) \geq \cdots$ converges to a local minimum.

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Example

Consider the problem of minimizing $f(x, y) = 2x^2 - 2xy + 1.5y^2$ using the gradient descent method.

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$$\nabla f(x, y) = \begin{bmatrix} 4x - 2y \\ -2x + 3y \end{bmatrix}$$

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We obtain

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Exercise. Write a python programe to find the local minima of the function

$$f(x) = x^2 - 2x + 8$$

starting from point $x_0 = 2$ by the gradient descent algorithm. Let us assume the learning rate 0.01 and the difference between x values from two consecutive iterations is less than 0.001.

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Gradient descent with momentum

Remembers the update $\Delta x_i = x_i - x_{i-1}$ at each iteration i and determines the next update as a linear combination of the current and previous gradients:

$$x_{i+1} = x_i - \gamma_i \left((\nabla f)(x_i) \right)^T + \alpha \Delta x_i$$
$$\Delta x_i = x_i - x_{i-1} = \alpha \Delta x_{i-1} - \gamma_{i-1} \left((\nabla f)(x_{i-1}) \right)^T .$$

Consider the constrained optimization problem

$$\min f(x)$$
$$\text{subject to } g_i(x) \leq 0, i = 1, \ldots, m$$

We can convert the constrained problem into an unconstrained one:

$$J(x) = f(x) + \sum_{i=1}^{m} \mathbb{1}(g_i(x)),$$

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The problem

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A set $\mathcal{C}$ is convex if for any $x, y \in C$ then the line segment joining x, y lie inside $\mathcal{C}$, i.e

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Definition

Let $f:\mathbb{R}^D \to \mathbb{R}$ be a function whose domain is a convex set. The function f is a convex function if for all x, y in the domain of f, we have

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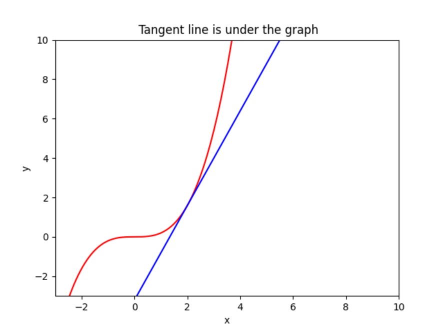
The function $f(x) = x \ln x$ is convex on $x > 0$.



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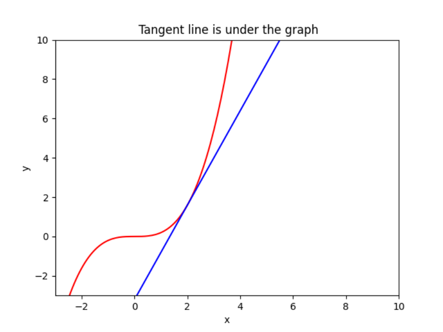

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The function $f(x, y) = x^2 + y^2$ is convex on $\mathbb{R}^2$since its Hessian

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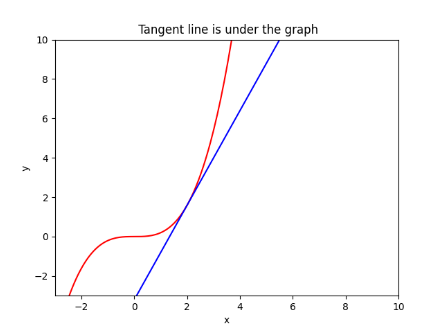

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A constrained optimization problem is called a convex optimization problem if

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where functions $f(x), g_i(x)$ are convex functions and $\{x : h_j(x) = 0\}$ are convex sets.

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7.3.1 Linear programming

We consider a special cases of convex optimization problem called linear program

$$\min_{x \in \mathbb{R}^n} c^T x \tag{1}$$

$$\text{subject to } Ax \le b$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. The Lagrangian is given by

$$\begin{aligned}\mathcal{L}(x, \lambda) &= c^T x + \lambda^T (Ax - b) \\ &= \left(c^T + A^T \lambda\right)^T x - \lambda^T b.\end{aligned}$$

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Hence the dual Lagrangian

$$\mathcal{D}(\lambda) = -\lambda^T b.$$

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where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$.The Lagrangian is given by

$$\begin{aligned}\mathcal{L}(x, \lambda) &= c^T x + \lambda^T (Ax - b) \\ &= \left(c^T + A^T \lambda\right)^T x - \lambda^T b.\end{aligned}$$

We have

$$\frac{\partial \mathcal{L}(x, \lambda)}{\partial x} = c^T + A^T \lambda = 0.$$

Hence the dual Lagrangian

$$\mathcal{D}(\lambda) = -\lambda^T b.$$

The dual optimization

$$\max_{\lambda \in (\mathbb{R}^+)^m} -b^T \lambda \quad (2)$$

$$\text{subject to } c + A^T \lambda = 0.$$

This is also a linear program, but with m variables. We can solve the primal (1) or the dual (2) program depending on whether m or n is larger.

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7.3.2 Quadratic Programming

Consider a quadratic program:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T Q x + c^T x \qquad (3)$$
$$\text{subject to } Ax \leq b$$

where $Q \in \mathbb{R}^{n \times n}$ is positive definite, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$.

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The Lagrange is given by

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The partial derivative

$$\frac{\mathcal{L}(x,\lambda)}{\partial x} = Qx + (c + A^T\lambda) = 0 \Rightarrow x = -Q^{-1}(c + A^T\lambda)$$

We get the dual Lagrangian

$$\mathcal{D}(\lambda) = -\frac{1}{2}(c + A^T\lambda)^T Q^{-1}(c + A^T\lambda) - \lambda^T b.$$

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Therefore, the dual optimization problem is given by

$$\max_{\lambda\in(\mathbb{R}^+)^m} -\frac{1}{2}(c + A^T\lambda)^T Q^{-1}(c + A^T\lambda) - \lambda^T b$$

subject to $\lambda \geq 0$.

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Summary

We have studied:

- optimization using gradient method;
- constrained optimization;
- convex optimization.

Exercises for practice: 7.1-7.8 (page 247, 248).